$$\sigma(x) = \frac{\exp(x)}{\exp(x) + \exp(0)} \tag{3.33}$$

$$\frac{d}{dx}\sigma(x) = \sigma(x)(1 - \sigma(x)) \tag{3.34}$$

$$1 - \sigma(x) = \sigma(-x) \tag{3.35}$$

$$\log \sigma(x) = -\zeta(-x) \tag{3.36}$$

$$\frac{d}{dx}\zeta(x) = \sigma(x) \tag{3.37}$$

$$\forall x \in (0, 1),\ \sigma^{-1}(x) = \log\left(\frac{x}{1-x}\right) \tag{3.38}$$

$$\forall x > 0,\ \zeta^{-1}(x) = \log\left(\exp(x) - 1\right) \tag{3.39}$$

$$\zeta(x) = \int_{-\infty}^{x} \sigma(y)dy \tag{3.40}$$

$$\zeta(x) - \zeta(-x) = x \tag{3.41}$$

The function $\sigma^{-1}(x)$ is called the **logit** in statistics, but this term is more rarely used in machine learning.

Equation 3.41 provides extra justification for the name "softplus." The softplus function is intended as a smoothed version of the **positive part** function, $x^+ = \max\{0, x\}$. The positive part function is the counterpart of the **negative part** function, $x^- = \max\{0, -x\}$. To obtain a smooth function that is analogous to the negative part, one can use $\zeta(-x)$. Just as x can be recovered from its positive part and negative part via the identity $x^+ - x^- = x$, it is also possible to recover x using the same relationship between $\zeta(x)$ and $\zeta(-x)$, as shown in equation 3.41.

3.11 Bayes' Rule

We often find ourselves in a situation where we know $P(\mathrm{y} \mid \mathrm{x})$ and need to know $P(\mathrm{x} \mid \mathrm{y})$. Fortunately, if we also know $P(\mathrm{x})$, we can compute the desired quantity using **Bayes' rule**:

$$P(\mathrm{x} \mid \mathrm{y}) = \frac{P(\mathrm{x})P(\mathrm{y} \mid \mathrm{x})}{P(\mathrm{y})}. \tag{3.42}$$

Note that while $P(\mathrm{y})$ appears in the formula, it is usually feasible to compute $P(\mathrm{y}) = \sum_x P(\mathrm{y} \mid x)P(x)$, so we do not need to begin with knowledge of $P(\mathrm{y})$.