which means that

$$E_{p(\boldsymbol{y})}\left[(J(\boldsymbol{y})-b(\boldsymbol{\omega}))\frac{\partial \log p(\boldsymbol{y})}{\partial \boldsymbol{\omega}}\right]=E_{p(\boldsymbol{y})}\left[J(\boldsymbol{y})\frac{\partial \log p(\boldsymbol{y})}{\partial \boldsymbol{\omega}}\right]-b(\boldsymbol{\omega})E_{p(\boldsymbol{y})}\left[\frac{\partial \log p(\boldsymbol{y})}{\partial \boldsymbol{\omega}}\right] \tag{20.66}$$

$$=E_{p(\boldsymbol{y})}\left[J(\boldsymbol{y})\frac{\partial \log p(\boldsymbol{y})}{\partial \boldsymbol{\omega}}\right]. \tag{20.67}$$

Furthermore, we can obtain the optimal $b(\boldsymbol{\omega})$ by computing the variance of $(J(\boldsymbol{y})-b(\boldsymbol{\omega}))\frac{\partial \log p(\boldsymbol{y})}{\partial \boldsymbol{\omega}}$ under $p(\boldsymbol{y})$ and minimizing with respect to $b(\boldsymbol{\omega})$. What we find is that this optimal baseline $b^*(\boldsymbol{\omega})_i$ is different for each element ω_i of the vector $\boldsymbol{\omega}$:

$$b^*(\boldsymbol{\omega})_i=\frac{E_{p(\boldsymbol{y})}\left[J(\boldsymbol{y})\frac{\partial \log p(\boldsymbol{y})}{\partial \omega_i}^2\right]}{E_{p(\boldsymbol{y})}\left[\frac{\partial \log p(\boldsymbol{y})}{\partial \omega_i}^2\right]}. \tag{20.68}$$

The gradient estimator with respect to ω_i then becomes

$$(J(\boldsymbol{y})-b(\boldsymbol{\omega})_i)\frac{\partial \log p(\boldsymbol{y})}{\partial \omega_i} \tag{20.69}$$

where $b(\boldsymbol{\omega})_i$ estimates the above $b^*(\boldsymbol{\omega})_i$. The estimate b is usually obtained by adding extra outputs to the neural network and training the new outputs to estimate $E_{p(\boldsymbol{y})}[J(\boldsymbol{y})\frac{\partial \log p(\boldsymbol{y})}{\partial \omega_i}^2]$ and $E_{p(\boldsymbol{y})}\left[\frac{\partial \log p(\boldsymbol{y})}{\partial \omega_i}^2\right]$ for each element of $\boldsymbol{\omega}$. These extra outputs can be trained with the mean squared error objective, using respectively $J(\boldsymbol{y})\frac{\partial \log p(\boldsymbol{y})}{\partial \omega_i}^2$ and $\frac{\partial \log p(\boldsymbol{y})}{\partial \omega_i}^2$ as targets when $\boldsymbol{y}$ is sampled from $p(\boldsymbol{y})$, for a given $\boldsymbol{\omega}$. The estimate b may then be recovered by substituting these estimates into equation 20.68. Mnih and Gregor (2014) preferred to use a single shared output (across all elements i of $\boldsymbol{\omega}$) trained with the target $J(\boldsymbol{y})$, using as baseline $b(\boldsymbol{\omega}) \approx E_{p(\boldsymbol{y})}[J(\boldsymbol{y})]$.

Variance reduction methods have been introduced in the reinforcement learning context (Sutton *et al.*, 2000; Weaver and Tao, 2001), generalizing previous work on the case of binary reward by Dayan (1990). See Bengio *et al.* (2013b), Mnih and Gregor (2014), Ba *et al.* (2014), Mnih *et al.* (2014), or Xu *et al.* (2015) for examples of modern uses of the REINFORCE algorithm with reduced variance in the context of deep learning. In addition to the use of an input-dependent baseline $b(\boldsymbol{\omega})$, Mnih and Gregor (2014) found that the scale of $(J(\boldsymbol{y})-b(\boldsymbol{\omega}))$ could be adjusted during training by dividing it by its standard deviation estimated by a moving average during training, as a kind of adaptive learning rate, to counter the effect of important variations that occur during the course of training in the