Algorithm 6.1 A procedure that performs the computations mapping n_i inputs $u^{(1)}$ to $u^{(n_i)}$ to an output $u^{(n)}$. This defines a computational graph where each node computes numerical value $u^{(i)}$ by applying a function $f^{(i)}$ to the set of arguments $\mathbb{A}^{(i)}$ that comprises the values of previous nodes $u^{(j)}$, $j < i$, with $j \in Pa(u^{(i)})$. The input to the computational graph is the vector $\boldsymbol{x}$, and is set into the first n_i nodes $u^{(1)}$ to $u^{(n_i)}$. The output of the computational graph is read off the last (output) node $u^{(n)}$.

for $i = 1, \ldots, n_i$ **do**
  $u^{(i)} \leftarrow x_i$
end for
for $i = n_i + 1, \ldots, n$ **do**
  $\mathbb{A}^{(i)} \leftarrow \{u^{(j)} \mid j \in Pa(u^{(i)})\}$
  $u^{(i)} \leftarrow f^{(i)}(\mathbb{A}^{(i)})$
end for
return $u^{(n)}$

using the chain rule with respect to scalar output $u^{(n)}$:

$$\frac{\partial u^{(n)}}{\partial u^{(j)}} = \sum_{i:j \in Pa(u^{(i)})} \frac{\partial u^{(n)}}{\partial u^{(i)}} \frac{\partial u^{(i)}}{\partial u^{(j)}} \tag{6.49}$$

as specified by algorithm 6.2. The subgraph $\mathcal{B}$ contains exactly one edge for each edge from node $u^{(j)}$ to node $u^{(i)}$ of $\mathcal{G}$. The edge from $u^{(j)}$ to $u^{(i)}$ is associated with the computation of $\frac{\partial u^{(i)}}{\partial u^{(j)}}$. In addition, a dot product is performed for each node, between the gradient already computed with respect to nodes $u^{(i)}$ that are children of $u^{(j)}$ and the vector containing the partial derivatives $\frac{\partial u^{(i)}}{\partial u^{(j)}}$ for the same children nodes $u^{(i)}$. To summarize, the amount of computation required for performing the back-propagation scales linearly with the number of edges in $\mathcal{G}$, where the computation for each edge corresponds to computing a partial derivative (of one node with respect to one of its parents) as well as performing one multiplication and one addition. Below, we generalize this analysis to tensor-valued nodes, which is just a way to group multiple scalar values in the same node and enable more efficient implementations.

The back-propagation algorithm is designed to reduce the number of common subexpressions without regard to memory. Specifically, it performs on the order of one Jacobian product per node in the graph. This can be seen from the fact that backprop (algorithm 6.2) visits each edge from node $u^{(j)}$ to node $u^{(i)}$ of the graph exactly once in order to obtain the associated partial derivative $\frac{\partial u^{(i)}}{\partial u^{(j)}}$.