PENDULUM WITH VIBRATING SUSPENSION (Translated from here)

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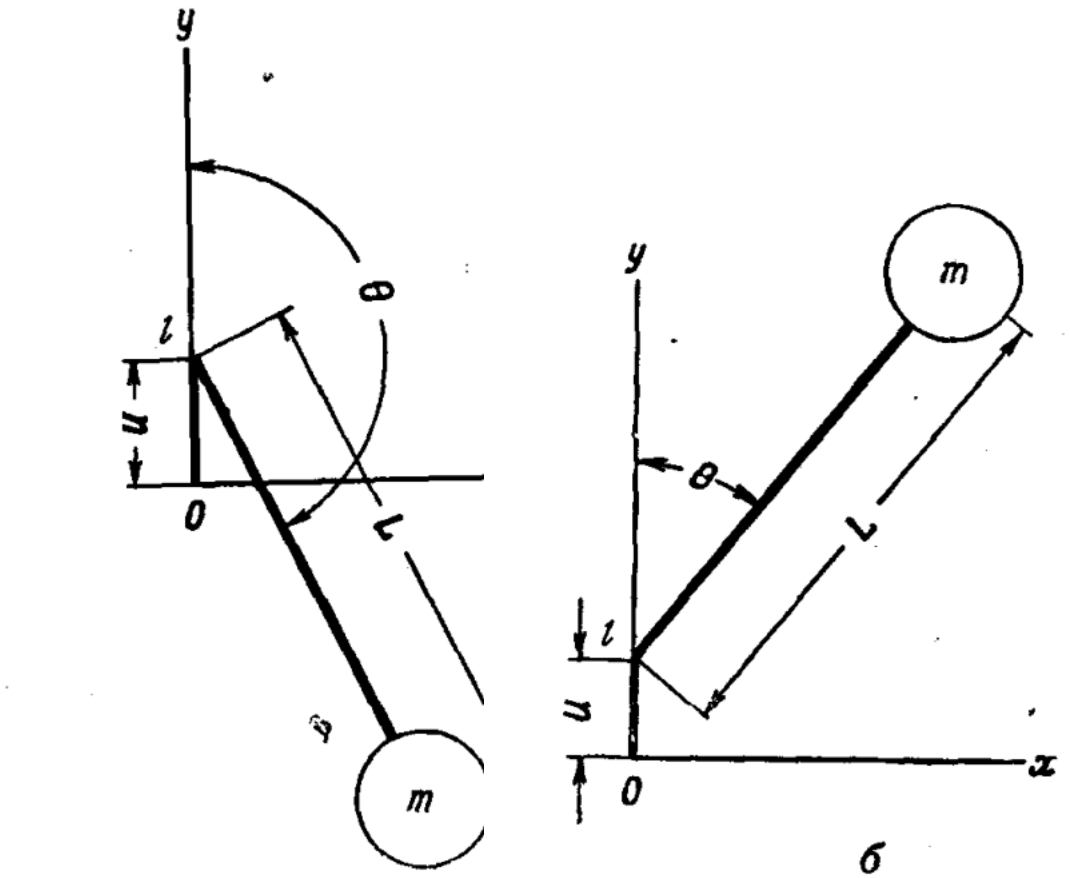
The development of mechanics is undoubtedly closely connected with the study of the pendulum. After Galileo drew attention to the isochronism of its oscillations, it became possible to create a very perfect mechanism for measuring time - a pendulum clock, the accuracy of which was only recently surpassed by quartz clocks. Thanks to the study of the pendulum, methods were found to measure time as accurately as length and mass were measured, which was necessary so that the development of mechanics could follow a solid path. Naturally, no mechanical system has received as much attention and comprehensive theoretical study as all varieties of pendulum motion. It would seem that in the 300 years that have passed since the time of Galileo, this question should have been exhausted and if anything remained for study, it should have been in the nature of polishing previously obtained results. But, apparently, the type of pendulum motion to which this article is devoted has not been given sufficient attention, and one of the very unique and interesting varieties of pendulum oscillations has remained almost completely unstudied. This article aims to draw attention to this type of motion and the possibilities that open up when studying it.

Fig. 1 shows a mathematical pendulum in two positions, which can oscillate at the suspension point; the mass m is concentrated at the end of the rod of length L. The position of the pendulum on the left side of the figure (1(a)), when the suspension point is above the center of gravity, we will call the normal position. In the figure on the right (1(b)), the suspension point of the pendulum is below the center of gravity; we will call this position the inverted position of the pendulum. The type of pendulum we shall consider has the peculiarity that the suspension point l moves along the y-axis near the origin O, with the distance being a periodic function of time; we also assume that the amplitude a is small compared to the length of the pendulum L. This is the well-known pendulum with an oscillating suspension.

In studying this type of pendulum, all attention was focused on the type of motion when the period of oscillation of the suspension T differed little from the period of oscillation of the pendulum itself τ. In this case, it was found that in those cases when 271 or a multiple of it is close to the period τ, the phenomenon of parametric resonance occurs. These studies were reduced to studying the properties of solutions of the Mathieu equation, which describes this motion at small oscillation amplitudes.



(a) Normal position (b) Inverted position

Figure 1: Pendulum in the normal position (left) and inverted position (right).

Furthermore, it was discovered that at small values of T compared to τ, the pendulum can acquire a special type of stability - it can stand without falling in an inverted position. The nature of the pendulum's motion in this position and the degree of its stability at high frequencies of oscillation of the suspension, apparently, remained completely unstudied. Thus, the beautiful and instructive phenomenon of the dynamic stability of the inverted pendulum not only was not included in modern manuals on mechanics, but is even almost unknown to a wide circle of specialists.

It can be assumed that such an undeserved attitude towards this phenomenon was a consequence of the fact that its study was connected with the solution of the Mathieu equation; it was produced by infinite determinants (Gill's method) or special functions, which led to a solution of a formal nature, not allowing the possibility of a visual description of the motion.

While studying this movement, I noticed that under the condition that the amplitude of the suspension oscillations a is small compared to the length of the pendulum L, there is a method for an approximate solution to the problem of movement that simply and clearly describes the phenomenon.

The ratio of the suspension oscillation amplitude to the pendulum length will be denoted by α:

$$\alpha = a/L \ll 1 \tag{1}$$

The value of α will play a very important role in this method, since the accuracy of the results obtained in the types of motion that interest us is mainly determined by it. We will mainly study those types of pendulum motion when the frequency of the suspension oscillations is high compared to the frequency of the pendulum oscillations, and, in addition, is not at all connected with it by any phase relationships, while the spectrum of the suspension oscillations themselves can be the sum of the spectrum and various frequencies. Therefore, in order to distinguish the motion we are studying from the motion of a pendulum with an oscillating suspension, we will call it the motion of a pendulum with a vibrating suspension.

The method we used to solve the problem is based on successive approximations together with the introduction of averaged coordinates over time. A detailed presentation of it and a study of the accuracy of the results obtained are given by us elsewhere.

Here, we will limit ourselves to a description of the main results obtained and the possibility of their practical application.

The method of successive approximation already at the first stage reduces the problem of the influence of a vibrating suspension point on the motion of a pendulum to a very simple physical picture: it turns out that this influence is equivalent to a moment of forces, which behaves in exactly the same way as a pair of ordinary forces, and tends to install the pendulum so that its mass is always in the direction of the vibrations of the suspension. We called this moment the vibration moment and denoted it by M. As will be seen from what follows, the introduction of the vibration moment makes solving the problems of the motion of this type of pendulums no more difficult than solving problems for ordinary pendulums. Below we will also describe a method for simply constructing a pendulum with a vibrating suspension, on which the obtained theoretical results can be demonstrated.

Let us write an equation for the general case of motion of the considered type of mathematical pendulum. If, as shown in Fig. 1, we denote the angle between the pendulum rod and the y-axis by θ, then the coordinates of the pendulum mass x and y will be

$$x = L\sin\theta; \quad y = U + L\cos\theta \tag{2}$$

where U represents the distance (along the y-axis) of the pendulum's suspension point l from the origin O. The forces acting on the mass m along the x and y axes we denote by F_x and F_y; then we obtain:

$$\begin{cases} F_x = m\ddot{x} = mL(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta), \\ F_y = m\ddot{y} = m\left[\ddot{U} - L\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\right]. \end{cases} \tag{3}$$

The moment of a pair of external forces acting on the mass of the pendulum[1], we will denote by M_θ; it will be equal to[2]:

$$M_\theta = L\left(F_x\cos\theta - F_y\sin\theta\right) \tag{4}$$

Substituting the values of F_x and F_y, we obtain:

$$\begin{aligned} M_\theta &= L\left(mL\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\cos\theta - m\left[\ddot{U} - L\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\right]\sin\theta\right) \\ &= mL^2\ddot{\theta} - mL\ddot{U}\sin\theta \end{aligned} \tag{5}$$

[1]Also called the "moment of Force", or simply the torque

[2]$\vec{r} = (x, y) = (L\sin\theta, U + L\cos\theta) \to \vec{r}\times\vec{F} = (L\sin\theta, U + L\cos\theta)\times(F_x, F_y) = L\left(F_x\cos\theta - F_y\sin\theta\right)$

This equation can easily be generalized for the case of a physical pendulum. To do this, consider m as an elementary mass and integrate the right-hand side of equation 5 over the entire volume of the pendulum mass; then instead of 5 we get[3]:

$$M_\theta = m\left(L^2 + K^2\right)\ddot{\theta} - mL\ddot{U}\sin\theta \tag{6}$$

where θ and L are the coordinates of the center of gravity of the pendulum mass, and K is the radius of inertia of the pendulum. Let the pendulum suspension perform simple harmonic oscillations with amplitude a and angular frequency ω; then we have:

$$U = a\sin(\omega t) \tag{7}$$

Differentiating this expression twice with respect to time and substituting the value of U into (6), we obtain:

$$M_\theta = m\left(L^2 + K^2\right)\ddot{\theta} + mLa\omega^2\sin(\omega t)\sin\theta \tag{8}$$

In the particular case when the moment of external forces is created by the force of gravity, it is equal to:

$$M_\theta = mgL\sin\theta \tag{9}$$

and the equation of motion will take the form:

$$\ddot{\theta} = \frac{L}{(L^3 + K^3)}(g - a\omega^2\sin\omega t)\sin\theta \tag{10}$$

This equation is usually simplified by limiting the problem to the consideration of small values of the angle θ and replacing the value of $\sin\theta$ with it. With this simplification, we obtain the Mathieu equation, with the help of which the problem of the motion of a pendulum with an oscillating suspension has been studied so far. When we apply the method of successive approximation, the consideration of the problem is not limited to small angles θ. The basic idea of this method is the assumption that during the period of

[3]Another way to think of this is to realize that the simple pendulum has a moment of inertia $I_\theta = mL^2$ and thus in the physical pendulum case where the center of mass is a distance K from the mass m the new moment of inertia becomes by the parallel-axis theorem (which is where this volume integral idea comes from) is $I_\theta + mK^2 = m(L^2 + K^2)$

rapid oscillation of the suspension the angle θ will change little, remaining close to some value φ. We assume:

$$\theta = \varphi + \beta \tag{11}$$

The angle β is a periodic quantity, but its value over the period of oscillation T always remains small. The angle φ can have any value, but over the same time T it will change little. If we average these quantities over time over the period T and denote this averaging by a line, then we have:

$$\overline{\theta} \cong \varphi;\ \overline{\beta} \cong \theta \tag{12}$$

When studying the motion of a pendulum with a vibrating suspension, we are mainly interested in the change in angle φ, which represents the position around which small vibrations occur.

Therefore, the solution method is constructed in such a way that by averaging the angle β is eliminated from the equation and θ is replaced by the angle φ. It turns out that this can be done if the problem is reduced to a motion in which a vibrational moment is involved, equal to (for a physical pendulum)[4]:

$$\overline{M} = -\frac{1}{4}(1 + K^2/L^2)^{-1} m a^2 \omega^2 \sin(2\varphi) \tag{13}$$

Then it can be shown with an accuracy of order a that in most types of motion of interest to us, in the first approximation (for determining the quantities α^3) the following simple equation of motion holds:

$$m(L^2 + K^3)\ddot{\varphi} = M_\varphi + \overline{M} \tag{14}$$

where the moment of external forces M_φ is obtained from M_Θ by simply replacing the angle Θ with φ. In this way, we obtain the same equations as if the suspension were at rest, but in addition to the external moment M_φ,

[4]From equation 8: $M_\theta = m\left(L^2 + K^2\right)\ddot{\theta} + mLa\omega^2 \sin(\omega t)\sin\theta$, $\theta = \varphi + \beta \rightarrow M_\theta = m\left(L^2 + K^2\right)\left(\ddot{\varphi} + \ddot{\beta}\right) + mLa\omega^2 \sin(\omega t)\sin(\varphi + \beta)$. Small $\beta \rightarrow \cos(\beta) \sim 1$, $\sin(\beta) \sim \beta \rightarrow \sin(\varphi + \beta) = \sin(\varphi)\cos(\beta) + \cos(\varphi)\sin(\beta) \sim \sin(\varphi) + \cos(\varphi)\beta \rightarrow M_\theta \sim m\left(L^2 + K^2\right)\left(\ddot{\varphi} + \ddot{\beta}\right) + mLa\omega^2 \sin(\omega t)\left(\sin(\varphi) + \cos(\varphi)\beta\right) \rightarrow M_{\varphi+\beta} = m(L^2 + K^2)\ddot{\varphi} + m(L^2 + K^2)\ddot{\beta} + mLa\omega^2 \sin(\omega t)\sin\varphi + mLa\omega^2 \sin(\omega t)\beta\cos\varphi$.

there was also an additional moment $\overline{M}$. It is easy to see that integrating the equation obtained in this way for the angle φ does not present any greater difficulties than in the case of the motion of ordinary pendulums with a fixed suspension. This is a consequence of the fact that the vibration moment $\overline{M}$, since it does not include time, acts in the same way as the moment of ordinary forces. From expression 13, it is clear that the vibration moment tends to establish the pendulum rod along the direction of the y-axis, i.e., the axis along which the suspension oscillates. $\overline{M}$ has its greatest value at $\varphi = 45°$. Further, from 13 it follows that the magnitude of the vibration moment does not depend on the length of the pendulum and is mainly determined by the kinetic energy imparted to the mass of the pendulum during the vibration of the suspension.

For a given and constant vibration of the suspension, the magnitude of the vibration moment $\overline{M}$ depends only on the angle φ, therefore, the semi-valuable solutions of mechanical problems of pendulum oscillation, as will be seen from what follows, take on a clear form.

Simplifying the solution to the pendulum problem by introducing a vibrational moment resembles similar simplifications of the problems of motion of various types of tops and gyroscopes by introducing the concept of a gyroscope moment. In this respect, there is a certain analogy between the gyroscope moment and the vibrational moment.

Let us give a number of examples of solutions to equation 14 that are of practical interest.

Let us first examine the problems of "static" equilibrium between the applied moment M_φ and the vibration moment $\overline{M}$. The solution to these problems is obtained from equation 14; assuming $\varphi = \text{const}$, we have:

$$M_\varphi + \overline{M} = 0 \tag{15}$$

Let us assume that M_φ is created by gravity, and assume that the y-axis along which the vibrations occur forms an angle γ with the plumb line. Then the moment of gravity is equal to:

$$M_\varphi = mgL\sin(\varphi + \gamma) \tag{16}$$

Substituting this value into 14, as well as the value for $\overline{M}$ (13), we obtain the following equation:

$$4\left(1+\frac{K^3}{L^3}\right)Lg\sin(\varphi_n+\gamma)-a^2\omega^2\sin(2\varphi_n)=0 \tag{17}$$

From this expression, one can determine those values of the angle φ_n at which the equilibrium position of the pendulum is possible.

Graphical analysis of the equation shows that, depending on the value of the parameters, φ_n can have 4 or 2 values, which are its roots. In the case where there are two roots, only at one of them is the pendulum in stable equilibrium, corresponding to its normal position. In the case where there are four roots, the pendulum is in stable equilibrium at two values of the angle φ_n — one of them corresponds to the normal position, and the other to the inverted one. Four roots are possible only when the value of $a^2\omega^3$ is large enough, i.e. the vibrations are sufficiently intense.

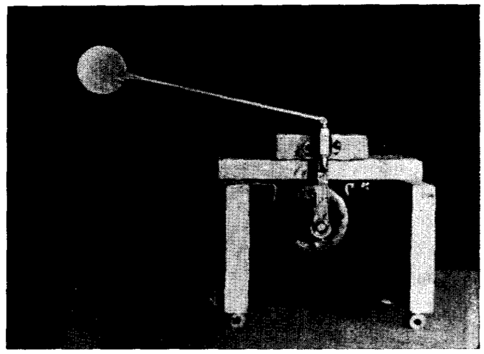
The two stability positions found can be demonstrated on the pendulum shown in Fig. 2 (2(a) and 2(b)). Two identical pendulums are symmetrically suspended at the end of the vibrating lever. With sufficient vibration intensity, they occupy positions corresponding to each of the two angles φ_n, determining the stability of equilibrium.

Giving light pushes to the pendulum, one can verify from experience that these positions do indeed correspond to stable equilibrium.

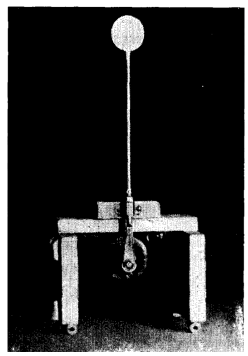
In the particular case when the vibration of the pendulum suspension occurs in the vertical direction, i.e. $\gamma=0$, it is clear that the equation is always satisfied when $\varphi_1=\pi$ and $\varphi_2=0$. The value $\varphi_2=0$, i.e. when the pendulum is in an inverted position, becomes stable only if there are also two values of angle $\varphi_n=\varphi_3$ and $\varphi_n=\varphi_4$ for unstable equilibrium. Equation 17, if we put $\gamma=0$ in it, gives:

$$\sin(\varphi_n)=0; \quad \cos(\varphi_n)=\frac{2gL}{a^2\omega^2}\left(1+\frac{K^2}{L^2}\right). \tag{18}$$

From this we obtain that $\varphi_1=\pi$, $\varphi_2=0$, and $\varphi_3=2\pi-\varphi_4$; the last angle determines the opening of the cone from which the pendulum will pass into a



(a) Normal position of the pendulum.



(b) Inverted position of the pendulum.

Figure 2: Pendulum in the normal position (left) and inverted position (right).

stable inverted position at $\varphi_2 = 0$. With the initial position of the pendulum with an angle greater than φ_3 , it will pass into a stable equilibrium with an angle $\varphi_1 = \pi$, i.e., into a normal position. The smaller the value of $\cos(\varphi_n)$, the wider the region in the inverted position in which the pendulum is stable. The initial condition necessary to obtain a stable position of the inverted pendulum is obtained from (18); it has the form:

$$\frac{1}{2}a^2\omega^2 \gg gL\left(1 + \frac{K^3}{L^3}\right) \tag{19}$$

This condition has already been obtained for the mathematical pendulum and, apparently, this is the only result for characterizing the behavior of the pendulum in the inverted position, which has been so far obtained from the consideration of the Mathieu equation. For this result, the following physical interpretation of stability in the inverted position was given: the value of the kinetic energy of the pendulum mass created during the oscillation of the suspension must be greater than the potential energy of the pendulum mass above the suspension point. As can be seen from our analysis, this interpretation is valid only for the mathematical pendulum; it does not hold for the physical pendulum.

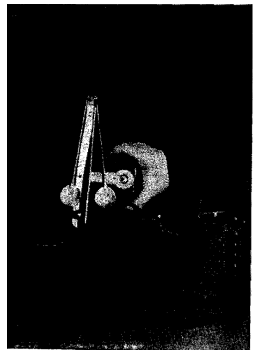
Let us now turn to the consideration of dynamic problems. Then in equation (14) the angle φ should be considered as a variable quantity. Let us consider the simplest case, when the external pair of forces M_φ is absent. Setting it equal to zero, from expression (14) we obtain the following equation of motion:

$$\left(1+\frac{K^2}{L^2}\right)^2 \ddot{\varphi} = -\frac{1}{4}a^2\omega^2\sin(2\varphi) \tag{20}$$

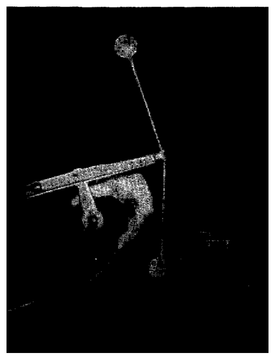
This equation is easily integrated and leads to elliptic integrals of the first kind; the pendulum, when the suspension vibrates, even in the absence of external forces, will perform a periodic oscillatory motion. If the period of oscillations with which the angle φ changes is denoted as τ, as before, and the period of oscillation of the suspension as T, then from the solved equation (20) we obtain:

$$\frac{\tau}{T} = \sqrt{2}\cdot\alpha^{-1}\left(1+\frac{K^2}{L^2}\right)F(k) \tag{21}$$

$F(k)$ is the complete elliptic integral of the first kind, $k = \sin(\varphi_a)$, where φ_a is the angular amplitude of the pendulum oscillations. For constant or small values of the amplitude φ_a, there is a simple proportionality between the period of oscillation of the pendulum and the period of vibration of the suspension. Since α is a small value, the period τ will be significantly greater than T. This type of oscillation can be reproduced on the pendulum shown in Fig. 3. In these experiments, the influence of gravity will be eliminated if the pendulum is placed horizontally. If the period τ is large enough to be determined by simply counting the oscillations, then knowing the proportionality coefficient from expression (21), we can determine the period of vibrations T. The described phenomenon can be used as a kind of simple tachometer. Let us now consider the oscillations of a conical pendulum in the absence of gravity. We assume that the rotation of the mass of the



(a) Pendulums at rest.



(b) Inverted position of the pendulum.

Figure 3: Pendulums during vibrations.

pendulum occurs about the y-axis with a constant angular velocity Ω; then for the moment created by the centrifugal force, we obtain the following expression:

$$M_\varphi = \frac{1}{2} m \Omega^3 L^3 \sin(2\varphi) \tag{22}$$

The magnitude of M_φ depends on the angle φ in the same way as the vibrational moment $\overline{M}$ (13). Therefore, the equilibrium between M_φ and $\overline{M}$ does not depend on the value of the angle φ, and we get the following simple relation:

$$\frac{\Omega^2}{\omega^2} = \frac{1}{2}\alpha^2 \left(1 + \frac{K^2}{L^2}\right)^{-1} \tag{23}$$

It follows that the angular velocity Ω of the rotation of a conical pendulum with a vibrating suspension does not depend on the angle φ. It is somewhat difficult to reproduce this type of motion experimentally, since the influence of gravity must be excluded. It can be approached by imparting powerful vibrations to the suspension so that the vibrational moment significantly exceeds the moment of gravity.

As a more detailed analysis shows, the degree of accuracy obtained for the period of oscillation of a vibrating pendulum is entirely determined by the value $\boldsymbol{\alpha}$, equal to the ratio of the length of the pendulum to the amplitude of the vibration of the suspension, and is of the order of α^2.

The solution to the problem of pendulum oscillations in a gravitational field with the suspension vibrating along the y-axis inclined to the vertical at an angle γ is obtained from the solution of equation (14) by substituting in it for M_φ the value given by expression (16). The resulting equation is integrated and the solution yields oscillatory motion in which the amplitude is an elliptical function of time. If the intensity of the vibrations is sufficient for equation (17) to have four roots, then the oscillatory process is possible for about two values of the angle φ. One corresponds to the inverted position of the pendulum, the other to the normal position. In this case, we obtain that in the inverted position the period of oscillation of the pendulum is greater than in the normal position. The period of oscillation of the same pendulum in the absence of suspension oscillations has an average value between these two periods. From the solution of the equation it follows that for any vibrations of the pendulum suspension, the period of oscillations in the normal position are always shortened. These phenomena are well demonstrated on the double pendulum, shown in Fig. 2. By tilting the device so that the direction of vibrations makes different angles with the vertical γ, the pendulums can be made to swing simultaneously so that one of them is in the normal position and the other in the inverted position. Then it is possible to clearly compare the periods of oscillations and verify the above conclusion.