

for networks of practical interest and whether optimization algorithms encounter them. For many years, most practitioners believed that local minima were a common problem plaguing neural network optimization. Today, that does not appear to be the case. The problem remains an active area of research, but experts now suspect that, for sufficiently large neural networks, most local minima have a low cost function value, and that it is not important to find a true global minimum rather than to find a point in parameter space that has low but not minimal cost (Saxe *et al.*, 2013; Dauphin *et al.*, 2014; Goodfellow *et al.*, 2015; Choromanska *et al.*, 2014).

Many practitioners attribute nearly all difficulty with neural network optimization to local minima. We encourage practitioners to carefully test for specific problems. A test that can rule out local minima as the problem is to plot the norm of the gradient over time. If the norm of the gradient does not shrink to insignificant size, the problem is neither local minima nor any other kind of critical point. This kind of negative test can rule out local minima. In high dimensional spaces, it can be very difficult to positively establish that local minima are the problem. Many structures other than local minima also have small gradients.

### 8.2.3 Plateaus, Saddle Points and Other Flat Regions

For many high-dimensional non-convex functions, local minima (and maxima) are in fact rare compared to another kind of point with zero gradient: a saddle point. Some points around a saddle point have greater cost than the saddle point, while others have a lower cost. At a saddle point, the Hessian matrix has both positive and negative eigenvalues. Points lying along eigenvectors associated with positive eigenvalues have greater cost than the saddle point, while points lying along negative eigenvalues have lower value. We can think of a saddle point as being a local minimum along one cross-section of the cost function and a local maximum along another cross-section. See figure 4.5 for an illustration.

Many classes of random functions exhibit the following behavior: in low-dimensional spaces, local minima are common. In higher dimensional spaces, local minima are rare and saddle points are more common. For a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  of this type, the expected ratio of the number of saddle points to local minima grows exponentially with  $n$ . To understand the intuition behind this behavior, observe that the Hessian matrix at a local minimum has only positive eigenvalues. The Hessian matrix at a saddle point has a mixture of positive and negative eigenvalues. Imagine that the sign of each eigenvalue is generated by flipping a coin. In a single dimension, it is easy to obtain a local minimum by tossing a coin and getting heads once. In  $n$ -dimensional space, it is exponentially unlikely that all  $n$  coin tosses will