A question in auction theory

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Introduction

This short article will go through a challenging question taken from the first round of the manual section of IMC's Prosperity 2 trading challenge. The question requires knowledge of probability distributions, some basic integration and also how to locate stationary points using partial differentiation. The main backstory to the Prosperity challenge is that you have been placed on a tropical island with lots of different objects and materials, from which you must try to trade and make as much profit as possible - in the currency of seashells.

The question for this article says that lots of goldfish have arrived at the island with scuba gear that they would like to sell. But only at the right price. We do not know how many goldfish there are, however, we do know that there is constant demand for scuba gear on the island, at the price of **1000** seashells. You have the ability to make **two** offers to the fish, a lower offer, and a higher offer. Each fish will have a reserve price and for each fish you will first pose your lower offer. If this offer is greater than there reserve price then a trade is made, and your profit is **1000 - the price of your lower offer**. However, if your lower offer is not greater than the fishes reserve price then you shall pose your higher offer, and again if this is greater than the reserve price, a trade is made and your profit is **1000 - the price of your higher offer**. The two offer prices are the same for every fish. *Your goal is to set prices that ensure a profitable trade with as many goldfish as possible.*

The only information we are given is the distribution of the fishes' reserve prices. **'The reserve price will be no lower than 900 and no higher than 1000. The probability scales linearly from least likely at 900 to most likely at 1000.'** And that **at the price of 900, the probability is 0**.

Note: the reserve prices are **real** values, while bid prices can only be **integer** values.

Solution

Firstly, we will define the probability density function of the reserve prices to be $P(x)$, where x is a reserve price in the interval $[900, 1000]$. We are told that

$P(x)$ scales **linearly**, and therefore it is in the form: $ax + b$. So now let's write down everything we know:

$$P(x) = ax + b \tag{1}$$

$$P(900) = 900a + b = 0 \tag{2}$$

$$\int_{900}^{1000} P(x)dx = 1 \tag{3}$$

The last equation here is simply stating that the sum of all the probabilities of the reserve prices will be equal to one.

Solving this integral:

$$\int_{900}^{1000} P(x)dx = 1 \tag{4}$$

$$= \int_{900}^{1000} (ax + b)dx = 1 \tag{5}$$

$$= \left[\frac{1}{2}ax^2 + bx\right]_{900}^{1000} = 1 \tag{6}$$

$$= (500000a + 1000b) - (405000a + 900b) = 1 \tag{7}$$

$$= 95000a + 100b = 1 \tag{8}$$

Now using equation (8) and (2) we can solve simultaneously to find the values of a and b.

First multiplying equation 2 by 100

$$90000a + 100b = 0 \tag{9}$$

Subtracting (9) from (8)

$$5000a = 1 \tag{10}$$

$$\implies a = \frac{1}{5000} \tag{11}$$

Now substituting back into 2

$$\frac{900}{5000} + b = 0 \tag{12}$$

$$\implies b = -\frac{9}{50} \tag{13}$$

Now that we have the unique values of a and b, we can redefine $P(x)$ in terms of these values.

$$P(x) = \frac{1}{5000}x - \frac{9}{50} \tag{14}$$

With the distribution now well-defined, we need a way to calculate our expected profit for any given bids. So we will define a profit function $Prof(b_1, b_2)$ where b_1 is our lower offer, and b_2 is our higher offer. We will also let Y be a random reserve price from the distribution.

$$Prof(b_1, b_2) = \left\{ \begin{array}{ll} 1000 - b_1, & Y < b_1 \\ 1000 - b_2, & b_1 < Y < b_2 \\ 0, & otherwise \end{array} \right\} \tag{15}$$

Now, in place of taking a random sample (Y) we can instead multiply the profit by the probability of Y being less than our offers. Which will give us an expected value of the profit function

$$\mathbb{E}[Prof(b_1, b_2)] = (1000 - b_1)(P(Y < b_1)) + (1000 - b_2)(P(b_1 < Y < b_2)) \tag{16}$$

Then, using $P(x)$ which we defined earlier, we can determine $P(Y < b_1)$

$$P(Y < b_1) = \int_{900}^{b_1} P(x)dx \tag{17}$$

$$= \int_{900}^{b_1} (\frac{1}{5000}x - \frac{9}{50})dx \tag{18}$$

$$= \left[\frac{1}{10000}x^2 - \frac{9}{50}x\right]_{900}^{b_1} \tag{19}$$

$$= \frac{1}{10000}b_1^2 - \frac{9}{50}b_1 + 81 \tag{20}$$

We can use the same method as above to also determine $P(b_1 < Y < b_2)$

$$P(b_1 < Y < b_2) = \left[\frac{1}{10000}x^2 - \frac{9}{50}x\right]_{b_1}^{b_2} \tag{21}$$

$$= \frac{1}{10000}b_2^2 - \frac{9}{50}b_2 - \frac{1}{10000}b_1^2 + \frac{9}{50}b_1 \tag{22}$$

Now using these expressions, we can rewrite $\mathbb{E}[Prof(b_1, b_2)]$

$$\begin{aligned} \mathbb{E}[Prof(b_1, b_2)] = (1000 - b_1)\left(\frac{1}{10000}b_1^2 - \frac{9}{50}b_1 + 81\right) \\ + (1000 - b_2)\left(\frac{1}{10000}b_2^2 - \frac{9}{50}b_2 - \frac{1}{10000}b_1^2 + \frac{9}{50}b_1\right) \end{aligned} \tag{23}$$

$\mathbb{E}[Prof(b_1, b_2)]$ is now our objective function to maximize, and I will refer to it as $O(b_1, b_2)$ for ease of notation.

Our goal was to maximize profits from our trading. We can therefore reframe this as maximizing our objective function with respect to b_1 and b_2. Firstly, we can actually plot this function as a surface, with x and y as our two bids, and z as the value of our objective function (expected profit) with these bid values and this will give us a good visualization of our problem.

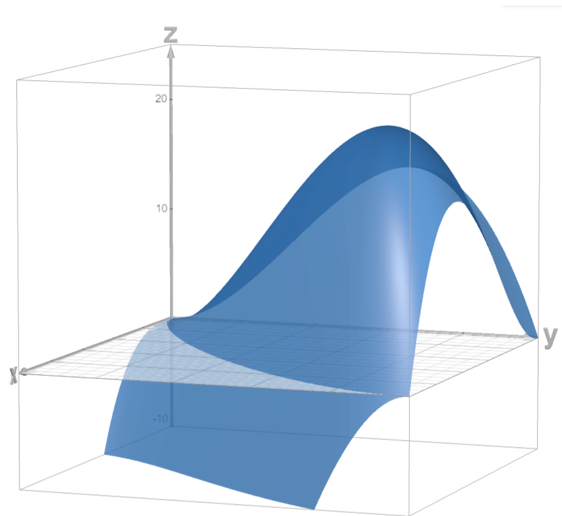

Figure 1: Plot of objective function with b_1 and b_2 values in the range $[900, 1000]$ in Desmos

Now, with this visualization, we can clearly see the shape of the objective function and the maxima that we are trying to locate. To determine the location of a stationary point on a surface, we can take the partial derivatives of $O(b_1, b_2)$ with respect to each input (b_1, b_2) and set both equal to zero.

$$\frac{\partial O}{\partial b_1} = -\frac{3}{10000}b_1^2 + \frac{2}{10000}b_1 b_2 + \frac{9}{25}b_1 - \frac{9}{50}b_2 - 81 \tag{24}$$

$$\frac{\partial O}{\partial b_2} = \frac{1}{10000}b_1^2 - \frac{3}{10000}b_2^2 - \frac{9}{50}b_1 + \frac{14}{25}b_2 - 180 \tag{25}$$

Setting both expressions equal to zero and solving simultaneously

$$\frac{\partial O}{\partial b_1} = 0 \tag{26}$$

$$\frac{\partial O}{\partial b_2} = 0 \tag{27}$$

Rearranging (24) for b_2

$$\frac{2}{10000}b_1b_2 - \frac{9}{50}b_2 = \frac{3}{10000}b_1^2 - \frac{9}{25}b_1 + 81 \tag{28}$$

$$2b_1b_2 - 1800b_2 = 3b_1^2 - 3600b_1 + 8100000 \tag{29}$$

$$b_2 = \frac{3b_1^2 - 3600b_1 + 8100000}{2(b_1 - 900)}, b_1 \neq 900 \tag{30}$$

$$b_2 = \frac{3(b_1^2 - 1200b_1 + 2700000)}{2(b_1 - 900)}, b_1 \neq 900 \tag{31}$$

$$b_2 = \frac{3((b_1 - 300)(b_1 - 900))}{2(b_1 - 900)}, b_1 \neq 900 \tag{32}$$

$$b_2 = \frac{3(b_1 - 300)}{2}, b_1 \neq 900 \tag{33}$$

Substituting back into (25)

$$\frac{1}{10000}b_1^2 - \frac{3}{10000}\left(\frac{3(b_1 - 300)}{2}\right)^2 - \frac{9}{50}b_1 + \frac{14}{25}\left(\frac{3(b_1 - 300)}{2}\right) - 180 = 0 \tag{34}$$

Simplifying and solving for b_1

$$b_1^2 - \frac{27(b_1 - 300)^2}{4} - 1800x + 8400(b_1 - 300) - 1800000 = 0 \tag{35}$$

$$-\frac{23b_1^2}{4} + 10650b_1 - 4927500 = 0 \tag{36}$$

$$b_1 = \frac{-10650 \pm 300}{2(-\frac{23}{4})} \tag{37}$$

$$b_1 = 900 \ \text{ or } \ b_1 = \frac{21900}{23} \tag{38}$$

When we defined b_2 in terms of b_1 in equation (33), we acknowledged that b_1 cannot equal 900. Therefore, our only solution for b_1 is $\frac{21900}{23} \approx 952.17$. Substituting this back into (24)

$$-\frac{3}{10000}\left(\frac{21900}{23}\right)^2 + \frac{2}{10000}\frac{21900}{23}b_2 + \frac{9}{25}\frac{21900}{23} - \frac{9}{50}b_2 - 81 = 0 \tag{39}$$

Simplifying and solving for b_2

$$-\frac{143883}{529} + \frac{219}{1150}b_2 + \frac{7884}{23} - \frac{9}{50}b_2 - 81 = 0 \tag{40}$$

$$\frac{6}{575}b_2 - \frac{5400}{529} = 0 \tag{41}$$

$$\implies b_2 = \frac{22500}{23} \tag{42}$$

$$b_2 \approx 978.26 \tag{43}$$

Checks

Now we have solved for our two offers which maximize our objective function. However, if you remember, our bids must be integers. Therefore, we could just round to the nearest integer, giving values of $b_1 = 952$ and $b_2 = 978$, however I will complete a small grid search to ensure the rounding produces the optimal values.

The lower value for b_1 will be 952, and the upper value will be 953. As for b_2, the lower value will be 978, and the upper value 979.

Creating our grid

$O(b_1, b_2)$	952	953
978		
979		

Now for each pair in the grid, we can substitute the values into our objective function as b_1 and b_2. Doing so produces the following results

$O(b_1, b_2)$	952	953
978	20.4152	20.4073
979	20.4069	20.4095

After completing this quick grid search, we get the final result that the optimal value for b_1 is 952, and the optimal value for b_2 is 978.

Simulation

After creating a simulation to run locally in the form of a basic python script, I completed an exhaustive grid search, as well as a SciPy optimization.

I first chose to run the simulation on 1 million samples from the distribution, and then generated random reserve prices by first generating random values in the range $[0, 1]$ and then using the inverse of the cumulative distribution function of reserve prices to generate the reserve price samples.

For the grid search, I then created a list of tuples which contained every combination of different low and high offers. This resulted in just over 5000 different combinations of prices. After evaluating each of these pairs on 1 million samples, the best performing pair was also $(952, 978)$. With a simulated profit per fish of 20.4195. This search took approx. 60.2 seconds.

Note: After rerunning this search with 10 samples, pair $(952, 978)$ was still the highest performing, scoring a profit per fish of 20.4218. With the simulation taking 828 seconds (13 minutes 48 seconds).

Now, using the minimize function from SciPy.optimize, with the metric being to minimize the negative profit per fish i.e. maximize profit per fish on 1 million samples, the optimal pair is yet again (952.2, 978.2). With a profit per fish of 20.3968.

Note: After running this simulation again with 10 million samples, the optimal pair was again $(952.2, 978.2)$

Conclusion

A lower bid of 952 and a higher bid of 978 is most certainly the optimal answer for this question, averaging a profit per fish of 20.4 on large sample sizes of reserve prices. Backed up by not only closed-form solutions, but also exhaustive grid-search and computational optimization.