5.1. Stopped martingales are martingales. Let $(M_n, n = 0, 1, 2, \dots)$ be a martingale in discrete time for the filtration $(\mathcal{F}_n, n \geq 0)$. Let τ be a stopping time for the same filtration. Use the martingale transform with the process

$$X_n(\omega) = \begin{cases} +1 & \text{if } n < \tau(\omega), \\ 0 & \text{if } n \geq \tau(\omega) \end{cases}$$

to show that the stopped martingale $(M_{\tau \wedge n}, n \geq 0)$ is a martingale.

define $M_n^\tau = M_0 + \sum_{k=1}^{n} X_{k-1}(M_k - M_{k-1})$

M is a martingale, X_n is bounded and predictable

So M_n^τ is a martingale

So the stopping process $M_{\tau\wedge n}$ is a martingale

5.3. Convergence in L^2 implies convergence of first and second moments. Let $(X_n, n \geq 0)$ be a sequence of random variables that converge to X in $L^2(\Omega, \mathcal{F}, \mathbf{P})$.

(a) Show that $\mathbf{E}[X_n^2]$ converges to $\mathbf{E}[X^2]$.
Hint: Write $X = (X - X_n) + X_n$. The Cauchy-Schwarz inequality might be useful.

(b) Show that $\mathbf{E}[X_n]$ converges to $\mathbf{E}[X]$.
Hint: Write $|\mathbf{E}[X_n] - \mathbf{E}[X]|$ and use Jensen's inequality twice.

(a) given $E[(X_n - X)^2] \to 0$, prove $E(X_n^2) \to E(X^2)$

$|E(X_n^2) - E(X^2)| = |E[(X_n - X)(X_n + X)]| \leq \sqrt{E[(X_n - X)^2]}\sqrt{E[(X_n + X)^2]}$

$\leq \sqrt{2E(X_n^2) + 2E(X^2)}\sqrt{E[(X_n - X)^2]}$

$E(X_n^2) = E[(X + (X_n - X))^2] \leq 2E(X^2) + \underbrace{2E[(X_n - X)^2]}_{\to 0}$

$E(X_n^2) \leq 2E(X^2) + 2,$

$\underbrace{E[(X_n + X)^2]}_{\text{have a upper limit}} \leq 2[2E(X^2) + 2] + 2E(X^2) = 6E(X^2) + 4$

$\lim_{n\to\infty} |E(X_n^2) - E(X^2)| = 0 \Rightarrow E(X_n^2) \to E(X^2)$

(b) $|E(X_n) - E(X)| = |E(X_n - X)| \leq E(|X_n - X|) \leq \underbrace{\sqrt{E[(X_n - X)^2]}}_{\to 0}$

$\Rightarrow E(X_n) \to E(X)$