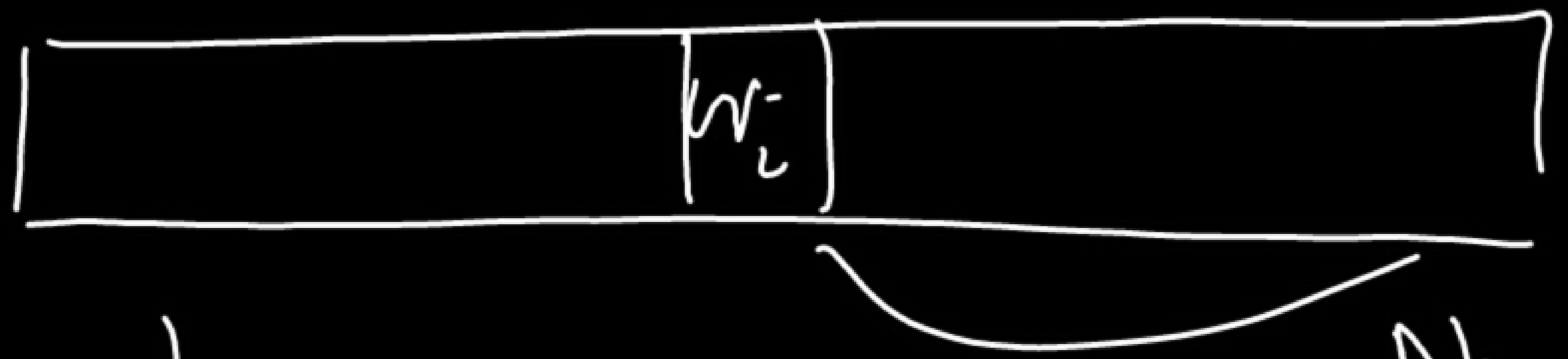
Philippe LAROQUE
laroque@cyu.fr



	① not sorted	② sorted	③ sorted, binary search
successful :	$N/2$	$N/2$	$u_n = \log_2 N$
unsuccessful :	N	$N/2$	

Ncomp	size
1	$N/2$
2	$N/4$
3	$N/8$
⋮	
i	$N/2^i$
? ($\log_2 N$)	1

$N = 10^6 \quad t_c = 10\,\mu s = 10^{-5}\,s$

sol 2: $\bar{T} = \frac{N}{2} t_c = 5 \cdot 10^5 \, 10^{-5} = 5\,s$

sol 3: $\bar{T} = \log 10^6 \times 10^{-5} = 6 \log 10 \cdot 10^{-5}$
$= 20 \cdot 10^{-5} = 0.2\,ms$

Sequences:

$$\begin{cases} u_n = 3n + 1 \\ u_0 = 1 \end{cases}$$

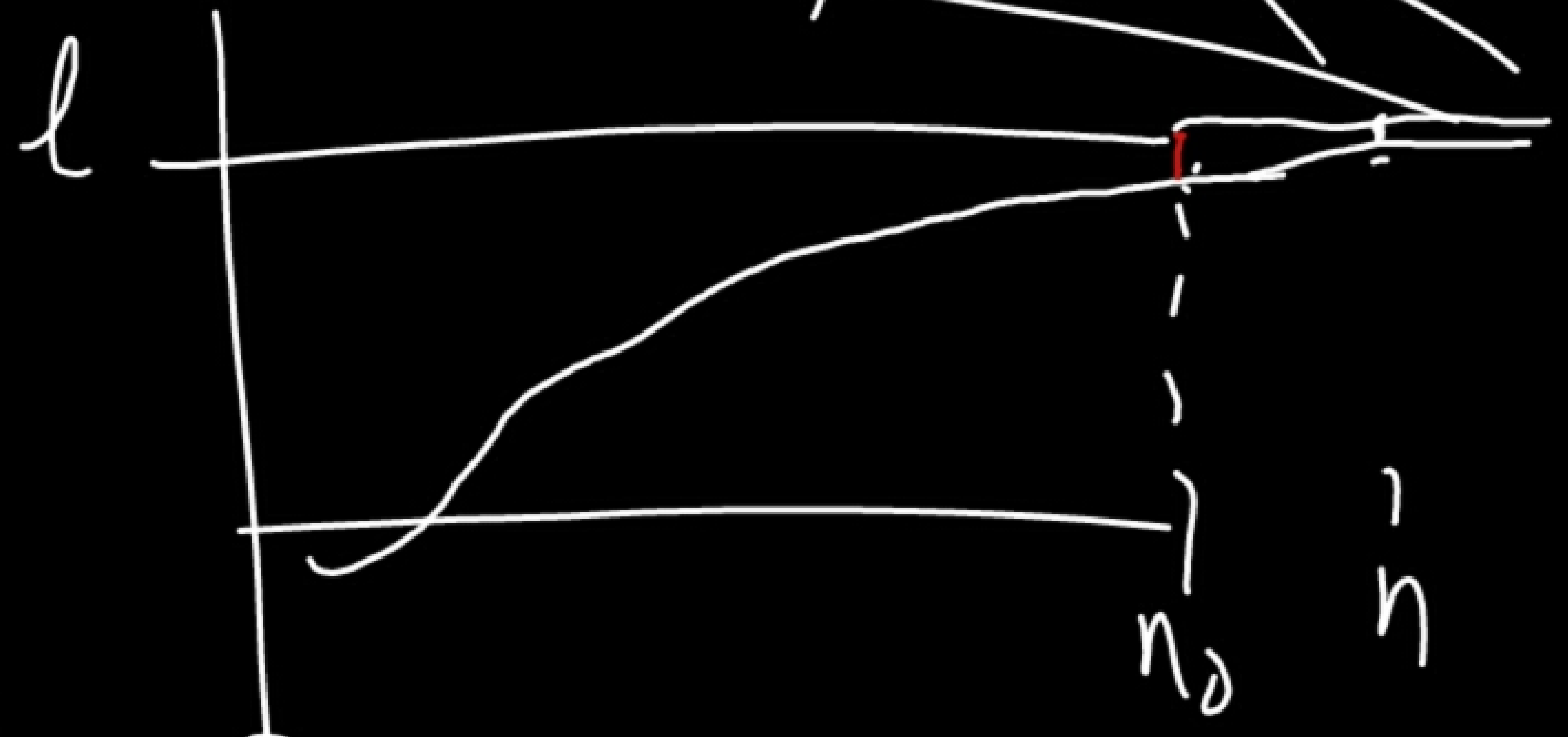
increasing sequences → upper bound (rare) $< 1\%$

→ not limited $> 99\%$

def: $\lim\limits_{n \to +\infty} u_n = l$

$\Updownarrow$

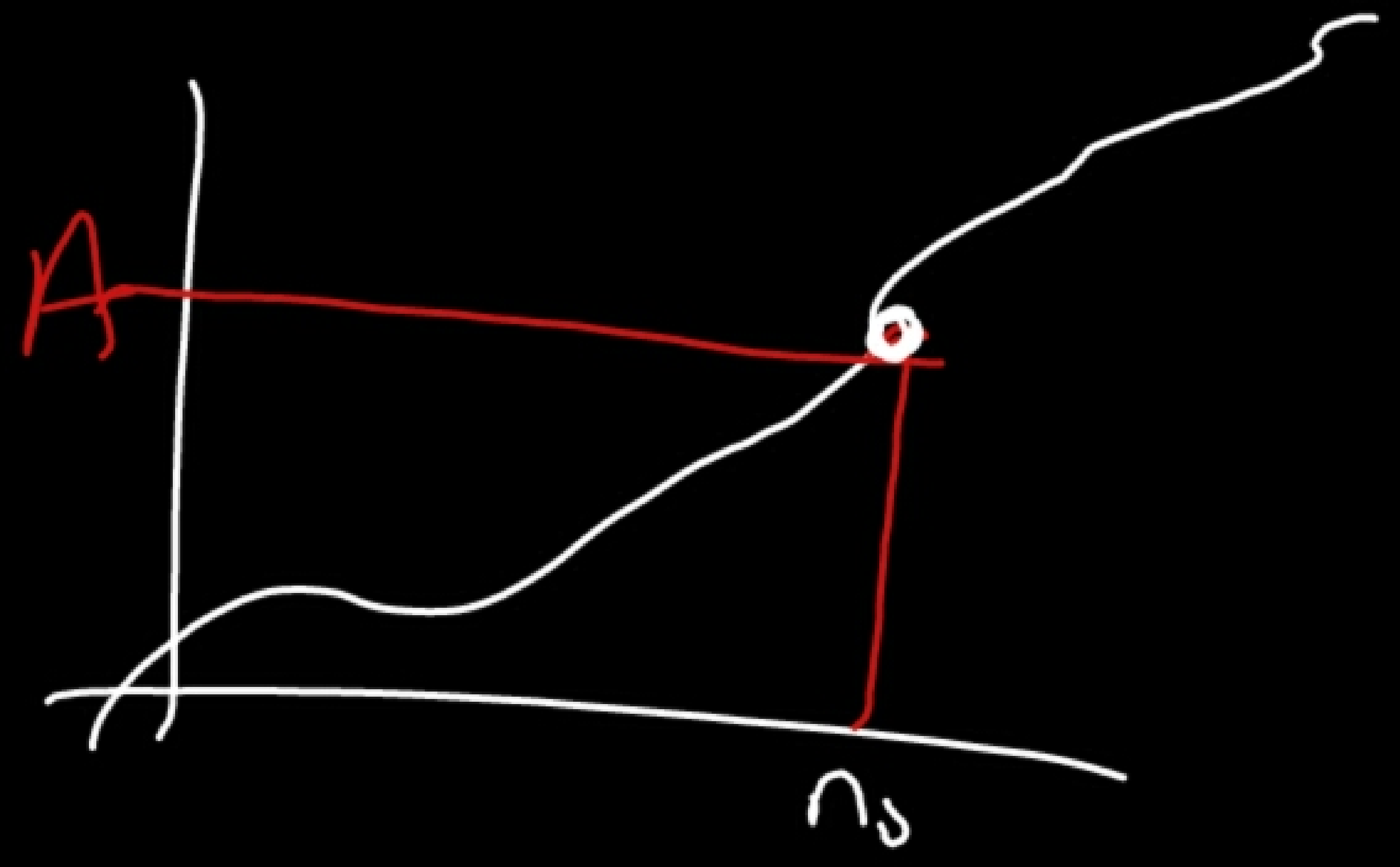
$\forall \varepsilon > 0, \exists n_0 \,/\, \forall n > n_0, \; l - u_n < \varepsilon$



def : $\lim\limits_{n \to +\infty} u_n = +\infty$

$\hookrightarrow \forall A > 0, \exists n_0 \,/\, \forall n > n_0, u_n > A$

A]



⚠ sol1, sol2 and sol3 are in this case

$\hookrightarrow$ need to define "interesting" subclasses

Landau notations

Context : increasing sequences

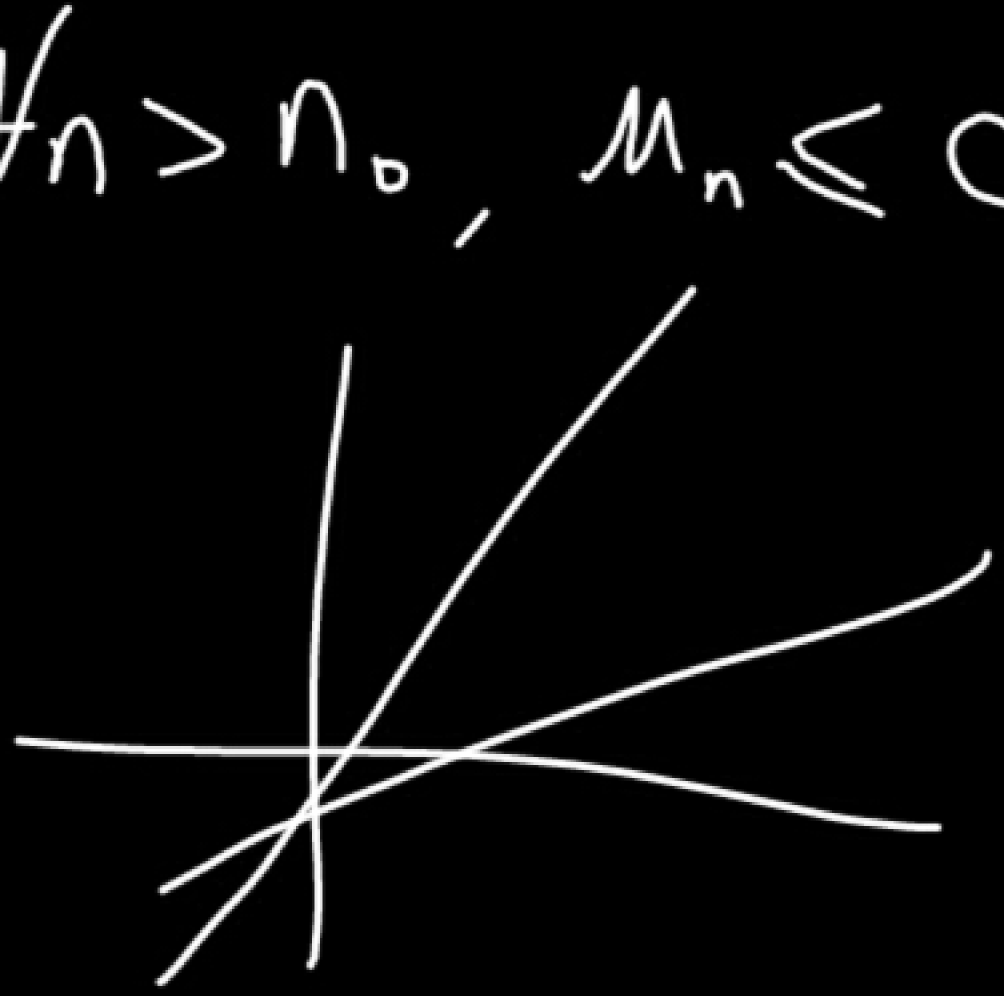
1) O : u, v sequences

$$u = O(v)$$

$$\Updownarrow$$

$$\exists\, n_0, c \,/\, \forall n > n_0,\ u_n \leq c\, v_n$$

"u is dominated by v"



$$u_n = n + 2$$

$$v_n = 5n - 1$$

$$v = O(u) ?$$

$6u_n$	n	u_n	v_n
18	1	3	4
24	2	4	9
30	3	5	14
36	4	6	19

$n_0 = 2,\ c = 6$

$$\begin{aligned} v_n &= 5n - 1 \\ &< 6n - 1 \\ &< 6n + 12 \\ &= 6\, u_n \end{aligned}$$

o : $u = o(v)$ "u is negligeable compared to v"

$\Updownarrow$

$\lim\limits_{n \to +\infty} \frac{u_n}{v_n} = 0$

example : $u_n = n+1$ $\quad u = o(v)$

$v_n = n^2 - 1$

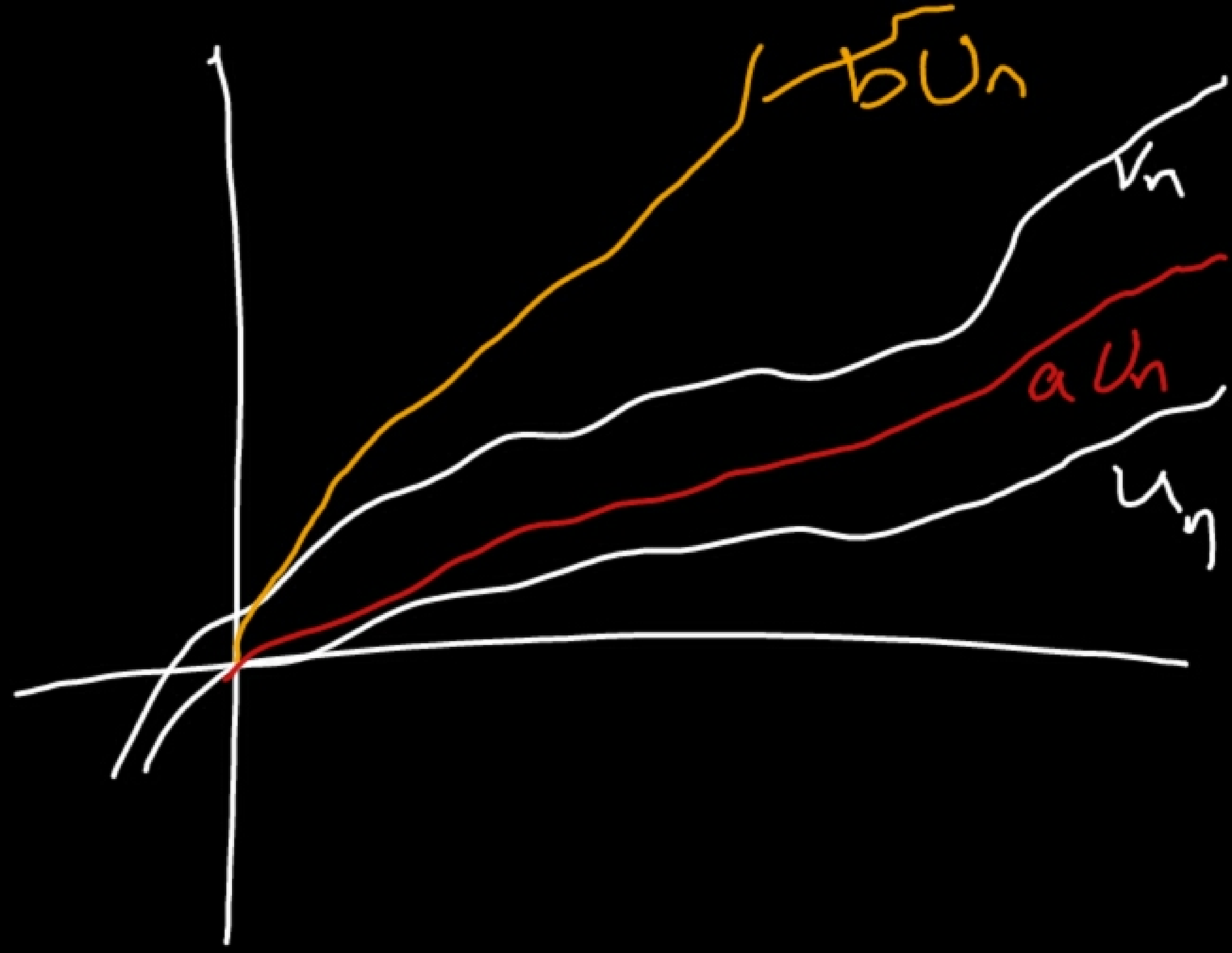
$\lim \frac{u_n}{v_n} = \lim \frac{n+1}{n^2-1} = \lim \frac{1}{n-1} = 0$

(4) $u = \Theta(v)$

$\Updownarrow$

$\exists n_0, a, b \,/\, \forall n > n_0, \quad a\,u_n \leq v_n \leq b\,u_n$

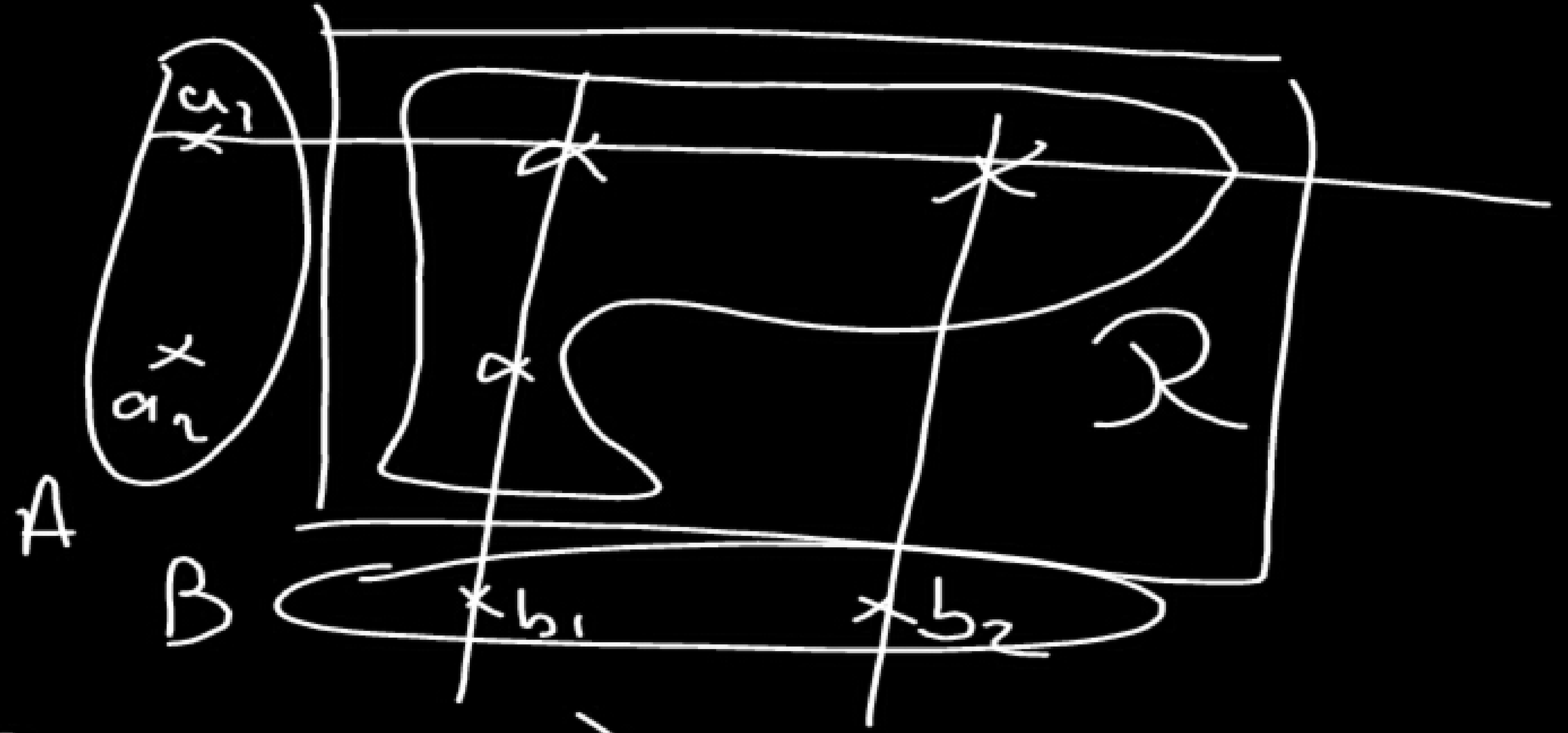
"u and v are of the same kind"



Relations

$R : A \longrightarrow B$

$x \longmapsto y$



$R \subset A \times B$

$A = B = \mathcal{S}$ (set of sequences)

$\emptyset$, σ and ④ are relations $\mathcal{S} \longrightarrow \mathcal{S}$

4 fundamental properties:

R is reflexive if $\forall u \in \mathcal{S}$, $u R u$

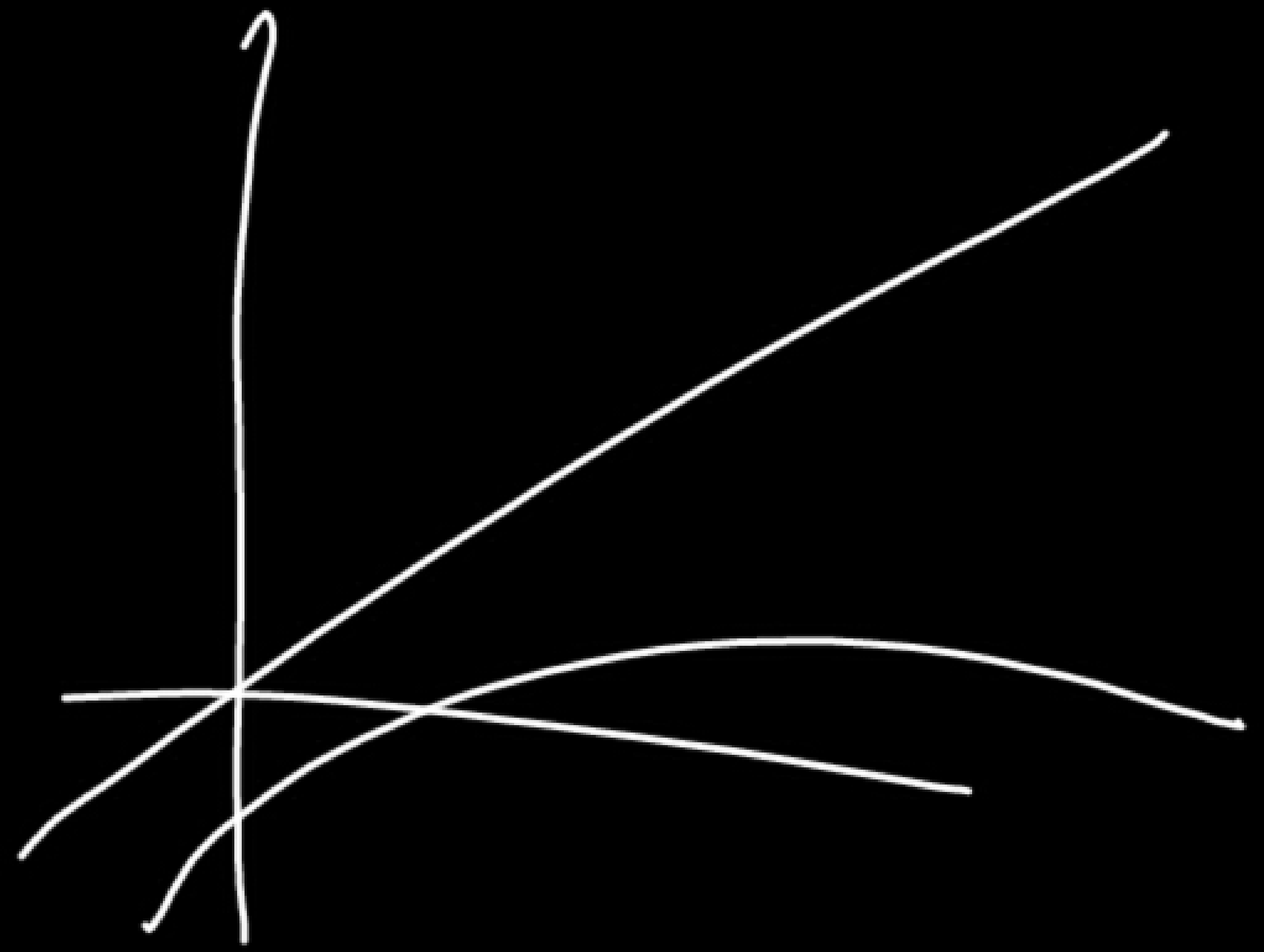
R is symmetric if $u R v \Longrightarrow v R u$

R is asymmetric if $\left\{\begin{array}{l} u R v \\ v R u \end{array}\right. \Longrightarrow u = v$

R is transitive if $\left\{\begin{array}{l} u R v \\ v R w \end{array}\right. \Longrightarrow u R w$

def: Order relation: R, A, T

Equivalence relation: R, S, T



	O	o	$\asymp$
R	Y	N	Y
S	N	N	Y
A	N	N/A	N
T	T	T	Y

$u = O(v)$ | $\exists n_1, c_1 / n > n_1 \Rightarrow u_n \leq c_1 v_n$

$v = O(w)$ | $\exists n_2, c_2 / n > n_2 \Rightarrow v_n \leq c_2 w_n$

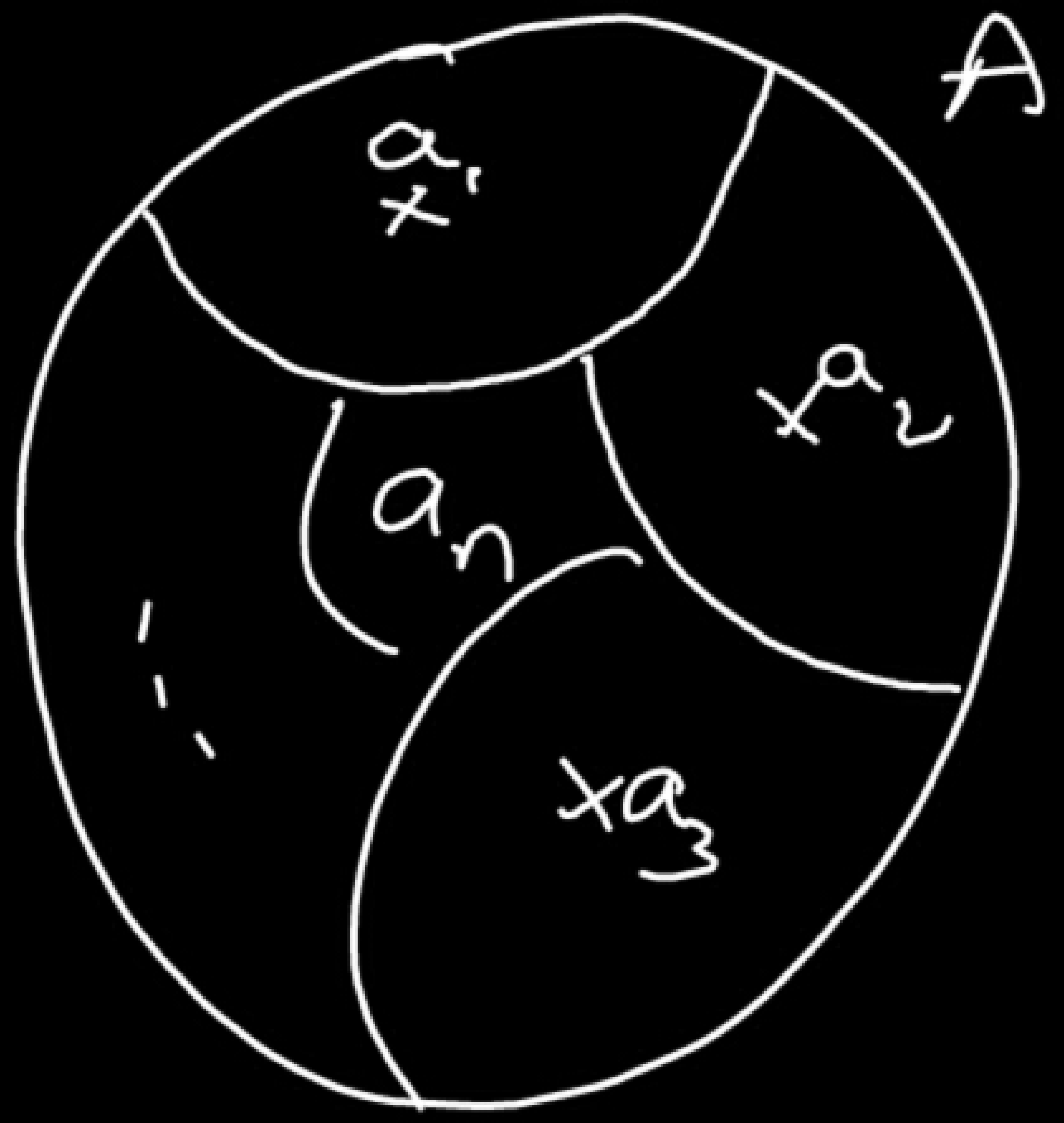
$n > \max(n_1, n_2) \Rightarrow u_n \leq c_1 v_n$

$\leq c_1 c_2 w_n$

$n_3 = \max(n_1, n_2)$

$c_3 = c_1 c_2$

$\asymp$ is an equivalence relation

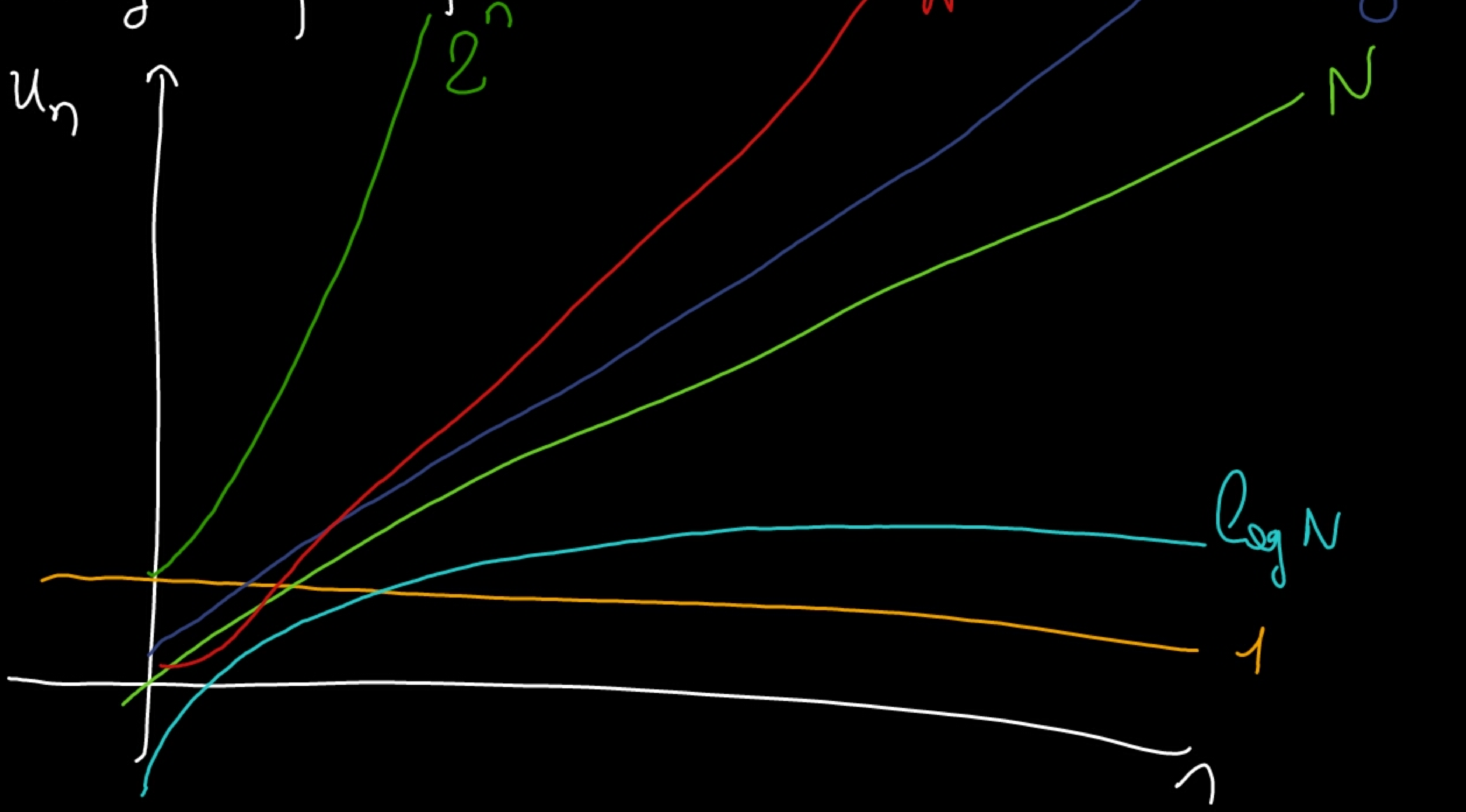


class of equivalence of a_1 : $\{x \,/\, x \mathcal{R} a_i\}$ $\dot{a}_1$

$$\left\{\begin{array}{l} \bigcup \dot{a}_i = A \\ \bigcap \dot{a}_i = \emptyset \end{array}\right.$$

$\{\dot{a}_i\}$ is a partition of A

Scale of complexity:



Evaluating complexity (iterative algorithms)

<u>steps:</u>

① define → the parameter(s)
→ the "unit operations": operation(s) whose complexity is (considered as) 1

② use <u>rules</u> to determine complexity

r1: $s = \{s_1, s_2, \dots, s_n\}$ $C(s) = \sum_{i=1}^{n} C(s_i)$

r2: loop: $C(\text{loop}) = \sum_{i=1}^{N} C(\text{iteration}_i)$

r3: function calls: ① compute function complexity ② replace call with the result

24 : conditional

$$C(\text{if cond then ifYes else ifNo})$$

1) "best case" :

$$C = C(\text{cond}) + \min\left(C(\text{ifYes}), C(\text{ifNo})\right)$$

2) "worst case"

$$C = C(\text{cond}) + \max\left(C(\text{ifYes}), C(\text{ifNo})\right)$$

3) "average case" : $p = \text{prob}(\text{cond} = \text{True})$

$$C = C(\text{cond}) + p\,C(\text{ifYes}) + (1-p)\,C(\text{ifNo})$$

```
getMax(a) {
1      res = a[0]
2      for i in 1..size(a)-1
3          if (a[i] > res)
4              res = a[i]
5      return res
}
```

param: N, size of a

unit op: comparison

$$C(N) = \cancel{C_1(N)} + C_2(N) + \cancel{C_5(N)}$$

$$= \sum_{i=1}^{N-1} C_3(i,N)$$

wc: $\sum_{i=1}^{N-1} \left(1 + \max(0,0)\right)$

$$= \sum_{i=1}^{N-1} 1 = N-1 = C_4(N)$$

bc: $\sum_{i=1}^{N-1} \left(1 + \min(0,0)\right) = N-1$

param: N (size of a)
unit op: c, a

4	2	8	5	7	1

```
sort(a) {
  for l in 0..size(a)-2
    for r in l+1..size(a)-1
      if (a[l] > a[r]) {
        temp = a[l]
        a[l] = a[r]
        a[r] = temp
      }
}
```

$k = N - 1 - \ell$

(avg case: $p(\text{cond}) = \frac{1}{2}$)

$$C(N) = C_1(N)$$

$$= \sum_{\ell=0}^{N-2} C_2(\ell, N)$$

$$= \sum_{\ell=0}^{N-2} \sum_{r=\ell+1}^{N-1} C_3(r, \ell, N)$$

$$= \sum_{\ell=0}^{N-2} \sum_{r=\ell+1}^{N-1} \left(c + \frac{1}{2} 3a + \frac{1}{2} 0 \right)$$

$$= \sum_{\ell=0}^{N-2} \left(c + \frac{3a}{2} \right) (N - 1 - \ell)$$

$$= \left(\sum_{k=0}^{N-1} k \right) \times \left(\frac{c + 3a}{2} \right)$$

$$= \frac{N(N-1)}{2} \left(\frac{c+3a}{2} \right)$$

$$= N(N-1) = \Theta(N^2)$$

Some "classical" sums:

1) $\sum_{i=0}^{N} i = \frac{N(N+1)}{2}$

$S_N = N + (N-1) + (N-2) + \dots + 2 + 1$

$S_N = 1 + 2 + 3 + \dots + (N-1) + N$

$2S_N = N(N+1)$

2) $\sum_{i=0}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$

3) $\sum_{i=0}^{N} 2^i = 2^{N+1} - 1$

$$\left[\sum_{i=0}^{N} a^i = \frac{a^{N+1} - 1}{a - 1} \right]$$

$p(N): \sum_{i=0}^{N} 2^i = 2^{N+1} - 1$

1) $p(0): \sum_{0}^{0} 2^i = 2^0 = 1$

$2^{0+1} - 1 = 2 - 1 = 1$

2) prove $p(n) \Rightarrow p(n+1)$

hyp : $\sum_{i=0}^{N} 2^i = 2^{N+1} - 1$

$$\sum_{i=0}^{N+1} 2^i = \sum_{i=0}^{N} 2^i + 2^{N+1}$$

$$= 2^{N+1} - 1 + 2^{N+1}$$

$$= 2 \cdot 2^{N+1} - 1$$

$$= 2^{N+2} - 1$$

QED

https://tiny.one/algo1

Thomas CORMEN, introduction to algorithms (3rd ed. ?)