GIST 4302/5302: Spatial Analysis and Modeling

Point Pattern Analysis

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Characteristics:

- set of n point locations with recorded "events", e.g., locations of trees, disease or crime incidents $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_i, \ldots, \mathbf{s}_n\}$
- point locations correspond to all possible events or to subsets of them
- attribute values also possible at same locations, e.g., tree diameter, magnitude of earthquakes (*marked point pattern*) $W = \{w_1, \ldots, w_i, \ldots, w_n\}$

Analysis objectives:

- detect spatial clustering or repulsion, as opposed to complete randomness, of event locations (in space and time)
- if clustering detected, investigate possible relations with nearby "sources"

Further issues:

- analysis of point patterns over large areas should take into account distance distortions due to map projections
- boundaries of study area should not be arbitrary
- analysis of sampled point patterns can be misleading
- one-to-one correspondence between objects in study area and events in pattern

Simple Descriptive Statistics

Mean center of a point pattern:

- point with coordinates $\bar{s} = (\bar{x}, \bar{y})$:

$$\bar{x} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} \qquad and \qquad \bar{y} = \frac{\sum_{i=1}^{n} w_i y_i}{\sum_{i=1}^{n} w_i}$$

- center of point pattern, or point with average x and y-coordinates

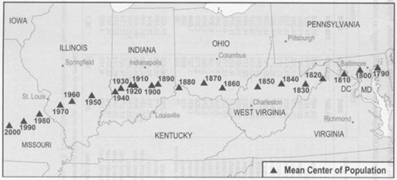
Median center of a point pattern:

- *both* of the following two centers are called *median centers*, although they are essentially different (confusing!)
  - the intersection between the median of the x and the y coordinates
  - *center for minimum distance:* $\boldsymbol{s_c} \in \{\boldsymbol{s_1}, \ldots, \boldsymbol{s_n}\} s.t. min \sum_{i=1}^{n} |\boldsymbol{s_i} - \boldsymbol{s_c}|$
- the first type of *median center* is not unique, and there is no closed form for the second type
- p-median problem (a typical problem in spatial optimization): the problem of locating p "facilities" relative to a set of "customers" such that the sum of the shortest demand weighted distance between "customers" and "facilities" is minimized

Changes of population center (year 1790-2000):

Year	Median center		Mean center		
	Latitude-N	Longitude	Latitude-N	Longitude-W	Approximate location
1790 (August 2)	(NA)	(NA)	39 16 30	76 11 12	In Kent County, MD, 23 miles E of Baltimore MD
1850 (June 1). .	(NA)	(NA)	38 59 00	81 19 00	In Wirt County, WV, 23 miles SE of Parkersburg, WV[1]
1900 (June 1). .	40 03 32	84 49 01	39 09 36	85 48 54	In Bartholomew County, IN, 6 miles SE of Columbus, IN
1950 (April 1) . .	40 00 12	84 56 51	38 50 21	88 09 33	In Richland County, IL, 8 miles NNW of Olney, IL
1960 (April 1) . .	39 56 25	85 16 60	38 35 58	89 12 35	In Clinton County, IL, 6.5 miles NW of Centralia, IL
1970 (April 1) .	39 47 43	85 31 43	38 27 47	89 42 22	In St. Clair County, IL, 5.3 miles ESE of Mascoutah, IL
1980 (April 1) . .	39 18 60	86 08 15	38 08 13	90 34 26	In Jefferson County, MO, .25 mile W of DeSoto, MO
1990 (April 1) . .	38 57 55	86 31 53	37 52 20	91 12 55	In Crawford County, MO, 10 miles SE of Steelville, MO
2000 (April 1) . .	38 45 23	86 55 51	37 41 49	91 48 34	In Phelps County, MO, 3 miles E of Edgar Springs, MO

NA Not available. [1]West Virginia was set off from Virginia, Dec. 31, 1862, and admitted as a state, June 19, 1863.



Descriptive Statistics

Standard distance of a point pattern:

- average squared deviations of x and y coordinates from their respective mean:

$$d_{std} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{i=1}^{n}(y_i - \bar{y})^2}{n-2}}$$

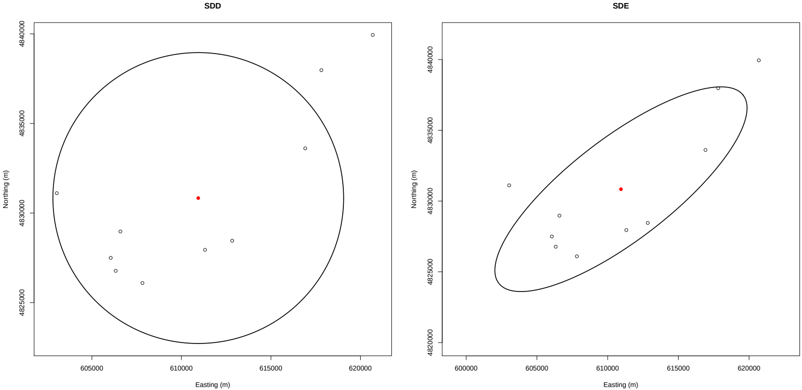
- related to standard deviation of coordinates, a summary circle (centered at $\bar{s}$ with radius d_{std}) of a point pattern

Standard deviational ellipse:

- Taking directional effects into account for *anisotropy* cases
- Please refer to Levine and Associates, 2004 for calculations

Descriptive Statistics

Examples:



Remarks:

- indicates overall shape and center of point pattern
- *do not suffice to fully specify a spatial point pattern*

Point Pattern Analysis Methods

1st order (i.e., intensity): absolute location of events on map:

- Quadrat methods
- Density Estimation (KDE)

2nd order (i.e., interactions): interaction of events:

- Nearest neighbor distance
- Distance functions G, K (or L)

Quadrat methods

Consider a point pattern with n events within a study region A of area $|A|$

Global intensity:

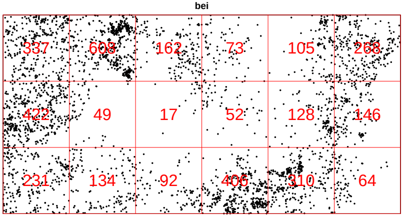
$$\hat{\lambda} = \frac{n}{|A|} = \frac{\#\text{of events within} A}{|A|}$$

Local intensity via quadrats

1. partition A into L sub-regions $A_l, l = 1, \ldots, L$ of equal area $|A_l|$ (also called quadrats)
2. count number of events $n(A_l)$ in each sub-region A_l
3. convert sample counts into estimated intensity rates as:

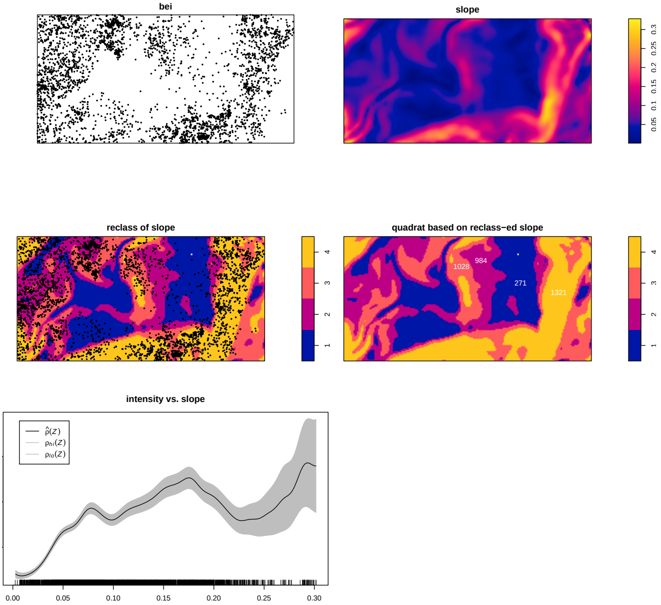
$$\hat{\lambda}(A_l) = \frac{n(A_l)}{|A_l|}$$

Quadrat methods



- estimated rates $\hat{\lambda}(A_l)$ over set of quadrats
- reveal large-scale patterns in intensity variation over A
- larger quadrats yield smoother intensity maps; smaller quadrats yield 'spiky' intensity maps
- size, origin, and shape of quadrats is critical (recall: *MAUP*)
- *only first-order effects are captured*

Dependence of intensity on a covariate (Inhomogeneous

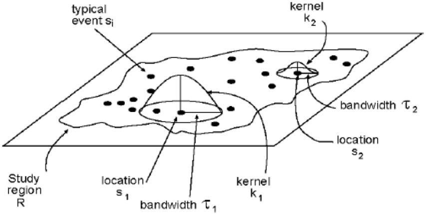


Procedure of Kernel Density Estimation (KDE)

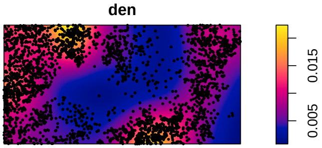
1. define a kernel $K(\boldsymbol{s}; r)$ of radius (or bandwidth) r centered at any arbitrary location $\boldsymbol{s}$
2. estimate local intensity at $\boldsymbol{s}$ as:

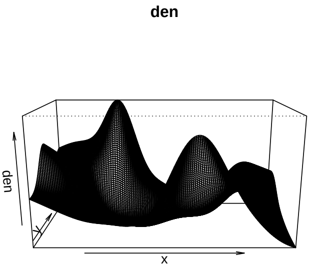
$$\hat{\lambda}(\boldsymbol{s}) = \frac{1}{n}\sum_{i=1}^{n} K(\boldsymbol{s}_i - \boldsymbol{s}; r)$$

3. repeat estimation for all points s in the study region to create a density map



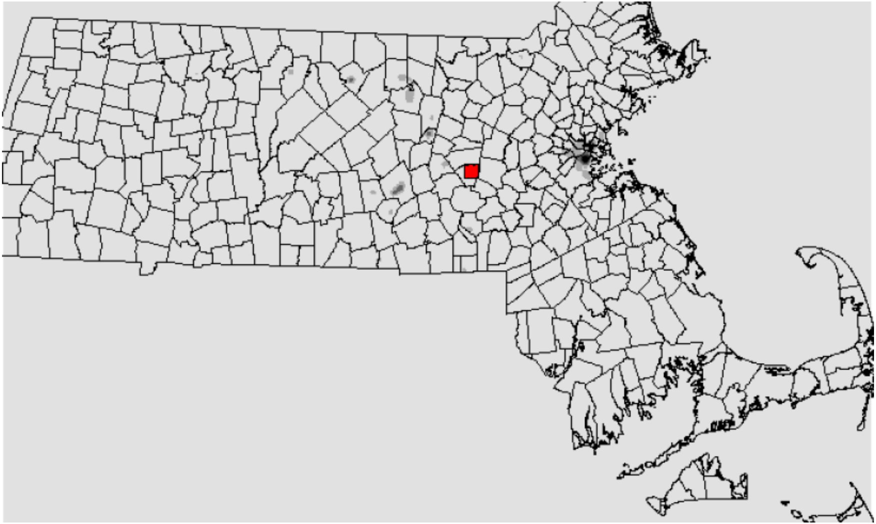
Example for the previous dataset:





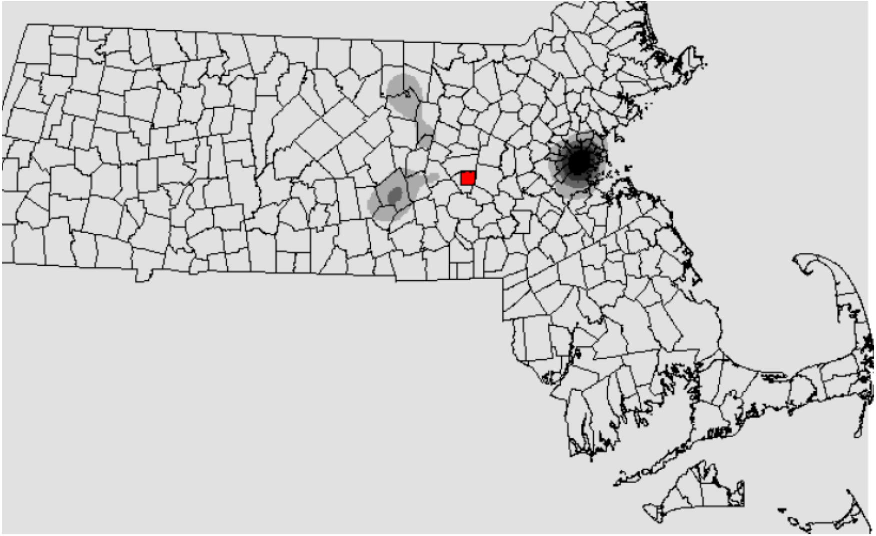
Kernel Density Estimation

Example with 2km bandwidth



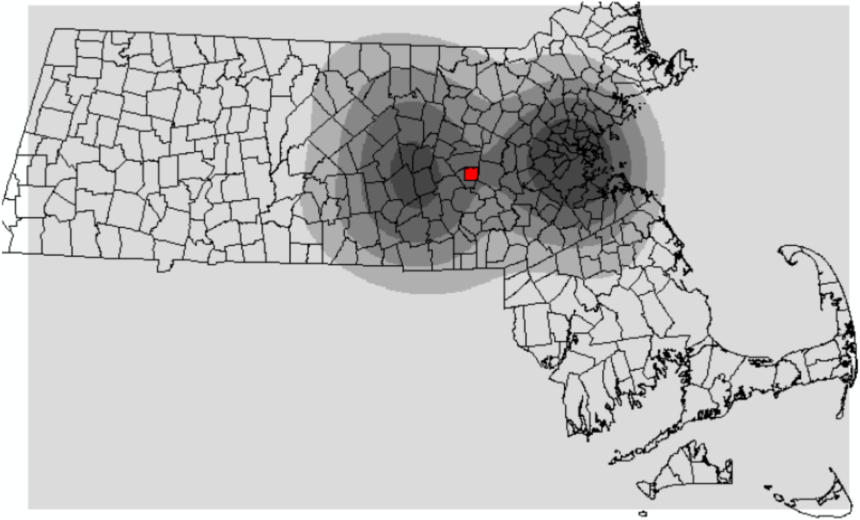
Kernel Density Estimation

Example with 10km bandwidth



Kernel Density Estimation

Example with 40km bandwidth



Comments

- Choice of kernel function is not critical (Diggle, 1985)
- Choice of bandwidth, or degree of smoothing critical:
  - Small bandwidth $\rightarrow$ spiky results
  - Large bandwidth $\rightarrow$ loss of detail
- Multi-scale analyses can use these bandwidth characteristics to investigate both broad trends and localized variation
- How to choose bandwidth: choose the degree of smoothing subjectively, by eye, or by formula (Diggle)
- could define local bandwidth based on function of presence of events in neighborhood of $\boldsymbol{s}$ (i.e., adaptive kernel estimation)

What does the output of KDE means?

Distance-based Descriptors of Point Patterns

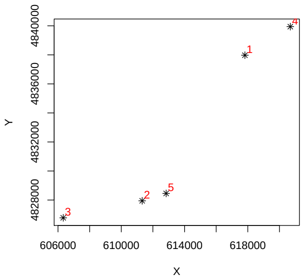
- Distances: accessing second order effects
  - Event-to-event distance: distance d_{ij} between event at arbitrary location s_i and another event at another arbitrary location s_j:

$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

  - Event-to-nearest-neighbour distance: distance $d_{min}(s_i)$ between an event at location s_i and its *nearest neighbor* event:

$$d_{min}(s_i) = min\{d_{ij}, j \neq i, j = 1, \ldots, n\}$$

Event-to-Nearest-Neighbor Distances

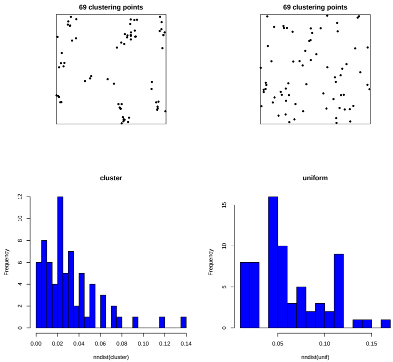


	1	2	3	4	5
1	0.00	11947.70	16042.65	3481.22	10742.98
2	11947.70	0.00	5126.79	15219.58	1599.07
3	16042.65	5126.79	0.00	19481.59	6720.59
4	3481.22	15219.58	19481.59	0.00	13913.70
5	10742.98	1599.07	6720.59	13913.70	0.00

Table: Euclidean distance matrix

Event-to-Nearest-Neighbor Distances

Nearest neighbour distances



- Mean nearest neighbour distance: Average of all $d_{min}(s_i)$ values

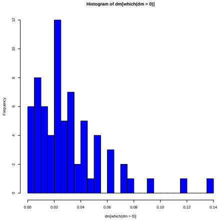
$$\bar{d}_{min} = \frac{1}{n} \sum_{i=1}^{n} d_{min}(s_i)$$

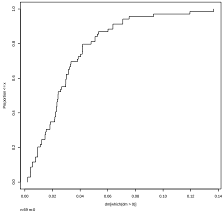
The G function

- Definition: nearest neighbour distance function, i.e., proportion of event-to-nearest-neighbor distances $d_{min}(s_i)$ no greater than given distance cutoff d, estimated as:

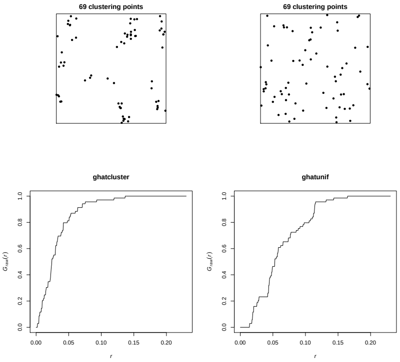
$$\hat{G}(d) = \frac{\#\{d_{min}(s_i) < d, i = 1, \ldots, n\}}{n}$$

- alternative definition: cumulative distribution function (CDF) of all n event-to-nearest-neighbor distances; instead of computing average $\bar{d}_{min}$ of d_{min} values, compute their CDF
- the G function provides information on event *proximity*
- example for previous clustering point pattern:





Examples of G function



Expected plot:

- for clustered events, $\hat{G}(d)$ rises sharply at short distances, and then levels off at larger d-values
- for randomly-spaced events, $\hat{G}(d)$ rises gradually up to the distance at which most events are spaced, and then increases sharply

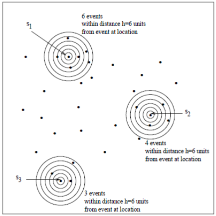
The K function

Working with pair-wise distances&looking beyond nearest neighours

Concept

1. construct set of concentric circles (of increasing radius d) around each event
2. count number of events in each distance "band"
3. cumulative number of events up to radius d around all events becomes the sample K function $\hat{K}(d)$

Example of K function estimation



The K function

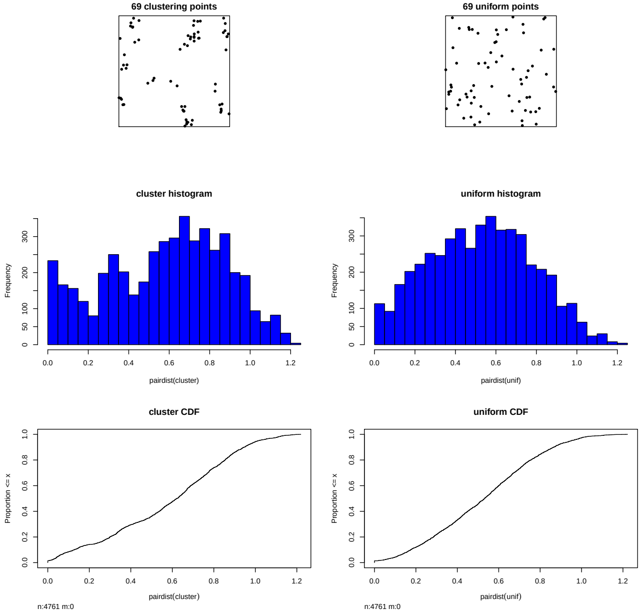
Working with pair-wise distances&looking beyond nearest neighours

- Formal definition:

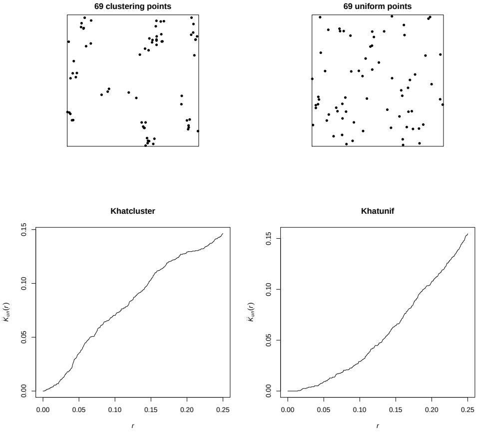
$$
\begin{aligned}
K(d) &= \frac{1}{\lambda} \frac{\#\{d_{ij} \leq d, i, j = 1, \ldots, n\}}{n} \\
&= \frac{|A|}{n} \frac{\#\{d_{ij} \leq d, i, j = 1, \ldots, n\}}{n} \\
&= |A|(\text{proportion of event-to-event distance} \leq d)
\end{aligned}
$$

- In other words, the $\hat{K}(d)$ is the sample cumulative distribution function (CDF) of all n^2 event-to-event distances, scaled by $|A|$

Examples of Event-to-Event Distance Histogram and CD



Examples of K functions



- the sample K function $\hat{K}(d)$ is monotonically increasing and is a scaled (by area $|A|$) version of the CDF of E2E distances

Recap

Spatial point patterns

- set of n point locations with recorded "events"

Describing the first-order effect

- overal intensity
- local intensity (quadrat count and kernel density estimation)

Describing the second-order effect

- nearest neighbour distances
  - the G function
- pair-wise distances
  - the K function

Descriptive vs Statistical Point Pattern Analysis

Descriptive analysis:

- set of quantitative (and graphical) tools for characterizing spatial point patterns
- different tools are appropriate for investigating first- or second-order effects (e.g., kernel density estimation versus sample G function)
- can shed light onto whether points are clustered or evenly distributed in space

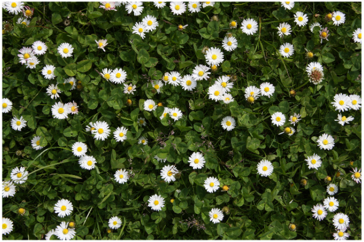
Limitation:

- no assessment of *how* clustered or *how* evenly-spaced is an observed point pattern
- no yardstick against which to compare observed values (or graph) of results

Statistical analysis:

- assessment of whether an observed point pattern can be regarded as one (out of many) realizations from a particular spatial process
- measures of confidence with which the above assessment can be made (how likely is that the observed pattern is a realization of a particular spatial process)

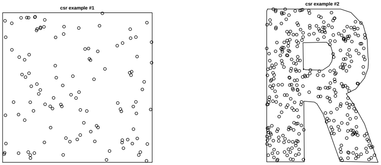
Are daisies randomly distributed in your garden?



Complete Spatial Randomness (CSR)

- yardstick, reference model that observed point patterns could be compared with, i.e., null hypothesis
- = *homogeneous (uniform) Poisson point process*
- basic properties:
  - the number of points falling in any region A has a Poisson distribution with mean $\lambda|A|$
  - given that there are n points inside region A, the locations of these points are i.i.d. and uniformly distributed inside A
  - the contents of two disjoint regions A and B are independent

Example:



Quadrat counting test

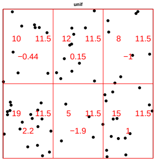
- partition study area A into L sub-regions (quadrats), $A_1, \dots, A_L$
- count number of events $n(A_l)$ in each sub-region A_l
- Under the null hypothesis of CSR, the $n(A_l)$ are i.i.d. Poisson random variables with the same expected value
- The Pearson χ^2 goodness-of-fit test can be used
  - test statistics: Pearson residual $\sum_l \epsilon(A_l)^2$

$$\epsilon(A_l) = \frac{n(A_l) - \mu(A_l)}{\sqrt{\mu(A_l)}},$$

    where $\mu(A_l)$ indicates the expected number of events in A_l
  - $\sum_{l=1} \epsilon(A_l)^2$ is assumed to follow χ^2 distribution

Quadrat counting test for CSR

Example



- three values indicate the number of observations, CSR-expected number of observations, and the Pearson residuals
- p-value = 0.617

Nearest neighbour index

- Compares the mean of the distance observed between each point and its nearest neighbor ($\bar{d}_{min}$) and the expected mean distance under CSR $E(d_{min})$

$$NNI = \frac{\bar{d}_{min}}{E(d_{min})}$$

- Under CSR, we have:

$$E(d_{min}) = \frac{1}{2\sqrt{\lambda}}$$

$$\sigma(d_{min}) = \frac{0.26136}{\sqrt{n^2/A}}$$

Nearest neighbour index test

- Test statistics:

$$z = \frac{\bar{d}_{min} - E(d_{min})}{\sigma(d_{min})},$$

- z is assumed to follow Gaussian distribution, thus, if $z < -1,96$ or $z > 1.96$, we are 95% confident that the distribution is <u>not</u> randomly distributed

The K Function under CSR

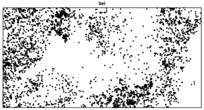
- The K function is a function of pair-wise distances
- For a homogeneous Poisson point process of intensity λ, the pair-wise distance distribution (the K function) is known to be:

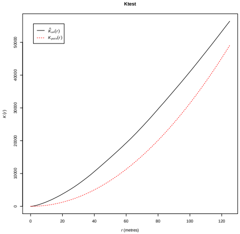
$$K(d) = \pi d^2$$

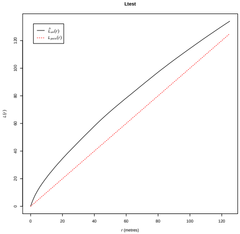
- A commonly-used transformation of K is the L-function:

$$L(d) = \sqrt{\frac{K(d)}{\pi}} = d$$

Example

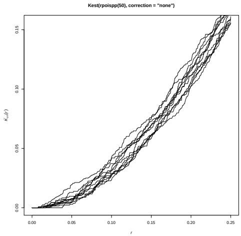






- because of random variability, we will never obtain perfect agreement between sample functions (say the K function) with theoretical functions (the theoretical K functions), even with a completely random pattern

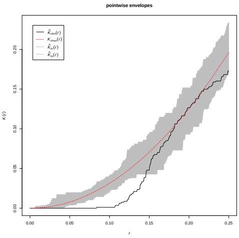
Example



Monte Carlo test

- A *Monte Carlo* test is a test based on simulations from the null hypothesis
- Basic procedures:
  - generate M independent simulations of CSR inside the study region A
  - compute the estimated K functions for each of these realisations, say $\hat{K}^{(j)}(r)$ for $j = 1, \ldots, M$
  - obtain the pointwise upper and lower envelopes of these simulated curves
  - not a confidence interval

Example



Recap

Statitsical analysis of spatial point patterns:

- allows to quantify departure of results obtained via exploratory tools, e.g., $\hat{G}(d)$, from expected such results derived under specific null hypotheses, here CSR hypothesis
- can be used to assess to what extent observed point patterns can be regarded as realizations from a particular spatial process (here CSR)
- Same concepts can be applied for hypothesis of other types of point processes (e.g., Poisson cluster process, Cox process)

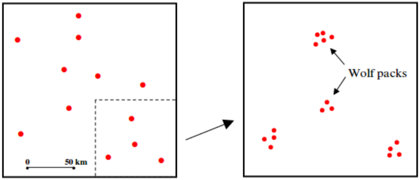
Sampling distribution of a test statistics

- lies at the heart of any statistical hypothesis testing procedure, and is tied to a particular null hypothesis
- simulation and analytical derivations are two alternative ways of computing such sampling distributions (the latter being increasingly replaced by the former)

Edge Effects

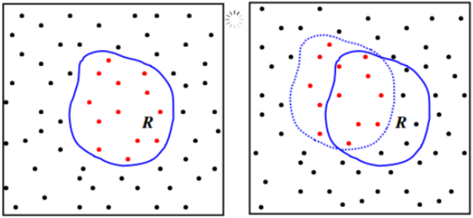
Scale effects

- Wolf pack example



- Nearest neighour distance (NN distance, G,F functions) vs K function

Edge effects



Recap

Extended into line processes

- Line density