Priority Queues: Binary Heaps

Alexander S. Kulikov

Steklov Institute of Mathematics at St. Petersburg
Russian Academy of Sciences

Data Structures Fundamentals
Algorithms and Data Structures

Outline

Definition

Binary max-heap is a binary tree (each node has zero, one, or two children) where the value of each node is at least the values of its children.

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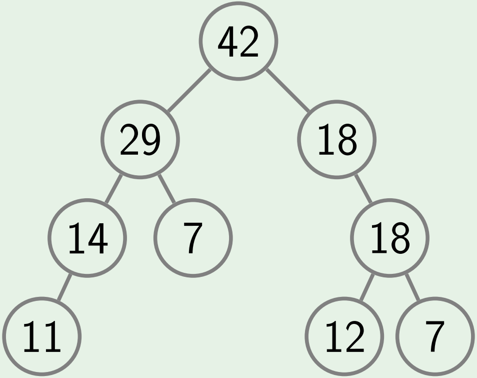
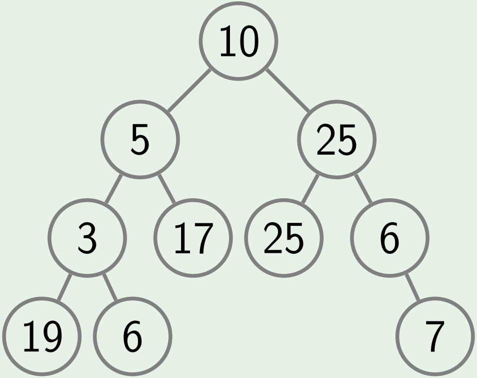
In other words

For each edge of the tree, the value of the parent is at least the value of the child.

Example: heap



Example: not a heap



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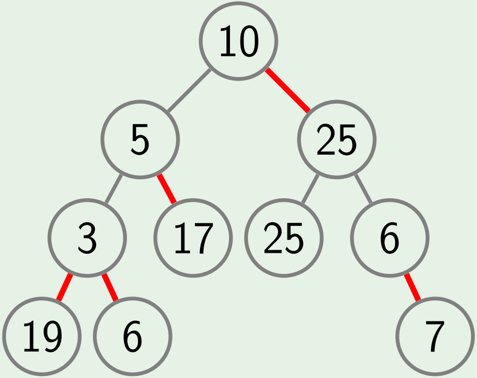
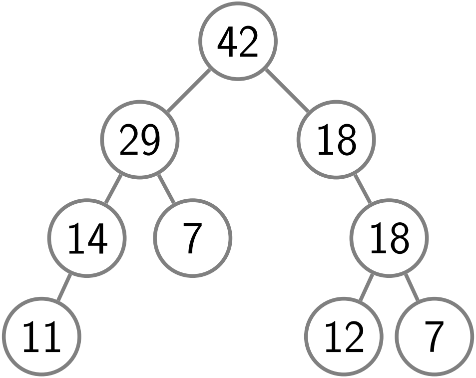


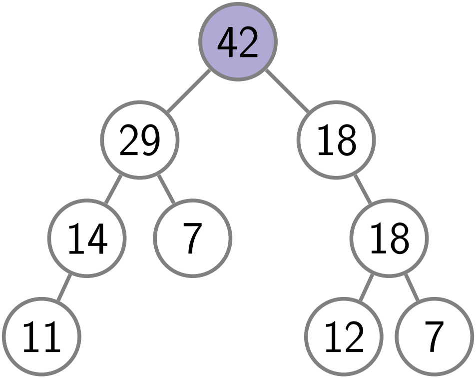
Outline

GetMax



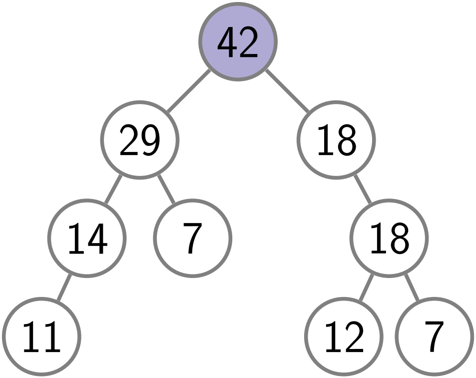
GetMax

return the root value



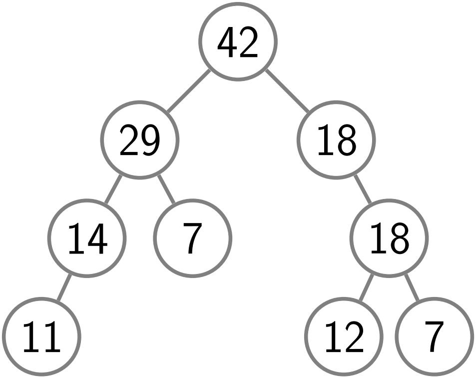
`GetMax`

return the root value



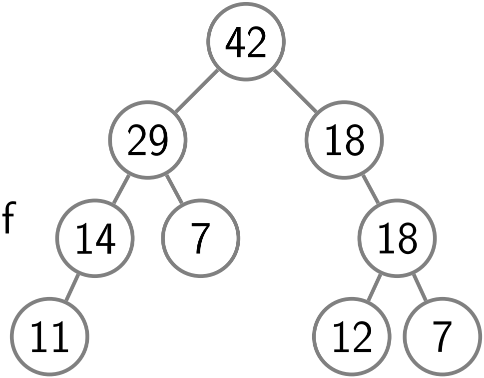
running time: $O(1)$

Insert



Insert

attach a new
node to any leaf



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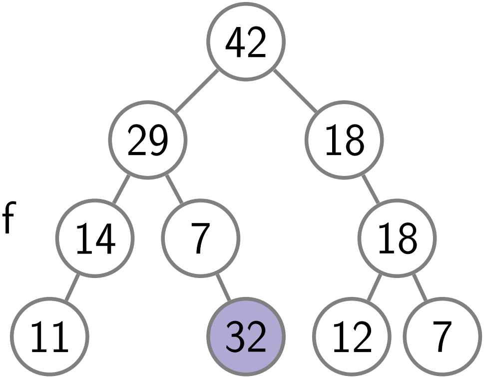
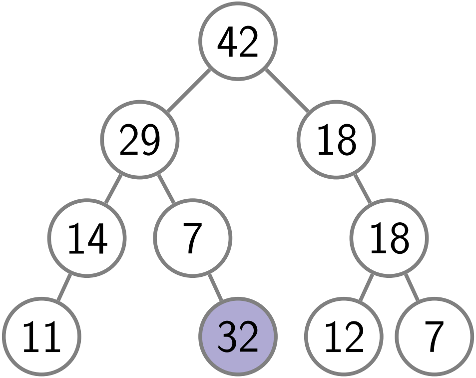


Insert

this may violate the heap property



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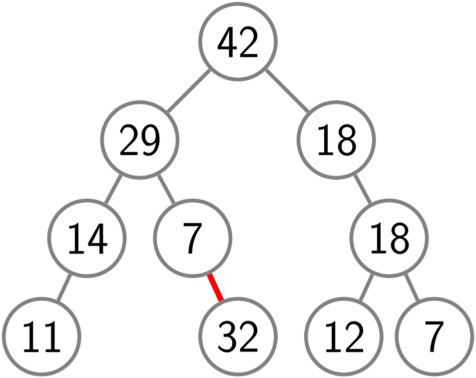
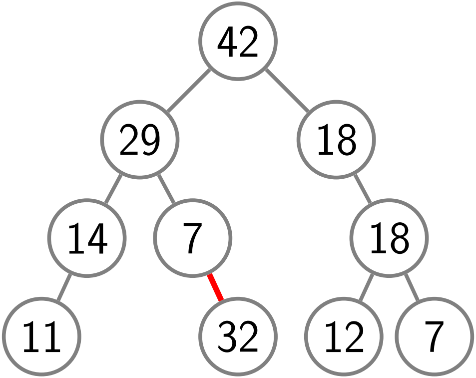


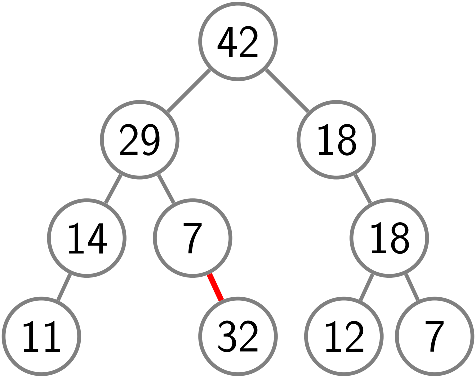
Insert

to fix this, we let the new node sift up



SiftUp

for this, we swap the problematic node with its parent until the property is satisfied



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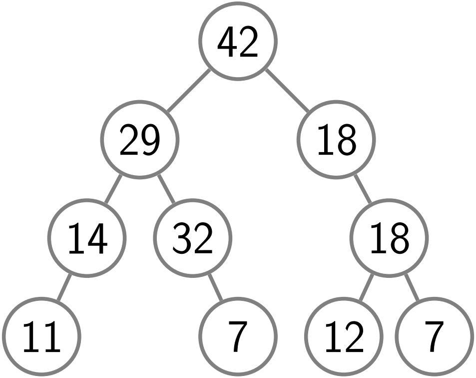

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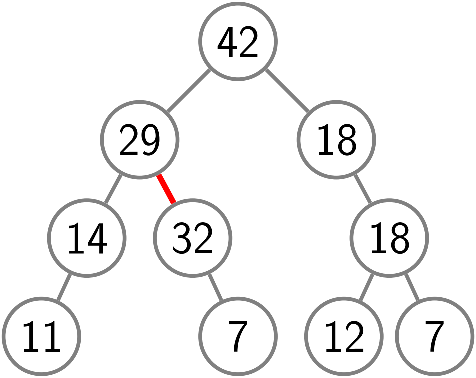

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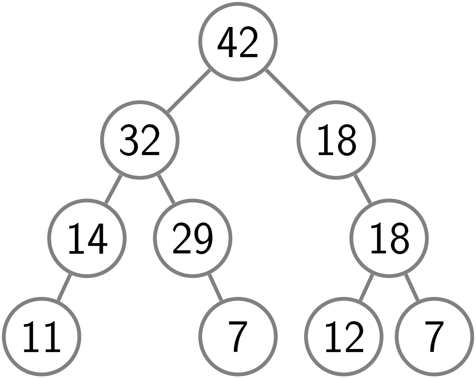
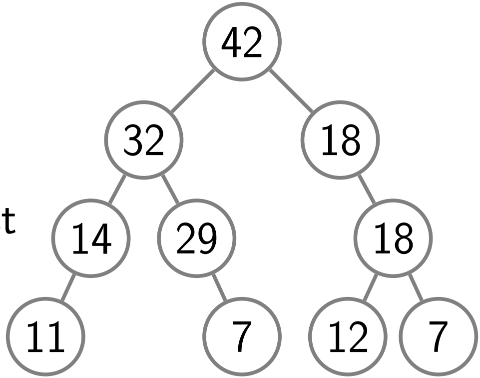


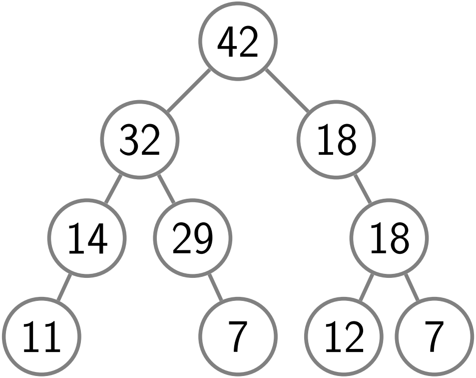
SiftUp

invariant: heap property is violated on at most one edge



SiftUp

this edge gets closer to the root while sifting up



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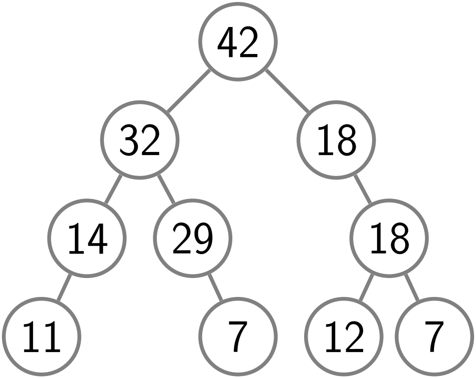
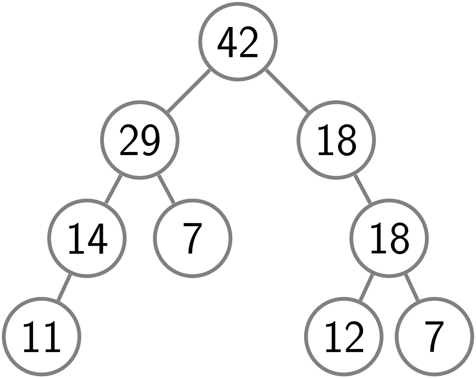


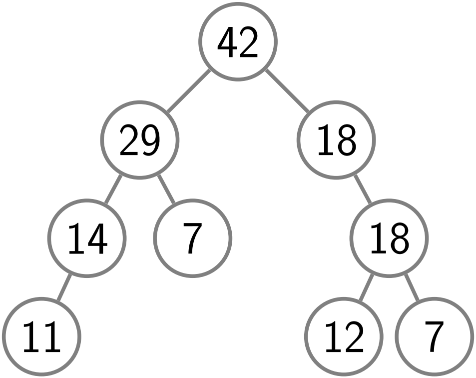
running time: O(tree height)

ExtractMax



ExtractMax

replace the root
with any leaf



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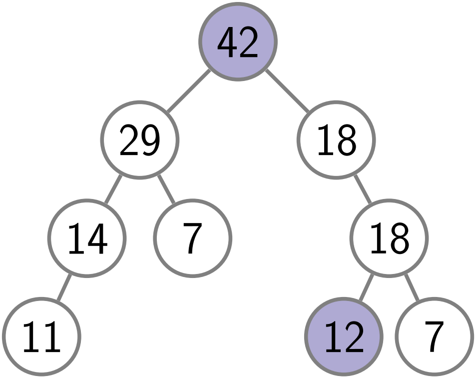

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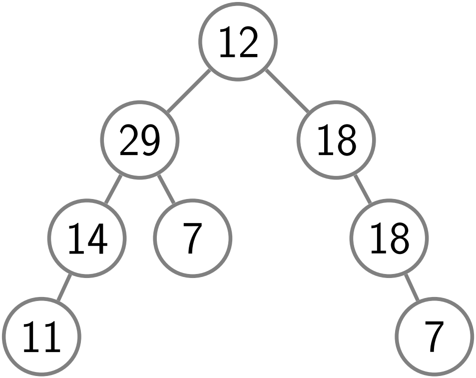
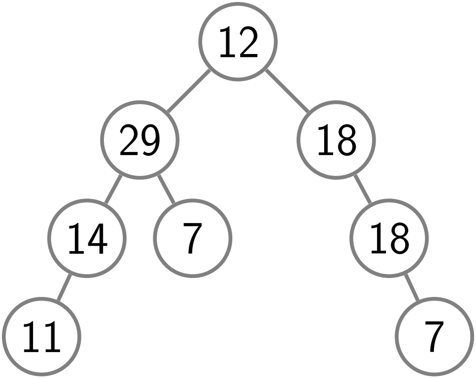


ExtractMax

again, this may violate the heap property



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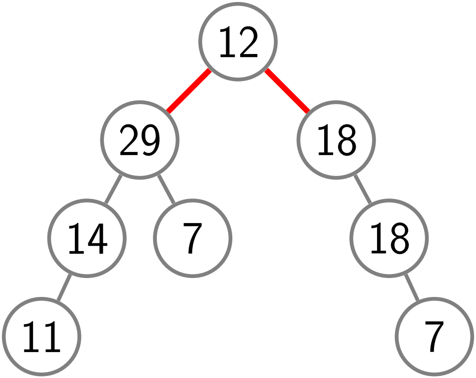
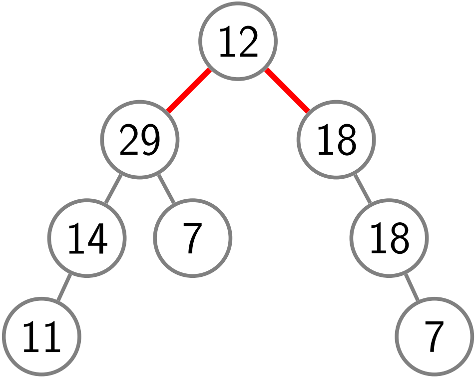


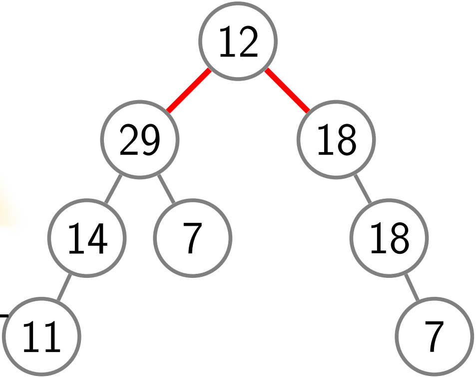
ExtractMax

to fix it, we let the problematic node sift down

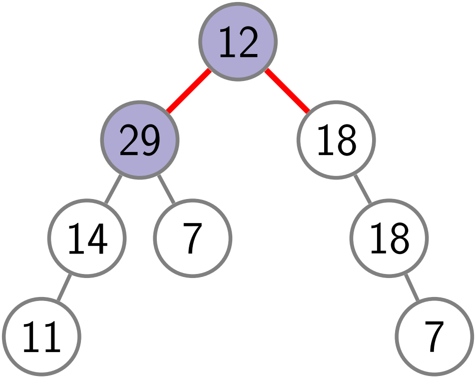


SiftDown

for this, we swap the problematic node with larger child until the heap property is satisfied



SiftDown



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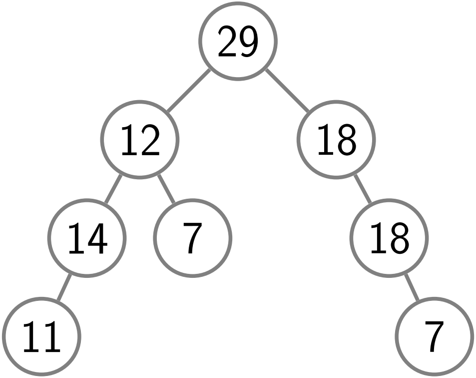

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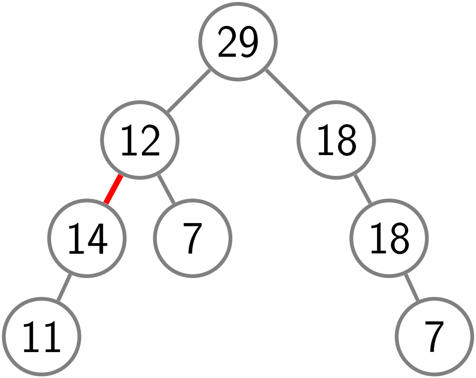

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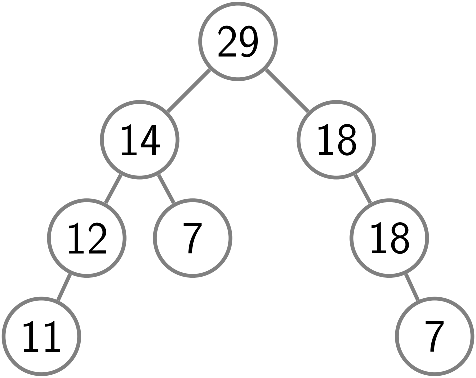
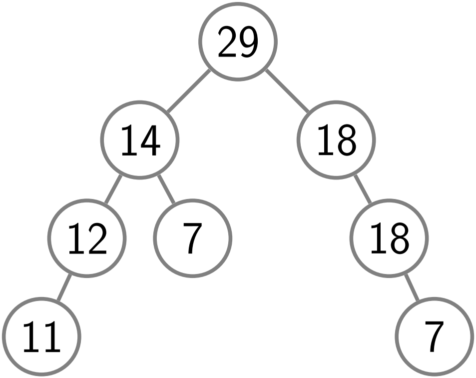


SiftDown

we swap with the larger child which automatically fixes one of the two bad edges



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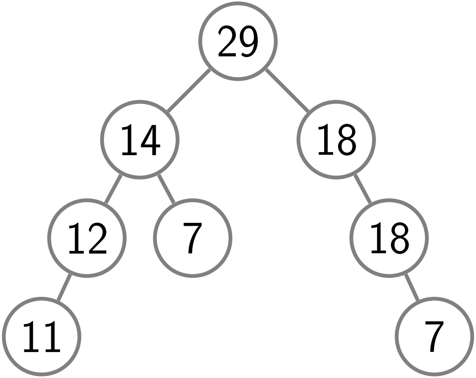
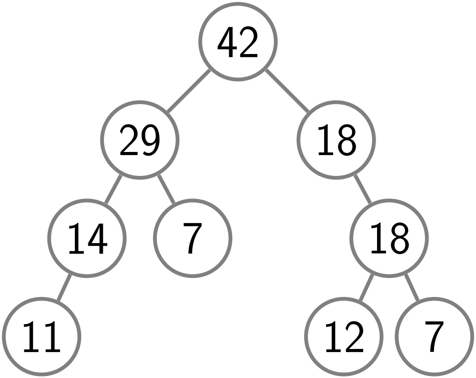


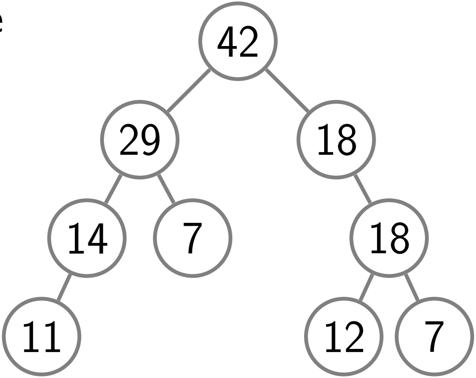
running time: $O(\text{tree height})$

ChangePriority



ChangePriority

change the priority and let the changed element sift up or down depending on whether its priority decreased or increased



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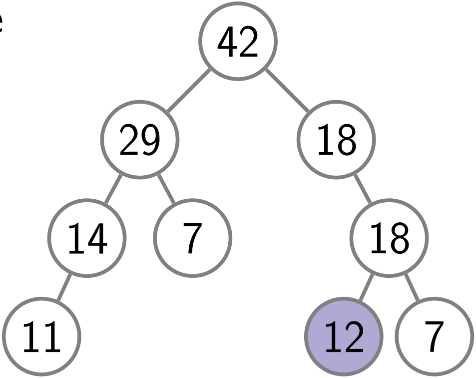

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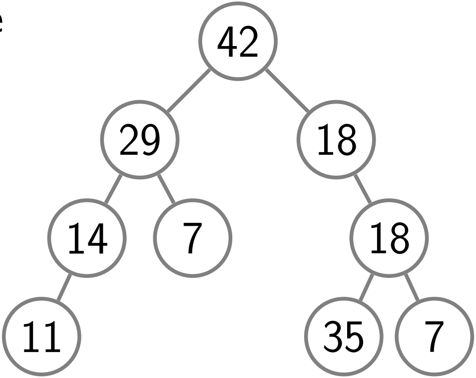

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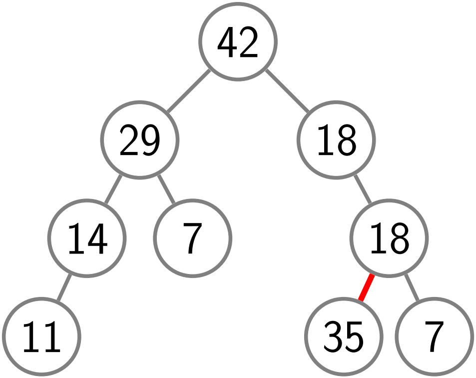

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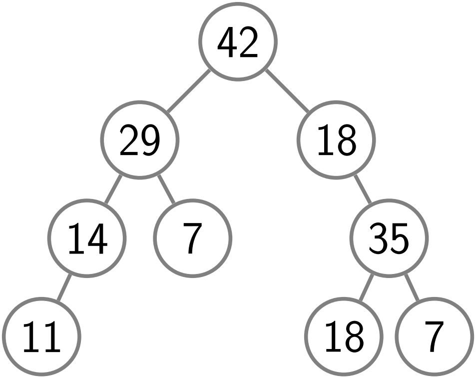

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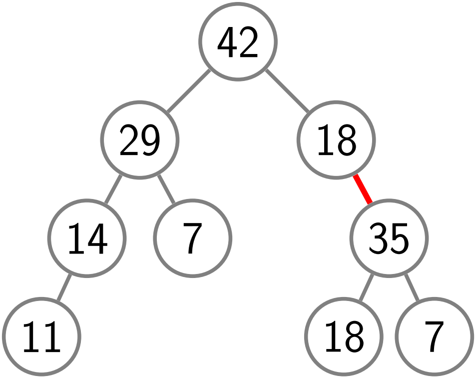

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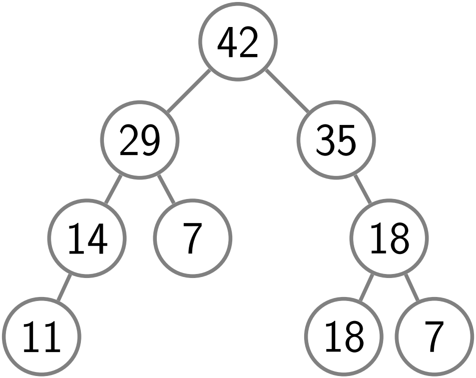

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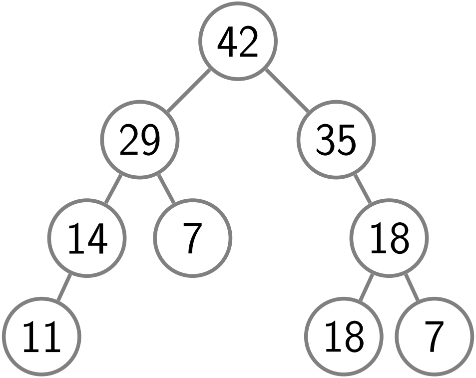
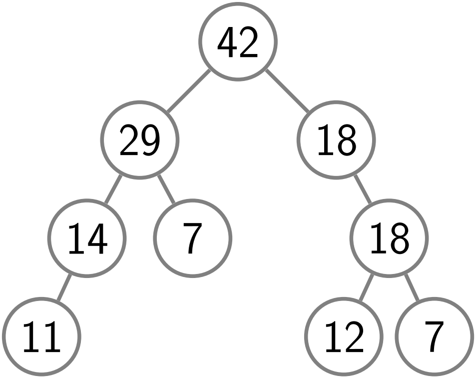


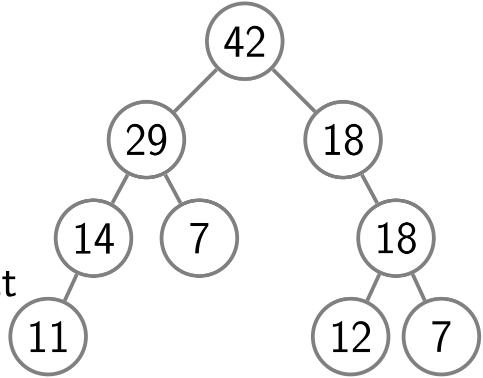
running time: O(tree height)

Remove



Remove

change the priority of the element to ∞, let it sift up, and then extract maximum



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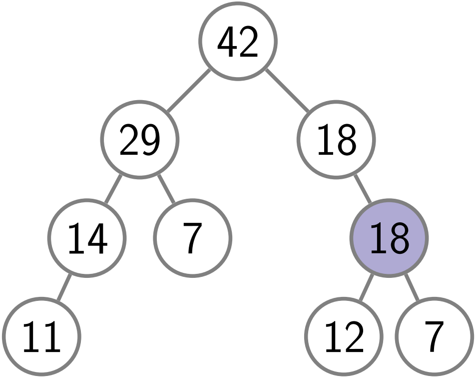

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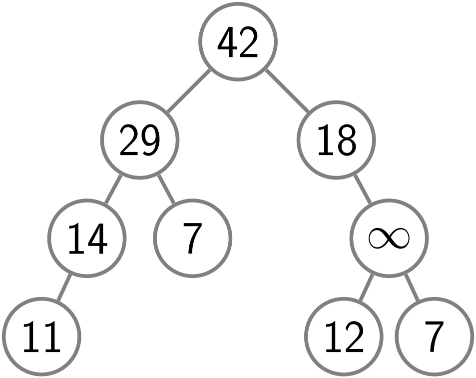

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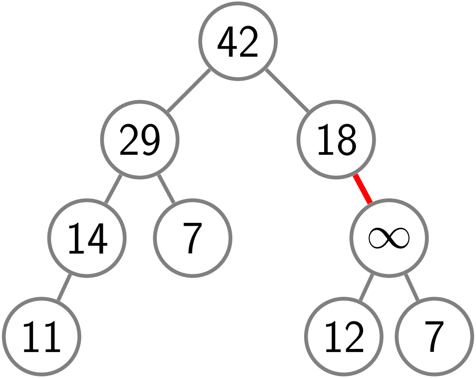

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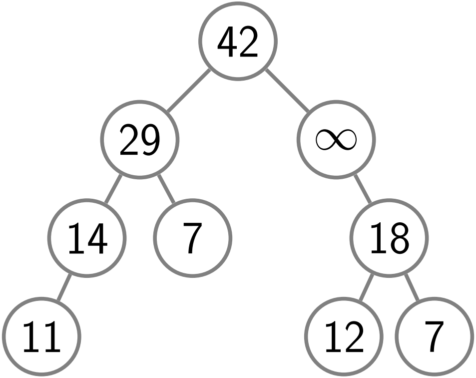

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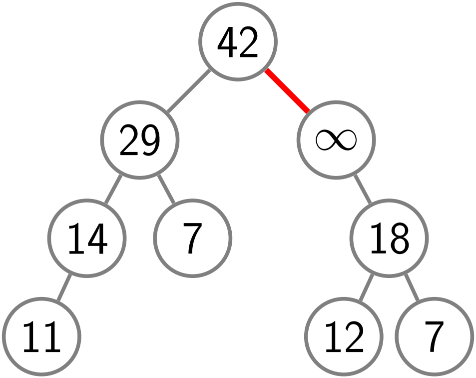

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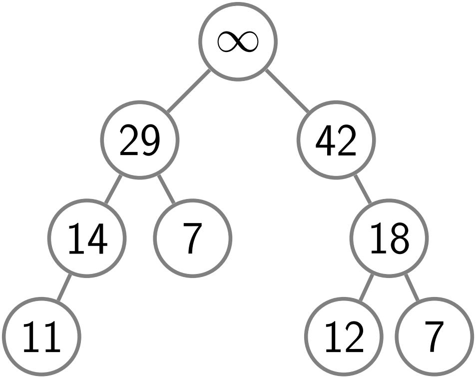
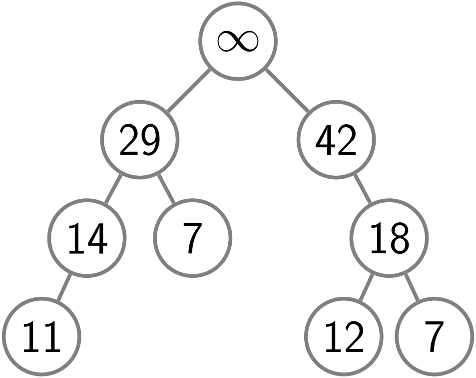


Remove

now, call
`ExtractMax()`



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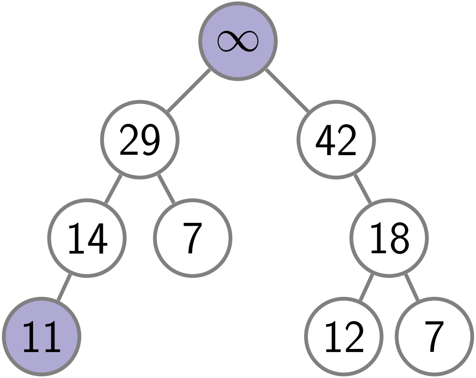

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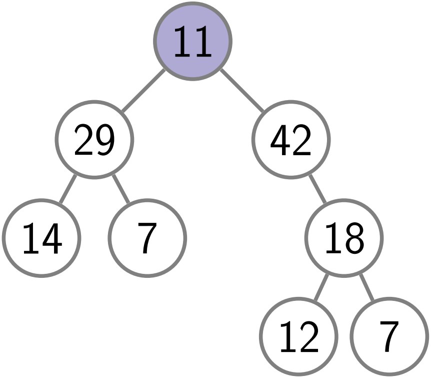

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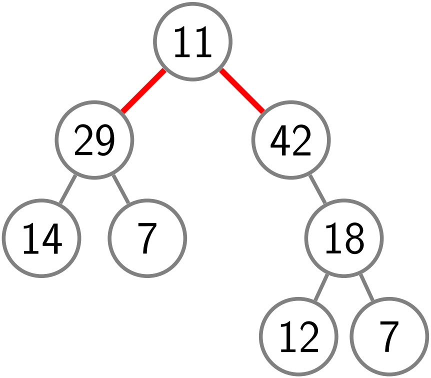

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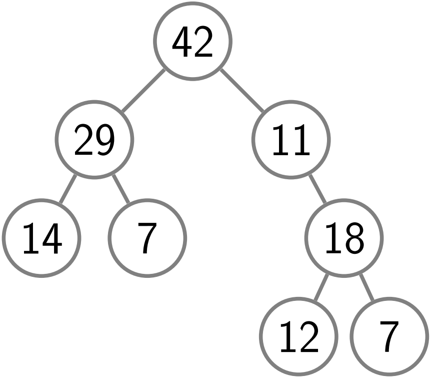

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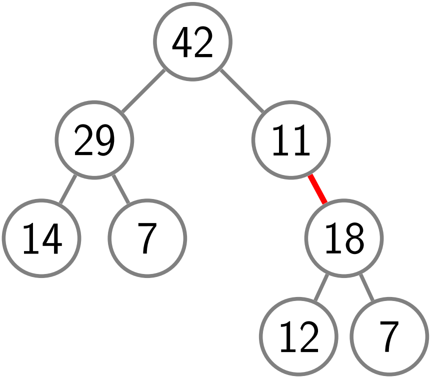

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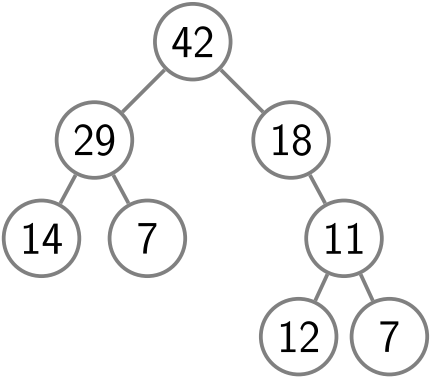

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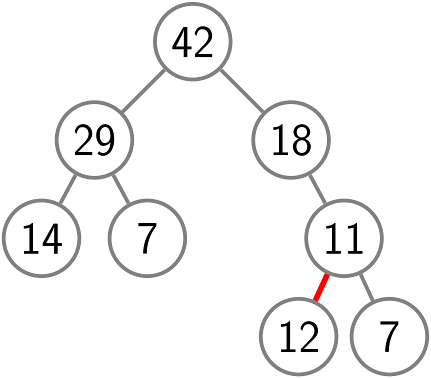

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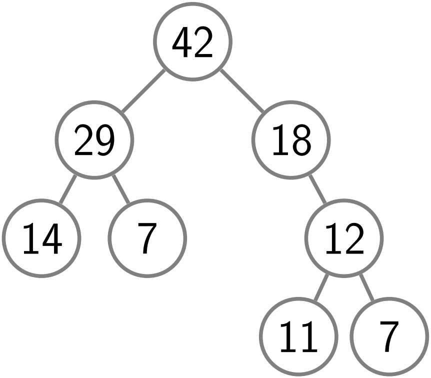

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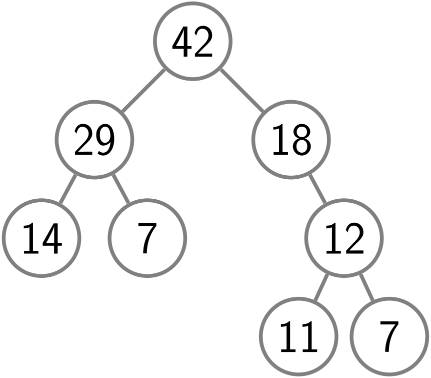


running time: $O(\text{tree height})$

Summary

- `GetMax` works in time $O(1)$, all other operations work in time $O(\text{tree height})$

Summary

- `GetMax` works in time $O(1)$, all other operations work in time $O(\text{tree height})$
- we definitely want a tree to be shallow

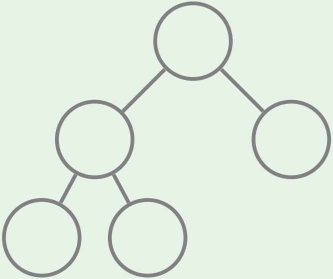
Outline

How to Keep a Tree Shallow?

Definition

A binary tree is complete if all its levels are filled except possibly the last one which is filled from left to right.

Example: complete binary tree



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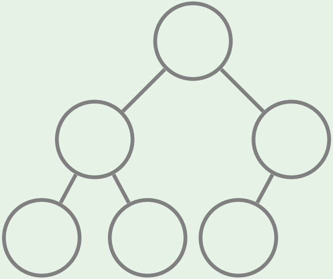

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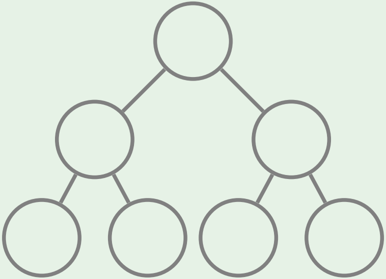

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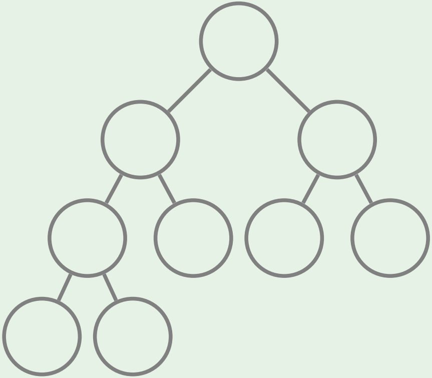
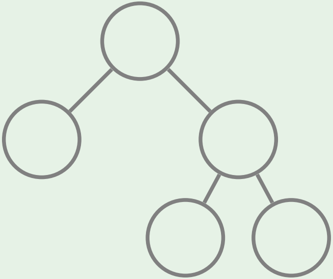


Example: not complete binary tree



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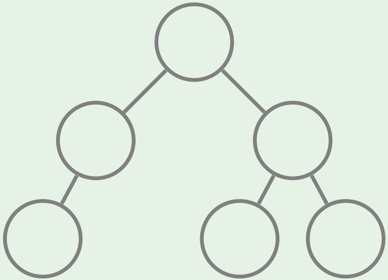

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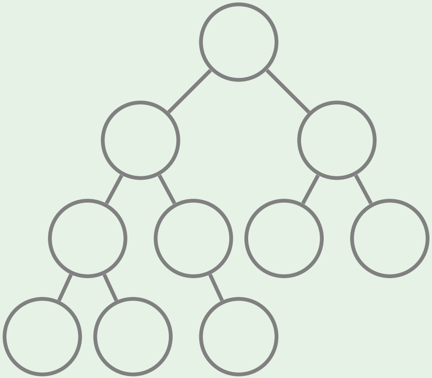

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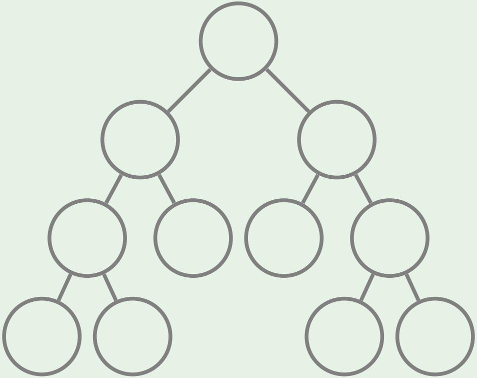


First Advantage: Low Height

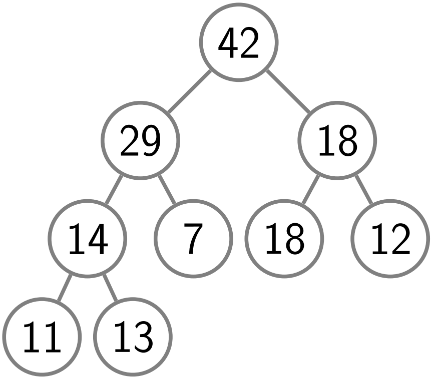
Lemma

A complete binary tree with n nodes has height at most $O(\log n)$.

Proof

- Complete the last level to get a full binary tree on $n' \geq n$ nodes and the same number of levels ℓ.
- Note that $n' \leq 2n$.
- Then $n' = 2^\ell - 1$ and hence $\ell = \log_2(n' + 1) \leq \log_2(2n + 1) = O(\log n)$. $\square$

Second Advantage: Store as Array



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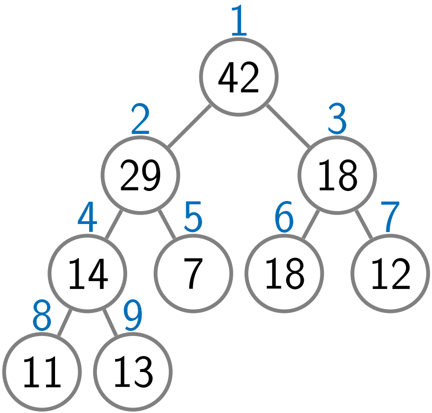

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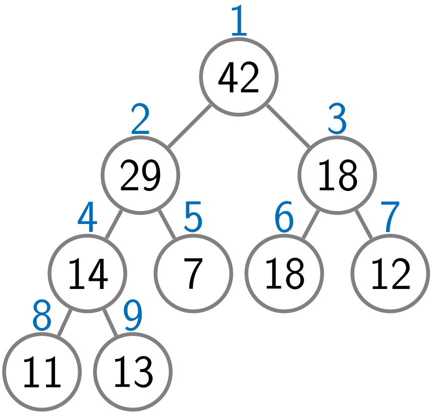

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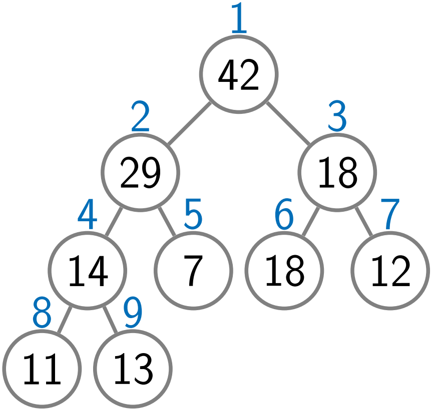

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1	2	3	4	5	6	7	8	9
42	29	18	14	7	18	12	11	13

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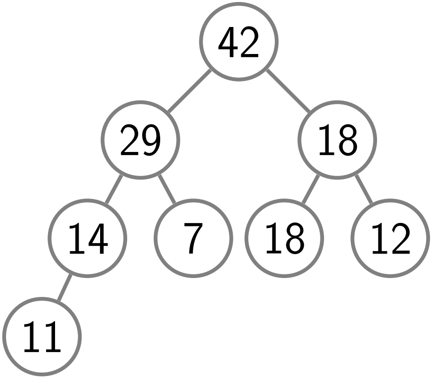
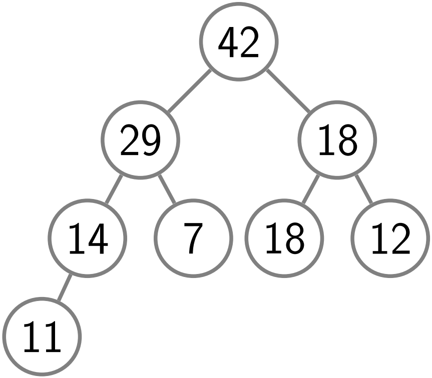
- Which binary heap operations modify the shape of the tree?
- Only `Insert` and `ExtractMax` (`Remove` changes the shape by calling `ExtractMax`).

Keeping the Tree Complete



Keeping the Tree Complete

to insert an element, insert it as a leaf in the leftmost vacant position in the last level and let it sift up



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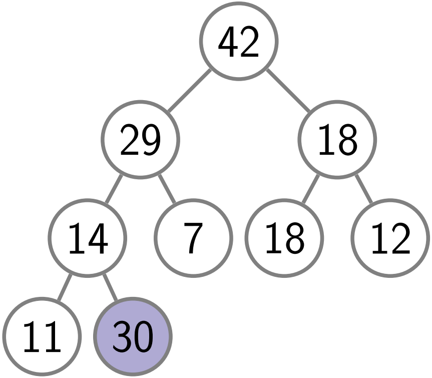

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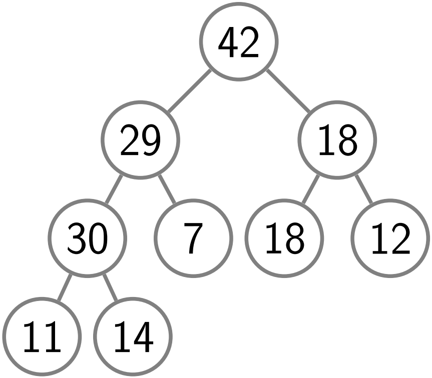

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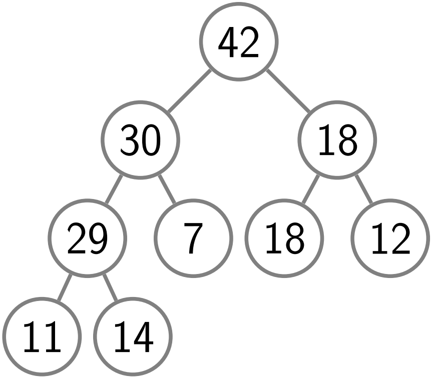
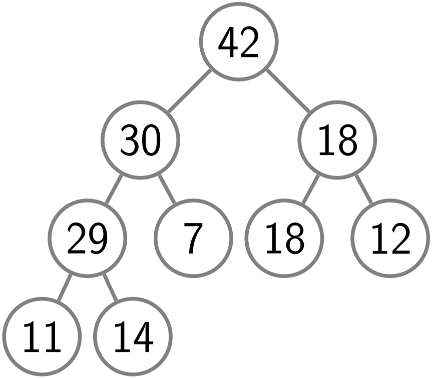


Keeping the Tree Complete

to extract the maximum value, replace the root by the last leaf and let it sift down



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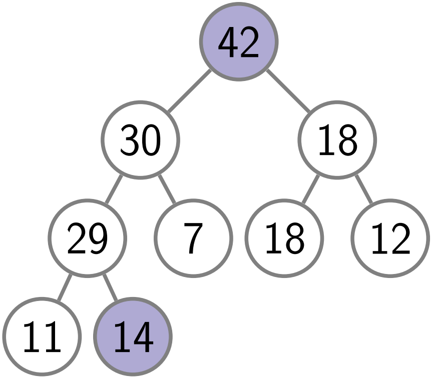

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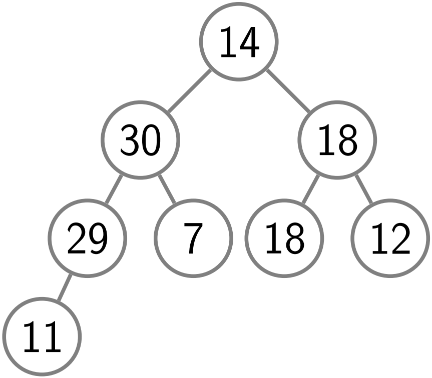

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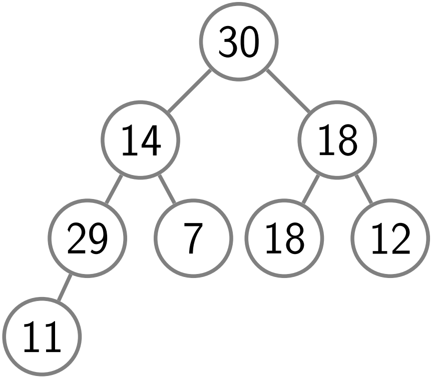

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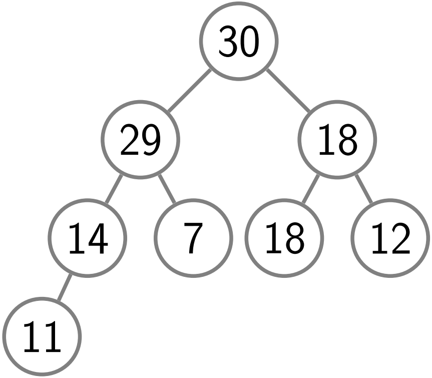

Outline

General Setting

- *maxSize* is the maximum number of elements in the heap

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- *size* is the size of the heap

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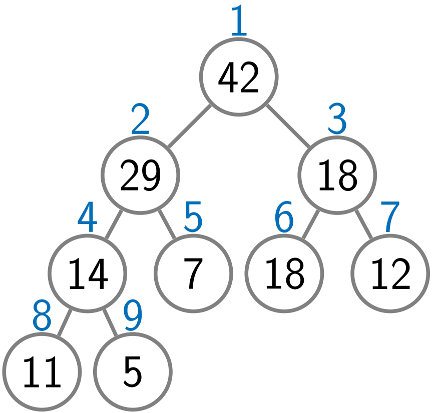
- $size$ is the size of the heap
- $H[1 \dots maxSize]$ is an array of length $maxSize$ where the heap occupies the first $size$ elements

Example



$size = 9$

$maxSize = 13$

	1	2	3	4	5	6	7	8	9	10	11	12	13
H	42	29	18	14	7	18	12	11	5	30	29	2	8

Parent(i)

return $\left\lfloor \frac{i}{2} \right\rfloor$

LeftChild(i)

return $2i$

RightChild(i)

return $2i+1$

SiftUp(i)

```
while i > 1 and H[Parent(i)] < H[i]:
    swap H[Parent(i)] and H[i]
    i ← Parent(i)
```

SiftDown(i)

```
maxIndex ← i
ℓ ← LeftChild(i)
if ℓ ≤ size and H[ℓ] > H[maxIndex]:
   maxIndex ← ℓ
r ← RightChild(i)
if r ≤ size and H[r] > H[maxIndex]:
   maxIndex ← r
if i ≠ maxIndex:
   swap H[i] and H[maxIndex]
   SiftDown(maxIndex)
```

$\texttt{Insert}(p)$

$\texttt{if}\ \ size = maxSize$:

  $\texttt{return ERROR}$

$size \leftarrow size + 1$

$H[size] \leftarrow p$

$\texttt{SiftUp}(size)$

ExtractMax()

$result \leftarrow H[1]$
$H[1] \leftarrow H[size]$
$size \leftarrow size - 1$
SiftDown(1)
return $result$

`Remove`(i)

$H[i] \leftarrow \infty$

`SiftUp`(i)

`ExtractMax`()

ChangePriority(i, p)

```
oldp ← H[i]
H[i] ← p
if p > oldp:
  SiftUp(i)
else:
  SiftDown(i)
```

Summary

The resulting implementation is

- fast: all operations work in time $O(\log n)$ (`GetMax` even works in $O(1)$)

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- fast: all operations work in time $O(\log n)$ (`GetMax` even works in $O(1)$)
- space efficient: we store an array of priorities; parent-child connections are not stored, but are computed on the fly
- easy to implement: all operations are implemented in just a few lines of code

Outline

Sort Using Priority Queues

HeapSort($A[1 \dots n]$)

create an empty priority queue
for i from 1 to n:
  Insert($A[i]$)
for i from n downto 1:
  $A[i] \leftarrow$ ExtractMax()

- The resulting algorithms is comparison-based and has running time $O(n \log n)$ (hence, asymptotically optimal!).

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- Natural generalization of selection sort: instead of simply scanning the rest of the array to find the maximum value, use a smart data structure.
- Not in-place: uses additional space to store the priority queue.

This lesson

In-place heap sort algorithm. For this, we will first turn a given array into a heap by permuting its elements.

Turn Array into a Heap

BuildHeap($A[1 \dots n]$)

```
size ← n
for i from ⌊n/2⌋ downto 1:
   SiftDown(i)
```

- We repair the heap property going from bottom to top.

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- When we reach the root, the heap property is satisfied in the whole tree.
- Online visualization
- Running time: $O(n \log n)$

In-place Heap Sort

HeapSort($A[1 \ldots n]$)

```
BuildHeap(A)                {size = n}
repeat (n − 1) times:
   swap A[1] and A[size]
   size ← size − 1
   SiftDown(1)
```

Building Running Time

- The running time of `BuildHeap` is $O(n \log n)$ since we call `SiftDown` for $O(n)$ nodes.

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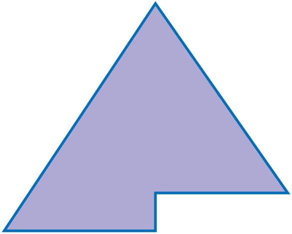
Building Running Time

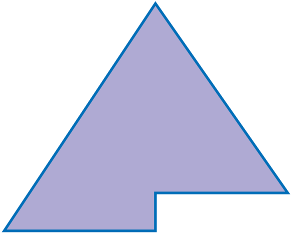
- The running time of `BuildHeap` is $O(n \log n)$ since we call `SiftDown` for $O(n)$ nodes.
- If a node is already close to the leaves, then sifting it down is fast.
- We have many such nodes!
- Was our estimate of the running time of `BuildHeap` too pessimistic?

Building Running Time



# nodes	$T(\texttt{SiftDown})$
1	$\log_2 n$
2	
$\vdots$	$\vdots$
$\leq n/4$	2
$\leq n/2$	1

Building Running Time



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1	$\log_2 n$
2	
$\vdots$	$\vdots$
$\leq n/4$	2
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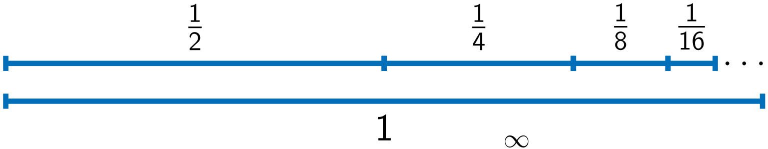
$$\begin{aligned} T(\texttt{BuildHeap}) &\leq \frac{n}{2}\cdot 1 + \frac{n}{4}\cdot 2 + \frac{n}{8}\cdot 3 + \dots \\ &\leq n \cdot \sum_{i=1}^{\infty} \frac{i}{2^i} = 2n \end{aligned}$$

Estimating the Sum



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

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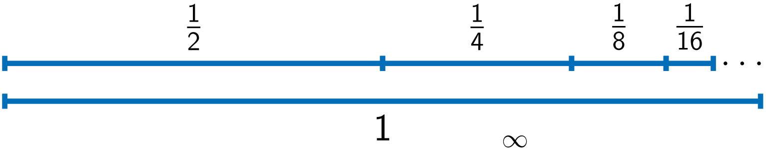

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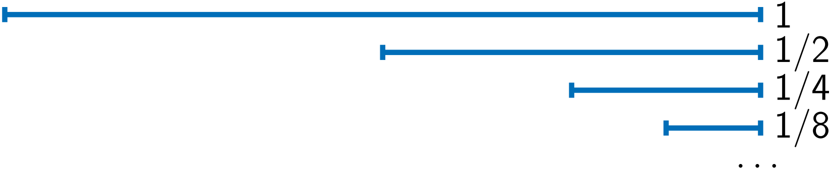
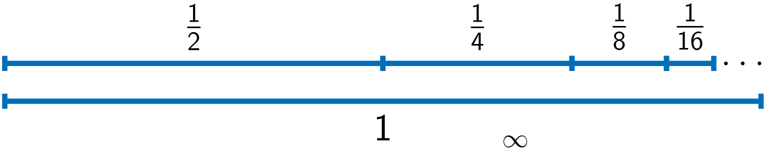


Estimating the Sum



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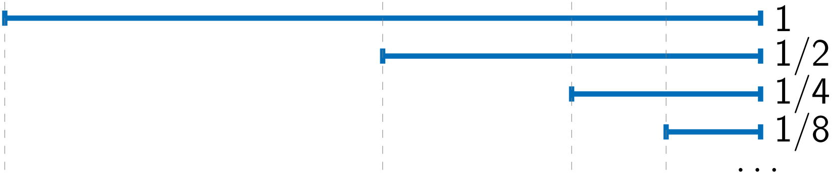

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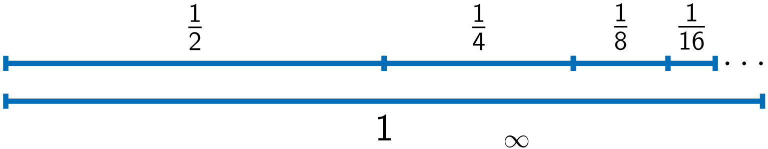

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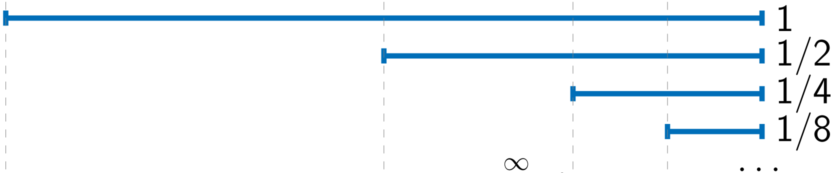

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots = \sum_{k=1}^{\infty} \frac{k}{2^k} = 2$$

Partial sorting

Input: An array $A[1 \dots n]$, an integer $1 \leq k \leq n$.

Output: The last k elements of a sorted version of A.

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Output: The last k elements of a sorted version of A.

Can be solved in $O(n)$ if $k = O(\frac{n}{\log n})$!

PartialSorting($A[1 \dots n], k$)

```
BuildHeap(A)
for i from 1 to k:
  ExtractMax()
```

$\texttt{PartialSorting}(A[1 \ldots n], k)$

```
BuildHeap(A)
for i from 1 to k:
  ExtractMax()
```

Running time: $O(n + k \log n)$

Summary

Heap sort is a time and space efficient comparison-based algorithm: has running time $O(n \log n)$, uses no additional space.

Outline

0-based Arrays

Parent(i)

return $\left\lfloor \frac{i-1}{2} \right\rfloor$

LeftChild(i)

return $2i + 1$

RightChild(i)

return $2i + 2$

Binary Min-Heap

Definition

Binary min-heap is a binary tree (each node has zero, one, or two children) where the value of each node is at most the values of its children.

Can be implemented similarly.

d-ary Heap

- In a d-ary heap nodes on all levels except for possibly the last one have exactly d children.

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d-ary Heap

- In a d-ary heap nodes on all levels except for possibly the last one have exactly d children.
- The height of such a tree is about $\log_d n$.
- The running time of `SiftUp` is $O(\log_d n)$.
- The running time of `SiftDown` is $O(d \log_d n)$: on each level, we find the largest value among d children.

Summary

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- Priority queue supports two main operations: `Insert` and `ExtractMax`.
- In an array/list implementation one operation is very fast ($O(1)$) but the other one is very slow ($O(n)$).
- Binary heap gives an implementation where both operations take $O(\log n)$ time.
- Can be made also space efficient.