Numerical Linear Algebra for Financial Applications

Refresher Seminar, Summer 2025

Homework 2

Assigned: August 29; Due: September 8

Linear Regression and Applications

(1) Assume that, for 15 consecutive trading days, the yields of the 2-year, 3-year, 5-year, and 10-year treasury bonds were, respectively:

2-year	3-year	5-year	10-year
1.69	2.58	3.57	4.63
1.81	2.71	3.69	4.73
1.81	2.72	3.70	4.74
1.79	2.78	3.77	4.81
1.79	2.77	3.77	4.80
1.83	2.75	3.73	4.79
1.81	2.71	3.72	4.76
1.81	2.72	3.74	4.77
1.83	2.76	3.77	4.80
1.81	2.73	3.75	4.77
1.82	2.75	3.77	4.80
1.82	2.75	3.76	4.80
1.80	2.73	3.75	4.78
1.78	2.71	3.72	4.73
1.79	2.71	3.71	4.73

Denote by T_2, T_3, T_5, and T_{10} the time series data vectors corresponding to the yield of the 2-year, 3-year, 5-year, and 10-year treasury bonds, respectively.

(i) Find the coefficients a, b_1, b_2, b_3 of the linear regression for the yield of the 3-year bond in terms of the yields of the 2-year, 5-year, and 10-year bonds, i.e., find a, b_1, b_2, b_3 corresponding to the solution to the ordinary least squares problem

$$T_3 \approx a\mathbf{1} + b_1 T_2 + b_2 T_5 + b_3 T_{10},$$

where $\mathbf{1}$ is the 15×1 column vector with all entries equal to 1. Let

$$T_{3,LR} = a\mathbf{1} + b_1 T_2 + b_2 T_5 + b_3 T_{10}.$$

Find the approximation error

$$\text{error}_{LR} = ||T_3 - T_{3,LR}||$$

of the linear regression.

(ii) Compute the linear interpolation values of the 3-year yield by doing linear interpolation between the 2-year yield and the 5-year yield at each data point. Denote by $T_{3,linear\_interp}$ the time series vector of these values. In other words,

$$T_{3,linear\_interp} = \frac{2}{3}T_2 + \frac{1}{3}T_5.$$

Find the approximation error

$$\text{error}_{linear\_interp} = ||T_3 - T_{3,linear\_interp}||$$

of the linear interpolation.

(iii) Compute the cubic interpolation values of the 3-year yield by doing cubic spline interpolation between the 2-year, 5-year, and 10-year yield at each data point. Denote by $T_{3,cubic\_interp}$ the time series vector of these values.
Find the approximation error

$$\text{error}_{cubic\_interp} = ||T_3 - T_{3,cubic\_interp}||$$

of the cubic interpolation.

(iv) Compare the approximation errors from (i), (ii), and (iii), and comment on the results.

(2) The file S&P500_ETF_Option_0917.xlsx contains the S&P 500 option prices with 9/29/2017 maturity as of March 16, 2017. The spot price of the index corresponding to these option prices was $2,381$.

(i) Use a least squares method to compute the annualized continuous dividend yield of the S&P 500 index and for the risk–free rate implied by these prices.

(ii) Compute the implied volatilities for each option.

(iii) How do the implied volatilities of calls and puts with the same strike compare to each other?

(iv) Use the explicit implied volatility formulas from the paper "An Explicit Implied Volatility Formula" from
https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2908494
(see Tables 1 and 2 for the pseudocodes) to compute approximate values for the implied volatilites of all these options. Report these approximate values and the relative errors with respect to the corresponding Black–Scholes implied volatilities computed in part (ii).

(3) The file *financials2016-8.xlsx* contains the end of week adjusted closing prices for the stocks of the following financial companies: JPM; GS; MS; BAC (Bank of America); RBS; CS; UBS; BCS (Barclays) between November 6, 2015, and September 16, 2016.

(i) Compute the weekly percentage returns of these stocks.

(ii) Find the linear regression of the JPM returns with respect to the returns of the other stocks. What is the approximation error of this linear regression?

(iii) Find the linear regression of the JPM returns with respect to the returns of the other American financial companies, i.e., with respect to GS, MS, and BAC. What is the approximation error of this linear regression?

(iv) Find the linear regression of the JPM stock prices with respect to the prices of the other stocks. What is the approximation error of this linear regression? How does it compare with the approximation error of the linear regression of the JPM returns computed at (ii)?

Optimal Portfolios

(4) Assume that the asset allocation for a \$100 million maximum return portfolio invested in three assets is \$25 million in the first asset, \$15 million in the second asset, \$45 million in the third asset, and \$15 million in cash. What is the asset allocation of the tangency portfolio made of these three assets?

(5) Assume that you invest \$20 million in two different assets and cash. The three-months returns of the two assets have expected values of 6% and 12%, respectively, and standard deviations of 15% and 25%, respectively. The correlation of the returns of the two assets is 30%. The risk–free interest rate is 2%.

(i) Find the asset allocation for the tangency portfolio.

(ii) Find the asset allocation for a minimum variance portfolio with 8% expected return, and the standard deviation of the return of this portfolio.

(iii) Find the asset allocation for a minimum variance portfolio with 15% expected return, and the standard deviation of the return of this portfolio.

(iv) Find the asset allocation for a maximum return portfolio with 20% standard deviation of return, and the expected return of this portfolio.

(v) Find the asset allocation for a maximum return portfolio with 30% standard deviation of return, and the expected return of this portfolio.

(vi) Assume that the risk–free interest rate changes to 2.5%.
How do you adjust the asset allocation of the minimum variance portfolio with 8% expected return in order to maintain a minimum variance portfolio with 8% expected return?
How do you adjust the asset allocation of the maximum return portfolio with 20% standard deviation of return in order to maintain a maximum return portfolio with 20% standard deviation of return?

(6) Consider four assets with the following expected returns over a fixed time period:

$$\mu_1 = 5.1\%;\ \mu_2 = 4.5\%;\ \mu_3 = 6.8\%;\ \mu_4 = 4.2\%,$$

and with the following covariance matrix of their returns over the same time period:

$$\begin{pmatrix} 0.09 & -0.01 & -0.03 & -0.02 \\ -0.01 & 0.0625 & 0.02 & -0.01 \\ -0.03 & 0.02 & 0.1225 & -0.015 \\ -0.02 & -0.01 & -0.015 & 0.0576 \end{pmatrix}.$$

Assume that the risk–free interest rate is 1.5%.

(i) Find the asset allocation for the tangency portfolio. Find the expected value and the standard deviation of the return of the tangency portfolio. What is the Sharpe ratio of the tangency portfolio?

(ii) Find the asset allocation for a minimum variance portfolio with 5% expected return, and the standard deviation of the return of this portfolio. What is the Sharpe ratio of this portfolio?

(iii) Find the asset allocation for a maximum return portfolio with 29% standard deviation of return, and the expected return of this portfolio. What is the Sharpe ratio of this portfolio?

(iv) Find the asset allocation for the minimum variance portfolio fully invested in the assets (i.e., with no cash position). What is the Sharpe ratio of this portfolio?