1.4.1 Proof by simple induction

a) Use simple induction to show that for every positive integer n, $5^n - 1$ is divisible by 4
$5^n - 1$ is divisible by 4
Base case:

$$
\begin{aligned}
5^1 - 1 &= 4j \\
&= 4 = 4j \\
5^k - 1 &= 4j \\
5^k &= 4j + 1
\end{aligned}
$$

Induction step:

$$
\begin{aligned}
&= 5^{k+1} - 1 \\
&= 5^k \cdot 5 - 1 \\
&= (4j + 1) \cdot 5 - 1 \\
&= (20j + 5) - 1 \\
&= 20j + 4 \\
&= 4 \cdot (5j + 1) \\
&\implies true\ for\ n = k + 1
\end{aligned}
$$

b) Use simple induction to show that for every positive integer n, $n^3 - n$ is divisible by 3. (Hint: In the induction step, you will need to make use of the arithmetic fact that $(k+1)^3 = k^3 + 3k^2 + 3k + 1$)
$n^3 - n$ is divisible by 3
Base case:

$$
\begin{aligned}
1^3 - 1 &= 3j \\
&= 0 = 3j \\
k^3 - k &= 3j \\
k^3 &= 3j + k
\end{aligned}
$$

Induction step:

$$
\begin{aligned}
&= (k+1)^3 - (k+1) \\
&= (k^3 + 3k^2 + 3k + 1) - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (3j + k) + 3k^2 + 2k \\
&= 3j + 3k^2 + 3k \\
&= 3 \cdot (j + k^2 + k) \\
&\implies true\ for\ n = k + 1
\end{aligned}
$$

c) Show by simple induction that for every natural number n, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$
$f(n) = 2^{n+1} - 1$
Base case:

$$
\begin{aligned}
f(1) &= 2^{1+1} - 1 \\
f(1) &= 4 - 1 \\
f(1) &= 3 \\
f(k) &= 2^{k+1} - 1
\end{aligned}
$$

Induction step:

$$
\begin{aligned}
f(k+1) &= 2^{k+2} - 1 \\
f(k+1) &= 2^1 + 2^2 ... + 2^k + 2^{k+1} \\
&= f(k) + 2^{k+1} \\
&= 2^{k+1} - 1 + 2^{k+1} \\
&= 2(2^{k+1}) - 1 \\
&= 2^1 \cdot 2^{k+1} - 1 \\
&= 2^{k+1+1} - 1 \\
&= 2^{k+2} - 1 \\
&\implies true\ for\ n = k+1
\end{aligned}
$$

1.4.2: Definition by simple recursion

a) Let $f : N \to N$ be the function defined by putting $f(0) = 0$ and $f(n+1) = n$ for all $n \in N$.

   i) Evaluate this function bottom-up for all arguments 0–5.

$$
\begin{aligned}
f(0) &= 0 \\
f(1) &= 0 \\
f(2) &= 1 \\
f(3) &= 2 \\
f(4) &= 3 \\
f(5) &= 4
\end{aligned}
$$

   ii) Explain what f does by expressing it in explicit terms (i.e. without a recursion).
   This function is not recursive, it only returns the antecesor of the current value

b) Let $f : N^+ \to N$ be the function that takes each positive integer n to the greatest natural number p with $2^p \leq n$. Define this function by a simple recursion. (Hint: You will need to divide the recursion step into two cases.)

$$
\begin{aligned}
f(n) &= 0 && , when\ n = 1 \\
f(n) &= f(n-1) && , when\ n > 1, \log_2(n) \notin N \\
f(n) &= f(n-1) + 1 && , when\ n > 1, \log_2(n) \in N
\end{aligned}
$$

c) Let $g : NXN \to N$ be defined by putting $g(m, 0) = m$ for all $m \in N$ and $g(m, n+1) = f(g(m, n))$ where f is the function defined in part (a) of this exercise.

   i) Evaluate $g(3, 4)$ top-down.

$$
\begin{aligned}
g(3,4) &= f(g(3,3)) \\
&= f(f(g(3,2))) \\
&= f(f(f(g(3,1)))) \\
&= f(f(f(f(g(3,0))))) \\
&= f(f(f(f(f(3))))) \\
&= f(f(f(f(2)))) \\
&= f(f(f(1))) \\
&= f(f(0)) \\
&= f(0) \\
&= 0
\end{aligned}
$$

ii) Explain what g does by expressing it in explicit terms (i.e. without a recursion).
It substracts the right item from the left, but if the remain is negative, it returns 0

1.4.3: Proof by cumulative induction

a) Use cumulative induction to show that any postage cost of four or more pence can be covered by two-pence and five-pence stamps.

b) Use cumulative induction to show that for every natural number n, $F(n) \leq 2^n - 1$, where F is the Fibonacci function.

c) Calculate F(5) top-down, and then again bottom-up, where again F is the Fibonacci function

d) Express each of the numbers 14, 15 and 16 as a sum of $3s$ and/or $8s$. Using this fact in your basis, show by cumulative induction that every positive integer $n \leq 14$ may be expressed as a sum of $3s$ and/or $8s$.

e) Show by induction that for every natural number n, $A(1, n) = n + 2$, where A is the Ackermann function.