(4.17)

By Levy's reflection principle, for $m>0$ and $a\leq m$

$$P(M_T > m, B_T \leq a) = P(B_T > 2m-a) \qquad (1)$$

We can partition $\{B_T \leq a\}$ as

$$\{B_T \leq a\} = \{M_T \leq m, B_T \leq a\} \,\dot\cup\, \{M_T > m, B_T \leq a\}$$

Let $F(m,a) :=$ Joint cdf of M_T, B_T

$$F(m,a) = P(M_T \leq m, B_T \leq a) = P(B_T \leq a) - P(M_T > m, B_T \leq a).$$

Using (1) we have $F(m,a) = P(B_T \leq a) - P(B_T > 2m-a) =$

$= P(B_T \leq a) - (1 - P(B_T < 2m-a)) =$ Since $B_T \sim N(0,T)$

since $\varphi_T(x) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}$

if we scale for variance we have ↗ if we define $Z = \frac{B_T}{\sqrt{T}}$ we see $Z \sim N(0,1)$

$$F(m,a) = \Phi\left(\frac{a}{\sqrt{T}}\right) - \left(1 - \Phi\left(\frac{2m-a}{\sqrt{T}}\right)\right) = \Phi\left(\frac{a}{\sqrt{T}}\right) + \Phi\left(\frac{2m-a}{\sqrt{T}}\right)$$

To obtain the joint pdf we have to derive on both m and a

so $f(m,a) = \frac{\partial F(m,a)}{\partial m \partial a}$. ~~[illegible]~~

~~[illegible]~~

After many calculations, we realized it's easier to just derive the original expression $P(B_T > 2m-a)$. Let $\varphi_T(x) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}$

$$\frac{\partial F(m,a)}{\partial a} = \frac{\partial}{\partial a} P(B_T > 2m-a) = \frac{\partial}{\partial a} \int_{2m-a}^{\infty} \varphi_T(x)\, dx =$$

$$= \varphi_T(2m-a)$$

$$\frac{\partial F(m,a)}{\partial m \partial a} = \frac{\partial\left[\varphi_T(2m-a)\right]}{\partial m} = \frac{2(2m-a)}{T} \varphi_T(2m-a) =$$

$$= \frac{2(2m-a)}{T} \cdot \varphi_T(2m-a) = \frac{2(2m-a)}{T} \cdot \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} =$$

$$= \frac{2(2m-a)}{T^{3/2}\sqrt{2\pi}} e^{-\frac{(2m-a)^2}{2T}}$$ as we wanted to prove

Gen AI was used to help for the rigor of the proof, and to write the first draft of the code in the last numerical project

Scansionato con CamScanner

5.1. Stopped martingales are martingales. Let $(M_n, n = 0, 1, 2, \dots)$ be a martingale in discrete time for the filtration $(\mathcal{F}_n, n \geq 0)$. Let τ be a stopping time for the same filtration. Use the martingale transform with the process

$$X_n(\omega) = \begin{cases} +1 & \text{if } n < \tau(\omega), \\ 0 & \text{if } n \geq \tau(\omega) \end{cases}$$

to show that the stopped martingale $(M_{\tau \wedge n}, n \geq 0)$ is a martingale.

define $M_n^\tau = M_0 + \sum_{k=1}^{n} X_{k-1}(M_k - M_{k-1})$

M is a martingale, X_n is bounded and predictable

so M_n^τ is a martingale

so the stopping process $M_{\tau \wedge n}$ is a martingale

5.3. Convergence in L^2 implies convergence of first and second moments. Let $(X_n, n \geq 0)$ be a sequence of random variables that converge to X in $L^2(\Omega, \mathcal{F}, \mathbf{P})$.

(a) Show that $\mathbf{E}[X_n^2]$ converges to $\mathbf{E}[X^2]$.
Hint: Write $X = (X - X_n) + X_n$. The Cauchy-Schwarz inequality might be useful.

(b) Show that $\mathbf{E}[X_n]$ converges to $\mathbf{E}[X]$.
Hint: Write $|\mathbf{E}[X_n] - \mathbf{E}[X]|$ and use Jensen's inequality twice.

(a) given $E[(X_n - X)^2] \to 0$, prove $E(X_n^2) \to E(X^2)$

$|E(X_n^2) - E(X^2)| = |E[(X_n - X)(X_n + X)]| \leq \sqrt{E[(X_n - X)^2]}\sqrt{E[(X_n + X)^2]}$

$\leq \sqrt{2E(X_n^2) + 2E(X^2)}\sqrt{E[(X_n - X)^2]}$

$E(X_n^2) = E[(X + (X_n - X))^2] \leq 2E(X^2) + \underbrace{2E[(X_n - X)^2]}_{\to 0}$

$E(X_n^2) \leq 2E(X^2) + 2$,

$\underbrace{E[(X_n + X)^2]}_{\text{have a upper limit}} \leq 2[2E(X^2) + 2] + 2E(X^2) = 6E(X^2) + 4$

$\lim_{n \to \infty} |E(X_n^2) - E(X^2)| = 0 \Rightarrow E(X_n^2) \to E(X^2)$

(b) $|E(X_n) - E(X)| = |E(X_n - X)| \leq E(|X_n - X|) \leq \underbrace{\sqrt{E[(X_n - X)^2]}}_{\to 0}$

$\Rightarrow E(X_n) \to E(X)$

5.4)

(a) $(M_t)_{t\geq 0}$ $\quad t_1 \leq t_2 \leq t_3 \leq t_4$

$$E\left[(Mt_2 - Mt_1)(M_{t_4} - M_{t_3})\right] =$$

$$E\left[E\left[(Mt_2 - Mt_1)\cdot(M_{t_4} - M_{t_3}) \mid F_3\right]\right]$$

Applying filtration on F_3

= Now Mt_2 and Mt_1 are F_3 measurable.

$$E\left[(Mt_2 - M_{t_1})\cdot E\left[(M_{t_4} - M_{t_3}) \mid F_3\right]\right]$$

$$E\left[M_{t_4} \mid F_3\right] = M_{t_3}$$

$$E\left[(M_{t_4} - M_{t_3}) \mid F_3\right] = 0$$

$$\Rightarrow E\left[(Mt_2 - Mt_1)\cdot 0\right] = 0$$

(b) B_t be brownion motion

$$X \in L^2_c(T)$$

$$M_t = \int_0^t X_s \, \partial B_s \qquad t \leq T$$

$t < t'$

$$\int_0^{t'} X_s \, \partial B_s = \underbrace{\int_0^t X_s \, \partial B_s}_{M_s} + \underbrace{\int_t^{t'} X_s \, \partial B_s}_{M_{t'} - M_t}$$

$$E\left[\left(\int_0^t X_s \, \partial B_s\right) \int_0^{t'} X_s \, \partial B_s\right.$$

$$= E[M_t^2] + E\left[M_t (M_{t'} - M_t)\right]$$

Now cross term $= 0$ from above

$$E[M_t^2] = E\left[\left(\int_0^t X_s \, \partial B_s\right)\right]^2 = \int_0^t E[X_s^2] \, \partial s$$

by ito's symmetry

$$E\left[\left(\int_0^t X_s \, \partial B_s\right)\left(\int_0^{t'} X_s \, \partial B_s\right)\right] = \int_0^t E[X_s^2] \, \partial s$$

5.6) To check $(M_t)_{t<T}$ is in $\mathcal{L}^2_c(T)$

B_t is adapted to paths $e^{\sigma B_t - \frac{1}{2}\sigma^2 t}$
is continous.

Square Integrable

$B_t \sim N(0, t)$

$$E\left[e^{\theta B_t}\right] = e^{\frac{1}{2}\theta^2 t} \quad \left\{ \begin{array}{l} \text{M.G.F} \\ \text{of Normal} \end{array} \right\}$$

$$E(M_t^2) = E\left[e^{2\sigma B_t - \sigma^2 t}\right]$$
$$= e^{-\sigma^2 t} E\left[e^{2\sigma B_t}\right]$$
$$= e^{-\sigma^2 t} e^{+2\sigma^2 t}$$
$$= e^{\sigma^2 t}$$

$$\int_0^T E(M^2 t) \cdot \partial t = \int_0^T e^{\sigma^2 t} \cdot \partial t$$

$$\boxed{= \frac{e^{\sigma^2 T} - 1}{\sigma^2} < \infty}$$

5.7)

$$M_t = e^{B_t^2} \qquad t \leq T.$$

$$E(M_t^2) = E\left(e^{2B_t^2}\right) \qquad B_t \sim N(0, t)$$

$$E\left(e^{2B_t^2}\right) = \int_{-\infty}^{\infty} \frac{e^{2x^2} \cdot e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \, \partial x$$

For this to be finite we need coeff. of $X^2 < 0$ otherwise it diverges

$$2 - \frac{1}{2t} < 0$$

$$2 < \frac{1}{2t} \qquad \{t > 0\}$$

$$\boxed{\text{Hence } t \leq \frac{1}{4}}$$

For $t \le \frac{1}{4}$

$\boxed{\text{From C.D.F}}$

$$E(M_t^2) = \frac{1}{\sqrt{1-4t}} \quad \left\{ \text{Since the integration is} = \frac{1}{\sqrt{1-4t}} \right\}$$

$$\int_0^T \frac{1}{\sqrt{1-4t}}\, dt < \infty \quad \boxed{T \le \frac{1}{4}}$$

$$\rightarrow \frac{1}{2}\left(1 - \sqrt{1-4T}\right) \quad \boxed{T \le \frac{1}{4}}$$