

2.4.1 First-Order Logic.

Prove the statements:

- $\forall x p(x) \wedge \forall x q(x) \rightarrow \forall x (p(x) \wedge q(x))$

$$\begin{array}{c}
 A = \forall x p(x) \wedge \forall x q(x) \\
 B = \forall x (p(x) \wedge q(x)) \\
 A \rightarrow B \\
 \neg(A \rightarrow B) \equiv A \wedge \neg B \\
 \begin{array}{c} | \\ \alpha \\ A, \neg B \end{array} \\
 \parallel \\
 \forall x p(x) \wedge \forall x q(x), \exists x (\neg p(x) \vee \neg q(x)) \\
 \vdots \text{ instantiation} \\
 p(a) \wedge q(a), \neg p(a) \vee \neg q(a) \\
 \begin{array}{c} | \\ \alpha \\ p(a), q(a), \neg p(a) \vee \neg q(a) \end{array} \\
 \begin{array}{cc} \beta / & \backslash \beta \\ p(a), q(a), \neg p(a) & p(a), q(a), \neg q(a) \end{array} \\
 \begin{array}{cc} \times & \times \end{array}
 \end{array}$$

By contradiction of the inverse we found out that it's valid

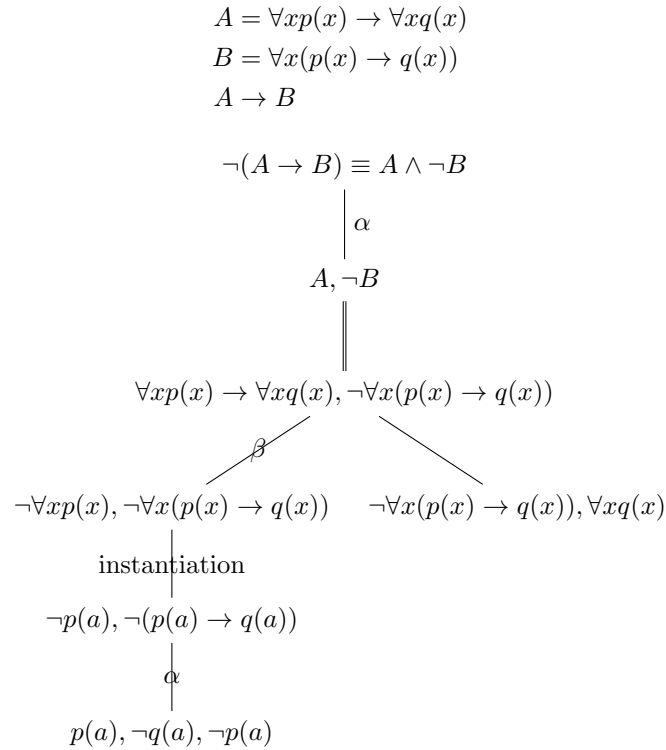
- $\forall x (p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x))$ is a valid formula (but its converse $(\forall x p(x) \rightarrow \forall x q(x)) \rightarrow \forall x (p(x) \rightarrow q(x))$ is not).

$$\begin{array}{c}
 A = \forall x (p(x) \rightarrow q(x)) \\
 B = \forall x p(x) \rightarrow \forall x q(x) \\
 A \rightarrow B \\
 \neg(A \rightarrow B) \equiv A \wedge \neg B \\
 \begin{array}{c} | \\ \alpha \\ A, \neg B \end{array} \\
 \parallel \\
 \forall x (p(x) \rightarrow q(x)), \forall x p(x) \wedge \exists x \neg q(x) \\
 \vdots \text{ instantiation} \\
 p(a) \rightarrow q(a), p(a) \wedge \neg q(a) \\
 \begin{array}{c} | \\ \alpha \\ p(a) \rightarrow q(a), p(a), \neg q(a) \end{array} \\
 \begin{array}{cc} \beta / & \backslash \beta \\ \neg p(a), p(a), \neg q(a) & q(a), p(a), \neg q(a) \end{array} \\
 \begin{array}{cc} \times & \times \end{array}
 \end{array}$$

By contradiction of the inverse we found out that it's valid

2.4.2 First-Order Logic.

Prove that the formula $(\forall x p(x) \rightarrow \forall x q(x)) \rightarrow \forall x(p(x) \rightarrow q(x))$ is not valid by constructing a semantic tableau for its negation.



2.4.3 First-Order Logic.

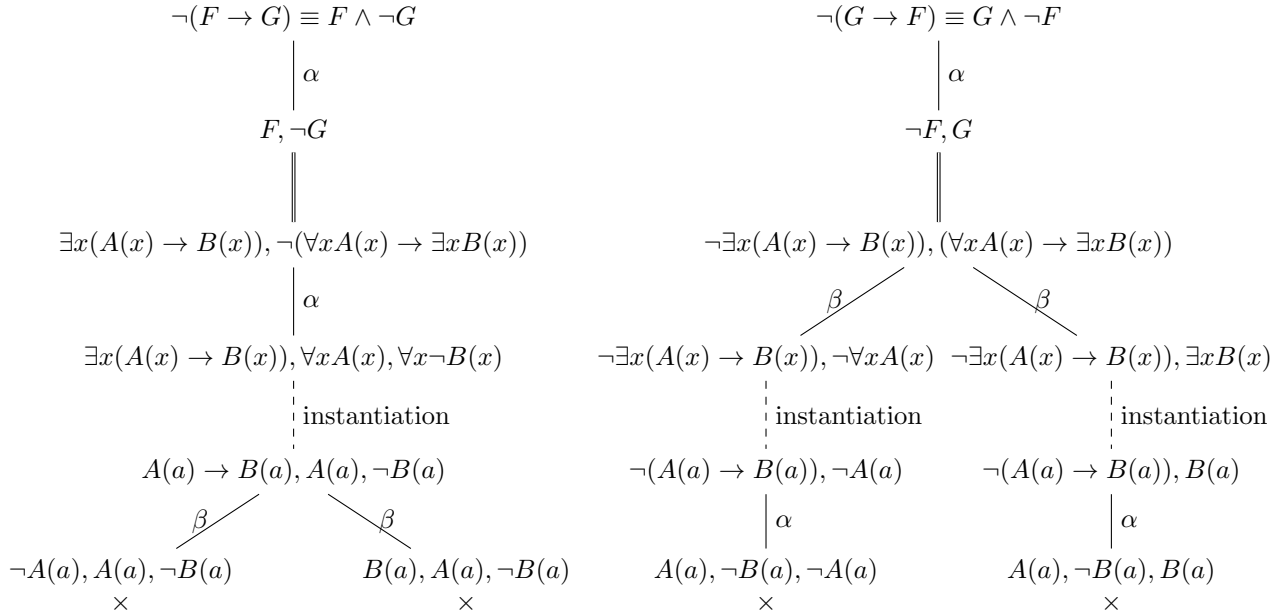
Prove that the following formulas are valid

- $\exists x(A(x) \rightarrow B(x)) \leftrightarrow (\forall x A(x) \rightarrow \exists x B(x))$

$$F = \exists x(A(x) \rightarrow B(x))$$

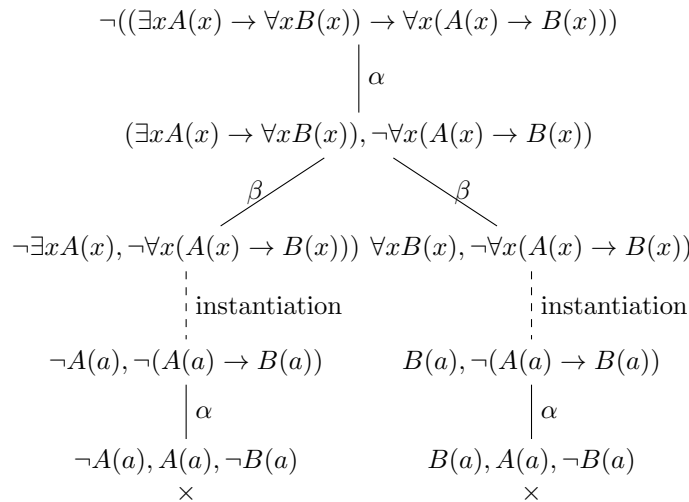
$$G = (\forall x A(x) \rightarrow \exists x B(x))$$

$$F \rightarrow G, G \rightarrow F$$



It's true by contradiction of the inverse implications

- $(\exists x A(x) \rightarrow \forall x B(x)) \rightarrow \forall x(A(x) \rightarrow B(x))$



It's true by contradiction of the inverse

- $\forall x(A(x) \vee B(x)) \rightarrow (\forall xA(x) \vee \exists xB(x))$

$$\begin{array}{c}
 \neg(\forall x(A(x) \vee B(x)) \rightarrow (\forall xA(x) \vee \exists xB(x))) \\
 \quad \quad \quad \downarrow \alpha \\
 \forall x(A(x) \vee B(x)), \neg(\forall xA(x) \vee \exists xB(x)) \\
 \quad \quad \quad \downarrow \alpha \\
 \forall x(A(x) \vee B(x)), \neg\forall xA(x), \neg\exists xB(x) \\
 \quad \quad \quad \vdots \text{ instantiation} \\
 (A(a) \vee B(a)), \neg A(a), \neg B(a) \\
 \quad \quad \swarrow \beta \quad \quad \searrow \beta \\
 A(a), \neg A(a), \neg B(a) \qquad \qquad B(a), \neg A(a), \neg B(a) \\
 \quad \quad \times \qquad \qquad \qquad \quad \times
 \end{array}$$

It's true by contradiction of the inverse

- $\forall x(A(x) \rightarrow B(x)) \rightarrow (\exists xA(x) \rightarrow \exists xB(x))$

$$\begin{array}{c}
\neg(\forall x(A(x) \rightarrow B(x)) \rightarrow (\exists xA(x) \rightarrow \exists xB(x))) \\
\downarrow \alpha \\
\forall x(A(x) \rightarrow B(x)), \neg(\exists xA(x) \rightarrow \exists xB(x)) \\
\downarrow \alpha \\
\forall x(A(x) \rightarrow B(x)), \exists xA(x), \neg\exists xB(x) \\
\vdots \text{ instantiation} \\
(A(a) \rightarrow B(a)), A(a), \neg B(a) \\
\swarrow \beta \quad \searrow \beta \\
\neg A(a), A(a), \neg B(a) \quad B(a), A(a), \neg B(a) \\
\times \quad \times
\end{array}$$

It's true by contradiction of the inverse