15.5 Exponential Gains from Depth

We have seen in section 6.4.1 that multilayer perceptrons are universal approximators, and that some functions can be represented by exponentially smaller deep networks compared to shallow networks. This decrease in model size leads to improved statistical efficiency. In this section, we describe how similar results apply more generally to other kinds of models with distributed hidden representations.

In section 15.4, we saw an example of a generative model that learned about the explanatory factors underlying images of faces, including the person's gender and whether they are wearing glasses. The generative model that accomplished this task was based on a deep neural network. It would not be reasonable to expect a shallow network, such as a linear network, to learn the complicated relationship between these abstract explanatory factors and the pixels in the image. In this and other AI tasks, the factors that can be chosen almost independently from each other yet still correspond to meaningful inputs are more likely to be very high-level and related in highly nonlinear ways to the input. We argue that this demands *deep* distributed representations, where the higher level features (seen as functions of the input) or factors (seen as generative causes) are obtained through the composition of many nonlinearities.

It has been proven in many different settings that organizing computation through the composition of many nonlinearities and a hierarchy of reused features can give an exponential boost to statistical efficiency, on top of the exponential boost given by using a distributed representation. Many kinds of networks (e.g., with saturating nonlinearities, Boolean gates, sum/products, or RBF units) with a single hidden layer can be shown to be universal approximators. A model family that is a universal approximator can approximate a large class of functions (including all continuous functions) up to any non-zero tolerance level, given enough hidden units. However, the required number of hidden units may be very large. Theoretical results concerning the expressive power of deep architectures state that there are families of functions that can be represented efficiently by an architecture of depth k, but would require an exponential number of hidden units (with respect to the input size) with insufficient depth (depth 2 or depth $k - 1$).

In section 6.4.1, we saw that deterministic feedforward networks are universal approximators of functions. Many structured probabilistic models with a single hidden layer of latent variables, including restricted Boltzmann machines and deep belief networks, are universal approximators of probability distributions (Le Roux and Bengio, 2008, 2010; Montúfar and Ay, 2011; Montúfar, 2014; Krause *et al.*, 2013).