

*GIST 4302/5302: Spatial Analysis and Modeling*  
*Point Pattern Analysis*

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Fall 2019



## Characteristics:

- set of  $n$  point locations with recorded “events”, e.g., locations of trees, disease or crime incidents  $S = \{s_1, \dots, s_i, \dots, s_n\}$
- point locations correspond to all possible events or to subsets of them
- attribute values also possible at same locations, e.g., tree diameter, magnitude of earthquakes (*marked point pattern*)  
 $W = \{w_1, \dots, w_i, \dots, w_n\}$

## Analysis objectives:

- detect spatial clustering or repulsion, as opposed to complete randomness, of event locations (in space and time)
- if clustering detected, investigate possible relations with nearby “sources”



## Further issues:

- analysis of point patterns over large areas should take into account distance distortions due to map projections
- boundaries of study area should not be arbitrary
- analysis of sampled point patterns can be misleading
- one-to-one correspondence between objects in study area and events in pattern



## Mean center of a point pattern:

- point with coordinates  $\bar{s} = (\bar{x}, \bar{y})$ :

$$\bar{x} = \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i} \quad \text{and} \quad \bar{y} = \frac{\sum_{i=1}^n w_i y_i}{\sum_{i=1}^n w_i}$$

- center of point pattern, or point with average  $x$  and  $y$ -coordinates

## Median center of a point pattern:

- both of the following two centers are called median centers, although they are essentially different (confusing!)
  - the intersection between the median of the  $x$  and the  $y$  coordinates
  - center for minimum distance*:  $s_c \in \{s_1, \dots, s_n\} \text{ s.t. } \min \sum_{i=1}^n |s_i - s_c|$
- the first type of *median center* is not unique, and there is no closed form for the second type
- p*-median problem (a typical problem in spatial optimization ): the problem of locating *p* "facilities" relative to a set of "customers" such that the sum of the shortest demand weighted distance between "customers" and "facilities" is minimized

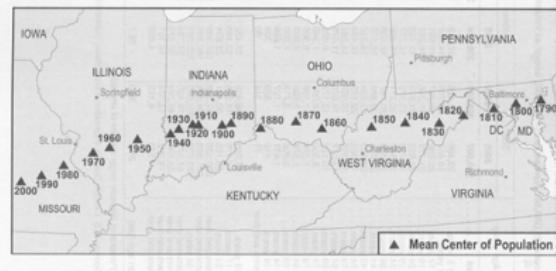


# Simple Descriptive Statistics

## Changes of population center (year 1790-2000):

Year	Median center		Mean center		Approximate location
	Latitude-N	Longitude	Latitude-N	Longitude-W	
1790 (August 2)	(NA)	(NA)	39° 16' 30"	76° 11' 12"	In Kent County, MD, 23 miles E of Baltimore MD
1800 (March 1)	NA	NA	38° 58' 00"	76° 50' 00"	In Kent County, MD, 23 miles E of Baltimore MD
1850 (March 1)	40° 03' 32"	84° 49' 01"	39° 09' 36"	85° 48' 54"	In Bartholomew County, IN, 6 miles S of Columbus, IN
1890 (June 1)	40° 00' 12"	84° 56' 51"	38° 50' 21"	85° 49' 33"	In Richland County, IL, 8 miles NNW of Olney, IL
1950 (April 1)	39° 56' 00"	85° 56' 60"	38° 58' 59"	89° 12' 35"	In Clinton County, IL, 6.5 miles NW of Mattoon, IL
1970 (April 1)	39° 43' 43"	85° 31' 20"	39° 27' 47"	89° 01' 53"	In St. Clair County, IL, 5.5 miles SW of Marion, IL
1980 (April 1)	39° 18' 60"	86° 08' 15"	38° 08' 13"	90° 34' 26"	In Jefferson County, MO, 25 miles W of DeSoto, MO
1990 (April 1)	38° 57' 55"	86° 31' 53"	37° 52' 20"	91° 12' 55"	In Crawford County, MO, 10 miles SE of Steeleville, MO
2000 (April 1)	38° 45' 20"	86° 55' 57"	37° 47' 49"	91° 48' 54"	In Phelps County, MO, 5 miles E of Edgar Springs, MO

NA Not available. \*West Virginia was set off from Virginia, Dec. 31, 1862, and admitted as a state, June 19, 1863.





## Standard distance of a point pattern:

- average squared deviations of  $x$  and  $y$  coordinates from their respective mean:

$$d_{std} = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{i=1}^n (y_i - \bar{y})^2}{n - 2}}$$

- related to standard deviation of coordinates, a summary circle (centered at  $\bar{s}$  with radius  $d_{std}$ ) of a point pattern

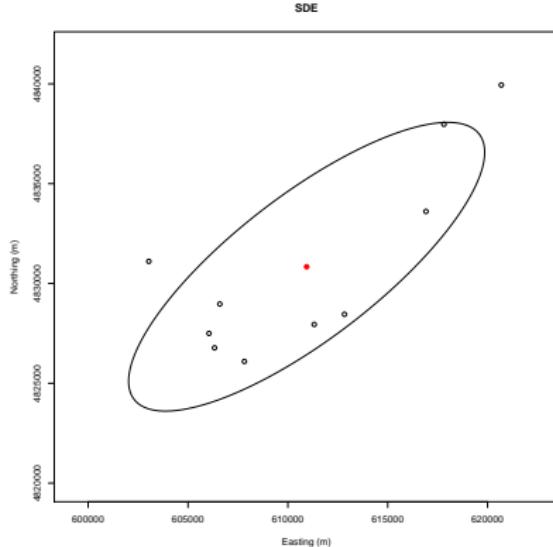
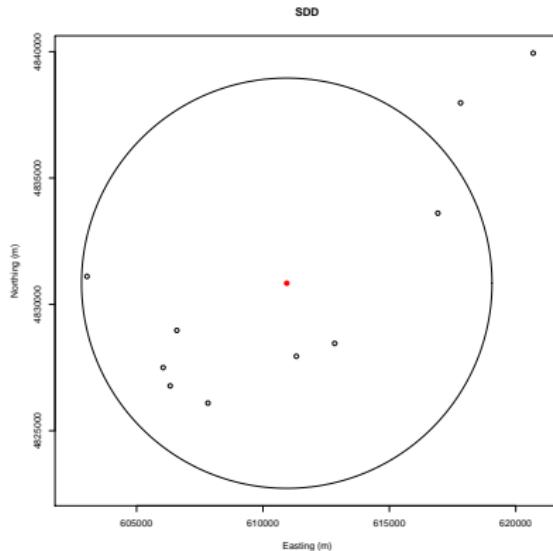
## Standard deviational ellipse:

- Taking directional effects into account for anisotropy cases
- Please refer to Levine and Associates, 2004 for calculations



# Descriptive Statistics

## Examples:



## Remarks:

- indicates overall shape and center of point pattern
- do not suffice to fully specify a spatial point pattern



1st order (i.e., intensity): absolute location of events on map:

- Quadrat methods
- Density Estimation (KDE)
- Moran's I and Geary's C

2nd order (i.e., interactions): interaction of events:

- Nearest neighbor distance
- Distance functions G, K, F, L
- Getis-Ord Gi\* and Anselin local Moran's I



## Quadrat methods

Consider a point pattern with  $n$  events within a study region  $A$  of area  $|A|$

Global intensity:

$$\hat{\lambda} = \frac{n}{|A|} = \frac{\text{#of events within } A}{|A|}$$

Local intensity via quadrats

1. partition  $A$  into  $L$  sub-regions  $A_I, I = 1, \dots, L$  of equal area  $|A_I|$  (also called quadrats)
2. count number of events  $n(A_I)$  in each sub-region  $A_I$
3. convert sample counts into estimated intensity rates as:

$$\hat{\lambda}(A_I) = \frac{n(A_I)}{|A_I|}$$



# Quadrat methods

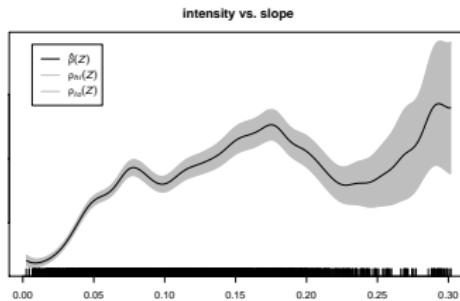
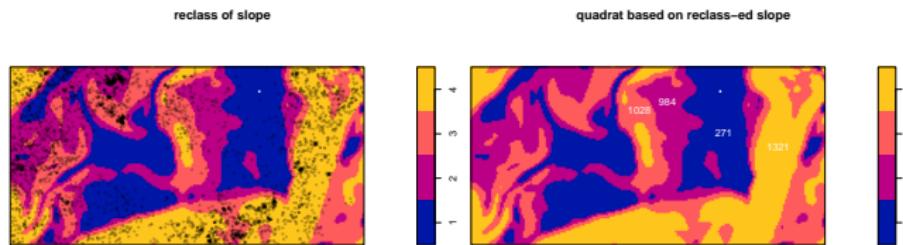
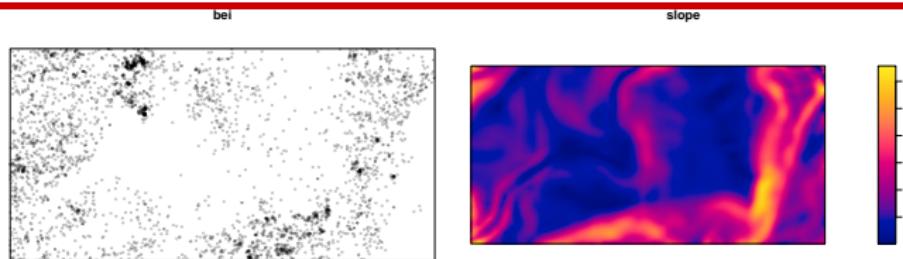
bei

337	608	162	73	105	268
422	49	17	52	128	146
231	134	92	406	310	64

- estimated rates  $\hat{\lambda}(A_I)$  over set of quadrats
- reveal large-scale patterns in intensity variation over  $A$
- larger quadrats yield smoother intensity maps; smaller quadrats yield ‘spiky’ intensity maps
- size, origin, and shape of quadrats is critical (recall: *MAUP*)
- only first-order effects are captured



# Dependence of intensity on a covariate (Inhomogeneous)





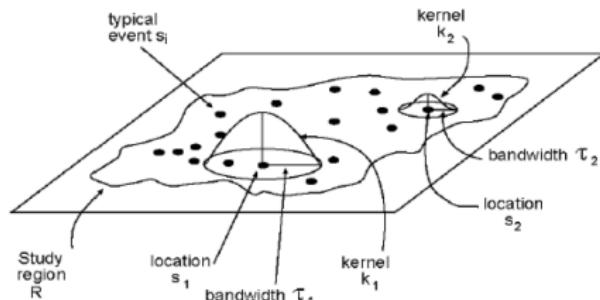
# Kernel Density Estimation

## Procedure of Kernel Density Estimation (KDE)

1. define a kernel  $K(s; r)$  of radius (or bandwidth)  $r$  centered at any arbitrary location  $s$
2. estimate local intensity at  $s$  as:

$$\hat{\lambda}(s) = \frac{1}{n} \sum_{i=1}^n K(s_i - s; r)$$

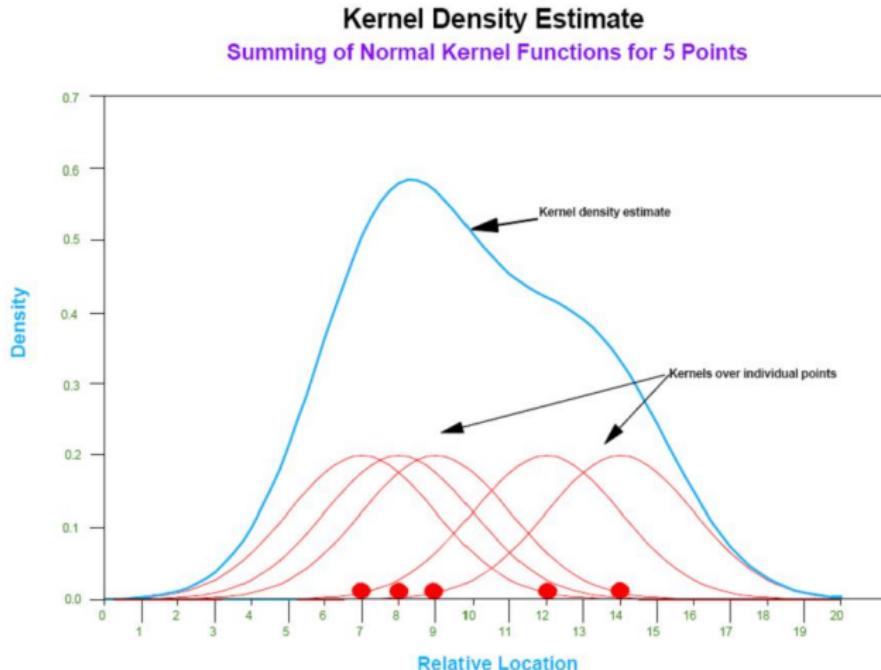
3. repeat estimation for all points  $s$  in the study region to create a density map





# Kernel Density Estimation

An illustration of the KDE procedure in 1D

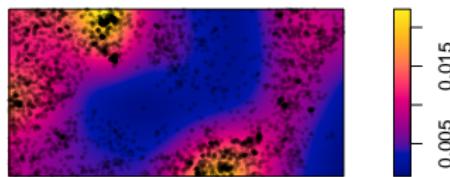




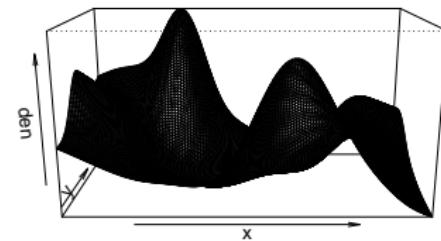
# Kernel Density Estimation

Example for the previous dataset:

den



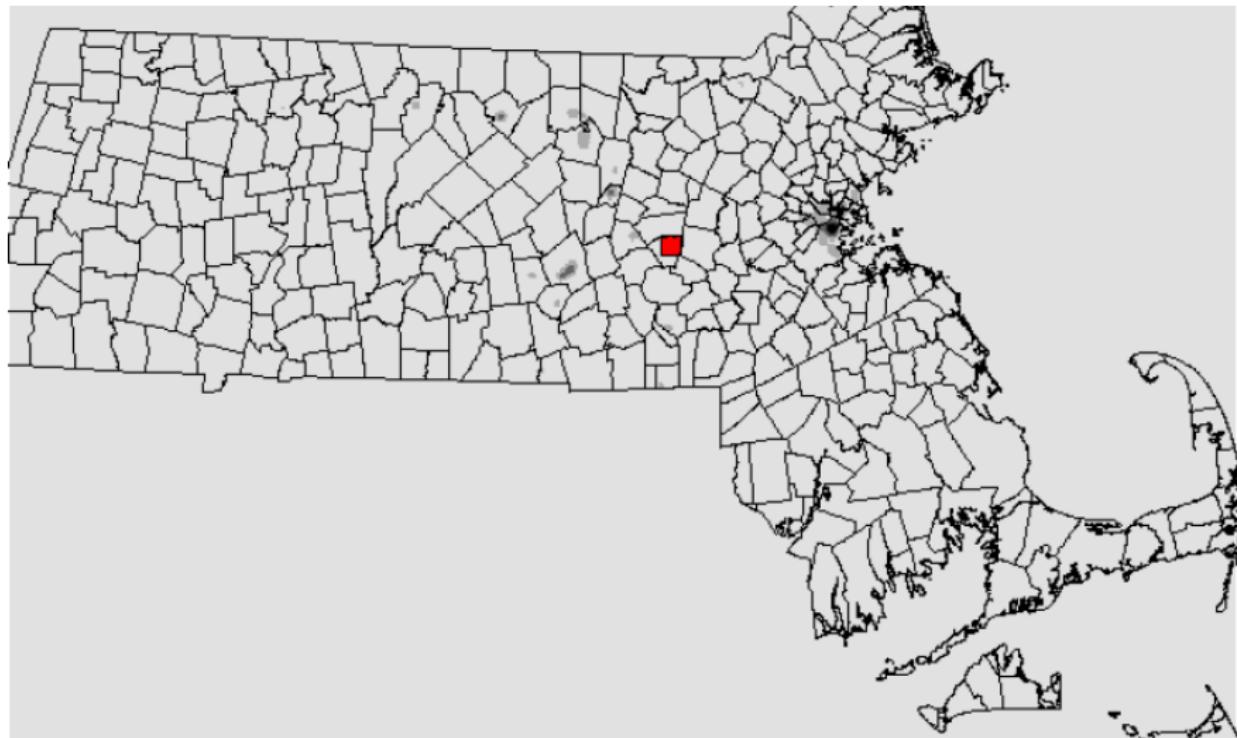
den





# Kernel Density Estimation

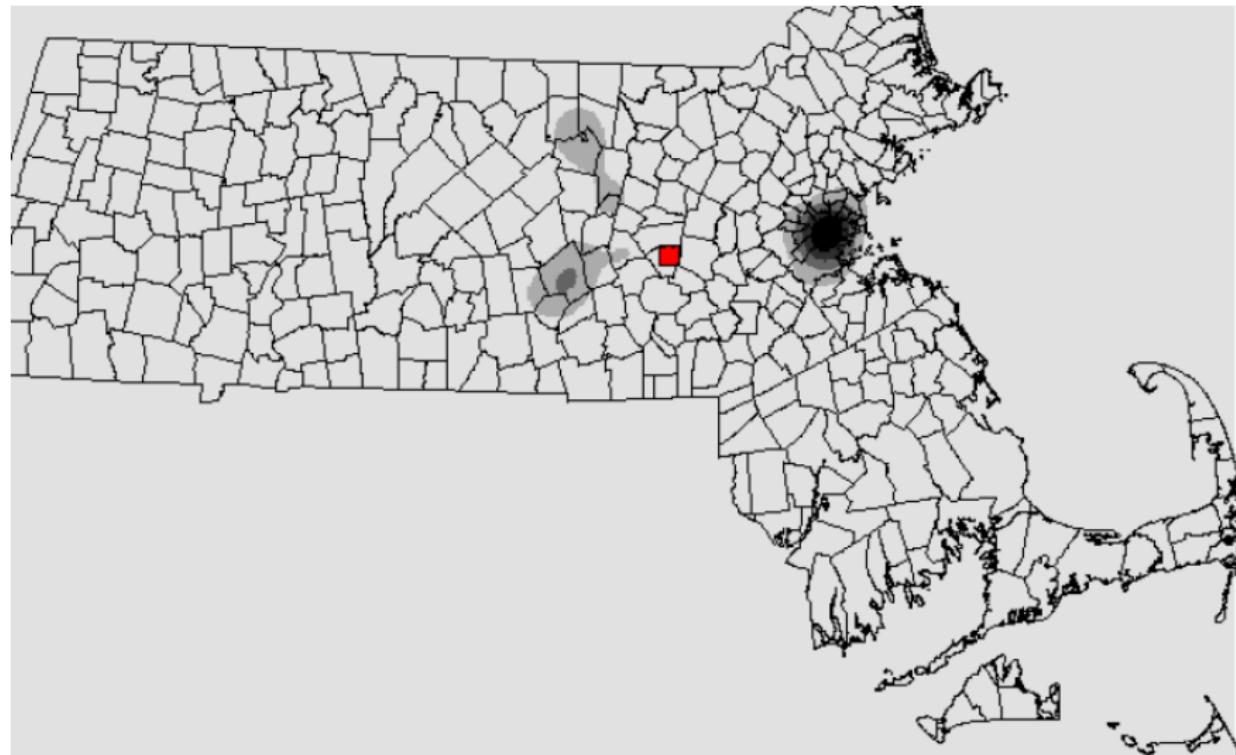
Example with 2km bandwidth





# Kernel Density Estimation

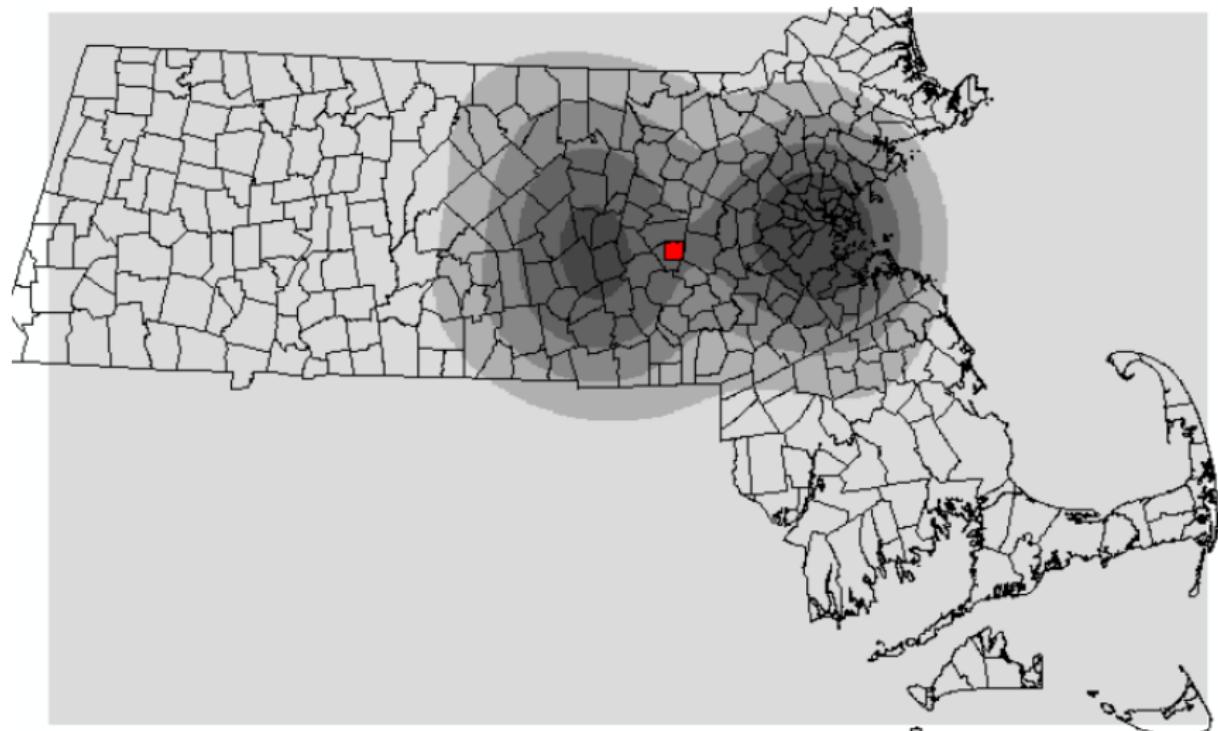
Example with 10km bandwidth





# Kernel Density Estimation

Example with 40km bandwidth





## Comments

- Choice of kernel function is not critical (Diggle, 1985)
- Choice of bandwidth, or degree of smoothing critical:
  - Small bandwidth → spiky results
  - Large bandwidth → loss of detail
- Multi-scale analyses can use these bandwidth characteristics to investigate both broad trends and localized variation
- How to choose bandwidth: choose the degree of smoothing subjectively, by eye, or by formula (Diggle)
- could define local bandwidth based on function of presence of events in neighborhood of  $s$  (i.e., adaptive kernel estimation)

What does the output of KDE means?



# Distance-based Descriptors of Point Patterns

- Distances: assessing second order effects
  - Event-to-event distance: distance  $d_{ij}$  between event at arbitrary location  $s_i$  and another event at another arbitrary location  $s_j$ :

$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

- Point-to-event distance: distance  $\tilde{d}_{pj}$  between a randomly chosen point at location  $\tilde{s}_p$  and an event at location  $s_j$ :

$$\tilde{d}_{pj} = \sqrt{(\tilde{x}_p - x_j)^2 + (\tilde{y}_p - y_j)^2}$$

- Event-to-nearest-neighbour distance: distance  $d_{min}(s_i)$  between an event at location  $s_i$  and its *nearest neighbor* event:

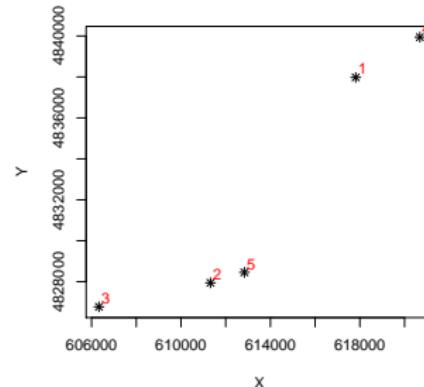
$$d_{min}(s_i) = \min\{d_{ij}, j \neq i, j = 1, \dots, n\}$$

- Point-to-nearest-neighbour distance (i.e., *empty space distance*): distance  $\tilde{d}_{min}(\tilde{s}_p)$  between a randomly chosen point at location  $\tilde{s}_p$  and its *nearest neighbor* event:

$$\tilde{d}_{min}(\tilde{s}_p) = \min\{\tilde{d}_{pj}, j = 1, \dots, n\}$$



# Event-to-Nearest-Neighbor Distances



	1	2	3	4	5
1	0.00	11947.70	16042.65	3481.22	10742.98
2	11947.70	0.00	5126.79	15219.58	1599.07
3	16042.65	5126.79	0.00	19481.59	6720.59
4	3481.22	15219.58	19481.59	0.00	13913.70
5	10742.98	1599.07	6720.59	13913.70	0.00

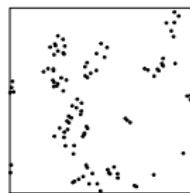
Table: Euclidean distance matrix



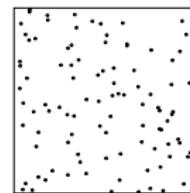
# Event-to-Nearest-Neighbor Distances

## Nearest neighbour distances

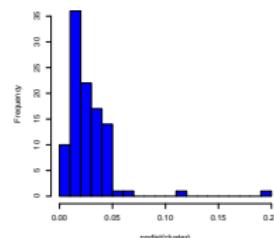
103 clustering points



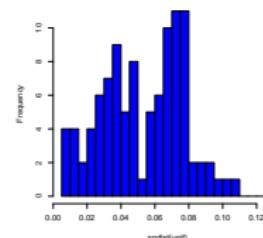
103 clustering points



cluster



uniform



- Mean nearest neighbour distance: Average of all  $d_{min}(s_i)$  values

$$\bar{d}_{min} = \frac{1}{n} \sum_{i=1}^n d_{min}(s_i)$$

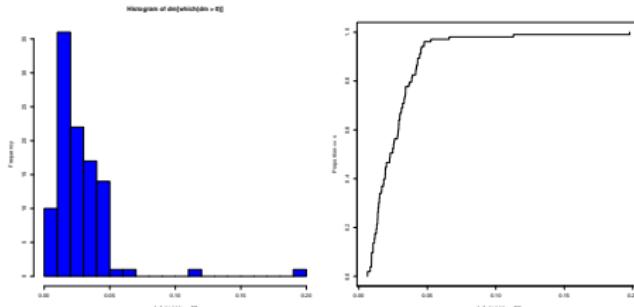


# The G function

- Definition: nearest neighbour distance function, i.e., proportion of event-to-nearest-neighbor distances  $d_{min}(s_i)$  no greater than given distance cutoff  $d$ , estimated as:

$$\hat{G}(d) = \frac{\#\{d_{min}(s_i) < d, i = 1, \dots, n\}}{n}$$

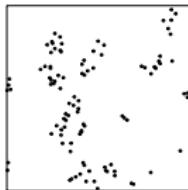
- alternative definition: cumulative distribution function (CDF) of all  $n$  event-to-nearest-neighbor distances; instead of computing average  $\bar{d}_{min}$  of  $d_{min}$  values, compute their CDF
- the G function provides information on event *proximity*
- example for previous clustering point pattern:



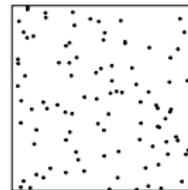


## Examples of $G$ function

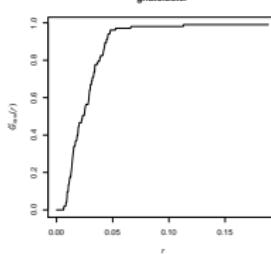
103 clustering points



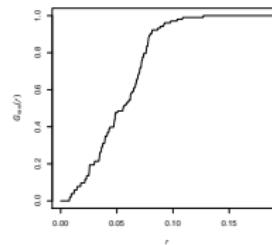
103 clustering points



ghatcluster



ghatunif



### Expected plot:

- for clustered events,  $\hat{G}(d)$  rises sharply at short distances, and then levels off at larger  $d$ -values
- for randomly-spaced events,  $\hat{G}(d)$  rises gradually up to the distance at which most events are spaced, and then increases sharply

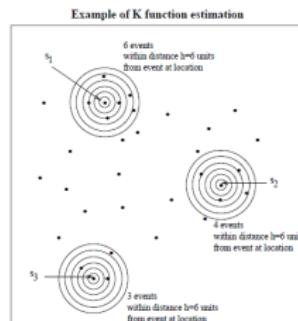


# The K function

Working with pair-wise distances & looking beyond nearest neighbours

## Concept

1. construct set of concentric circles (of increasing radius  $d$ ) around each event
2. count number of events in each distance “band”
3. cumulative number of events up to radius  $d$  around all events becomes the sample  $K$  function  $\hat{K}(d)$





## The K function

Working with pair-wise distances & looking beyond nearest neighbours

- Formal definition:

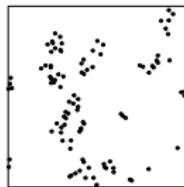
$$\begin{aligned}K(d) &= \frac{1}{\lambda} \frac{\#\{d_{ij} \leq d, i, j = 1, \dots, n\}}{n} \\&= \frac{|A|}{n} \frac{\#\{d_{ij} \leq d, i, j = 1, \dots, n\}}{n} \\&= |A|(\text{proportion of event-to-event distance} \leq d)\end{aligned}$$

- In other words, the  $\hat{K}(d)$  is the sample cumulative distribution function (CDF) of all  $n^2$  event-to-event distances, scaled by  $|A|$

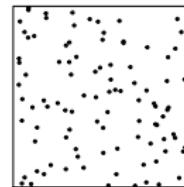


# Examples of Event-to-Event Distance Histogram and CD

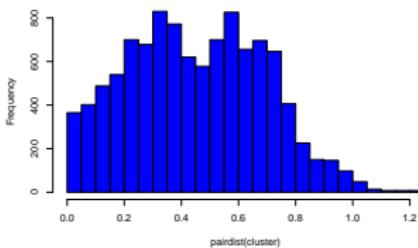
103 clustering points



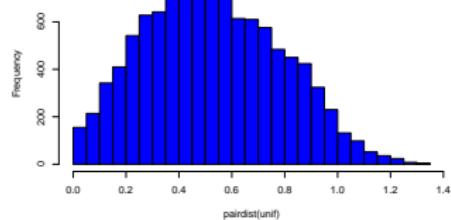
103 uniform points



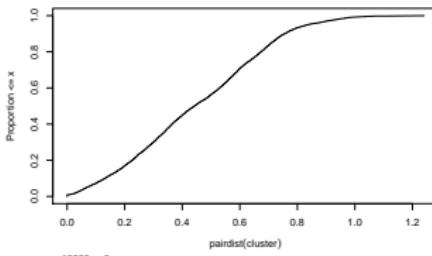
cluster histogram



uniform histogram

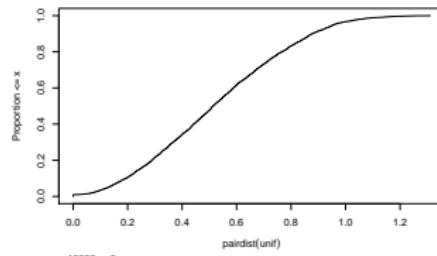


cluster CDF



n:10609 m:0

uniform CDF

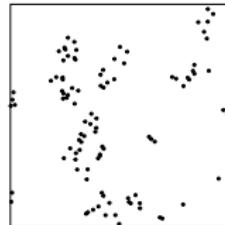


n:10609 m:0

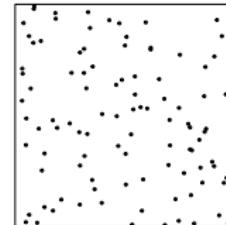


# Examples of $K$ functions

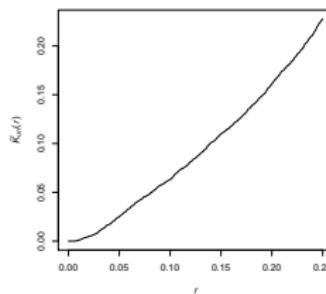
103 clustering points



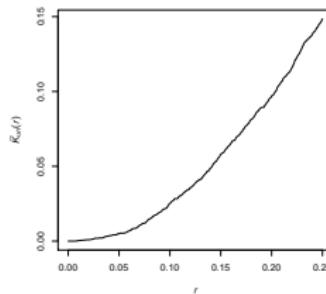
103 uniform points



Khatcluster



Khatunif



- the sample  $K$  function  $\hat{K}(d)$  is monotonically increasing and is a scaled (by area  $|A|$ ) version of the CDF of E2E distances



## Spatial point patterns

- set of  $n$  point locations with recorded “events”

### Describing the first-order effect

- overall intensity
- local intensity (quadrat count and kernel density estimation)

### Describing the second-order effect

- nearest neighbour distances
  - the G function
- pair-wise distances
  - the K function

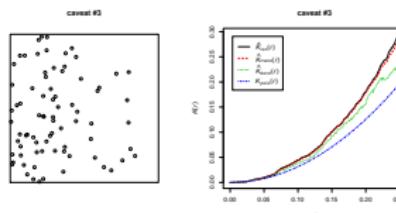
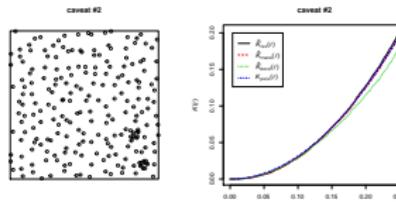


# Caveats

## Caveats:

- theoretical G, F, K functions are defined and estimated under the *assumption that the point process is stationary (homogeneous)*
- these summary functions *do not completely characterise the process*
- if the process is not stationary, deviations between the empirical and theoretical functions (e.g.  $\hat{K}$  and  $K$ ) are not necessarily evidence of interpoint interaction, since they may also be attributable to variations in intensity

## Example





## Descriptive analysis:

- set of quantitative (and graphical) tools for characterizing spatial point patterns
- different tools are appropriate for investigating first- or second-order effects (e.g., kernel density estimation versus sample G function)
- can shed light onto whether points are clustered or evenly distributed in space

## Limitation:

- no assessment of *how* clustered or *how* evenly-spaced is an observed point pattern
- no yardstick against which to compare observed values (or graph) of results



# *Descriptive vs Statistical Point Pattern Analysis*

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## Statistical analysis:

- assessment of whether an observed point pattern can be regarded as one (out of many) realizations from a particular spatial process
- measures of confidence with which the above assessment can be made (how likely is that the observed pattern is a realization of a particular spatial process)

Are daisies randomly distributed in your garden?



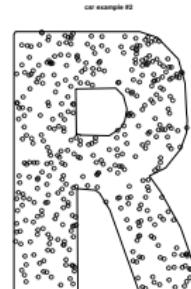
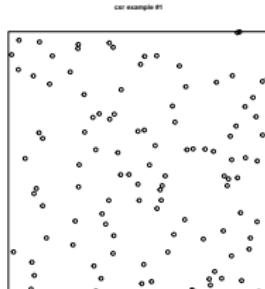


# Complete Spatial Randomness (CSR)

## Complete Spatial Randomness (CSR)

- yardstick, reference model that observed point patterns could be compared with, i.e., null hypothesis
- = *homogeneous (uniform) Poisson point process*
- basic properties:
  - the number of points falling in any region  $A$  has a Poisson distribution with mean  $\lambda|A|$
  - given that there are  $n$  points inside region  $A$ , the locations of these points are i.i.d. and uniformly distributed inside  $A$
  - the contents of two disjoint regions  $A$  and  $B$  are independent

Example:





## Quadrat counting test

- partition study area  $A$  into  $L$  sub-regions (quadrats),  $A_1, \dots, A_L$
- count number of events  $n(A_I)$  in each sub-region  $A_I$
- Under the null hypothesis of CSR, the  $n(A_I)$  are i.i.d. Poisson random variables with the same expected value
- The Pearson  $\chi^2$  goodness-of-fit test can be used
  - test statistics: Pearson residual  $\sum_I \epsilon(A_I)^2$

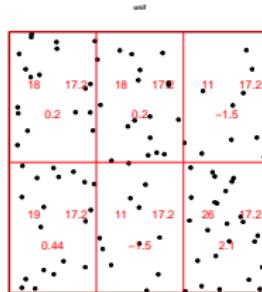
$$\epsilon(A_I) = \frac{n(A_I) - \mu(A_I)}{\sqrt{\mu(A_I)}},$$

- where  $\mu(A_I)$  indicates the expected number of events in  $A_I$
- $\sum_{I=1} \epsilon(A_I)^2$  is assumed to follow  $\chi^2$  distribution



# Quadrat counting test for CSR

## Example



- three values indicate the number of observations, CSR-expected number of observations, and the Pearson residuals
- $p$ -value = 0.617



## Nearest neighbour index

- Compares the mean of the distance observed between each point and its nearest neighbor ( $\bar{d}_{min}$ ) and the expected mean distance under CSR  $E(\bar{d}_{min})$

$$NNI = \frac{\bar{d}_{min}}{E(\bar{d}_{min})}$$

- Under CSR, we have:

$$E(\bar{d}_{min}) = \frac{1}{2\sqrt{\lambda}}$$

$$\sigma(\bar{d}_{min}) = \frac{0.26136}{\sqrt{n^2/A}}$$



## Nearest neighbour index test

- Test statistics:

$$z = \frac{\bar{d}_{min} - E(d_{min})}{\sigma(d_{min})},$$

- $z$  is assumed to follow Gaussian distribution, thus, if  $z < -1.96$  or  $z > 1.96$ , we are 95% confident that the distribution is not randomly distributed

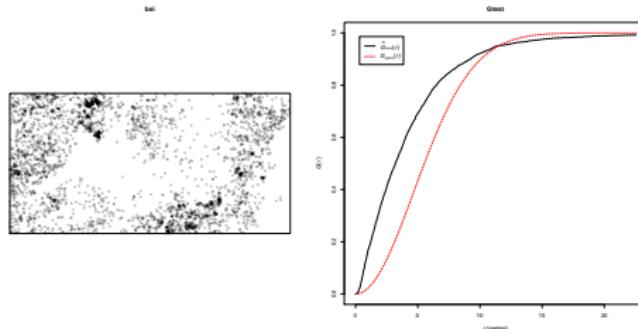


# The G Function under CSR

- The G function is a function of nearest-neighbour distances
- For a homogeneous Poisson point process of intensity  $\lambda$ , the nearest-neighbour distance distribution (the G function) is known to be:

$$G(d) = 1 - \exp\{-\lambda\pi d^2\}$$

## Example





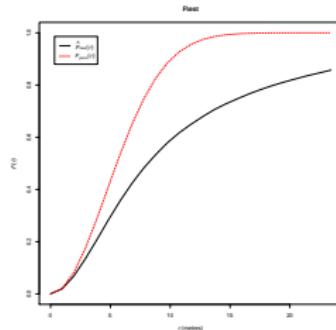
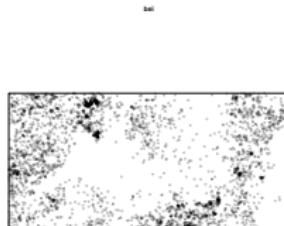
# The F Function under CSR

- The F function is a function of empty space distances
- For a homogeneous Poisson point process of intensity  $\lambda$ , the empty space distance distribution (the F function) is known to be:

$$F(d) = 1 - \exp\{-\lambda\pi d^2\}$$

- Equivalent to the G function
- Intuitively, because points (events) of the Poisson process are independent of each other, the knowledge that a random point is an event of a point pattern does not affect any other event of the process

## Example





# The K Function under CSR

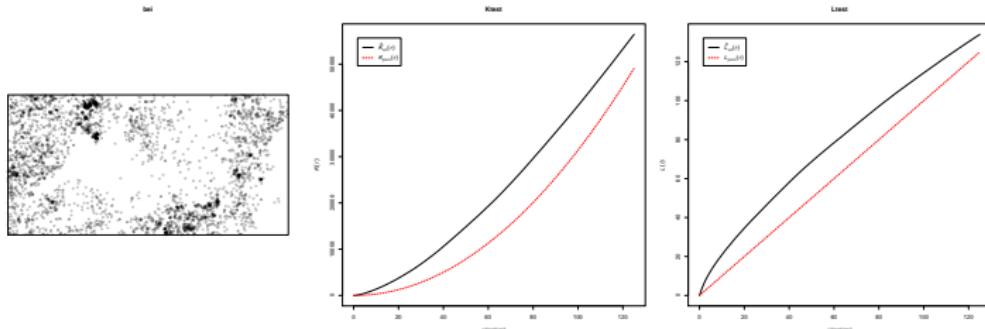
- The K function is a function of pair-wise distances
- For a homogeneous Poisson point process of intensity  $\lambda$ , the pair-wise distance distribution (the K function) is known to be:

$$K(d) = \pi d^2$$

- A commonly-used transformation of K is the L-function:

$$L(d) = \sqrt{\frac{K(d)}{\pi}} = d$$

## Example

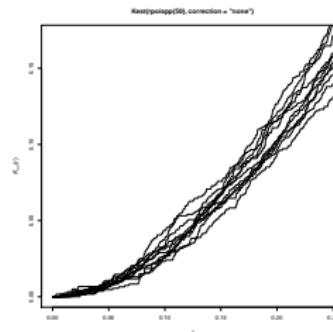




# Monte Carlo test

- because of random variability, we will never obtain perfect agreement between sample functions (say the K function) with theoretical functions (the theoretical K functions), even with a completely random pattern

## Example

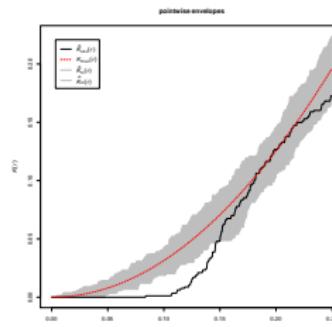




# Monte Carlo test

- A *Monte Carlo* test is a test based on simulations from the null hypothesis
- Basic procedures:
  - generate  $M$  independent simulations of CSR inside the study region  $A$
  - compute the estimated K functions for each of these realisations, say  $\hat{K}^{(j)}(r)$  for  $j = 1, \dots, M$
  - obtain the pointwise upper and lower envelopes of these simulated curves
  - not a confidence interval

## Example





## Recap

### Statistical analysis of spatial point patterns:

- allows to quantify departure of results obtained via exploratory tools, e.g.,  $\hat{G}(d)$ , from expected such results derived under specific null hypotheses, here CSR hypothesis
- can be used to assess to what extent observed point patterns can be regarded as realizations from a particular spatial process (here CSR)
- Same concepts can be applied for hypothesis of other types of point processes (e.g., Poisson cluster process, Cox process)

### Sampling distribution of a test statistics

- lies at the heart of any statistical hypothesis testing procedure, and is tied to a particular null hypothesis
- simulation and analytical derivations are two alternative ways of computing such sampling distributions (the latter being increasingly replaced by the former)

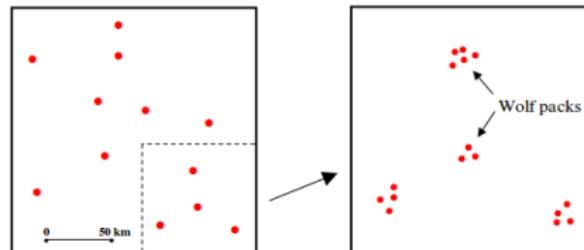
### Edge Effects



## Recap

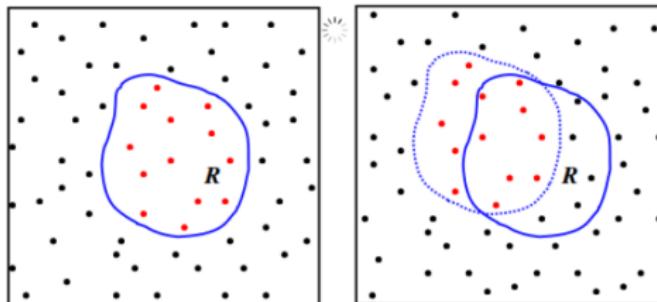
### Scale effects

- Wolf pack example



- Nearest neighbour distance (NN distance, G,F functions) vs K function

### Edge effects





# Recap

Extended into line processes

- Line density

