

### 1.2.1

(a) Show that  $A \times (B \cap C) = (A \times B) \cap (A \times C)$

$$\begin{aligned}
 (x, y) \in A \times (B \cap C) &\implies x \in A \wedge y \in (B \cap C) \\
 &\implies x \in A \wedge y \in B \wedge y \in C \\
 (x, y) \in (A \times B) \cap (A \times C) &\implies (x, y) \in (A \times B) \wedge (x, y) \in (A \times C) \\
 &\implies x \in A \wedge y \in B \wedge x \in A \wedge y \in C \\
 &\implies x \in A \wedge y \in B \wedge y \in C
 \end{aligned}$$

(b) Show that  $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$$\begin{aligned}
 (x, y) \in A \times (B \cup C) &\implies x \in A \wedge y \in (B \cup C) \\
 &\implies x \in A \wedge (y \in B \vee y \in C) \\
 (x, y) \in (A \times B) \cup (A \times C) &\implies (x, y) \in (A \times B) \vee (x, y) \in (A \times C) \\
 &\implies (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \\
 &\implies x \in A \wedge (y \in B \vee y \in C)
 \end{aligned}$$

### 1.2.2

(a) Consider the relations  $R = \{(1, 7), (3, 3), (13, 11)\}$  and  $S = \{(1, 1), (1, 7), (3, 11), (13, 12), (15, 1)\}$  over the positive integers. Identify  $\text{dom}(R \cap S)$ ,  $\text{range}(R \cap S)$ ,  $\text{dom}(R \cup S)$ ,  $\text{range}(R \cup S)$

$$\begin{aligned}
 \text{dom}(R \cap S) &= \{1\} \\
 \text{range}(R \cap S) &= \{7\} \\
 \text{dom}(R \cup S) &= \{1, 3, 13, 15\} \\
 \text{range}(R \cup S) &= \{7, 3, 11, 1, 12\}
 \end{aligned}$$

(b) In the same example, identify  $\text{join}(R, S)$ ,  $\text{join}(S, R)$ ,  $S \circ R$ ,  $R \circ S$ ,  $R \circ R$ ,  $S \circ S$ .

$$\begin{aligned}
 \text{join}(R, S) &= \{(3, 3, 11)\} \\
 \text{join}(S, R) &= \{(1, 1, 7), (15, 1, 1)\} \\
 S \circ R &= \{(1, 7), (15, 1)\} \\
 R \circ S &= \{(3, 11)\} \\
 R \circ R &= \{(3, 3)\} \\
 S \circ S &= \{(1, 1), (1, 7), (15, 1), (15, 7)\}
 \end{aligned}$$

(c) In the same example, identify  $R(X)$  and  $S(X)$  for  $X = \{1, 3, 11\}$  and  $X = \emptyset$ .

$$\begin{array}{ll}
 X = \{1, 3, 11\} & X = \emptyset \\
 R(X) = \{(1, 7), (3, 3)\} & R(X) = \emptyset \\
 S(X) = \{(1, 1), (1, 7), (3, 11)\} & S(X) = \emptyset
 \end{array}$$

(d) Explain how to carry out composition by means of join and projection.

Composition is the result of first applying the *join* and then using the *projection* to eliminate the common item

### 1.2.3

(a) Show that  $R$  is reflexive over  $A$  iff  $I_A \subseteq R$ . Here  $I_A$  is the identity relation over  $A$ , defined in an exercise in Sect. 2.1.3. (b) Show that the converse of a relation  $R$  that is reflexive over a set  $A$  is also reflexive over  $A$ . (c) Show that  $R$  is transitive iff  $R \circ R \subseteq R$ .

### 1.2.4

(a) Show that the following three conditions are equivalent: (i)  $R$  is symmetric, (ii)  $R \subseteq R^{-1}$ , (iii)  $R = R^{-1}$ . (b) Show that if  $R$  is reflexive over  $A$  and also transitive, then the relation  $S$  defined by  $(a, b) \in S$  iff both  $(a, b) \in R$  and  $(b, a) \in R$  is an equivalence relation. (c) Enumerate all the partitions of  $A = \{1, 2, 3\}$  and draw a Hasse diagram for them under fineness.

### 1.2.5

Let  $R$  be any transitive relation over a set  $A$ . Define  $S$  over  $A$  by putting  $(a, b) \in S$  iff either  $a = b$  or both  $(a, b) \in R$  and  $\neg(b, a) \in R$ . Show that  $S$  partially orders  $A$ .