may not be able to find the value of the parameters that corresponds to the desired function. Second, the training algorithm might choose the wrong function due to overfitting. Recall from section 5.2.1 that the "no free lunch" theorem shows that there is no universally superior machine learning algorithm. Feedforward networks provide a universal system for representing functions, in the sense that, given a function, there exists a feedforward network that approximates the function. There is no universal procedure for examining a training set of specific examples and choosing a function that will generalize to points not in the training set.

The universal approximation theorem says that there exists a network large enough to achieve any degree of accuracy we desire, but the theorem does not say how large this network will be. Barron (1993) provides some bounds on the size of a single-layer network needed to approximate a broad class of functions. Unfortunately, in the worse case, an exponential number of hidden units (possibly with one hidden unit corresponding to each input configuration that needs to be distinguished) may be required. This is easiest to see in the binary case: the number of possible binary functions on vectors $\boldsymbol{v} \in \{0, 1\}^n$ is $2^{2^n}$ and selecting one such function requires $2^n$ bits, which will in general require $O(2^n)$ degrees of freedom.

In summary, a feedforward network with a single layer is sufficient to represent any function, but the layer may be infeasibly large and may fail to learn and generalize correctly. In many circumstances, using deeper models can reduce the number of units required to represent the desired function and can reduce the amount of generalization error.

There exist families of functions which can be approximated efficiently by an architecture with depth greater than some value $d$, but which require a much larger model if depth is restricted to be less than or equal to $d$. In many cases, the number of hidden units required by the shallow model is exponential in $n$. Such results were first proved for models that do not resemble the continuous, differentiable neural networks used for machine learning, but have since been extended to these models. The first results were for circuits of logic gates (Håstad, 1986). Later work extended these results to linear threshold units with non-negative weights (Håstad and Goldmann, 1991; Hajnal *et al.*, 1993), and then to networks with continuous-valued activations (Maass, 1992; Maass *et al.*, 1994). Many modern neural networks use rectified linear units. Leshno *et al.* (1993) demonstrated that shallow networks with a broad family of non-polynomial activation functions, including rectified linear units, have universal approximation properties, but these results do not address the questions of depth or efficiency—they specify only that a sufficiently wide rectifier network could represent any function. Montufar *et al.*