Ediz Ferit Kula 16.02.2021
2017405087 ME353 Computational Homework Q3

Question 3:

Introduction

Flow characteristics:

- Laminar boundary layer
- Flat plate aligned with flow

Symbolic variables:

- Stream velocity = U
- Position along x axis = x
- Kinematic viscosity = v
- Density = ρ
- Dynamic viscosity = η
- Width of the plate = b
- Reynolds number = Re $\left(\frac{U * x}{\nu}\right)$

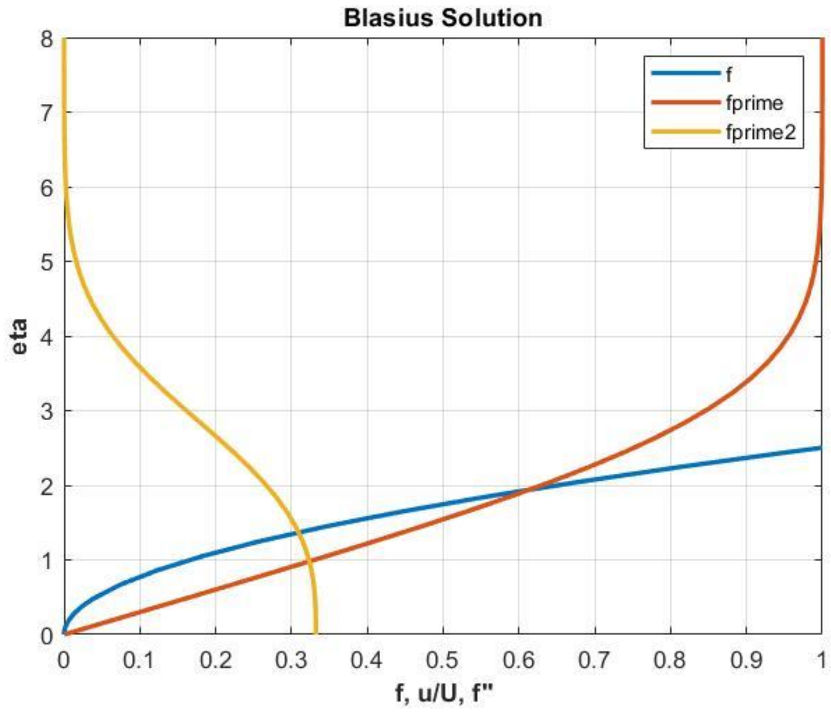
(*Note: All Reynolds numbers used in this report is* Re_x.)

Blasius Solution:

I solved the following differential equation by applying change of variables to make it first order and by using ode45. This process and the resulting plot gave me a good idea about the underlying concept of boundary layer phenomenon.

$2f''' + f * f'' = 0$

Plot of the result:



Part a:

Boundary layer thickness

Boundary layer thickness is defined as the y value when u/U reaches 0.99. Eta (η) is 5 when u/U is greater than 0.99 first time. Hence, by leaving alone y in the following equation, we can get boundary layer thickness.

$$\eta = y * \sqrt{\frac{U}{\nu * x}} \quad \delta = \frac{5}{\sqrt{\text{Re}}} * x$$

Boundary layer displacement thickness

The displacement thickness is defined by:

$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right)\text{dy}$ *this can be written as* $\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right)\text{dy}$

Apply change of variables:

- $f' = \frac{u}{U}$
- $\eta = y * \sqrt{\frac{U}{\nu * x}}$
- $y = \eta * \sqrt{\frac{\nu * x}{U}}$, $y = \delta$, $\eta = 5$
- $\text{dy} = d\eta * \sqrt{\frac{\nu * x}{U}}$

The displacement thickness after change of variables:

$$\delta^* = \frac{x}{\sqrt{\text{Re}}} \int_0^5 (1 - f')d\eta$$

Numeric Integral with Trapezoidal Rule for Displacement Thickness

Numeric integration is applied by using Trapezoidal rule and the values from the table. The result is:

$$T_D = 1.7277$$

The estimated integration error is calculated via the following formula:

$$E_T(f, h) = \frac{-(b-a) * f^{(2)}(c) * h^2}{12} = O(h^2)$$

The resulting error is:

$$\text{error}_{T_d} = -0.0187$$

In conclusion, the displacement thickness is equal to the following:

$$\delta^* = 1.7277 * \frac{x}{\sqrt{\text{Re}}}$$

Boundary layer momentum thickness

The momentum thickness is defined by:

$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right)\mathrm{dy}$ *this can be written as* $\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)\mathrm{dy}$

Apply change of variables:

- $f' = \frac{u}{U}$
- $\eta = y * \sqrt{\frac{U}{\nu * x}}$
- $y = \eta * \sqrt{\frac{\nu * x}{U}}$, $y = \delta$, $\eta = 5$
- $\mathrm{dy} = d\eta * \sqrt{\frac{\nu * x}{U}}$

The momentum thickness after change of variables:

$$\theta = \frac{x}{\sqrt{\mathrm{Re}}}\int_0^5 f'(1 - f')d\eta$$

Numeric Integral with Trapezoidal Rule for Momentum Thickness

Numeric integration is applied by using Trapezoidal rule and the values from the table. The result is:

$T_M = 0.6572$

The estimated integration error is calculated via the following formula:

$$E_T(f, h) = \frac{-(b-a) * f^{(2)}(c) * h^2}{12} = O(h^2)$$

The resulting error is:

$\mathrm{error}_{T_d} = -0.03705$

In conclusion, the momentum thickness is equal to the following:

$$\theta = 0.6572 * \frac{x}{\sqrt{\mathrm{Re}}}$$

Part b:

Shear stress

Shear stress is defined by Newton's Law of Viscosity in the surface of the plate.

$$\tau = \mu * \frac{\mathrm{du}}{\mathrm{dy}}$$

So, we need $\frac{\mathrm{du}}{\mathrm{dy}}$.

- $f' = \frac{u}{U}$
- $f'' = \frac{d}{d\eta} f' = \frac{d}{\mathrm{d}\left(\frac{y}{x} * \sqrt{\mathrm{Re}}\right)} \left(\frac{u}{U}\right) = \frac{x}{U * \sqrt{\mathrm{Re}}} * \frac{\mathrm{du}}{\mathrm{dy}}$
- $\frac{\mathrm{du}}{\mathrm{dy}} = U * \frac{\sqrt{\mathrm{Re}}}{x} * f''$

We found $\frac{\mathrm{du}}{\mathrm{dy}}$:

$\tau = \frac{\mu U \sqrt{\mathrm{Re}}}{x} * f''$ f'' *is calculated at y=0 which means* $\eta = 0$.

In conclusion, shear stress is equal to the following:

$$\tau_w = 0.3321 * \frac{\mu U \sqrt{\mathrm{Re}}}{x}$$

Drag Force and friction drag coefficient

Drag force can be calculated with two different ways. Either using momentum thickness or integrating the shear stress over the surface of the plate.

- First, we will use momentum thickness to calculate drag force and friction drag coefficient.

$F_D = \rho * b * U^2 * \theta$ *(Von-Karman expression)*

Replace momentum thickness with the value we found before:

$$F_D = 0.6572 * \rho b U^2 \frac{x}{\sqrt{\mathrm{Re}}}$$

The friction drag coefficient:

$$C_{\mathrm{Df}} = \frac{F_D}{\frac{1}{2}\rho U^2 \mathrm{bx}} = 1.3144 * \frac{1}{\sqrt{\mathrm{Re}}}$$

- Secondly, we will use shear stress to calculate drag force and friction drag coefficient.

$$F_D = b * \int_0^x \tau_w * \mathrm{dx} = 0.332 * \mu U^{\frac{3}{2}} \frac{b}{\sqrt{\nu}} * \int_0^x \frac{1}{\sqrt{x}} \mathrm{dx}$$

Calculate the integral:

$$F_D = 0.664 * b \rho U^2 \frac{x}{\sqrt{\mathrm{Re}}}$$

The friction drag coefficient:

$$C_{Df} = \frac{F_D}{\frac{1}{2}\rho U^2 bx} = 1.328 * \frac{1}{\sqrt{Re}}$$

Local friction coefficient

We will use the following formula to calculate local friction coefficient:

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

Replace the shear stress that we calculated above:

$$c_f = 0.6642 * \frac{1}{\sqrt{Re}}$$

Compare the results with table 9.2

$\tau_w = 0.3321 * \frac{\mu U \sqrt{Re}}{x}$ As we can see, our results are

$C_{Df} = 1.3144 * \frac{1}{\sqrt{Re}}$ almost same as table 9.2

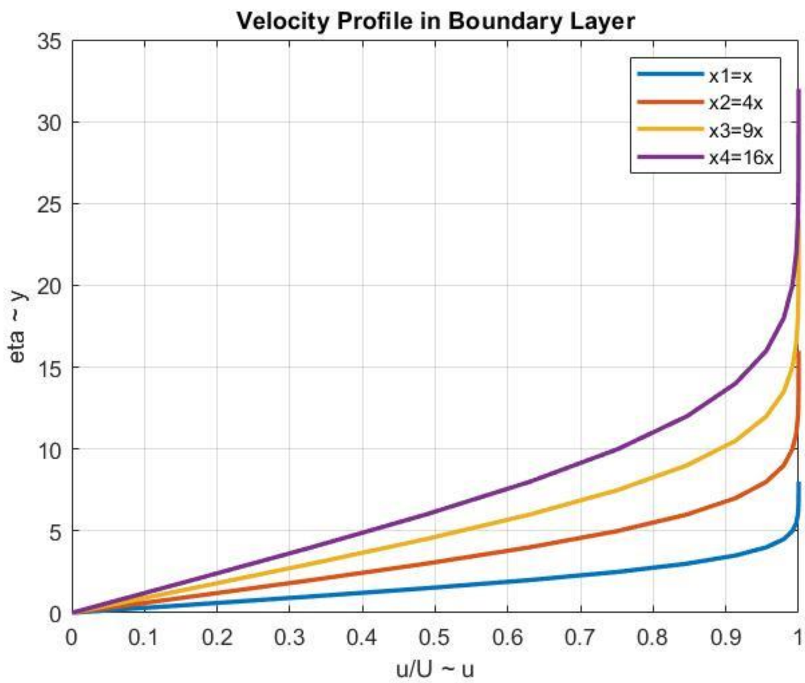
$c_f = 0.6642 * \frac{1}{\sqrt{Re}}$

Profile Character $\delta \frac{\sqrt{Re}}{x}$ $c_f \sqrt{Re}$ $C_{Df} \sqrt{Re}$

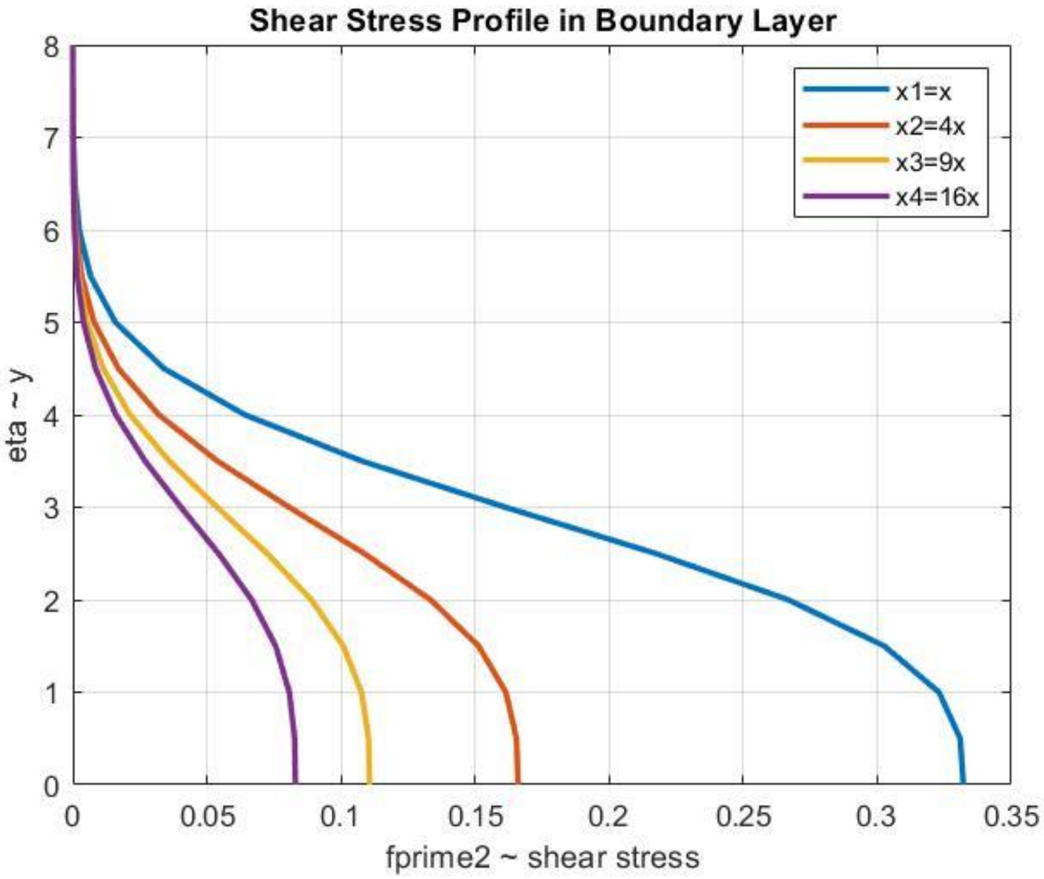
a. Blasius solution 5.00 0.664 1.328

Part c:

The velocity distribution in the boundary layer



The shear stress distribution in the boundary layer



Codes:

```
% FLUID MECHANICS COMPUTATIONAL HOMEWORK QUESTION-3
clear all, close all, clc

% Laminar boundary layer
%
% U - free stream velocity in x direction
% v - kinematic viscosity
% d - density
% delta - boundary layer thickness
% deltaStar - boundary layer displacement thickness
% teta - boundary layer momentum thickness

% Symbolic Values
syms Re;
syms x;
syms U;
syms density;
syms b;          % width
syms nu;         % dynamic viscosity

% Table
blasius_table = [0 0 0 0.3321; 0.5 0.0415 0.1659 0.3309;...
1 0.1656 0.3298 0.323; 1.5 0.3701 0.4868 0.3026;...
2 0.65 0.6298 0.2668; 2.5 0.9963 0.7513 0.2174;...
3 1.3968 0.846 0.1614; 3.5 1.8377 0.913 0.1078;...
4 2.3057 0.9555 0.0642; 4.5 2.7901 0.9795 0.034;...
5 3.2833 0.9915 0.0159; 5.5 3.7806 0.9969 0.0066;...
```

```
6 4.2796 0.999 0.0024; 6.5 4.7793 0.9997 0.0008;...
7 5.2792 0.9999 0.0002; 7.5 5.7792 1 0.0001;...
8 6.2792 1 0];

% From table
eta = blasius_table(:,1);
f = blasius_table(:,2);
fprime = blasius_table(:,3);
fprime2 = blasius_table(:,4);
deta = 0.5; % step size
N = length(eta);


% Solve Differential Equation
tspan = [0 8];
y0 = [0 0 0.3321];
[t,y] = ode45(@(t,y) odecfn(y),tspan,y0);

% Plot Blasius solution
figure
plot(y(:,1),t,'linewidth',2)
hold on, grid on
plot(y(:,2),t,'linewidth',2)
plot(y(:,3),t,'linewidth',2)
xlabel('f, u/U, f"','Fontweight','bold')
ylabel('eta','Fontweight','bold')
title('Blasius Solution')
xlim([0 1])
legend('f','fprime','fprime2')


%%
%======= BOUNDARY LAYER THICKNESS =======%
% y = delta when u/U = 0.99
delta = 5/sqrt(Re)*x


%%
%======= DISPLACEMENT THICKNESS =======%
% The displacement thickness is defined by:
% deltaStar = integral of (1-u/U)dy from 0 to infinity.
% deltaStar = integral of (1-u/U)dy from 0 to delta.
% Changing variables:
% f'=u/U, eta=y*sqrt(U/v/x), y=eta*sqrt(v*x/U), dy=deta*sqrt(v*x/U)
% deltaStar = x*sqrt(1/Re)*(integral of (1-f')deta from 0 to 5)

% Numeric Integral with Trapezoidal Rule for Displacement Thickness
T_d = 0;
for i=1:N-1
    T_d = T_d + deta/2*(1-fprime(i+1)+1-fprime(i));
end

T_d     % print the result of numeric integration

deltaStar = T_d*x/sqrt(Re)

%========== ERROR ANALYSIS ==========%
% Apply numeric differentiation to find f'''.
% Maximum value of f''' will be used in error calculation.

fprime3 = zeros(size(fprime2));
```

```
for i=1:N-1
    fprime3(i+1) = (fprime2(i+1)-fprime2(i))/(eta(i+1)-eta(i));
end

% Estimated integration error is:
error_T_d = -(eta(end)-eta(1)).*max(abs(fprime3)).*deta.^2/12


%%
%======= MOMENTUM THICKNESS =======%
% The momentum thickness is defined by:
% teta = integral of u/U(1-u/U)dy from 0 to infinity.
% teta = integral of u/U(1-u/U)dy from 0 to delta.
% Changing variables:
% f'=u/U, eta=y*sqrt(U/v/x), y=eta*sqrt(v*x/U), dy=deta*sqrt(v*x/U)
% deltaStar = x*sqrt(1/Re)*(integral of f'(1-f')deta from 0 to 5)

% Numeric Integral with Trapezoidal Rule for Momentum Thickness
T_m = 0;
for i=1:N-1
    T_m = T_m + deta/2*(fprime(i+1)*(1-fprime(i+1))+...
        fprime(i)*(1-fprime(i)));
end

T_m     % print the result of numeric integration

teta = T_m*x/sqrt(Re)

%========== ERROR ANALYSIS ==========%
% Apply numeric differentiation to find second derivative of f'(1-f').
% Maximum value of this derivative will be used in error calculation.

f_error = zeros(size(fprime2));
for i=1:N-1
    f_error(i+1) = (fprime2(i+1)-2*fprime(i+1)*fprime2(i+1) - ...
        (fprime2(i)-2*fprime(i)*fprime2(i)))/(eta(i+1)-eta(i));
end

% Estimated integration error is:
error_T_m = -(eta(end)-eta(1)).*max(abs(f_error)).*deta.^2/12


%=============== SHEAR STRESS ================%
shear = nu*U*sqrt(Re)/x

%=============== DRAG FORCE ================%
dragForce_momentum = density*b*U^2*teta      % from momentum
dragForce_shear = 0.664*b*density*U^2*x/sqrt(Re)

%=============== LOCAL FRICTION COEFFICIENT ===============%
localFrictionCoefficient_shear = shear/(density*U^2/2)
localFrictionCoefficient_momentum = 0.6572*2/sqrt(Re)


%========== VELOCITY PROFILE and SHEAR PROFILE ========%
figure
plot(fprime,eta,'linewidth',2)
grid on, hold on
plot(fprime,eta*2,'linewidth',2)
plot(fprime,eta*3,'linewidth',2)
plot(fprime,eta*4,'linewidth',2)
```

```
xlabel('u/U ~ u')
ylabel('eta ~ y')
title('Velocity Profile in Boundary Layer')
legend('x1=x','x2=4x','x3=9x','x4=16x')

figure
plot(fprime2,eta,'linewidth',2)
grid on, hold on
plot(fprime2./2,eta,'linewidth',2)
plot(fprime2./3,eta,'linewidth',2)
plot(fprime2./4,eta,'linewidth',2)
xlabel('fprime2 ~ shear stress')
ylabel('eta ~ y')
title('Shear Stress Profile in Boundary Layer')
legend('x1=x','x2=4x','x3=9x','x4=16x')


%========== DIFFERENTIAL EQUATIONS ==========%
% This function, stores the 3 first order differential equations which will
% be used in ode45. These equations come from 2f''' + ff'' = 0.
function dydt = odecfn(y)
dydt = zeros(3,1);
dydt(1) = y(2);
dydt(2) = y(3);
dydt(3) = -y(1)*y(3)/2;
end
```