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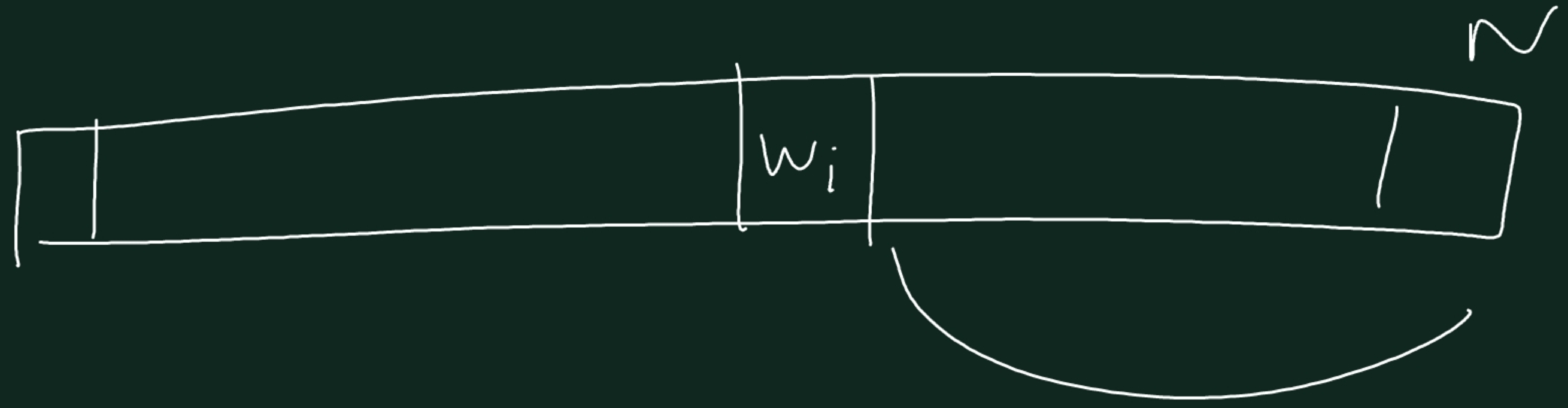


① not sorted

unsuccessful : N

successful : $N/2$

② sorted

$N/2$

$N/2$

③ sorted, binary search

ncomp	size
1	$N/2$
2	$N/4$
3	$N/8$
⋮	
i	$N/2^i$
⋮	
$\log n$	1

$$\boxed{t = N/2 \times t_c} = 5\,10^5\;10^{-5} = 5\,s$$

$N = 10^6$

$t_c = 10\,\mu s = 10^{-5}\,s$

$t = \log 10^6 \cdot 10^{-5}$

$= 6 \log 10 \; 10^{-5}$

$= 2 . 10^{-5}$

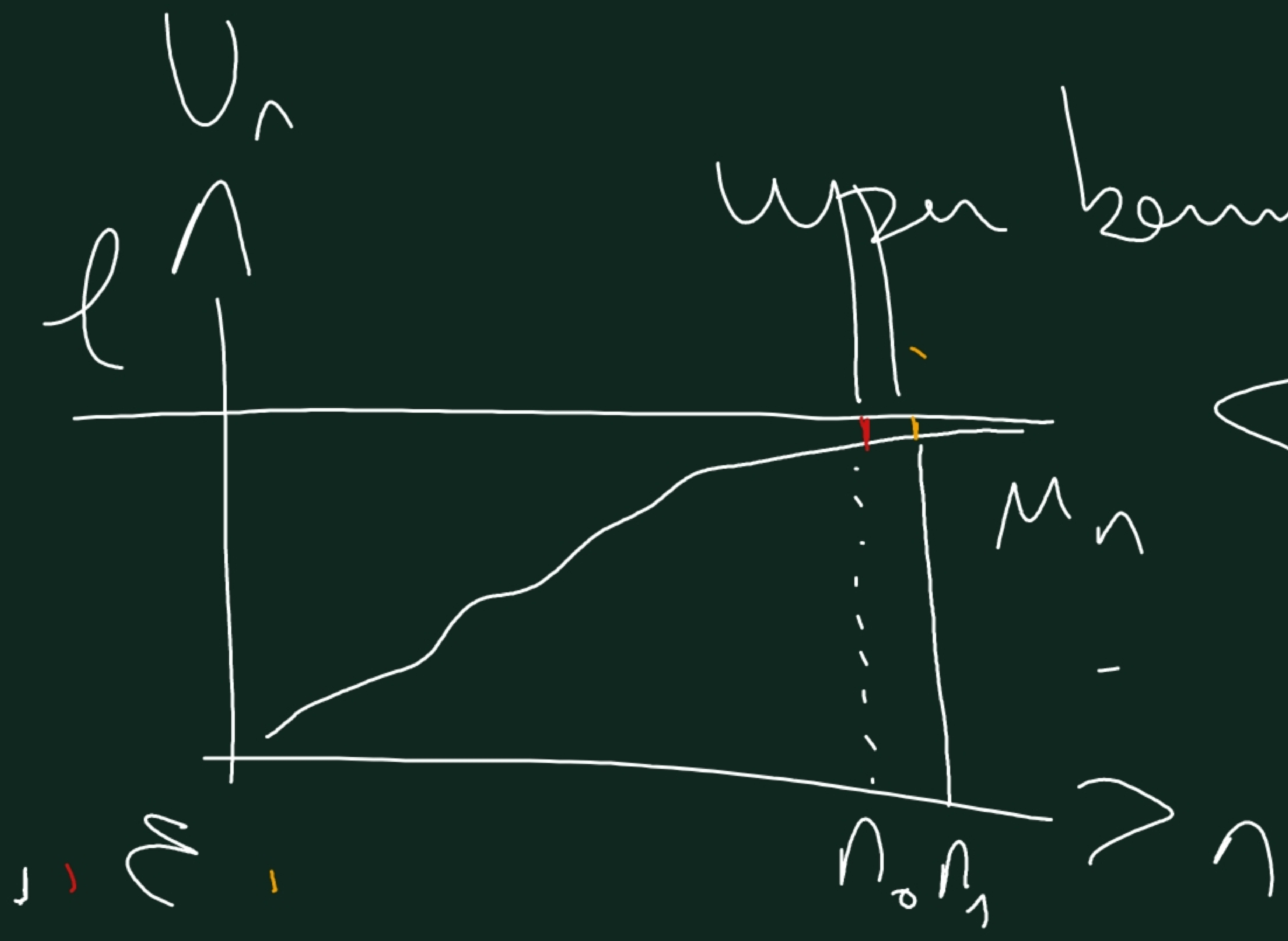
$= 0.2\,ms$

real function $\longrightarrow$ real sequence

$$\begin{cases} u_n = 3n+2 \\ u_0 = 1 \end{cases}$$

* increasing

upper bound



$< 1\%$

$\forall \varepsilon > 0, \exists n_0 / \forall n > n_0, \ell - u_n < \varepsilon$

$y = f(x)$

$$\lim_{n \to +\infty} u_n = \ell$$

unlimited

$$\lim_{n \to +\infty} u_n = +\infty$$

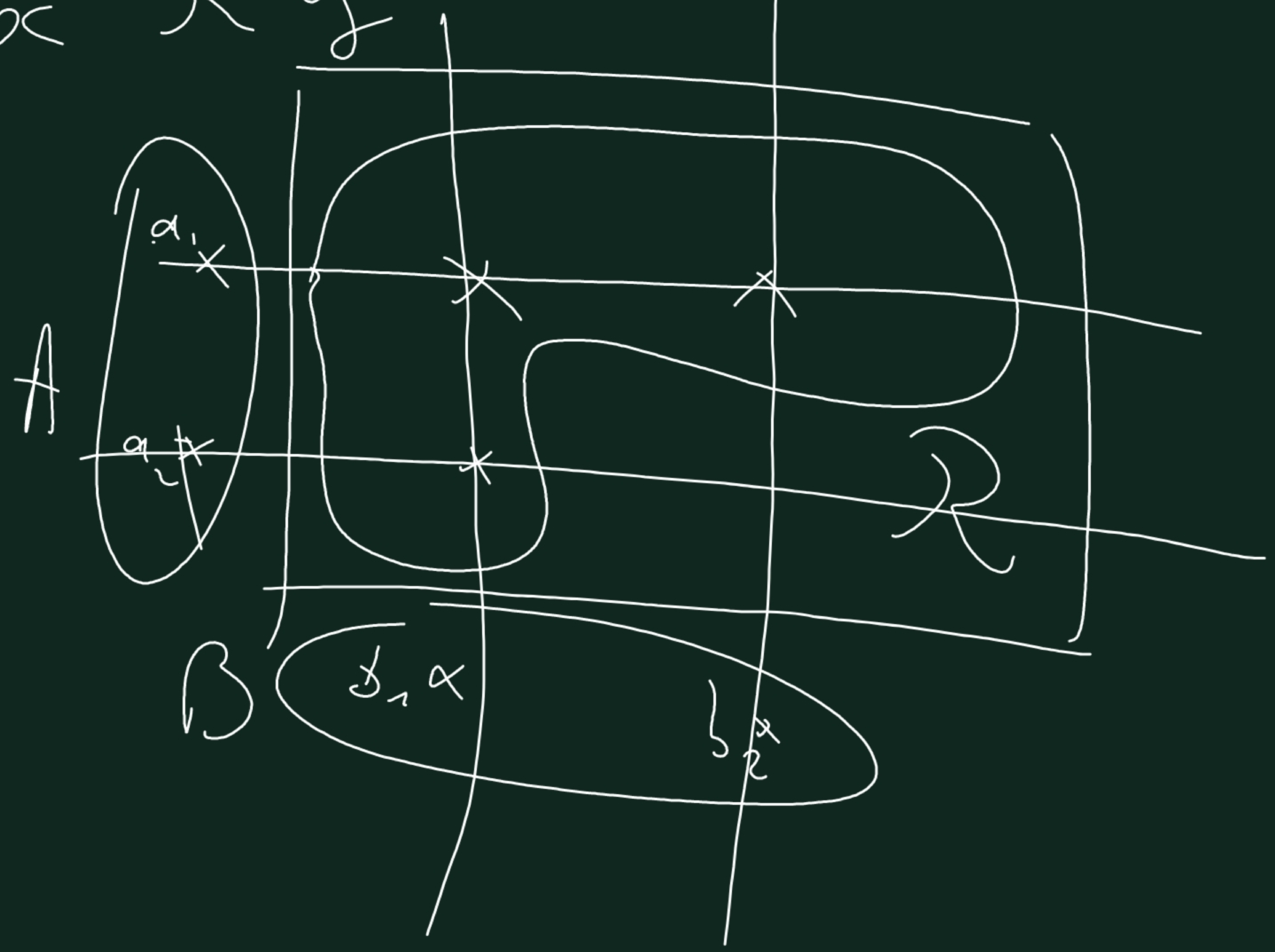
$> 99\%$

$\forall A > 0, \exists n_0 / \forall n > n_0, u_n > A$

Relation

$$R : A \longrightarrow B \quad \subset A \times B$$
$$x \longmapsto y$$

$x \, R \, y$



$\mathcal{S}$: set of increasing sequences

Landau notations

$\mathcal{O}$: u, v sequences

$u = \mathcal{O}(v)$ "u is dominated by v"

iff $\exists\, n_0, c \,/\, \forall n > n_0,\ u_n \leq c \cdot v_n$

$u_n = 3n + 2$

$v_n = 2n + 5$

n	u_n	v_n
1	5	7
2	8	9
3	11	11
4	14	13

- $u = o(v)$ "u is negligeable compared to v"

iff $\lim\limits_{n \to +\infty} \dfrac{u_n}{v_n} = 0$

$u_n = n + 1$

$v_n = n^2 - 1$

$$\lim_{n \to +\infty} \left(\frac{u_n}{v_n} \right) = \lim_{n \to +\infty} \left(\frac{n+1}{n^2-1} \right) = \lim_{n \to +\infty} \left(\frac{1}{n-1} \right) = 0$$

Θ $u = \Theta(v)$ "u is of the same kind as v"

iff

$\exists n_0, a, b \,/\, \forall n > n_0, \quad a \cdot u_n \leq v_n \leq b \cdot u_n$

($\longrightarrow$ reciprocal O)

ex $u_n = n+1$

$v_n = n^2 - 1$

Main properties of relations:

reflexivity: R is reflexive iff $\forall u \; u R u$

symmetry: ——— symmetric iff $\forall u, v, \; u R v \Longrightarrow v R u$

asymmetry: ——— asymmetric iff $\begin{cases} u R v \\ v R u \end{cases} \Longrightarrow u = v$

transitivity: ——— transitive iff $\begin{cases} u R v \\ v R w \end{cases} \Longrightarrow u R w$

equivalence relations: R, S, T

order relations: R, A, T

	R	S	A	T
O	Y	N	N	Y
o	N	N	N/A	Y
4	Y	Y	N	Y

$a u_n \leq v_n \leq b u_n$

$\frac{1}{b} v_n \leq u_n \leq \frac{1}{a} v_n$

$\longrightarrow$ 4 is an equivalence relation!

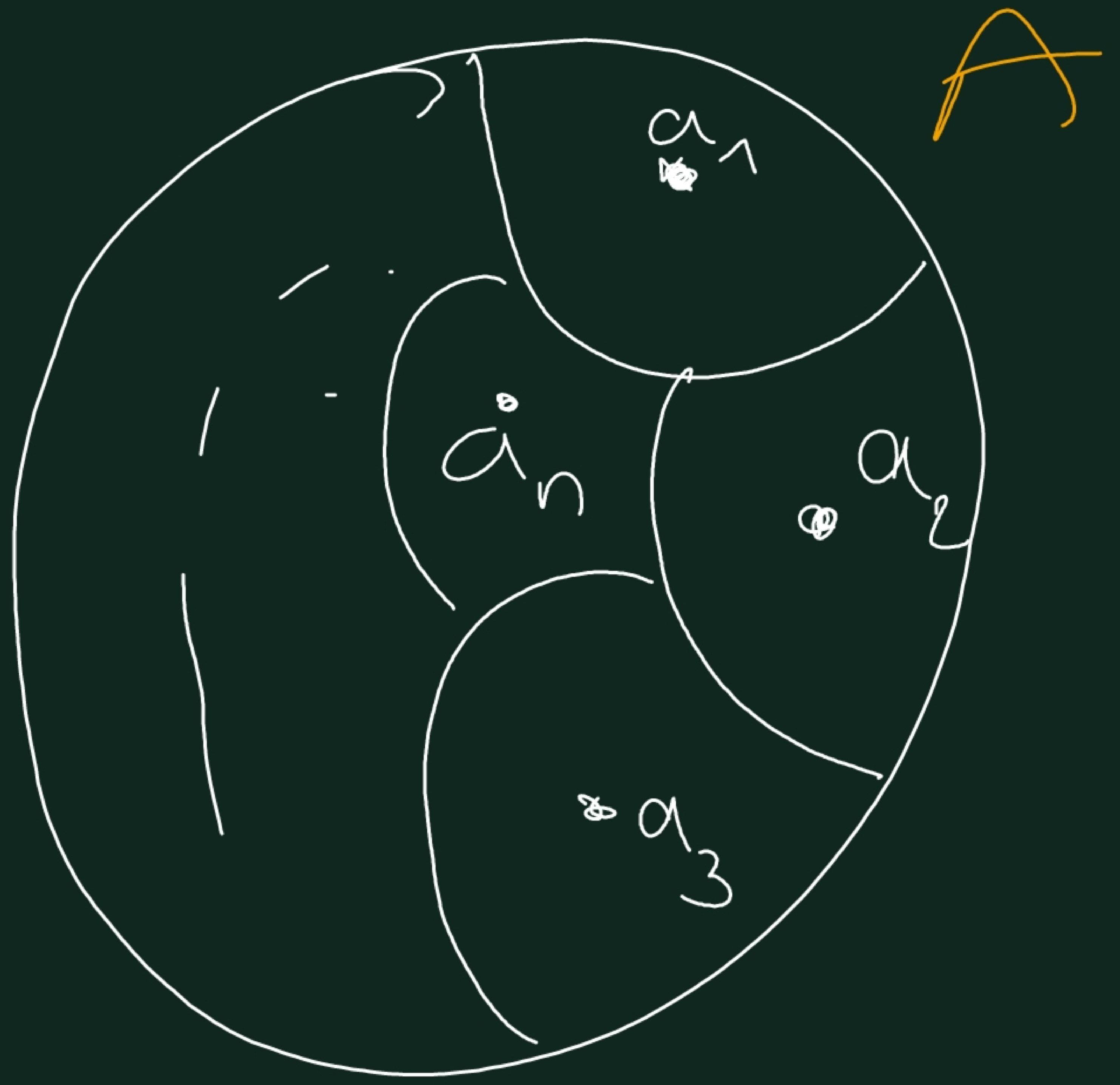
$u = O(v)$

$\exists n_1, c_1 \,/\, \forall n > n_1,\ u_n \leq c_1 v_n$

$v = O(w)$

$\exists n_2, c_2 \,/\, \forall n > n_2,\ v_n \leq c_2 w_n$

$\forall n > \underbrace{\max(n_1, n_2)}_{n_3}\ u_n \leq c_1 v_n \leq \underbrace{c_1 c_2}_{c_3} w_n$



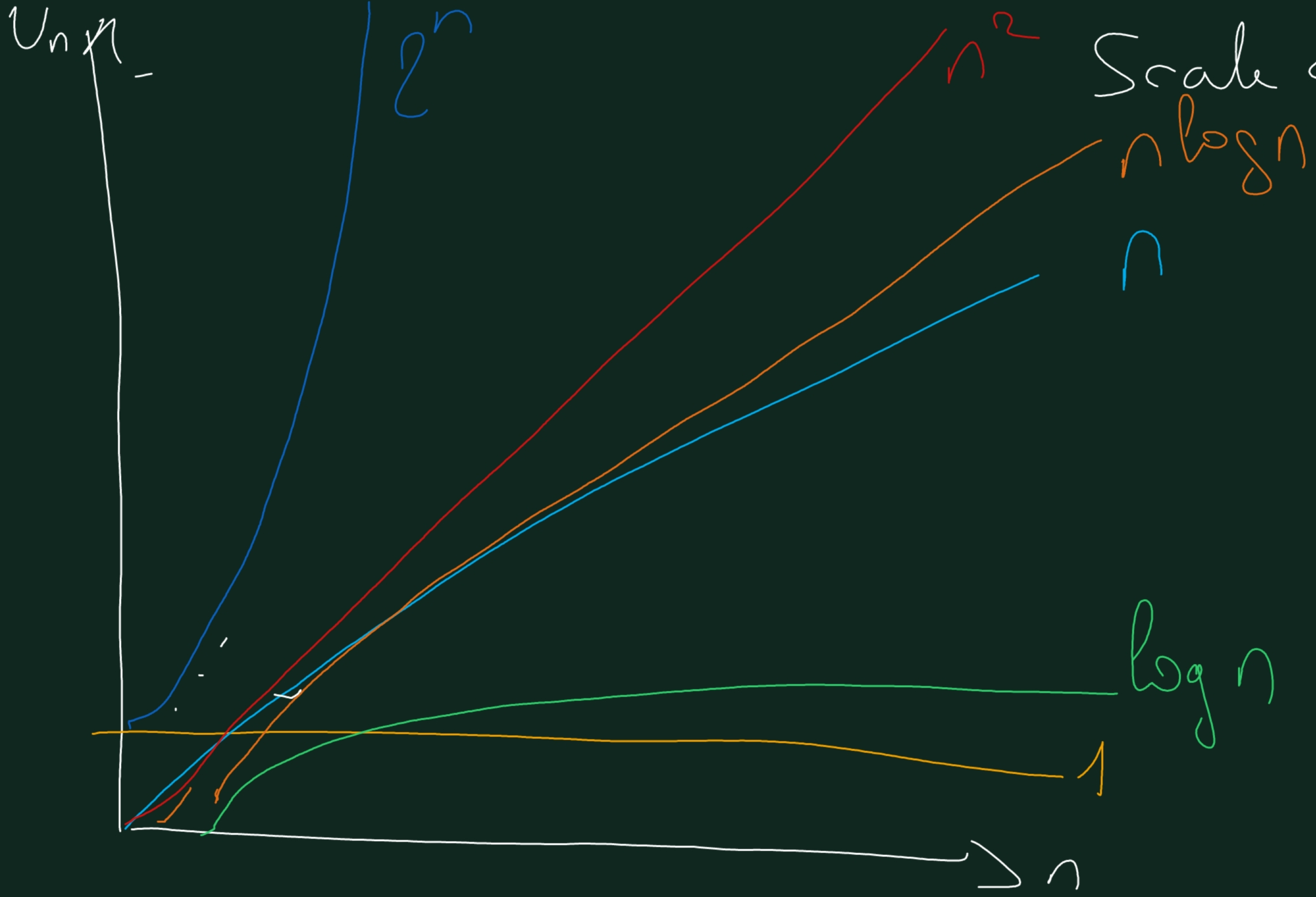
$\mathring{a}_1 = \{ x \in A \,/\, x \mathrel{R} a_1 \}$

"class of equivalence of a_1"

$\bigcup_i \mathring{a}_i = A$ and $\bigcap_i \mathring{a}_i = \varnothing$

classes of equivalence form a <u>partition</u> of A

Scale of complexity



Computing complexity (iterative algorithms)

step 1 : setup the context

- choose parameter(s)
- choose unit operation(s)

step 2 : apply rules :

r1 : $C(\{s1; s2; \dots; s_N\}) = \sum_{i=1}^{N} C(s_i)$

r2 : loops $C(\text{for } p \text{ in } (min \dots max)\ s_p) = \sum_{i=min}^{max} C(\text{iteration}_i)$

r3 : function call : $C(f(x)) = \sum_i s_i,\ s_i \in f$

r4 : conditional

3 possible situations

1) "best case"

$$C(\text{if (cond) ifYes else ifNo})$$

$$= C(\text{cond}) + \min(C(\text{ifYes}), C(\text{ifNo}))$$

2) "worst case"

$$C(\qquad)$$

$$= C(\text{cond}) + \max(\qquad)$$

3) "average case" : $p = \text{prob}(\text{cond} = \text{True})$

$$C(\qquad)$$

$$= C(\text{cond}) + p \times C(\text{ifYes}) + (1-p)\, C(\text{ifNo})$$

```
getMax(a) {
① res = a[0]
② for i in 1.. size(a)-1
③     if (res < a[i])
④         res = a[i]
⑤ return res
}
```

Param: N, size of a

unit op: <

$$C(N) = \cancel{C_1(N)} + C_2(N) + \cancel{C_5(N)}$$

$$= \sum_{i=1}^{N-1} C_3(i, N)$$

worst case

$$= \sum_{i=1}^{N-1} 1 + \cancel{\max(0, 0)}$$

$$= N - 1 = \Theta(N)$$

ex 2 : sorting an array

~~2~~	~~8~~	~~5~~	~~7~~	~~1~~	~~4~~
↑ l	↑ r	8	~~2~~	5	
1	~~5~~ ~~8~~ 2		~~5~~ 7		
	~~7~~ ~~5~~ 4				

```
sort (a) {
① for l in 0 .. size(a) - 2
②     for r in l+1 .. size(a) - 1
③         if (a[l] > a[r]) {
④             temp = a[l]
⑤             a[l] = a[r]
⑥             a[r] = temp
          }
}
```

param : N, size of array
unit ops : a, c

$$C(N) = C_1(N) = \sum_{l=0}^{N-2} C_2(l,N) = \sum_{l=0}^{N-2}\sum_{r=l+1}^{N-1} C_3(r,l,N) = \sum_{l=0}^{N-2}\sum_{r=l+1}^{N-1} c + \frac{1}{2}(3a) + \frac{1}{2}(0)$$

avg case $\Rightarrow p = 0.5\ (1/2)$

$$= \left(c + \frac{3a}{2}\right) \sum_{l=0}^{N-2} \sum_{r=l+1}^{N-1} 1 = \left(c + \frac{3a}{2}\right) \sum_{l=0}^{N-2} N-1-l \qquad k = N-1-l$$

$$= \left(c + \frac{3a}{2}\right) \sum_{k=1}^{N-1} k = \boxed{\left(c + \frac{3a}{2}\right) \frac{N(N-1)}{2} = C(N)}$$

$$= \frac{5}{2}\left(\frac{N^2 - N}{2}\right) = \Theta(N^2)$$

$$s_n = \sum_{i=1}^{n} i \overset{?}{=} \frac{n(n+1)}{2}$$

n	s_n
1	1
2	3
3	6
4	10
5	15
6	21

$\vdots$

$$S_N = 1 + 2 + 3 + \dots + (n-1) + n$$
$$S_N = n + (n-1) + (n-2) + \dots + 2 + 1$$
$$2S_N = n(n+1)$$

"Classical" sums:

1) $\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$

2) $\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$

3) $\sum_{i=0}^{N} 2^i = 2^{N+1} - 1$

n	s_n
0	1
1	3
2	7
3	15
4	31

by induction: $p(n): \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

step 1: $p(0)$ $\sum_{i=0}^{0} 2^i = 2^0 = 1$ $\quad 2^{0+1} - 1 = 2 - 1 = 1$

step 2: $p(n) \Rightarrow p(n+1)$ ok

hyp: $\sum_{i=0}^{N} 2^i = 2^{n+1} - 1$

$$\sum_{i=0}^{n+1} 2^i = \sum_{i=0}^{n} 2^i + 2^{n+1} = 2^{n+1} - 1 + 2^{n+1}$$

$$= 2 \cdot 2^{n+1} - 1$$

$$= 2^{n+2} - 1$$

qed

https://tiny.one/algo1

Thomas CORMEN, introduction to algorithms