From the point of view of learning via maximization with respect to $\boldsymbol{w}$, we can ignore the $\log \alpha - \log 2$ terms because they do not depend on $\boldsymbol{w}$.

7.2 Norm Penalties as Constrained Optimization

Consider the cost function regularized by a parameter norm penalty:

$$\tilde{J}(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{y}) = J(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{y}) + \alpha\Omega(\boldsymbol{\theta}). \tag{7.25}$$

Recall from section 4.4 that we can minimize a function subject to constraints by constructing a generalized Lagrange function, consisting of the original objective function plus a set of penalties. Each penalty is a product between a coefficient, called a Karush–Kuhn–Tucker (KKT) multiplier, and a function representing whether the constraint is satisfied. If we wanted to constrain $\Omega(\boldsymbol{\theta})$ to be less than some constant k, we could construct a generalized Lagrange function

$$\mathcal{L}(\boldsymbol{\theta}, \alpha; \boldsymbol{X}, \boldsymbol{y}) = J(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{y}) + \alpha(\Omega(\boldsymbol{\theta}) - k). \tag{7.26}$$

The solution to the constrained problem is given by

$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}} \max_{\alpha, \alpha \geq 0} \mathcal{L}(\boldsymbol{\theta}, \alpha). \tag{7.27}$$

As described in section 4.4, solving this problem requires modifying both $\boldsymbol{\theta}$ and α. Section 4.5 provides a worked example of linear regression with an L^2 constraint. Many different procedures are possible—some may use gradient descent, while others may use analytical solutions for where the gradient is zero—but in all procedures α must increase whenever $\Omega(\boldsymbol{\theta}) > k$ and decrease whenever $\Omega(\boldsymbol{\theta}) < k$. All positive α encourage $\Omega(\boldsymbol{\theta})$ to shrink. The optimal value α^* will encourage $\Omega(\boldsymbol{\theta})$ to shrink, but not so strongly to make $\Omega(\boldsymbol{\theta})$ become less than k.

To gain some insight into the effect of the constraint, we can fix α^* and view the problem as just a function of $\boldsymbol{\theta}$:

$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}} \mathcal{L}(\boldsymbol{\theta}, \alpha^*) = \arg\min_{\boldsymbol{\theta}} J(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{y}) + \alpha^*\Omega(\boldsymbol{\theta}). \tag{7.28}$$

This is exactly the same as the regularized training problem of minimizing $\tilde{J}$. We can thus think of a parameter norm penalty as imposing a constraint on the weights. If Ω is the L^2 norm, then the weights are constrained to lie in an L^2 ball. If Ω is the L^1 norm, then the weights are constrained to lie in a region of