analytically. However, in a software implementation, machine rounding error could result in 0 or 1 values. In software, we may wish to implement binary sparse coding using an unrestricted vector of variational parameters $\boldsymbol{z}$ and obtain $\hat{\boldsymbol{h}}$ via the relation $\hat{\boldsymbol{h}} = \sigma(\boldsymbol{z})$. We can thus safely compute $\log \hat{\boldsymbol{h}}_i$ on a computer by using the identity $\log \sigma(z_i) = -\zeta(-z_i)$ relating the sigmoid and the softplus.

To begin our derivation of variational learning in the binary sparse coding model, we show that the use of this mean field approximation makes learning tractable.

The evidence lower bound is given by

$$\mathcal{L}(\boldsymbol{v}, \boldsymbol{\theta}, q) \tag{19.29}$$

$$=\mathbb{E}_{\mathbf{h}\sim q}[\log p(\boldsymbol{h}, \boldsymbol{v})] + H(q) \tag{19.30}$$

$$=\mathbb{E}_{\mathbf{h}\sim q}[\log p(\boldsymbol{h}) + \log p(\boldsymbol{v} \mid \boldsymbol{h}) - \log q(\boldsymbol{h} \mid \boldsymbol{v})] \tag{19.31}$$

$$=\mathbb{E}_{\mathbf{h}\sim q}\left[\sum_{i=1}^{m} \log p(h_i) + \sum_{i=1}^{n} \log p(v_i \mid \boldsymbol{h}) - \sum_{i=1}^{m} \log q(h_i \mid \boldsymbol{v})\right] \tag{19.32}$$

$$=\sum_{i=1}^{m} \left[\hat{h}_i(\log \sigma(b_i) - \log \hat{h}_i) + (1 - \hat{h}_i)(\log \sigma(-b_i) - \log(1 - \hat{h}_i))\right] \tag{19.33}$$

$$+ \mathbb{E}_{\mathbf{h}\sim q}\left[\sum_{i=1}^{n} \log \sqrt{\frac{\beta_i}{2\pi}} \exp\left(-\frac{\beta_i}{2}(v_i - \boldsymbol{W}_{i,:}\boldsymbol{h})^2\right)\right] \tag{19.34}$$

$$=\sum_{i=1}^{m} \left[\hat{h}_i(\log \sigma(b_i) - \log \hat{h}_i) + (1 - \hat{h}_i)(\log \sigma(-b_i) - \log(1 - \hat{h}_i))\right] \tag{19.35}$$

$$+\frac{1}{2}\sum_{i=1}^{n}\left[\log \frac{\beta_i}{2\pi} - \beta_i \left(v_i^2 - 2v_i \boldsymbol{W}_{i,:}\hat{\boldsymbol{h}} + \sum_j \left[W_{i,j}^2 \hat{h}_j + \sum_{k\neq j} W_{i,j} W_{i,k} \hat{h}_j \hat{h}_k\right]\right)\right]. \tag{19.36}$$

While these equations are somewhat unappealing aesthetically, they show that $\mathcal{L}$ can be expressed in a small number of simple arithmetic operations. The evidence lower bound $\mathcal{L}$ is therefore tractable. We can use $\mathcal{L}$ as a replacement for the intractable log-likelihood.

In principle, we could simply run gradient ascent on both $\boldsymbol{v}$ and $\boldsymbol{h}$ and this would make a perfectly acceptable combined inference and training algorithm. Usually, however, we do not do this, for two reasons. First, this would require storing $\hat{\boldsymbol{h}}$ for each $\boldsymbol{v}$. We typically prefer algorithms that do not require per-example memory. It is difficult to scale learning algorithms to billions of examples if we must remember a dynamically updated vector associated with each example.