Column Generation and Dantzig-Wolfe Decomposition

For the last decades, my professors François Soumis, Guy Desaulniers and their colleagues at GERAD in Montreal (where I'm doing my PhD) have made millions of dollars by exploiting two clever mathematical tricks to solve industrial problems. These tricks have become a cornerstone of **operations research**, the subfield of applied mathematics which involves computational approaches to help decision-making.

GERAD
GROUP FOR RESEARCH IN DECISION ANALYSIS

What's the tricks?

The tricks are called **column generation** and the **Dantzig-Wolfe decomposition**. They date back from the 1950s. Let's see how they work.

We'll first draft the philosophy of column generation through an example of sports team lineup. Then, we'll describe the column generation approach per se, with various insightful interpretations. Next, we'll get to the Dantzig-Wolfe decomposition which enables column generation to be very powerful.

Sports Team Lineup

Let's start with an example. Imagine you were the newly appointed president of the GERAD football team. A Qatari foundation has just provided you with seemingly unlimited funds and you are to design the best starting eleven for your football team. In other words, in the huge pool of professional football players, you need to find the best combination of eleven players. And what's tricky is that these players need to complement one another.

What do you mean?

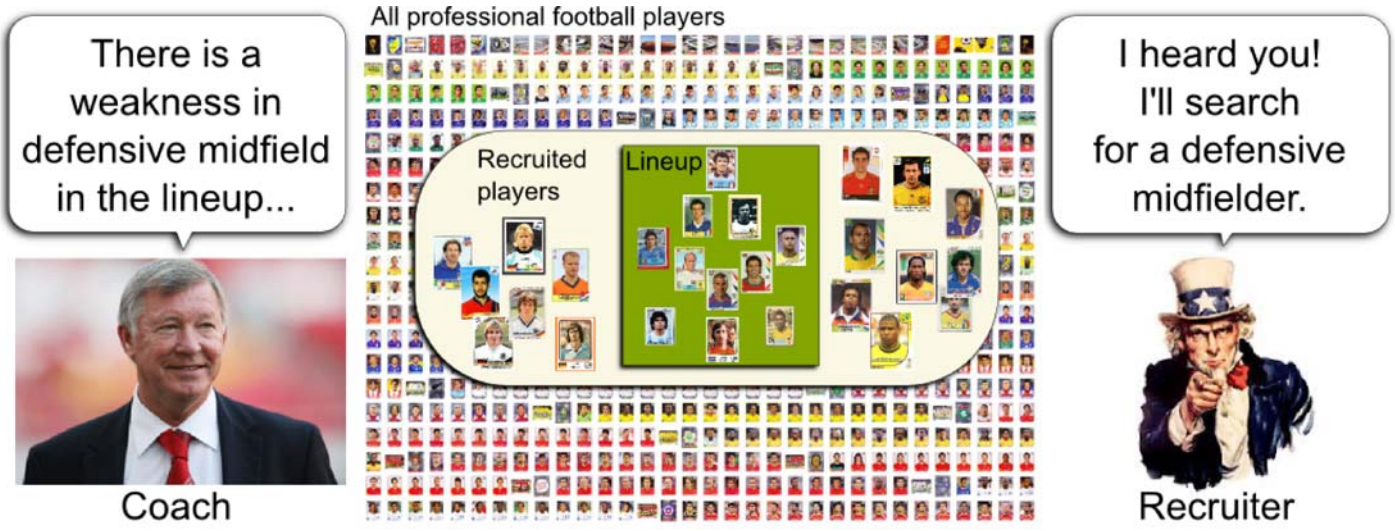
I mean that it's not good enough to hire the *best* players. As any football fan knows it, an equilibrium in the team must be met, and complementarities between players must be found. In mathematical words, these nonlinear aspects make the lineup design nontrivial. In more straightforward terms, we need to be smart about how to design the starting eleven. This is where column generation enters in play.

How does that work?

The trouble is that the set of combinations of eleven players is extremely huge. To face this issue, column generation proposes to divide the job of searching for good players (let's call that recruitment and give it to a **recruiter**) and the job of selecting among recruited players the best starting eleven (let's call that the lineup and have a **coach** doing it). Crucially, because the coach picks up players out of a small set of recruited players, it will be a much easier job for him to find the best starting eleven.

Can't the coach have a say in recruitment?

Not only he can. He should. Given the weaknesses of his starting eleven, the coach can direct the recruiter towards the profiles of players that would definitely improve his team. Typically, if a quality defender is missing, the coach should say so, hence signaling the recruiter that he should focus on defenders. Let me recapitulate the column generation procedure with the following figure.



Is that it? Is that really a breakthrough?

Amazingly, it is. First, because the coach can now really optimize his lineup by testing nearly all possible combinations. The coach's job is called the **master problem**. But also and more importantly, his remarks will greatly help the recruiter to pinpoint the promising players who could strictly improve the lineup. The recruiter's job with the coach's directives is known as the **subproblem**. This division of problems into master and sub-problems has enabled great reductions of computation times of optimization algorithms both in exact methods and heuristics. For instance, its applications to integer programming are countless, from logistics to transport, all the way through scheduling and network design.

But why is column generation called so?

To go further, I need to tell you about the setting in which column generation was invented and has successfully surpassed all other methods. This context is that of linear programming.

For the sequel, my articles on linear algebra and linear programming are prerequisites. It may also be useful to read my articles on duality, simplex methods and integer programming.

Algebraic Column Generation

Let's write our linear program as follows:

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{Minimize}} \quad & c^T x \\ \text{Subject to} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

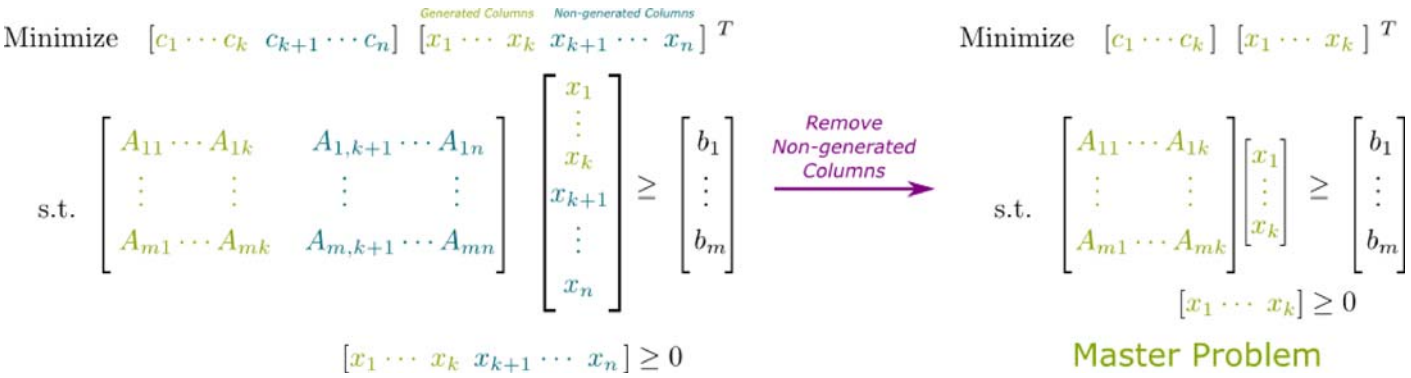
We assume that A is a $m \times n$ matrix, which means that our linear program has n variables and $m+n$ inequality constraints. The usefulness of column generation appears when the number of variables is very large. Typically, in vehicle routing, we assign a variable for every possible route. Thus, the number of variables is of the order of the factorial of the number of places to visit.

So what do we do then?

As illustrated through the sports team lineup problem above, we will define a **master problem** which will only deal with a small subset of variables. Algebraically, this corresponds to dividing variables into two groups. The first group will be dealt with by the master problem and its elements are called *generated columns*. Others fall in the second group of non-generated columns.

Why don't we call these "generated variables" and "non-generated variables"?

Algebraically, by reordering variables of x accordingly to the division of variables, we can rewrite the linear program as follows:



Now, computationally, you have to keep in mind that the left complete formulation never appears. The master problem only knows about the right formulation with no non-generated columns. Whenever the subproblem proposes a new variable to add to the formulation, then the main modification to the master problem is the addition of a column to the matrix A. This is why column generation is called so.

How does the subproblem manage to recruit new promising variables?

Through a sensitivity analysis! One powerful aspect of linear programming is the ability to identify the bottlenecks of an optimization, just as our coach has managed to identify the weaknesses of his lineup. This identification can be done pretty straightforwardly using duality. In essence, given the optimal dual variables, the master problem can relate the potential improvement of a non-generated variable x_j by merely looking at the cost c_j of the variable and the column A_{*j} of matrix A corresponding to that variable.

The variation of the objective value when we add x_j to the base is given by its reduced cost $\hat{c}_j = c_j - y^T A_{*j}$, where y is the optimal dual variable of the master problem. It's interesting to note that this reduced cost doesn't give us the full story, as it might not be able to make a lot of steps in the direction of improvement. This is particularly the case when we face degeneracy.

One beautiful aspect of linear programming is that we have different angles to look at it. For one thing, you might wonder what column generation looks like from a dual perspective. Since columns become rows in the dual program, column generation naturally becomes row generation in the dual, which is better known as the **cutting plane method** that is widely used in integer programming. Plus, if the dual has a Benders decomposition structure with some stochasticity, and we see that the *L-shaped method* is nothing else than the dual of column generation.

Note that in integer linear programming, the program is equivalent to the optimization over the convex hull of feasible integer solution. Yet, this convex hull is a polyhedron, and can thus always be described as a linear program. But this linear program has possibly exponentially many constraints. This is why a row generation method (also known as a cutting plane method) is so adequate.

But there's more than the dual! I now propose to look at geometrical and combinatorial interpretations of column generation.

Geometrical Column Generation

Recall that linear programming corresponds to determining the South pole of a polyhedron, with the Northwards direction given by the vector c, as explained below in my talk A Trek through 20th Century Mathematics:

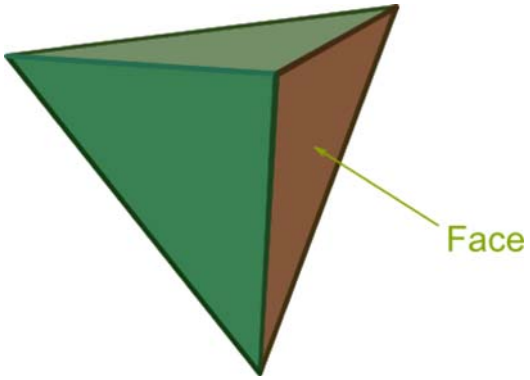
The illustration of column generation given after the end of the video extract above is actually misleading. It is true that column generation removes some extreme points. However, as we shall see, it does so in a very specific geometrical way.

The trouble is that, for high-dimensional polyhedra, the path to the South pole may be a long one.

So how does column generation simplify the problem?

Algebraically, removing variables from the master problem is actually equivalent to fixing them to zero. In other words, it' s exactly as if we added equality constraints $x_{k+1} = x_{k+2} = \ldots = x_n = 0$ to the initial linear program. Now, geometrically, such equality constraints represents a vector subspace of $\mathbb{R}^n$. Thus, the master program corresponds to a focus on the intersection of the initial polyhedron with a vector subspace.

So, can we say that the master problem restricts the problem to a sub-polyhedron?



Yes. But not just any sub-polyhedron. Indeed, it' s a vector subspace obtained by saturating some of the inequality constraints defining the polyhedron. Thus, the sub-polyhedron is a k-face, that is, a generalization of what 2-dimensional faces of 3-dimensional polyhedra (note that k can be any number between 1 and n). Meanwhile, the Northward direction corresponding to this k-face is easily derived as the orthogonal projection of c on the vector subspace.

So, to recapitulate, the master problem of column generation is a restriction of a linear optimization over a polyhedron to the optimization over one of the faces of the polyhedron.

What about the subproblem? What does it correspond to?

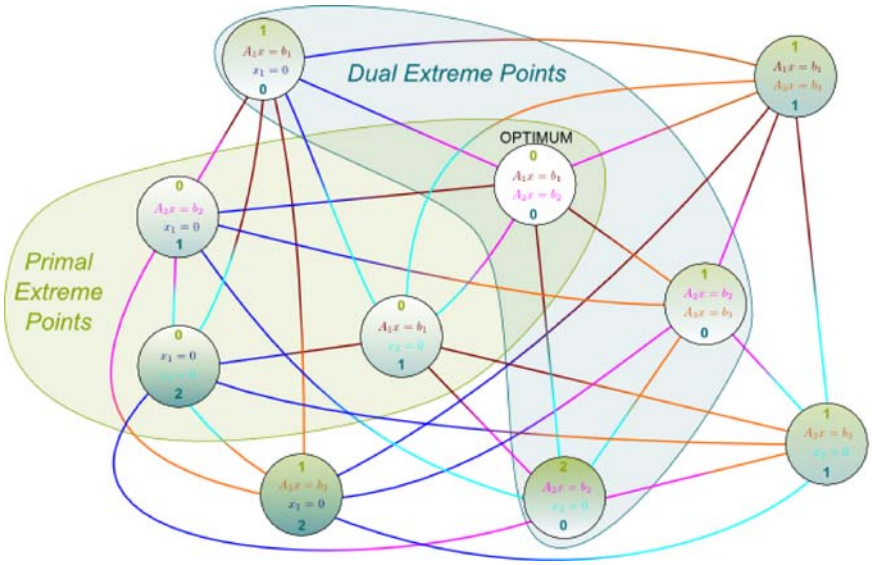
Recall that the master problem focuses on a face defined as the intersection of the initial polyhedron with a vector subspace. Then, the subproblem consists in identifying a direction orthogonal to this vector subspace which may improve the objective value. Crucially, to know whether such a direction x_j is likely to improve the objective value, we need to know additional information about constraints and the Northward direction in this orthogonal direction. These are describes by the cost c_j and the column A_{*j}.

Combinatorial Column Generation

Duality teaches us that the concept of **base solutions** is more fundamental and insightful than that of *extreme points*. This remark leads me to give you a combinatorial interpretation of column generation.

How does it go?

Disregarding rank issues, base solutions are choices of n saturated constraints out of the $m+n$ constraints. Let's represent each base solution by a node of a graph. Next, let's link two nodes, if a single pivot is needed to move from the first to the latter base solution. Below is an example where $n=2$ and $m=3$, which also displays the primal and dual feasibilities of base solutions. Note that primal base solutions are exactly primal extreme points.



Nodes are base solutions. Edges are pivots.

The upper green digit of a node counts the number of primal constraints violated.

The lower blue digit of a node counts the number of dual constraints violated.

A node is optimal iff it violates no constraint.

$$\begin{array}{lll} \text{Max} & x_1+x_2 & \\ \text{s.t.} & 3x_1+x_2 \leq 3 & \\ & x_1+3x_2 \leq 3 & \\ & x_1-x_2 \leq 2 & \\ & x_1 \geq 0 & \\ & x_2 \geq 0 & \end{array} \qquad \begin{array}{lll} \text{Min} & 3y_1+3y_2+2y_3 & \\ \text{s.t.} & y_1 \geq 0 & \\ & y_2 \geq 0 & \\ & y_3 \geq 0 & \\ & 3y_1+y_2+y_3 \geq 1 & \\ & y_1+3y_2-y_3 \geq 1 & \end{array}$$

You may notice that the subgraph made of primal extreme point circles along with edges between these extreme point circles (called the *induced subgraph*) is a quadrilateral. Indeed, diagonals are not in the graph. This means that the primal polyhedron is a 2-dimensional quadrilateral. Conversely, the dual extreme points are these of an unbounded 3-dimensional polyhedron, which has one triangle face and one edge between its extreme points (as well as several edges pointing towards infinity).

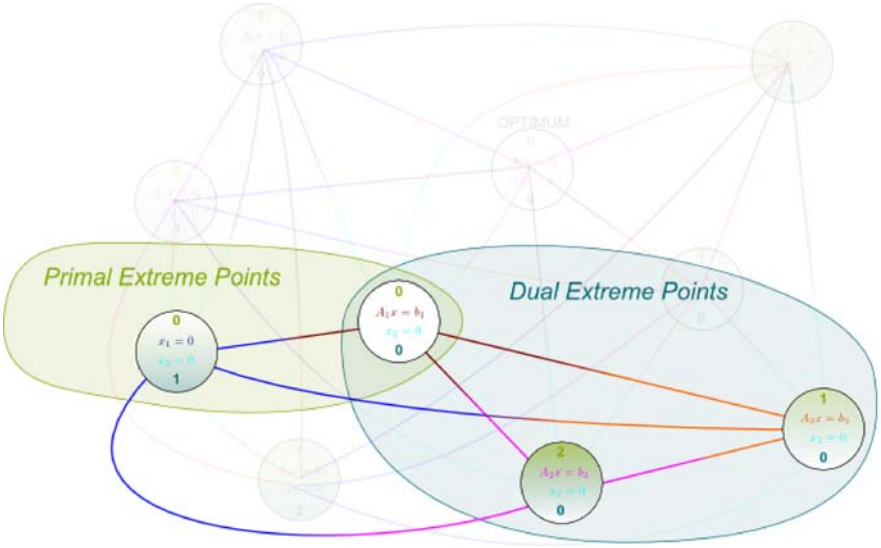
> Hummm... That is a troubling interpretation of linear programming!

I don't know how troubling it is, but it is definitely unconventional. Computationally though, it is far from being useful. After all, graphs like the ones above have $\binom{n+m}{n}$ nodes. This is way too many to ever be implemented entirely. However, I do find this representation of linear programming very insightful. For instance, in this representation, the generalized simplex method merely consists of a path along which the numbers of primal and dual violated constraints both never increase. Crucially, while it has no knowledge of the entire graph, the simplex method can enumerate the neighbors of a node to find the best next node to visit. Eventually, it finishes at a node where both numbers are zero, in which case we are at a base solution that is both primal and dual feasible. Such base solution is necessarily primal and dual optimal.

While I've often had more or less this combinatorial image of linear programming in mind, here is the first time I've actually made it explicit. This raises some questions I don't have the answer to. Namely, could it be that, from any node, there always is a path along which the total number of primal and dual violated constraints decreases? This is an important question as, if the answer is yes, it'd yield a variant of the simplex method which runs in polynomial time (the primal simplex method is not polynomial)...

> I had never seen the simplex methods that way!

I know! Now, let me finish with what column generation does to this graph. Basically, by forcing non-generated variables to equal zero, column generation focuses on particular induced subgraphs. Crucially, this enables to dramatically reduce the size of the graph we work with. For instance, if we do not generate variable x_2 in the example above, we obtain the following induced subgraph:



In this example, the primal polyhedron is just a 1-dimensional interval, while the dual polyhedron is once again unbounded with merely a triangle as a bounded face (other faces are unbounded).

Notice how column generation affects the dual program: It corresponds to removing a dual constraint. As a result, column generation decreases the number of violated constraints of all nodes. Hence, in the master program, some nodes become dual feasible.

What about the subproblem?

It's not clear how to illustrate it through our example, as we only had two primal variables here. But the key role of this subproblem is really to find which family of nodes to add such that the optimal node of the master problem is no longer a dual extreme points. In cutting plane method terms, this corresponds to determining a dual constraint which is violated by the dual optimum of the master problem.

Dantzig-Wolfe Decomposition

An article on column generation would not be complete without mentioning the Dantzig-Wolfe decomposition. In fact, from my experience, these two concepts are so often combined that students tend to confuse them. Yet, they are in fact very different ideas. For one thing, column generation is quite universal and applies to all linear programs.

Wait. Didn't you say that it was particularly adequate for problems with a large number of variables?

I did. Indeed, column generation always works, but it doesn't add much unless we have problems with a large number of variables. And by large, I mean exponentially large.

So, it's pointless to apply column generation to other problems?

Hehe… There is a clever twist here. In integer linear programming, I'd say that there are 3 main indicators of the difficulty of a problem: The number of variables, the number of constraints and the integrality gap.

What's the integrality gap?

The integrality gap is the relative increase in the objective function when we include the integrality condition on variables, as opposed to when we don't. The greater the integrality gap, the more time it will take to solve an integer linear program.

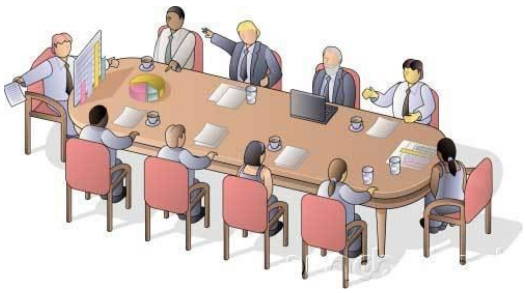
So what can be done?

When the problem has the right structure (and this happens a lot in practice!), it is possible restate the optimization problem to redirect the integrality gap difficulty towards a number-of-variable difficulty, which we can then fight with column generation. This restatement is the so-called **Dantzig-Wolfe decomposition**.

How does it work?

My professor François Soumis likes to illustrate this Dantzig-Wolfe decomposition with an old project he did. Decades ago, he was asked to infer who voted what from the results of a vote by a board of directors. Given that each director of the board has a known weight in the vote, it is usually possible to do so. For instance, if there are three directors with weights 40%, 35% and

25%, and if the results yield a 65% versus 35%, it is not to hard to guess that the first and third directors voted the inverse of what voted the second one. However, things are not that easy if we now have 15 directors.



What does this even have to do with integer linear programming?

We can model the set of possible votes as the integer points of a polyhedron! Let's call green, blue, orange... etc the alternatives of the vote. Also, we number the directors by numbers 1, 2, 3, ... etc. Denoting w_i the weight of director i in the vote, v_c the total of votes for color c and x_{ci} the binary variable which equals 1 if director i votes for color c, the fact that all votes for color c add up to v_c corresponds to the constraint $w_1x_{c1} + w_2x_{c2} + w_3x_{c3} + \ldots = v_c$. Then, for any values of parameters α_{ci}, a consistent vote is necessarily an optimal solution of the following integer linear program:

$Minimize$	$\sum_{c,i} \alpha_{ci} x_{ci}$		
subject to:	$\sum_i w_i x_{ci} = v_c$	$\forall c,$	(The votes of directors add up consistently with the observed results)
	$\sum_c x_{ci} = 1$	$\forall i,$	(Each director votes exactly once)
	$x_{ci} \geq 0$	$\forall i, c,$	(The last two conditions guarantee that variables are binary)
	$x_{ci} \in \mathbb{Z}$	$\forall i, c,$	

You might be tempted to fix $\alpha_{ci} = 0$ for all c and i. It would be right. But interestingly, by now varying these coefficients, we can determine all the possible votes by directors which led to the result we observed!

Waw! That's pretty cool!

Plus, as you can see, there is a relatively small number of constraints and variables. Indeed, the number of variables for instance is the product of the number of colors by the number of directors. We're far from the factorial number of routes in vehicle routing problems! Yet...

Let me guess... The integrality gap is huge.

Yes. And this means that this apparently simple problem can be unbelievably hard to solve.

Unless we use the Dantzig-Wolfe decomposition...

Exactly! The key idea of the Dantzig-Wolfe decomposition lies in the Minkowski theorem, which says that all bounded polyhedra are convex combinations of their extreme points. In other words, given any polyhedron P, denoting $extr(P)$ the set of its extreme points, then points of P can be written $x = \sum \lambda_e e$ with $\sum \lambda_e = 1$ and $\lambda_e \geq 0$ for all extreme points $e \in extr(P)$. In particular, $x_{ci} = \sum \lambda_e e_{ci}$. However, this new formulation of the polyhedron P now has as many variables as the number of extreme points of P. And that's a lot. Usually, that's too many. Even for column generation.

So what can be done?

It's here that it is essential to exploit the structure of the problem. Let's rewrite the integer linear program by now highlighting where the different variables appear, as we group them accordingly to the colors they correspond to:

Linear combination of green variables $= v_{green}$

Linear combination of orange variables $= v_{blue}$

$\vdots$

Linear combination of orange variables $= v_{orange}$

Linear combinations of diverse variables $= 1$

$\vdots$

$= 1$

The lower black constraints of the representation above are called **linking constraints**. Let' s forget them for a while. Then, crucially, the matrix A has a striking diagonal structure. Each diagonal block is unrelated to any other diagonal block. The core idea of the Dantzig-Wolfe decomposition is to remark that the whole polyhedron described by all the constraints is the Minkowski sum of the polyhedra that each block describes.

What' s a Minkowski sum?

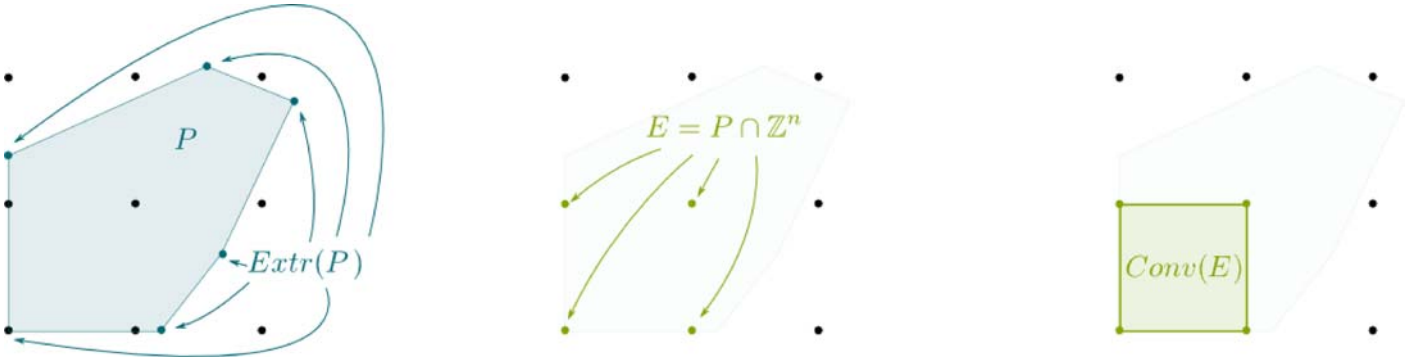
The Minkowski sum $P = \sum P_c$ of polyhedra P_c is the set of vectors x which are sums $x = \sum x_c$ where each $x_c \in P_c$. Importantly, the fact that P is the sum of polyhedra P_c means that it suffices to apply the Minkowski theorem to each polyhedron P_c.

So, for instance, for the green block, we need to determine the extreme points the green equation defines?

Basically, yes.

Actually, P_c is the polyhedron containing solutions x^c which satisfy the equation for color c while guaranteeing $x^c_{di} = 0$ for $d \neq c$. If we take $x^c \in P_c$ for all colors c, it is then easy to see that $x = \sum x^c$ will satisfy all equations for colors.

Now, crucially, we can take the integrality condition into account right here. More precisely, instead of describing extreme points $e^c \in Extr(P_c)$, we are going to list the integral points e^c which are extreme points of $E_c = Conv(P_c \cap \mathbb{Z}^n)$. This key subtlety is where lies the true power of the Dantzig-Wolfe formulation. Indeed, by considering integrality conditions at this stage, we will greatly reduce the integrality gap, hence making the problem much easier to solve.



But how can we determine the integral extreme points $e^c \in E_c$?

That is the question we leave to the subproblem. Actually, given the dual optimum of the master problem, a clever subproblem will even make sure to only generate the promising extreme points. But in our case, it' s actually not too hard to determine all extreme points. Indeed, as it turns out, any decomposition we can do of a vote v_c as an exact sum of some of the weights w_i defines an extreme point $e \in E_c$. This extreme point e is characterized by its coordinates $e_{ci} \in \{0, 1\}$ which equal 1 if they include a vote by director i for color c. It is also characterized by the color c it corresponds to. We model this latter fact by defining $\delta_{ec} = 1$, by opposition to $\delta_{ed} = 0$ if $d \neq c$.

So, that was the subproblem... What about the master problem?

Using the Minkowski theorem, any point of the polyhedron $Conv(E_c)$ is written $x^c = \sum \lambda_{e^c} e^c$, where the sum is taken over $e^c \in E_c$ and where $\sum \lambda_{e^c} = 1$. All of this can be reformulated as $x^c = \sum \lambda_e \delta_{ec} e$, with $\sum \delta_{ec} \lambda_e = 1$, where the sums are now taken for all $e \in E = \bigcup E_c$. But then, we know the total polyhedron $Conv(P \cap \mathbb{Z}^n)$ is the Minkowski sum $\sum Conv(E_c)$. Therefore, any vector x satisfying all the block constraints is actually $x = \sum x_c = \sum \lambda_e e$, where the weights λ_e must satisfy constraints $\sum \delta_{ec} \lambda_e = 1$ for all $c \in C$.

By replacing x_{ci} by $\sum e_{ci} \lambda_e$, we can finally rewrite entirely the integer linear program for variables x as a integer linear program for variables λ as follows:

$$
\begin{array}{lll}
Minimize & \sum_e \left(\sum_{c,i} \alpha_{ci} e_{ci} \right) \lambda_e & \\
\text{subject to:} & \sum_e \delta_{ec} \lambda_e = 1 & \forall c, \quad (\text{Exactly one weight } \lambda_e \text{ corresponding to color } c \text{ must equal } 1) \\
& \sum_e \left(\sum_c e_{ci} \right) \lambda_e = 1 & \forall i, \quad (\text{Exactly one weight } \lambda_e \text{ corresponding to director } i \text{ must equal } 1) \\
& \lambda_e \in \{0, 1\} & \forall e,
\end{array}
$$

Note that all the coefficients of the constraints are either 0 or 1. In fact, the constraints can be rewritten $B\lambda = \mathbf{1}$, where B is a matrix with binary entries and $\mathbf{1}$ is the column vector of 1s. This means that we have a set partitioning problem: Each column

(or variable) defines a subset of constraints, and any feasible solution corresponds to a set of columns which partitions the set of constraints.

So what' s so great about this rewriting of the integer linear program?

Crucially, now, the integrality gap can be expected to be much smaller. Indeed, any partition is integral and an extreme points. This hints at the fact that many extreme points are integral. Plus, even though we have a large number of variables, our formulation is now much more manageable by column generation. And, as the story goes, using this Dantzig-Wolfe decomposition, my professor François Soumis managed to give his clients valuable answers to the question of who voted what at a board meeting vote...

What are the applications of the Dantzig-Wolfe decomposition?

They are quite numerous. Most importantly, the combination Dantzig-Wolfe decomposition + column generation has been a powerful approach to solving vehicle routing, shift scheduling and assignment problems.

Let' s Conclude

To recapitulate, column generation relies on a separation of tasks in an optimization algorithm. Namely, it separates the task of proposing interesting alternatives from the task of making the best out of a small subset of alternatives. Amazingly, this simple idea has been monumentally powerful to address industrial problems, and it won my professors millions of dollars. But to reveal the full potential of column generation also requires the use of another clever idea called the Dantzig-Wolfe decomposition. In integer linear programs with great integrality gaps, this decomposition has the great advantage of cutting much of the non-integer points by redescribing polyhedra in terms of convex combinations of (integral) extreme points.

Are there more developments? What are the current researches on?

In the last decade, new advancements have come from a clever grouping of constraints. As we enumerate only a tiny fraction of all columns, many constraints become redundant. This means that the problem would remain the same if some of these constraints were simply removed. Because of the very specific set partitioning structure of the Dantzig-Wolfe decomposition, it often turns out that a huge number of constraints are indeed redundant. The **dynamic constraint aggregation** method is precisely about removing these redundant constraints from the master problem to speed it up.

So, dynamic constraint aggregation boils down to a management of constraint redundancy?

Exactly. Now, the trouble though, is that, as we generate new columns, we need to figure out which redundant constraints are no longer redundant. At this point, it may be even more clever to redirect the subproblem towards columns which will not require to re-add many no-longer-redundant constraints. This has led to classify non-generated columns into classes of so-called *compatibility* (which indicates the number of no-longer-redundant constraints one would have to add). The general framework to work this out is called the **improved primal simplex** (IPS). Finally, one interesting variant of IPS called **Integral Simplex Using Decomposition** (ISUD) focuses on adding columns in such a way that the optimum of the master problem always remains integral. And all of this is being developed right here in my lab, the GERAD.

I' d like to dedicate this article to all my labmates who have been guiding me through column generation and its developments over the last 4 years. These labmates include Samuel, Jean-Bertrand, Marilène, Charles, Hocine, Jérémy, Claudio, Rémi, Antoine, Pascal, Matthieu and a bunch of others I' m forgetting...

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Comments

June 10, 2014 at 10:14 am

Maybe you can look at Random Forests next.

johnfoley

February 19, 2016 at 10:06 am

Thanks for your posts on this website, they are very intuitive thus comprehensive! On this post, I got a questions: in the picture of the section Algebraic Column Generation, why the constraints of the master problems are still valid after the removal of several columns?

Léo

February 21, 2016 at 2:45 pm

I think you mean “feasible” instead of “valid”, in which case you’re totally right. In practice though, a heuristic is usually used to generate sufficiently columns to start with an initial feasible (good) solution. In fact, in practice, having a good initial point is essential for fast computations.

★Lê Nguyên Hoang