5.8. **Practice on Itô integrals.** Consider the two processes

$$X_t = \int_0^t (1-s)\,\mathrm{d}B_s, \qquad Y_t = \int_0^t (1+s)\,\mathrm{d}B_s.$$

(a) Find the mean and the covariance of the process $(X_t, t \geq 0)$. What is its distribution?
(b) Find the mean and the covariance of the process $(Y_t, t \geq 0)$. What is its distribution?
(c) For which time t, if any, do we have that X_t and Y_t are uncorrelated? Are X_t and Y_t independent at these times?

(a) $E(X_t) = \int_0^t E[(1-s)dB_s] = 0$

$Cov(X_t, X_s) = E(X_t X_s) = \int_0^{\min(s,t)} (1-u)^2 du = \min(s,t) - \min^2(s,t) + \frac{1}{3}\min^3(s,t)$

X_t is Gaussian distribution, $X_t \sim N(0, t - t^2 + \frac{1}{3}t^3)$

(b) $E(Y_t) = \int_0^t E[(1+s)dB_s] = 0$

$Cov(Y_t, Y_s) = E(Y_t Y_s) = \int_0^{\min(s,t)} (1+u)^2 du = \min(s,t) + \min^2(s,t) + \frac{1}{3}\min^3(s,t)$

Y_t is Gaussian distribution, $Y_t \sim N(0, t + t^2 + \frac{1}{3}t^3)$

(c) $Cov(X_t, Y_t) = E(X_t Y_t) = \int_0^t (1-t)(1+t)dt = t - \frac{1}{3}t^3 = t(1 - \frac{1}{3}t^2)$

$t > 0$, $Cov(X_t, Y_t) = 0$ when $t = 0$ or $t = \sqrt{3}$,

$\Updownarrow$

$Corr(X_t, Y_t) = 0$

X_t, Y_t are joint Gaussian distribution. so uncorrelated $\Leftrightarrow$ independent

5.11. **Practice on Itô integrals.** Let $(B_t, t \geq 0)$ be a Brownian motion defined on $(\Omega, \mathcal{F}, \mathbf{P})$. We define for $t \geq 0$ the process

$$X_t = \int_0^t \operatorname{sgn}(B_s)\,\mathrm{d}B_s,$$

where $\operatorname{sgn}(x) = -1$ if $x < 0$ and $\operatorname{sgn}(x) = +1$ if $x \geq 0$.
The integral is well-defined even though $s \mapsto \operatorname{sgn}(B_s)$ *is not continuous.*
(a) Compute the mean and the covariance of the process $(X_t, t \geq 0)$.
(b) Show that X_t and B_t are uncorrelated for all $t \geq 0$.

(a) $E(X_t) = \int_0^t E[\operatorname{sgn}(B_s)dB_s] = 0$

$Cov(X_t, X_s) = E(X_t X_s] = E(\int_0^{\min(s,t)} \operatorname{sgn}(B_u^2) dB_u) = \min(s,t)$

(b) $Cov(X_t, B_t) = E(X_t B_t) = E(\int_0^t \operatorname{sgn}(B_s) ds) = 0, \ \forall t \geq 0$

(c) Show that X_t and B_t are not independent. (Use $B_t^2 = 2\int_0^t B_s\,\mathrm{d}B_s + t$.)
It turns out that $(X_t, t \geq 0)$ *is a standard Brownian motion. See Theorem 7.26.*

$B_t^2 = 2\int_0^t B_s dB_s + t$, if X_t and B_t are independent

$E(X_t B_t^2) = 2E[\int_0^t \operatorname{sgn}(B_s) dB_s \int_0^t B_u dB_u] + tE(X_t)$

$= 2\int_0^t E(|B_s|) ds = 2\int_0^t \sqrt{\frac{2s}{\pi}} ds = \frac{4}{3}\sqrt{\frac{2}{\pi}} t^{\frac{3}{2}} \neq 0$

$\Rightarrow$ X_t and B_t are not independent $\square$