(1) Show that for a one period binomial model for an asset paying dividends continuously at rate q to be arbitrage-free, the following condition must be satisfied:

$$d < e^{(r-q)\delta t} < u.$$

Hint: Recall that having a long position between time 0 and time δt in one unit of an asset paying dividends continuously at rate q is equivalent to having a long position between time 0 and time δt in $e^{-q\,\delta t}$ units of a non–dividend–paying asset with the same drift and volatility.

$\hat{S}_t := e^{-qt} S_t$, $\hat{S}_0 = S_0$, $\hat{S}_{\delta t} = e^{-q\delta t} S_{\delta t} = \begin{cases} u\hat{S}_0 \\ d\hat{S}_0 \end{cases}$

$\hat{S}_0 = e^{-r\delta t}[\hat{p}^* u\hat{S}_0 + (1-\hat{p}^*)d\hat{S}_0)$, $\hat{p}^* = \frac{e^{r\delta t}-d}{u-d}$, $\hat{p}^* \in (0,1)$. $d < e^{r\delta t} < u$

$E^Q[S_{\delta t}|S_0] = S_0 e^{(r-q)\delta t} \Rightarrow d < e^{(r-q)\delta t} < u$ $\square$