1.2.1

(a) Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$

$$\begin{aligned}
(x,y) \in A \times (B \cap C) &\implies x \in A \wedge y \in (B \cap C) \\
&\implies x \in A \wedge y \in B \wedge y \in C \\
(x,y) \in (A \times B) \cap (A \times C) &\implies (x,y) \in (A \times B) \wedge (x,y) \in (A \times C) \\
&\implies x \in A \wedge y \in B \wedge x \in A \wedge y \in C \\
&\implies x \in A \wedge y \in B \wedge y \in C
\end{aligned}$$

(b) Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$$\begin{aligned}
(x,y) \in A \times (B \cup C) &\implies x \in A \wedge y \in (B \cup C) \\
&\implies x \in A \wedge (y \in B \vee y \in C) \\
(x,y) \in (A \times B) \cup (A \times C) &\implies (x,y) \in (A \times B) \vee (x,y) \in (A \times C) \\
&\implies (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \\
&\implies x \in A \wedge (y \in B \wedge y \in C)
\end{aligned}$$

1.2.2

(a) Consider the relations $R = \{(1,7),(3,3),(13,11)\}$ and $S = \{(1,1),(1,7),(3,11),(13,12),(15,1)\}$ over the positive integers. Identify $dom(R \cap S)$, $range(R \cap S)$, $dom(R \cup S)$, $range(R \cup S)$

$$dom(R \cap S) = \{1\}$$

$$range(R \cap S) = \{7\}$$

$$dom(R \cup S) = \{1, 3, 13, 15\}$$

$$range(R \cup S) = \{7, 3, 11, 1, 12\}$$

(b) In the same example, identify $join(R,S)$, $join(S,R)$, $S \circ R$, $R \circ S$, $R \circ R$, $S \circ S$.

$$join(R,S) = \{(3,3,11)\}$$

$$join(S,R) = \{(1,1,7),(15,1,1)\}$$

$$S \circ R = \{(1,7),(15,1)\}$$

$$R \circ S = \{(3,11)\}$$

$$R \circ R = \{(3,3)\}$$

$$S \circ S = \{(1,1),(1,7),(15,1),(15,7)\}$$

(c) In the same example, identify $R(X)$ and $S(X)$ for $X = \{1,3,11\}$ and $X = \emptyset$.

$$\begin{aligned}
X &= \{1,3,11\} & X &= \emptyset \\
R(X) &= \{(1,7),(3,3)\} & R(X) &= \emptyset \\
S(X) &= \{(1,1),(1,7),(3,11)\} & S(X) &= \emptyset
\end{aligned}$$

(d) Explain how to carry out composition by means of join and projection.
Composition is the result of first applying the *join* and thenn using the *projection* to eliminate the common item

1.2.3

(a) Show that R is reflexive over A iff $I_A \subseteq R$. Here I_A is the identity relation over A, defined in an exercise in Sect. 2.1.3.

$$\begin{aligned} R\ is\ reflexive\ in\ A &= \forall x \in A : xRx \\ I_A &= \{(a,a) : a \in A\} \\ I_A &\subseteq R \\ \forall a \in A &\implies \exists (a,a) \in I_A \\ I_A \subseteq R &\implies \exists (a,a) \in R \\ &\implies R\ is\ reflexive\ on\ A \end{aligned}$$

(b) Show that the converse of a relation R that is reflexive over a set A is also reflexive over A.

$$\begin{aligned} R\ is\ reflexive\ over\ A &= \forall x \in A : xRx \\ R^{-1} &= \{(a,b) : (b,a) \in R\} \\ R = \{(a,b) : a = b \wedge\ a, b \in A\} &\implies (a,b) = (b,a) \\ &\implies R^{-1} = R \\ &\implies R^{-1}\ is\ reflexive\ over\ A \end{aligned}$$

(c) Show that R is transitive iff $R \circ R \subseteq R$.

$$\begin{aligned} R\ is\ transitive &\iff R \circ R \subseteq R \\ R \circ R &= \{(a,c) : aRb \wedge bRc\} \\ R \circ R \subseteq R &\implies (a,c) \in R \\ &\implies R(R(a,b),c) = R(a, R(b,c)) \end{aligned}$$

1.2.4

(a) Show that the following three conditions are equivalent: (i) R is symmetric, (ii) $R \subseteq R^{-1}$, (iii) $R = R^{-1}$.

$$R\ is\ symetric \equiv R \subseteq R^{-1} \equiv R = R^{-1}$$

(b) Show that if R is reflexive over A and also transitive, then the relation S defined by $(a,b) \in S$ iff both $(a,b) \in R$ and $(b,a) \in R$ is an equivalence relation.

$$\begin{aligned} reflexive &= (a,a) \in R \forall a \in A \\ S &= \{(a,b) : (a,b) \in R \wedge (b,a) \in R\} \\ (a,b) \in R \wedge (b,a) \in R &\implies a = b \\ &\implies S\ is\ equivalent \end{aligned}$$

(c) Enumerate all the partitions of $A = \{1,2,3\}$ and draw a Hasse diagram for them under fineness.

$$\begin{aligned} Partition(A) = \{&\{\{1\},\{2\},\{3\}\}, \\ &\{\{1,2\},\{3\}\}, \\ &\{\{1,3\},\{2\}\}, \\ &\{\{2,3\},\{1\}\}, \\ &\{\{1,2,3\}\}\} \end{aligned}$$

$$\begin{tikzcd} & \{1,2,3\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \{\{1,2\},\{3\}\} \arrow[dr, dash] & \{\{1,3\},\{2\}\} \arrow[d, dash] & \{\{1\},\{2,3\}\} \arrow[dl, dash] \\ & \{\{1\},\{2\},\{3\}\} & \end{tikzcd}$$

1.2.5

Let R be any transitive relation over a set A. Define S over A by putting $(a, b) \in S$ iff either $a = b$ or both $(a, b) \in R$ and $\neg(b, a) \in R$. Show that S partially orders A.

$$
\begin{aligned}
Partial\ order &= reflexive,\ transitive\ and\ antisymmetric \\
reflexive &= (a, a) \in R \forall a \in A \\
antisymmetric &= (a, b) \in R \land (b, a) \notin R \\
S &= \{(a, b) : a = b \lor [(a, b) \in R \land (b, a) \notin R]\} \\
\{(a, b) : a = b\} &\implies S\ is\ reflexive \\
\{(a, b) : (a, b) \in R \land (b, a) \notin R\} &\implies S\ is\ transitive \\
&\implies S\ is\ antisymmetric
\end{aligned}
$$