1.3.1 Functions: Image, Closure

a) The floor function from $\mathbb{R}_+$ into $\mathbb{N}$ is defined by putting $\lfloor x \rfloor$ to be the largest integer less than or equal to x. What are the images under the floor function of the sets

i) $[0, 1] = \{x \in R : 0 \leq x \leq 1\}$

$$Image(Floor([0, 1])) = \{0, 1\}$$

ii) $[0, 1) = \{x \in R : 0 \leq x < 1\}$

$$Image(Floor([0, 1))) = \{0\}$$

iii) $(0, 1] = \{x \in R : 0 < x \leq 1\}$

$$Image(Floor((0, 1])) = \{0, 1\}$$

iv) $(0, 1) = \{x \in R : 0 < x < 1\}$

$$Image(Floor((0, 1))) = \{0\}$$

b) Let $f : A \to A$ be a function from set A into itself. Show that for all $X \subseteq A$, $f(X) \subseteq f[X]$, and give a simple example of the failure of the converse inclusion.

$$\begin{aligned} f[X] &= \{a \in A : f(x) = a, x \in X\} \\ \forall x \in X, x \in A \implies f[X] &= \{a \in A : f(x) = a, x \in A\} \\ \implies f(x) &\to A \end{aligned}$$

c) Show that when $f(A) \subseteq A$ then $f[A] = A$

$$\begin{aligned} f(A) \subseteq A &\implies f[A] = \{a \in A : f(x) = a\} \\ &\implies \forall a \in f[A], a \in A \end{aligned}$$

d) Show that for any partition of A, the function f taking each element $a \in A$ to its cell is a function on A into the power set $\mathcal{P}(A)$ of A with the partition as its range.

$$\begin{aligned} f &: A \to \mathcal{P}(A) \\ range(f) &= \mathcal{P}(A) \\ \forall X &\in \mathcal{P}(A) \end{aligned}$$

e) Let $f : A \to B$ be a function from set A into set B. Recall the 'abstract inverse' function$f^{-1} : B \to \mathcal{P}(A)$ defined at the end of Slide 52 by putting $f^{-1}(b) = \{a \in A : f(a) = b\}$ for each $b \in B$.

i) Show that the collection of all sets for $b \in f(A) \subseteq B$ is a partition of A in the sense defined in Chapter 2 of the David Makinson's book.

$$\begin{aligned} \mathcal{P}(A) &= \{A_i : i \in I, \forall a_i \in A_i, A_i \neq \emptyset, A_i \cap A_{i'} = \emptyset\} \\ \{a \in A : f(a) = b, b \in B\} &= \mathcal{P}(A) \\ \forall a \in A, \exists b \in B : f(a) = b &\implies f^{-1}(b_i) \cap f^{-1}(b_{i'}) = \emptyset \\ &\implies f^{-1}(B) = \mathcal{P}(A) \end{aligned}$$

ii) Is this still the case if we include in the collection the sets $f^{-1}(b)$ for $b \in B \setminus f(A)$?
Depends in if f is bijective, if it is, B then would just add an empty set, which would not affect the partition, but if B is not srujective, then f^{-1} would be greater that $\mathcal{P}(A)$

1.3.2 Injections, surjections, bijections

a) Is the floor function from $\mathbb{R}_+$ into $\mathbb{N}$ injective?
No, because several values from $\mathbb{R}_+$ map to the same values in $\mathbb{N}$, An exaple is that 1.01, 1.1, 1.9 all map to 1

b) Show that the composition of two bijections is a bijection. You may make use of results of exercises in the previous slides on injectivity and surjectivity.

$$f : A \to B \qquad\qquad g : B \to C$$
$$\forall a \in A, \exists b \in B : f(a) = b \qquad\qquad \forall b \in B, \exists c \in C : g(b) = c$$
$$f(a) = f(a') \iff a = a' \qquad\qquad g(b) = g(b') \iff b = b'$$

$$\begin{aligned} g \circ f &\implies \forall a \in A,\ \exists c \in C : g(f(a)) = c \\ &\implies g(f(a)) = g(f(a')) \iff a = a' \\ &\implies g \circ f \textit{ is bijective} \end{aligned}$$

c) Use the equinumerosity principle to show that there is never any bijection between a finite set and any of its proper subsets.

$$\begin{aligned} \#(A) = \#(B) &\iff f : A \to B \textit{ is bijective} \\ B \subset A &\implies B = \{\forall b \in B, b \in A\},\ A \setminus B \neq \emptyset \\ B \subset A &\implies \exists a \in A : f(a) \notin B \\ &\implies f \textit{ is not bijective} \\ &\implies \#(A) \neq \#(B) \end{aligned}$$

d) Give an example to show that there can be a bijection between an infinite set and certain of its proper subsets.

$$A = \mathbb{N}, B = \mathbb{N}_+$$
$$f : A \to B = f(a) = a + 1$$
$$f(0) = 1, f(1) = 2 ... f(n) = n + 1$$

e) Use the principle of comparison to show that for finite sets A, B, if there are injective functions $f : A \to B$ and $g : B \to A$, then there is a bijection from A to B. (Hint: Consider the superposition $g \circ f$ and establish that it provides for a desired bijection).

$$f : A \to B \qquad\qquad g : B \to A$$
$$\forall a \in A, \exists b \in B : f(a) = b \qquad\qquad \forall b \in B, \exists a \in A : g(b) = a$$
$$f(a) = f(a') \iff a = a' \qquad\qquad g(b) = g(b') \iff b = b'$$

$$\begin{aligned} &g \circ f : A \to A \\ &f \circ g : B \to B \\ &\implies \forall a \in A,\ \exists b \in B \\ &\implies A \textit{ and } B \textit{ are bijective} \end{aligned}$$

1.3.3 Pigeonhole principle

a) If a set A is partitioned into n cells, how many distinct elements of A need to be selected to guarantee that at least two of them are in the same cell?
$n+1$

b) Let $K = \{1, 2, 3, 4, 5, 6, 7, 8\}$. How many distinct numbers must be selected from K to guarantee that there are two of them that sum to 9? (Hint: Let A be the set of all unordered pairs (x, y) with $x, y \in K$ and $x + y = 9$. Check that this set forms a partition of K and apply the preceding part of the exercise).

$$\begin{aligned}
A &= \{x, y : x, y \in K, x + y = 9\} \\
A &= \{\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}\} \\
A &= \mathcal{P}(K) \\
\#(A) = 4 &\implies You\ need\ 5\ numbers
\end{aligned}$$

1.3.4 Handy functions

a) Let $f : A \to B$ and $g : B \to C$.

i) Show that if at least one of f, g is a constant function, then $g \circ f : A \to C$ is a constant function.

$$\begin{aligned}
f\ is\ constant &\implies \exists b \in B \forall a \in A : f(a) = b \\
&\implies f(A) = b \\
g \circ f &\implies \exists c \in C \forall a \in A : g(f(a)) = c
\end{aligned}$$

ii) If $g \circ f : A \to C$ is a constant function, does it follow that at least one of f,g is a constant function (give a verification or a counterexample).

$$\begin{aligned}
g \circ f &\implies \exists c \in C \forall a \in A : g(f(a)) = c \\
&\implies f(A) = b \vee g(B) = c \\
&\implies \exists b \in B \forall a \in A : f(a) = b \vee \exists c \in C \forall b \in B : g(b) = c \\
&\implies f\ is\ constant\ or\ g\ is\ constant
\end{aligned}$$