Convex Optimization

Assessing data, controlling stuff, and making decisions.

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UC Berkeley, Edge Effect

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Goals

Today's Goals

1. Optimization is interesting.
2. Optimization is broadly applicable.
3. Optimization is relevant to members of our group.

This is not intended to be a robust and rigorous explanation of convex optimization.

1. Convex Optimization Theory
   - Optimization standard forms
   - Convex sets and functions
   - Building convex optimization problems

2. Applications
   - Portfolio Optimization

Optimization

What is optimization?

- The solution to an optimization problem represents the "best choice" (objective) of $x \in \mathbb{R}^n$ among all choices that meet firm requirements (constraints).
- In general, there are no analytical solutions to general or convex optimization problems.
- For convex optimization problems, we do have efficient iterative algorithms to find the solution, and the solution comes with strong guarantees such as unicity and constraint satisfaction.

Optimization General Formulation

$$\begin{aligned}
\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\
\text{subject to} \quad & g_i(x) = 0 && i = 1, \ldots, l \\
& h_j(x) \leq 0 && j = 1, \ldots, m
\end{aligned}$$

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- g_i, h_j are the constraints for the problem. These are usually known or obvious.
- No restrictions on f, g_i, h_j at this point.

Convex Optimization Formulation

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- g_i must all be **affine** (linear + DC offset).

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- f, the objective function, must be *convex*.
- g_i must all be **affine** (linear + DC offset).
- g_i, h_j must define a convex set.

Convex Sets

Convex Sets

The set $\mathcal{X} \subseteq \mathbb{R}^n$ is convex if, for any $x, y \in \mathcal{X}$, the segment $[x, y]$ lies in $\mathcal{X}$:

$$\{\alpha x + (1-\alpha)y : 0 \le \alpha \le 1\} \subseteq \mathcal{X}$$

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Epigraph

The *epigraph* of a function f, denoted Epi f is the set of points lying above the graph:

$$\{(x, y) \in \mathbb{R}^{n+1} : x \in \mathcal{X}, y \geq f(x)\}$$

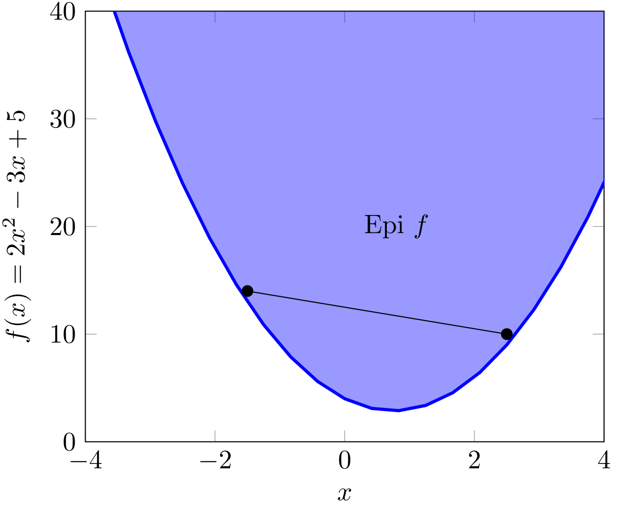
Convex Functions

Convex Functions

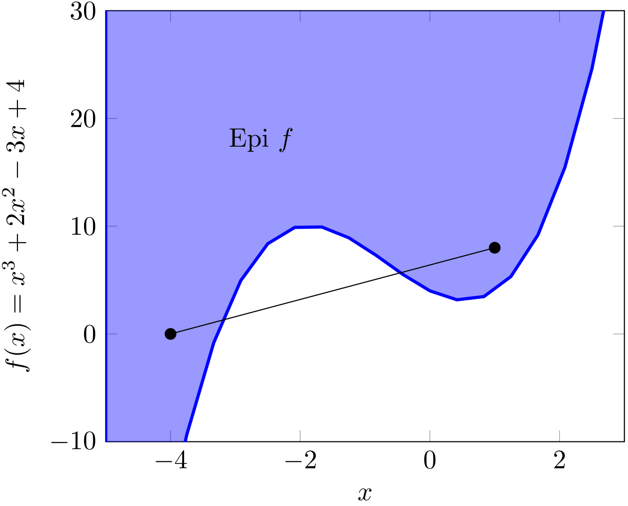
Let $\mathcal{X}$ be a convex set. A function f is convex *iff* Epi f is a convex set, or equivalently, if the segment $[f(x), f(y)]$ always lies above the function $\forall x, y \in \mathcal{X} \subseteq \mathrm{dom}\ f$:

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

Convexity



Non-Convexity



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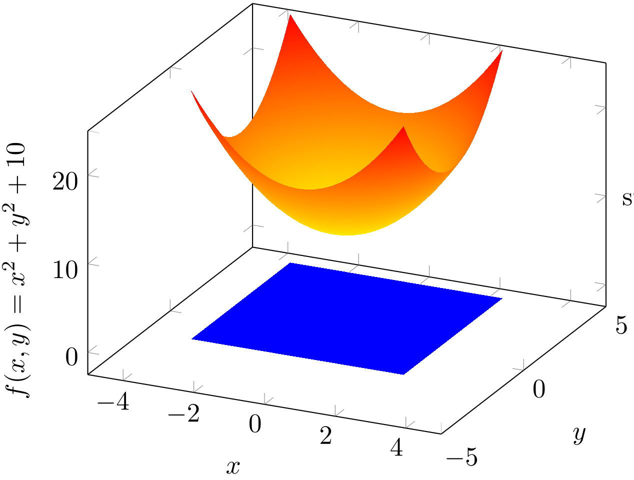
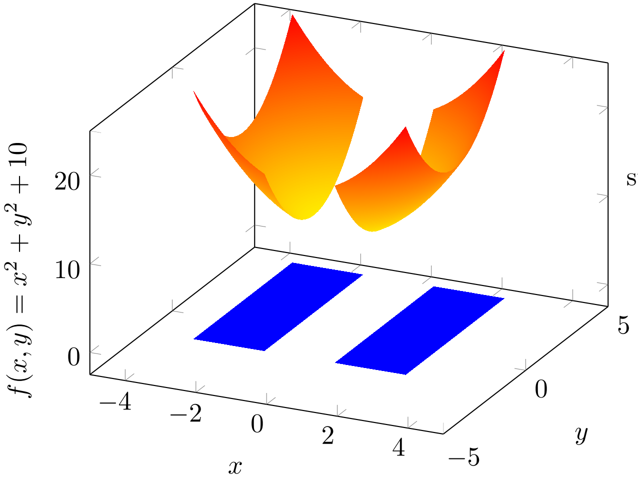


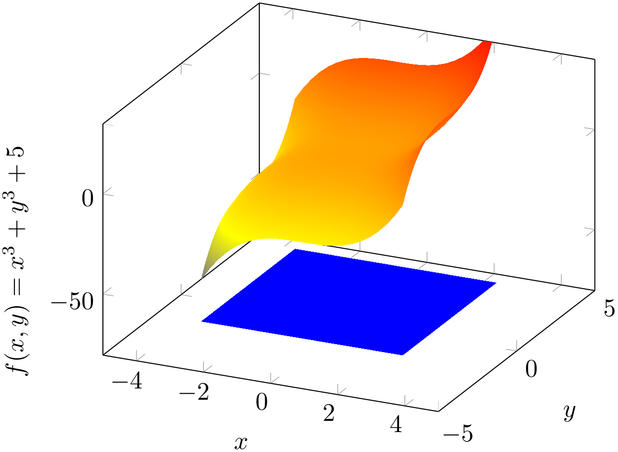
$$\begin{aligned} \underset{x,y}{\text{minimize}} \quad & x^2 + y^2 + 10 \\ \text{subject to} \quad & -3 \leq x, y \leq 3 \end{aligned}$$

Non-Convex



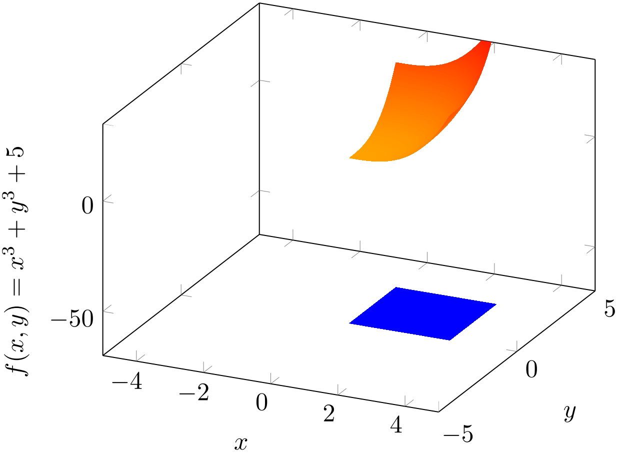
$$\begin{aligned} \underset{x,y}{\text{minimize}} \quad & x^2 + y^2 + 10 \\ \text{subject to} \quad & -3 \leq x \leq -1 \\ & \cup\, 1 \leq x \leq 3 \\ & -3 \leq y \leq 3 \end{aligned}$$

Non-Convex



$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & x^3 + y^3 + 5 \\ \text{subject to} & -3 \leq x, y \leq 3 \end{array}$$

Convex



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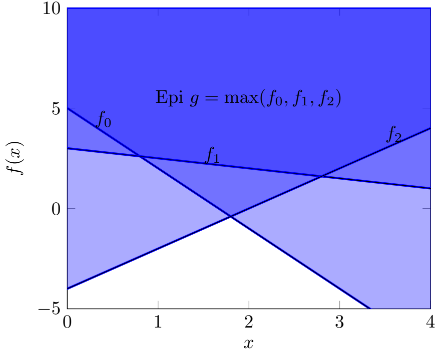
Convex Building Blocks

- Affine functions (convex and concave).
- Functions in quadratic form: $x^\top Q x : Q \succeq 0$.
- Any $f : \nabla^2 f(x) \succeq 0$.
- All norms: $\|\cdot\|$.
- Point-wise maximum.

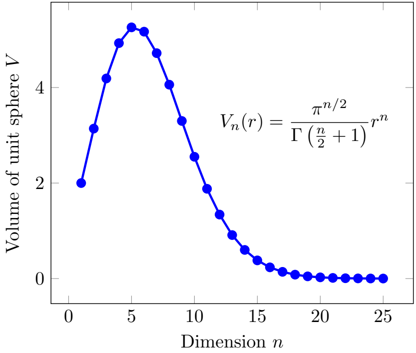


Operations that Preserve Convexity

- Intersection of any number of convex sets.
- Non-negative weighted sum of convex functions.
- Any composition of a convex function with an affine function.
- More general function results: Let $h : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$, and $f(x \in \mathbb{R}^n) = h(g(x))$.
  - h convex and non-decreasing, g convex $\Rightarrow f$ is convex.
  - h convex and non-increasing, g concave $\Rightarrow f$ is convex.

Using Intuition

- Geometrical intuition can be powerful.
- Beware the intuition traps for large dimensional problems.



Linear Algebra Equivalence

$$\text{Solve } Ax = y \Leftrightarrow \underset{x}{\text{minimize}} \, \|Ax - y\|_2$$

- A is square, full rank $\Rightarrow x = A^{-1}y$.
- A is full column rank $\Rightarrow x = (A^\top A)^{-1}A^\top$ (least norm solution).
- A is full row rank $\Rightarrow x = A^\top(AA^\top)^{-1}$.

Linear Algebra Equivalence

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \|Ax - y\|_2 \\ \text{subject to} \quad & x \succeq 0 \end{aligned}$$

- What do we do now??

Linear Algebra Equivalence

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \|Ax - y\|_2 \\ \text{subject to} \quad & x \succeq 0 \end{aligned}$$

- What do we do now??
- One of optimization's greatest attributes is the rigorous treament of constraints.

Convex Optimization

Implications of convexity for optimization:

- Provably global solution.
- Richness in modeling phenomena beyond linear systems.
- Explicit accounting for constraints.
- Efficient algorithms.

Convex Optimization

Challenges for convex optimization:

- Integer programs (e.g., binary allocation) pervasive.
- Real constraints often not defined as convex sets.
- As yet, no strategy for incorporating uncertainty.

Convex Optimization

Options?

- All non-convex problems can be *relaxed* to convex approximations.
- Sometimes non-convex problems can be fully transformed to convex ones (e.g., geometric programming).
- *Robust programming* for uncertainty.

Convex Optimization Classes

Some major classes of convex optimization problems:

- Linear programs (LP).
- Quadratic programs (QP).
- Second-order cone programs (SOCP).
- Semi-definite programs (SDP).
- Geometric programs (GP).

$$\text{LP} \subset \text{QP} \subset \text{SOCP} \subset \text{SDP}.$$

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Portfolio Optimization in Finance

Classic portfolio problem: How to invest across n assets.

- x_i is the amount of money placed in the i-th stock.
- p_i is the price change over the investment period.
- $p \in \mathbb{R}^n$ is a random vector.
- $\Sigma \in \mathbb{R}^{n \times n}$ is the covariance matrix for p.
- We have varying levels of information for p and Σ.
- $r \in \mathbb{R} = p^\top x$ is the total return in dollars.
- How best to choose $x \in \mathbb{R}^n$ (allocate money)?
- The choice in x is a tradeoff between return and variance (risk).

Markowitz QP for Portfolio Optimization

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & x^\top \Sigma x \\
\text{subject to} \quad & \bar{p}^\top x \geq r_{\min} \\
& \mathbf{1}^\top x = 1 \\
& x \succeq 0
\end{aligned}
$$

Assumptions

- We know $\bar{p} = \mathbb{E}[p]$.
- We know the matrix Σ (e.g., model from history).

SOCP Portfolio Optimization

$$
\begin{array}{ll}
\underset{x}{\text{maximize}} & \bar{p}^\top x \\
\text{subject to} & \bar{p}^\top x + \Phi^{-1}(\beta) \left\| \Sigma^{1/2} x \right\|_2 \geq \alpha \\
& \mathbf{1}^\top x = 1 \\
& x \succeq 0
\end{array}
$$

$\mathbf{prob}(r \leq \alpha) \leq \beta \;\Leftrightarrow\; \bar{p}^\top x + \Phi^{-1}(\beta) \left\| \Sigma^{1/2} x \right\|_2 \geq \alpha.$

Assumptions

- $\bar{p}$ and Σ known.
- $p \in \mathbb{R}^n$ is a Gaussian random variable.

Convex Optimization

Should I put term in objective or constraints?

- Constraints are reserved for strict limits.
- Can tradeoff different entities in the objective.
- Can put something in the constraints as a threshold, then optimize over something else in the objective.
- With multiple design parameters, can mix and match between objective and constraints.
- Ultimately, you decide based on application.

SDP Portfolio Assessment

Consider incomplete knowledge of Σ, e.g.:

$$\Sigma = \begin{bmatrix} 0.1 & + & - \\ + & 0.03 & ? \\ - & ? & 0.6 \end{bmatrix}$$

We can assess the bounds on the risk of a given x with an SDP.

$$\text{Let } \mathbf{P} = \{\Sigma = \Sigma^\top \in \mathbf{S}^n : \Sigma_{11} = 0.1, \Sigma_{22} = 0.3 \\ \Sigma_{33} = 0.6, \Sigma_{12} \geq 0, \Sigma_{13} \leq 0\}$$

SDP Portfolio Assessment

$$
\begin{array}{ll}
\underset{\Sigma}{\text{maximize}} & x^\top \Sigma x \\
\text{subject to} & \Sigma \in \mathbf{P} \\
& \Sigma \succeq 0
\end{array}
$$

- SDP optimization is over a matrix variable.
- The solution is the maximum variance for a given portfolio distribution x (worst case scenario).
- Replacing the max above with a min gives the most optimistic variance (best case scenario).

Questions?

References

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