Project 2

MGMTMFE 405
Instructor: L. Goukasian

You will need to write codes for all the parts of the project. Make sure the codes work properly and understand the ideas behind each problem below. You may be asked to demonstrate how the codes work, by running them, and interpret the results. Code quality, speed, and accuracy will determine the grades.

1. Generate a series $(X_i, Y_i) for i = 1, \dots, n$ of Bivariate-Normally distributed random vectors, with the mean vector of (0,0) and the variance – covariance matrix of $\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$. Compute the following by simulation:

$$\rho(a) = \frac{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \acute{X})(Y_i - \acute{Y})}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \acute{X})^2}\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \acute{Y})^2}}$$

where $\acute{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\acute{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Take $n = 1000$ and $a = -0.7$.
Inputs: *seed*
Outputs: Values: ρ.

2. Evaluate the following expected values by using Monte Carlo simulation:

$$\mathrm{E}\,[\max(0, (X^3 + \sin(Y) + X^2 Y))]$$

where X and Y have $N(0,1)$ distribution and a correlation of $\rho = 0.6$.
Inputs: *seed*
Outputs: Values: *E*

3. (a) Estimate the following expected values by simulation:

$$E\left(W_5{}^2 + sin(W_5)\right) \text{ and } E\left(e^{\frac{t}{2}}cos(W_t)\right) \text{for } t = 0.5, 3.2, 6.5.$$

Here W_t is a Standard Wiener Process.

(b) How are the values of the last three integrals ($for\ the\ cases\ \ t = 0.5,\ 3.2,\ 6.5$) related?

(c) Now use a variance reduction technique (whichever you want) to compute the expected values in part (a). Do you see any improvements? Comment.
Inputs: *seed*
Outputs:

i. Values: *Ea1*, *Ea2*, *Ea3*, *Ea4* for part (a); *Eb1*, *Eb2*, *Eb3*, *Eb4* for part (b)

ii. Writeup: comments in a .pdf file for part (c)

4. Let S_t be a Geometric Brownian Motion process: $S_t = S_0 e^{\left(\sigma W_t + \left(r - \frac{\sigma^2}{2}\right)t\right)}$, where $r = 0.04, \sigma = 0.2, S_0 = \$88$, W_t is a Standard Brownian Motion process (Standard Wiener process).

(a) Estimate the price **c** of a European Call option on the stock with $T = 5, X = \$100$ *by using Monte Carlo simulation*.

(b) Compute the exact value of the option c by the Black-Scholes formula. Now use variance reduction techniques (whichever you want) to estimate the price in part (a) again. Did the accuracy improve? Comment.

Inputs: *seed*

Outputs:

i. Values: *Ca1*, *Ca2* for part (a); *Cb1*, *Cb2* for part (b)
ii. Writeup: comments in a .pdf file for part (b)

5. (a) For each integer number n from 1 to 10, use 1000 simulations of S_n to estimate ES_n, where S_t is a Geometric Brownian Motion process: $S_t = S_0 e^{\left(\sigma W_t + \left(r - \frac{\sigma^2}{2}\right)t\right)}$, where $r = 0.04, \sigma = 0.18, S_0 =$ \$88. Plot all of the above $E(S_n)$, for n ranging from 1 to 10, in one graph.

(b) Now simulate 6 paths of S_t for $0 \leq t \leq 10$ (defined in part (a)) by dividing up the interval [0, 10] into 1,000 equal parts.

(c) Plot your data from parts (a) and (b) in one graph.

(d) What would happen to the ES_n graph if you increased σ from 18% to 35%? What would happen to the 6 plots of S_t for $0 \leq t \leq 10$, if you increased σ from 18% to 35%?

Inputs: *seed*

Outputs:

i. Graphs: plots in a .png file for part (c)
ii. Writeup: comments in a .pdf file for part (d)

6. Consider the following integral for computing the number π: $4\int_0^1 \sqrt{1-x^2}dx = \pi$.

(a) The integral above can be estimated by a simple numerical integration using, say Euler's discretization (or any other discretization) scheme. Estimate the integral by using the Euler's discretization scheme.
(b) Estimate the integral by Monte Carlo simulation.
(c) Now try the Importance Sampling method to improve the estimate of π in part (b). Comment on errors and improvements.

Inputs: *seed*

Outputs:

i. Values: *Ia* for part (a), *Ib* for part (b)
ii. Writeup: comments in a .pdf file

7. (***Optional, NOT for grading***) (Very popular interview questions on stochastic calculus)
What can you say about the distributions of the following integrals? Can you compute them explicitly?

$$\int_0^T W_t dt \text{ and } \int_0^T W_t dW_t$$

8. (***Optional, NOT for grading***)
We (two of us) are to play on a table in the next room. We each have bags of an infinite number of identical quarters (American 25-cents). We will take it in turns to put a quarter on the table. Quarters may not overlap on the table. When there is no more room left on the table to put another quarter, then the winner is the last person to put a quarter on the table. Does there exist a winning strategy for the starter of the game? What about the second player? Assume the table is of rectangular shape. What if the table was of a circular shape?