Characterizing elements of Burau's representation of B_4

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Definition 0.1. We define the braid group in 4 strings by

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \; ; \; \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \sigma_1\sigma_3 = \sigma_3\sigma_1 \rangle .$$

We define the Burau representation of B_4 by

$$\begin{aligned} \rho : B_4 &\to GL_3(\mathbb{Z}[t,t^{-1}]) \\ \sigma_1 &\mapsto \begin{pmatrix} -t & 0 & 0 \\ -t & 1 & 0 \\ -t & 0 & 1 \end{pmatrix} \\ \sigma_2 &\mapsto \begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \sigma_3 &\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

For the following calculations, we will use

$$t_n := \sum_{i=0}^{|n|} (-t)^i .$$

Proposition 0.2. If $n \in \mathbb{Z}_{>0}$, then

$$\rho(\sigma_1^n) = \begin{pmatrix} (-t)^n & 0 & 0 \\ t_n - 1 & 1 & 0 \\ t_n - 1 & 0 & 1 \end{pmatrix}, \; \rho(\sigma_1^{-n}) = \begin{pmatrix} (-t)^{-n} & 0 & 0 \\ -t_{-n} + (-t)^{-n} & 1 & 0 \\ -t_{-n} + (-t)^{-n} & 0 & 1 \end{pmatrix}$$

$$\rho(\sigma_2^n) = \begin{pmatrix} t_n & -t_n + 1 & 0 \\ t_n - (-t)^n & -t_n + 1 + (-t)^n & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \rho(\sigma_2^{-n}) = \begin{pmatrix} -t_{-n} + (-t)^{-n} + 1 & t_{-n} - (-t)^{-n} & 0 \\ -t_{-n} + 1 & t_{-n} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho(\sigma_3^n) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_n & -t_n + 1 \\ 0 & t_n - (-t)^n & -t_n + 1 + (-t)^n \end{pmatrix}, \; \rho(s_3^{-n}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -t_{-n} + 1 + (-t)^{-n} & t_{-n} - (-t)^{-n} \\ 0 & -t_{-n} + 1 & t_{-n} \end{pmatrix}.$$

Proof. We proceed by induction. Note that

$$\rho(\sigma_1^1) = \begin{pmatrix} -t & 0 & 0 \\ -t & 1 & 0 \\ -t & 0 & 1 \end{pmatrix}, \; \rho(\sigma_1^{-1}) = \begin{pmatrix} -t^{-1} & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\rho(\sigma_2^1) = \begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \rho(\sigma_2^{-1}) = \begin{pmatrix} 0 & 1 & 0 \\ t^{-1} & 1-t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho(\sigma_3^1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{pmatrix},\ \rho(\sigma_3^{-1}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & t^{-1} & 1-t^{-1} \end{pmatrix}.$$

Suppose that each formula in the statement is true for $n = k$, then we have that:

$$\rho(\sigma_1^{k+1}) = \rho(\sigma_1^k\sigma) = \begin{pmatrix} (-t)^k & 0 & 0 \\ t_k - 1 & 1 & 0 \\ t_k - 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -t & 0 & 0 \\ -t & 1 & 0 \\ -t & 0 & 1 \end{pmatrix} = \begin{pmatrix} (-t)^{k+1} & 0 & 0 \\ -(t_k-1)t - t & 1 & 0 \\ -(t_k-1)t - t & 0 & 1 \end{pmatrix} = \begin{pmatrix} (-t)^{k+1} & 0 & 0 \\ t_{k+1} - 1 & 1 & 0 \\ t_{k+1} - 1 & 0 & 1 \end{pmatrix},$$

$$\begin{aligned}\rho(\sigma_1^{-(k+1)}) = \rho(\sigma_1^{-k}\sigma_1^{-1}) &= \begin{pmatrix} (-t)^{-k} & 0 & 0 \\ -t_{-k} + (-t)^{-k} & 1 & 0 \\ -t_{-k} + (-t)^{-k} & 0 & 1 \end{pmatrix}\begin{pmatrix} -t^{-1} & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} (-t)^{-k-1} & 0 & 0 \\ (-t_{-k})(-t^{-1}) + (-t)^{-k-1} - 1 & 1 & 0 \\ (-t_{-k})(-t^{-1}) + (-t)^{-k-1} & 0 & 1 \end{pmatrix} = \begin{pmatrix} (-t)^{-(k+1)} & 0 & 0 \\ -t_{-(k+1)} + (-t)^{-(k+1)} & 1 & 0 \\ -t_{-(k+1)} + (-t)^{-(k+1)} & 0 & 1 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\rho(\sigma_2^{k+1}) = \rho(\sigma_2^k\sigma_2) &= \begin{pmatrix} t_k & t_k + 1 & 0 \\ t_k - (-t)^k & -t_k + 1 + (-t)^k & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - t\cdot t_k & -t_{k+1} + 1 & 0 \\ 1 - t\cdot t_k + t(-t)^k & -t_{k+1} + 1 - (-t)^{k+1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} t_{k+1} & -t_{k+1} + 1 & 0 \\ t_{k+1} - (-t)^{k+1} & -t_{k+1} + 1 + (-t)^{k+1} & 0 \\ 0 & 0 & 1 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\rho(\sigma_2^{-(k+1)}) &= \begin{pmatrix} -t_{-n} + (-t)^{-n} + 1 & t_{-n} - (-t)^{-n} & 0 \\ -t_{-n} + 1 & t_{-n} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ t^{-1} & 1 - t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} t^{-1}\cdot t_{-n} - t^{-1}(-t)^n & -t_{-n} + 1 + t_{-n} - t^{-1}t_{-n} + t^{-1}(-t)^{-n} & 0 \\ t_{-n}t^{-1} & 1 - t_{-n}t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -t_{-(n+1)} + (-t)^{-(n+1)} + 1 & t_{-(n+1)} - (-t)^{-(n+1)} & 0 \\ -t_{-(n+1)} & t_{-(n+1)} & 0 \\ 0 & 0 & 1 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\rho(\sigma_3^{k+1}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_k & -t_k + 1 \\ 0 & t_k - (-t)^k & -t_k + 1 + (-t)^k \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -t\cdot t_k + 1 & t_k\cdot t \\ 0 & 1 - t\cdot t_k - (-t)(-t)^k & t\cdot t_k - t(-t)^k \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_{(k+1)} & -t_{(k+1)} + 1 \\ 0 & t_{(k+1)} - (-t)^{(k+1)} & -t_{(k+1)} + 1 + (-t)^{(k+1)} \end{pmatrix},\end{aligned}$$

$$\rho(\sigma_3^{-(k+1)}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -t_{-k}+1+(-t)^{-k} & t_{-k}-(-t)^{-k} \\ 0 & -t_{-k}+1 & t_{-k} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & t^{-1} & 1-t^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^{-1}\cdot t_{-k} - t^{-1}(-t)^{-k} & 1-t^{-1}\cdot t_{-k} - (-t)^{-1}(-t)^{-k} \\ 0 & t^{-1}\cdot t_{-k} & 1-t^{-1}\cdot t_{-k} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -t_{-(k+1)}+1+(-t)^{-(k+1)} & t_{-(k+1)}-(-t)^{-(k+1)} \\ 0 & -t_{-(k+1)}+1 & t_{-(k+1)} \end{pmatrix}.$$

□

With the above proposition, we obtain as a corollary that if $n, l, k \in \mathbb{Z}_{>0}$, then

$$\rho(\sigma_1^n \sigma_2^l \sigma_3^k) = (a_{i,j}) \quad \text{y} \quad \rho(s_1^{-n} \sigma_2^{-l} \sigma_3^{-k}) = (b_{i,j}),$$

where

- $a_{1,1} = (-t)^n t_l$
- $a_{1,2} = (-t)^n(-t_l+1)t_k$
- $a_{1,3} = (-t)^n(-t_l+1)(-t_k+1)$
- $a_{2,1} = (t_n-1)t_l + (t_l - (-t)^l)$
- $a_{2,2} = ((t_n-1)(-t_l+1) - t_l + 1 + (-t)^l)t_k$
- $a_{2,3} = ((t_n-1)(-t_l+1) - t_l + 1 + (-t)^l)(-t_k+1)$
- $a_{3,1} = (t_n-1)t_l$
- $a_{3,2} = (t_n-1)(-t_l+1)t_k + t_k - (-t)^k$
- $a_{3,3} = (t_n-1)(-t_l+1)(-t_k+1) - t_k + 1 + (-t)^k$
- $b_{1,1} = (-t)^{-n}(-t_{-l} + (-t)^{-l} + 1)$
- $b_{1,2} = (-t)^{-n}(t_{-l} - (-t)^{-l})(-t_{-k} + 1 + (-t)^{-k})$
- $b_{1,3} = (-t)^{-n}(t_{-l} - (-t)^{-l})(t_{-k} - (-t)^{-k})$
- $b_{2,1} = (-t_{-n} + (-t)^{-n})(-t_{-l} + (-t)^{-l} + 1) - t_{-l} + 1$
- $b_{2,2} = ((-t_{-n} + (-t)^{-n})(t_{-l} - (-t)^{-l}) + t_{-l})(-t_{-k} + 1 + (-t)^{-k})$
- $b_{2,3} = ((-t_{-n} + (-t)^{-n})(t_{-l} - (-t)^{-l}) + t_{-l})(t_{-k} - (-t)^{-k})$
- $b_{3,1} = (-t_{-n} + (-t)^{-n})(-t_{-l} + (-t)^{-l} + 1)$
- $b_{3,2} = (-t_{-n} + (-t)^{-n})(t_{-l} - (-t)^{-l})(-t_{-k} + 1 + (-t)^{-k}) - t_{-k} + 1$
- $b_{3,3} = (-t_{-n} + (-t)^{-n})(t_{-l} - (-t)^{-l})(t_{-k} - (-t)^{-k}) + t_{-k}$

In practice, the entries of these matrices behave similarly to rows of Pascal's triangle. For example,

$$\rho(\sigma_1^1\sigma_2^2\sigma_3^3) = \begin{pmatrix} -t+t^2-t^3 & -t^2+2t^3-2t^4+2t^5-t^6 & -t^3+2t^4-2t^5+t^6 \\ 1-2t+t^2-t^3 & t-2t^2+3t^3-3t^4+2t^5-t^6 & t^2-2t^3+3t^4-2t^5+t^6 \\ -t+t^2-t^3 & 1-t+2t^3-2t^4+2t^5-t^6 & t-t^2-t^3+2t^4-2t^5+t^6 \end{pmatrix}.$$

Proposition 0.3. If $k \in \mathbb{Z}_{\geq 2}$, then

$$\rho\left((\sigma_1\sigma_2\sigma_3)^{2^k}\right) = \begin{pmatrix} t^{2^k} & 0 & 0 \\ 0 & t^{2^k} & 0 \\ 0 & 0 & t^{2^k} \end{pmatrix} \quad \text{y} \quad \rho\left((\sigma_1\sigma_2\sigma_3)^{-2^k}\right) = \begin{pmatrix} t^{-2^k} & 0 & 0 \\ 0 & t^{-2^k} & 0 \\ 0 & 0 & t^{-2^k} \end{pmatrix}.$$

Proof. We proceed with induction. For $k = 2$, we have that

$$\rho\left((\sigma_1\sigma_2\sigma_3)^{2^2}\right) = \begin{pmatrix} t^4 & 0 & 0 \\ 0 & t^4 & 0 \\ 0 & 0 & t^4 \end{pmatrix} \quad \text{y} \quad \rho\left((\sigma_1\sigma_2\sigma_3)^{-2^2}\right) = \begin{pmatrix} t^{-4} & 0 & 0 \\ 0 & t^{-4} & 0 \\ 0 & 0 & t^{-4} \end{pmatrix}.$$

Suppose the result is true for $k \in \mathbb{Z}_{\geq 2}$, then

$$\begin{aligned}
\rho\left((\sigma_1\sigma_2\sigma_3)^{2^{k+1}}\right) &= \rho\left(\sigma_1\sigma_2\sigma_3\right)^{2^{k+1}} = \rho\left(\sigma_1\sigma_2\sigma_3\right)^{2^k 2} = \left(\rho\left(\sigma_1\sigma_2\sigma_3\right)^{2^k}\right)^2 \\
&= \begin{pmatrix} t^{2^k} & 0 & 0 \\ 0 & t^{2^k} & 0 \\ 0 & 0 & t^{2^k} \end{pmatrix}\begin{pmatrix} t^{2^k} & 0 & 0 \\ 0 & t^{2^k} & 0 \\ 0 & 0 & t^{2^k} \end{pmatrix} \\
&= \begin{pmatrix} t^{2^k}t^{2^k} & 0 & 0 \\ 0 & t^{2^k}t^{2^k} & 0 \\ 0 & 0 & t^{2^k}t^{2^k} \end{pmatrix} \\
&= \begin{pmatrix} t^{2\cdot 2^k} & 0 & 0 \\ 0 & t^{2\cdot 2^k} & 0 \\ 0 & 0 & t^{2\cdot 2^k} \end{pmatrix} \\
&= \begin{pmatrix} t^{2^{k+1}} & 0 & 0 \\ 0 & t^{2^{k+1}} & 0 \\ 0 & 0 & t^{2^{k+1}} \end{pmatrix},
\end{aligned}$$

and similarly,

$$\begin{aligned}
\rho\left((\sigma_1\sigma_2\sigma_3)^{-2^{k+1}}\right) &= \rho\left(\sigma_1\sigma_2\sigma_3\right)^{-2^{k+1}} = \rho\left(\sigma_1\sigma_2\sigma_3\right)^{-2^k 2} = \left(\rho\left(\sigma_1\sigma_2\sigma_3\right)^{-2^k}\right)^2 \\
&= \begin{pmatrix} t^{-2^k} & 0 & 0 \\ 0 & t^{-2^k} & 0 \\ 0 & 0 & t^{-2^k} \end{pmatrix}\begin{pmatrix} t^{-2^k} & 0 & 0 \\ 0 & t^{-2^k} & 0 \\ 0 & 0 & t^{-2^k} \end{pmatrix} \\
&= \begin{pmatrix} t^{-2^k}t^{-2^k} & 0 & 0 \\ 0 & t^{-2^k}t^{-2^k} & 0 \\ 0 & 0 & t^{-2^k}t^{-2^k} \end{pmatrix} \\
&= \begin{pmatrix} t^{2\cdot -2^k} & 0 & 0 \\ 0 & t^{2\cdot -2^k} & 0 \\ 0 & 0 & t^{2\cdot -2^k} \end{pmatrix} \\
&= \begin{pmatrix} t^{-2^{k+1}} & 0 & 0 \\ 0 & t^{-2^{k+1}} & 0 \\ 0 & 0 & t^{-2^{k+1}} \end{pmatrix}.
\end{aligned}$$

$\square$

Note that the above proposition gives us a subset of faithful representation of B_4, that is, $\left\langle (\sigma_1\sigma_2\sigma_3)^{\pm 2^k} : k > 2 \right\rangle$.

Next we correct Lemma 2.7 of [1].

Definition 0.4. Let us consider Burau's representation $\rho' : B_4 \to GL_3\left(\mathbb{Z}[q, q^{-1}]\right)$ given for

$$\sigma_1 \mapsto \begin{pmatrix} q & 0 & 0 \\ -q & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_2 \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & q & 0 \\ 0 & -q & 1 \end{pmatrix}$$

$$\sigma_3 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & q \end{pmatrix}.$$

We define the Garside element of B_4 by $\Delta = (\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2)\sigma_1$.

Proposition 0.5. Let $n \in \mathbb{Z}$. Then

$$\rho'(\Delta^n) = \begin{cases} \begin{pmatrix} q^{2n} & 0 & 0 \\ 0 & q^{2n} & 0 \\ 0 & 0 & q^{2n} \end{pmatrix}, & n \text{ par} \\ \begin{pmatrix} 0 & 0 & q^{2n-1} \\ 0 & -q^{2n} & 0 \\ q^{2n+1} & 0 & 0 \end{pmatrix}, & n \text{ impar} \end{cases}$$

Proof. We proceed by induction. In the even case, for $n = 2$ we have that

$$\rho'(\Delta^2) = \rho'(\Delta)^2 = \begin{pmatrix} 0 & 0 & q \\ 0 & -q^2 & 0 \\ q^3 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & q \\ 0 & -q^2 & 0 \\ q^3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} q^4 & 0 & 0 \\ 0 & q^4 & 0 \\ 0 & 0 & q^4 \end{pmatrix},$$

and similarly for $n = -2$, we have that

$$\rho'(\Delta^{-2}) = \begin{pmatrix} 0 & 0 & q^{-3} \\ 0 & -q^{-2} & 0 \\ q^{-1} & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & q^{-3} \\ 0 & -q^{-2} & 0 \\ q^{-1} & 0 & 0 \end{pmatrix} = \begin{pmatrix} q^{-4} & 0 & 0 \\ 0 & q^{-4} & 0 \\ 0 & 0 & q^{-4} \end{pmatrix}.$$

If the statement is true for $n = 2k$ positive pair, then for the next positive pair $2(k+1)$ we have that

$$\begin{aligned} \rho'(\Delta^{2k+2}) &= \rho'(\Delta)^{2k+2} = \rho'(\Delta)^{2k}\rho(\Delta)^2 \\ &= \begin{pmatrix} q^{2(2k)} & 0 & 0 \\ 0 & q^{2(2k)} & 0 \\ 0 & 0 & q^{2(2k)} \end{pmatrix}\begin{pmatrix} q^4 & 0 & 0 \\ 0 & q^4 & 0 \\ 0 & 0 & q^4 \end{pmatrix} \\ &= \begin{pmatrix} q^{2(2k+2)} & 0 & 0 \\ 0 & q^{2(2k+2)} & 0 \\ 0 & 0 & q^{2(2k+2)} \end{pmatrix}, \end{aligned}$$

while if the statement is true for $n = -2k$ negative even, then for $-2(k+1)$ we have that

$$\begin{aligned}
\rho'(\Delta^{-2k-2}) &= \rho'(\Delta)^{-2k-2} = \rho'(\Delta)^{-2k}\rho'(\Delta)^{-2} \\
&= \begin{pmatrix} q^{-2(2k)} & 0 & 0 \\ 0 & q^{-2(2k)} & 0 \\ 0 & 0 & q^{-2(2k)} \end{pmatrix} \begin{pmatrix} q^{-4} & 0 & 0 \\ 0 & q^{-4} & 0 \\ 0 & 0 & q^{-4} \end{pmatrix} \\
&= \begin{pmatrix} q^{-2(2(k+1))} & 0 & 0 \\ 0 & q^{-2(2(k+1))} & 0 \\ 0 & 0 & q^{-2(2(k+1))} \end{pmatrix}.
\end{aligned}$$

If n is odd, for $n = \pm 1$ we have that

$$\rho'(\Delta) = \begin{pmatrix} 0 & 0 & q \\ 0 & -q^2 & 0 \\ q^3 & 0 & 0 \end{pmatrix} \quad \text{y} \quad \rho'(\Delta^{-1}) = \begin{pmatrix} 0 & 0 & q^{-3} \\ 0 & -q^{-2} & 0 \\ q^{-1} & 0 & 0 \end{pmatrix}.$$

If we assume that the statement is true for positive odd $n = 2k+1$, then for the next odd $2(k+1)+1$ we have the following:

$$\begin{aligned}
\rho'(\Delta^{2(k+1)+1}) &= \rho'(\Delta^{2k+3}) = \rho'(\Delta)^{2k+3} = \rho'(\Delta)^{2k+1}\rho'(\Delta)^{2} \\
&= \begin{pmatrix} 0 & 0 & q^{2(2k+1)-1} \\ 0 & -q^{2(2k+1)} & 0 \\ q^{2(2k+1)+1} & 0 & 0 \end{pmatrix} \begin{pmatrix} q^{4} & 0 & 0 \\ 0 & q^{4} & 0 \\ 0 & 0 & q^{4} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & q^{2(2k+1)+3} \\ 0 & -q^{2(2k+1)+4} & 0 \\ q^{2(2k+1)+5} & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & q^{2(2(k+1)+1)-1} \\ 0 & -q^{2(2(k+1)+1)} & 0 \\ q^{2(2(k+1)+1)+1} & 0 & 0 \end{pmatrix}.
\end{aligned}$$

And assuming that it is true for negative odd $n = -2k-1$, then for the next negative odd $-2(k+1)-1$ we have the following:

$$\begin{aligned}
\rho'(\Delta^{-2(k+1)-1}) &= \rho'(\Delta^{-2k-3}) = \rho'(\Delta)^{-2k-3} = \rho'(\Delta)^{-2k-1}\rho'(\Delta)^{-2} \\
&= \begin{pmatrix} 0 & 0 & q^{2(-2k-1)-1} \\ 0 & -q^{-2(-2k-1)} & 0 \\ q^{2(-2k-1)+1} & 0 & 0 \end{pmatrix} \begin{pmatrix} q^{4} & 0 & 0 \\ 0 & q^{4} & 0 \\ 0 & 0 & q^{4} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & q^{2(-2k-1)-1-4} \\ 0 & -q^{2(-2k-1)-4} & 0 \\ q^{2(-2k-1)+1-4} & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & q^{2(-2(k+1)-1)} \\ 0 & -q^{2(-2(k+1)-1)} & 0 \\ q^{2(-2(k+1)-1)+1} & 0 & 0 \end{pmatrix}.
\end{aligned}$$

□

From various observations, we can make conjectures regarding powers of $\rho(\sigma_1\sigma_2\sigma_3)$:

Conjecture 0.6. Let p be an odd prime and $k \in \mathbb{Z}_{\geq 2}$, then

$$\rho(\sigma_1\sigma_2\sigma_3)^{p^k} = \begin{pmatrix} -t^{p^k} + t^{p^k+1} & -t^{p^k+1} + t^{p^k+2} & t^{p^k+2} \\ t^{p^k-1} - t^{p^k} + t^{p^k+1} & -t^{p^k+1} + t^{p^k+2} & t^{p^k+2} \\ -t^{p^k} + t^{p^k+1} & t^{p^k-1} - t^{p^k+1} + t^{p^k+2} & t^{p^k+2} \end{pmatrix},$$

$$\rho(\sigma_1\sigma_2\sigma_3)^{-p^k} = \begin{pmatrix} -t^{-(p^k-1)} & t^{-(p^k-1)} & 0 \\ -t^{-(p^k-1)} & 0 & t^{-(p^k-1)} \\ -t^{-(p^k-1)} + t^{-(p^k+1)} + t^{-(p^k+2)} & t^{-p^k} - t^{-(p^k+1)} & t^{-(p^k-1)} - t^{-p^k} \end{pmatrix}.$$

Note that the conjectured matrices have entries that we can view in terms of t_n, that is,

$$\rho(\sigma_1\sigma_2\sigma_3)^{p^k} = \begin{pmatrix} t_{p^k+1} - t_{p^k-1} & -t_{p^k+2} + t_{p^k} & t^{p^k+2} \\ t_{p^k+1} - t_{p^k-2} & -t_{p^k+2} + t_{p^k} & t^{p^k+2} \\ t_{p^k+1} - t_{p^k-1} & t_{-p^k+2} + t_{p^k-2} - t^{p^k} & t^{p^k+2} \end{pmatrix},$$

$$\rho(\sigma_1\sigma_2\sigma_3)^{p^k} = \begin{pmatrix} t^{-(p^k-1)} & t^{-(p^k-1)} & 0 \\ t^{-(p^k-1)} & 0 & t^{-(p^k-1)} \\ -t_{-(p^k+2)} + t_{-p^k} & -t_{-(p^k+1)} + t_{-(p^k-1)} & t_{-p^k} - t_{-(p^k-2)} \end{pmatrix}.$$

This might lead to future research, since the way the conjecture is proven is not that straightforward, but it can reveal the behavior of the representation for several elements of B_4. On the other hand, we can also notice that $\rho(\sigma_1^n\sigma_2^l\sigma_3^k)$ shows a new combinatorics that reveals the behavior of all elements of B_4 via $\langle\sigma_1^n\sigma_2^l\sigma_3^k\rangle$ for $n, l, k \in \mathbb{Z}$.

References

[1] Amitesh Datta. A strong characterization of the entries of the burau matrices of 4-braids: The burau representation of the braid group b_4 is faithful almost everywhere, 2022.