

1.2.1

(a) Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$

$$\begin{aligned}
 (x, y) \in A \times (B \cap C) &\implies x \in A \wedge y \in (B \cap C) \\
 &\implies x \in A \wedge y \in B \wedge y \in C \\
 (x, y) \in (A \times B) \cap (A \times C) &\implies (x, y) \in (A \times B) \wedge (x, y) \in (A \times C) \\
 &\implies x \in A \wedge y \in B \wedge x \in A \wedge y \in C \\
 &\implies x \in A \wedge y \in B \wedge y \in C
 \end{aligned}$$

(b) Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$$\begin{aligned}
 (x, y) \in A \times (B \cup C) &\implies x \in A \wedge y \in (B \cup C) \\
 &\implies x \in A \wedge (y \in B \vee y \in C) \\
 (x, y) \in (A \times B) \cup (A \times C) &\implies (x, y) \in (A \times B) \vee (x, y) \in (A \times C) \\
 &\implies (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \\
 &\implies x \in A \wedge (y \in B \vee y \in C)
 \end{aligned}$$

1.2.2

(a) Consider the relations $R = \{(1, 7), (3, 3), (13, 11)\}$ and $S = \{(1, 1), (1, 7), (3, 11), (13, 12), (15, 1)\}$ over the positive integers. Identify $\text{dom}(R \cap S)$, $\text{range}(R \cap S)$, $\text{dom}(R \cup S)$, $\text{range}(R \cup S)$

$$\begin{aligned}
 \text{dom}(R \cap S) &= \{1\} \\
 \text{range}(R \cap S) &= \{7\} \\
 \text{dom}(R \cup S) &= \{1, 3, 13, 15\} \\
 \text{range}(R \cup S) &= \{7, 3, 11, 1, 12\}
 \end{aligned}$$

(b) In the same example, identify $\text{join}(R, S)$, $\text{join}(S, R)$, $S \circ R$, $R \circ S$, $R \circ R$, $S \circ S$.

$$\begin{aligned}
 \text{join}(R, S) &= \{(3, 3, 11)\} \\
 \text{join}(S, R) &= \{(1, 1, 7), (15, 1, 1)\} \\
 S \circ R &= \{(1, 7), (15, 1)\} \\
 R \circ S &= \{(3, 11)\} \\
 R \circ R &= \{(3, 3)\} \\
 S \circ S &= \{(1, 1), (1, 7), (15, 1), (15, 7)\}
 \end{aligned}$$

(c) In the same example, identify $R(X)$ and $S(X)$ for $X = \{1, 3, 11\}$ and $X = \emptyset$.

$$\begin{array}{ll}
 X = \{1, 3, 11\} & X = \emptyset \\
 R(X) = \{(1, 7), (3, 3)\} & R(X) = \emptyset \\
 S(X) = \{(1, 1), (1, 7), (3, 11)\} & S(X) = \emptyset
 \end{array}$$

(d) Explain how to carry out composition by means of join and projection.

Composition is the result of first applying the *join* and then using the *projection* to eliminate the common item

1.2.3

(a) Show that R is reflexive over A iff $I_A \subseteq R$. Here I_A is the identity relation over A , defined in an exercise in Sect. 2.1.3.

$$\begin{aligned}
 R \text{ is reflexive in } A &= \forall x \in A : xRx \\
 I_A &= \{(a, a) : a \in A\} \\
 I_A &\subseteq R \\
 \forall a \in A &\implies \exists (a, a) \in I_A \\
 I_A \subseteq R &\implies \exists (a, a) \in R \\
 &\implies R \text{ is reflexive on } A
 \end{aligned}$$

(b) Show that the converse of a relation R that is reflexive over a set A is also reflexive over A .

$$\begin{aligned}
 R \text{ is reflexive over } A &= \forall x \in A : xRx \\
 R^{-1} &= \{(a, b) : (b, a) \in R\} \\
 R = \{(a, b) : a = b \wedge a, b \in A\} &\implies (a, b) = (b, a) \\
 &\implies R^{-1} = R \\
 &\implies R^{-1} \text{ is reflexive over } A
 \end{aligned}$$

(c) Show that R is transitive iff $R \circ R \subseteq R$.

$$\begin{aligned}
 R \text{ is transitive} &\iff R \circ R \subseteq R \\
 R \circ R &= \{(a, c) : aRb \wedge bRc\} \\
 R \circ R \subseteq R &\implies (a, c) \in R \\
 &\implies R(R(a, b), c) = R(a, R(b, c))
 \end{aligned}$$

1.2.4

(a) Show that the following three conditions are equivalent: (i) R is symmetric, (ii) $R \subseteq R^{-1}$, (iii) $R = R^{-1}$. (b) Show that if R is reflexive over A and also transitive, then the relation S defined by $(a, b) \in S$ iff both $(a, b) \in R$ and $(b, a) \in R$ is an equivalence relation. (c) Enumerate all the partitions of $A = \{1, 2, 3\}$ and draw a Hasse diagram for them under fineness.

1.2.5

Let R be any transitive relation over a set A . Define S over A by putting $(a, b) \in S$ iff either $a = b$ or both $(a, b) \in R$ and $\neg(b, a) \in R$. Show that S partially orders A .