Mixed Strategies

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Why Mixed Strategies?

Motivating Example: Matching Pennies

	Heads	**Tails**
Heads	$(1, -1)$	$(-1, 1)$
Tails	$(-1, 1)$	$(1, -1)$

A is the row player and B is the column player.

- Zero-sum game: one player wins what the other loses.
- No pure strategy Nash equilibrium:
  - If A plays Heads, B prefers Tails; but then A prefers Tails, and B switches again.
  - The cycle continues with no stable strategy.

Why Mixed Strategies?

Real-World Analogy: Serving in Tennis

- A player can serve to the opponent's forehand or backhand.
- Always serving to one side becomes predictable.
- Best response depends on opponent's expectations.

Key Insight: Strategic Uncertainty

- Players benefit from being unpredictable.
- Mixed strategies involve randomizing over actions.
- Keeps the opponent guessing and prevents exploitation.

What is a Mixed Strategy?

Definition: A *mixed strategy* is a probability distribution over a player's available pure strategies.

- Instead of choosing a single action, a player randomizes.
- Example: Player A plays Heads 30% of times, and Tails 70% of times.
- Player B plays Heads 65% of times, and Tails 35% of times.

We need to introduce the concept of "Expected Payoff" before solving a game like this.

Expected Payoff Definition

Definition: The *expected payoff* is the weighted average of a player's payoffs, where the weights are given by the probabilities of each outcome.

Simple Example:

- Suppose a lottery pays:
  - \$10 with probability 0.4
  - \$2 with probability 0.6
- Then the expected payoff is:

$$\mathbb{E}[\text{Payoff}] = 0.4 \cdot 10 + 0.6 \cdot 2 = 4 + 1.2 = 5.2$$

If you played this lottery many times, you'd win \$5.20 on average per play.

Expected Payoff

Suppose:

- Player A plays **Heads** with probability $p = 0.3$, and **Tails** with 0.7.
- What are Player B's expected payoffs if she chooses:

1. Play Heads (pure strategy):

$$\mathbb{E}[u_B^{\text{Heads}} \mid p = 0.3] = p\cdot(-1)+(1-p)\cdot(+1) = 0.3\cdot(-1)+0.7\cdot 1 = 0.4$$

2. Play Tails (pure strategy):

$$\mathbb{E}[u_B^{\text{Tails}} \mid p = 0.3] = p\cdot(+1)+(1-p)\cdot(-1) = 0.3\cdot 1+0.7\cdot(-1) = -0.4$$

Expected Payoff

Recap:

- Player A plays Heads with $p = 0.3$, Tails with 0.7.
- Player B's expected payoff from:
  - Playing **Heads**: 0.4
  - Playing **Tails**: -0.4
- Player B randomizes: plays Heads with $q = 0.65$, Tails with 0.35.

Player B's expected payoff:

$$\mathbb{E}[u_B] = q \cdot 0.4 + (1 - q) \cdot (-0.4) = 0.65 \cdot 0.4 + 0.35 \cdot (-0.4) = 0.12$$

Evert vs. Navratilova: The Tennis Game

Players:

- Evert and Navratilova choose between: **DL (Down the Line)**, **CC (Cross Court)**

Navratilova

		DL	CC
Evert	DL	(50, 50)	(80, 20)
	CC	(90, 10)	(20, 80)

Let:

- p: probability that **Evert** plays DL (so $1 - p$ is CC)
- q: probability that **Navratilova** plays DL (so $1 - q$ is CC)

Navratilova's Best Response to Evert's Mixed Strategy

Evert plays:

- DL with probability p
- CC with probability $1 - p$

Navratilova's Expected Payoffs:

- **If she plays DL (pure strategy):**

$$\mathbb{E}[u_N \mid \text{DL}] = 50p + 10(1 - p) = 10 + 40p$$

- **If she plays CC (pure strategy):**

$$\mathbb{E}[u_N \mid \text{CC}] = 20p + 80(1 - p) = 80 - 60p$$

Navratilova's best response: Choose DL if $10 + 40p > 80 - 60p$, CC if $10 + 40p < 80 - 60p$, and is indifferent when $p = 0.7$.

Evert's Best Response to Navratilova's Mixed Strategy

Navratilova plays:

- DL with probability q
- CC with probability $1 - q$

Evert's Expected Payoffs:

- **If she plays DL (pure strategy):**

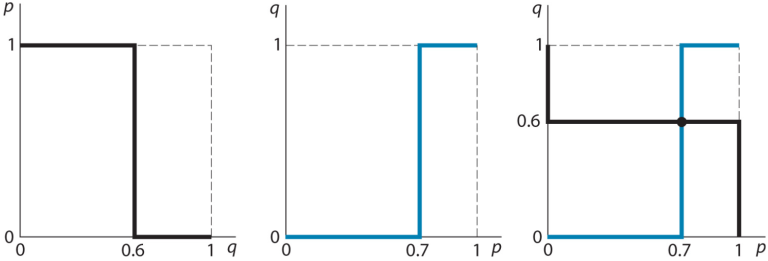
$$\mathbb{E}[u_E \mid \text{DL}] = 50q + 80(1 - q) = 80 - 30q$$

- **If she plays CC (pure strategy):**

$$\mathbb{E}[u_E \mid \text{CC}] = 90q + 20(1 - q) = 20 + 70q$$

Evert's best response: Choose DL if $80 - 30q > 20 + 70q$, CC if $80 - 30q < 20 + 70q$, and is indifferent when $q = 0.6$.

Game Graph



Interpreting the Best Response Graph

- The three graphs show the best response strategies for Evert and Navratilova as functions of the opponent's mixing probability.

- **Left:** Evert's best response curve:
  - Evert plays DL when $q < 0.6$, CC when $q > 0.6$, and is indifferent at $q = 0.6$.

- **Middle:** Navratilova's best response curve:
  - She plays CC when $p < 0.7$, DL when $p > 0.7$, and is indifferent at $p = 0.7$.

Interpreting the Best Response Graph

- **Right:** The intersection point at $(p = 0.7, q = 0.6)$ shows the unique Nash equilibrium:
  - Each player is indifferent between their actions given the other's strategy.
  - No pure strategy equilibrium exists; mixed strategies are essential.

- This equilibrium is **exploitation-proof**: neither player can improve unilaterally.

- As an exercise, find the Mixed NE at the Heads-Tails game.

Non-Zero-Sum Games

Holmes and Watson must choose between location A and location B. They cannot communicate, but they want to meet.

		Watson	
		A	B
Holmes	A	(1, 1)	(0, 0)
	B	(0, 0)	(1, 1)

- Two pure strategy Nash equilibria: (A, A) and (B, B).
- But without communication, coordination is difficult.
- Randomization can prevent systematic failure to coordinate. What is the NE in mixed strategies?

Non-Zero-Sum Games

Scenario: Same as before, but meeting at location A is more rewarding.

		Watson	
		A	B
Holmes	A	(2, 2)	(0, 0)
	B	(0, 0)	(1, 1)

– Still two pure strategy NE: (A, A) and (B, B).

– But (A, A) strictly dominates (B, B).

– Location A becomes a **focal point**—a natural solution that stands out. In this case, it stands out because it *strictly dominates* the alternative. But focal points can also arise from conventions, labels, symmetry, or any feature that draws mutual attention.

Exercises

		Player 2	
		Swerve	Straight
Player 1	Swerve	(0, 0)	(0, 2)
	Straight	(2, 0)	(-1, -1)

		Wife	
		Opera	Football
Husband	Opera	(2, 1)	(0, 0)
	Football	(0, 0)	(1, 2)

Find all pure and mixed strategy Nash equilibria in both games.
What kind of strategic tension do these games reveal?

Modified Tennis Game

Consider the following modified version of the tennis game:

Navratilova

		DL	CC
Evert	DL	(30, 70)	(80, 20)
	CC	(90, 10)	(20, 80)

Find the Nash Equilibrium in mixed strategies for this game.
Why might the equilibrium result seem counterintuitive?

- Remember that equilibrium outcomes are driven by the *strategic interaction* of both players.
- Even if one action appears dominant, the outcome depends on how each player responds to the other's choices.

Counterintuitive Outcomes

Monetary Policy as a Strategic Game

When the central bank lowers interest rates, the goal is to boost growth by encouraging borrowing and investment.

But the outcome depends not only on policy, but on how private agents—especially firms—*expect* the economy to react.

If firms anticipate higher inflation, they may raise prices and wages in advance. This can drive inflation up, offsetting the intended stimulus.

Thus, a policy meant to support growth can **undermine it**, through a feedback loop of expectations and price-setting.

Key insight: outcomes depend on strategic interaction, not just one side's intentions.

Penalty Kick Game: 3 Strategies Each

Payoffs: First number = Kicker's chance of scoring, second number = Goalie's chance of saving.

		Goalie		
		Left	Center	Right
Kicker	Left	(45, 55)	(90, 10)	(90, 10)
	Center	(85, 15)	(0, 100)	(85, 15)
	Right	(95, 5)	(95, 5)	(60, 40)

No Dominant Strategy Exists

Observation:

- The Kicker does not have a dominant strategy — the best action depends on where the Goalie dives.
- Similarly, the Goalie's best move depends on the Kicker's shot.

Implication: We are in a setting where players must form beliefs about the other's strategy. This is a classic case for using **mixed strategies**.

Key Question: What probabilities should each player assign to their actions to be unpredictable and maximize their payoff?

Mixed Strategy NE: Step-by-Step

Step 1: Confirm there are no dominant strategies.

- The Kicker's best action depends on the Goalie's move.
- The Goalie's best response depends on the Kicker's choice.

Conclusion: No dominant strategies $\rightarrow$ we must use **mixed strategies**.

Goal: Find mixed strategies such that each player is indifferent across their own actions.

Step 2: Make the Kicker Indifferent

Let the Goalie mix: Choose q_L, q_C, q_R with $q_L + q_C + q_R = 1$.
To simplify: set $q_R = 1 - q_L - q_C$

Kicker's expected payoffs:

$$\begin{aligned} U_L &= 45q_L + 90q_C + 90(1 - q_L - q_C) \\ U_C &= 85q_L + 0q_C + 85(1 - q_L - q_C) \\ U_R &= 95q_L + 95q_C + 60(1 - q_L - q_C) \end{aligned}$$

Set $U_L = U_C = U_R$ and solve the system for q_L and q_C.

Step 3: Make the Goalie Indifferent

Let the Kicker mix: Choose p_L, p_C, p_R with $p_L + p_C + p_R = 1$.
To simplify: set $p_R = 1 - p_L - p_C$

Goalie's expected payoffs:

$$V_L = 55p_L + 15p_C + 5(1 - p_L - p_C)$$
$$V_C = 10p_L + 100p_C + 5(1 - p_L - p_C)$$
$$V_R = 10p_L + 15p_C + 40(1 - p_L - p_C)$$

Set $V_L = V_C = V_R$ and solve for p_L and p_C.

One player with 3 strategies (Optional)

Consider the following game

		Navratilova	
		DL	CC
Evert	DL	(50, 50)	(80, 20)
	CC	(90, 10)	(20, 80)
	Lob	(70, 30)	(60, 40)

Find the NE in mixed strategies.