1.4.1 Proof by simple induction

a) Use simple induction to show that for every positive integer n, $5^n - 1$ is divisible by 4
$5^n - 1$ is divisible by 4
Base case:

$$\begin{aligned} 5^1 - 1 &= 4j \\ &= 4 = 4j \\ 5^k - 1 &= 4j \\ 5^k &= 4j + 1 \end{aligned}$$

Induction step:

$$\begin{aligned} &= 5^{k+1} - 1 \\ &= 5^k \cdot 5 - 1 \\ &= (4j+1) \cdot 5 - 1 \\ &= (20j+5) - 1 \\ &= 20j + 4 \\ &= 4 \cdot (5j+1) \\ &\implies true\ for\ n = k+1 \end{aligned}$$

b) Use simple induction to show that for every positive integer n, $n^3 - n$ is divisible by 3. (Hint: In the induction step, you will need to make use of the arithmetic fact that $(k+1)^3 = k^3 + 3k^2 + 3k + 1$)
$n^3 - n$ is divisible by 3
Base case:

$$\begin{aligned} 1^3 - 1 &= 3j \\ &= 0 = 3j \\ k^3 - k &= 3j \\ k^3 &= 3j + k \end{aligned}$$

Induction step:

$$\begin{aligned} &= (k+1)^3 - (k+1) \\ &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (3j + k) + 3k^2 + 2k \\ &= 3j + 3k^2 + 3k \\ &= 3 \cdot (j + k^2 + k) \\ &\implies true\ for\ n = k+1 \end{aligned}$$

c) Show by simple induction that for every natural number n, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$
$f(n) = 2^{n+1} - 1$
Base case:

$$\begin{aligned} f(1) &= 2^{1+1} - 1 \\ f(1) &= 4 - 1 \\ f(1) &= 3 \\ f(k) &= 2^{k+1} - 1 \end{aligned}$$

Induction step:

$$
\begin{aligned}
f(k+1) &= 2^{k+2} - 1 \\
f(k+1) &= 2^1 + 2^2 ... + 2^k + 2^{k+1} \\
&= f(k) + 2^{k+1} \\
&= 2^{k+1} - 1 + 2^{k+1} \\
&= 2(2^{k+1}) - 1 \\
&= 2^1 \cdot 2^{k+1} - 1 \\
&= 2^{k+1+1} - 1 \\
&= 2^{k+2} - 1 \\
&\implies true\ for\ n = k+1
\end{aligned}
$$

1.4.2: Definition by simple recursion

a) Let $f : N \to N$ be the function defined by putting $f(0) = 0$ and $f(n+1) = n$ for all $n \in N$.

  i) Evaluate this function bottom-up for all arguments 0–5.

$$
\begin{aligned}
f(0) &= 0 \\
f(1) &= 0 \\
f(2) &= 1 \\
f(3) &= 2 \\
f(4) &= 3 \\
f(5) &= 4
\end{aligned}
$$

  ii) Explain what f does by expressing it in explicit terms (i.e. without a recursion).
  This function is not recursive, it only returns the antecesor of the current value

b) Let $f : N^+ \to N$ be the function that takes each positive integer n to the greatest natural number p with $2^p \leq n$. Define this function by a simple recursion. (Hint: You will need to divide the recursion step into two cases.)

$$
\begin{aligned}
f(n) &= 0 && , when\ n = 1 \\
f(n) &= f(n-1) && , when\ n > 1, \log_2(n) \notin N \\
f(n) &= f(n-1) + 1 && , when\ n > 1, \log_2(n) \in N
\end{aligned}
$$

c) Let $g : NXN \to N$ be defined by putting $g(m, 0) = m$ for all $m \in N$ and $g(m, n+1) = f(g(m, n))$ where f is the function defined in part (a) of this exercise.

  i) Evaluate $g(3, 4)$ top-down.

$$
\begin{aligned}
g(3,4) &= f(g(3,3)) \\
&= f(f(g(3,2))) \\
&= f(f(f(g(3,1)))) \\
&= f(f(f(f(g(3,0))))) \\
&= f(f(f(f(f(3))))) \\
&= f(f(f(f(2)))) \\
&= f(f(f(1))) \\
&= f(f(0)) \\
&= f(0) \\
&= 0
\end{aligned}
$$

ii) Explain what g does by expressing it in explicit terms (i.e. without a recursion).
It substracts the right item from the left, but if the remain is negative, it returns 0

1.4.3: Proof by cumulative induction

a) Use cumulative induction to show that any postage cost of four or more pence can be covered by two-pence and five-pence stamps.
$2x + 5y = n$, when $n \geq 4$
Base case:

$$n = 4 \qquad\qquad 4 = 2(2) + 5(0)$$

Induction step:

$$\begin{aligned} hypothesis &\rightarrow \forall j < k,\ j = 2x + 5y \\ goal &\rightarrow k = 2x + 3y \\ case(1) &\rightarrow k\ is\ multiple\ of\ two \implies y = 0\ and\ x \in N \\ case(2) &\rightarrow k\ is\ multiple\ of\ five \implies x = 0\ and\ y \in N \\ case(3) &\rightarrow k\ is\ not\ multiple\ of\ any \implies y \in N\ and\ x \in N \end{aligned}$$

b) Use cumulative induction to show that for every natural number n, $F(n) \leq 2^n - 1$, where F is the Fibonacci function.
$F(n) \leq 2^n - 1$,
Base case:

$$n \leq 1 \qquad\qquad 1 \leq 2^1 - 1$$

Induction step:

$$\begin{aligned} hypothesis &\rightarrow \forall j < k,\ F(j) \leq 2^j - 1 \\ goal &\rightarrow F(k) \leq 2^k - 1 \\ case(1) &\rightarrow F(k)\ is\ greater\ than\ 2^k - 1 \implies 2^{k+1} - 1 < F(j+1) \\ &\quad Impossible\ given\ that\ F\ grows\ at\ F(k)\ and\ 2^k - 1\ grows\ at\ double\ rate \\ case(2) &\rightarrow F(k)\ is\ less\ or\ equal\ than\ 2^k - 1e \implies This\ must\ be\ true\ then \end{aligned}$$

c) Calculate F(5) top-down, and then again bottom-up, where again F is the Fibonacci function
Top-down:

$$\begin{aligned} F(5) &= F(4) + F(3) \\ F(5) &= (F(3) + F(2)) + (F(2) + F(1)) \\ F(5) &= ((F(2) + F(1)) + (F(1) + F(0))) + ((F(1) + F(0)) + 1) \\ F(5) &= (((F(1) + F(0)) + 1) + (1 + 0)) + ((1 + 0) + 1) \\ F(5) &= (((1 + 0) + 1) + (1 + 0)) + ((1 + 0) + 1) \\ F(5) &= ((1 + 1) + (1)) + (1 + 1) \\ F(5) &= (2 + 1) + (2) \\ F(5) &= 3 + 2 \\ F(5) &= 5 \end{aligned}$$

Bottom-up

$$
\begin{aligned}
&F(0) = 0 \\
&F(1) = 1 \\
&F(2) = F(1) + F(0) = 1 + 0 = 1 \\
&F(3) = F(2) + F(1) = 1 + 1 = 2 \\
&F(4) = F(3) + F(2) = 2 + 1 = 3 \\
&F(5) = F(4) + F(3) = 3 + 2 = 5
\end{aligned}
$$

d) Express each of the numbers 14, 15 and 16 as a sum of $3s$ and/or $8s$. Using this fact in your basis, show by cumulative induction that every positive integer $n \leq 14$ may be expressed as a sum of $3s$ and/or $8s$.
$n = 3x + 8y$, when $n > 14$
Base case:

$$
\begin{aligned}
n &= 14 & 14 &= 3(2) + 8(1) \\
n &= 15 & 15 &= 3(5) + 8(0) \\
n &= 16 & 16 &= 3(0) + 8(2)
\end{aligned}
$$

Induction step:

$$
\begin{aligned}
hypothesis &\rightarrow \forall j < k,\ j = 3x + 8y \\
goal &\rightarrow k = 3x + 8y \\
case(1) &\rightarrow k\ is\ multiple\ of\ three \implies y = 0\ and\ x \in N \\
case(2) &\rightarrow k\ is\ multiple\ of\ eigth \implies x = 0\ and\ y \in N \\
case(3) &\rightarrow k\ is\ not\ multiple\ of\ any \implies y \in N\ and\ x \in N
\end{aligned}
$$

e) Show by induction that for every natural number n, $A(1, n) = n + 2$, where A is the Ackermann function.

$$
A(m,n) = \begin{cases} n+1 & if\ m = 0 \\ A(m-1, 1) & if\ m > 0\ and\ n = 0 \\ A(m-1, A(m, n-1)) & if\ m > 0\ and\ n > 0 \end{cases}
$$

Base case:

$$
\begin{aligned}
n &= 0 & A(1,0) &= A(0,1) \\
& & &= 2 \\
& & &= (0) + 2 \\
n &= 1 & A(1,1) &= A(0, A(1,0)) \\
& & &= A(0,2) \\
& & &= 3 \\
& & &= (1) + 2
\end{aligned}
$$

Induction step:

$$
\begin{aligned}
hypothesis &\rightarrow \forall j < k,\ A(1,j) = j + 2 \\
goal &\rightarrow A(1,k) = k + 2 \\
A(1, k+1) &= (k+1) + 2 \\
&= A(0, A(1,k)) \\
&= A(1,k) + 1 \\
&= (k+2) + 1 \\
A(1, k+1) &= k + 3 \\
&\therefore A(1,k) = k + 2
\end{aligned}
$$