Pricing of Return-Adjusted POV Shares Disposal Contract

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Abstract
In this article, the pricing of a deposition contract associated with volume-weighted average price disposal strategies will be evaluated.

Keywords: Shares disposal, Execution algorithm, Optimal execution

1. Introduction

When investors plan to unwind a large number of shares to the market, this act will create a huge market impact that drags the share price down, the loss in the selldown incurred due to the market impact is called the slippage. To minimize the slippage, investors will enter a structured or block trade with a bank, to all their shares at a pre-determined strike price. Products like reserve accelerated shares repurchase, hereafter reverse ASR, and decumulator, are designed to serve this purpose. However, investors are capped with a limited upside or crossed the underlying shares at a discount when entering these structured trades. The other resort that investors can leverage from the bank is to use disposal strategies embedded in the execution algorithm of the trading platform that the bank provides. A variety of disposition strategies, like the percentage of volume (POV), the volume-weighted average price (VWAP), and the time-weighted average price (TWAP), are designed for clients to choose from. These strategies perform slightly different from each other but all of them aim to minimize the slippage of the deposition. The VWAP and TWAP at discount are used as the reference price to cross the block trade in which still incurs to a certain extent of loss to clients. Therefore, a disposal product that provides potential upside to the client while carrying out a deposition will be attractive to investors. The POV strategy which adds in the consideration of market-to-market (MTM) stock return will be a desirable strategy to achieve this goal.

Given a pre-determined period of time, the bank will sell shares on behalf of the client to the market. Shares to dispose of per day can be calculated as the total number of shares to dispose of divided by the number of periods within the agreed timeframe.

2. Disposal Strategy

The two important features in a share deposition are the ability to fully unwind the shares and to minimize the slippage cost incurred. This section will discuss and compare various strategies that could be used in the product.

2.1. Percentage Of Volume (POV)

POV is a simple execution strategy that executes the order quantity as a percentage of the trade volume of the stock in a given time interval. Shares are disposed to the market at discrete timestamps from period t_0 to period t_N. At timestamp t, n_t shares are sold at a price F_t and there will be N periods in total. In a simple POV model, n_t is calculated by the following formula,

$$n_t = APTV_t * POV$$

where

$APTV_t$ = Average Period Volume at time t

POV = Pre-determined Percentage Of Volume for disposal

However, the total number of period N cannot be fixed at t_0 as the number of shares sold in each period is not the same. Period N will be the timestamp that remaining shares for disposition R_t equal 0,

$$R_t = D - \sum_{i=0}^{t} n_t$$

$$t_N = \min(\{t \mid R_t = 0\}, t_{mat})$$

where

D = The total number of shares to be disposed

The POV strategy for shares disposal can be treated as a basket of forward contracts with maturity at every timestamp up to period t_N. Still, as clients using the shares disposal contract using the POV strategy are exposed to both upside award and downside risk of the underlying stock price, the clients should not be charged any premium. Assuming the underlying stock's price follows a log-normal distribution, the probability for the underlying stock price to go above or below the strike price K (decided by clients, can be either the average cost or the initial spot when entering the shares disposal contract) are approximately the same.

2.2. Return-adjusted Percentage Of Volume (RPOV)

2.2.1. Vanilla Return-adjusted Percentage Of Volume

By taking stock return into account in the POV strategy,

$$n_t = APTV_t * POV * \mathbb{1}_{\{S_t > K\}}$$

Shares will only be sold when stock price at period t is higher than the strike price.

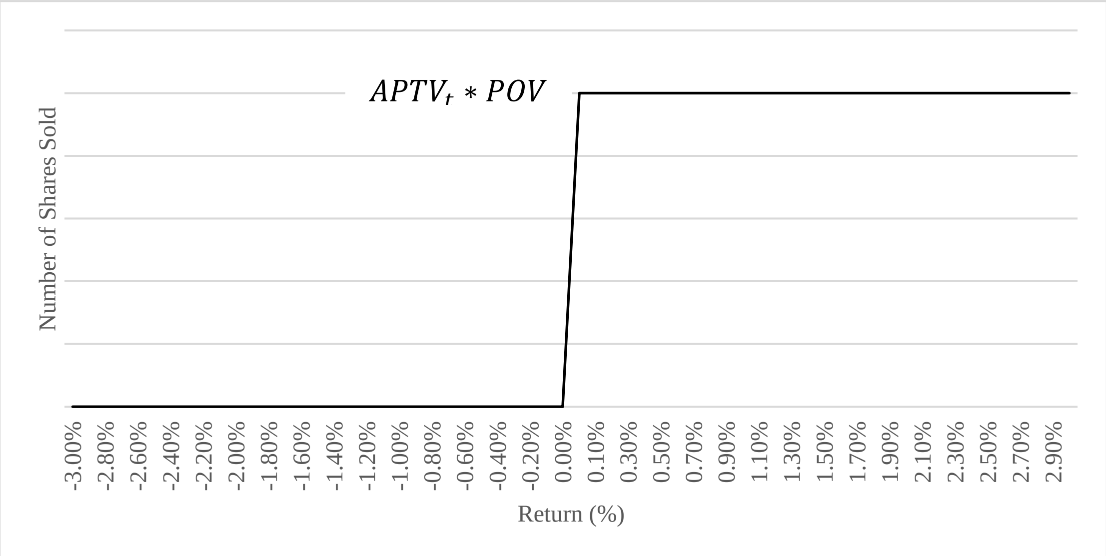


Figure 1. RPOV strategy

The RPOV can be simply described as the POV strategy with activation function on stock return at timestamp t. This return adjustment allows the POV strategy to keep the upside potential and ditch the downside risk in the disposition contract. Implying that the pricing of the disposal strategy is then changed from pricing of a basket of futures to a basket of put option.

Under risk neutral measure Q, at filtration time t :

$$V_0 = POV * E^Q\left[\sum_{i=0}^{N} APTV_t * \mathbb{1}_{\{S_t > K\}} \mid \mathcal{F}_t\right]$$

However, the main drawback is that period t_N is an unknown at the trade date of the contract time t_0. The pricing of this product with unknown maturity would be challenging and no close form solution can be deduced. Another issue arise on the return adjustment is that there will be possibility that there are remaining shares at the end of disposal period t_{mat}. Thus, the disposal contract should consist of a forward contract at maturity to unwind the remaining shares to the bank. To keep the pricing simple, the assumption is made on both options and forward contract sharing the same strike price K.

2.2.2. Logistic Function Return-adjusted Percentage Of Volume (LFRPOV)

Instead of selling a fixed percentage of market average volume, it is more reasonable to sell more shares when the stock return is higher at timestamp t. Still, the maximum number of shares allowed to be sold at each timestamp should be capped with the average market volume limit to keep the POV strategy's feature on minimizing the market impact.

In the LFRPOV strategy, we first denote the average number of shares to be disposed per period as α calculated as the total number of shares to be sold divided by the number of periods between the start and end period:

$$\alpha = \frac{D}{Mat}$$

Then, the multiplier m_t for return adjustment on the number of shares sold at each timestamp t in the LFRPOV strategy is defined as:

$$m_t = \frac{\mathrm{e}^{TP_t} + 1}{e^{-(r_t - TP_t)} + 1} * \mathbb{1}_{\{S_t > K\}}$$

where

$$r_t = \frac{S_t}{K} - 1 \; or \; \ln\left(\frac{S_t}{K}\right)$$

$$TP_t = Turning\ Point\ of\ the\ Logistic\ Function\ at\ time\ t$$

And the actual number of shares sold in each timestamp is defined as:

$$n_t = m_t * \alpha$$

which gives the contract value V_0 equals:

$$V_0 = E^Q\left[\sum_{i=0}^{N} n_t \mid \mathcal{F}_t\right]$$

TP_t is the turning point of the logistic function to ensure the logistic function will be capped at the pre-defined APTV percentage (equivalent as the $APTV_t * POV$ in vanilla POV model).

Denote $\max\left(\frac{APTV_t * POV}{\alpha}, 2\right)$ as the maximum multiplier mm_t at timestamp t. The assumption on the maximum multiplier is that the $APTV_t * POV$ should be at least 2 times larger than the α, the lower bound 2 is set to ensure this assumption hold.

When the stock return tends to infinite, the multiplier m_t should be equal to mm_t such that the actual number of shares sold at timestamp t is less than the volume limit.

$$\lim_{x\to\infty} \frac{\mathrm{e}^{TP_t} + 1}{e^{-(x-TP_t)} + 1} = mm_t$$
$$\mathrm{TP_t} = \ln(mm_t - 1)$$

By checking the second derivatives of the function at $r_t = TP_t$,

$$\frac{\partial}{\partial x}\left(\frac{\mathrm{e}^{TP_t} + 1}{e^{-(x-TP_t)} + 1}\right) = \frac{(\mathrm{e}^{TP_t} + 1)\left(\mathrm{e}^{-(x-TP_t)}\right)}{(e^{-(x-TP_t)} + 1)^2}$$
$$\frac{\partial^2}{\partial x^2}\left(\frac{\mathrm{e}^{TP_t} + 1}{e^{-(x-TP_t)} + 1}\right) = \frac{(\mathrm{e}^{TP_t} + 1)(\mathrm{e}^{x+TP_t})(\mathrm{e}^{TP_t} - \mathrm{e}^{x})}{(e^{x} + e^{TP_t})^3}$$
$$\frac{\partial^2}{\partial x^2}\left(\frac{\mathrm{e}^{TP_t} + 1}{e^{-(x-TP_t)} + 1}\right)|_{x=TP_t} = \frac{(\mathrm{e}^{TP_t} + 1)(\mathrm{e}^{TP_t+TP_t})(\mathrm{e}^{TP_t} - \mathrm{e}^{TP_t})}{(e^{TP_t} + e^{TP_t})^3} = 0$$

e^{TP_t} is a straightly positive function as TP_t is a non-negative for all t. Based on this fact, $\frac{(\mathrm{e}^{TP_t}+1)(\mathrm{e}^{TP_t+TP_t})}{(e^{TP_t}+e^{TP_t})^3}$ will also be straightly positive, thus, the sign of second derivatives of mm_t will be solely affected the term $(\mathrm{e}^{TP_t} - \mathrm{e}^{x})$.

If $x = TP_t^-$,

$$\frac{\partial^2}{\partial x^2}\left(\frac{\mathrm{e}^{TP_t} + 1}{e^{-(x-TP_t)} + 1}\right)|_{x=TP_t^-} > 0, \qquad convex\ at\ x < TP_t$$

Else if $x = TP_t^+$,

$$\frac{\partial^2}{\partial x^2}\left(\frac{\mathrm{e}^{TP_t} + 1}{e^{-(x-TP_t)} + 1}\right)|_{x=TP_t^+} < 0, \qquad concave\ at\ x > TP_t$$

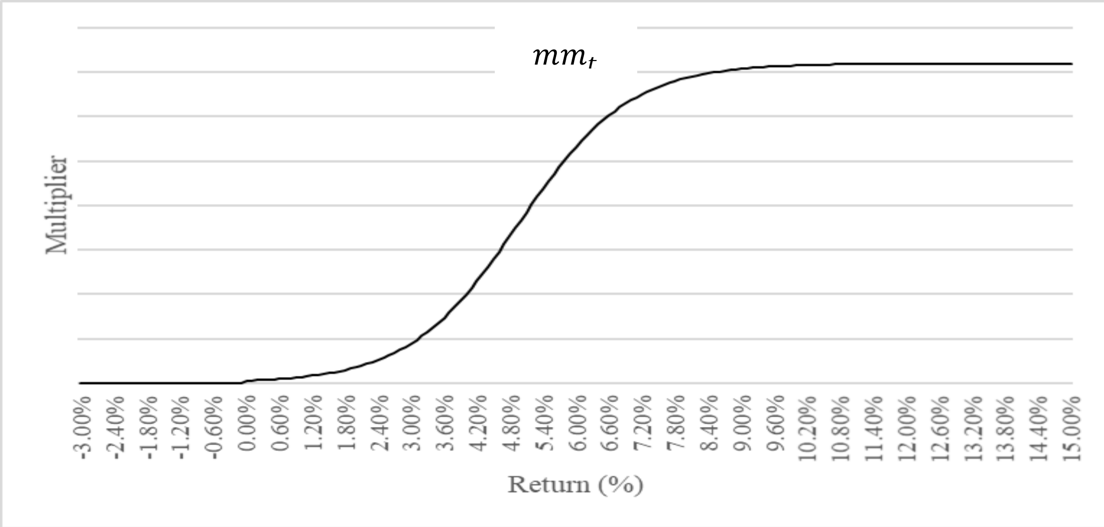


Figure 2. LFRPOV strategy

2.2.3. Hyperbolic Tangent Return-adjusted Percentage Of Volume (HTRPOV)

The multiplier m_t for HTRPOV is defined as:

$$m_t = \frac{\tanh(x - TP_t) + 1}{\tanh(-TP_t) + 1} * \mathbb{I}_{\{S_t > K\}}$$

Similarly, number of shares sold at each timestamp t will be defined as:

$$n_t = m_t * \alpha$$

where

$$\alpha = \frac{D}{Mat}$$

and contract value will be:

$$V_0 = E^Q \left[\sum_{i=0}^{N} n_t \mid \mathcal{F}_t \right]$$

TP_t for HTRPOV follows the same assumption and mm_t will be the same as LFRPOV strategy.

$$mm_t = \max\left(\frac{APTV_t * POV}{\alpha}, 2\right)$$

By setting m_t tends to mm_t when stock return tends to infinity,

$$\lim_{x\to\infty} \frac{\tanh(x - TP_t) + 1}{\tanh(-TP_t) + 1} = mm_t$$

$$\lim_{x\to\infty} \frac{\tanh(x) + \tanh(-TP_t)}{1 + \tanh(x)\tanh\,(-TP_t)} = ADTV \cdot \tanh(-TP_t) + (ADTV - 1)$$

$$ADTV \cdot \tanh^2(TP_t) - 2(ADTV - 1)\tanh(TP_t) + (ADTV - 2) = 0$$

$Denote$ $\tanh(TP_t)$ $be\ y$

$$ADTV \cdot\ y^2 - 2(ADTV - 1)\ \cdot y + (ADTV - 2) = 0$$

$$\mathrm{y} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = ADTV;\ b = -2(ADTV - 1);\ c = (ADTV - 2)$$

$$\sqrt{b^2 - 4ac} = \sqrt{\left(-2(ADTV - 1)\right)^2 - 4 * ADTV * (ADTV - 2)} = 2$$

$$y = \frac{2(ADTV - 1) \pm 2}{2ADTV} = \frac{(ADTV - 1) \pm 1}{ADTV}$$

$$\tanh(TP_t) = y = 1\ or\ \frac{ADTV - 2}{ADTV}$$

$$TP_t = \operatorname{arctanh}\left(\frac{ADTV - 2}{ADTV}\right)$$

By checking the second derivatives of the function at $r_t = TP_t$,

$$\frac{\partial}{\partial x}\left(\frac{\tanh(x - TP_t) + 1}{\tanh(-TP_t) + 1}\right) = \frac{\operatorname{sech}^2(x - TP_t)}{1 - \tanh(TP_t)}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\tanh(x - TP_t) + 1}{\tanh(-TP_t) + 1}\right) = \frac{1}{1 - \tanh(TP_t)}\left(-2\tanh(x - TP_t)\operatorname{sech}^2(x - TP_t)\right)$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\tanh(x - TP_t) + 1}{\tanh(-TP_t) + 1}\right) = -\frac{2}{1 - \tanh(TP_t)}\tanh(x - TP_t)\operatorname{sech}^2(x - TP_t)$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\tanh(x - TP_t) + 1}{\tanh(-TP_t) + 1}\right)|_{x=TP_t} = 0, \qquad \tanh(0) = 0$$

As TP_t is a non-negative variable, $\tanh(TP_t)$ is bounded between 0 and 1, $\frac{2}{1-\tanh(TP_t)}$ must be a positive term and so as the $\operatorname{sech}^2(x - TP_t)$ due to the square.

Therefore, the sign for the secondary derivative is solely impacted by the $\tanh(x - TP_t)$ term.

If $x = TP_t^-$,

$$-\tanh(x - TP_t) > 0$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\tanh(x - TP_t) + 1}{\tanh(-TP_t) + 1}\right)\Big|_{x=TP_t^-} > 0, \qquad convex\ at\ x < TP_t$$

Else if $x = TP_t^+$,

$$-\tanh(x - TP_t) < 0$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\tanh(x - TP_t) + 1}{\tanh(-TP_t) + 1}\right)\Big|_{x=TP_t^+} < 0, \qquad concave\ at\ x > TP_t$$

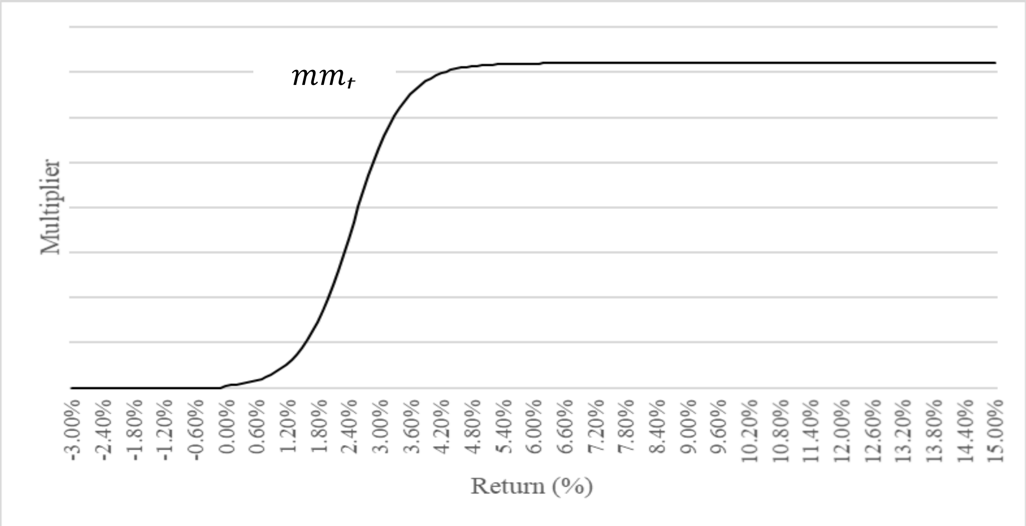


Figure 3. HTRPOV strategy

The idea for HTRPOV is same as the LFRPOV but the properties between the two functions will be slightly different. The main difference will be the stiffness of the multiplier function to reach the maximum multiplier mm_t. HTRPOV will converge to the mm_t at a more rapid speed meaning that more shares will be sold at a lower return compared to LFRPOV. HTRPOV tends to lock down small profits and has higher probability to fully unwind the shares without using the makeup forward contract whereas LFRPOV preserves more shares when the stock return is at a low level and these preserved shares can be sold when reaching a high return.

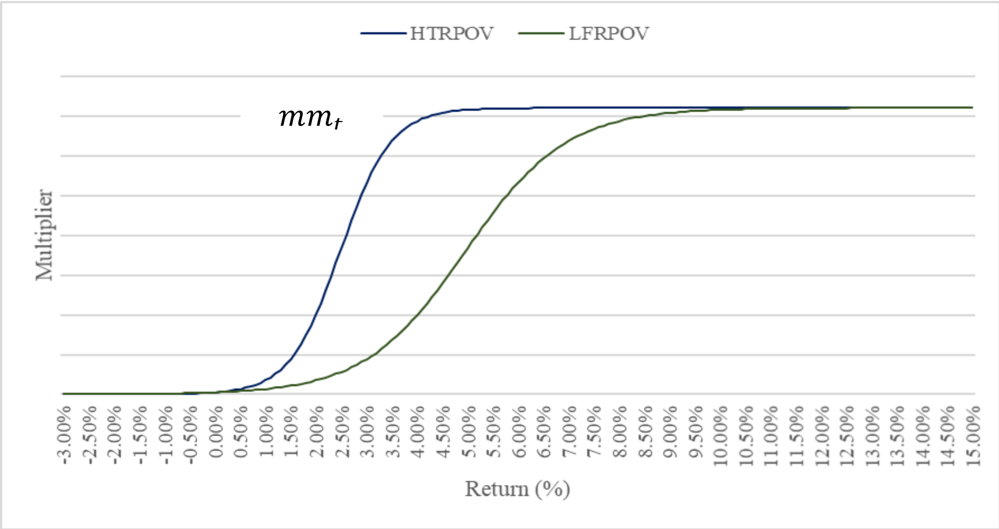


Figure 3. Comparison between the LFRPOV and HTRPOV strategy

2.2.4. Additional features

2.2.4.1. Excess return to sold

An addition feature in the disposal strategies is the targeted excess return to sell μ it can be interpreted as the strike shift of the options.

For Vanilla RPOV,

$$n_t = APTV_t * POV * \mathbb{1}_{\{S_t > K*(1+\mu)\}}$$

For LFRPOV,

$$m_t = \frac{e^{TP_t} + 1}{e^{-(r_t - TP_t - \mu)} + 1} * \mathbb{1}_{\{S_t > K*(1+\mu)\}}$$

For HTRPOV,

$$m_t = \frac{\tanh(x - TP_t - \mu) + 1}{\tanh(-TP_t) + 1} * \mathbb{1}_{\{S_t > K*(1+\mu)\}}$$

<u>3. Pricing and Hedging</u>

<u>(will continue working on this in Christmas holiday and Lunar New Year)</u>

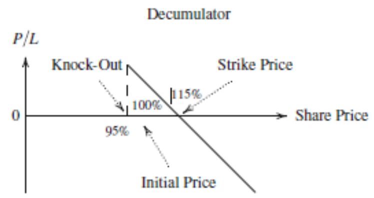
4. Conclusion

(will continue working on this in Christmas holiday and Lunar New Year)

5. Appendix

Decumulator

The opposite trade to Accumulator is the so-called Decumulator which consists of selling shares at a premium to the initial share price. It is also known as Reverse Knock-Out Forward or SAM (Sell Above Market). Similar to the Accumulator, the investor will stop selling shares if the share price trades below a pre-defined Knock-Out barrier level on the downside.



Accelerated share repurchase (ASR)

It refers to a method that publicly traded companies may use to buy back shares of its stock from the market.

The ASR method involves the company buying its shares from an investment bank (who in turn borrowed them from their clients), and paying cash to the investment bank while entering into a forward contract. The investment bank will then seek to purchase shares of the company from the market to return to its clients. At the end of the transaction, the company may receive even more shares than it initially received, which are then retired. This method can be contrasted with a typical open market repurchase, where the company simply announces that it is repurchasing shares on the market, and then does so.

A firm might choose the ASR method as a way of reducing the number of shares outstanding for a fixed cost, transferring the risk to the investment bank (which is now short the stock) for a negotiated premium.

By purchasing the shares in this way, it immediately exchanges a fixed amount of money for shares of its stock. It is currently being theorized that such arrangements are used by management to manipulate earnings figures for incentive compensation and reporting reasons. There are also earnings reporting black-out periods during which companies are not allowed to announce repurchasing of stock, so the ASR product from an investment bank could allow a company to essentially buy back shares during a blackout.

6. Reference

Guo, Y. Z. Y., Lee, S. T., & Li, X. (2019). Popular Structured Products. In *Financial mathematics, derivatives and structured products* (pp. 309–350). essay, Springer.

Percentage of volume. Algoji. (n.d.). Retrieved December 12, 2021, from https://algoji.com/percentage-of-volume/