1.3.1 Functions: Image, Closure

a) The floor function from R_+ into N is defined by putting $\lfloor x \rfloor$ to be the largest integer less than or equal to x. What are the images under the floor function of the sets

  i) $[0, 1] = \{x \in R : 0 \leq x \leq 1\}$

  ii) $[0, 1) = \{x \in R : 0 \leq x < 1\}$

  iii) $(0, 1] = \{x \in R : 0 < x \leq 1\}$

  iv) $(0, 1) = \{x \in R : 0 < x < 1\}$

b) Let $f : A \to A$ be a function from set A into itself. Show that for all $X \subseteq A$, $f(X) \subseteq f[X]$, and give a simple example of the failure of the converse inclusion.

c) Show that when then $f(A) \subseteq A$ then $f[A] = A$

d) Show that for any partition of A, the function f taking each element $a \in A$ to its cell is a function on A into the power set $P(A)$ of A with the partition as its range.

e) Let $f : A \to B$ be a function from set A into set B. Recall the 'abstract inverse' function$f^{-1} : B \to P(A)$ defined at the end of Slide 52 by putting $f^{-1}(b) = \{a \in A : f(a) = b\}$ for each $b \in B$.

  i) Show that the collection of all sets for $b \in f(A) \subseteq B$ is a partition of A in the sense defined in Chapter 2 of the David Makinson's book.

  ii) Is this still the case if we include in the collection the sets $f^{-1}(b)$ for $b \in B \setminus f(A)$?

1.3.2 Injections, surjections, bijections

a) Is the floor function from R_+ into N injective? (ii) Is it onto N?

b) Show that the composition of two bijections is a bijection. You may make use of results of exercises in the previous slides on injectivity and surjectivity.

c) Use the equinumerosity principle to show that there is never any bijection between a finite set and any of its proper subsets.

d) Give an example to show that there can be a bijection between an infinite set and certain of its proper subsets.

e) Use the principle of comparison to show that for finite sets A, B, if there are injective functions $f : A \to B$ and $g : B \to A$, then there is a bijection from A to B. (Hint: Consider the superposition $g \circ f$ and establish that it provides for a desired bijection).

1.3.3 Pigeonhole principle

a) If a set A is partitioned into n cells, how many distinct elements of A need to be selected to guarantee that at least two of them are in the same cell?

b) Let $K = \{1, 2, 3, 4, 5, 6, 7, 8\}$. How many distinct numbers must be selected from K to guarantee that there are two of them that sum to 9? (Hint: Let A be the set of all unordered pairs (x, y) with $x, y \in K$ and $x + y = 9$. Check that this set forms a partition of K and apply the preceding part of the exercise).

1.3.4 Handy functions

a) Let $f : A \to B$ and $g : B \to C$.

  i) Show that if at least one of f, g is a constant function, then $g \circ f : A \to C$ is a constant function.

ii) If $g \circ f : A \to C$ is a constant function, does it follow that at least one of f,g is a constant function (give a verification or a counterexample).