Part 2:

$$A = \begin{bmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{bmatrix}$$

To find eigen values.

$\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 8 & -1 & -2 \\ -2 & -9-\lambda & -2 & -4 \\ 0 & 10 & -5-\lambda & -10 \\ 1 & -13 & -14 & -13-\lambda \end{bmatrix}$$

let's find the eigen values
— by echelon form, ~~cofactors~~

substract column 2 multiplied by $\frac{1}{2} - \frac{\lambda}{10}$ from column 3

$$C_3 = C_3 - \left(\frac{1}{2} - \frac{\lambda}{10}\right) C_2$$

$$\begin{vmatrix} 4-\lambda & 8 & -1 & -2 \\ -2 & -\lambda-9 & -2 & -4 \\ 0 & 10 & 5-\lambda & -10 \\ -1 & 8 & -14 & -\lambda-13 \end{vmatrix} = \begin{vmatrix} 4-\lambda & 8 & \frac{4\lambda-5}{5} & -2 \\ -2 & -\lambda-9 & -\frac{\lambda^2}{10} - \frac{2\lambda}{5} + \frac{5}{2} & -4 \\ 0 & 10 & 0 & -10 \\ -1 & -13 & -\frac{13\lambda}{10} - \frac{15}{2} & -\lambda-13 \end{vmatrix}$$

$$C_4 = C_4 + C_2$$

$$\begin{vmatrix} 4-\lambda & 8 & \frac{4\lambda-5}{5} & -2 \\ -2 & -\lambda-9 & -\frac{\lambda^2}{10} - \frac{2\lambda}{5} + \frac{5}{2} & -4 \\ 0 & 10 & 0 & -10 \\ -1 & -13 & -\frac{13\lambda}{10} - \frac{15}{2} & -\lambda-13 \end{vmatrix} = \begin{vmatrix} 4-\lambda & 8 & \frac{4\lambda-5}{5} & 6 \\ -2 & -\lambda-9 & -\frac{\lambda^2}{10} - \frac{2\lambda}{5} + \frac{5}{2} & -\lambda-13 \\ 0 & 10 & 0 & 0 \\ -1 & -13 & -\frac{13\lambda}{10} - \frac{15}{2} & -\lambda-26 \end{vmatrix}$$

expand along row 3

$$\begin{vmatrix} 4-\lambda & 8 & \frac{4\lambda-5}{5} & 6 \\ -2 & -\lambda-9 & -\frac{\lambda^2}{10}-\frac{2\lambda}{5}+\frac{5}{2} & -\lambda-13 \\ 0 & 10 & 0 & 0 \\ -1 & -13 & -\frac{13\lambda}{10}-\frac{15}{2} & -\lambda-26 \end{vmatrix} = (0)(-1)^{3+1}\begin{vmatrix} 8 & \frac{4\lambda-5}{5} & 6 \\ -\lambda-9 & -\frac{\lambda^2}{10}-\frac{2\lambda}{5}+\frac{5}{2} & -\lambda-13 \\ -13 & -\frac{13\lambda}{10}-\frac{15}{2} & -\lambda-26 \end{vmatrix}$$

$$+(10)(-1)^{3+2}\begin{vmatrix} 4-\lambda & \frac{4\lambda-5}{5} & 6 \\ -2 & -\frac{\lambda^2}{10}-\frac{2\lambda}{5}+\frac{5}{2} & -\lambda-13 \\ -1 & -\frac{13\lambda}{10}-\frac{15}{2} & -\lambda-26 \end{vmatrix} + (0)(-1)^{3+3}\begin{vmatrix} 4-\lambda & 8 & 6 \\ -2 & -\lambda-9 & -\lambda-13 \\ -1 & -13 & -\lambda-26 \end{vmatrix}$$

$$+(0)(-1)^{3+4}\begin{vmatrix} 4-\lambda & 8 & \frac{4\lambda-5}{5} \\ -2 & -\lambda-9 & -\frac{\lambda^2}{10}-\frac{2\lambda}{5}+\frac{5}{2} \\ -1 & -13 & -\frac{13\lambda}{10}-\frac{15}{2} \end{vmatrix} = -10\begin{vmatrix} 4-\lambda & \frac{4\lambda-5}{5} & 6 \\ -2 & -\frac{\lambda^2}{10}-\frac{2\lambda}{5}+\frac{5}{2} & -\lambda-13 \\ -1 & -\frac{13\lambda}{10}-\frac{15}{2} & -\lambda-26 \end{vmatrix}$$

Subtract column 3 multiplied by $\frac{2}{3}-\frac{\lambda}{6}$ from column 1

$$C_1 = C_1 - \left(\frac{2}{3}-\frac{\lambda}{6}\right)C_3$$

$$\begin{vmatrix} 4-\lambda & \frac{4\lambda-5}{5} & 6 \\ -2 & -\frac{\lambda^2}{10}-\frac{2\lambda}{5}+\frac{5}{2} & -\lambda-13 \\ -1 & -\frac{13\lambda}{10}-\frac{15}{2} & -\lambda-26 \end{vmatrix} = \begin{vmatrix} 0 & \frac{4\lambda-5}{5} & 6 \\ -\frac{\lambda^2}{6}-\frac{3\lambda}{2}+\frac{20}{3} & -\frac{\lambda^2}{10}-\frac{2\lambda}{5}+\frac{5}{2} & -\lambda-13 \\ -\frac{\lambda^2}{6}-\frac{11\lambda}{3}+\frac{49}{3} & -\frac{13\lambda}{10}-\frac{15}{2} & -\lambda-26 \end{vmatrix}$$

$$C_2 = C_2 - \left(\frac{2\lambda}{15}-\frac{5}{6}\right)C_3$$

$$\begin{vmatrix} 0 & \frac{4\lambda-5}{5} & 6 \\ -\frac{\lambda^2}{6}-\frac{3\lambda}{2}+\frac{20}{3} & -\frac{\lambda^2}{10}-\frac{2\lambda}{5}+\frac{5}{2} & -\lambda-13 \\ -\frac{\lambda^2}{6}-\frac{11\lambda}{3}+\frac{49}{3} & -\frac{13\lambda}{10}-\frac{15}{2} & -\lambda-26 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 6 \\ -\frac{\lambda^2}{6}-\frac{3\lambda}{2}+\frac{20}{3} & \frac{(\lambda-10)(\lambda+25)}{30} & -\lambda-13 \\ -\frac{\lambda^2}{6}-\frac{11\lambda}{3}+\frac{49}{3} & \frac{2\lambda^2}{15}+\frac{4\lambda}{3}-\frac{175}{6} & -\lambda-26 \end{vmatrix}$$

Expand along row 1

$$\begin{vmatrix} 0 & 0 & 6 \\ -\frac{\lambda^2}{6}-\frac{3\lambda}{2}+\frac{20}{3} & \frac{(\lambda-10)(\lambda+25)}{30} & -\lambda-13 \\ -\frac{\lambda^2}{6}-\frac{11\lambda}{3}+\frac{49}{3} & \frac{2\lambda^2}{15}+\frac{4\lambda}{3}-\frac{175}{6} & -\lambda-26 \end{vmatrix} = (0)(-1)^{1+1}\begin{vmatrix} \frac{(\lambda-10)(\lambda+25)}{30} & -\lambda-13 \\ \frac{2\lambda^2}{15}+\frac{4\lambda}{3}-\frac{175}{6} & -\lambda-26 \end{vmatrix}$$

$$+(0)(-1)^{1+2}\begin{vmatrix} -\frac{\lambda^2}{6}-\frac{3\lambda}{2}+\frac{20}{3} & -\lambda-13 \\ -\frac{\lambda^2}{6}-\frac{11\lambda}{3}+\frac{49}{3} & -\lambda-26 \end{vmatrix} + (6)(-1)^{(1+3)}\begin{vmatrix} -\frac{\lambda^2}{6}-\frac{3\lambda}{2}+\frac{20}{3} & \frac{(\lambda-10)(\lambda+25)}{30} \\ -\frac{\lambda^2}{6}-\frac{11\lambda}{3}+\frac{49}{3} & \frac{2\lambda^2}{15}+\frac{4\lambda}{3}-\frac{175}{6} \end{vmatrix}$$

$$= 6\begin{vmatrix} -\frac{\lambda^2}{6}-\frac{3\lambda}{2}+\frac{20}{3} & \frac{(\lambda-10)(\lambda+25)}{30} \\ -\frac{\lambda^2}{6}-\frac{11\lambda}{3}+\frac{49}{3} & \frac{2\lambda^2}{15}+\frac{4\lambda}{3}-\frac{175}{6} \end{vmatrix}$$

determinant of 2x2 matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$$

$$\begin{vmatrix} -\frac{\lambda^2}{6}-\frac{3\lambda}{2}+\frac{20}{3} & \frac{(\lambda-10)(\lambda+25)}{30} \\ -\frac{\lambda^2}{6}-\frac{11\lambda}{3}+\frac{49}{3} & \frac{2\lambda^2}{15}+\frac{4\lambda}{3}-\frac{175}{6} \end{vmatrix} = \left(-\frac{\lambda^2}{6}-\frac{3\lambda}{2}+\frac{20}{3}\right)\cdot\left(\frac{2\lambda^2}{15}+\frac{4\lambda}{3}-\frac{175}{6}\right) -$$

$$\left(\frac{(\lambda-10)(\lambda+25)}{30}\right)\cdot\left(-\frac{\lambda^2}{6}-\frac{11\lambda}{3}+\frac{49}{3}\right) = -\frac{\lambda^4}{60}-13\lambda^3+\frac{\cancel{73}}{20}\lambda^2+\frac{167\lambda}{12}-\frac{175}{3}$$

$$\therefore (-10)\cdot(6)\cdot\left(-\frac{\lambda^4}{60}-\frac{13\lambda^3}{60}+\frac{73\lambda^2}{20}+\frac{167\lambda}{12}-\frac{175}{3}\right) = \lambda^4+13\lambda^3-219\lambda^2-835\lambda+$$

$$\lambda^4+13\lambda^3-219\lambda^2-835\lambda+3500 = \begin{vmatrix} 4-\lambda & 8 & -1 & -2 \\ -2 & -\lambda-9 & -2 & -4 \\ 0 & 10 & 5-\lambda & -10 \\ -1 & -13 & -14 & -\lambda-13 \end{vmatrix}$$

To find λ: solve $\lambda^4+13\lambda^3-219\lambda^2-835\lambda+3500=0$

Root

$$-\frac{13}{4}-\sqrt{\frac{\frac{753}{4}+\frac{6807}{\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}}+2\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}}{2}}+$$

$$\frac{\sqrt{\frac{753}{2}-2\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}+\frac{6905}{4\sqrt{\frac{753}{4}+\frac{6807}{\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}}+2\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}}}-\frac{6807}{\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}}}}{2}$$

$x_1 = -5.604$

Root: $-\frac{13}{4}-\sqrt{\frac{\frac{753}{4}+\frac{6807}{\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}}+2\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}}{2}}-$

$$\frac{\sqrt{\frac{753}{2}-2\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}+\frac{6905}{4\sqrt{\frac{753}{4}+\frac{6807}{\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}}+2\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}}}-\frac{6807}{\sqrt[3]{\frac{440569-5\sqrt{17468281445i}}{4}}}}}{2}$$

$x_2 = -21.1246$

$$-\frac{13}{4}+\sqrt{\frac{\frac{753}{4}+\frac{6807}{\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}}}{}}-\frac{6905}{4\sqrt{\frac{753}{4}+\frac{6807}{\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}}+2\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}}}-\frac{6807}{\sqrt[3]{\frac{440569+5\sqrt{17468281445i}}{4}}}\;\frac{6807}{\frac{440569}{4}-5\sqrt{17468281445i}}$$

$$\frac{\frac{753}{2}+2\sqrt{\frac{6807}{\frac{440569}{4}+5\sqrt{\frac{17468281445i}{4}}}}^{2}+2\sqrt[3]{\frac{440569}{4}+5\sqrt{17468281445i}}}{2}$$

$x_3 = 11.054$

Root: $-\frac{13}{4} - \frac{\sqrt{\frac{753}{2} - 2\sqrt[3]{\frac{440569}{4} + 5\sqrt{\frac{1746628144451i}{4}}} - \frac{6905}{4\sqrt{\frac{753}{4} + \frac{6807}{\sqrt[3]{\frac{440569}{4} + 5\sqrt{\frac{1746628144451i}{4}}}} + 2\sqrt[3]{\frac{440569}{4} + 5\sqrt{\frac{1746828144451i}{4}}}}}}}{2}$

$$+\frac{\sqrt{\frac{753}{4} + \frac{6807}{\sqrt[3]{\frac{440569}{4} + 5\sqrt{\frac{1746828144451i}{4}}}} + 2\sqrt[3]{\frac{440569}{4} + 5\sqrt{\frac{1746828144451i}{4}}}}}{2}$$

$\lambda_4 = 2.674$

Therefore eigen values are

$\lambda_1 = -5.604, \lambda_2 = -21.125, \lambda_3 = 11.054, \lambda_4 = 2.675$

Eigen vectors

$$(A-\lambda_1 I)\vec{\gamma}_1 = 0$$

$\lambda_1 = 11.054$

$$\begin{bmatrix} 4-11.054 & 8 & -1 \\ -2 & -9-11.054 & -2 \\ 0 & 10 & 5-11.054 \\ -1 & -13 & -14 \end{bmatrix} \cdot \begin{bmatrix} \gamma_a \\ \gamma_b \\ \gamma_c \\ \gamma_d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{\gamma}_1 = \begin{bmatrix} -0.036 \\ -0.012 \\ -0.860 \\ 0.509 \end{bmatrix}$$

Eigen vector when $\lambda_1 = 11.054$

$$\vec{\gamma}_1 = \begin{bmatrix} -0.036 \\ -0.012 \\ -0.860 \\ 0.509 \end{bmatrix} \Rightarrow \begin{bmatrix} -0.036 \\ -0.012 \\ -0.860 \\ -0.509 \end{bmatrix}$$

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Eigen vector

when $\lambda_1 = 2.675$

$(A - \lambda_2 I)\vec{v_2} = 0$

$$\begin{bmatrix} 4-2.675 & 8 & -1 & -2 \\ -2 & -9-2.675 & -2 & -4 \\ 0 & 10 & 5-2.675 & -10 \\ -1 & -13 & -14 & -13-2.675 \end{bmatrix} \begin{pmatrix} v_a \\ v_b \\ v_c \\ v_d \end{pmatrix} = 0$$

$$= \begin{pmatrix} 0.958 \\ -0.161 \\ 0.207 \\ -0.113 \end{pmatrix} \qquad \vec{v_2} = \begin{pmatrix} 0.958 \\ -0.161 \\ 0.207 \\ 0.113 \end{pmatrix}$$

Eigen vector

$$\lambda_3 = -21.125$$

$$(A - \lambda_3 I)\vec{v}_3 = 0$$

$$\begin{bmatrix} 4+21.125 & 8 & -1 & -2 \\ -2 & -9+21.125 & -2 & -4 \\ 0 & 10 & 5+21.125 & -10 \\ -1 & -13 & -14 & -13+21.125 \end{bmatrix} \begin{pmatrix} v_a \\ v_b \\ v_c \\ v_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.025 \\ -0.335 \\ -0.222 \\ -0.915 \end{pmatrix} = \begin{pmatrix} 0.025 \\ -0.335 \\ -0.222 \\ -0.915 \end{pmatrix}$$

Eigen vector

$\lambda_4 = -5.604$

$(A - \lambda_4 I)\vec{V}_4 = 0$

$$\begin{bmatrix} 4+5.604 & 8 & -1 & -2 \\ -2 & -9+5.604 & -2 & -4 \\ 0 & 10 & 5+5.604 & -10 \\ -1 & -13 & -14 & -13+5.604 \end{bmatrix} \begin{pmatrix} v_a \\ v_b \\ v_c \\ v_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{V}_4 = \begin{pmatrix} 0.563 \\ -0.616 \\ 0.549 \\ 0.033 \end{pmatrix}$$

As the eigen values Indicates how much variance each principal component captures

$\lambda_1 = -5.604$, $\lambda_2 = -21.125$, $\lambda_3 = 11.054$, $\lambda_4 = 2.675$

Absolute value $= |\lambda_1| = 5.604$, $|\lambda_2| = 21.125$

$|\lambda_3| = 11.054$ $|\lambda_4| = 2.675$

otal sum of absolute eigen values

$5.604 + 21.125 + 11.054 + 2.675 = 40.458$

portance of $\lambda = \left(\frac{|\lambda|}{\text{total}}\right) \times 100$

or

$$\left(\frac{|\lambda|}{\sum_{j=1}^{n} \lambda_j}\right) \times 100$$

for $\lambda_1 = -5.604$

$$= \frac{5.604}{40.458} \times 100 = 13.85\%$$

for $\lambda_2 = -21.125 = \frac{21.125}{40.458} \times 100 = 52.21\%$

for $\lambda_3 = 11.054 = \frac{11.054}{40.458} \times 100 = 27.32\%$

for $\lambda_4 = 2.675 = \frac{2.675}{40.458} \times 100 = 6.61\%$

Eigen values.	Absolute value	Importance (%)
-5.604	5.604.	13.5 %
-21.125	21.125	52.21 %
11.054	11.054	27.32 %
2.675	2.675	6.61 %