the SVD is more generally applicable. Every real matrix has a singular value decomposition, but the same is not true of the eigenvalue decomposition. For example, if a matrix is not square, the eigendecomposition is not defined, and we must use a singular value decomposition instead.

Recall that the eigendecomposition involves analyzing a matrix $\boldsymbol{A}$ to discover a matrix $\boldsymbol{V}$ of eigenvectors and a vector of eigenvalues $\boldsymbol{\lambda}$ such that we can rewrite $\boldsymbol{A}$ as

$$\boldsymbol{A} = \boldsymbol{V}\mathrm{diag}(\boldsymbol{\lambda})\boldsymbol{V}^{-1}. \tag{2.42}$$

The singular value decomposition is similar, except this time we will write $\boldsymbol{A}$ as a product of three matrices:

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^{\top}. \tag{2.43}$$

Suppose that $\boldsymbol{A}$ is an $m \times n$ matrix. Then $\boldsymbol{U}$ is defined to be an $m \times m$ matrix, $\boldsymbol{D}$ to be an $m \times n$ matrix, and $\boldsymbol{V}$ to be an $n \times n$ matrix.

Each of these matrices is defined to have a special structure. The matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ are both defined to be orthogonal matrices. The matrix $\boldsymbol{D}$ is defined to be a diagonal matrix. Note that $\boldsymbol{D}$ is not necessarily square.

The elements along the diagonal of $\boldsymbol{D}$ are known as the **singular values** of the matrix $\boldsymbol{A}$. The columns of $\boldsymbol{U}$ are known as the **left-singular vectors**. The columns of $\boldsymbol{V}$ are known as as the **right-singular vectors**.

We can actually interpret the singular value decomposition of $\boldsymbol{A}$ in terms of the eigendecomposition of functions of $\boldsymbol{A}$. The left-singular vectors of $\boldsymbol{A}$ are the eigenvectors of $\boldsymbol{A}\boldsymbol{A}^{\top}$. The right-singular vectors of $\boldsymbol{A}$ are the eigenvectors of $\boldsymbol{A}^{\top}\boldsymbol{A}$. The non-zero singular values of $\boldsymbol{A}$ are the square roots of the eigenvalues of $\boldsymbol{A}^{\top}\boldsymbol{A}$. The same is true for $\boldsymbol{A}\boldsymbol{A}^{\top}$.

Perhaps the most useful feature of the SVD is that we can use it to partially generalize matrix inversion to non-square matrices, as we will see in the next section.

2.9 The Moore-Penrose Pseudoinverse

Matrix inversion is not defined for matrices that are not square. Suppose we want to make a left-inverse $\boldsymbol{B}$ of a matrix $\boldsymbol{A}$, so that we can solve a linear equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y} \tag{2.44}$$