6

Linear Programming

After optimizing a function of single or multiple variables (considered for review in Chapter 9) , the least sophisticated optimization problems are concerned with either minimizing or maximizing a linear function which satisfy conditions modeled by linear inequalities. There are numerous practical problems which fall into this category and for these reasons we begin our study of Optimization here.

6.1 A Geometric Approach to Linear Programming in Two Dimensions

To illustrate the material, let us consider the following example:

6.1.1 Example

Example 6.1.1 (Lincoln Outdoors)**.** Lincoln Outdoors*, a camping merchandise manufacturer, makes two types of sleeping bags: the Cabin Model for light camping and the Frontier Model for more rugged use. Each Cabin sleeping bag requires 1 labor-hour from the cutting department and 2 labor hours from the assembly department whereas each Frontier model requires 2 labor-hour from the cutting department and 3 labor hours from the assembly department. The per day maximum amount of labor hours for the cutting department is 40 labor-hours where the assembly department has 72 labor-hours available per day. The company makes a profit of \$60 on each Cabin model it sells and a profit of \$90 on each Frontier model sold. Assuming that all sleeping bags that are manufactured will sell, how many bags of each type should* Lincoln Outdoors *manufacture per day in order to maximize the total daily profit?*

Solution. We may summarize the information as follows:

The first step will be to identify the ***decision variables*** for the model. In this situation, let us put

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TABLE 6.1
Manufacturing Data for Lincoln Outdoors in Example 6.1.1

Labor-Hours	Cabin Model	Frontier Model	Max Hours per Day
Cutting Dept.	1	2	40
Assembly Dept.	2	3	72
Profit per Bag	\$60	\$90	

x_1 = the number of Cabin Model sleeping bags manufactured per day and

x_2 = the number of Frontier Model sleeping bags manufactured per day.

We next form our ***objective function***, that is the function we wish to optimize. In this situation our objective function is

$$P(x_1, x_2) = 60x_1 + 90x_2. \tag{6.1}$$

According to this function, profit can be made arbitrarily large by letting x_1 or x_2 grow without bound. Unfortunately, the situation has restrictions due to the amount of available labor-hours. Thus we have the following ***problem constraints***:

$$\begin{aligned} &\text{Cutting Department Constraints:} && x_1 + 2x_2 \leq 40 \quad \text{and} && (6.2)\\ &\text{Assembly Department Constraints:} && 2x_1 + 3x_2 \leq 72. && (6.3)\end{aligned}$$

As well, we have the ***non-negativity constraints***[1] that

$$\begin{aligned} x_1 &\geq 0 \text{ and} && (6.4)\\ x_2 &\geq 0. && (6.5)\end{aligned}$$

These constraints are usually expressed with the single statement

$$x_1, x_2 \geq 0.$$

Thus the mathematical model for the problem we are considering is

$$\begin{aligned} \text{Maximize: } P(x_1, x_2) &= 60x_1 + 90x_2 && (6.6)\\ \text{Subject to: } \quad x_1 + 2x_2 &\leq 40 && (6.7)\\ 2x_1 + 3x_2 &\leq 72 && (6.8)\\ x_1, x_2 &\geq 0. && (6.9)\end{aligned}$$

The graph of this system of linear inequalities given by the constraints is known as the ***feasible region***.

[1]Certainly we also have the natural constraint that the number of sleeping bags be integer-valued, but this is a matter for Chapter 8.

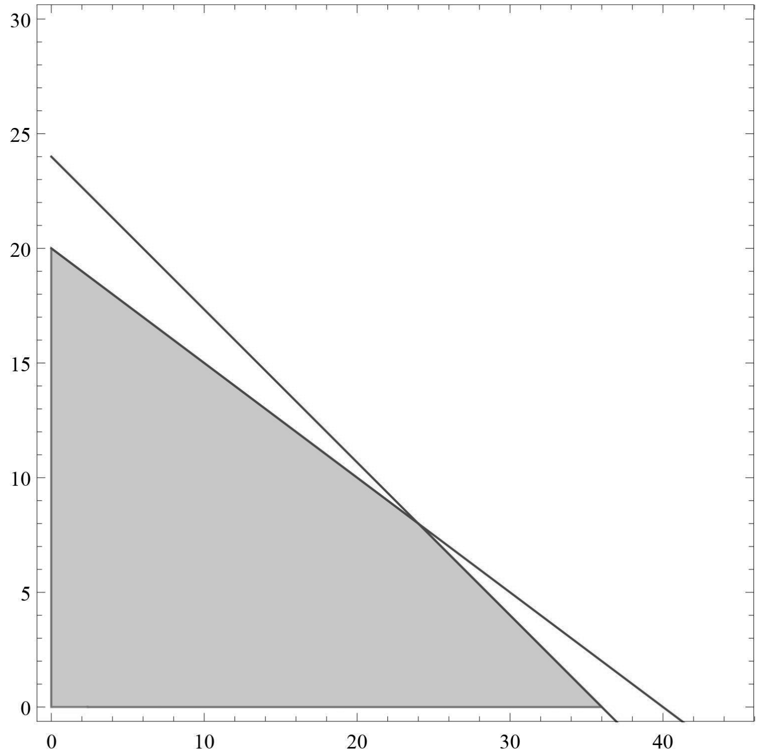

FIGURE 6.1
Feasible region for Lincoln Outdoors.

Now it is a wonderful thing that we are able to graph the feasible region and thus know the set of solutions to the system of linear inequalities, but which pair maximizes the profit function? This is a daunting task as there are infinitely many possible points (unless we only consider integer solutions; more on this later).

Our aim is to maximize $P(x_1, x_2) = 60x_1 + 90x_2$ so let us consider this function. If we fix a value for the profit, call it K, we then have a linear equation in two variables. In particular, if we solve for x_2 we have

$$x_2 = -\frac{2}{3}x_1 + \frac{K}{90}. \tag{6.10}$$

Notice that as K increases, this line moves further away from the origin (see Figure 6.2).

We wish to increase K as much as possible, but recall there are restrictions. Specifically, the line representing the profit must intersect the feasible region. So to maximize profit but also satisfy the constraints, we move the line as far away from the origin as possible but still have at least one point on the line in the feasible region. By this reasoning, we may conclude that an optimal solution to a linear programming problem occurs at a corner point (as mentioned in Section 4.1, a corner point of feasible regions is often called a *vertex* of the feasible region). ■

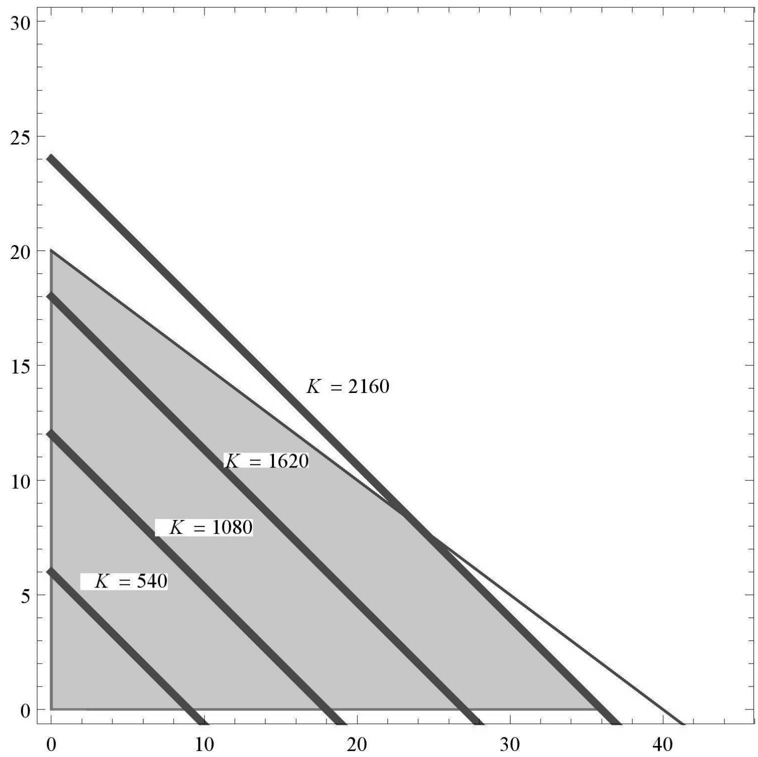

FIGURE 6.2
Graphs of the objective function for Lincoln Outdoors.

Though we have not formally proved[2] it, we have

Theorem 6.1.2 (The Fundamental Theorem of Linear Programming[3])**.** *If the optimal value of the objective function in a linear programming problem exists, then that value (known as the* optimal solution*) must occur at one or more of the corner points of the feasible region.*

Also from our exploration we have

Remark 6.1.3. *The possible classifications of the solutions to Linear Programming problems are:*

- *If the feasible region of a linear programming problem is bounded, then there exists a maximum and a minimum value for the objective function.*
- *If the feasible region of a linear programming problem is unbounded and if the coefficients of the objective function are positive[4], then there exists a minimum value for the objective function, but there does not exist a maximum value for this function. (An analogous exists for a feasible region not bounded below and the associated min and max... but these are seldom encountered in applications.)*

[2]Formal proofs of the theorems in this section will be offered in Chapter 17.
[3]The formal statement of this theorem is given in Theorem 17.2.2.
[4]Scenarios with negative coefficients are vary rare in applications.

- *If the feasible region is empty, then there does not exist a maximum or a minimum value for the objective function.*

Given the Fundamental Theorem of Linear Programming, we may now answer the question we considered in Example 6.1.1. We accomplish this by evaluating the objective function at each corner point of the feasible region and the work is shown in Table 6.2. We see that profit is maximized at \$2160 which occurs when either 24 Cabin Model and 8 Frontier Model bags are made or when 36 Cabin Model and no Frontier Model bags are produced. This leads to

TABLE 6.2
$P(x_1, x_2)$ Evaluated at Corner Points

Corner Point (x_1, x_2)	$P(x_1, x_2)$
$(0, 0)$	0
$(0, 20)$	1800
$(24, 8)$	2160
$(36, 0)$	2160

Remark 6.1.4. *Note that it is possible that the optimal solution occurs at more than one corner point. If this situation occurs, then* any point of the line segment joining the corner points is also an optimal solution.

The situation addressed in Remark 6.1.4 is what we have in the Lincoln Outdoors example and hence we conclude that the objective function $P = 60x_1 + 90x_2$ subjected to the given constraints is maximized at all points on the line segment connecting the points $(24, 8)$ and $(36, 0)$, that is the maximum of \$2160 occurs over $\{(t, -\frac{2}{3}t + 24) \mid 24 \leq t \leq 36\}$. This solution set is represented in Figure 6.2 where the objective function P intersects the constraint boundary $2x_1 + 3x_2 = 72$.

Of course, in this situation, we will be concerned with integer solutions and the multiple integer solutions for Lincoln Outdoors are $(24, 8)$, $(27, 6)$, $(30, 4)$, $(33, 2)$, and $(36, 0)$ which appear in Figure 6.3. In Chapter 8 we will explore how to find integer solutions and in Chapter 7 we will show how to use Excel to find other possible solutions. In the meantime, we can be aware that multiple solutions exist not only from the corner point analysis but also by the observation that the objective function $P(x_1, x_2) = 60x_1 + 90x_2$ is parallel to the assembly constraint $2x_1 + 3x_2 \leq 72$ (as seen in Figure 6.2).

6.1.2 Summary

We may summarize our techniques as follows:

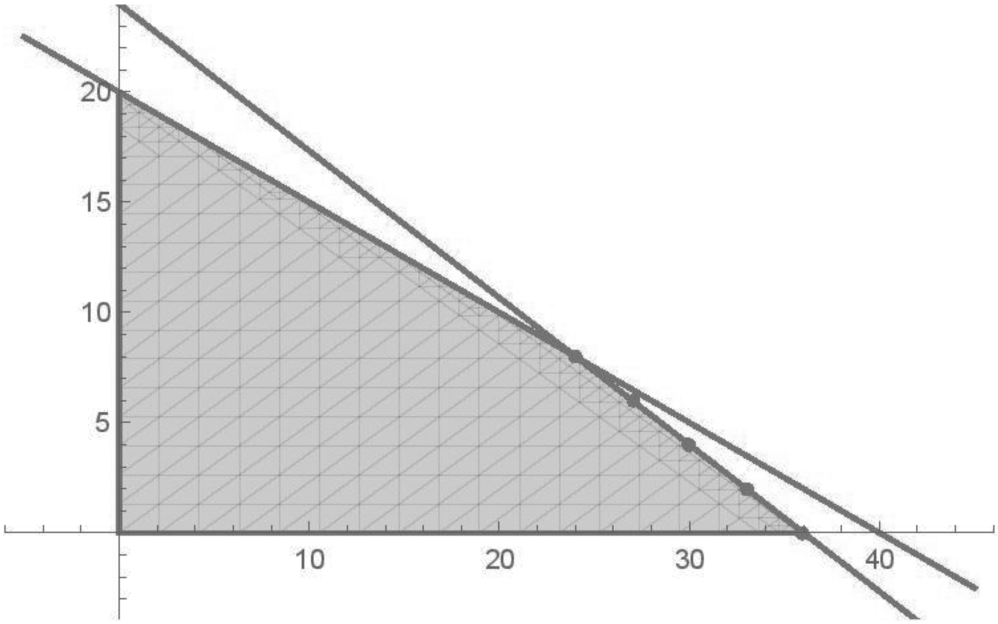

FIGURE 6.3
The multiple integer solutions for Lincoln Outdoors.

1. Summarize the data in table form (see Table 6.1).
2. Form a mathematical model for the problem by
   - introducing decision variables,
   - stating the objective function,
   - listing the problem constraints, and
   - writing the nonnegative constraints.
3. Graph the feasible region.
4. Make a table listing the value of the objective function at each corner point.
5. The optimal solutions will be the largest and smallest values of the points in this table.

6.1.3 Keywords

- *solution set*
- *bounded/unbounded*
- *corner point*
- *decision variables*
- *objective function*
- *problem constraints*
- *nonnegative constraints*
- *feasible region*
- *optimal solution*

6.2 The Simplex Method: Max LP Problems with Constraints of the Form $\leq$

6.2.1 Introduction

We have thus far looked at a geometric means of solving a linear programming problem. Our previous method works fine when we have two unknowns and are thus working with a feasible region that is a subset of the real plane (in other words, is a two-dimensional object). This will also work with three variables, but the graphs of the feasible regions are much more complicated (they would be three dimensional objects). Of course, things get terribly messy if we have four or more unknowns (we could not draw their graphs). As such we must develop another means of tackling linear programming problems.

The technique introduce in this section is due to George Bernard Dantzig (b. November 8, 1914, d. May 13, 2005). Dantzig developed it while on leave from his Ph.D. at Berkeley working for the Army Air Force during World War II. His algorithm was developed while serving and kept secret until it was published in 1951.

The approach begins with examining what we did in Section 4.2 when we considered a system of linear *equalities* of more than one variable. In particular, we were faced with solving the system

$$\begin{cases} 3x_1 - 5x_2 + x_3 &= 18 \\ -2x_1 + x_2 - x_3 &= 5 \\ x_1 - x_2 + 3x_3 &= -4 \end{cases}$$

We considered isolating a variable, substituting, and working to reduce the system to two equations in two unknowns, but that was too much work. Instead, we introduced an augmented matrix

$$\left[\begin{array}{rrr|r} 3 & -5 & 1 & 18 \\ -2 & 1 & -1 & 5 \\ 1 & -1 & 3 & -4 \end{array}\right]$$

and used Gauss-Jordan Elimination (row operations) to get the matrix in reduced form. It is this method that we adapt to solve linear programming problems.

6.2.2 Slack Variables

The careful reader may have observed that in linear programming problems the constraints are expressed as *inequalities* and not *equalities*. To modify the problem from one we do not know how to solve into one we do know how to solve, we introduce **slack variables**.

Recall that in Example 6.1.1 we had the problem constraints that

$$1x_1 + 2x_2 \leq 40, \tag{6.11}$$
$$2x_1 + 3x_2 \leq 72, \text{ and} \tag{6.12}$$
$$x_1, x_2 \geq 0. \tag{6.13}$$

We now introduce nonnegative ***slack variables*** s_1 and s_2 to "pick up the slack", i.e.

$$1x_1 + 2x_2 + s_1 = 40, \tag{6.14}$$
$$2x_1 + 3x_2 + s_2 = 72, \text{ and} \tag{6.15}$$
$$x_1, x_2, s_1, s_2 \geq 0. \tag{6.16}$$

Unfortunately, we are now in a situation where we have four unknowns and two equations. This means that we now have infinitely many solutions to the system (we do know something about these solutions, though; namely that we can fix two variables[5] and express the other two variables as a function of the fixed variables). We get around this little problem of an infinite number of solutions by introducing *basic* and *nonbasic* variables.

Definition 6.2.1 (Basic and Nonbasic Variables)**.**
Basic variables *are chosen arbitrarily but with the restriction that there are exactly the same number of basic variables as there are constraint equations (we will see in Highlight 6.2.2 that there is a clever way to choose which variables are basic). We then say that the remaining variables are* **nonbasic variables***.*

We may now present the idea of a basic solution. We **put the nonbasic variables equal to 0** and then the solution of the resulting system of linear equations is called a ***basic solution***. A basic solution is said to be a ***basic feasible solution*** if it lies within the feasible region of the problem (i.e. satisfies all constraints).

Let us revisit Example 6.1.1 with the inclusion of the slack variables. We then have the linear programming problem:

$$\text{Maximize:} \quad P(x_1, x_2) = 60x_1 + 90x_2 \tag{6.17}$$
$$\text{Subject to:} \quad x_1 + 2x_2 + s_1 \qquad = 40 \tag{6.18}$$
$$2x_1 + 3x_2 \qquad + s_2 = 72 \tag{6.19}$$
$$x_1, x_2, s_1, s_2 \geq 0. \tag{6.20}$$

With reference to our original linear programming problem, we may refer to this as the ***modified linear programming problem***.

[5] In this statement we are assuming the constraints are linearly independent, which is almost always the case in applications.

6.2.3 The Method

We begin by writing the model as an augmented matrix with the objective function as the last line:

$$\left[\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & P & \\ 1 & 2 & 1 & 0 & 0 & 40 \\ 2 & 3 & 0 & 1 & 0 & 72 \\ \hline -60 & -90 & 0 & 0 & 1 & 0 \end{array}\right] \tag{6.21}$$

where the last line refers to rewriting the objective function as $-60x_1 + -90x_2 + P = 0$.[6] Recall from Definition 6.2.1 that there are to be as many basic variables as there are equations. Hence in the example we are to have three basic variables which leaves two to be nonbasic variables. By definition these may be chosen arbitrarily, but some careful thought will make the technique more friendly to use. Note specifically the last three columns of 6.21; they are the vectors $[1,0,0]^T$, $[0,1,0]^T$, and $[0,0,1]^T$. Realizing this matrix has rank 3 we see that our choice of vectors will nicely serve as a basis for the column space of the matrix. For this reason we choose these vectors to be our three basic variables (and this is, in fact, why they are called basic). Hence the decision variables x_1 and x_2 are left to be nonbasic variables. By convention, these are set equal to 0 which means row 1 of our augmented matrix tells us $0+0+s_1+0+0 = 40$, i.e. $s_1 = 40$. Likewise the second row gives $s_2 = 72$ and the last row $P = 0$.

We now rewrite 6.21 to reflect this:

$$\left[\begin{array}{cccccc|c} (basic) & x_1 & x_2 & s_1 & s_2 & P & \\ (s_1) & 1 & 2 & 1 & 0 & 0 & 40 \\ (s_2) & 2 & 3 & 0 & 1 & 0 & 72 \\ \hline (P) & -60 & -90 & 0 & 0 & 1 & 0 \end{array}\right] \tag{6.22}$$

This form of the matrix representing our linear programming problem is called a ***tableau***.

Highlight 6.2.2 (Selecting Basic Variables)**.** *The variables represented by columns with exactly one nonzero entry (always a 1) are selected to be the basic variables.*

Note in 6.22 we have $\{s_1, s_2, P\}$ as basic variables and $\{x_1, x_2\}$ as nonbasic, hence this corresponds to the solution $x_1 = 0, x_2 = 0, s_1 = 40, s_2 = 72, P = 0$

[6] Note that others choose to put the objective function in the first row of their matrix representing the Linear Programming problem. Also, instead of $-60x_1 + -90x_2 + P = 0$ some choose to write $60x_1 + 90x_2 - P = 0$, thus their row representing the objective function will have positive coefficients and their method is the corresponding adapted version of what follows. Hence other explanations of the Simplex Method may appear different than what we are doing, but really are the same procedure just written differently. This subject is a relatively young one extending over multiple disciplines and notation and procedures are not yet standardized.

or, specifically, $x_1 = 0, x_2 = 0, P = 0$. Though this does not optimize P, it does satisfy the constraints and is therefore feasible. Hence we have that $(0, 0, 40, 72, 0)$ is a basic feasible solution and we now refer to 6.22 as the ***initial Simplex tableau***.

Definition 6.2.3 (Initial or Canonical Simplex Tableau)**.** *If the augmented matrix representing the modified linear programming problem has a solution that is feasible, it is called the* ***initial or canonical Simplex tableau****.*

This is all very well and good, but our aim is to maximize profit. Where to go next? Well, since the Frontiersman model produces the most per-unit profit, it seems reasonable that letting x_2 be as large as possible will lead to maximizing the profit. x_2 corresponds to the second column and notice that this column has the largest negative entry in last row (the row corresponding to the objective function). We will choose to work with this column and refer to it as the ***pivot column***.

Highlight 6.2.4 (Selecting the PIVOT COLUMN)**.** *To select the* pivot column*, choose the column with the largest negative entry in the bottom row. If there is a tie, choose either column.* ***If there are no negative entries, we are done and an optimal solution has been found****.*

Some reflection will reveal how we know in Highlight 6.2.4 that if a pivot column cannot be selected then the optimal value as been obtained. No decision variable having a negative coefficient in the final row of the corresponding Simplex tableau means that there is no decision variable that can be increased and result in a larger objective function. Thus no changes in any decision variable (unless we leave the feasible region) will lead to a more optimal value of the objective function. Hence the maximum over the feasible region has been obtained and the process terminates.

Once a pivot column has been selected, what is to be done next? Since we are focusing on making as many Frontiersman models as possible (this choice over the Cabin model increases the profit the most), let us recall the constraints. In particular, we know that the cutting department needs 2 labor-hours to cut each sleeping bag and that the assembly department needs 3 labor-hours to assemble a sleeping bag. The cutting department only has 40 labor-hours available which means they can cut for at most 20 sleeping bags. The assembly department has only 72 labor-hours available, so they can assemble at most 24 sleeping bags. Hence we have the restriction that we can make at most 20 Frontiersman models of the sleeping bag in a single day. Hence we choose the first row as the ***pivot row***. Notice that these restrictions correspond to *dividing the each value in the last column by the corresponding value in the pivot column and then selecting the smallest positive ratio.*

Highlight 6.2.5 (Selecting the PIVOT ROW)**.** *To select the* pivot row*, choose the row with the smallest positive ratio of the entry in the last column divided by the corresponding entry in the pivot column. If there is a tie, choose either* *row.* ***If there are no positive entries in the pivot column above the last row, the linear program has no optimal solution and we are done.***

We refer to the element of the tableau that is in the pivot row and the pivot column as the ***pivot element***. In our example, the pivot element is 2. Our job is to now perform legal row operations to make the pivot element 1 and every other entry in the pivot column 0. These row operations are commonly called ***pivot operations***.

Highlight 6.2.6 (Pivot Operations)**.** *There are two:*

- *Multiply the pivot row by the reciprocal of the pivot element. This transforms the pivot element into a 1. Symbolically, if k is the pivot element and R_t is the pivot row: $\frac{1}{k}R_t \to R_t$.*
- *Add multiples of the pivot row to all other rows in the tableau in order to annihilate (transform to 0) all other entries in the pivot column. Symbolically, something like $aR_s + R_t \to R_t$.*

Again, some thought sheds light on why not being able to select a pivot row in Highlight 6.2.5 leads to a linear programming problem not having a max. If no entry in the selected pivot column has a positive coefficient, then increasing the corresponding decision variable will lead to a *decrease* in the left-hand side of the constraint that the row represents. As the constraint is of the form $\leq$, the decision variable can be made arbitrarily large – thus increasing the objective function without end – and still satisfy the constraint. Hence when a pivot row cannot be selected, it is the case that the feasible region is unbounded.

Back to the example, we have decided that the second column (the column representing the variable x_2) is the pivot column and the first row (currently representing the basic variable s_1) is the pivot row. Hence we should begin our row operations by doing

$$\frac{1}{2}R_1 \to R_1$$

which gives us

$$\left[\begin{array}{cccccc|c} (basic) & x_1 & x_2 & s_1 & s_2 & P & \\ (s_1) & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 20 \\ (s_2) & 2 & 3 & 0 & 1 & 0 & 72 \\ \hline (P) & -60 & -90 & 0 & 0 & 1 & 0 \end{array}\right].$$

Next, we do the row operations

$$-3R_1 + R_2 \to R_2 \text{ and } 90R_1 + R_3 \to R_3$$

which gives

$$\left[\begin{array}{cccccc|c} (basic) & x_1 & x_2 & s_1 & s_2 & P & \\ (x_2) & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 20 \\ (s_2) & \frac{1}{2} & 0 & -\frac{3}{2} & 1 & 0 & 12 \\ \hline (P) & -15 & 0 & 45 & 0 & 1 & 1800 \end{array}\right].$$

Observe that we now have that the basic variables are given by the set $\{x_2, s_2, P\}$ and the non-basic variables are $\{x_1, s_1\}$. It is also worthwhile to point out that we currently have $x_1 = 0$ (since it is currently a non-basic variable) and $x_2 = 20$. This corresponds to a corner point in the graph of the feasible solution and, in fact, **the Simplex Method after one iteration has moved us from the origin to the point** $(0, 20)$ where P has increased from \$0 to \$1800.

Since we still have a negative entry in the bottom row, we repeat the process. Column 1 will now be the pivot column. As well, $20/\frac{1}{2} = 40$ and $12/\frac{1}{2} = 24$, hence we choose the second row as the pivot row. Since the pivot element is $\frac{1}{2}$ we initially perform the row operation

$$2R_2 \to R_2$$

to get

$$\left[\begin{array}{lcccccc|c} (basic) & x_1 & x_2 & s_1 & s_2 & P & & \\ (x_2) & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & & 20 \\ (s_2) & 1 & 0 & -3 & 2 & 0 & & 24 \\ \hline (P) & -15 & 0 & 45 & 0 & 1 & & 1800 \end{array}\right].$$

then the row operations

$$-\frac{1}{2}R_2 + R_1 \to R_1 \text{ and } 15R_2 + R_3 \to R_3$$

which gives us

$$\left[\begin{array}{lcccccc|c} (basic) & x_1 & x_2 & s_1 & s_2 & P & & \\ (x_2) & 0 & 1 & 2 & -1 & 0 & & 8 \\ (x_1) & 1 & 0 & -3 & 2 & 0 & & 24 \\ \hline (P) & 0 & 0 & 0 & 30 & 1 & & 2160 \end{array}\right].$$

Again, notice that x_1 has now entered as a basic variable (selected column 1 as the pivot column) while s_2 has excited the set of basic variables (selected row 2 as the pivot row). As well, since there are no more negative entries in the bottom row, the Simplex Method terminates and we have found the optimal solution; in particular that producing 24 Weekend model tents (which is what the variable x_1 represents) and producing 8 Backcountry model tents (represented by x_2) produces a maximum daily profit of \$2160. Specifically, **the Simplex Method during the second iteration moved us from the** $(0, 20)$ **to** $(24, 8)$ and P has increased from \$1800 to \$2160 and any other move will not increase P.

It is worthwhile to emphasize a very important property of the Simplex Method.

Discussion 6.2.7. *The Simplex method is designed in such a way that as long as the method begins at a feasible solution (hence the importance of the* initial simplex tableau *over just a tableau), are bounded, and we do not have* *degeneracy (see Section 6.5), the algorithm will efficiently move from corner point to corner point until it terminates at the optimal value. This is important because it guarantees*

1. *the process always terminates and*
2. *the process always produces a feasible solution.*

6.2.4 Summary

To summarize the Simplex Method,

1. Introduce slack variables into the mathematical model and write the initial tableau.
2. Are there any negative entries in the bottom row?
   - Yes – go to step 3.
   - No – the optimal solution has been found.
3. Select the pivot column.
4. Are there any positive elements above the last row (above the solid line)?
   - Yes – go to step 5.
   - No – no optimal solution exists (the feassible region is unbounded).
5. Select the pivot row and thus the pivot element. Perform the appropriate pivot operations then return to step 2.

   Example 6.1.1 summarized:

$$\begin{aligned}
\text{Maximize: } & P(x_1, x_2) = 60x_1 + 90x_2 \\
\text{Subject to: } & x_1 + 2x_2 + s_1 = 40 \\
& 2x_1 + 3x_2 + s_2 = 72 \\
& x_1, x_2, s_1, s_2 \geq 0.
\end{aligned}$$

$$\overbrace{\left[\begin{array}{ccccccc|c}
(basic) & x_1 & x_2 & s_1 & s_2 & P & \\
(s_1) & 1 & \boxed{2} & 1 & 0 & 0 & 40 \\
(s_2) & 2 & 3 & 0 & 1 & 0 & 72 \\
\hline
(P) & -60 & -90 & 0 & 0 & 1 & 0
\end{array}\right]}^{\textit{Notes i, ii.}}$$

$$\xrightarrow{\frac{1}{2}R_1 \to R_1}
\left[\begin{array}{cccccc|c}
 & x_1 & x_2 & s_1 & s_2 & P & \\
(s_1) & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 20 \\
(s_2) & 2 & 3 & 0 & 1 & 0 & 72 \\
\hline
(P) & -60 & -90 & 0 & 0 & 1 & 0
\end{array}\right]$$

$$\xrightarrow[90R_1+R_3\to R_3]{-3R_1+R_2\to R_2} \overbrace{\begin{array}{cc} & \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{c} (x_2) \\ (s_2) \\ (P) \end{array} & \left[\begin{array}{ccccc|c} \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 20 \\ \boxed{0.5} & 0 & -\frac{3}{2} & 1 & 0 & 12 \\ \hline -15 & 0 & 45 & 0 & 1 & 1800 \end{array}\right] \end{array}}^{Note\ iii.}$$

$$\xrightarrow{2R_2\to R_2} \begin{array}{cc} & \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{c} (x_2) \\ (s_2) \\ (P) \end{array} & \left[\begin{array}{ccccc|c} \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 20 \\ \boxed{1} & 0 & -3 & 2 & 0 & 24 \\ \hline -15 & 0 & 45 & 0 & 1 & 1800 \end{array}\right] \end{array}$$

$$\xrightarrow[15R_2+R_3\to R_3]{-\frac{1}{2}R_2+R_1\to R_1} \overbrace{\begin{array}{cc} & \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{c} (x_2) \\ (x_1) \\ (P) \end{array} & \left[\begin{array}{ccccc|c} 0 & 1 & 1 & -\frac{3}{2} & 0 & 8 \\ 1 & 0 & -3 & 2 & 0 & 24 \\ \hline 0 & 0 & 0 & 30 & 1 & 2160 \end{array}\right] \end{array}}^{Note\ iv.}.$$

Notes:
i. As this matrix's solution set $x_1 = 0, x_2 = 0, s_1 = 40, s_2 = 72$ satisfies all constraints, the solution set is in the feasible region and thus we have an *initial Simplex tableau.*
ii. Since $|-90| > |-60|$ and $0 < \frac{40}{2} < \frac{72}{3}$, the 2 is the pivot element.
iii. Since the first column is the only column with a negative entry and since $0 < 12/\frac{1}{2} < 20/\frac{1}{2}$, the circled $\frac{1}{2}$ is the pivot element.
iv. As there are no negative entries in the bottom row, the process terminates and we have an optimal value.

6.2.5 Keywords

- *slack variables*
- *basic variables*
- *nonbasic variables*
- *basic solution*
- *basic feasible solution*
- *pivot row*
- *pivot column*
- *pivot element*
- *pivot operation*
- *Simplex Method*
- *Simplex Tableau*
- *Initial Simplex Tablea*

6.3 The Dual: Minimization with Problem Constraints of the Form $\geq$

6.3.1 How It Works

Instead of maximizing a particular objective function, let us are now consider a situation where we want to minimize an objective function, e.g. minimizing costs. These problems are of the form:

$$\begin{aligned} \text{Minimize:} \quad C(y_1, y_2) &= 40y_1 + 72y_2 && (6.23)\\ \text{Subject to:} \quad y_1 + 2y_2 &\geq 60 && (6.24)\\ 2y_1 + 3y_2 &\geq 90 && (6.25)\\ y_1, y_2 &\geq 0. && (6.26) \end{aligned}$$

As we will see in Section 6.3.2, each problem of this form can be associated with a corresponding maximization problem which we refer to as the ***dual problem***.

Our first step in forming the dual problem will be to form a matrix from the problem constraints and the objective function. The appropriate matrix for our example is

$$A = \left[\begin{array}{cc|c} 1 & 2 & 60 \\ 2 & 3 & 90 \\ \hline 40 & 72 & 1 \end{array}\right].$$

Please note that A is not the matrix associated with the initial simplex tableau as in Example 6.1.1. We now consider A^T, the ***transpose*** of matrix A. Hence, for our example:

$$A^T = \left[\begin{array}{cc|c} 1 & 2 & 40 \\ 2 & 3 & 72 \\ \hline 60 & 90 & 1 \end{array}\right].$$

Given A^T we may now form the dual problem, namely we now have a maximization problem with constraints of the form $\leq$. In particular, we have the dual problem:

$$\begin{aligned} \text{Maximize:} \quad P(x_1, x_2) &= 60x_1 + 90x_2 && (6.27)\\ \text{Subject to:} \quad x_1 + 2x_2 &\leq 40 && (6.28)\\ 2x_1 + 3x_2 &\leq 72 && (6.29)\\ x_1, x_2 &\geq 0. && (6.30) \end{aligned}$$

Look familiar? This is the example we have considered previously. This time, though, let us name our slack variables y_1 and y_2. Our initial Simplex tableau is

$$\left[\begin{array}{ccccc|c} x_1 & x_2 & y_1 & y_2 & P & \\ 1 & 2 & 1 & 0 & 0 & 40 \\ 2 & 3 & 0 & 1 & 0 & 72 \\ \hline -60 & -90 & 0 & 0 & 1 & 0 \end{array}\right]. \tag{6.31}$$

and the Simplex method gives us as the final tableau:

$$\left[\begin{array}{ccccc|c} x_1 & x_2 & y_1 & y_2 & P & \\ 0 & 1 & 1 & -\frac{3}{2} & 0 & 8 \\ 1 & 0 & -3 & 2 & 0 & 24 \\ \hline 0 & 0 & 0 & 30 & 1 & 2160 \end{array}\right]. \tag{6.32}$$

We note that the bottom row of the simplex tableau gives us the solution to our minimization problem, namely that $y_1 = 0$ and $y_2 = 30$ minimizes $C(y_1, y_2) = 60y_1 + 90y_2$ under the stated constraints. I.e.

Highlight 6.3.1. *An optimal solution to a minimization problem is obtained from the bottom row of the final simplex tableau for the dual maximization problem.*

One word of caution, though.

Warning!! 6.3.2. *Never multiply the inequality representing a problem constraint in a maximization problem by a number if that maximization problem is being used to solve a corresponding minimization problem.*

Example 6.3.3.

6.3.2 Why It Works

The type of problems considered in Section 6.2 were problems of the form

$$\begin{aligned} \text{maximize} \quad & P(\mathbf{x}) = \mathbf{c}^{\mathbf{T}}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{6.33}$$

Here we have abbreviated the statement of the problem by using notation from Linear Algebra. By writing vectors together, e.g. $\mathbf{uv}$, we mean to multiply the vectors using the dot product. By the inequality of (6.33) we mean that for each component of the resulting vectors we have $a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$. Linear programming problems the form (6.33) ("**maximize**" together with "$A\mathbf{x} \leq \mathbf{b}$") are said to be of ***first primal form***, which was the class of problems we studied in Section 6.2. The ***dual of the first primal form*** is

$$\begin{aligned} \text{minimize} \quad & C(\mathbf{y}) = \mathbf{b}^{\mathbf{T}}\mathbf{y} \\ \text{subject to} \quad & A^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned} \tag{6.34}$$

where this class of problems is characterized by "**minimize**" together with "$A\mathbf{y} \leq \mathbf{c}$".

These two forms are joined by the following theorem:

Theorem 6.3.4 (The Fundamental Principle of Duality)**.** *A minimization problem has a solution if and only if the corresponding dual maximization problem has a solution.*

More precisely

Theorem 6.3.5 (Weak Duality)**.** *If* $\mathbf{x}$ *satisfies the constraints of a linear programming problem in first primal form and if* $\mathbf{y}$ *satisfies the constraints of the corresponding dual, then*

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}. \tag{6.35}$$

Proof. Since we have a problem of first primal form, we know that $\mathbf{x} \geq \mathbf{0}$ and $A\mathbf{x} \leq \mathbf{b}$. Likewise, by the dual $\mathbf{y} \geq \mathbf{0}$ and $A^T\mathbf{y} \geq \mathbf{c}$ are satisfied. Hence

$$\mathbf{c}^T\mathbf{x} \leq (A^T\mathbf{y})^T\mathbf{x} = \mathbf{y}^T A\mathbf{x} \leq \mathbf{y}^T\mathbf{b} = \mathbf{b}^T\mathbf{y} \tag{6.36}$$

where the last equality holds because both expressions are dot products. □

Showing that $max\{\mathbf{c}^T\mathbf{x}\}$ and $min\{\mathbf{b}^T\mathbf{y}\}$ exist and are equal in this case is known as *Strong Duality* and this proof is left as an exercise.

The significance of 6.36 in Theorem 6.3.2 is clear once we realize we are trying to maximize the LHS and minimize the RHS. Thus we have a solution to each problem exactly when $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$; in other words if $\mathbf{x}$ is a solution to the first primal LP problem, then $\mathbf{y}$ is a solution to its dual and vice versa.

6.3.3 Keywords

- *dual problem*
- *the Fundamental Theorem of Duality*

6.4 The Big M Method: Max/Min LP Problems with Varying Constraints

6.4.1 Maximization Problems with the Big M Method

We motivate a technique for solving maximization problems with mixed constraints by considering the following example:

$$\begin{aligned} \text{Maximize: } P(x_1, x_2) &= 2x_1 + x_2 && (6.37)\\ \text{Subject to: } \quad x_1 + x_2 &\le 10 && (6.38)\\ -x_1 + x_2 &\ge 2 && (6.39)\\ x_1, x_2 &\ge 0. && (6.40) \end{aligned}$$

As before, since the first inequality involves a $\le$, we introduce a slack variable s_1:

$$x_1 + x_2 + s_1 = 10. \tag{6.41}$$

Note that this slack variable is necessarily nonnegative.

We need the second inequality to be an equality as well, so we introduce the notion of a *surplus variable*, s_2 (remember, the left hand side exceeds the 2, so the s_2 makes up for the difference). It would be natural to make this variable nonpositive, but, to be consistent with the other variables, let us require the surplus variable s_2 to as well be nonnegative. Hence the second inequality is rewritten

$$-x_1 + x_2 - s_2 = 2. \tag{6.42}$$

Hence the modified problem is:

$$\begin{aligned} \text{Maximize:} \quad & P(x_1, x_2) = 2x_1 + x_2 && (6.43)\\ \text{Subject to:} \quad & x_1 + x_2 + s_1 \qquad = 10 && (6.44)\\ & -x_1 + x_2 \qquad - s_2 = 2 && (6.45)\\ & x_1, x_2, s_1, s_2 \ge 0. && (6.46) \end{aligned}$$

and the preliminary tableau for the Simplex Method is

$$\left[\begin{array}{cccccc|c} (basic) & x_1 & x_2 & s_1 & s_2 & P & \\ (s_1) & 1 & 1 & 1 & 0 & 0 & 10 \\ (s_2) & -1 & 1 & 0 & -1 & 0 & 2 \\ \hline (P) & -2 & -1 & 0 & 0 & 1 & 0 \end{array}\right]. \tag{6.47}$$

which has as its basic solution

$$x_1 = 0, x_2 = 0, s_1 = 10, s_2 = -2. \tag{6.48}$$

Unfortunately, this is not feasible (s_2 fails to satisfy the nonnegativity constraint). Hence this cannot be the initial Simplex tableau[7] and some work most be done to modify the problem so that we have an initial feasible basic solution (if the basic solution is not feasible, we eventually reach a step in the

[7] Some literature refers to an initial tableau which gives a basic feasible solution as a **canonical Simplex tableau**.

process where we do not yet have the optimal value for P, but we are unable to select a next pivot element – try it!).

We thus introduce an *artificial variable* for each surplus variable. These artificial variables are included in the constraint equations involving the respective surplus variable with the purpose of the surplus variable can now be nonnegative and the artificial variable will pick up the slack:

$$-x_1 + x_2 - s_2 + a_1 = 2. \tag{6.49}$$

The introduction of the artificial variable is very clever, but it is "artificial", so we do not want it to be part of an optimal solution. In particular, we want the artificial variable to be 0 in the solution. As such, we introduce a very large *penalty* into the objective function if the, artificial variable is anything other than 0 namely:

$$P(x_1, x_2) = 2x_1 + x_2 - Ma_1 \tag{6.50}$$

where M is an arbitrary number (hence the name "Big M" Method).

Hence the modified problem becomes:

$$\begin{aligned} \text{Maximize:} \quad & P(x_1, x_2) = 2x_1 + x_2 - Ma_1 && (6.51)\\ \text{Subject to:} \quad & x_1 + x_2 + s_1 \qquad\qquad = 10 && (6.52)\\ & -x_1 + x_2 \quad - s_2 + a_1 = 2 && (6.53)\\ & x_1, x_2, s_1, s_2, a_1 \geq 0. && (6.54) \end{aligned}$$

and the tableau is

$$\left[\begin{array}{lrrrrrr|r} (basic) & x_1 & x_2 & s_1 & s_2 & a_1 & P & \\ (s_1) & 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ (s_2) & -1 & 1 & 0 & -1 & 1 & 0 & 2 \\ \hline (P) & -2 & -1 & 0 & 0 & M & 1 & 0 \end{array}\right] \tag{6.55}$$

which has as its basic solution

$$x_1 = 0, x_2 = 0, s_1 = 10, s_2 = -2, a_1 = 0. \tag{6.56}$$

Note we still have the same problem... a nonfeasible basic solution and therefore the tableau in 6.55 is not the initial Simplex tableau. To remedy this, let us make a_1 a basic variable:
The row operation $(-M)R_2 + R_3 \rightarrow R_3$ gives us

$$\left[\begin{array}{lcccccc|c} (basic) & x_1 & x_2 & s_1 & s_2 & a_1 & P & \\ (s_1) & 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ (a_1) & -1 & 1 & 0 & -1 & 1 & 0 & 2 \\ \hline (P) & M-2 & -M-1 & 0 & M & 0 & 1 & -2M \end{array}\right]. \tag{6.57}$$

which has as its basic solution

$$x_1 = 0, x_2 = 0, s_1 = 10, s_2 = 0, a_1 = 2, \tag{6.58}$$

which is feasible.

Now we are able to employ the Simplex method where M is some large *fixed* positive number. The Simplex Method yields:

$$\left[\begin{array}{lcccccc|c} (basic) & x_1 & x_2 & s_1 & s_2 & a_1 & P & \\ (x_1) & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 4 \\ (x_2) & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 6 \\ \hline (P) & 0 & 0 & \frac{3}{2} & \frac{1}{2} & M-\frac{1}{2} & 1 & 14 \end{array}\right]. \tag{6.59}$$

which has as its basic solution

$$x_1 = 4, x_2 = 6, s_1 = 0, s_2 = 0, a_1 = 0, P = 14, \tag{6.60}$$

which is feasible.

To summarize:

THE BIG M METHOD

SET-UP:

1. Multiply any constraints that have negative constants on the right by -1 (this is so that the notions of slack and surplus variables will be consistent).
2. Introduce a slack variable for every constraint that has a $\leq$.
3. Introduce a surplus variable and an artificial variable for every constraint that has a $\geq$.
4. For every artificial variable a_i that has been introduced, add $-Ma_i$ to the objective function.

SOLUTION:

1. Form the preliminary tableau to use the Simplex Method on the modified problem.
2. Do the necessary row operations to make each artificial variable a basic variable (i.e. make sure each column that represents an artificial variable has exactly 1 nonzero entry).
3. Apply the Simplex method to obtain an optimal solution to the modified problem.
4. a) If the modified problem has no optimal solution, then the original problem has no solution.
   b) If all artificial variables are 0, an optimal solution to the original problem has been found.
   c) If any artificial variable is nonzero, then the original problem has no optimal solution.

6.4.2 Minimization Problems with the Big M Method

In Exercise 2.2, you showed that for a real-valued function $f(\mathbf{x})$, $f(\mathbf{x}^*)$ is a maximum of f if and only if $-f(\mathbf{x}^*)$ is a minimum of $-f$. In other words, minimizing $f(x)$ is the same as maximizing $-f(x)$; they can be different values, but the location of $max f(\mathbf{x})$ and $min f(\mathbf{x})$ are the same: $\mathbf{x}^*$. Therefore if one is to minimize $C(x_1, x_2, ..., x_n)$, merely apply the above procedures to maximize $-C(x_1, x_2, ..., x_n)$.

Highlight 6.4.1. *Note that when using the Big M Method to solve a minimization problem, the penalty $+Ma_1$ (add since we are minimizing) is to be introduced* ***before*** *negating the objective function C.*

Example 6.4.2.

$$\begin{aligned} &\textit{Minimize:} \quad && C = 5x_1 + 3x_2 && (6.61)\\ &\textit{Subject to:} \quad && 3x_1 + 4x_2 \geq 12 && (6.62)\\ & && 2x_1 + 5x_2 \leq 20 && (6.63)\\ & && x_1, x_2 \geq 0. && (6.64) \end{aligned}$$

Thus the modified problem is

$$\begin{aligned} &\textit{Minimize:} \quad && C = 5x_1 + 3x_2 + Ma_1 && (6.65)\\ &\textit{Subject to:} \quad && 3x_1 + 4x_2 - s_1 \quad + a_1 = 12 && (6.66)\\ & && 2x_1 + 5x_2 \quad + s_2 \quad = 20 && (6.67)\\ & && x_1, x_2 \geq 0 && (6.68) \end{aligned}$$

with surplus variable s_1, slack variable s_2, and artificial variable a_1. Note that we have added the penalty Ma_1 because this is a minimization problem and we want a_1 having any value above 0 to take us away from the minimum.

As the Big M Method is designed for maximization problems, we adapt the given problem in the following way:

$$\begin{aligned} &\textit{Maximize:} \quad && P = -C = -5x_1 - 3x_2 - Ma_1 && (6.69)\\ &\textit{Subject to:} \quad && 3x_1 + 4x_2 - s_1 \quad + a_1 = 12 && (6.70)\\ & && 2x_1 + 5x_2 \quad + s_2 \quad = 20 && (6.71)\\ & && x_1, x_2, s_1, s_2 \geq 0. && (6.72) \end{aligned}$$

The corresponding tableau is

$$\left[\begin{array}{lcccccc|c} (\textit{basic}) & x_1 & x_2 & s_1 & s_2 & a_1 & P & \\ (s_1) & 3 & 4 & -1 & 0 & 1 & 0 & 12 \\ (s_2) & 2 & 5 & 0 & 1 & 0 & 0 & 20 \\ \hline (P) & 5 & 3 & 0 & 0 & M & 1 & 0 \end{array}\right]. \qquad (6.73)$$

which has $x_1 = x_2 = a_1 = 0$ and $s_1 = -12, s_2 = 20$ as its basic solution.

Note this is not feasible, so we perform the row operation $-MR_1 + R_3 \to R_3$ to obtain

$$\left[\begin{array}{lcccccc|c} (\textit{basic}) & x_1 & x_2 & s_1 & s_2 & a_1 & P & \\ (a_1) & 3 & 4 & -1 & 0 & 1 & 0 & 12 \\ (s_2) & 2 & 5 & 0 & 1 & 0 & 0 & 20 \\ \hline (P) & 5-3M & 3-4M & M & 0 & 0 & 1 & -12M \end{array}\right]. \qquad (6.74)$$

TABLE 6.3
Summary of Applying the Simplex Method to LP Problems

LP	Constraints	RHS Constants	Coeff. of P	Solution
Max	$\leq$	nonnegative	any	Simplex w/ slack variables
Min	$\geq$	any	nonnegative	the dual
Max	$\geq$ or mixed	nonnegative	any	Big M
Min	$\leq$ or mixed	nonnegative	any	Big M max negative obj. func.

which has $x_1 = x_2 = s_1 = 0$ and $a_1 = 12, s_2 = 20$ as its basic solution. Since this is feasible, we may now proceed with the Simplex Method.

$$\left[\begin{array}{ccccccc|c} (basic) & x_1 & x_2 & s_1 & s_2 & a_1 & P & \\ (a_1) & 3 & \boxed{4} & -1 & 0 & 1 & 0 & 12 \\ (s_2) & 2 & 5 & 0 & 1 & 0 & 0 & 20 \\ \hline (P) & 5-3M & 3-4M & M & 0 & 0 & 1 & -12M \end{array}\right] \tag{6.75}$$

Using row operations $\frac{1}{4}R_1 \to R_1$, $-\frac{5}{4}R_1 + R_2 \to R_2$, and $\frac{4M-3}{4}R_1 + R_3 \to R_3$, we obtain:

$$\left[\begin{array}{ccccccc|c} & x_1 & x_2 & s_1 & s_2 & a_1 & P & \\ (x_2) & \frac{3}{4} & 1 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 3 \\ (s_2) & -\frac{7}{4} & 0 & \frac{5}{4} & 1 & -\frac{5}{4} & 0 & 5 \\ \hline (P) & \frac{11}{4} & 0 & \frac{3}{4} & 0 & \frac{4M-3}{4} & 1 & -9 \end{array}\right] \tag{6.76}$$

As there are no negative entries in the columns involving the variables, the process terminates and we see that max $P = -C - 9$ therefore min $C = 9$ and this occurs at $x_1 = 0, x_2 = 3$.

6.5 Degeneracy and Cycling in the Simplex Method

We have seen that the notion of a basic feasible solution is fundamental to the Simplex Method. The name *basic* comes from the fact that the column vectors representing the basic variables form a basis for the column space ofthe linear programming problem's corresponding matrix. The Simplex Method can fail when at some iteration in the process a basic variable is 0. The pivot that introduces the basic variable into the feasible solution is referred to as a *degenerate pivot* and we refer to the method in this case as being *degenerate*.

Degenerate pivots can happen and not be a death blow to the algorithm accomplishing its task. Unfortunately, there are times when a degenerate pivot leads to further application of the algorithm causing old decision variables to leave the set of basic variables and new ones enter, but the value of the objective function does not change. In this situation, it is possible for the Simplex Method to cycle through these sets in some maddening infinite loop and the process thus not terminate. Such a situation is referred to as *cycling* and causes a problem only in that the algorithm will not stop. Including an anti-cycling rule such as *Bland's Smallest Subscript Rule* (or just *Bland's Rule*) [3] when employing the Simplex Method guarantees that the algorithm will terminate.

6.6 Exercises

For Exercises 6.1 through 6.5, do each of the following:

I. Solve the following linear programming problems by graphing the feasible region then evaluating the objective function at each corner point. "Solve" means state the optimal value of the objective function and ***all points*** in the feasible region at which this optimal value occurs.
II. Solve each problem using the Simplex Method, it's dual, or the Big M Method. Your work should contain a clear statement of the model after the introduction of slack, surplus, and artificial variables. You may use a calculator or computer to do the row operations, but write down the obtained simplex tableau after each iteration of the method. At each iteration identify the pivot element.
III. Check your work using a software package of your choice (Solver, Matlab, etc.). Print and submit your answer screen and please make clear what software you have used.

Exercise 6.1.

$$\begin{aligned}
\textit{Minimize } P(x,y) &= 5x + 2y \\
\textit{Subject to } x + y &\geq 2 \\
2x + y &\geq 4 \\
x, y &\geq 0
\end{aligned}$$

For this question only *(i.e. Exercise 6.1), when the Simplex method (part II above), at each iteration state which variables are basic and which are non-basic. Also, at each iteration state the value of the objective function.*

Exercise 6.2.

$$\begin{aligned} \textit{Maximize } P(x,y) &= 5x+2y \\ \textit{Subject to } x+y &\geq 2 \\ 2x+y &\geq 4 \\ x,y &\geq 0 \end{aligned}$$

Exercise 6.3.

$$\begin{aligned} \textit{Maximize } P(x,y) &= 20x+10y \\ \textit{Subject to } x+y &\geq 2 \\ x+y &\leq 8 \\ 2x+y &\leq 10 \\ x,y &\geq 0 \end{aligned}$$

Exercise 6.4.

$$\begin{aligned} \textit{Maximize } P(x,y) &= 20x+10y \\ \textit{Subject to } 2x+3y &\geq 30 \\ 2x+y &\leq 26 \\ -2x+5y &\leq 34 \\ x,y &\geq 0 \end{aligned}$$

Exercise 6.5.

$$\begin{aligned} \textit{Minimize } P(x,y) &= 20x+10y \\ \textit{Subject to } 2x+3y &\geq 30 \\ 2x+y &\leq 26 \\ -2x+5y &\leq 34 \\ x,y &\geq 0 \end{aligned}$$

Exercise 6.6. *In this problem, there is a tie for the choice of the first pivot column. When you do your work using the simplex method use the method twice to solve the problem two different ways; first by choosing column 1 as the first pivot column and then for your second solution effort, solve by choosing column 2 as the first pivot column. You may use a computer or calculator to perform the Simplex Method, but do write down the results of each iteration.*

$$
\begin{aligned}
\textit{Maximize } & P(x,y) = x + y \\
\textit{Subject to } & 2x + y \leq 16 \\
& x \leq 6 \\
& y \leq 10 \\
& x, y \geq 0
\end{aligned}
$$

Exercise 6.7. *This problem has multiple parts.*

1. *Solve the following by using the dual of the Simplex Method. Your work should contain each the details of each iteration.*

$$
\begin{aligned}
\textit{Minimize } & C(x_1,x_2,x_3) = 40x_1 + 12x_2 + 40x_3 \\
\textit{Subject to } & 2x_1 + x_2 + 5x_3 \geq 20 \\
& 4x_1 + x_2 + x_3 \geq 30 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

2. *The dual problem has as its first constraint*

$$2y_1 + 4y_2 \leq 40. \tag{6.77}$$

Replace this constraint by its simplified version

$$y_1 + 2y_2 \leq 20 \tag{6.78}$$

then proceed with the Simplex Method.

3. *Compare your answers from the first two parts. Why are they different?*

7

Sensitivity Analysis

7.1 Motivation

Recall Lincoln Outdoors manufacturing sleeping bags from Example 6.1.1 with the details of the situation summarized in Table 7.1.

We sought to help Lincoln Outdoors by determining what manufacturing levels maximized profit and modeled the situation with the linear programming problem:

$$\text{Maximize:} \quad P(x_1, x_2) = 60x_1 + 90x_2 \tag{7.1}$$

$$\text{Subject to:} \quad x_1 + 2x_2 = 40 \tag{7.2}$$

$$2x_1 + 3x_2 = 72 \tag{7.3}$$

$$x_1, x_2 \geq 0. \tag{7.4}$$

We found that any of $(24, 8)$, $(27, 6)$, $(30, 4)$, $(33, 2)$, or $(36, 0)$ would give a maximum profit of \$2,160.

In many real life situations, the vales used for the coefficients of the constraints and the objective functions are often (hopefully) good estimates of what is taking place (or even a good prediction of what will take place) and, as such, management may be skeptical of the analysis. Also, situations may change in that the cost of materials might fluctuate as may the efficiency of our machines or labor force. For example, it may be that due to a change in the cost of materials the profit realized on the production and sale of a Cabin Model sleeping bag is \$58.77 instead of \$60 or by an employee's clever suggestion assembling a Frontier Model sleeping bag only now takes 2.67 hours instead of 3. *Sensitivity analysis* can help convince someone that the model is reliable as well as give insight into the how a solution may change if any of the contributing factors vary slightly.

7.2 An Excel Example

We will illustrate the ideas of this chapter by considering the Lincoln Outdoors problem. We can enter it in an Excel worksheet as in Figure 7.1.

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TABLE 7.1
Manufacturing Data for Lincoln Outdoors in Example 6.1.1

Labor-Hours	Cabin Model	Frontier Model	Max Hours per Day
Cutting Dept.	1	2	40
Assembly Dept.	2	3	72
Profit per Bag	$60	$90	

Cabin Model				
Frontier Model				
		Cabin	Frontier	Available
Profit	0	60	90	
Cutting	0	1	2	40
Assembly	0	2	3	72

FIGURE 7.1
The Lincoln Outdoors problem in Excel.

Cabin Model	24			
Frontier Model	8			
		Cabin	Frontier	Available
Profit	2160	60	90	
Cutting	40	1	2	40
Assembly	72	2	3	72

FIGURE 7.2
An Excel solution for Lincoln Outdoors.

Using Excel's add-in Solver and selecting "Simplex LP" as the Solving Method option gives the result shown in Figure 7.2.

Before accessing the result, Solver permits accessing some optional reports as in Figure 7.3. As we are exploring sensitivity analysis, we are going to want Solver to show us an Answer Report, Sensitivity Report, and Limits Report and we get these options by clicking on the appropriate words in the upper right of the window.

After highlighting the Answer, Sensitivity, and Limits reports, select the "OK" tab and Solver returns to the workbook displaying the solution with tabs at the bottom of the sheet. Sheet 1 (or whatever name you change it to) displays Figure 7.4.

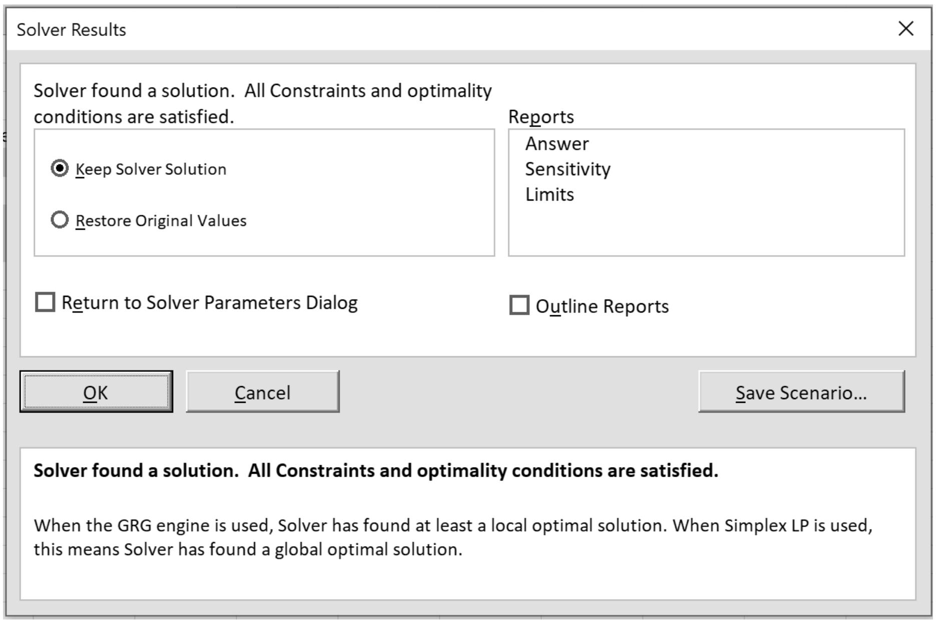
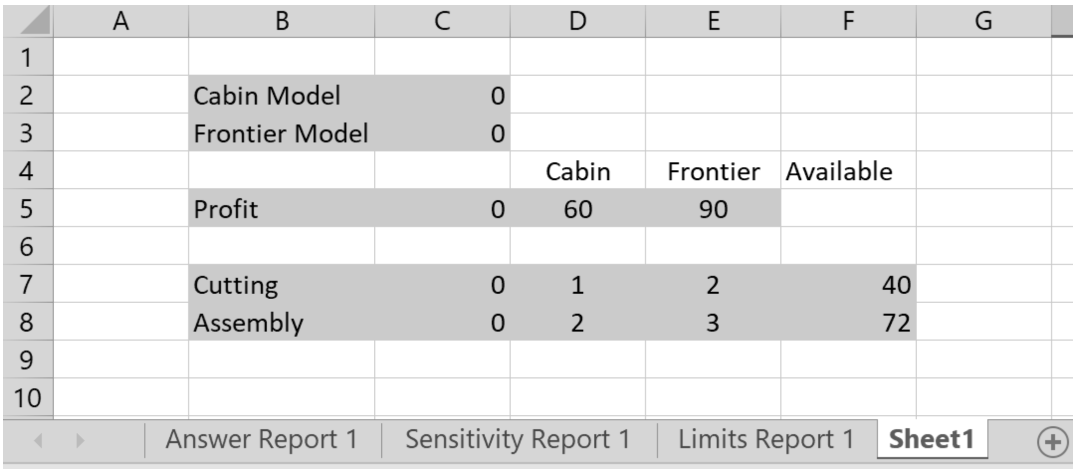


FIGURE 7.3
Options in Excel's solution for Lincoln Outdoors.



	A	B	C	D	E	F	G
1							
2		Cabin Model	0				
3		Frontier Model	0				
4				Cabin	Frontier	Available	
5		Profit	0	60	90		
6							
7		Cutting	0	1	2	40	
8		Assembly	0	2	3	72	
9							
10							

FIGURE 7.4
Report options displayed in Excel's solution for Lincoln Outdoors.

7.2.1 Solver's Answer Report

Let us select the Answer Report tab. Excel displays what is shown in Figure 7.5. The Objective Cell (Max) part of the report tells us that our Profit cell (C5) started with a value of 0 (the default value for a blank cell) and reached its max at $2,160. When using the Simplex LP solution option, the starting value should be at the origin as the technique is designed to start there, though in practice any value should work as the search is very quick and (barring

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$C$5	Profit	0	2160

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$C$2	Cabin Model	0	24	Contin
$C$3	Frontier Model	0	8	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$C$7	Cutting	40	$C$7<=$F$7	Binding	0
$C$8	Assembly	72	$C$8<=$F$8	Binding	0

FIGURE 7.5
The answer report for Lincoln Outdoors.

irregularities that seldom occur in application) the process is guaranteed to converge.

The next part of the report, Variable Cells, gives us the initial and solution values of the decision variables as well as the additional information that we found the solution over the real numbers ("Contin" = "Continuous" as compared to an integer or binary solution).

Lastly the Constraints section reports the value of the constraints when the solution is reached. For the situation when $x = \textit{Cabin Models} = 24$ and $y = \textit{Frontier Models} = 8$, the Cutting constraint has a value of 40 while the Assembly constraint has a value of 72; both of which are at their respective limits hence the report that for each variable the Slack is 0 and therefore each constraint is "Binding" meaning there is no room to move either constraint any higher.

The previous report serves its purpose, but for the sake of this chapter we want to focus on the reports in the second and third tabs, namely the Sensitivity Report (Figure 7.6) and the Limits Report (Figure 7.8).

7.2.2 Solver's Sensitivity Report

In the Sensitivity Report (Figure 7.6), the Variable Cells section refers to the decision variables. The Final Value column displays the value of the decision variables returned by Solver for yields the optimal solution. Recall that in this example there are multiple solutions: $(24, 8)$, $(27, 6)$, $(30, 4)$, $(33, 2)$, and $(36, 0)$. Solver returns $(24, 8)$ because it used the Simplex Method which begins

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$2	Cabin Model	24	0	60	0	15
$C$3	Frontier Model	8	0	90	30	0

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$C$7	Cutting	40	0	40	8	4
$C$8	Assembly	72	30	72	8	12

FIGURE 7.6
The sensitivity report for Lincoln Outdoors.

at $(0,0)$, one iteration takes us to $(0,20)$ and the next iteration to $(24,8)$. As no selection of a pivot column would improve the objective function's value (no negative values in the bottom row as per Highlight 6.2.4), the algorithm terminates here. The Reduced Cost requires a little bit of work and we will address that after we understand Shadow Prices. The Objective Coefficient is (as the name states) the coefficient of the decision variable in the objective function. The Allowable Increase and Allowable Decrease columns tell us how much range we have for a change in the particular coefficient to not affect the solution **provided all other coefficients remain unchanged**. For example, in this situation, if we think of the objective function as being $P(x_1,x_2) = Ax_1 + 90x_2$ then the stated global solution [the $(24,8)$; *not the others!*] will remain the same as long as $45 \leq A \leq 60$. Likewise, $P(x_1,x_2) = 60x_1 + Bx_2$ has also has the solution unchanged as long as $90 \leq B \leq 120$.

You are encouraged to experiment with the ranges of the decision variables in a spreadsheet using Solver. Geometrically, Figure 7.7 shows the idea behind these numbers: for instance, once $A < 45$, the furthest corner point the objective function would intersect as it leaves the origin (as in Figure 6.2) would be $(0,20)$. If the A goes above 60, the objective function becomes steep enough that the last corner point it would intersect leaving the feasible region would be $(36,0)$. Each of these situations is illustrated in Figure 7.7. Again, it is important to restate that this range of values only holds if the other decision variable's constant – the 90 – remains unchanged.

In situations where other solutions of a Linear Programming problem exist, just like in Lincoln Outdoors, Solver's Sensitivity Report has another use.

Highlight 7.2.1 (Other LP Solutions in Solver)**.** *Apart from degenerate cases, when using Solver to find the optimum in a Linear Programming problem, a* 0 *in either the Allowable Increase or Allowable Decrease in the Variable Cells section of the Sensitivity Report signals that other solutions exists.*

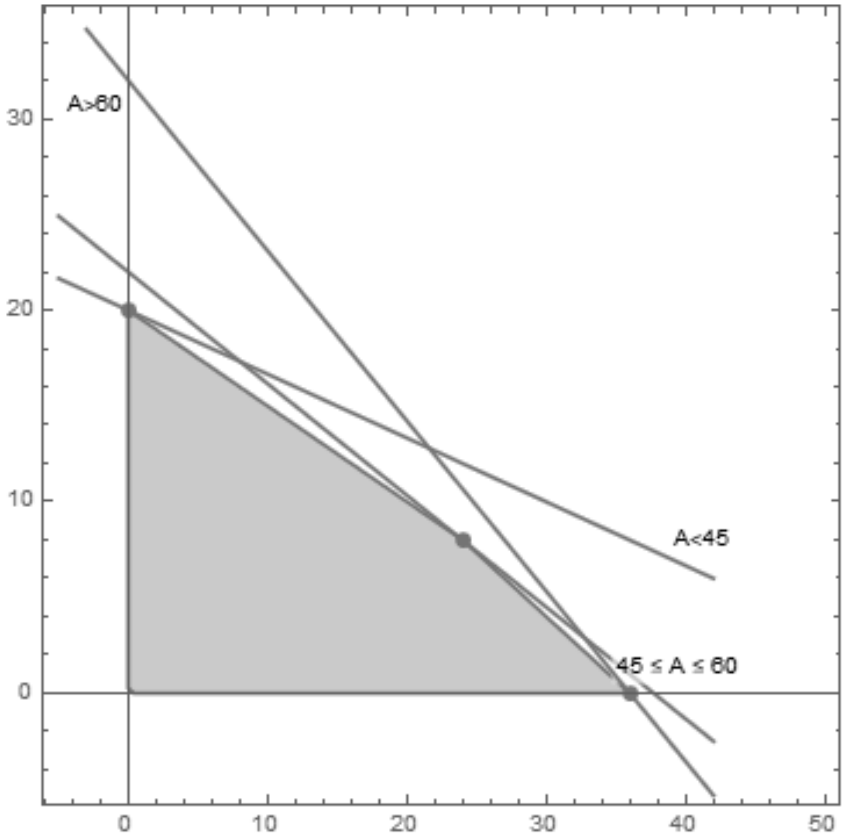


FIGURE 7.7
Solution changes for different A in the objective function $P = Ax_1 + 90x_2$ with Lincoln Outdoors.

In this situation we can use Solver to attempt to find other solutions by using Algorithm 7.2.1.

Algorithm 7.2.1 Finding Additional Non-Degenerate LP Solutions Using Solver.

Input: Solved LP problem in Solver with decision variables $x_1, \ldots, x_n$.

Add a constraint to the model that holds the objective function at the optimal value.
for $i = 1$ to k **do**
    if Allowable Decrease $= 0$ for x_i **then**
        run Solver to minimize x_i
    end if
    if Allowable Increase $= 0$ **then**
        run Solver to maximize x_i
    end if
end for

Output: Additional non-degenerate LP solutions via Solver (if they exist).

Cell	Objective Name	Value
\$C\$5	Profit	2160

Cell	Variable Name	Value	Lower Limit	Objective Result	Upper Limit	Objective Result
\$C\$2	Cabin Model	24	0	720	24	2160
\$C\$3	Frontier Model	8	0	1440	8	2160

FIGURE 7.8
The limits report for Lincoln Outdoors.

Note that in Algorithm 7.2.1, we minimize the decision variable if its Allowable Decrease $= 0$ due to the fact that there is an Allowable Increase and a move in that direction would not change the current optimal solution (hence an "*allowable*" increase). Likewise, if the Allowable Increase $= 0$ we would not want to minimize as there is a positive allowable decrease that would not change the stated solution.

In the Lincoln Outdoors example, this would mean we add $2160 = 60x_1 + 90x_2$ to the constraints. Then if we choose to work with the decision variable representing the number of Cabin Models, x_1, we would make the objective function $Max\ x_1$; or, if we rather choose to focus on the number of Frontier Models, we would make the objective function $Min\ x_2$. The models to explore would then be

$$\text{Maximize:} \quad f(x_1, x_2) = x_1 \tag{7.5}$$
$$\text{Subject to: } 60x_1 + 90x_2 = 2160 \tag{7.6}$$
$$x_1 + 2x_2 = 40 \tag{7.7}$$
$$2x_1 + 3x_2 = 72 \tag{7.8}$$
$$x_1, x_2 \geq 0 \tag{7.9}$$

and

$$\text{Minimize:} \quad f(x_1, x_2) = x_2 \tag{7.10}$$
$$\text{Subject to: } 60x_1 + 90x_2 = 2160 \tag{7.11}$$
$$x_1 + 2x_2 = 40 \tag{7.12}$$
$$2x_1 + 3x_2 = 72 \tag{7.13}$$
$$x_1, x_2 \geq 0. \tag{7.14}$$

Next we turn to the Constraints section of the Sensitivity Report. The Final Value is the value of the constraint when it is evaluated at the point given in the solution and the Constraint R.H. Side is the constant on the right of the constraint. Note that in the Lincoln Outdoors example these columns agree for both constraints. This is exactly because both constraints are binding and there are no labor-hours available for either at the production level given in the solution.

The Shadow Price returns the marginal value the problem constraints (usually resources). Specifically, this value is the amount that the objective function will change with a unit increase in the constant on the right hand side of the constraint; a positive shadow price means an increase in the objective function's optimal value where a negative shadow price means the optimal value of the objective function would decrease. For example, the shadow price of the assembly constraint in the example is 30; this means that if we make one more labor-hour available in the assembly department, the objective function will increase by \$30 (note *this is a theoretical return as 1 more assembly hour does not return a full sleeping bag and we will only be selling an integer amount of sleeping bags*). The shadow price of 0 for the cutting constraint means a unit increase in the labor-hours available for cutting will result in no change in the optimal value of the objective function. The Allowable Increase and Allowable Decrease columns that follow provide a range of the changes in the constraint's bound, i.e. the constant on the right hand side of the constraint, for which these shadow prices hold. For example, in the Lincoln Outdoors solution any amount of cutting room labor hours from 36 to 48 would still have a shadow price (marginal value) of 0.

Now that we understand Shadow Price, we can address Reduced Cost. The Reduced Cost of a variable is calculated by

$$\begin{aligned}\textsf{Reduced Cost} &= \textsf{coefficient of variable in objective function}\\ &\quad - \textsf{value per unit of resources used}\end{aligned} \tag{7.15}$$

where *the resources are valued at their shadow price.*

For example, the reduced cost of the Cabin Model for Lincoln Outdoors is (all values are *per unit*)

$$\begin{aligned}&\text{contribution to objective function} - \text{cutting hours} \cdot \text{shadow price}\\ &\quad - \text{assembly hours} \cdot \text{shadow price}\\ &\qquad = 60 - 1 \cdot 0 - 2 \cdot 30 = 0.\end{aligned}$$

As well, the reduced cost of the Frontier Model is

$$90 - 2 \cdot 0 - 3 \cdot 30 = 0.$$

It is not a coincidence that both of these values are 0, as the only time Reduced Cost of a decision variable is non-zero is if the variable is at either its lower or upper bound of the feasible region. For example, based on the cutting and assembly constraints we will only be able to produce between 0 and 36 Cabin Model sleeping bags. The solution $x_1 = 24$ is easily within this range.

7.3 Exercises

Exercise 7.1. *Solve the linear programming problem in Exercise 6.1 using Solver. Provide the Answer Report and Limits Report generated by Solver. Explain all the details given in these reports.*

Exercise 7.2. *Explain why Highlight 7.2.1 is true for non-degenerate Linear Programming problems with multiple solutions.*