Newton's Method

Bisection Method

- Let's say we have a function, f(**x**)
- We want to find an **x** such that f(**x**) = 0
- If we know two points **a**,**b** such that **a***__b__ < 0, then due to the Intermediate Value Theorem we know that there is an **x** such that
  - **a** < **x** < **b** and f(**x**) = 0
- So let's cut our search in half, let **c** = (**a**+**b**)/2
- If **a***__c__ < 0 then we know **a** < **x** < **c**, likewise with **b**
- Repeat this process until the we are satified enough and chose the midpoint

Linear Approximation Refresh

- Lines are easy to work with
- I can easily solve $0 = m x + b$, $x = -b/m$
- So, if I have a complicated function and want to find $f(x) = 0$
  - This sounds pretty difficult
- Let's just pretend it's a line, then it is easy

Newton's Method

- If we have a function, f(**x**) and its derivative, f'(**x**), then we can calulate its linear approximation near **a** as:
- $\mathbf{y} = f'(\mathbf{a})\ (\mathbf{x} – \mathbf{a}) + f(\mathbf{a})$
- So to find when the approximation is 0 we get:
- $\mathbf{x} = \mathbf{a} – (f(\mathbf{a})\ /\ f'(\mathbf{a}))$
- Now we have a better guess to where f(**x**) = 0
- Repeat this process with our better guess until we are satified enough

Satified Enough?

- So, when are we satified enough?
- Ideally it is when $|\mathbf{x} - \mathbf{r}| < \boldsymbol{\varepsilon}$, where $\mathbf{x}$ is our guess, $\mathbf{r}$ is the real answer, and $\boldsymbol{\varepsilon}$ is the max error that we can tolerate
- But sadly usually we don't know r
  - If we did then why are we doing this?
- So instead we use the residual, the difference between the current guess and the last guess

Example 1

- Find where $x^3 – 2x^2 + 1/2x - 5$ and $-x^2 + 1$ intersect
- $f(x) = (x^3 – 2x^2 + 1/2x – 5) - (-x^2 + 1)$
- $f(x) = x^3 – x^2 + 1/2x – 6$
- $f'(x) = 3x^2 -2x + 1/2$
- $x = a – (f(a) / f'(a))$
- $x = a - (a^3 – a^2 + 1/2 a – 6)/(3 a^2 -2 a + 1/2)$

Example 1

- Let's guess a = 2 (1 accurate digit)
- x = 2 - (2^3 – 2^2 + 1/2*2 - 6)/(3*2^2 -2*2 + 1/2)
- x = 2 - (8 – 4 + 1 – 6)/(3*4 - 4 + 1/2)
- x = 2 - (-1)/(8.5)
- x ≈ 2.1176470588235294117.... (3 accurate digits)
- Next iterations:
- x ≈ 2.1103582611609714207... (5 accurate digits)
- x ≈ 2.1103288013410904930... (9 accurate digits)
- r ≈ 2.1103288008611313326355 14...

Example 2

- Use Newton's method to find sqrt(2)
- Rewrite to: $f(x) = x^2 - 2 = 0$
- $f'(x) = 2x$
- $x = a - (a^2 - 2)/2a$
- If $a = 1.5$, $x = 1.41667$
- $a = 1.41667$, $x = 1.41422$
- $a = 1.41422$, $x = 1.41421$

Questions?