2.4.1 First-Order Logic.

Prove the statements:

- $\forall x p(x) \wedge \forall x q(x) \rightarrow \forall x (p(x) \wedge q(x))$

$$\begin{aligned} &A = \forall x p(x) \wedge \forall x q(x) \\ &B = \forall x (p(x) \wedge q(x)) \\ &A \rightarrow B \end{aligned}$$

$$\neg(A \rightarrow B) \equiv A \wedge \neg B$$

$$\Big| \; \alpha$$

$$A, \neg B$$

$$\Big\|$$

$$\forall x p(x) \wedge \forall x q(x), \exists x (\neg p(x) \vee \neg q(x))$$

$$\vdots \; \text{instantiation}$$

$$p(a) \wedge q(a), \neg p(a) \vee \neg q(a)$$

$$\Big| \; \alpha$$

$$p(a), q(a), \neg p(a) \vee \neg q(a)$$

$$\begin{array}{cc} \beta \diagup & \diagdown \beta \\ p(a), q(a), \neg p(a) & p(a), q(a), \neg q(a) \\ \times & \times \end{array}$$

By contradiction of the inverse we found out that it's valid

- $\forall x (p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x))$ is a valid formula (but its converse $(\forall x p(x) \rightarrow \forall x q(x)) \rightarrow \forall x (p(x) \rightarrow q(x))$ is not).

$$\begin{aligned} &A = \forall x (p(x) \rightarrow q(x)) \\ &B = \forall x p(x) \rightarrow \forall x q(x) \\ &A \rightarrow B \end{aligned}$$

$$\neg(A \rightarrow B) \equiv A \wedge \neg B$$

$$\Big| \; \alpha$$

$$A, \neg B$$

$$\Big\|$$

$$\forall x (p(x) \rightarrow q(x)), \forall x p(x) \wedge \exists x \neg q(x)$$

$$\vdots \; \text{instantiation}$$

$$p(a) \rightarrow q(a), p(a) \wedge \neg q(a)$$

$$\Big| \; \alpha$$

$$p(a) \rightarrow q(a), p(a), \neg q(a)$$

$$\begin{array}{cc} \beta \diagup & \diagdown \beta \\ \neg p(a), p(a), \neg q(a) & q(a), p(a), \neg q(a) \\ \times & \times \end{array}$$

By contradiction of the inverse we found out that it's valid

2.4.2 First-Order Logic.

Prove that the formula $(\forall x p(x) \rightarrow \forall x q(x)) \rightarrow \forall x(p(x) \rightarrow q(x))$ is not valid by constructing a semantic tableau for its negation.

$$A = \forall x p(x) \rightarrow \forall x q(x)$$
$$B = \forall x(p(x) \rightarrow q(x))$$
$$A \rightarrow B$$

$$\neg(A \rightarrow B) \equiv A \wedge \neg B$$
$$\Big|\ \alpha$$
$$A, \neg B$$
$$\|$$
$$\forall x p(x) \rightarrow \forall x q(x), \neg \forall x(p(x) \rightarrow q(x))$$
$$\beta$$
$$\neg \forall x p(x), \neg \forall x(p(x) \rightarrow q(x)) \qquad \neg \forall x(p(x) \rightarrow q(x)), \forall x q(x)$$

$$\Big|\ \text{instantiation}$$
$$\neg p(a), \neg(p(a) \rightarrow q(a))$$
$$\Big|\ \alpha$$
$$p(a), \neg q(a), \neg p(a)$$

2.4.3 First-Order Logic.

Prove that the following formulas are valid

- $\exists x(\mathrm{A}(x) \to \mathrm{B}(x)) \leftrightarrow (\forall x\mathrm{A}(x) \to \exists x\mathrm{B}(x))$

$$F = \exists x(\mathrm{A}(x) \to \mathrm{B}(x))$$
$$G = (\forall x\mathrm{A}(x) \to \exists x\mathrm{B}(x))$$
$$F \to G, G \to F$$

$$\neg(F \to G) \equiv F \wedge \neg G$$
$$\Big| \ \alpha$$
$$F, \neg G$$
$$\|$$
$$\exists x(A(x) \to B(x)), \neg(\forall x A(x) \to \exists x B(x))$$
$$\Big| \ \alpha$$
$$\exists x(A(x) \to B(x)), \forall x A(x), \forall x \neg B(x)$$
$$\vdots \ \text{instantiation}$$
$$A(a) \to B(a), A(a), \neg B(a)$$
$$\beta \swarrow \quad \searrow \beta$$
$$\neg A(a), A(a), \neg B(a) \qquad B(a), A(a), \neg B(a)$$
$$\times \qquad \times$$

$$\neg(G \to F) \equiv G \wedge \neg F$$
$$\Big| \ \alpha$$
$$\neg F, G$$
$$\|$$
$$\neg\exists x(A(x) \to B(x)), (\forall x A(x) \to \exists x B(x))$$
$$\beta \swarrow \quad \searrow \beta$$
$$\neg\exists x(A(x) \to B(x)), \neg\forall x A(x) \qquad \neg\exists x(A(x) \to B(x)), \exists x B(x)$$
$$\vdots \ \text{instantiation} \qquad \vdots \ \text{instantiation}$$
$$\neg(A(a) \to B(a)), \neg A(a) \qquad \neg(A(a) \to B(a)), B(a)$$
$$\Big| \ \alpha \qquad \Big| \ \alpha$$
$$A(a), \neg B(a), \neg A(a) \qquad A(a), \neg B(a), B(a)$$
$$\times \qquad \times$$

It's true by contradiction of the inverse implications

- $(\exists x A(x) \to \forall x B(x)) \to \forall x(A(x) \to B(x))$

$$\neg((\exists x A(x) \to \forall x B(x)) \to \forall x(A(x) \to B(x)))$$
$$\Big| \ \alpha$$
$$(\exists x A(x) \to \forall x B(x)), \neg\forall x(A(x) \to B(x))$$
$$\beta \swarrow \quad \searrow \beta$$
$$\neg\exists x A(x), \neg\forall x(A(x) \to B(x))) \qquad \forall x B(x), \neg\forall x(A(x) \to B(x))$$
$$\vdots \ \text{instantiation} \qquad \vdots \ \text{instantiation}$$
$$\neg A(a), \neg(A(a) \to B(a)) \qquad B(a), \neg(A(a) \to B(a))$$
$$\Big| \ \alpha \qquad \Big| \ \alpha$$
$$\neg A(a), A(a), \neg B(a) \qquad B(a), A(a), \neg B(a)$$
$$\times \qquad \times$$

It's true by contradiction of the inverse

- $\forall x(A(x) \lor B(x)) \to (\forall x A(x) \lor \exists x B(x))$

$$\neg(\forall x(A(x) \lor B(x)) \to (\forall x A(x) \lor \exists x B(x)))$$

$$\Big| \; \alpha$$

$$\forall x(A(x) \lor B(x)), \neg(\forall x A(x) \lor \exists x B(x))$$

$$\Big| \; \alpha$$

$$\forall x(A(x) \lor B(x)), \neg\forall x A(x), \neg\exists x B(x)$$

$$\vdots \; \text{instantiation}$$

$$(A(a) \lor B(a)), \neg A(a), \neg B(a)$$

$$\overset{\beta}{\swarrow} \qquad \overset{\beta}{\searrow}$$

$$\begin{array}{cc} A(a), \neg A(a), \neg B(a) & B(a), \neg A(a), \neg B(a) \\ \times & \times \end{array}$$

It's true by contradiction of the inverse

- $\forall x(A(x) \to B(x)) \to (\exists x A(x) \to \exists x B(x))$

$$\neg(\forall x(A(x) \to B(x)) \to (\exists x A(x) \to \exists x B(x)))$$

$$\Big| \; \alpha$$

$$\forall x(A(x) \to B(x)), \neg(\exists x A(x) \to \exists x B(x))$$

$$\Big| \; \alpha$$

$$\forall x(A(x) \to B(x)), \exists x A(x), \neg\exists x B(x)$$

$$\vdots \; \text{instantiation}$$

$$(A(a) \to B(a)), A(a), \neg B(a)$$

$$\overset{\beta}{\swarrow} \qquad \overset{\beta}{\searrow}$$

$$\begin{array}{cc} \neg A(a), A(a), \neg B(a) & B(a), A(a), \neg B(a) \\ \times & \times \end{array}$$

It's true by contradiction of the inverse