Games with Sequential Moves

Edicson Luna

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Basic definitions

What Is a Strategy?

Strategy: A complete plan of action that a player will follow in a game, depending on the circumstances they face.

Some everyday examples:

- The route I choose to go to work.
- The offer I decide to make to a client.
- The way I ask my mom for something (depending on her mood).

In strategic games, a strategy defines what action to take in every possible situation a player might face.

Types of Strategies and Game Structure

Simultaneous-Move Games

- All players choose actions at the same time.
- **Pure strategy:** A single action from their action set: $s_i \in S_i$
- **Example:** In Rock–Paper–Scissors, $S_i = \{\text{Rock}, \text{Paper}, \text{Scissors}\}$
- In this case, S_i is the complete set of strategies of the individual i. Strategies are indicated by s_i (like rocks). We could also write it as $S_i = \{s_{i1}, ..., s_{in}\}$ for an individual with n possible strategies.

Types of Strategies and Game Structure

Sequential-Move Games

- Players make decisions one after another, observing previous moves.
- **Pure strategy:** A complete contingent plan: s_i specifies an action at every node where player i may move.
- **Example:** In Chess, a strategy tells you how to respond to every possible opponent move.

In both cases, a strategy fully specifies how a player will act—either in a single moment or across multiple stages.

Strategy Profiles and Outcomes

Strategy Profile: A complete list of strategies chosen by all players in the game.
In Rock-Paper-Scissors, the set of all possible strategy profiles are: *(Rock, Rock), (Rock, Paper), (Rock, Scissors), (Paper, Rock), (Paper, Paper), (Paper, Scissors), (Scissors, Rock), (Scissors, Paper), (Scissors, Scissors)*.

Outcome: The result of the game, typically represented by the final chosen single strategy profile (e.g., *(Rock, Scissors)*). For the moment, in sequential games, we can refer to outcomes as a *path*, whose definition will come in some slides (this will change once we start talking about equilibrium).

We will use these definitions throughout the course. You will get used to it.

Payoffs

Payoffs: The payoff is the reward that each player receives at the end of the game, depending on the actions taken by all players.

At first, it's helpful to think of a payoff simply as a "win" or a "loss." But in most games, outcomes are more complex: players might gain money, utility, reputation, or other benefits.

To analyze games formally, we assign numerical values to each outcome. In many cases, this is straightforward (e.g., dollars), while in others we use numbers to reflect preferences.

We assume that **higher payoffs are preferred.**
Players aim to maximize their own payoff.

Payoffs

Example (Non–Zero-Sum Game):

- Two classmates can choose to help each other with homework or not.
- If both help, they each understand the material better and get **3, 3**.
- If one helps and the other doesn't, the helper loses time and gets **0**, while the other benefits and gets **4**.
- If neither helps, they each struggle and get **1, 1**.

Note: The sum of payoffs is not constant — this is not a zero-sum game. Cooperation may lead to better outcomes for both.

Rationality

Rationality Each player has a consistent payoff ranking of all the logically possible outcomes of the game and chooses the strategy that best serves their interests.

But rationality can be limited by:

- **Asymmetric information:** When players have access to different information, strategic reasoning becomes harder. This is a key topic we will explore in this class.
- **Behavioral biases:** Players may rely on instincts, fixed rules, or heuristics instead of calculated reasoning.

However, rational behavior becomes more likely when:

- The game is repeated over time.
- Players understand the structure and incentives of the game well.

Equilibrium?

An equilibrium is a stable outcome of a strategic interaction, where the choices made by all players are mutually consistent. Each player's strategy fits with the situation they face, and there is no pressure to adjust behavior given the current environment.

Key Point: An equilibrium is not necessarily good for society.

Example: In highly corrupt countries, parents may need to bribe officials to get their children into good schools. If many do it, others are forced to follow — creating an undesirable but stable equilibrium.

Repetition Helps: In repeated games, better outcomes may arise through trust, reputation, and social punishments.

Sequential games

Definition

Sequential-move games involve strategic situations where there is a **clear order of play**.

Players take turns making decisions, and each player observes the actions of those who moved before them.

Examples:

- Most board games (e.g., Chess, Checkers)
- Auctions where bids are made one after another
- Bargaining or negotiation rounds
- Entry games in markets (e.g., one firm decides whether to enter, the other responds)

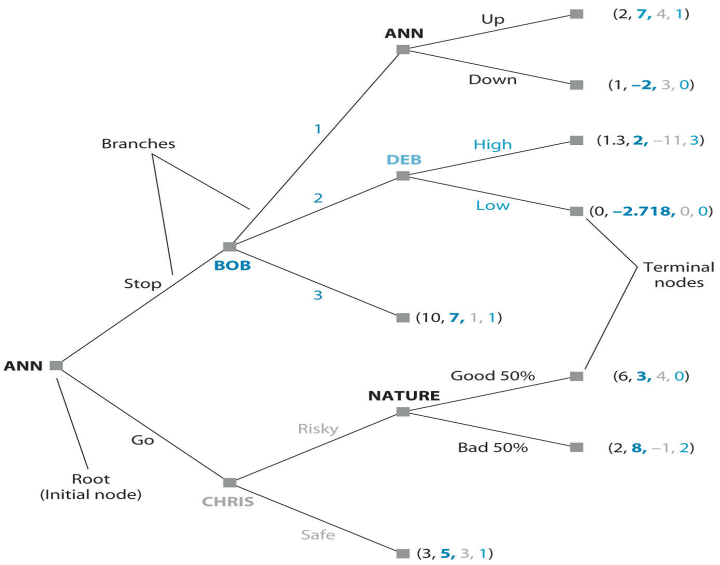
Game Tree

Game trees are the **extensive-form** representation of sequential games. They provide a full picture of the game's structure — including the order of play, possible actions at each decision point, and outcomes.

They are particularly useful for:

- Visualizing who moves when
- Clarifying information sets
- Analyzing strategies and subgames

Game Tree


ANN
Up
(2, 7, 4, 1)
Down
(1, −2, 3, 0)
1
Branches
DEB
High
(1.3, 2, −11, 3)
2
Low
(0, −2.718, 0, 0)
BOB
Stop
Terminal nodes
3
(10, 7, 1, 1)
ANN
Good 50%
(6, 3, 4, 0)
NATURE
Risky
Go
Bad 50%
(2, 8, −1, 2)
Root (Initial node)
CHRIS
Safe
(3, 5, 3, 1)

Nodes, Branches, and Paths of Play

Node: A point in the game where something happens. There are different types of nodes:

- A decision node is a node where a player chooses an action.
- The root node is the starting point of the game, and also a decision node.
- A terminal node is the endpoint that shows the outcome and associated payoffs.

Branch: A line connecting nodes, representing an available action.

Path of Play: A sequence of actions from the root to a terminal node.

In the diagram:

- ANN starts at the root and chooses Stop or Go.
- One path: ANN $\rightarrow$ Go, CHRIS $\rightarrow$ Safe $\rightarrow$ (3, 5, 3, 1).

Uncertainty and Nature's Moves

Sometimes, outcomes depend on chance rather than a player's choice.
Nature: A hypothetical player who moves based on fixed probabilities.
Example: In the game tree, Nature chooses "Good" or "Bad" with 50% probability after CHRIS chooses "Risky".

This type of uncertainty is modeled explicitly in game trees to reflect randomness or incomplete control. *Note:* Players account for these probabilities when forming their strategies.

Outcomes and Payoffs

Outcome: The final set of actions chosen by all players — i.e., a complete path of play. It corresponds to a specific strategy profile.

Payoffs: The numerical values assigned to each player at the end of the game, reflecting how much they value that outcome.

Convention: Payoffs are listed in a fixed order — e.g., (Player 1, Player 2, Player 3, Player 4)

Example: If the outcome is: ANN $\rightarrow$ Stop, BOB $\rightarrow$ 1, ANN $\rightarrow$ Down, then the resulting payoffs are: *(1, –2, 3, 0)*.

Game Tree: Writing Payoffs Example

Strategies

A **strategy** is a complete plan of action for a player, specifying what they would do at every decision point where they might move.

In sequential games, this means listing a choice for every node assigned to that player, even if some of those nodes won't be reached in equilibrium.

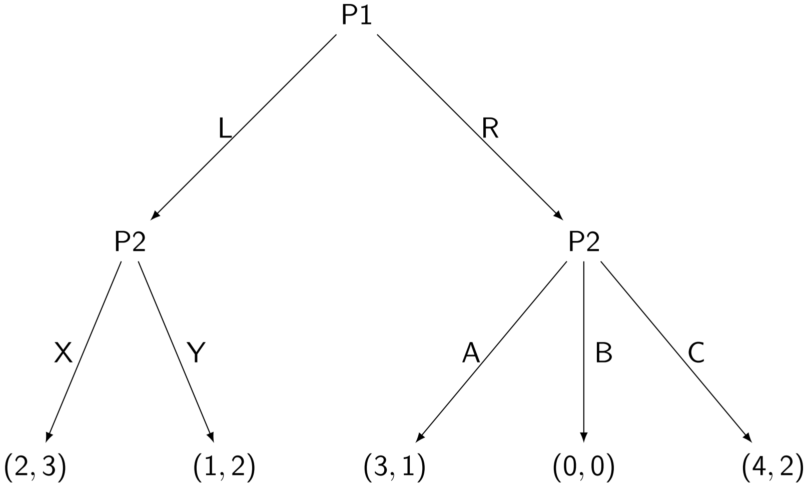
Example: ANN's strategy might be:

- Stop in the first node
- If BOB chooses 1, Up; otherwise, Down. (Notice that we define what she does, even if the node is not reached.)

Strategies are made of individual moves, but they must cover all possible situations.

Exercise

Name the set of strategies for P1 and P2, the set of strategy profiles, and the set of paths.



Exercise: Solution

– **Player 1**: The set of possible strategies for Player 1 is $\{$L, R$\}$.
– **Player 2**: The set of possible strategies for Player 2 depends on the choice of Player 1:

  If Player 1 chooses **L**, Player 2 chooses between **X** and **Y**.
  If Player 1 chooses **R**, Player 2 chooses between **A**, **B**, and **C**.

  Therefore, the overall set of possible strategies for Player 2 is $\{$XA, XB, XC, YA, YB, YC$\}$. Notice that this player's total number of strategies is $2 \times 3 = 6$.
– The set of strategy profiles, which includes the combination of strategies chosen by both players, is given by: $\{$(L, XA), (L, XB), (L, XC), (L, YA), (L, YB), (L, YC), (R, XA), (R, XB), (R, XC), (R, YA), (R, YB), (R, YC)$\}$

Exercise: Solution

The possible paths in the game are:

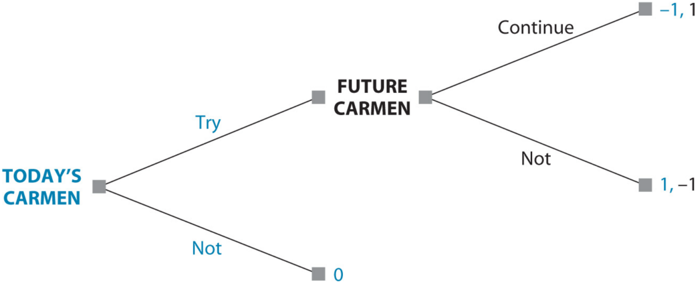
- **Path 1**: P1 chooses **L**, then P2 chooses **X**, leading to the payoff $(2, 3)$.
- **Path 2**: P1 chooses **L**, then P2 chooses **Y**, leading to the payoff $(1, 2)$.
- **Path 3**: P1 chooses **R**, then P2 chooses **A**, leading to the payoff $(3, 1)$.
- **Path 4**: P1 chooses **R**, then P2 chooses **B**, leading to the payoff $(0, 0)$.
- **Path 5**: P1 chooses **R**, then P2 chooses **C**, leading to the payoff $(4, 2)$.

Notice that there are different strategy profiles with the same paths.

Solving Sequential Games

Should I smoke today?

Consider the following example about smoking. Carmen is deciding whether to start smoking or not, depending on the continuation of her behavior.



Solving

Understanding the payoffs:

- If she does not smoke, she is neither happier nor sadder.
- She would be better if she smoked for a while. That is, she achieves the greatest utility today as long as it is something momentary.
- If she gets addicted, in the future she would be better if she continues smoking, and worse if she stops.

How should today's Carmen proceed strategically?
We need to introduce the method of *rollback / backward induction*. (Both terms mean the same.)

Rollback

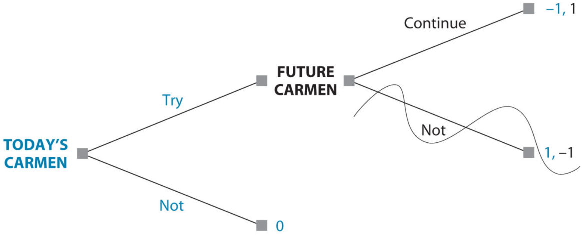
1. Begin at the decision nodes that point directly to terminal nodes, and determine the optimal actions for the final mover.
2. Move backward through each decision node, pruning non-optimal branches for each mover.
3. Identify and retain only optimal strategies at every stage.

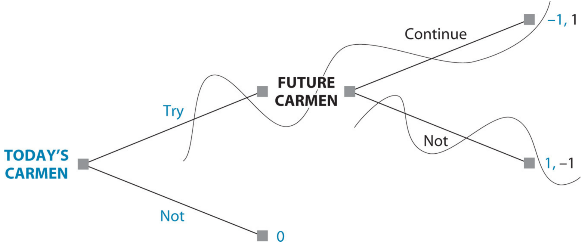
Rollback Equilibrium:

- The **strategy profile** resulting from applying rollback.
- Represents the optimal strategies for all players at each node of the game.
- Ensures that each player's choice is optimal, given the future optimal actions of other players.

Meaning: A rollback equilibrium ensures rational play in sequential games, leaving no incentive to deviate at any stage.

Applying rollback





Conclusion

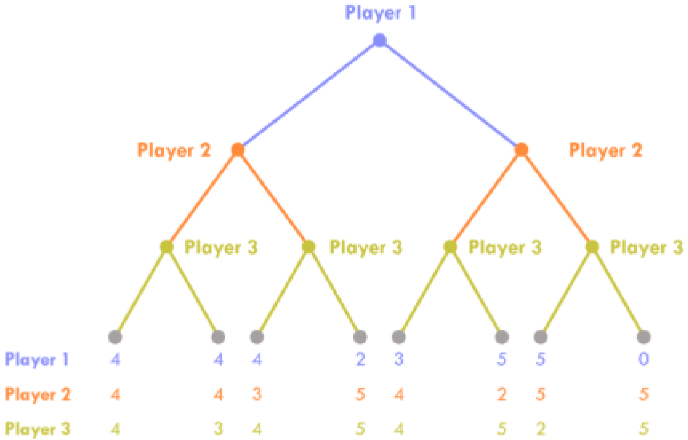
In summary, Carmen anticipates her future actions and preferences, and based on this foresight, she chooses not to start smoking today.

This logic of backward reasoning extends naturally to larger games and to games involving more players. As an exercise, solve the sequential game presented in the next slide.

Importantly, any finite sequential game with perfect information always has at least one rollback equilibrium.

Exercise

Solve the following three using rollback.



Order Advantages

First-Mover Advantage:

- The first mover can *commit* to a strategy that shapes the behavior of others. This commitment can be used to secure a more favorable outcome.
- *Example:* In a price-setting game, a firm that moves first can set a high price and deter aggressive pricing from competitors (Stackelberg competition).

Second-Mover Advantage:

- The second mover benefits from *flexibility*, reacting optimally to the first mover's action. Allows the player to avoid mistakes or exploit the first mover's commitment.
- *Example:* In a technology adoption game, a firm may wait to see if a new technology succeeds before investing, avoiding unnecessary risk.

More Complex Games

As games become more complex, the game tree can grow exponentially, making it nearly impossible to draw or analyze exhaustively.

- Even simple strategic settings can lead to large trees with hundreds of nodes.
- In real-world games like:
  - **Checkers:** $\sim 10^{20}$ possible positions.
  - **Chess:** Estimated $> 10^{40}$ different games.
  - **Go:** More possible configurations than atoms in the universe ($> 10^{170}$).
- These games are too complex to solve using full rollback or brute-force methods.

For complex games, we rely on abstraction, heuristics, and algorithms — not full game tree enumeration.

Evidence: Fairness in Ultimatum and Dictator Games

Ultimatum Game:

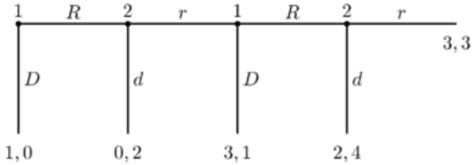
- The **proposer** (who makes the offer) typically offers 40–50% of the pie.
- The **responder** often *rejects* offers below 20–30%, even though this means getting nothing.
- Both behaviors challenge the standard prediction of self-interested rationality.

Dictator Game:

- Even when the responder has no power to reject, many proposers still give away a positive amount.
- Indicates that people care about *fairness*, *empathy*, or how they are perceived by others.

In conclusion, people are not purely egoistic—social preferences matter.

Evidence: Cooperation in the Centipede Game



- The **Centipede Game** predicts players will stop the game at the first move (via backward induction).
- But in experiments, players often *continue* the game for several rounds.
- This suggests trust, expectations of cooperation, or norms of reciprocity.
- People might believe their opponent is not strictly rational or may want to reward cooperative behavior.

Bargaining

Introduction to Bargaining

Bargaining situations occur when two or more parties must agree on how to divide a benefit or surplus.

A common example: An employer negotiating with a potential employee over a salary. Hiring the worker would yield a total profit of M, and they will bargain about how to split it.

We start with a simple case:

- The employer makes a single salary offer (w).
- The employee can accept or reject.

This is known as a **take-it-or-leave-it offer**.

Outside Options and Gains from Trade

Outside Option (or Outside Payoff): The value each player can secure if no agreement is reached. We will call a and b to the outside option values of the employer and the employee, respectively.

We require $M > a + b$ for an agreement to happen. This is usually called *gains from trade*.

Numerical Example: Take-it-or-leave-it Offer

Suppose:

- The profit generated by the employee is $M = \$100$.
- The employer gains nothing from not hiring the agent ($a = 0$).
- The employee has an outside option worth $b = 30$ (think of it as his leisure utility).

Are there incentives to build an agreement? **Yes**, because $M > a + b$ (i.e., $\$100 > \$30$).

For instance, $w = \$40$:

- ◦ The employee gets $\$40 > \$30 \Rightarrow$ better than his outside option.
- ◦ The employer keeps $\$60 > \$0 \Rightarrow$ better than her outside option.

This is a **Pareto improvement**: both parties are better off than without agreement.

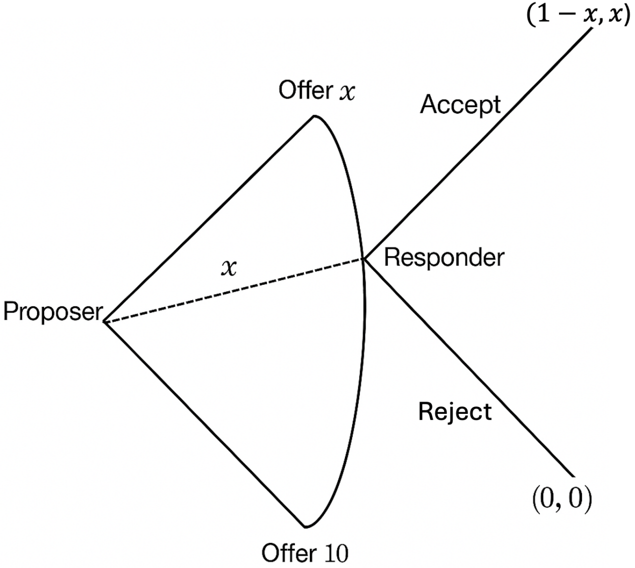
Solution by Rollback

- The employer anticipates the employee will accept any offer $w \geq 30$.
- To maximize her own payoff, the employer offers the smallest amount the employee is willing to accept: $w = 30$.
- The employee is indifferent between accepting the offer (30) and taking the outside option (30), so he accepts.

Note: In this class, we will assume that players accept offers when they are indifferent between accepting and rejecting.

Result: The employer gets \$70, and the employee gets \$30.

Graph representation



Extending to Two-Sided Bargaining

Now suppose both players can make offers.

We introduce **alternating offers**: One player proposes a split, the other can accept or counteroffer.

The logic becomes more strategic:

- Time preferences (discounting) matter.
- Players may accept suboptimal offers to avoid delays.

We now move beyond take-it-or-leave-it dynamics.

Graph representation: Extended Version

Other Bargaining Scenarios

Bargaining is not limited to hiring — it arises in many real-world situations:

- **Client–Supplier Negotiations:** A supplier and buyer negotiate price and delivery terms. Each side has outside options — alternative buyers or providers.
- **Trade Agreements:** Countries negotiate tariffs or quotas. Bargaining power may depend on market size or political urgency.
- **Personal Contexts:** Roommates deciding how to split chores, or friends deciding which movie to watch. Even these involve implicit offers and concessions.
- **Divorce Settlements:** Spouses bargain over asset division, custody, and support, often with lawyers acting as strategic agents.