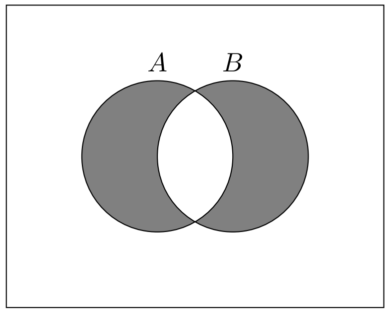
a) Show that for any x, $x \in A + B$ iff x is an element of exactly one of A, B

$$x \in A + B \implies (x \in A \cap x \notin B) \cup (x \in B \cap x \notin A)$$

$$x \in A \cap B \implies x \in A \cap x \in B \implies \textit{Not in } A + B$$

b) Draw a Venn diagram for the operation



(c) Show that $A + B \subseteq A \cup B$

$$x \in A + B \implies (x \in A \cap x \notin B) \cup (x \in B \cap x \notin A)$$

$$x \in A \cup B \implies x \in A \cup x \in B$$

$$\begin{aligned} x \in A \cap x \notin B &\implies x \in A \\ &\implies x \in A \cup B \end{aligned}$$

$$\begin{aligned} x \in B \cap x \notin A &\implies x \in B \\ &\implies x \in A \cup B \end{aligned}$$

(d) Show that $A + B$ is disjoint from $A \cap B$.

$$\textit{Show that } (A \cap B) \cap (A + B) = \emptyset$$

$$x \in A + B \implies (x \in A \cap x \notin B) \cup (x \in B \cap x \notin A)$$

$$\begin{aligned} x \in (A \cap B) \cap (A + B) &\implies (x \in A \cap x \in B) \cap ((x \in A \cap x \notin B) \cup (x \in B \cap x \notin A)) \\ &\implies ((x \in A \cap x \notin B) \cap (x \in A \cap x \in B)) \cup ((x \in B \cap x \notin A) \cap (x \in A \cap x \in B)) \\ &\implies (x \in A \cap x \notin B \cap x \in B) \cup (x \in B \cap x \notin A \cap x \in A) \\ &\implies (x \in A \cap \emptyset) \cup (xinB \cap \emptyset) \\ &\implies \emptyset \cup \emptyset \\ &\implies \emptyset \end{aligned}$$

(e) Show that $A + B = (A \cup B) \setminus (A \cap B)$.

$$A + B \implies (A \cap B') \cup (A' \cap B)$$

$$\begin{aligned} (A \cup B) \setminus (A \cap B) &\implies ((A \cup B) \setminus A) \cup ((A \cup B) \setminus B) \\ &\implies (B \setminus A) \cup (A \setminus B) \\ &\implies (A \cap B') \cup (A' \cap B) \end{aligned}$$

$$A + B = (A \cup B) \setminus (A \cap B)$$

(f) For each of the following properties of $\cup$, check out whether or not it also holds for $+$, giving a proof or a counter example as appropriate:

i commutativity

$$A + B = B + A$$

$$x \in A + B \implies (x \in A \cap x \notin B) \cup (x \in B \cap x \notin A)$$

$$x \in B + A \implies (x \in B \cap x \notin A) \cup (x \in A \cap x \notin B)$$

$$(x \in A \cap x \notin B) \cup (x \in B \cap x \notin A) = (x \in B \cap x \notin A) \cup (x \in A \cap x \notin B)$$

ii associativity

$$(A + B) + C = A + (B + C)$$

$$\begin{aligned}
(A + B) + C &\implies (((A \cap B') \cup (A' \cap B)) \cap C') \cup (((A \cap B') \cup (A' \cap B))' \cap C) \\
&\implies (((A \cap B') \cup (A' \cap B)) \cap C') \cup (((A \cap B')' \cap (A' \cap B)') \cap C) \\
&\implies (((A \cap B') \cup (A' \cap B)) \cap C') \cup (((A' \cup B) \cap (A \cup B')) \cap C) \\
&\implies (A \cap B' \cap C') \cup (A' \cap B \cap C') \cup ((A' \cup B) \cap (A \cup B') \cap C) \\
&\implies (A \cap B' \cap C') \cup (A' \cap B \cap C') \cup ((A' \cup B) \cap ((A \cap C) \cup (B' \cap C))) \\
&\implies (A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (((A' \cup B) \cap (A \cap C)) \cup ((A' \cup B) \cap (B' \cap C))) \\
&\implies (A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (B \cap A \cap C) \cup (A' \cap B' \cap C) \\
&\implies (A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C) \cup (A \cap B \cap C)
\end{aligned}$$

$$\begin{aligned}
(B + C) + A &\implies (((B \cap C') \cup (B' \cap C)) \cap A') \cup (((B \cap C') \cup (B' \cap C))' \cap A) \\
&\implies (((B \cap C') \cup (B' \cap C)) \cap A') \cup (((B \cap C')' \cap (B' \cap C)') \cap A) \\
&\implies (((B \cap C') \cup (B' \cap C)) \cap A') \cup (((B' \cup C) \cap (B \cup C')) \cap A) \\
&\implies (B \cap C' \cap A') \cup (B' \cap C \cap A') \cup ((B' \cup C) \cap (B \cup C') \cap A) \\
&\implies (B \cap C' \cap A') \cup (B' \cap C \cap A') \cup ((B' \cup C) \cap ((B \cap A) \cup (C' \cap A))) \\
&\implies (B \cap C' \cap A') \cup (B' \cap C \cap A') \cup (((B' \cup C) \cap (B \cap A)) \cup ((B' \cup C) \cap (C' \cap A))) \\
&\implies (B \cap C' \cap A') \cup (B' \cap C \cap A') \cup (C \cap B \cap A) \cup (B' \cap C' \cap A) \\
&\implies (B \cap C' \cap A') \cup (B' \cap C \cap A') \cup (B' \cap C' \cap A) \cup (B \cap C \cap A) \\
&\implies (A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C) \cup (A \cap B \cap C)
\end{aligned}$$

$$\begin{aligned}
x \in (A + B) + C \implies & (x \in A \cap x \notin B \cap x \notin C) \cup (x \notin A \cap x \in B \cap x \notin C) \cup \\
& (x \notin A \cap x \notin B \cap x \in C) \cup (x \in A \cap x \in B \cap x \in C)
\end{aligned}$$

$$\begin{aligned}
x \in A + (B + C) \implies & (x \in A \cap x \notin B \cap x \notin C) \cup (x \notin A \cap x \in B \cap x \notin C) \cup \\
& (x \notin A \cap x \notin B \cap x \in C) \cup (x \in A \cap x \in B \cap x \in C)
\end{aligned}$$

iii distribution of $\cap$ over $+$

$$A \cap (B + C) = (A \cap B) + (A \cap C)$$

$$\begin{aligned}
(A \cap B) + (A \cap C) &\implies ((A \cap B) \cap (A \cap C)') \cup ((A \cap B)' \cap (A \cap C)) \\
&\implies ((A \cap B) \cap (A' \cup C')) \cup ((A' \cup B') \cap (A \cap C)) \\
&\implies ((A \cap B) \cap A') \cup ((A \cap B) \cap C') \cup (A' \cap (A \cap C)) \cup (B' \cap (A \cap C)) \\
&\implies ((\emptyset) \cup ((A \cap B) \cap C') \cup (\emptyset) \cup (B' \cap (A \cap C)) \\
&\implies (A \cap B \cap C') \cup (A \cap B' \cap C) \\
&\implies A \cap ((B \cap C') \cup (B' \cap C)) \\
&\implies A \cap (B + C)
\end{aligned}$$

iv distribution of + over $\cap$

$$A + (B \cap C) \neq (A + B) \cap (A + C)$$

$$\begin{aligned}
A + (B \cap C) &\implies (A \cap (B \cap C)') \cup (A' \cap (B \cap C)) \\
&\implies (A \cap (B' \cup C')) \cup (A' \cap B \cap C) \\
&\implies (A \cap B') \cup (A \cap C') \cup (A' \cap B \cap C)
\end{aligned}$$

$$\begin{aligned}
(A + B) \cap (A + C) &\implies ((A \cap B') \cup (A' \cap B)) \cap ((A \cap C') \cup (A' \cap C)) \\
&\implies ((A \cap B') \cup A') \cap ((A \cap B') \cup B) \cap ((A \cap C') \cup A') \cap ((A \cap C') \cup C) \\
&\implies (A \cup A') \cap (B' \cup A') \cap (A \cup B) \cap (B' \cup B) \cap (A \cup A') \cap (C' \cup A') \cap (A \cup C) \cap (C' \cup C) \\
&\implies (U) \cap (B' \cup A') \cap (A \cup B) \cap (U) \cap (U) \cap (C' \cup A') \cap (A \cup C) \cap (U) \\
&\implies (A \cup B) \cap (A \cup C) \cap (B' \cup A') \cap (C' \cup A') \\
&\implies (A \cup (B \cap C)) \cap (A' \cup (B' \cap C'))
\end{aligned}$$

$$\begin{aligned}
x \in A \cap B \cap C' &\implies x \in A + (B \cap C) \\
&\implies x \notin (A + B) \cap (A + C)
\end{aligned}$$

(g) Express $-(A + B)$ using union, intersection and complement.

$$\begin{aligned}
-(A + B) &\implies -((A|B) \cup (B|A)) \\
&\implies ((A \cap B') \cup (B \cap A'))' \\
&\implies (A \cap B')' \cap (B \cap A')' \\
&\implies (A' \cup B) \cap (B' \cup A) \\
&\implies ((A' \cup B) \cap B') \cup ((A' \cup B) \cap A) \\
&\implies (A' \cap B') \cup (B \cap B') \cup (A' \cap A) \cup (B \cap A) \\
&\implies (A' \cap B') \cup (\emptyset) \cup (\emptyset) \cup (B \cap A) \\
&\implies (A \cap B) \cup (A' \cap B')
\end{aligned}$$

(h) We have seen that each of intersection, union and difference corresponds to a truth-functional logical connective. To what connective mentioned in this lecture does symmetric difference correspond? Draw its truth table.

A	**B**	$A + B$
1	1	0
1	0	1
0	1	1
0	0	0