HW 1

3.6.

(a) Let $t>0$.

We know that

$B_{1/t} \sim \mathcal{N}(0, 1/t)$

Thus $tB_{1/t} \sim \mathcal{N}(0,t)$

Then by linearity,

$tB_{1/t}$ is a gaussian process

because $(t_1 B_{1/t_1}, \dots, t_n B_{1/t_n})$ is

gaussian,

Moreover, $cov(tB_{1/t}, sB_{1/s}) = st \min(\frac{1}{s}, 1/t)$

$\underset{\text{case by case}}{=} \min(t,s)$

and $\mathbb{E}(tB_{1/t}) = 0 \quad \forall t>0$

Thus, $tB_{1/t}$ is a brownian motion for $t>0$

(b) Let $t>0$.

We have $\mathbb{E}((tB_{1/t})^2) = t^2 \int_{\mathbb{R}} \frac{x^2 e^{-\frac{x^2 t}{2}}}{\sqrt{\frac{2\pi}{t}}} dx$

$\underset{\substack{x^2 t = y^2 \\ dx\sqrt{t} = dy}}{=} t^2 \int_{\mathbb{R}} \frac{y^2/t \, e^{-y^2/2}}{\sqrt{2\pi}} \frac{\sqrt{t}}{\sqrt{t}} dy$

Thus $\mathbb{E}(\|tB_{1/t}\|^2) = t^{2} \times \underbrace{\int_{\mathbb{R}} \frac{x^2 e^{-x^2/2}}{\sqrt{2\pi}} dx}_{=C} = tC.$

Thus, $\mathbb{E}(\|tB_{1/t}\|^2) \xrightarrow[t\to 0]{} 0$

Then $tB_{1/t} \xrightarrow[t\to 0]{L^2} 0$

~~(c) let $X \geq 0$ we have~~

~~$\frac{X_t}{t} = B_{1/t}$.~~

~~Thus $\lim_{t\to\infty} \frac{X_t}{t} = \lim_{x\to 0} B_x = B_0$ a.s thanks to~~

(c) Let $t>0$,

$\lim_{t\to\infty} \frac{B_t}{t} = \lim_{t\to\infty} X_{1/t} = \lim_{t\to 0^+} X_t$

Thus we need to compute

$\lim_{t\to 0^+} X_t \underset{a.s}{=} 0$ thanks to the statement.

3.7.

(a) We have $U_{\frac{t}{t+1}} \sim \mathcal{N}(0, \frac{t}{t+1}(1-\frac{t}{t+1}))$.

Thus $\forall t_1 \dots t_n$, $(U_{\frac{t_1}{t_1+1}}, \dots U_{\frac{t_n}{t_n+1}})$ is a gaussian process.

Moreover, $\mathbb{E}(B_t) = (1+t)\,\mathbb{E}(U_{t/1+t}) \underset{\uparrow}{=} 0$

$B_{\cdot}$ is a Brownian Bridge.

Finally, $\forall t, s \in \mathbb{R}_+$,

$Cov(B_t, B_s) = (1+t)(1+s)\left[\min\left(\frac{s}{1+s}, \frac{t}{1+t}\right) - \frac{st}{(1+s)(1+t)}\right]$

if $s<t$. then $\frac{s}{1+s} \underset{\downarrow}{<} \frac{t}{1+t}$

derivative $x \mapsto \frac{x}{x+1}$ is $\frac{1}{(x+1)^2} \geq 0$

then, $Cov(B_t, B_s) = (1+t)(1+s)\left(\frac{s}{1+s} - \frac{st}{(1+s)(1+t)}\right)$

$= (1+t)s - st = s$.

Thus $Cov(B_t, B_s) = \min(s, t)$

Thus B_t is a B-Mo.

(b) We have

$\lim_{t\to\infty} \frac{B_t}{t} = \lim_{t\to\infty} \frac{(1+t)}{t} U_{t/1+t} = \lim_{t\to 1} U_x \underset{\downarrow}{=} 0$

a.s by definition of Brownian Bridge.

3.11) Let $n \geq 0$

We use $f_n = \sum_{i=1}^{n} X_i$

we have $f_n \leq f_{n+1}$.

Thus, by the monotone convergence theorem,

$$\lim_{n \to \infty} \mathbb{E}(f_n) = E(S) = \mathbb{E}\left(\sum_{n=1}^{\infty} X_n\right)$$

But $\mathbb{E}(f_n) = \sum_{i=1}^{n} \mathbb{E}(X_i)$

Thus, $\mathbb{E}\left(\sum_{n=1}^{\infty} X_n\right) = \lim_{n \to \infty} \sum_{i=1}^{n} \mathbb{E}(X_i) = \sum_{i=1}^{\infty} \mathbb{E}(X_i)$

Thus $\mathbb{E}\left(\sum_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} \mathbb{E}(X_n)$