# Basic Linear Algebra Refresher

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## Vector

**Dot product** is the sum of the product of two vectors' pairwise components:

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$$

**Angle** between two vectors

$$cos\, \theta = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{||\boldsymbol{x}||||\boldsymbol{y}||}$$

**Length** of a vector: $||\boldsymbol{x}|| = \sqrt{\boldsymbol{x}\boldsymbol{x}}$

**Orthogonal vectors** if their dot product is zero.
i.e. the angle $\theta = 90° \Leftrightarrow cos\, \theta = 0$.

**Projection** of vector $\boldsymbol{x}$ onto vector $\boldsymbol{y}$ results the vector

$$(\boldsymbol{x} \cdot \boldsymbol{y}) \frac{\boldsymbol{y}}{||\boldsymbol{y}||}$$

## Linear Transformations

### Scalar and vectors

$$T(k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \ldots + k_r\boldsymbol{v}_r) = k_1 T(\boldsymbol{v}_1) + k_2 T(\boldsymbol{v}_2) + \ldots + k_r T(\boldsymbol{v}_r)$$

### Matrix tmultiplication $\boldsymbol{A} = \boldsymbol{BC}$

$$a_{ij} = \sum_{k=1}^{N} b_{ik} c_{kj}$$

# Matrix

## Transpose

- $\boldsymbol{A}^T_{i,j} = \boldsymbol{A}_{j,i}$
- $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$
- $(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$
- $(\boldsymbol{A}\boldsymbol{B})^T = \boldsymbol{B}^T \boldsymbol{A}^T$
- If $\boldsymbol{A}$ has only real entries, $\boldsymbol{A}^T\boldsymbol{A}$ is positive semi-definite.

**Symmetric**: $\boldsymbol{A}^T = \boldsymbol{A}$
**Orthogonal**: $\boldsymbol{A}^T = \boldsymbol{A}^{-1}$

# Square Matrix Properties

## Determinant

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow det(\boldsymbol{A}) = ad - bc$$

$$\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$det(\boldsymbol{A}) = a \times det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \times det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \times det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

### Properties

(1) If $\boldsymbol{B}$ is obtained by multiplying any row or column in $\boldsymbol{A}$ by a scalar $\lambda$, $det(\boldsymbol{B}) = \lambda det(\boldsymbol{A})$
(2) If $\boldsymbol{B}$ is obtained by interchanging two rows or columns in $\boldsymbol{A}$, $det(\boldsymbol{B}) = det(\boldsymbol{A})$
(3) $det(\boldsymbol{A}^T) = det(\boldsymbol{A})$
(4) $\boldsymbol{B}_{n\times n}$ and $\boldsymbol{A}_{n\times n} \Rightarrow det(\boldsymbol{A}\boldsymbol{B}) = det(\boldsymbol{A})det(\boldsymbol{B})$

## Inverse

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \boldsymbol{A}^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{det(\boldsymbol{A})}$$

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

If $\boldsymbol{A}_{(m \times n)}$ and $\boldsymbol{B}_{(m \times n)}$ are both invertible:

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

If $\boldsymbol{A}$ is invertible:

- $\boldsymbol{A}^{-1} = \frac{adj(\boldsymbol{A})}{det(\boldsymbol{A})}$
- $det(\boldsymbol{A}^{-1}) = \frac{1}{det(\boldsymbol{A})}$

**Invertability**

$$det(\boldsymbol{A}) = \begin{cases} 0, & \text{singular} \Rightarrow \text{not invertible} \\ \text{else}, & \text{nonsingular} \Rightarrow \text{invertible} \end{cases}$$

## Quadratic Forms and Positive Definitive Matrix

**Quadratic Form** of real, symmetric matrix $\boldsymbol{A}_{n \times n}$: $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$

| Quadratic form $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ | Matrix |
|---|---|
| $>0$ for all $\boldsymbol{x} \neq 0$ | positive definite |
| $\geq 0$ | positive semi-definite |
| $<0$ | negative definite |
| $\neq 0$ | negative semi-definite |
| $>0$ for some $\boldsymbol{x}$ and $< 0$ for others | indefinite matrix |

## Necessary and sufficient conditions for $\boldsymbol{A}$ to be positive definite

- All eigenvalues of $\boldsymbol{A}$ are positive, or
- determinant of every principal submatrix is positive.

# References


[1] https://www.cs.utexas.edu/~dana/MLClass/

[2] https://www.cs.utexas.edu/~dana/MLClass/MT2013.pdf

[3] https://www.cs.utexas.edu/~dana/MLClass/MT2012.pdf

[4] https://www.cs.utexas.edu/~dana/MLClass/practice-midterm-1.pdf

[5] https://www.cs.utexas.edu/~dana/MLClass/practice-midterm-2.pdf