

### 2.4.1 First-Order Logic.

Prove the statements:

- $\forall x p(x) \wedge \forall x q(x) \rightarrow \forall x (p(x) \wedge q(x))$

$$\begin{array}{c}
 A = \forall x p(x) \wedge \forall x q(x) \\
 B = \forall x (p(x) \wedge q(x)) \\
 A \rightarrow B \\
 \neg(A \rightarrow B) \equiv A \wedge \neg B \\
 \begin{array}{c} | \\ \alpha \\ A, \neg B \end{array} \\
 \parallel \\
 \forall x p(x) \wedge \forall x q(x), \exists x (\neg p(x) \vee \neg q(x)) \\
 \vdots \text{ instantiation} \\
 p(a) \wedge q(a), \neg p(a) \vee \neg q(a) \\
 \begin{array}{c} | \\ \alpha \\ p(a), q(a), \neg p(a) \vee \neg q(a) \end{array} \\
 \begin{array}{cc} \beta / & \backslash \beta \\ p(a), q(a), \neg p(a) & p(a), q(a), \neg q(a) \end{array} \\
 \begin{array}{cc} \times & \times \end{array}
 \end{array}$$

By contradiction of the inverse we found out that it's valid

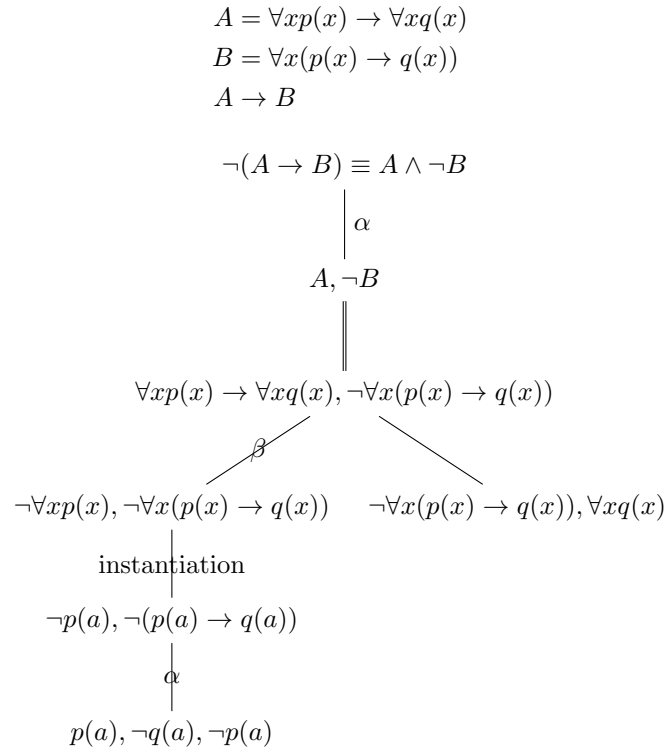
- $\forall x (p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x))$  is a valid formula (but its converse  $(\forall x p(x) \rightarrow \forall x q(x)) \rightarrow \forall x (p(x) \rightarrow q(x))$  is not).

$$\begin{array}{c}
 A = \forall x (p(x) \rightarrow q(x)) \\
 B = \forall x p(x) \rightarrow \forall x q(x) \\
 A \rightarrow B \\
 \neg(A \rightarrow B) \equiv A \wedge \neg B \\
 \begin{array}{c} | \\ \alpha \\ A, \neg B \end{array} \\
 \parallel \\
 \forall x (p(x) \rightarrow q(x)), \forall x p(x) \wedge \exists x \neg q(x) \\
 \vdots \text{ instantiation} \\
 p(a) \rightarrow q(a), p(a) \wedge \neg q(a) \\
 \begin{array}{c} | \\ \alpha \\ p(a) \rightarrow q(a), p(a), \neg q(a) \end{array} \\
 \begin{array}{cc} \beta / & \backslash \beta \\ \neg p(a), p(a), \neg q(a) & q(a), p(a), \neg q(a) \end{array} \\
 \begin{array}{cc} \times & \times \end{array}
 \end{array}$$

By contradiction of the inverse we found out that it's valid

## 2.4.2 First-Order Logic.

Prove that the formula  $(\forall x p(x) \rightarrow \forall x q(x)) \rightarrow \forall x (p(x) \rightarrow q(x))$  is not valid by constructing a semantic tableau for its negation.



### 2.4.3 First-Order Logic.

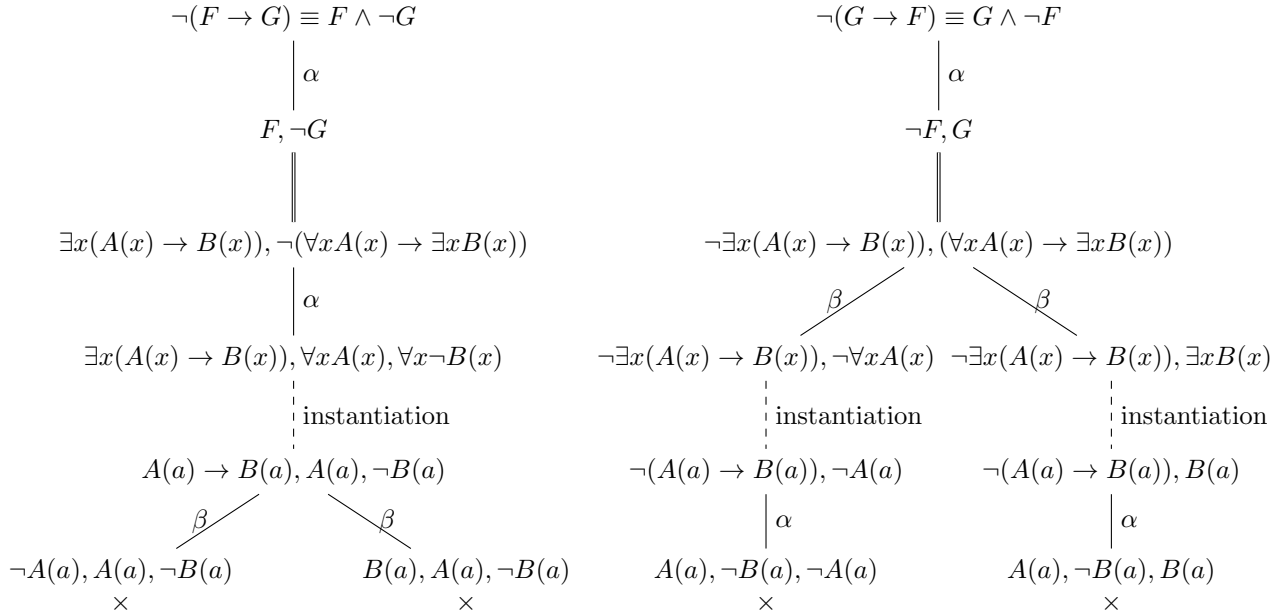
Prove that the following formulas are valid

- $\exists x(A(x) \rightarrow B(x)) \leftrightarrow (\forall x A(x) \rightarrow \exists x B(x))$

$$F = \exists x(A(x) \rightarrow B(x))$$

$$G = (\forall x A(x) \rightarrow \exists x B(x))$$

$$F \rightarrow G, G \rightarrow F$$



It's true by contradiction of the inverse implications

- $(\exists x A(x) \rightarrow \forall x B(x)) \rightarrow \forall x(A(x) \rightarrow B(x))$
- $\forall x(A(x) \vee B(x)) \rightarrow (\forall x A(x) \vee \exists x B(x))$
- $\forall x(A(x) \rightarrow B(x)) \rightarrow (\exists x A(x) \rightarrow \exists x B(x))$