4.12)

(a) $M_t = tB_t - \frac{1}{3}B_t^3$

$E(M_t \mid F_s) = Z$ {Expectation of M_t given filtration F_s}

$t > s$

$B_t = B_s + (B_t - B_s)$

$B_t^3 = (B_s)^3 + 3B_s^2(B_t - B_s) + 3(B_t - B_s)^2 B_s + B_t^3$ — ①

Taking expetation of $B_t^3 \mid F_s$

$E(B_s^3 \mid F_s) + 3E(B_s^2(B_t - B_s) \mid F_s) + 3E[B_s(B_t - B_s)^2 \mid F_s] + E((B_t - B_s)^3 \mid F_s)$

$\downarrow$

$B_s^3 + 3B_s^2 \cdot 0 + 3 \cdot B_s(t-s) + 0$

{$(B_t - B_s) \to$ odd moments are zero $\sim N(0, t-s)$

$(B_t - B_s)^2 = t-s$

$$E(t.B_t \mid F_s) = t.E(B_t \mid F_s) = t.B_s$$

$$Z = t.B_s - \frac{1}{3}\left(B_s^3 + 3B_s(t-s)\right)$$

$$= t.B_s - \frac{1}{3}B_s^3 - B_s.t + B_s.s$$

$$= B_s.s - \frac{1}{3}B_s^3 = X_s$$

$$\boxed{E(X_t \mid F_s) = X_s}$$ → So a martingale.

(b) By stopping time.

First → $P(a) = \frac{b}{a+b}$ and $P(b) = \frac{a}{a+b}$.

Now $X_t = tB_t - \frac{1}{3}B_t^3$

$$X_0 = 0.B_0 - \frac{1}{3}B_0^3 = 0$$

$$E\left(\tau B_\tau - \frac{1}{3}B_\tau^3\right) = 0$$

$$E(\tau B_\tau) = E\left(\frac{1}{3}B_\tau^3\right)$$

$$= \frac{b.a^3}{3(a+b)} + \frac{a.(-b)^3}{3(a+b)}$$

$$= \frac{ab}{3}\left[\frac{a^2-b^2}{a+b}\right] = \frac{ab(a-b)}{3}$$

(c) $E\left(e^{aB_\tau - a^2\tau/2}\right)$

$$= e^{-a^2\tau/2} E\left[e^{aB_\tau}\right] \quad — (1)$$

Since $e^{-a^2\tau/2}$ is constant for any τ,

$= E\left(e^{aB_\tau}\right) \Rightarrow$ M.G.F of B.M

$B_\tau \sim N(0,\tau)$; $E\left(e^{aB_\tau}\right) =$

$e^{\frac{a^2\tau}{2}}$ [M.G.F of a Normal]

So ① becomes

$$e^{-\frac{a^2\tau}{2}+\frac{a^2\tau}{2}} = 1$$

So $E\left(e^{+aB_\tau-\frac{a^2\tau}{2}}\right)=1$

(iv) $e^{\alpha B_\tau - \frac{1}{2}\alpha^2 \tau}$

$$= e^{\alpha B_\tau} e^{-\frac{1}{2}\alpha^2\tau}$$

$$= \Big(1+\alpha B_\tau + \frac{\alpha^2 B_\tau^2}{2} + \frac{\alpha^3 B_\tau^3}{6} + O(\alpha^4)\Big)$$

$$\cdot\left(1-\frac{1}{2}\alpha^2\tau + \frac{1}{4}\alpha^4\tau^2 + O(\alpha^6)\right)$$

Collecting terms upto α^3

$$1+\alpha B_\tau + \frac{\alpha^2}{2}(B^2-\tau) + \alpha^3\left(\frac{1}{6}B_\tau^3 - \frac{1}{2}\tau B_\tau\right)$$

$$+ O(\alpha^4)$$

$$E(B_\tau) = 0 \ , \ E(B_\tau^2) = \tau$$

and E (the whole term $= 1$)

$$E(1) + \alpha E(\cancel{B_\tau})^{0} + \frac{\alpha^2}{2}\left(\cancel{E(B_\tau^2)} - \tau\right)^{0}$$

$\downarrow$

1

$$+ \alpha^3\left(E\left(\frac{1}{6}B_\tau^3 - \frac{1}{2}\tau B_\tau\right)\right)$$

$$= 1$$

$$E\left(\frac{1}{6}B_\tau^3 - \frac{1}{2}\tau B_\tau\right) = 0$$

$$E(\tau B_\tau) = E\left(\frac{1}{3}B_\tau^3\right)$$

Using that $E(B_\tau^3) = \dfrac{a^3 b}{a+b} - \dfrac{b^3 a}{a+b} = ab(a-b)$

$$E(\tau B_\tau) = \frac{ab(a-b)}{3}$$

4.11)

$$X_t = e^{\alpha N_t - \lambda t(e^{\alpha}-1)}$$

for $t \geq s$

$$N_t = (N_t - N_s) + N_s$$

$$X_t = e^{\alpha(N_t - N_s) + \alpha N_s - \lambda t(e^{\alpha}-1)}$$

$$E(X_t | f_s) = E\left[e^{\alpha(N_t-N_s) + \alpha N_s - \lambda t(e^{\alpha}-1)} / f_s\right]$$

Now We know

$N_t - N_s \approx \text{Poisson}(\lambda(t-s))$

$N_t - N_s$ is independent of N_s

[Memorylessness of Poission]

$$\Rightarrow e^{-\lambda t(e^{\alpha}-1)} \cdot \underbrace{E\left[e^{\alpha(N_t-N_s)}\right]}_{\text{M.G.F of } N_t} \cdot E\left(e^{\alpha N_s} / f_s\right)$$

MGF of $N = e^{\lambda(e^t-1)}$

$\Rightarrow e^{-\lambda t(e^\alpha - 1)} e^{\lambda(t-s)(e^\alpha - 1)} \cdot e^{\alpha N_s}$

$\Rightarrow \boxed{e^{\alpha N_s - \lambda s(e^\alpha - 1)}} \Rightarrow E(S)$

Hence a martingale since $E(X_t | F_s) = E(X_s)$

and also it is integrable since it's a finite random variable with finite mean.

4.6) $C = \begin{bmatrix} 3/16 & 1/8 \\ 1/8 & 1/4 \end{bmatrix}$

If it's positive semidefinite then it's not a degenerate random vector.

$$\text{Det}(C) = \frac{3}{16} \times \frac{1}{4} - \frac{1}{8} \times \frac{1}{8}$$

$$= \frac{2}{64} = \frac{1}{32} > 0$$

and all leading principal minors > 0.

Hence it's not degenerate

(b) $E(Y|X) = \left(\frac{8}{Var(X)}\right)X = \frac{1}{8\times 3}\times 16X$

$$\boxed{= \frac{2}{3}X}$$

(c) $Y = W + \frac{2}{3}X$

$Cov(Y,Y) = Cov(W,Y) + \frac{2}{3}Cov(Y,X)$

As $cov(X,W) = 0$

$\frac{1}{4} = Cov(W,Y) + \frac{2}{3}\times\frac{1}{8}$

$\frac{1}{4} - \frac{1}{12} = Cov(W,Y)$

$$\boxed{\frac{1}{6} = Cov(W,Y)}$$

$$\overset{0}{E}(Y) = E(W) + \frac{2}{3}\overset{0}{E}(X)$$

$$\boxed{E(W) = 0}$$

Now

$$\mathrm{Cov}(W,Y) = \mathrm{Cov}(W,W) + \overset{\nearrow 0}{\mathrm{Cov}(W,X)}$$

$$W = N\left(0, \frac{1}{6}\right)$$

$$E(Y|X) = E(W|X) + \frac{2}{3}E(X|X)$$

$$E(Y|X) = \underset{\substack{\downarrow \\ 0}}{E(W)} + \frac{2}{3}X$$

$$E(Y|X) = \frac{2}{3}X.$$

$$\mathrm{Var}(Y|X) = \mathrm{Var}(W) = \frac{1}{6}$$

or $\mathrm{PDF}(Y|X) \sim N\left(\frac{2}{3}X, \frac{1}{6}\right)$

(d) Now

$$y - \frac{2}{3}X = W \qquad W \sim N\left(0, \frac{1}{6}\right)$$

$$\sqrt{6}\left(y - \frac{2}{3}X\right) = Z_1 \qquad Z \sim N(0,1)$$

$$= \sqrt{6}W$$

$$Z_2 = \frac{X}{\sqrt{\frac{3}{16}}} = \frac{4}{\sqrt{3}}X$$

$Z_2 \cdot Z_1$ are independent since

W and X are independent.