O17e

June 26, 2024

1 O17e Diffraction

Group #13

NG Kuok Fung | #3797903

Edgar Harutyunyan | #3775171

Overview of Tasks

Task 0

- Analytically calculate the Fourier transform of a single-slit and a double-slit function, and compare these with the familiar Fraunhofer diffraction-patterns for these objects.
- Using FFT (Fast-Fourier-Transform), determine the Fourier transforms of the functions $F(k) = sinc(2\pi k) = sin(2\pi k)/(2\pi k)$ and $F^2(k)$. Discuss this result in the light of the convolution theorem.

Task 1

Measure the diffraction pattern of a single-slit in the Fraunhofer regime, and for three distances z between slit and camera in the Fresnel regime.

In the Fraunhofer regime, determine the slit width b:

(a) From the position of the minima for various diffraction orders

(b) From the FFT of the intensity profile.

In the Fresnel regime,

(c) Compare the measured diffraction patterns with the theory.

Task 2

Measure the diffraction pattern of a double-slit in the Fraunhofer regime, and for three distances between slit and camera in the Fresnel regime.

In the Fraunhofer regime, determine:

(a) the slit distance g from the position of the minima for various diffraction orders

(b) the slit distance g and the slit width b from the fit of the Fraunhofer diffraction-pattern to the data

(c) the slit distance g and the slit width b from the FFT of the intensity profile.

In the Fresnel regime,

(d) compare the measured diffraction patterns with the theory.

1.0.1 Task 0

Task Definition

Analytically calculate the Fourier transform of a single-slit and a double-slit function and compare these with the familiar Fraunhofer diffraction-patterns for these objects. Using FFT (Fast-Fourier-Transform), determine the Fourier transforms of the functions $F(k) = sinc(2\pi k) = sin(2\pi k)/(2\pi k)$ and $F^2(k)$. Discuss this result in the light of the convolution theorem.

For a single-slit aperture of width b:

$$f(x) = rect\left(\frac{x}{b}\right) = \begin{cases} 1, -b/2 \leq x \leq b/2 \\ 0, \text{otherwise} \end{cases} \tag{i.1}$$

For a double-slit aperture with slit separation g and slit width b:

$$f(x) = rect\left(\frac{x+g/2}{b}\right) + rect\left(\frac{x-g/2}{b}\right) = \begin{cases} 1, -(g+b)/2 \leq x \leq -(g-b)/2 \\ 1, (g-b)/2 \leq x \leq (g+b)/2 \\ 0, \text{otherwise} \end{cases} \tag{i.2}$$

The Fourier transform $F(k)$ of $f(x)$ is given by:

$$F(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \tag{i.3}$$

For the single-slit function Eq i.1:

$$F(k) = \int_{-b/2}^{b/2} e^{-ikx}dx = \frac{e^{-ikx}}{-ik}\Bigg|_{x=-b/2}^{x=b/2} = \frac{b}{kb/2}\frac{e^{-ikb/2}-e^{ikb/2}}{-2i} = \frac{b\sin(kb/2)}{kb/2}$$

$$F(k) = bsinc\left(k\frac{b}{2}\right) \tag{i.4}$$

For the double-slit function Eq i.2:

$$F(k) = \int_{-(g+b)/2}^{-(g-b)/2} e^{-ikx}dx + \int_{(g-b)/2}^{(g+b)/2} e^{-ikx}dx = \frac{e^{-ikx}}{-ik}\Bigg|_{x=-(g+b)/2}^{x=-(g-b)/2} + \frac{e^{-ikx}}{-ik}\Bigg|_{x=(g-b)/2}^{x=(g+b)/2}$$

$$= \frac{e^{ikg/2}}{k/2}\frac{e^{ikb/2}-e^{-ikb/2}}{2i} + \frac{e^{-ikg/2}}{k/2}\frac{e^{ikb/2}-e^{-ikb/2}}{2i}$$

$$= b\frac{\sin(kb/2)}{kb/2}\left(e^{ikg/2}+e^{-ikg/2}\right)$$

$$= b\frac{\sin(kb/2)}{kb/2} 2\cos(kg/2)$$

$$F(k) = 2bsinc\left(k\frac{b}{2}\right)\cos\left(k\frac{g}{2}\right) \tag{i.5}$$

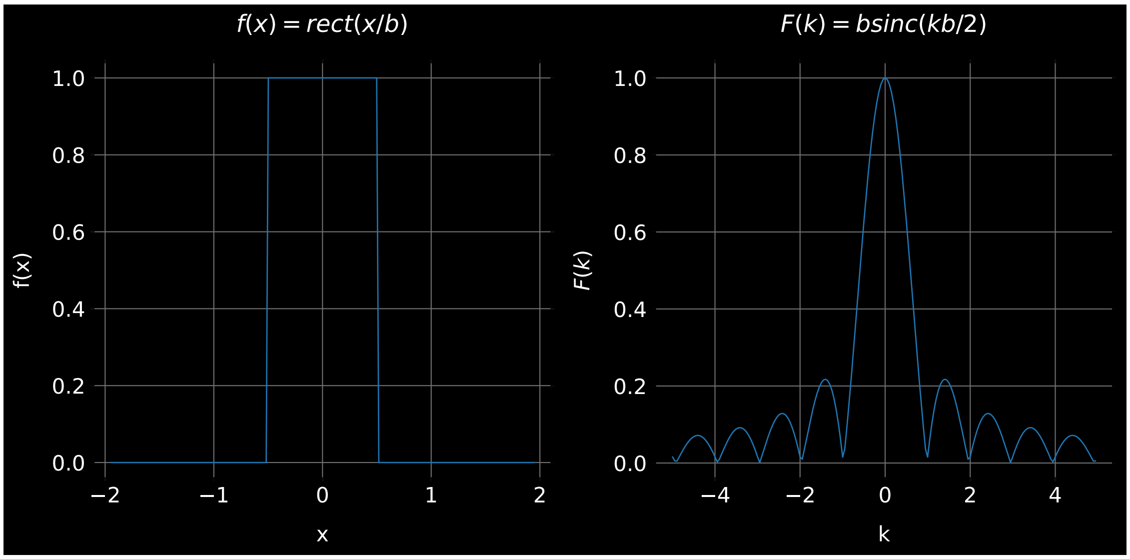


Figure i.1: Single Slit Aperture $f(x)$ and $F(k) = FFT(f(x))$.

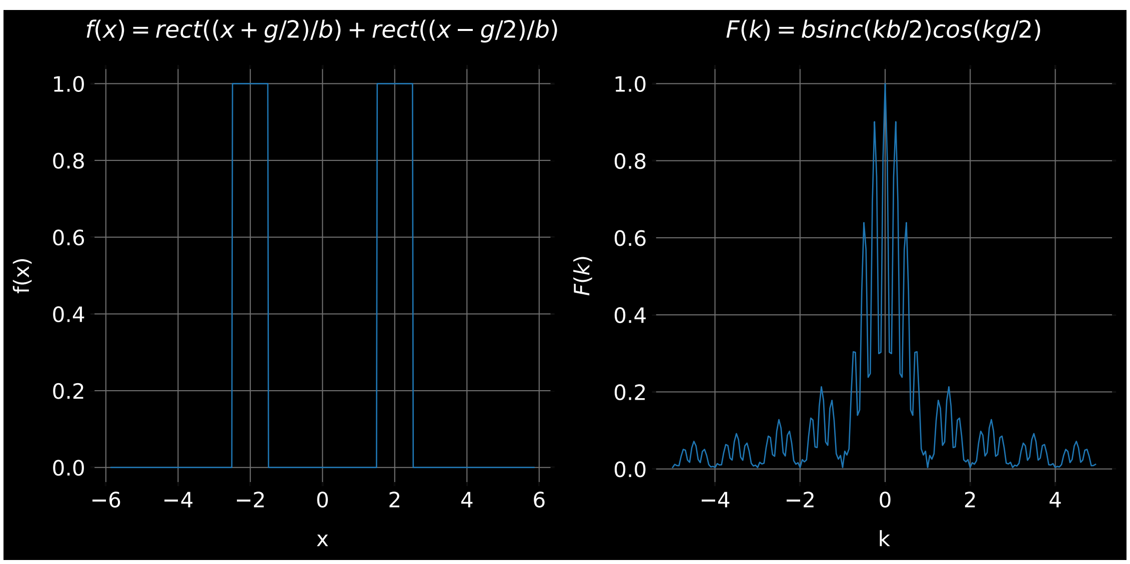
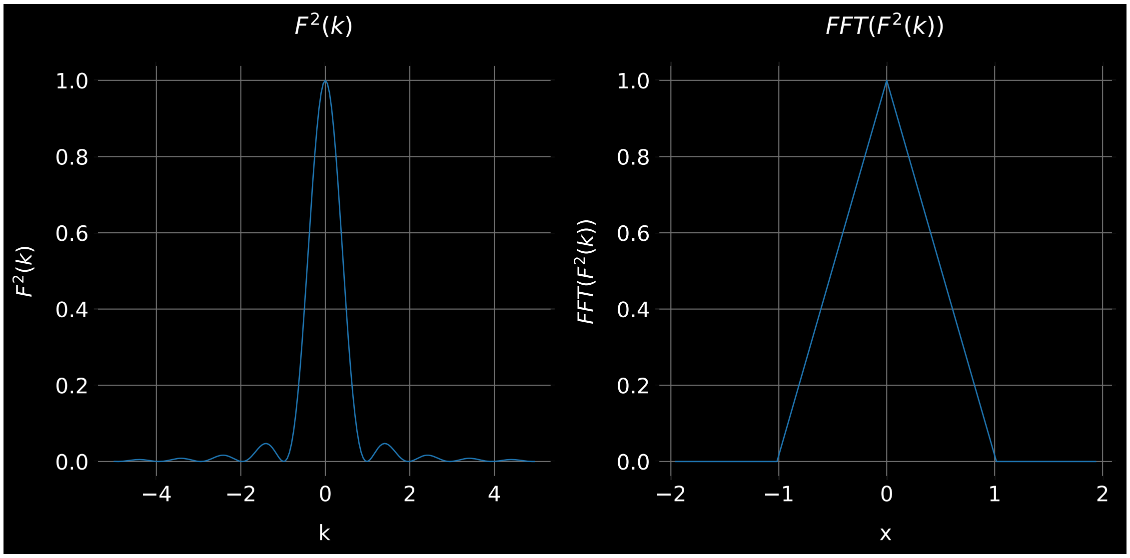
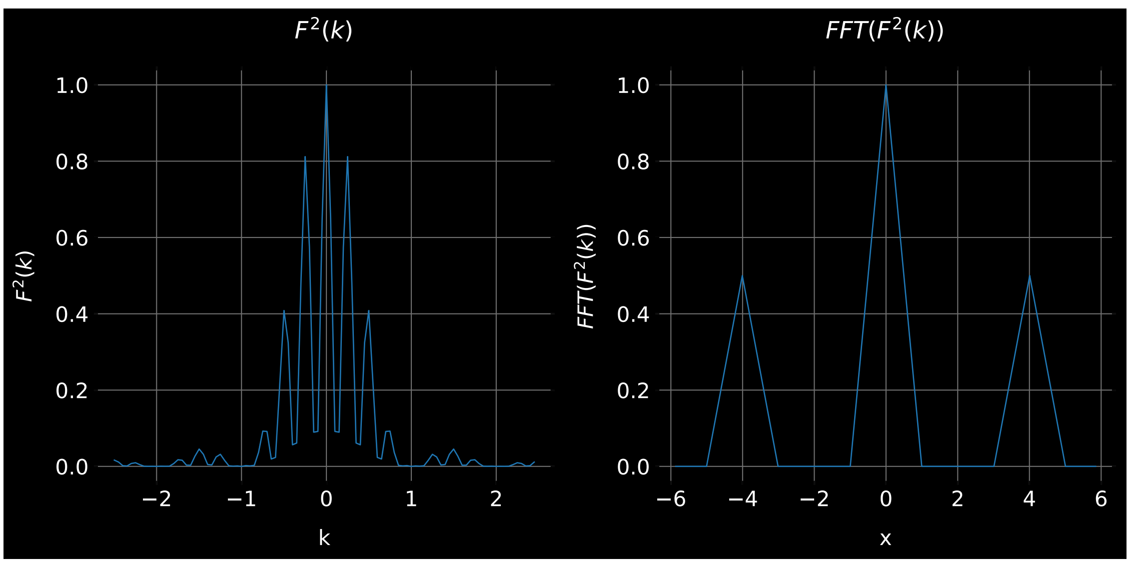


Figure i.2: Double Slit Aperture $f(x)$ and $F(k) = FFT(f(x))$.



$Figure\,i.3$: Single Slit Intensity $I \propto F^2(k)$ and $FFT(F^2(k))$.



$Figure\,i.4$: Double Slit Intensity $I \propto F^2(k)$ and $FFT(F^2(k))$.

Convolution theorem:

1. $F(k)$ is the Fourier Transform of the rectangular function $f(x)$ in the spatial domain.
2. According to the convolution theorem, the Fourier transform of the product of two functions F and G in the frequency domain or spatial frequency domain is the convolution of their Fourier transforms f and g in the spatial domain:

$$\mathcal{F}\{F(k) \cdot G(k)\} = f(x) * g(x)$$

3. By the convolution theorem, the FFT $F(k)^2$ is equivalent to taking the convolution of $f(x)$ with itself in the spatial domain.

$$\mathcal{F}\{F(k) \cdot F(k)\} = f(x) * f(x)$$

4. The convolution results in a triangular function as seen in Fig i.4.

1.1 Task 1

1.1.1 Task 1a

Task Definition

In the Fraunhofer regime, determine the slit width b from the position of the minima for various diffraction orders

Theoretical Basis

Light passsing through a single slit creates a diffraction pattern on a camera placed behind the slit. This pattern consists of bright spots (maxima) where light interferes constructively and dark spots (minima) where it interferes destructively.

The positions of the minima of arbitrary order (n) obey the following equation:

$$b\ sin\alpha_n = n\lambda$$

- Slit width: b
- Order of minima: n
- Wavelength of laser light: $\lambda = 635nm$
- Angle between optical axis and n-th order minima: α_n

Under small angle approximation $\alpha_n \approx 0$:

$$b\ \alpha_n \approx n\lambda \tag{1.1}$$

Geometrically, the relationship between x_n and α_n is derived as:

$$x_n = f\ tan\alpha_n$$

- Focal length of lens: $f = 300.8\ mm$
- Distance between optical axis and n-th order minima: x_n

Under small angle approximation $\alpha_n \approx 0$:

$$x_n \approx f\ \alpha_n \tag{1.2}$$

Substituing Eq 1.1 into Eq 1.2 yields:

$$x_n \approx f\frac{n\lambda}{b} \tag{1.3}$$

$$k \equiv \frac{f\lambda}{b}$$

$$b = \frac{f\lambda}{k}$$

The uncertainty is given by:

$$\mu_b = \left|\frac{\partial b}{\partial k}\right| \mu_k = \left|-\frac{b}{k}\right| \mu_k$$

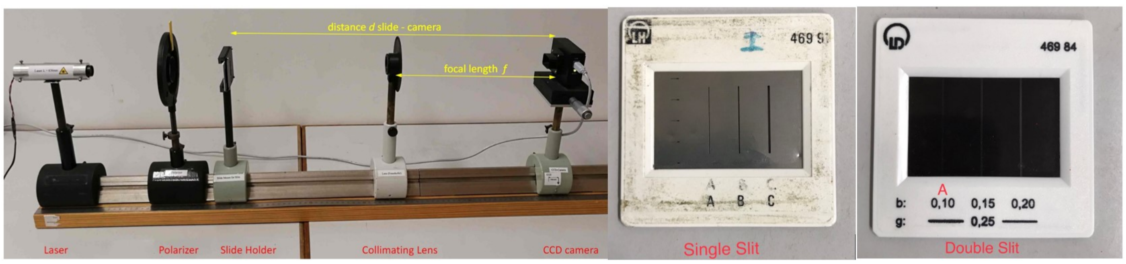


Figure 1.1: Setup

Procedure

1. A single slit "469-91-A" was inserted into the slide holder, and an old camera recorded the diffraction pattern **Fig1.1**.
2. During the experiment, the height of the optical elements were aligned to achieve the best diffraction pattern recording.
3. Subsequently, the pixel data was processed and used to plot a graph of x_n against n.
4. The slit width b was determined from linear regression using Eq 1.3.

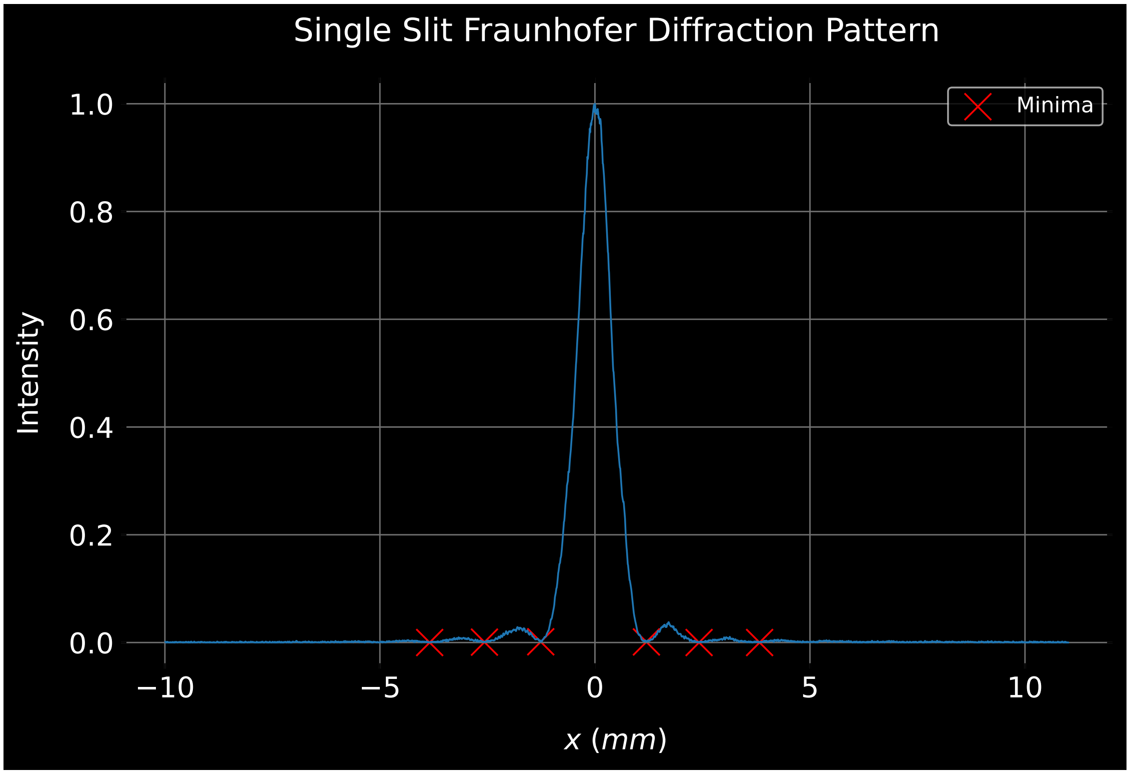


Figure 1.2: Single Slit Fraunhofer Diffraction Pattern

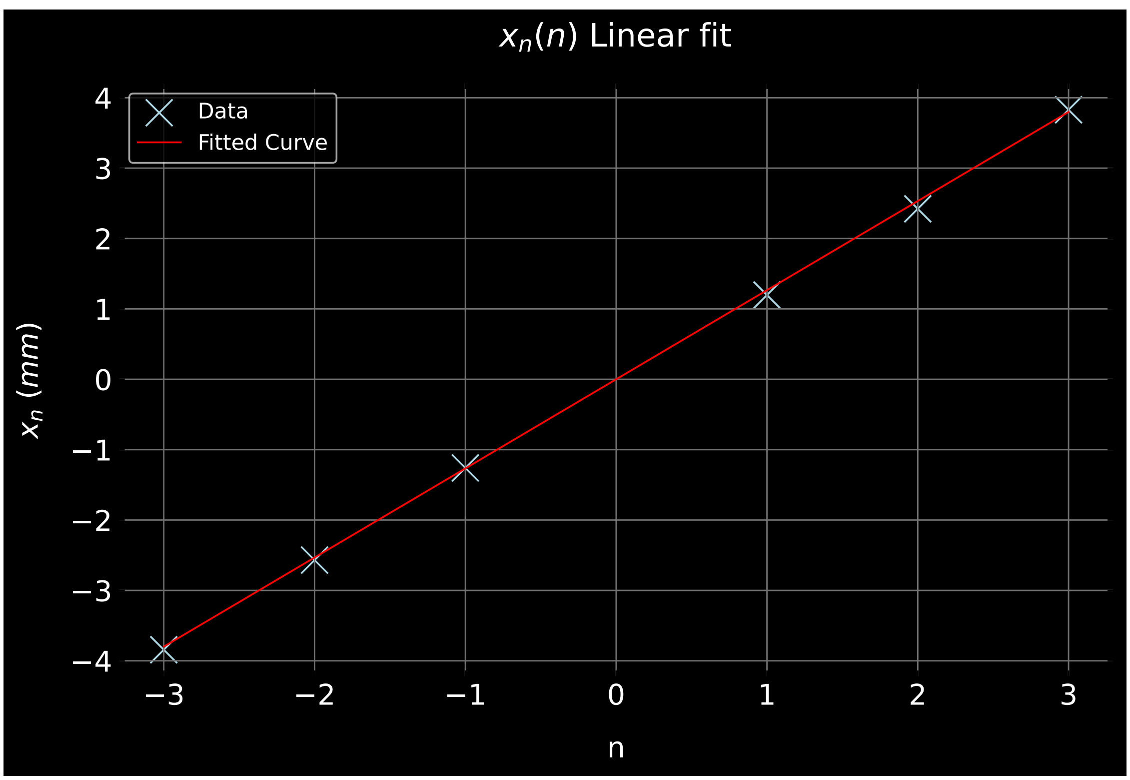


Figure 1.3: Linear fit of position with respect to minimum diffraction order.

$$b = (0.15 \pm 0.001)\ mm$$

1.1.2 Task 1b

Task Definition

In the Fraunhofer regime, determine the slit width b from the Fast Fourier Transform (FFT) of the intensity profile.

Theoretical Basis

The camera records both the intensity (I) and the position (x) of the diffraction pattern.

The inverse Fourier transform of is given by:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_x) e^{ik_x x}\, dk_x$$

It is given that intensity I is proportional to $F(k_x)$, which implies that I is a function of k_x.

Hence, the aperture function f(x) in real space may be found by supplying the values of I into the inverse Fourier Transform, which is acheived by applying an FFT to the discrete values of I.

Subsequently, the values of f(x) against x is plotted, and used to determine the slit width b.

Additionally, a graph of I against k_x may be plotted for completeness using the transformation:

$$k_x = \frac{2\pi}{\lambda} \frac{x}{\sqrt{x^2 + f^2}}$$

We expect the graph to match the given thereotical equation of $I(k_x)$ for single slits:

$$I = \text{sinc}^2\left(\frac{k_x b}{2}\right)$$

- Focal length of lens: f = 300.8 mm
- Wavelength of laser light: $\lambda = 635 nm$

Procedure

1. The intensity profile in real space is first shifted along to abscissa such that the peak intensity corresponding to the principal maximum is located at x=0.
2. Subseqeuntly, a graph of I against k_x is plotted for completeness using Eq 1.4.
3. An FFT is applied to values of I to attain the values of the aperture function $f(x)$.
4. The slit width b is determined from the graph

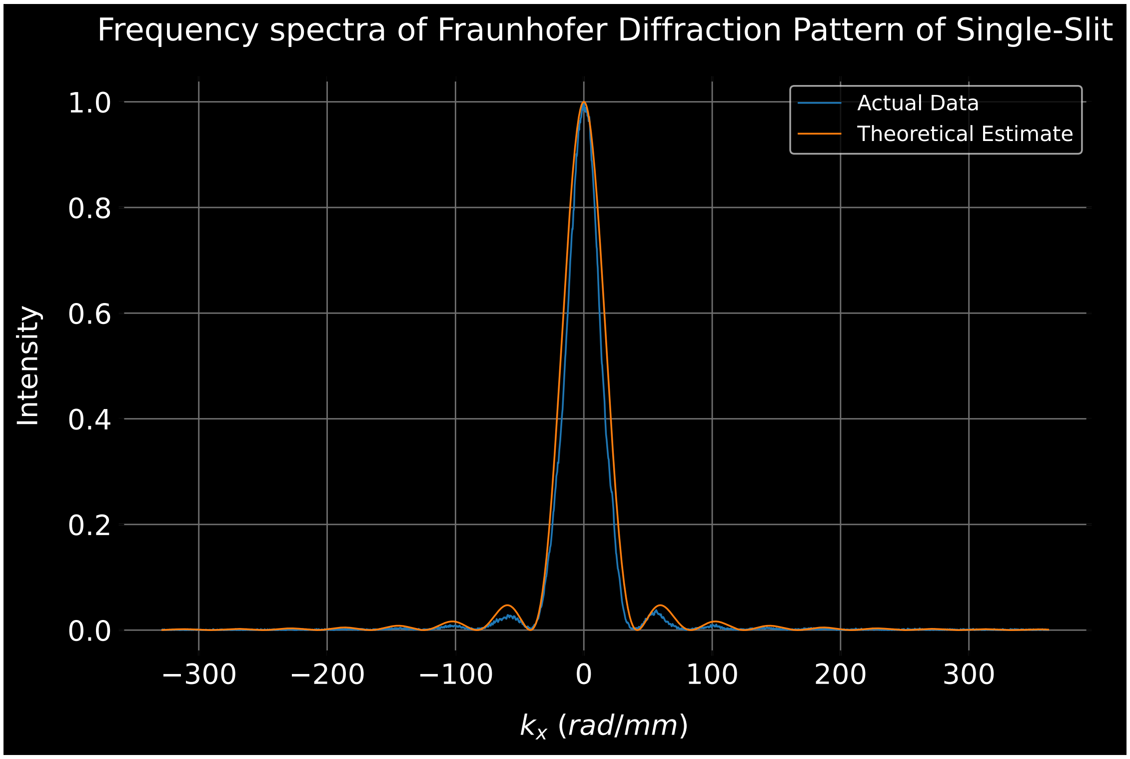


Figure 1.4: Frequency spectra of Fraunhofer Diffraction Pattern of Single-Slit.

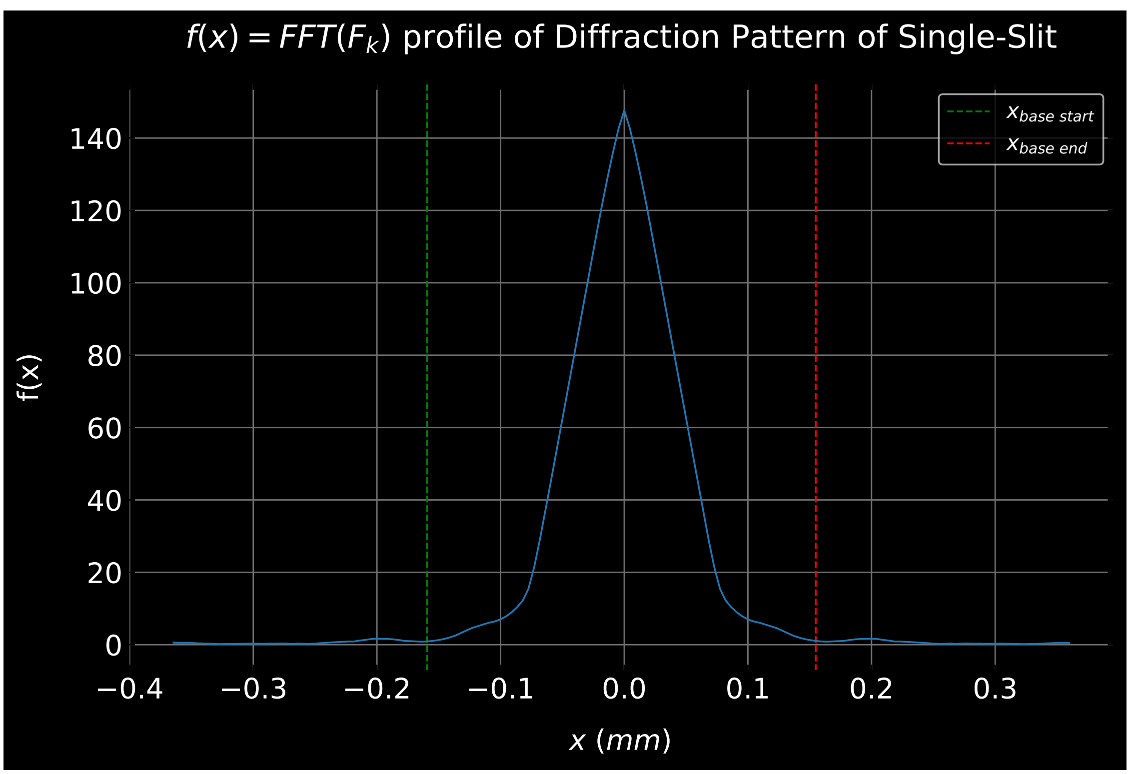


Figure 1.5: Fourier transformed profile of intensity of Single-Slit.

$$b = \frac{x_{base\ end} - x_{base\ start}}{2} \approx 0.16\ mm$$

1.1.3 Task 1c

Task Definition:

In the Fresnel regime, measure the diffraction pattern of a single slit for three distances z between slit and camera. Comapre the measured diffraction patterns with theory.

Theoretical Basis

In the Fresnel regime, only the lens is removed, and the procedure from Task 1a is repeated.

The theoretical graphs were plotted using the following formula:

$$I_{theory} = \frac{1}{2}\left(\left|F\left(\frac{b+2x}{\sqrt{2\lambda z}}\right)\right| + \left|F\left(\frac{b-2x}{\sqrt{2\lambda z}}\right)\right|\right)^2$$

$$\int_0^u F(u)\exp\left(\frac{i\pi x}{2}\right)\,dx$$

A quantitative comparison between theoretical and actual graphs is made by analysing the coefficient of determination (R^2) value which is a measure of variance between the 2 graphs.

$$\mathrm{SS}_{\mathrm{total}} = \sum_{i=1}^{n}(y_i - \overline{y})^2$$

$$\mathrm{SS}_{\mathrm{residual}} = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

$$R^2 = 1 - \frac{\mathrm{SS}_{\mathrm{residual}}}{\mathrm{SS}_{\mathrm{total}}}$$

- $\overline{y}$: Mean of actual data
- y_i : Actual values
- $\hat{y}_i$: Predicted Values

A higher value of R^2 implies a better correlation between theoretical and actual values, up to a perfect correlation when $R^2 = 1$

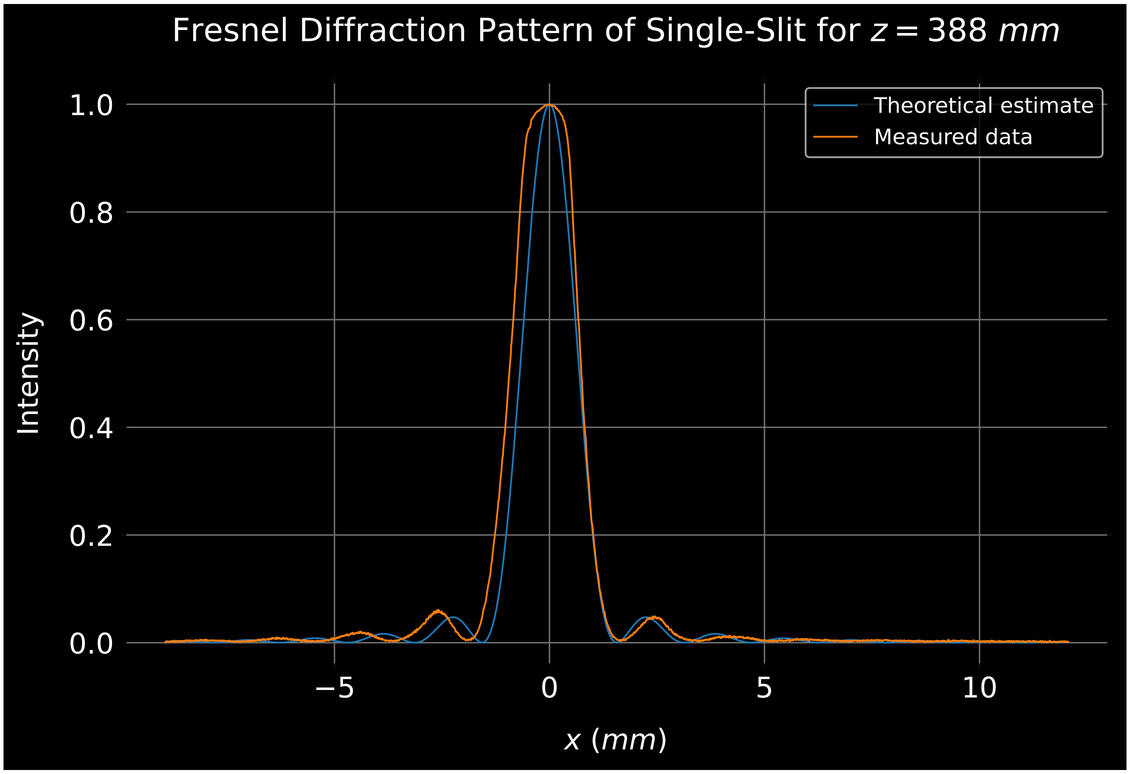


Figure 1.6: Fresnel Diffraction Pattern of Single-Slit ($z = 388$ mm). $R^2 = 0.95$

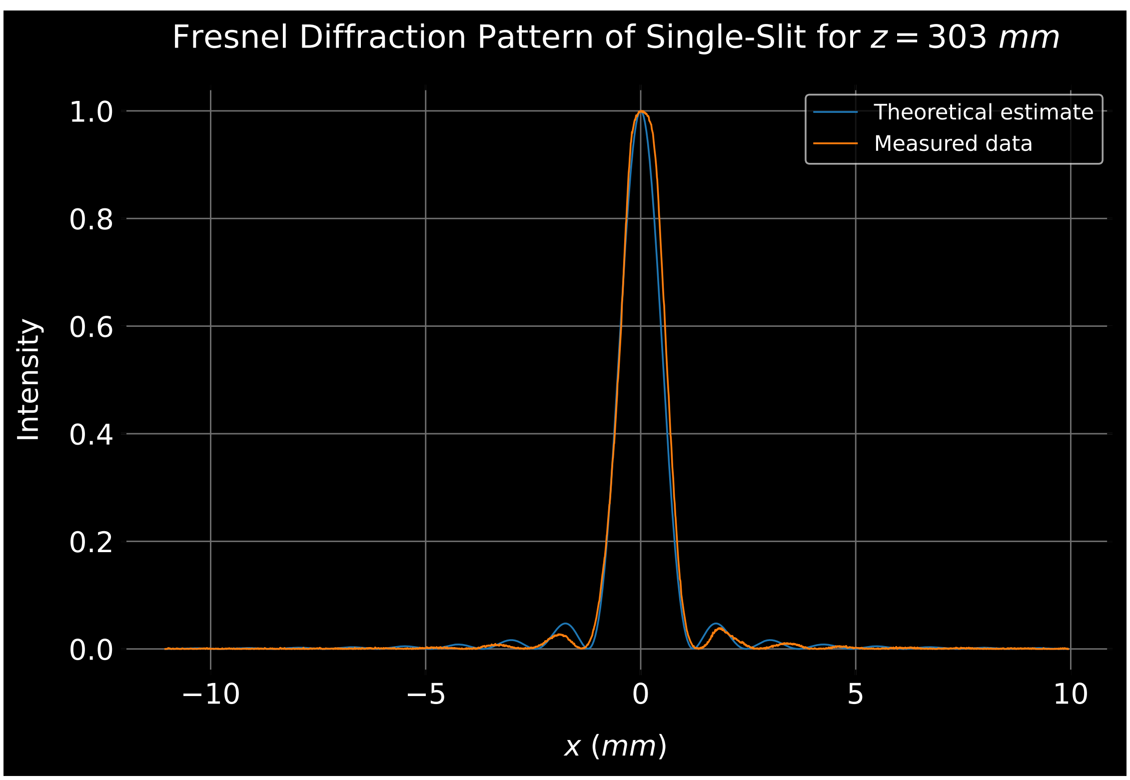


Figure 1.7: Fresnel Diffraction Pattern of Single-Slit ($z = 303\ mm$). $R^2 = 0.99$

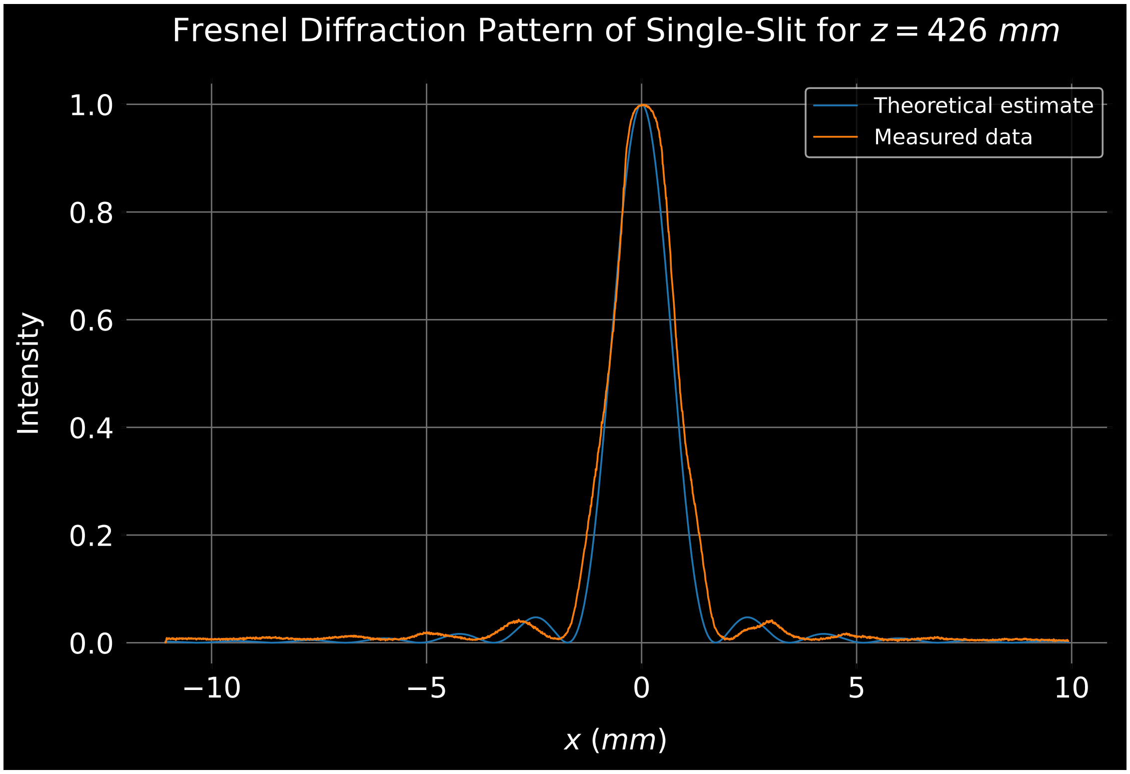


Figure 1.8: Fresnel Diffraction Pattern of Single-Slit ($z = 426\ mm$). $R^2 = 0.98$

1.2 Task 2

1.2.1 Task 2a

Task Definition:

Measure the diffraction pattern of a double-slit in the Fraunhofer regime, and determine the slit distance g from the position of the minima for various diffraction orders.

Theoretical Basis

Light passsing through a double slit creates a diffraction pattern on a camera placed behind the slit.

The positions of the minima of arbitrary order (n) obey the following equation:

$$g \sin(\alpha_n) = \left(n + \frac{1}{2}\right) \lambda$$

- Slit distance : b

- Order of minima: n
- Wavelength of laser light: $\lambda = 635nm$
- Angle between optical axis and n-th order minima: α_n

Under small angle approximation $\alpha_n \approx 0$:

$$x_n \approx \frac{f\lambda}{2g}(2n+1) \tag{2.1}$$

- Focal length of lens: $f = 300.8\ mm$
- Distance between optical axis and n-th order minima: x_n

$$k \equiv \frac{f\lambda}{2g}$$

$$g = \frac{f\lambda}{2k}$$

The uncertainty is given by:

$$\mu_g = \left|\frac{\partial g}{\partial k}\right| \mu_k = \left|-\frac{g}{k}\right| \mu_k$$

Procedure

1. The procedure from Task 1a is repeated using the double slit "469-84-A" instead.
2. The slit distance g was determined from linear regression using Eq 2.1.

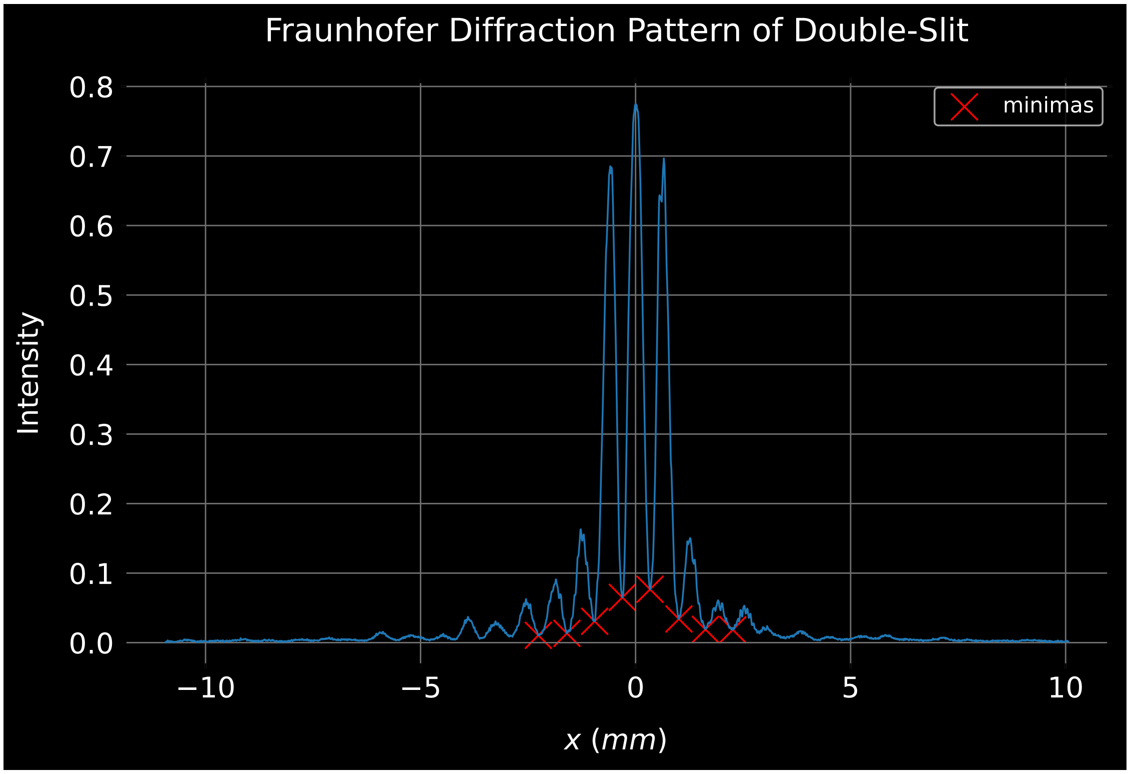


Figure 2.1: Fraunhofer Diffraction Pattern of Double-Slit.

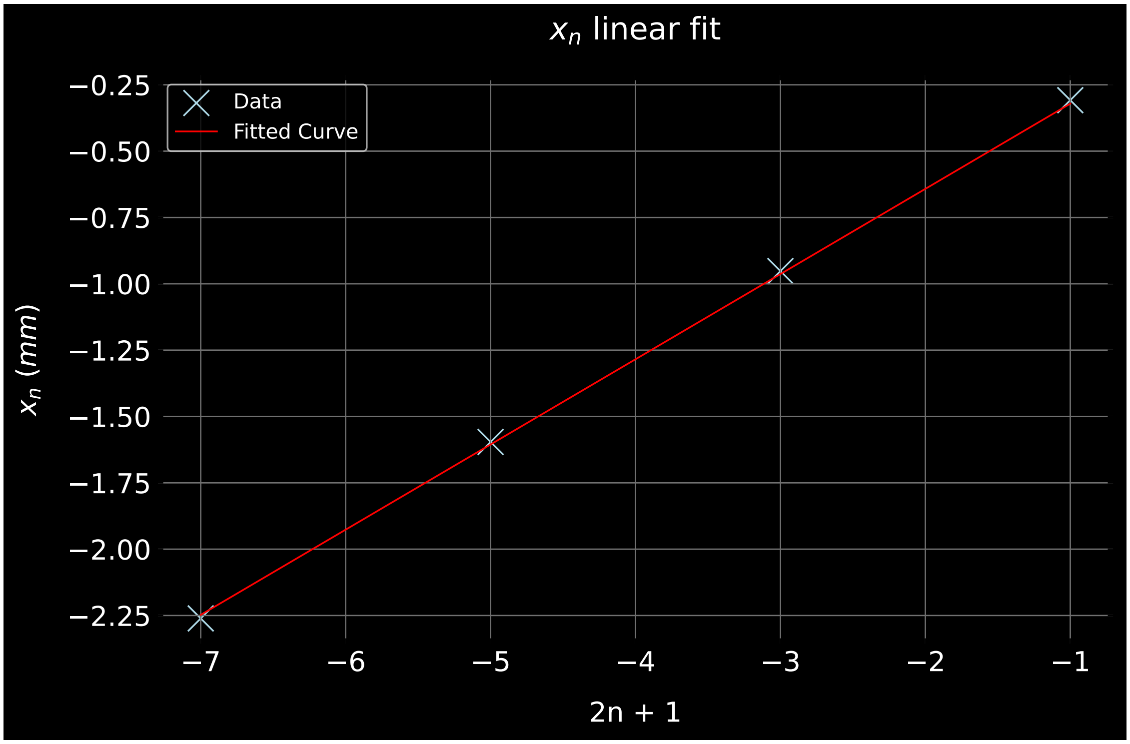


Figure 2.2: Linear fit of position with respect to minimum diffraction order.

$$g = (0.30 \pm 0.001)\ mm$$

1.2.2 Task 2b

Task Definition:

Measure the diffraction pattern of a double-slit in the Fraunhofer regime, and determine the slit distance g and the slit width b from the fit of the Fraunhofer diffraction-pattern to the data

Theoretical Basis:

The theoretical Franuhofer diffraction pattern for double slits is given by:

$$I = \mathrm{sinc}^2\left(\frac{k_x b}{2}\right) cos^2\left(\frac{k_x g}{2}\right)$$

$$k_x = \frac{2\pi}{\lambda}\frac{x}{\sqrt{x^2 + f^2}}$$

The actual data is fitted to the theoretical equation, and the values of b and g are determined from the fit.

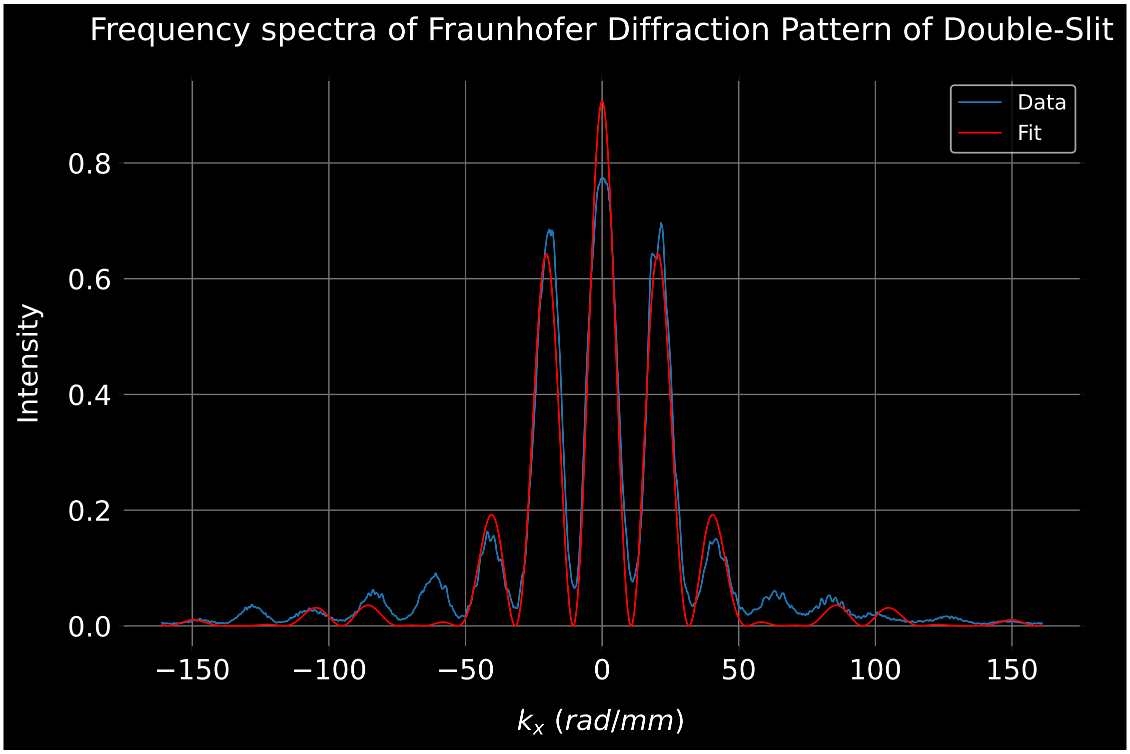


Figure 2.3: Frequency spectra of Fraunhofer Diffraction Pattern of Double-Slit.

$$b = (30.57 \pm 0.118)\ \mu m = 0.03\ mm$$

$$g = (296.63 \pm 0.320)\ \mu m = 0.30\ mm$$

1.2.3 Task 2c

Task Definition

In the Fraunhofer regime, determine the slit distance g and the slit width b from the FFT of the intensity profile.

Procedure

1. The procedure from Task 1b is repeated using the double slit "469-84-A" instead.

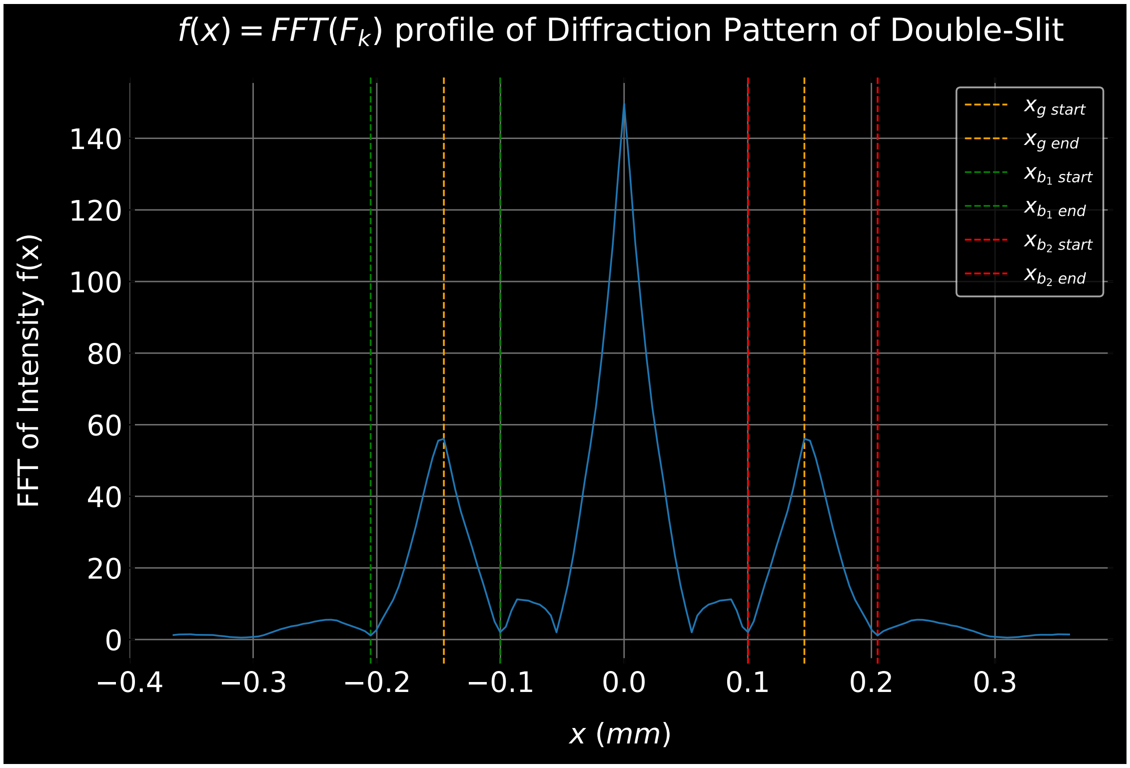


Figure 2.4: Fourier transformed profile of intensity of Double-Slit.

$$b = \frac{1}{2}\left(\frac{x_{1_{base\ end}} - x_{1_{base\ start}}}{2} + \frac{x_{2_{base\ end}} - x_{2_{base\ start}}}{2}\right) \approx 0.05\ mm$$

$$g = x_{g_2} - x_{g_1} \approx 0.29\ mm$$

1.2.4 Task 2d

Task Definition:

In the Fresnel regime, measure the diffraction pattern of a double slit for three distances z between slit and camera. Comapre the measured diffraction patterns with theory.

Theoretical Basis

The procedure from Task 1d is repeated using the double slit instead.

The theoretical graphs for the double slit were plotted using the following formula:

$$I_{theory} = \frac{1}{2}\left(\left|F\left(\frac{g+b+2x}{\sqrt{2\lambda z}}\right)\right| + \left|F\left(\frac{g+b-2x}{\sqrt{2\lambda z}}\right)\right| - \left|F\left(\frac{g-b+2x}{\sqrt{2\lambda z}}\right)\right| - \left|F\left(\frac{g-b-2x}{\sqrt{2\lambda z}}\right)\right|\right)^2$$

A similar quantitative analysis is performed using the R^2 value.

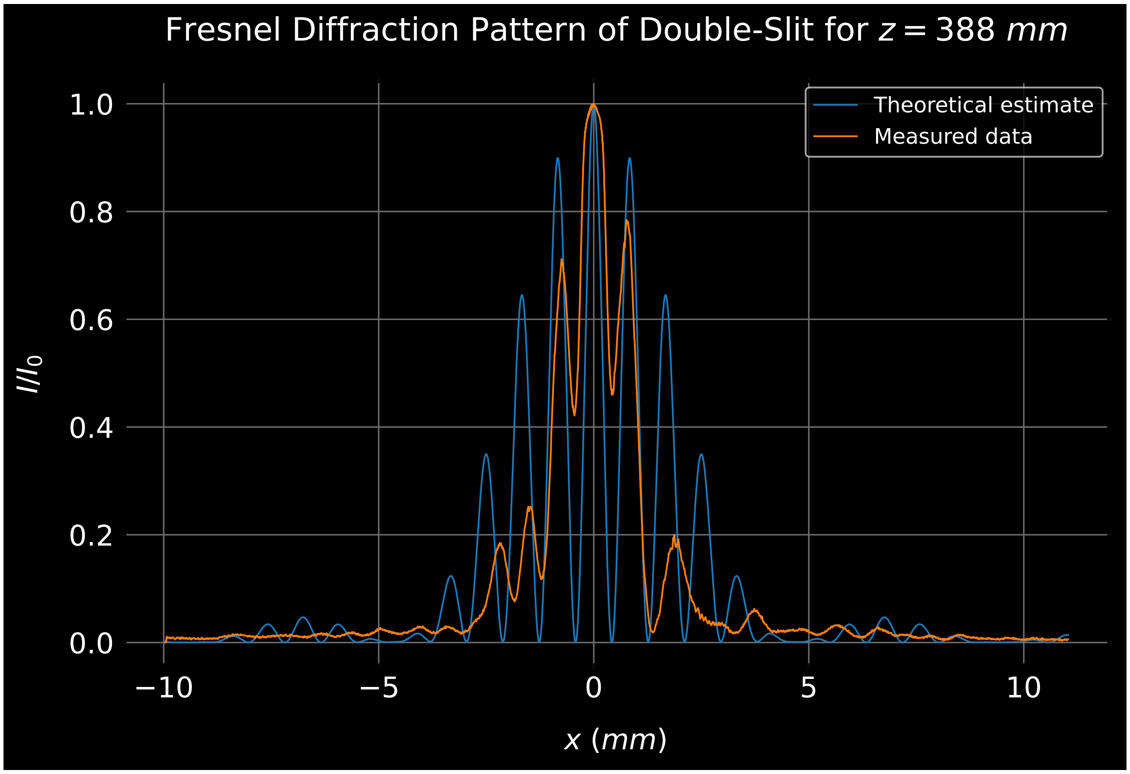


Figure 2.5: Fresnel Diffraction Pattern of Double-Slit ($z = 388$ mm). $R^2 = 0.55$

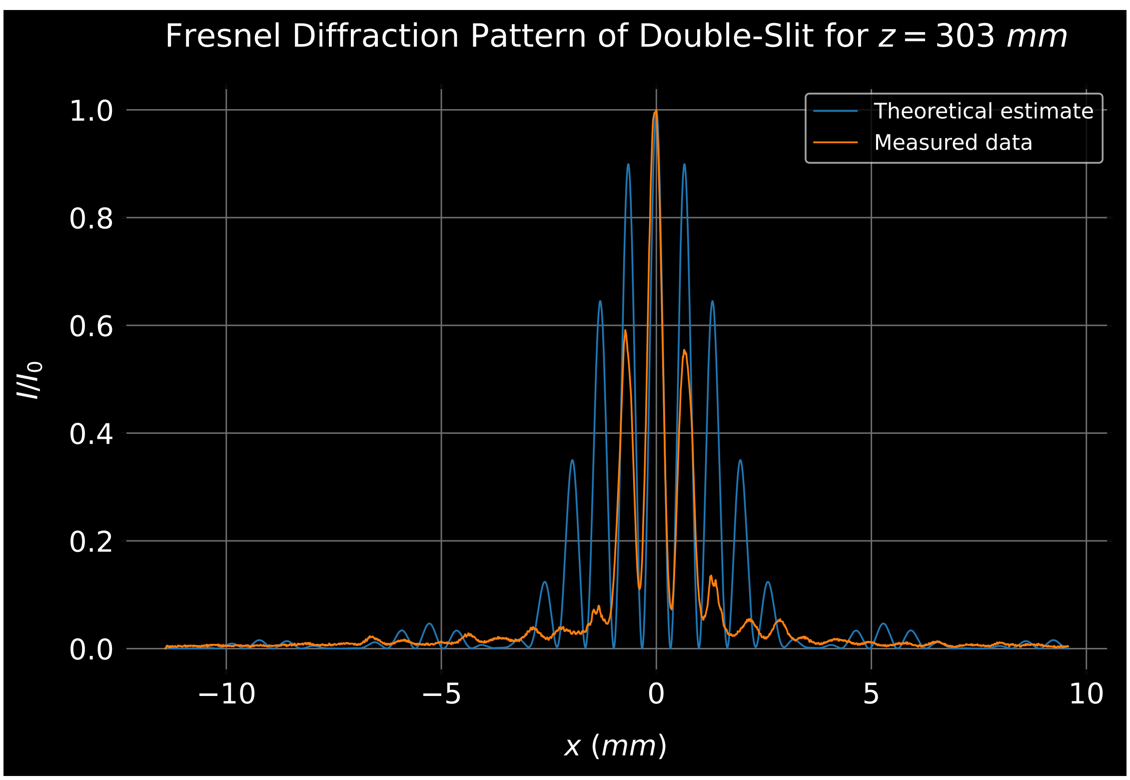


Figure 2.6: Fresnel Diffraction Pattern of Double-Slit ($z = 303\ mm$). $R^2 = 0.39$

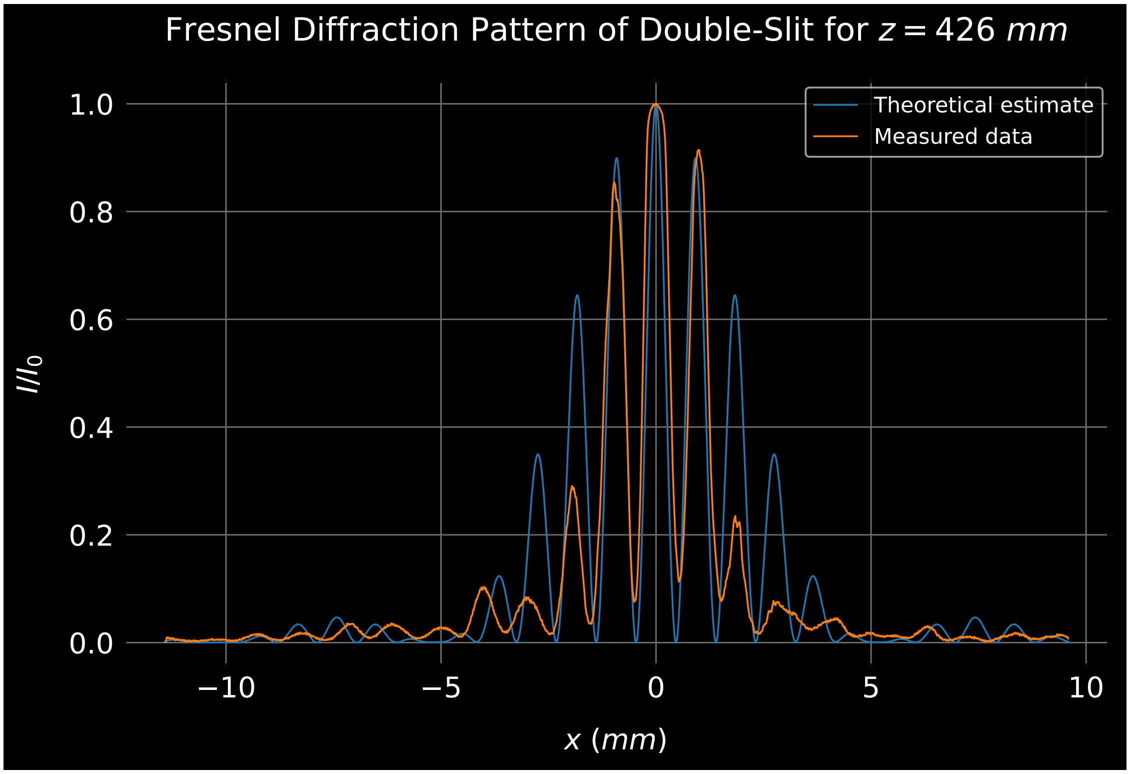


Figure 2.7: Fresnel Diffraction Pattern of Double-Slit ($z = 426\ mm$). $R^2 = 0.73$

References

1) O17e Lab instruction