

GIST 4302/5302: Spatial Analysis and Modeling

Point Pattern Analysis

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Spatial Point Patterns

Characteristics:

- set of n point locations with recorded “events”, e.g., locations of trees, disease or crime incidents $S = \{s_1, \dots, s_i, \dots, s_n\}$
- point locations correspond to all possible events or to subsets of them
- attribute values also possible at same locations, e.g., tree diameter, magnitude of earthquakes (*marked point pattern*)
 $W = \{w_1, \dots, w_i, \dots, w_n\}$

Analysis objectives:

- detect spatial clustering or repulsion, as opposed to complete randomness, of event locations (in space and time)
- if clustering detected, investigate possible relations with nearby “sources”



Further issues:

- analysis of point patterns over large areas should take into account distance distortions due to map projections
- boundaries of study area should not be arbitrary
- analysis of sampled point patterns can be misleading
- one-to-one correspondence between objects in study area and events in pattern



Simple Descriptive Statistics

Mean center of a point pattern:

- point with coordinates $\bar{s} = (\bar{x}, \bar{y})$:

$$\bar{x} = \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i} \quad \text{and} \quad \bar{y} = \frac{\sum_{i=1}^n w_i y_i}{\sum_{i=1}^n w_i}$$

- center of point pattern, or point with average x and y -coordinates

Median center of a point pattern:

- both of the following two centers are called median centers, although they are essentially different (confusing!)
 - the intersection between the median of the x and the y coordinates
 - center for minimum distance: $s_c \in \{s_1, \dots, s_n\}$ s.t. $\min \sum_{i=1}^n |s_i - s_c|$
- the first type of *median center* is not unique, and there is no closed form for the second type
- p -median problem (a typical problem in spatial optimization): the problem of locating p “facilities” relative to a set of “customers” such that the sum of the shortest demand weighted distance between “customers” and “facilities” is minimized

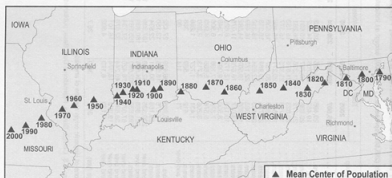


Simple Descriptive Statistics

Changes of population center (year 1790-2000):

| Year | Median center | | Mean center | | Approximate location |
|------------------|---------------|-----------|-------------|-------------|---|
| | Latitude-N | Longitude | Latitude-N | Longitude-W | |
| 1790 (August 23) | (NA) | (NA) | 39 16 30 | 78 11 12 | In Kent County, MD, 23 miles E of Baltimore, MD |
| 1850 (June 1) | (NA) | (NA) | 38 59 00 | 81 19 00 | In Wirt County, WV, 23 miles SE of Parkersburg, WV ¹ |
| 1900 (June 1) | 40 03 22 | 84 49 01 | 39 05 36 | 85 48 54 | In Bartholomew County, IN, 6 miles SE of Columbus, IN |
| 1950 (April 1) | 40 00 12 | 84 56 51 | 38 50 21 | 88 09 33 | In Richland County, IL, 8 miles NNW of Olney, IL |
| 1950 (April 1) | 39 56 25 | 85 16 60 | 38 35 58 | 89 12 35 | In Clinton County, IL, 6.5 miles NW of Centralia, IL |
| 1970 (April 1) | 38 47 43 | 85 31 43 | 38 27 47 | 89 40 22 | In St. Clair County, IL, 9.3 miles ESE of Macdonald, IL |
| 1980 (April 1) | 39 18 60 | 86 08 15 | 38 08 13 | 90 34 26 | In Jefferson County, MO, 25 mile W of DeSoto, MO |
| 1990 (April 1) | 38 57 55 | 86 31 53 | 37 52 20 | 91 12 55 | In Crawford County, MO, 10 miles E of Steelville, MO |
| 2000 (April 1) | 38 45 23 | 86 56 51 | 37 41 49 | 91 48 34 | In Phelps County, MO, 3 miles E of Edgar Springs, MO |

NA Not available. ¹West Virginia was set off from Virginia, Dec. 31, 1862, and admitted as a state, June 19, 1863.





Standard distance of a point pattern:

- average squared deviations of x and y coordinates from their respective mean:

$$d_{std} = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{i=1}^n (y_i - \bar{y})^2}{n - 2}}$$

- related to standard deviation of coordinates, a summary circle (centered at \bar{s} with radius d_{std}) of a point pattern

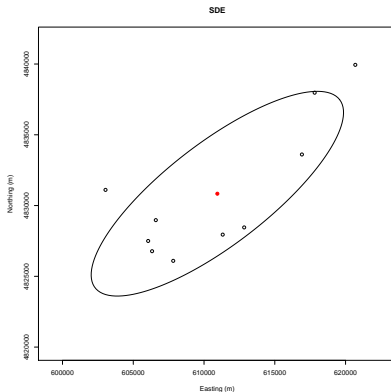
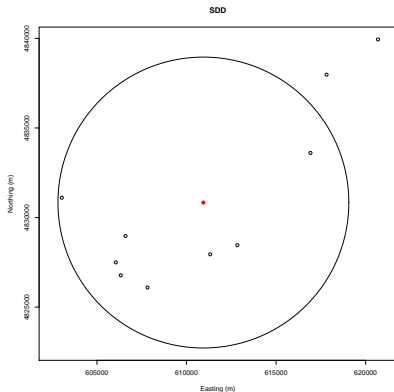
Standard deviational ellipse:

- Taking directional effects into account for anisotropy cases
- Please refer to Levine and Associates, 2004 for calculations



Descriptive Statistics

Examples:



Remarks:

- indicates overall shape and center of point pattern
- do not suffice to fully specify a spatial point pattern



1st order (i.e., intensity): absolute location of events on map:

- Quadrat methods
- Density Estimation (KDE)
- Moran's I and Geary's C

2nd order (i.e., interactions): interaction of events:

- Nearest neighbor distance
- Distance functions G, K, F, L
- Getis-Ord G_i^* and Anselin local Moran's I



Quadrat methods

Consider a point pattern with n events within a study region A of area $|A|$

Global intensity:

$$\hat{\lambda} = \frac{n}{|A|} = \frac{\text{\#of events within } A}{|A|}$$

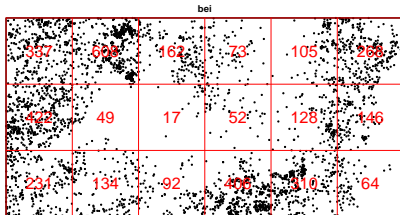
Local intensity via quadrats

1. partition A into L sub-regions $A_l, l = 1, \dots, L$ of equal area $|A_l|$ (also called quadrats)
2. count number of events $n(A_l)$ in each sub-region A_l
3. convert sample counts into estimated intensity rates as:

$$\hat{\lambda}(A_l) = \frac{n(A_l)}{|A_l|}$$



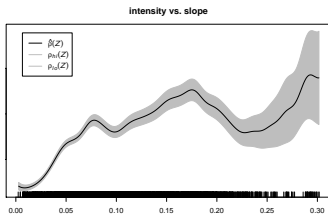
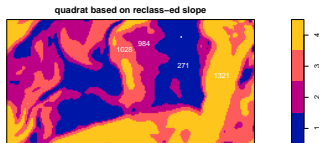
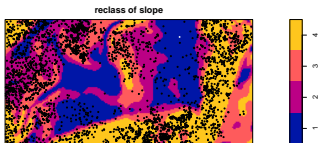
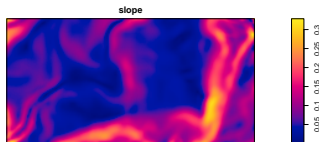
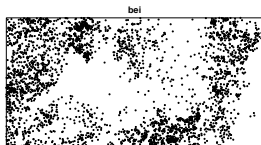
Quadrat methods



- estimated rates $\hat{\lambda}(A_i)$ over set of quadrats
- reveal large-scale patterns in intensity variation over A
- larger quadrats yield smoother intensity maps; smaller quadrats yield 'spiky' intensity maps
- size, origin, and shape of quadrats is critical (recall: *MAUP*)
- only first-order effects are captured



Dependence of intensity on a covariate (Inhomogeneous Poisson process)





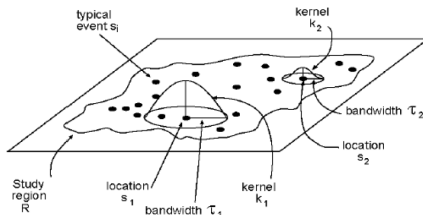
Kernel Density Estimation

Procedure of Kernel Density Estimation (KDE)

1. define a kernel $K(\mathbf{s}; r)$ of radius (or bandwidth) r centered at any arbitrary location \mathbf{s}
2. estimate local intensity at \mathbf{s} as:

$$\hat{\lambda}(\mathbf{s}) = \frac{1}{n} \sum_{i=1}^n K(\mathbf{s}_i - \mathbf{s}; r)$$

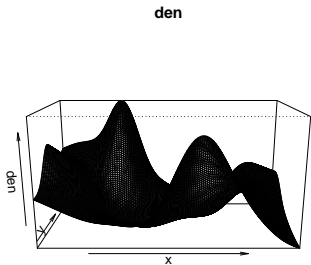
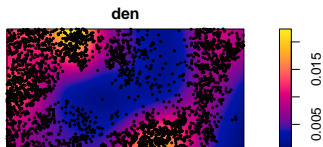
3. repeat estimation for all points \mathbf{s} in the study region to create a density map





Kernel Density Estimation

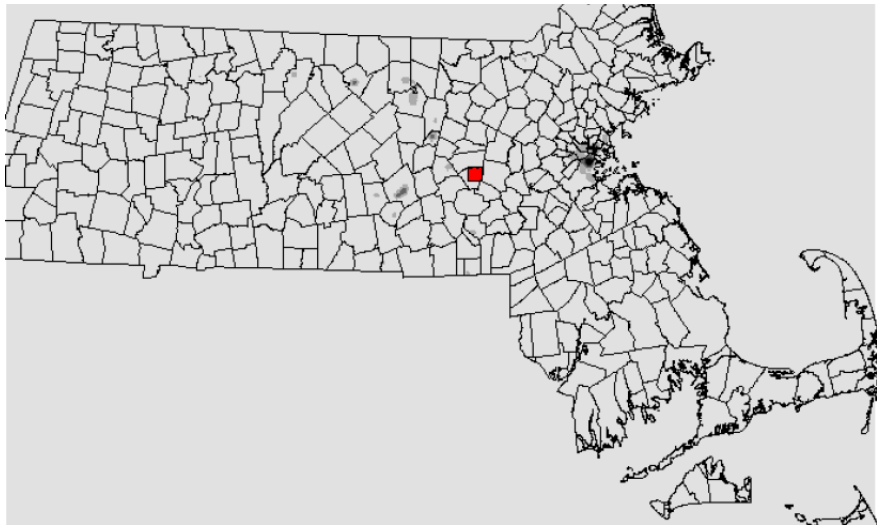
Example for the previous dataset:





Kernel Density Estimation

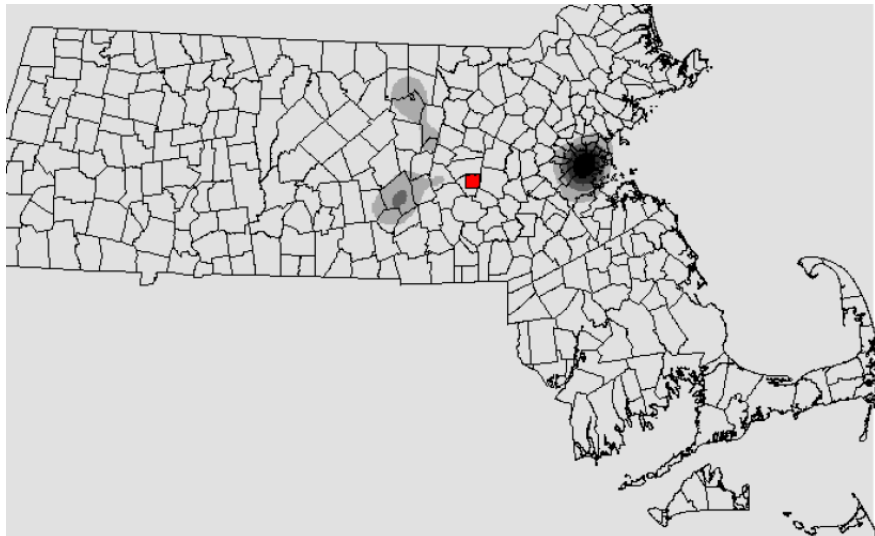
Example with 2km bandwidth





Kernel Density Estimation

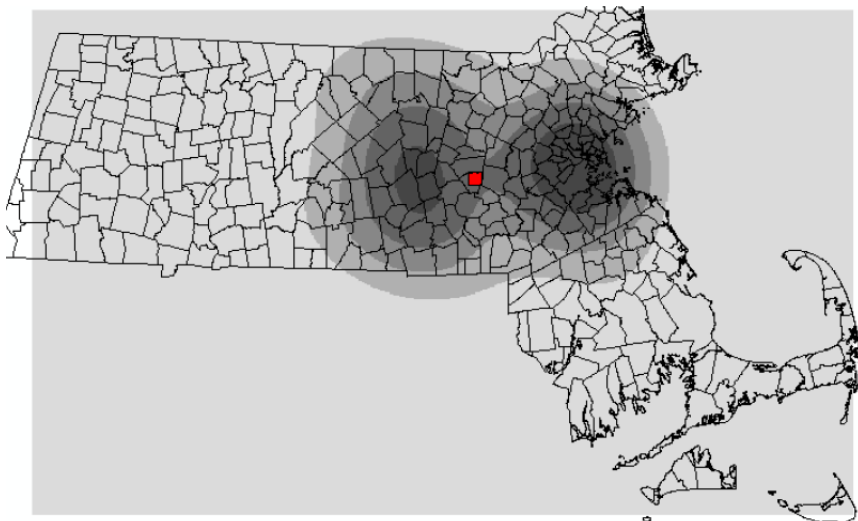
Example with 10km bandwidth





Kernel Density Estimation

Example with 40km bandwidth





Comments

- Choice of kernel function is not critical (Diggle, 1985)
- Choice of bandwidth, or degree of smoothing critical:
 - Small bandwidth \rightarrow spiky results
 - Large bandwidth \rightarrow loss of detail
- Multi-scale analyses can use these bandwidth characteristics to investigate both broad trends and localized variation
- How to choose bandwidth: choose the degree of smoothing subjectively, by eye, or by formula (Diggle)
- could define local bandwidth based on function of presence of events in neighborhood of **s** (i.e., adaptive kernel estimation)

What does the output of KDE means?



Distance-based Descriptors of Point Patterns

- Distances: accessing second order effects
 - Event-to-event distance: distance d_{ij} between event at arbitrary location s_i and another event at another arbitrary location s_j :

$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

- Point-to-event distance: distance \tilde{d}_{pj} between a randomly chosen point at location \tilde{s}_p and an event at location s_j :

$$\tilde{d}_{pj} = \sqrt{(\tilde{x}_p - x_j)^2 + (\tilde{y}_p - y_j)^2}$$

- Event-to-nearest-neighbour distance: distance $d_{min}(s_i)$ between an event at location s_i and its *nearest neighbor* event:

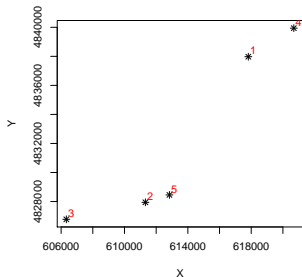
$$d_{min}(s_i) = \min\{d_{ij}, j \neq i, j = 1, \dots, n\}$$

- Point-to-nearest-neighbour distance (i.e., *empty space distance*): distance $\tilde{d}_{min}(\tilde{s}_p)$ between a randomly chosen point at location \tilde{s}_p and its *nearest neighbor* event:

$$\tilde{d}_{min}(\tilde{s}_p) = \min\{\tilde{d}_{pj}, j = 1, \dots, n\}$$



Event-to-Nearest-Neighbor Distances



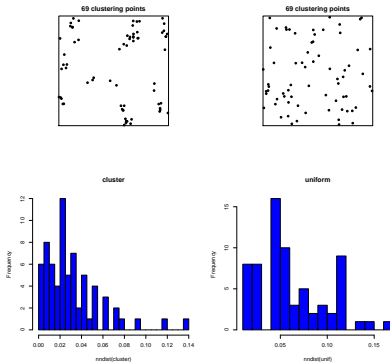
| | 1 | 2 | 3 | 4 | 5 |
|---|----------|----------|----------|----------|----------|
| 1 | 0.00 | 11947.70 | 16042.65 | 3481.22 | 10742.98 |
| 2 | 11947.70 | 0.00 | 5126.79 | 15219.58 | 1599.07 |
| 3 | 16042.65 | 5126.79 | 0.00 | 19481.59 | 6720.59 |
| 4 | 3481.22 | 15219.58 | 19481.59 | 0.00 | 13913.70 |
| 5 | 10742.98 | 1599.07 | 6720.59 | 13913.70 | 0.00 |

Table: Euclidean distance matrix



Event-to-Nearest-Neighbor Distances

Nearest neighbour distances



- Mean nearest neighbour distance: Average of all $d_{min}(s_i)$ values

$$\bar{d}_{min} = \frac{1}{n} \sum_{i=1}^n d_{min}(s_i)$$

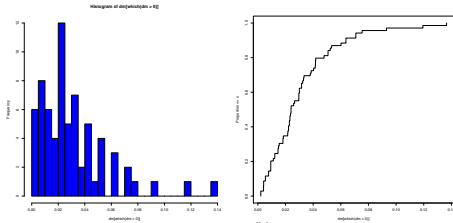


The G function

- Definition: nearest neighbour distance function, i.e., proportion of event-to-nearest-neighbor distances $d_{min}(s_i)$ no greater than given distance cutoff d , estimated as:

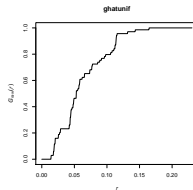
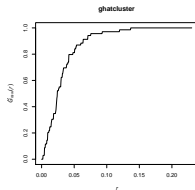
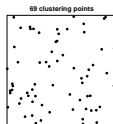
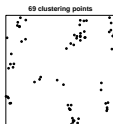
$$\hat{G}(d) = \frac{\#\{d_{min}(s_i) < d, i = 1, \dots, n\}}{n}$$

- alternative definition: cumulative distribution function (CDF) of all n event-to-nearest-neighbor distances; instead of computing average \bar{d}_{min} of d_{min} values, compute their CDF
- the G function provides information on event *proximity*
- example for previous clustering point pattern:





Examples of G function



Expected plot:

- for clustered events, $\hat{G}(d)$ rises sharply at short distances, and then levels off at larger d -values
- for randomly-spaced events, $\hat{G}(d)$ rises gradually up to the distance at which most events are spaced, and then increases sharply



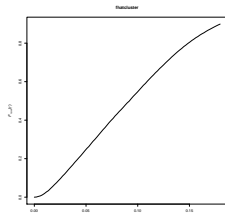
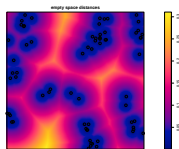
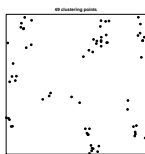
The F function

Definition

- proportion of point-to-nearest-neighbor distances (i.e., *empty space distances*) $\tilde{d}_{min}(s_p)$ no greater than given distance cutoff d , estimated as:

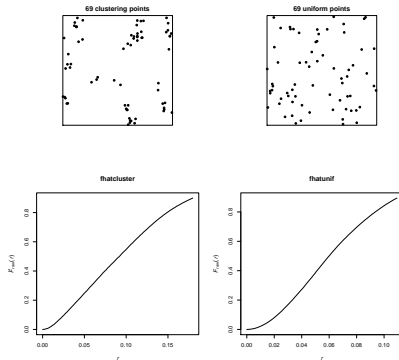
$$\hat{F}(d) = \frac{\#\{\tilde{d}_{min}(\tilde{s}_p) < d, p = 1, \dots, m\}}{m}$$

- alternative definition: cumulative distribution function (CDF) of all m point-to-nearest-neighbor distances
- the F function provides information on event proximity to voids
- Examples for previous clustering point pattern:





Examples of F function



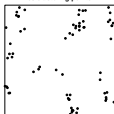
Expected plot:

- for clustered events, $\hat{F}(d)$ rises sharply at short distances, and then levels off at larger d -values
- for randomly-spaced events, $\hat{F}(d)$ rises rapidly up to the distance at which most events are spaced, and then levels off (there are more nearest neighbors at small distances from randomly placed points)

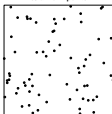


Comparing G and F functions

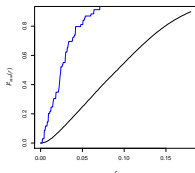
69 clustering points



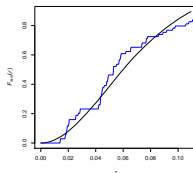
69 uniform points



Ghat vs. Fhat



Ghat vs. Fhat



Expected plot:

- for clustered events, $\hat{G}(d)$ rises faster
- for randomly-spaced events, $\hat{F}(d)$ tends to be close to $\hat{G}(d)$



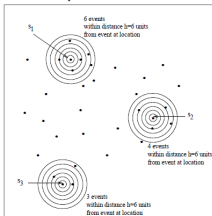
The K function

Working with pair-wise distances & looking beyond nearest neighbours

Concept

1. construct set of concentric circles (of increasing radius d) around each event
2. count number of events in each distance “band”
3. cumulative number of events up to radius d around all events becomes the sample K function $\hat{K}(d)$

Example of K function estimation





The K function

Working with pair-wise distances&looking beyond nearest neighbours

- Formal definition:

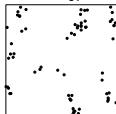
$$\begin{aligned} K(d) &= \frac{1}{\lambda} \frac{\#\{d_{ij} \leq d, i, j = 1, \dots, n\}}{n} \\ &= \frac{|A|}{n} \frac{\#\{d_{ij} \leq d, i, j = 1, \dots, n\}}{n} \\ &= |A|(\text{proportion of event-to-event distance} \leq d) \end{aligned}$$

- In other words, the $\hat{K}(d)$ is the sample cumulative distribution function (CDF) of all n^2 event-to-event distances, scaled by $|A|$

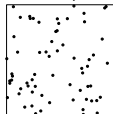


Examples of Event-to-Event Distance Histogram and CD

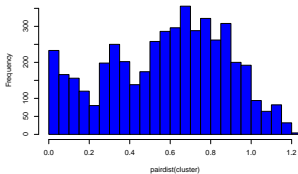
69 clustering points



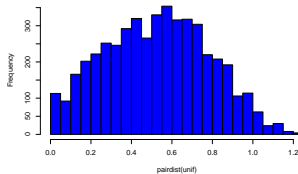
69 uniform points



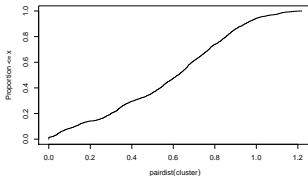
cluster histogram



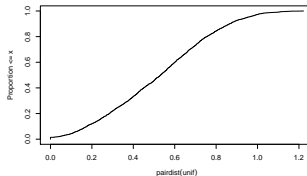
uniform histogram



cluster CDF



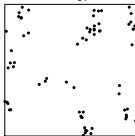
uniform CDF



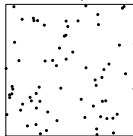


Examples of K functions

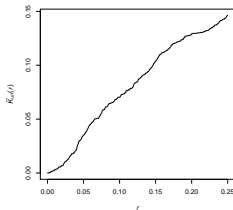
69 clustering points



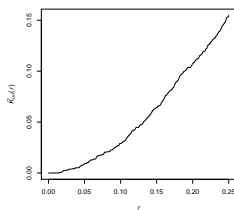
69 uniform points



Khatcluster



Khatunif



- the sample K function $\hat{K}(d)$ is monotonically increasing and is a scaled (by area $|A|$) version of the CDF of E2E distances



Spatial point patterns

- set of n point locations with recorded “events”

Describing the first-order effect

- overall intensity
- local intensity (quadrat count and kernel density estimation)

Describing the second-order effect

- nearest neighbour distances
 - the G function
- empty space distances
 - the F function
- pair-wise distances
 - the K function

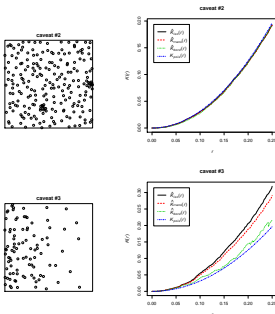


Caveats

Caveats:

- theoretical G, F, K functions are defined and estimated under the *assumption that the point process is stationary (homogeneous)*
- these summary functions *do not completely characterise the process*
- if the process is not stationary, deviations between the empirical and theoretical functions (e.g. \hat{K} and K) are not necessarily evidence of interpoint interaction, since they may also be attributable to variations in intensity

Example





Descriptive vs Statistical Point Pattern Analysis

Descriptive analysis:

- set of quantitative (and graphical) tools for characterizing spatial point patterns
- different tools are appropriate for investigating first- or second-order effects (e.g., kernel density estimation versus sample G function)
- can shed light onto whether points are clustered or evenly distributed in space

Limitation:

- no assessment of how clustered or how evenly-spaced is an observed point pattern
- no yardstick against which to compare observed values (or graph) of results

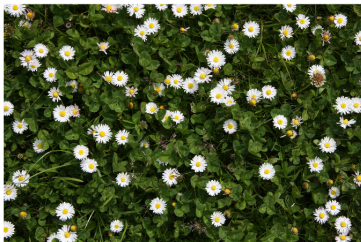


Descriptive vs Statistical Point Pattern Analysis

Statistical analysis:

- assessment of whether an observed point pattern can be regarded as one (out of many) realizations from a particular spatial process
- measures of confidence with which the above assessment can be made (how likely is that the observed pattern is a realization of a particular spatial process)

Are daisies randomly distributed in your garden?



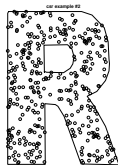
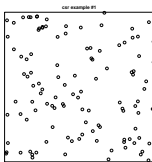


Complete Spatial Randomness (CSR)

Complete Spatial Randomness (CSR)

- yardstick, reference model that observed point patterns could be compared with, i.e., null hypothesis
- = *homogeneous (uniform) Poisson point process*
- basic properties:
 - the number of points falling in any region A has a Poisson distribution with mean $\lambda|A|$
 - given that there are n points inside region A , the locations of these points are i.i.d. and uniformly distributed inside A
 - the contents of two disjoint regions A and B are independent

Example:





Quadrat counting test for CSR

Quadrat counting test

- partition study area A into L sub-regions (quadrats), A_1, \dots, A_L
- count number of events $n(A_I)$ in each sub-region A_I
- Under the null hypothesis of CSR, the $n(A_I)$ are i.i.d. Poisson random variables with the same expected value
- The Pearson χ^2 goodness-of-fit test can be used
 - test statistics: Pearson residual $\sum_I \epsilon(A_I)^2$

$$\epsilon(A_I) = \frac{n(A_I) - \mu(A_I)}{\sqrt{\mu(A_I)}},$$

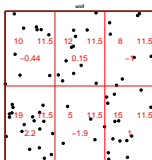
where $\mu(A_I)$ indicates the expected number of events in A_I

- $\sum_{I=1} \epsilon(A_I)^2$ is assumed to follow χ^2 distribution



Quadrat counting test for CSR

Example



- three values indicate the number of observations, CSR-expected number of observations, and the Pearson residuals
- p -value = 0.617



Nearest Neighbour Index (NNI) test under CSR

Nearest neighbour index

- Compares the mean of the distance observed between each point and its nearest neighbor (\bar{d}_{min}) and the expected mean distance under CSR $E(d_{min})$

$$NNI = \frac{\bar{d}_{min}}{E(d_{min})}$$

- Under CSR, we have:

$$E(d_{min}) = \frac{1}{2\sqrt{\lambda}}$$

$$\sigma(d_{min}) = \frac{0.26136}{\sqrt{n^2/A}}$$



Nearest neighbour index test

- Test statistics:

$$z = \frac{\bar{d}_{min} - E(d_{min})}{\sigma(d_{min})},$$

- z is assumed to follow Gaussian distribution, thus, if $z < -1.96$ or $z > 1.96$, we are 95% confident that the distribution is not randomly distributed

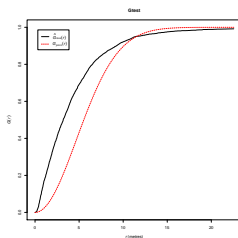
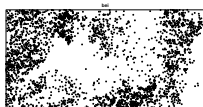


The G Function under CSR

- The G function is a function of nearest-neighbour distances
- For a homogeneous Poisson point process of intensity λ , the nearest-neighbour distance distribution (the G function) is known to be:

$$G(d) = 1 - \exp\{-\lambda\pi d^2\}$$

Example





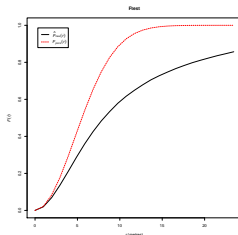
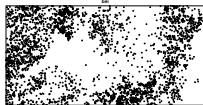
The F Function under CSR

- The F function is a function of empty space distances
- For a homogeneous Poisson point process of intensity λ , the empty space distance distribution (the F function) is known to be:

$$F(d) = 1 - \exp\{-\lambda\pi d^2\}$$

- Equivalent to the G function
- Intuitively, because points (events) of the Poisson process are independent of each other, the knowledge that a random point is a event of a point pattern does not affect any other event of the process

Example





The K Function under CSR

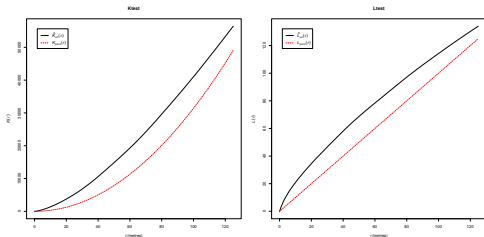
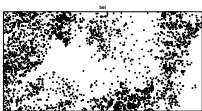
- The K function is a function of pair-wise distances
- For a homogeneous Poisson point process of intensity λ , the pair-wise distance distribution (the K function) is known to be:

$$K(d) = \pi d^2$$

- A commonly-used transformation of K is the L-function:

$$L(d) = \sqrt{\frac{K(d)}{\pi}} = d$$

Example

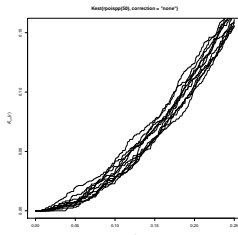




Monte Carlo test

- because of random variability, we will never obtain perfect agreement between sample functions (say the K function) with theoretical functions (the theoretical K functions), even with a completely random pattern

Example

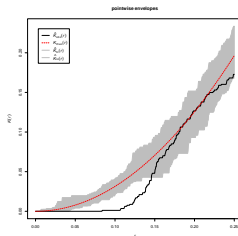




Monte Carlo test

- A *Monte Carlo* test is a test based on simulations from the null hypothesis
- Basic procedures:
 - generate M independent simulations of CSR inside the study region A
 - compute the estimated K functions for each of these realisations, say $\hat{K}^{(j)}(r)$ for $j = 1, \dots, M$
 - obtain the pointwise upper and lower envelopes of these simulated curves
 - not a confidence interval

Example





Recap

Statistical analysis of spatial point patterns:

- allows to quantify departure of results obtained via exploratory tools, e.g., $\hat{G}(d)$, from expected such results derived under specific null hypotheses, here CSR hypothesis
- can be used to assess to what extent observed point patterns can be regarded as realizations from a particular spatial process (here CSR)
- Same concepts can be applied for hypothesis of other types of point processes (e.g., Poisson cluster process, Cox process)

Sampling distribution of a test statistics

- lies at the heart of any statistical hypothesis testing procedure, and is tied to a particular null hypothesis
- simulation and analytical derivations are two alternative ways of computing such sampling distributions (the latter being increasingly replaced by the former)

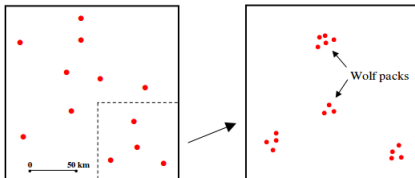
Edge Effects



Recap

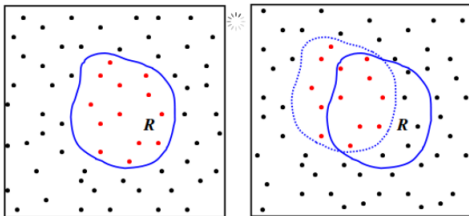
Scale effects

- Wolf pack example



- Nearest neighbour distance (NN distance, G,F functions) vs K function

Edge effects





Recap

Extended into line processes

- Line density

