MTH 9821 Numerical Methods for Finance

Fall 2025

Homework 1

Assigned: August 27; Due: September 8

This homework is to be done as a group. Each team will hand in one homework solution, and each member of the team should write at least one problem. On the cover page of the homework, please indicate the members of the team and who wrote each problem.

Pseudorandom number generators

In what follows, we will use the following pseudorandom number generators to price and calculate sensitivities of various options.

Linear Congruential Generator

Generate N independent samples from the uniform distribution on $[0,\ 1]$ by using the Linear Congruential Generator

$$\begin{aligned} x_{i+1} &= a x_i + c \pmod{k} \\ u_{i+1} &= \frac{x_{i+1}}{k}, \end{aligned}$$

with $x_0 = 1$, $a = 39373$, $c = 0$, and $k = 2^{31} - 1$ to generate $u_1, u_2, \ldots, u_N$.

Inverse Transform Method

Generate N independent samples z_i, $i = 1 : N$ from the standard normal distribution by using the independent uniform samples u_i, $i = 1 : N$ obtained above. Let

$$z_i = F^{-1}(u_i), \quad \forall\, i = 1 : N,$$

where $F^{-1}(u)$ is the inverse function of the cumulative distribution function $F(x)$ of the standard normal, i.e.,

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt.$$

Use the Beasley–Springer–Moro algorithm to compute $F^{-1}(u)$, as given in Figure 2.13, page 68, from Glasserman. (Note: the constants are given in Figure 2.12, same page.)

Acceptance–Rejection Method

Generate $N_{A-R} < N$ independent samples from the standard normal distribution by using the independent uniform samples u_i, $i = 1 : N$ obtained above. Generate samples from the double exponential random variable (with density function $g(x) = \frac{1}{2} e^{-|x|}$) and accept them with probability $\frac{f(x)}{cg(x)}$, where $c = \sqrt{\frac{2e}{\pi}}$.

Use the following algorithm:

<u>Step 1:</u> Generate $u_1, u_2, u_3 \in U([0,1])$

<u>Step 2:</u> Let $X = -\ln(u_1)$

<u>Step 3:</u>

If $u_2 > \exp\left(-\frac{1}{2}(X-1)^2\right)$

    go to Step 1
else
if $u_3 \leq 0.5$
    X = -X
return X

Box–Muller Method

Generate $N_{B-M} < N$ independent samples from the standard normal distribution by using the independent uniform samples u_i, $i = 1 : N$ obtained above.
The Marsaglia–Bray algorithm:

while $X > 1$
    Generate $u_1, u_2 \in U([0, 1])$
    $u_1 = 2u_1 - 1$; $u_2 = 2u_2 - 1$
    $X = u_1^2 + u_2^2$
end
$Y \;=\; \sqrt{-2\frac{\ln(X)}{X}}$
$Z_1 = u_1 Y; \quad Z_2 = u_2 Y$
return Z_1, Z_2

Monte Carlo Pricing and Greeks Estimations
for
Plain Vanilla European Options

Goal: Use Monte Carlo simulation to value and to compute the Greeks of plain vanilla European options on lognormally distributed underlying assets.

Consider a nine months European call and put options with strike \$42 on a lognormally distributed underlying asset with spot price \$41 and volatility 25%, paying 1% dividends continuously. Assume that the risk-free rate is constant at 3%. (In other words, $S(0) = 41$, $K = 42$, $T = 0.75$, $\sigma = 0.25$, $q = 0.01$, and $r = 0.03$.)

We simulate N possible values of the underlying asset at maturity. To do this, N independent samples of the standard normal distribution must be generated.

1. Generate N independent samples from the standard normal distribution by using the Inverse Transform Method.

2. Compute the Black–Scholes values of the call and put options, as well as the values of the Delta and vega of the options. Denote these values by C_{BS}, $\Delta_{BS}(C)$, $\text{vega}_{BS}(C)$, and P_{BS}, $\Delta_{BS}(P)$, $\text{vega}_{BS}(P)$, respectively.

3. Use the N independent samples z_i, $i = 1 : N$ from the standard normal distribution obtained above to find approximate values of the options and of their Delta and vega using pathwise derivative estimates.

For $i = 1 : N$, let

$$
\begin{aligned}
S_i &= S(0)\,\exp\left(\left(r - q - \frac{\sigma^2}{2}\right)T \;+\; \sigma\sqrt{T}z_i\right); \\
C_i &= e^{-rT}\max(S_i - K, 0); \\
\Delta_i(C) &= \mathbb{I}_{(S_i > K)} \cdot e^{-rT}\frac{S_i}{S(0)}; \\
\text{vega}_i(C) &= \mathbb{I}_{(S_i > K)} \cdot S_i e^{-rT}\left(-\sigma T + \sqrt{T}z_i\right); \\
P_i &= e^{-rT}\max(K - S_i, 0); \\
\Delta_i(P) &= -\mathbb{I}_{(K > S_i)} \cdot e^{-rT}\frac{S_i}{S(0)}; \\
\text{vega}_i(P) &= -\mathbb{I}_{(K > S_i)} \cdot S_i e^{-rT}\left(-\sigma T + \sqrt{T}z_i\right).
\end{aligned}
$$

Let $N = 10,000 \cdot 2^k$, for $k = 0 : 9$, and compute the following pathwise derivative estimates for the options and their Greeks:

$$
\begin{aligned}
\widehat{C}(N) &= \frac{1}{N}\sum_{i=1}^{N} C_i; \quad \widehat{\Delta(C)}(N) = \frac{1}{N}\sum_{i=1}^{N}\Delta_i(C); \quad \widehat{\text{vega}(C)}(N) = \frac{1}{N}\sum_{i=1}^{N}\text{vega}_i(C); \\
\widehat{P}(N) &= \frac{1}{N}\sum_{i=1}^{N} P_i; \quad \widehat{\Delta(P)}(N) = \frac{1}{N}\sum_{i=1}^{N}\Delta_i(P); \quad \widehat{\text{vega}(P)}(N) = \frac{1}{N}\sum_{i=1}^{N}\text{vega}_i(P).
\end{aligned}
$$

For each value $N = 10,000 \cdot 2^k$, for $k = 0 : 9$, report the Monte Carlo simulation approximate values and the corresponding approximation errors for the options and their Greeks.

Check whether the convergence of the Monte Carlo pricer is of order $O\left(\frac{1}{\sqrt{N}}\right)$, i.e., compute, for each N, the value of $\sqrt{N}|P_{BS} - \widehat{P}(N)|$ and try to find a convergence pattern.

Summarizing, report the following values:

N	10,000	20,000	40,000	80,000	160,000	320,000	640,000	1,280,000	2,560,000	5,1
$\widehat{C}(N)$										
$\sqrt{N}\lvert C_{BS} - \widehat{C}(N)\rvert$										
$\widehat{\Delta(C)}(N)$										
$\sqrt{N}\lvert\Delta_{BS}(C) - \widehat{\Delta(C)}(N)\rvert$										
$\widehat{\text{vega}(C)}(N)$										
$\sqrt{N}\lvert\text{vega}_{BS}(C) - \widehat{\text{vega}(C)}(N)\rvert$										

Note: for each column, start the random number generator at $x_0 = 1$. This means that your samples z_i generated for column $10,000$ will be the first half of the samples generated for column $20,000$, etc.

Because of this, you might find it convenient to generate all $z_1, z_2 \ldots z_{5,120,000}$ in the beginning, and use its initial segments for $N = 10,000 \cdot 2^k$ for $k < 9$.

Follow this convention for the rest of the assignment as well.

4. Use the same samples N independent samples z_i, $i = 1 : N$ from the standard normal distribution to find finite difference Monte Carlo approximations for the Greeks.

Let $\delta S = 0.01$. For every z_i, compute

$$\begin{aligned}
S_i(T) &= S(0)\ \exp\left(\left(r - q - \frac{\sigma^2}{2}\right)T \ + \ \sigma\sqrt{T}z_i\right); \\
S_i^{(\delta S)}(T) &= (S(0) + \delta S)\ \exp\left(\left(r - q - \frac{\sigma^2}{2}\right)T \ + \ \sigma\sqrt{T}z_i\right); \\
S_i^{(-\delta S)}(T) &= (S(0) - \delta S)\ \exp\left(\left(r - q - \frac{\sigma^2}{2}\right)T \ + \ \sigma\sqrt{T}z_i\right),
\end{aligned}$$

and compute the options values

$$\begin{aligned}
C_i &= e^{-rT}\max(S_i(T) - K, 0); \\
C_i^{(\delta S)}(T) &= e^{-rT}\max(S_i^{(\delta S)}(T) - K, 0); \\
C_i^{(-\delta S)}(T) &= e^{-rT}\max(S_i^{(-\delta S)}(T) - K, 0).
\end{aligned}$$

Let

$$\begin{aligned}
\widehat{C}(N) &= \frac{1}{N}\sum_{i=1}^{N} C_i; \\
\widehat{C}^{(\delta S)}(N) &= \frac{1}{N}\sum_{i=1}^{N} C_i^{(\delta S)}(T); \\
\widehat{C}^{(-\delta S)}(N) &= \frac{1}{N}\sum_{i=1}^{N} C_i^{(-\delta S)}(T).
\end{aligned}$$

Let $N = 10,000 \cdot 2^k$, for $k = 0 : 9$, and compute the central finite difference Monte Carlo estimates for Delta and Gamma:

$$\Delta_c(N) = \frac{C_i^{(\delta S)}(T) - C_i^{(-\delta S)}(T)}{2\delta S};$$

$$\Gamma_c(N) = \frac{C_i^{(\delta S)}(T) - 2C_i + C_i^{(-\delta S)}(T)}{(\delta S)^2}.$$

For each value $N = 10,000 \cdot 2^k$, for $k = 0 : 9$, report the finite difference Monte Carlo estimates for Delta and Gamma and their approximation errors with respect to the corresponding Black–Scholes values. Check whether the convergence of the Monte Carlo pricer is of order $O\left(\frac{1}{\sqrt{N}}\right)$.

Summarizing, report the following values:

N	10,000	20,000	40,000	80,000	160,000	320,000	640,000	1,280,000	2,560,000	5,120,000
$\Delta_c(N)$										
$\sqrt{N}\lvert\Delta_{BS}(C) - \Delta_c(N)\rvert$										
$\Gamma_c(N)$										
$\sqrt{N}\lvert\Gamma_{BS}(C) - \Gamma_c(N)\rvert$										

Monte Carlo Pricing of a Path–Dependent Option

Goal: Use Monte Carlo simulation to value a path–dependent option on a lognormally distributed underlying asset.

Consider a nine months down–and–out call with strike \$39 and barrier \$36 on a lognormally distributed underlying asset with spot price \$39 and volatility 25%, paying 1% dividends continuously. Assume that the risk-free rate is constant at 2%. (In other words, $S(0) = 39$, $K = 39$, $B = 36$, $T = 0.75$, $\sigma = 0.25$, $q = 0.01$, and $r = 0.02$.)

We simulate the risk neutral random path of the asset on n different paths, each one discretized by m time steps corresponding to $\delta t = \frac{T}{m}$. To do this, $N = nm$ independent samples of the standard normal distribution must be generated.

1. Generate n different paths for the evolution of the underlying asset

Use the independent samples $z_1, z_2, \ldots, z_N$ of the standard normal distribution obtained previously using the Inverse Transform Method.

Discretize the time to maturity using m time steps corresponding to $\delta t = \frac{T}{m}$. Let $t_j = j\delta t$, for $j = 0 : m$. Use the multiplicative formula for the evolution of the underlying, i.e., on every path, compute

$$S(t_{j+1}) \ = \ S(t_j) \exp\left(\left(r - \frac{\sigma^2}{2}\right)\delta t \ + \ \sigma\sqrt{\delta t} z_{im+j}\right), \ \forall\, j = 0 : (m-1).$$

2. Compute the exact value C_{dao} using the closed formula for valuing down-and-out calls.

The value of the down–and–out call with barrier B less than the strike K is

$$V(S,K) \ = \ C(S,K) \ - \ \left(\frac{B}{S}\right)^{2a} C\left(\frac{B^2}{S}, K\right), \tag{1}$$

where

$$a \ = \ \frac{r-q}{\sigma^2} - \frac{1}{2},$$

$C(S,K)$ is the value at time 0 of a plain vanilla call option with strike K and maturity T on the same underlying asset, and $C\left(\frac{B^2}{S}, K\right)$ is the Black–Scholes value at time 0 of a plain vanilla call with strike K and maturity T on an asset having spot price $\frac{B^2}{S}$ (and the same volatility as the underlying).

3. Use the $N = 10,000 \cdot 2^9$ independent samples from the standard normal distribution obtained previously to generate n different paths for the evolution of the underlying. Let V_i, $i = 1 : n$, the value of the down and out call provided the underlying evolves along the path i.

Compute an approximate value

$$\widehat{V}(n) \ = \ \frac{1}{n}\sum_{i=1}^{n} V_i,$$

and the corresponding approximation error $|C_{dao} - \widehat{V}(n)|$.

3.1. Consider a fixed number of time intervals corresponding to approximately one day, i.e., choose $m = 200$. Use $n = 50 \cdot 2^k$ paths, where $k = 0 : 9$, i.e., $n = \frac{N_k}{m}$, where $N_k = 10,000 \cdot 2^k$, $k = 0 : 9$.

3.2. Consider optimal values for the number of time intervals and paths for each $N_k = 10,000 \cdot 2^k$, $k = 0 : 9$, i.e., let

$$m_k \ = \ \text{ceil}\left(N_k^{1/3} T^{2/3}\right); \quad n_k \ = \ \text{floor}\left(\frac{N_k}{m_k}\right),$$

where $T = 0.75$, corresponding to a nine months maturity.

Report the results in the following format:

N_k	$m = 200$	n	$\widehat{V}(n)$	$\lvert C_{dao} - \widehat{V}(n) \rvert$	m_k	n_k	$\widehat{V}(n_k)$	$\lvert C_{dao} - \widehat{V}(n_k) \rvert$
10,000								
20,000								
40,000								
80,000								
160,000								
320,000								
640,000								
1,280,000								
2,560,000								
5,120,000								

Comparison of Random Number Generators

Monte Carlo Valuation of Plain Vanilla Options

1. Consider a six months European put option with strike 55 on a non–dividend–paying underlying asset with spot price 50 following a lognormal distribution with volatility 30%. Assume that the risk-free rate is constant at 4%.

Compute the Black–Scholes value V_{BS} of the option.

2. Find an approximate option values using Monte Carlo simulations.
   Let

$$S_i = S(0)\,\exp\left(\left(r - q - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z_i\right), \; i = 1:M;$$
$$V_i = e^{-rT}\max(K - S_i, 0), \; i = 1:M,$$

where z_i, $i = 1:M$, are independent samples of the standard normal variable obtained from N uniform samples.

Note that number of samples $M = N_{A-R}$ and $M = N_{B-M}$ corresponding to the Acceptance–Rejection method and to the Marsaglia–Bray version of the Box–Muller method are smaller than N.

Report the approximate values

$$\widehat{V}(n) = \frac{1}{n}\sum_{i=1}^{n} V_i$$

and the corresponding approximation errors $|V_{BS} - \widehat{V}(n)|$ in the tables below:

Inverse Transform

N	$\widehat{V}(N)$	$\lvert V_{BS} - \widehat{V}(N)\rvert$
10,000		
20,000		
...		
5,120,000		

Acceptance–Rejection

N	N_{A-R}	$\widehat{V}(N_{A-R})$	$\lvert V_{BS} - \widehat{V}(N_{A-R})\rvert$
10,000			
20,000			
...			
5,120,000			

Box–Muller

N	N_{B-M}	$\widehat{V}(N_{B-M})$	$\lvert V_{BS} - \widehat{V}(N_{B-M})\rvert$
10,000			
20,000			
$\dots$			
5,120,000			