PENDULUM WITH VIBRATING SUSPENSION (Translated from here)

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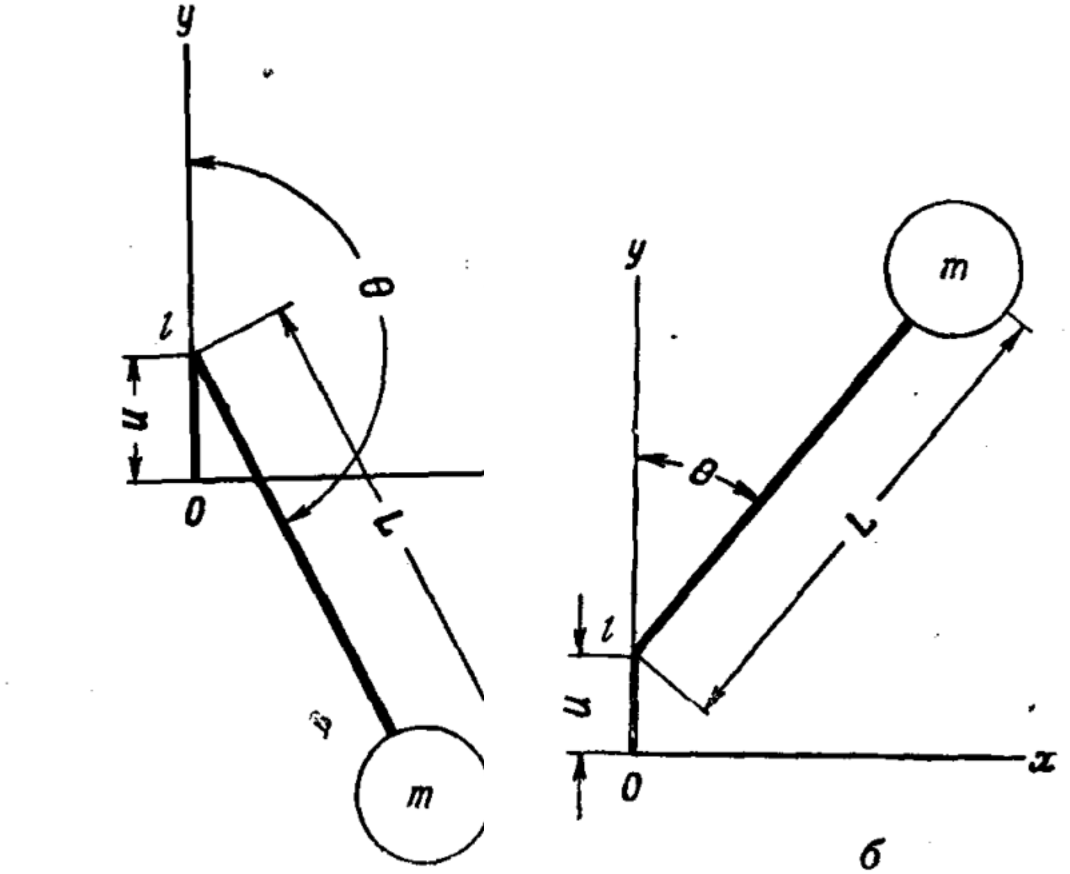
The development of mechanics is undoubtedly closely connected with the study of the pendulum. After Galileo drew attention to the isochronism of its oscillations, it became possible to create a very perfect mechanism for measuring time - a pendulum clock, the accuracy of which was only recently surpassed by quartz clocks. Thanks to the study of the pendulum, methods were found to measure time as accurately as length and mass were measured, which was necessary so that the development of mechanics could follow a solid path. Naturally, no mechanical system has received as much attention and comprehensive theoretical study as all varieties of pendulum motion. It would seem that in the 300 years that have passed since the time of Galileo, this question should have been exhausted and if anything remained for study, it should have been in the nature of polishing previously obtained results. But, apparently, the type of pendulum motion to which this article is devoted has not been given sufficient attention, and one of the very unique and interesting varieties of pendulum oscillations has remained almost completely unstudied. This article aims to draw attention to this type of motion and the possibilities that open up when studying it.

Fig. 1 shows a mathematical pendulum in two positions, which can oscillate at the suspension point; the mass m is concentrated at the end of the rod of length L. The position of the pendulum on the left side of the figure (1(a)), when the suspension point is above the center of gravity, we will call the normal position. In the figure on the right (1(b)), the suspension point of the pendulum is below the center of gravity; we will call this position the inverted position of the pendulum. The type of pendulum we shall consider has the peculiarity that the suspension point l moves along the y-axis near the origin O, with the distance being a periodic function of time; we also assume that the amplitude a is small compared to the length of the pendulum L. This is the well-known pendulum with an oscillating suspension.

In studying this type of pendulum, all attention was focused on the type of motion when the period of oscillation of the suspension T differed little from the period of oscillation of the pendulum itself τ. In this case, it was found that in those cases when 271 or a multiple of it is close to the period τ, the phenomenon of parametric resonance occurs. These studies were reduced to studying the properties of solutions of the Mathieu equation, which describes this motion at small oscillation amplitudes.



(a) Normal position (b) Inverted position

Figure 1: Pendulum in the normal position (left) and inverted position (right).

Furthermore, it was discovered that at small values of T compared to τ, the pendulum can acquire a special type of stability - it can stand without falling in an inverted position. The nature of the pendulum's motion in this position and the degree of its stability at high frequencies of oscillation of the suspension, apparently, remained completely unstudied. Thus, the beautiful and instructive phenomenon of the dynamic stability of the inverted pendulum not only was not included in modern manuals on mechanics, but is even almost unknown to a wide circle of specialists.

It can be assumed that such an undeserved attitude towards this phenomenon was a consequence of the fact that its study was connected with the solution of the Mathieu equation; it was produced by infinite determinants (Gill's method) or special functions, which led to a solution of a formal nature, not allowing the possibility of a visual description of the motion.

While studying this movement, I noticed that under the condition that the amplitude of the suspension oscillations a is small compared to the length of the pendulum L, there is a method for an approximate solution to the problem of movement that simply and clearly describes the phenomenon.

The ratio of the suspension oscillation amplitude to the pendulum length will be denoted by α:

$$\alpha = a/L \ll 1 \tag{1}$$

The value of α will play a very important role in this method, since the accuracy of the results obtained in the types of motion that interest us is mainly determined by it. We will mainly study those types of pendulum motion when the frequency of the suspension oscillations is high compared to the frequency of the pendulum oscillations, and, in addition, is not at all connected with it by any phase relationships, while the spectrum of the suspension oscillations themselves can be the sum of the spectrum and various frequencies. Therefore, in order to distinguish the motion we are studying from the motion of a pendulum with an oscillating suspension, we will call it the motion of a pendulum with a vibrating suspension.

The method we used to solve the problem is based on successive approximations together with the introduction of averaged coordinates over time. A detailed presentation of it and a study of the accuracy of the results obtained are given by us elsewhere.

Here, we will limit ourselves to a description of the main results obtained and the possibility of their practical application.

The method of successive approximation already at the first stage reduces the problem of the influence of a vibrating suspension point on the motion of a pendulum to a very simple physical picture: it turns out that this influence is equivalent to a moment of forces, which behaves in exactly the same way as a pair of ordinary forces, and tends to install the pendulum so that its mass is always in the direction of the vibrations of the suspension. We called this moment the vibration moment and denoted it by M. As will be seen from what follows, the introduction of the vibration moment makes solving the problems of the motion of this type of pendulums no more difficult than solving problems for ordinary pendulums. Below we will also describe a method for simply constructing a pendulum with a vibrating suspension, on which the obtained theoretical results can be demonstrated.

Let us write an equation for the general case of motion of the considered type of mathematical pendulum. If, as shown in Fig. 1, we denote the angle between the pendulum rod and the y-axis by θ, then the coordinates of the pendulum mass x and y will be

$$x = L\sin\theta; \quad y = U + L\cos\theta \tag{2}$$

where U represents the distance (along the y-axis) of the pendulum's suspension point l from the origin O. The forces acting on the mass m along the x and y axes we denote by F_x and F_y; then we obtain:

$$\begin{cases} F_x = m\ddot{x} = mL(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta), \\ F_y = m\ddot{y} = m\left[\ddot{U} - L\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\right]. \end{cases} \tag{3}$$

The moment of a pair of external forces acting on the mass of the pendulum[1], we will denote by M_θ; it will be equal to[2]:

$$M_\theta = L\left(F_x\cos\theta - F_y\sin\theta\right) \tag{4}$$

Substituting the values of F_x and F_y, we obtain:

$$\begin{aligned} M_\theta &= L\left(mL\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\cos\theta - m\left[\ddot{U} - L\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\right]\sin\theta\right) \\ &= mL^2\ddot{\theta} - mL\ddot{U}\sin\theta \end{aligned} \tag{5}$$

[1] Also called the "moment of Force", or simply the torque

[2] $\vec{r} = (x, y) = (L\sin\theta, U + L\cos\theta) \rightarrow \vec{r}\times\vec{F} = (L\sin\theta, U + L\cos\theta)\times(F_x, F_y) = L\left(F_x\cos\theta - F_y\sin\theta\right)$

This equation can easily be generalized for the case of a physical pendulum. To do this, consider m as an elementary mass and integrate the right-hand side of equation 5 over the entire volume of the pendulum mass; then instead of 5 we get[3]:

$$M_\theta = m\left(L^2 + K^2\right)\ddot{\theta} - mL\ddot{U}\sin\theta \tag{6}$$

where θ and L are the coordinates of the center of gravity of the pendulum mass, and K is the radius of inertia of the pendulum. Let the pendulum suspension perform simple harmonic oscillations with amplitude a and angular frequency ω; then we have:

$$U = a\sin(\omega t) \tag{7}$$

Differentiating this expression twice with respect to time and substituting the value of U into (6), we obtain:

$$M_\theta = m\left(L^2 + K^2\right)\ddot{\theta} + mLa\omega^2\sin(\omega t)\sin\theta \tag{8}$$

In the particular case when the moment of external forces is created by the force of gravity, it is equal to:

$$M_\theta = mgL\sin\theta \tag{9}$$

and the equation of motion will take the form:

$$\ddot{\theta} = \frac{L}{(L^3 + K^3)}(g - a\omega^2\sin\omega t)\sin\theta \tag{10}$$

This equation is usually simplified by limiting the problem to the consideration of small values of the angle θ and replacing the value of $\sin\theta$ with it. With this simplification, we obtain the Mathieu equation, with the help of which the problem of the motion of a pendulum with an oscillating suspension has been studied so far. When we apply the method of successive approximation, the consideration of the problem is not limited to small angles θ. The basic idea of this method is the assumption that during the period of

[3] Another way to think of this is to realize that the simple pendulum has a moment of inertia $I_\theta = mL^2$ and thus in the physical pendulum case where the center of mass is a distance K from the mass m the new moment of inertia becomes by the parallel-axis theorem (which is where this volume integral idea comes from) is $I_\theta + mK^2 = m(L^2 + K^2)$

rapid oscillation of the suspension the angle θ will change little, remaining close to some value φ. We assume:

$$\theta = \varphi + \beta \tag{11}$$

The angle β is a periodic quantity, but its value over the period of oscillation T always remains small. The angle φ can have any value, but over the same time T it will change little. If we average these quantities over time over the period T and denote this averaging by a line, then we have:

$$\overline{\theta} \cong \varphi;\ \overline{\beta} \cong \theta \tag{12}$$

When studying the motion of a pendulum with a vibrating suspension, we are mainly interested in the change in angle φ, which represents the position around which small vibrations occur.

Therefore, the solution method is constructed in such a way that by averaging the angle β is eliminated from the equation and θ is replaced by the angle φ. It turns out that this can be done if the problem is reduced to a motion in which a vibrational moment is involved, equal to (for a physical pendulum)[4]:

$$\overline{M} = -\frac{1}{4}(1 + K^2/L^2)^{-1} m a^2 \omega^2 \sin(2\varphi) \tag{13}$$

[4]From equation 8: $M_\theta = m\left(L^2 + K^2\right)\ddot{\theta} + mLa\omega^2 \sin(\omega t)\sin\theta$, $\theta = \varphi + \beta \rightarrow M_\theta = m\left(L^2+K^2\right)\left(\ddot{\varphi} + \ddot{\beta}\right) + mLa\omega^2\sin(\omega t)\sin(\varphi+\beta)$. Small $\beta \rightarrow \cos(\beta) \sim 1$, $\sin(\beta) \sim \beta \rightarrow \sin(\varphi+\beta) = \sin(\varphi)\cos(\beta) + \cos(\varphi)\sin(\beta) \sim \sin(\varphi) + \cos(\varphi)\beta \rightarrow M_\theta \sim m\left(L^2+K^2\right)\left(\ddot{\varphi}+\ddot{\beta}\right) + mLa\omega^2\sin(\omega t)\left(\sin(\varphi) + \cos(\varphi)\beta\right) \rightarrow M_{\varphi+\beta} = m(L^2+K^2)\ddot{\varphi} + m(L^2+K^2)\ddot{\beta} + mLa\omega^2\sin(\omega t)\sin\varphi + mLa\omega^2\sin(\omega t)\beta\cos\varphi$.