1.4.1 Proof by simple induction

a) Use simple induction to show that for every positive integer n, $5^n - 1$ is divisible by 4

b) Use simple induction to show that for every positive integer n, $n^3 - n$ is divisible by 3. (Hint: In the induction step, you will need to make use of the arithmetic fact that $(k+1)^3 = k^3 + 3k^2 + 3k + 1$)

c) Show by simple induction that for every natural number n, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

1.4.2: Definition by simple recursion

a) Let $f : N \to N$ be the function defined by putting $f(0) = 0$ and $f(n+1) = n$ for all $n \in N$.

   i) Evaluate this function bottom-up for all arguments 0–5.

   ii) Explain what f does by expressing it in explicit terms (i.e. without a recursion).

b) Let $f : N^+ \to N$ be the function that takes each positive integer n to the greatest natural number p with $2^p \leq n$. Define this function by a simple recursion. (Hint: You will need to divide the recursion step into two cases.)

c) Let $g : NXN \to N$ be defined by putting $g(m, 0) = m$ for all $m \in N$ and $g(m, n+1) = f(g(m, n))$ where f is the function defined in part (a) of this exercise.

   i) Evaluate $g(3, 4)$ top-down.

   ii) Explain what g does by expressing it in explicit terms (i.e. without a recursion).

1.4.3: Proof by cumulative induction

a) Use cumulative induction to show that any postage cost of four or more pence can be covered by two-pence and five-pence stamps.

b) Use cumulative induction to show that for every natural number n, $F(n) \leq 2^n - 1$, where F is the Fibonacci function.

c) Calculate F(5) top-down, and then again bottom-up, where again F is the Fibonacci function

d) Express each of the numbers 14, 15 and 16 as a sum of $3s$ and/or $8s$. Using this fact in your basis, show by cumulative induction that every positive integer $n \leq 14$ may be expressed as a sum of $3s$ and/or $8s$.

e) Show by induction that for every natural number n, $A(1, n) = n + 2$, where A is the Ackermann function.