CHAPTER

11

GRAPHS OF TRIGONOMETRIC FUNCTIONS

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Music is an integral part of the lives of most people. Although the kind of music they prefer will differ, all music is the effect of sound waves on the ear. Sound waves carry the energy of a vibrating string or column of air to our ears. No matter what vibrating object is causing the sound wave, the frequency of the wave (that is, the number of waves per second) creates a sensation that we call the pitch of the sound. A sound wave with a high frequency produces a high pitch while a sound wave with a lower frequency produces a lower pitch. When the frequencies of two sounds are in the ratio of 2 : 1, the sounds differ by an octave and produce a pleasing combination. In general, music is the result of the mixture of sounds that are mathematically related by whole-number ratios of their frequencies.

Sound is just one of many physical entities that are transmitted by waves. Light, radio, television, X-rays, and microwaves are others. The trigonometric functions that we will study in this chapter provide the mathematical basis for the study of waves.

11-1 GRAPH OF THE SINE FUNCTION

The sine function is a set of ordered pairs of real numbers. Each ordered pair can be represented as a point of the coordinate plane. The domain of the sine function is the set of real numbers, that is, every real number is a first element of one pair of the function.

To sketch the graph of the sine function, we will plot a portion of the graph using the subset of the real numbers in the interval $0 \leq x \leq 2\pi$. We know that

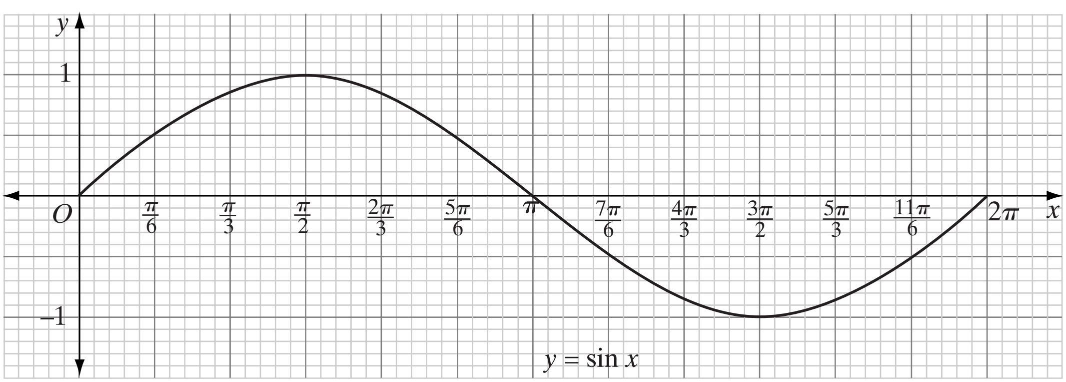
$$\sin \frac{\pi}{6} = \frac{1}{2} = 0.5$$

and that $\frac{\pi}{6}$ is the measure of the reference angle for angles with measures of $\frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$. We also know that

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} = 0.866025 \ldots$$

and that $\frac{\pi}{3}$ is the measure of the reference angle for angles with measures of $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots$. We can round the rational approximation of $\sin \frac{\pi}{3}$ to two decimal places, 0.87.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
sin x	0	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0

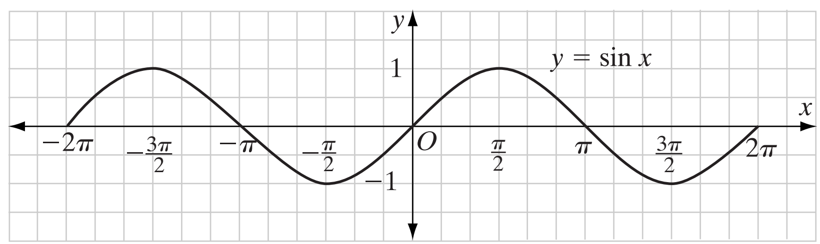


On the graph, we plot the points whose coordinates are given in the table. Through these points, we draw a smooth curve. Note how x and y change.

- As x increases from 0 to $\frac{\pi}{2}$, y increases from 0 to 1.
- As x increases from $\frac{\pi}{2}$ to π, y decreases from 1 to 0.
- As x increases from π to $\frac{3\pi}{2}$, y continues to decrease from 0 to -1.
- As x increases from $\frac{3\pi}{2}$ to 2π, y increases from -1 to 0.

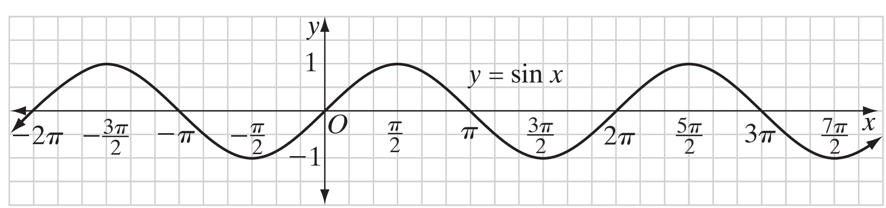
When we plot a larger subset of the domain of the sine function, this pattern is repeated. For example, add to the points given above the point whose x-coordinates are in the interval $-2\pi \leq x \leq 0$.

x	-2π	$-\frac{11\pi}{6}$	$-\frac{5\pi}{3}$	$-\frac{3\pi}{2}$	$-\frac{4\pi}{3}$	$-\frac{7\pi}{6}$	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0
sin x	0	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0



Each time we increase or decrease the value of the x-coordinates by a multiple of 2π, the basic sine curve is repeated. Each portion of the graph in an interval of 2π is one **cycle** of the sine function.

The graph of the function $y = \sin x$ is its own image under the translation $T_{2\pi,0}$. The function $y = \sin x$ is called a **periodic function** with a **period** of 2π because for every x in the domain of the sine function, $\sin x = \sin (x + 2\pi)$.



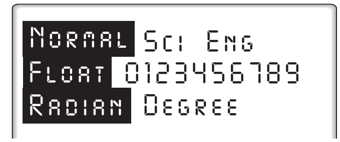
► **The period of the sine function $y = \sin x$ is 2π.**

Each cycle of the sine curve can be separated into four quarters. In the first quarter, the sine curve increases from 0 to the maximum value of the function. In the second quarter, it decreases from the maximum value to 0. In the third quarter, it decreases from 0 to the minimum value, and in the fourth quarter, it increases from the minimum value to 0.

A graphing calculator will display the graph of the sine function.

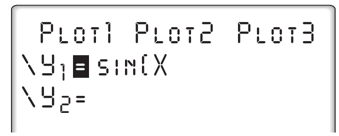
STEP 1. Put the calculator in radian mode.

ENTER: MODE ▼ ▼ ENTER

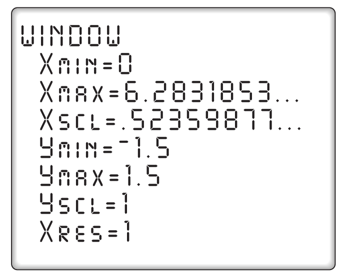


STEP 2. Enter the equation for the sine function.

ENTER: Y= SIN X,T,Θ,*n* ENTER



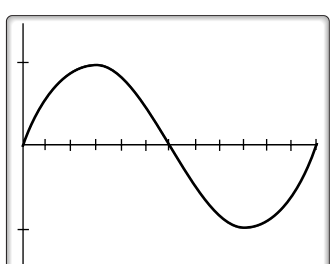
STEP 3. To display one cycle of the curve, let the window include values from 0 to 2π for x and values slightly smaller than -1 and larger than 1 for y. Use the following viewing window: *Xmin* $= 0$, *Xmax* $= 2\pi$, *Xscl* $= \frac{\pi}{6}$, *Ymin* $= -1.5$, *Ymax* $= 1.5$. (Note: *Xscl* changes the scale of the x-axis.)

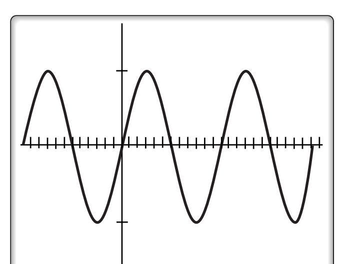


ENTER: WINDOW 0 ENTER 2 2nd π ENTER 2nd π ÷ 6 ENTER -1.5 ENTER 1.5 ENTER

STEP 4. Finally, graph the sin curve by pressing GRAPH. To display more than one cycle of the curve, change *Xmin* or *Xmax* of the window.

ENTER: WINDOW -2 2nd π ENTER 4 2nd π ENTER GRAPH

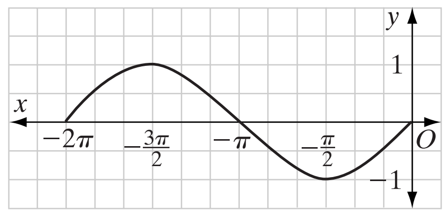




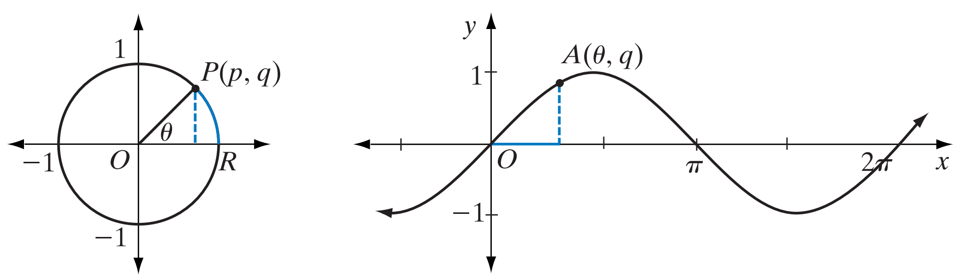
EXAMPLE 1

In the interval $-2\pi \leq x \leq 0$, for what values of x does $y = \sin x$ increase and for what values of x does $y = \sin x$ decrease?

Solution The graph shows that $y = \sin x$ increases in the interval $-2\pi \leq x \leq -\frac{3\pi}{2}$ and in the interval $-\frac{\pi}{2} \leq x \leq 0$ and decreases in the interval $-\frac{3\pi}{2} \leq x \leq -\frac{\pi}{2}$.

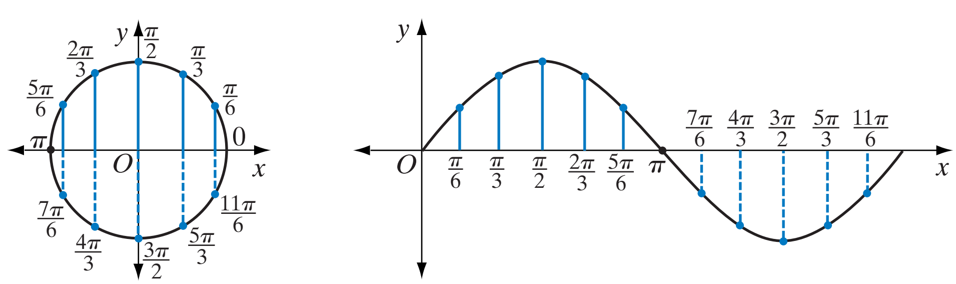


The Graph of the Sine Function and the Unit Circle



Recall from Chapter 9 that if $\angle ROP$ is an angle in standard position with measure θ and $P(p, q)$ is a point on the unit circle, then $(p, q) = (\cos\theta, \sin\theta)$ and $A(\theta, q)$ is a point on the graph of $y = \sin x$. Note that the x-coordinate of A on the graph of $y = \sin x$ is θ, the length of $\overset{\frown}{RP}$.

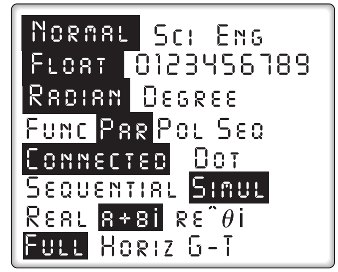
Compare the graph of the unit circle and the graph of $y = \sin x$ in the figures below for different values of θ.



Hands-On Activity: Unwrapping the Unit Circle

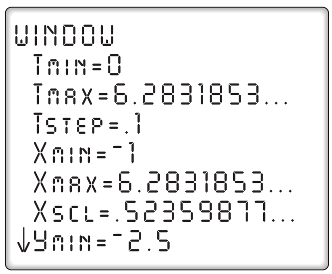
We can use the graphing calculator to explore the unit circle and its relationship to the sine and cosine functions.

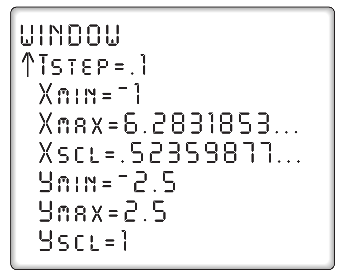
STEP 1. Press MODE . Select RADIAN mode, PAR graphing mode, and SIMUL graphing mode.



STEP 2. Press WINDOW to enter the window screen. Use the following viewing window:

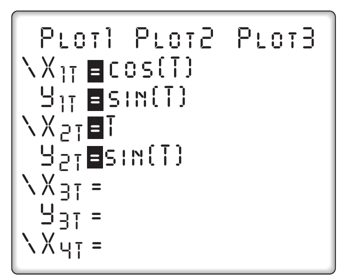
$Tmin = 0$, $Tmax = 2\pi$, $Tstep = 0.1$, $Xmin = -1$, $Xmax = 2\pi$, $Xscl = \frac{\pi}{6}$, $Ymin = -2.5$, $Ymax = 2.5$





Use the ▲ and ▼ arrow keys to display *Ymin* and *Ymax*.

STEP 3. Recall from Chapter 9 that a point P on the unit circle has coordinates $(\cos\theta, \sin\theta)$ where θ is the measure of the standard angle with terminal side through P. We can define a function on the graphing calculator that consists of the set of ordered pairs $(\cos\theta, \sin\theta)$. Its graph will be the unit circle.

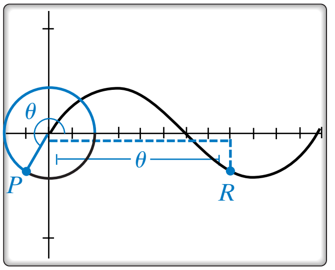


ENTER: Y= COS X,T,Θ,*n*) ▼ SIN X,T,Θ,*n*)

This key sequence defines the function consisting of the set of ordered pairs $(\cos T, \sin T)$. The variable T represents θ on the graphing calculator.

STEP 4. Similarly, we can define a function consisting of the set of ordered pairs $(\theta, \sin\theta)$.

ENTER: ▼ X,T,Θ,*n* ▼ SIN X,T,Θ,*n*)



STEP 5. Press GRAPH to watch the unit circle "unwrap" into the sine function. As the two functions are plotted, press ENTER to pause and resume the animation. You will see the unit circle swept out by point P (represented by a dot). Simultaneously, the graph of $y = \sin \theta$ will be plotted to the right of the y-axis by a point R. Notice that as point P is rotated about the unit circle, the x-coordinate of R is equal to θ, the length of the arc from the positive ray of the x-axis to P. The y-coordinate of R is equal to the y-coordinate of P.

Exercises

Writing About Mathematics

1. Is the graph of $y = \sin x$ symmetric with respect to a reflection in the origin? Justify your answer.

2. Is the graph of $y = \sin x$ symmetric with respect to the translation $T_{-2\pi,0}$? Justify your answer.

Developing Skills

3. Sketch the graph of $y = \sin x$ in the interval $0 \leq x \leq 4\pi$.

a. In the interval $0 \leq x \leq 4\pi$, for what values of x is the graph of $y = \sin x$ increasing?

b. In the interval $0 \leq x \leq 4\pi$, for what values of x is the graph of $y = \sin x$ decreasing?

c. How many cycles of the graph of $y = \sin x$ are in the interval $0 \leq x \leq 4\pi$?

4. What is the maximum value of y on the graph of $y = \sin x$?

5. What is the minimum value of y on the graph of $y = \sin x$?

6. What is the period of the sine function?

7. Is the sine function one-to-one? Justify your answer.

8. a. Point P is a point on the unit circle. The y-coordinate of P is $\sin \frac{\pi}{3}$. What is the x-coordinate of P?

b. Point A is a point on the graph $y = \sin x$. The y-coordinate of A is $\sin \frac{\pi}{3}$. What is the x-coordinate of A?

Applying Skills

9. A function f is **odd** if and only if $f(x) = -f(-x)$ for all x in the domain of the function. Note that a function is odd if it is symmetric with respect to the origin. In other words, the function is its own image under a reflection about the origin.

a. Draw a unit circle and any first-quadrant angle ROP in standard position, with point P on the unit circle. Let $m\angle ROP = \theta$.

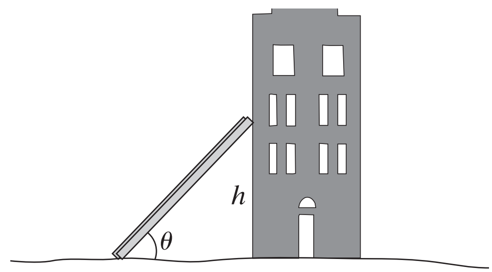
b. On the same set of axes, draw an angle in standard position with measure $-\theta$. What is the relationship between θ and $-\theta$? Between $\sin\theta$ and $\sin(-\theta)$?

c. Repeat steps **a** and **b** for second-, third-, and fourth-quadrant angles. Does $\sin\theta = -\sin(-\theta)$ for second-, third-, and fourth-quadrant angles? Justify your answer.

d. Does $\sin\theta = -\sin(-\theta)$ for quadrantal angles? Explain.

e. Do parts **a–d** show that $y = \sin x$ is an odd function? Justify your answer.

10. City firefighters are told that they can use their 25-foot long ladder provided the measure of the angle that the ladder makes with the ground is at least 15° and no more than 75°.



a. If θ represents the measure of the angle that the ladder makes with the ground *in radians*, what is a reasonable set of values for θ? Explain.

b. Express as a function of θ, the height h of the point at which the ladder will rest against a building.

c. Graph the function from part **b** using the set of values for θ from part **a** as the domain of the function.

d. What is the highest point that the ladder is allowed to reach?

11. In later courses, you will learn that the sine function can be written as the sum of an infinite sequence. In particular, for x in radians, the sine function can be approximated as the finite series:

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

a. Graph $Y_1 = \sin x$ and $Y_2 = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ on the graphing calculator. For what values of x does Y_2 seem to be a good approximation for Y_1?

b. The next term of the sine approximation is $-\frac{x^7}{7!}$. Repeat part **a** using Y_1 and $Y_3 = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$. For what values of x does Y_3 seem to be a good approximation for Y_1?

c. Use Y_2 and Y_3 to find approximations to the sine function values below. Which function gives a better approximation? Is this what you expected? Explain.

(1) $\sin\frac{\pi}{6}$ (2) $\sin\frac{\pi}{4}$ (3) $\sin\pi$

11-2 GRAPH OF THE COSINE FUNCTION

The cosine function, like the sine function, is a set of ordered pairs of real numbers. Each ordered pair can be represented as a point of the coordinate plane. The domain of the cosine function is the set of real numbers, that is, every real number is a first element of one pair of the function.

To sketch the graph of the cosine function, we plot a portion of the graph using a subset of the real numbers in the interval $0 \leq x \leq 2\pi$. We know that

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} = 0.866025\ldots$$

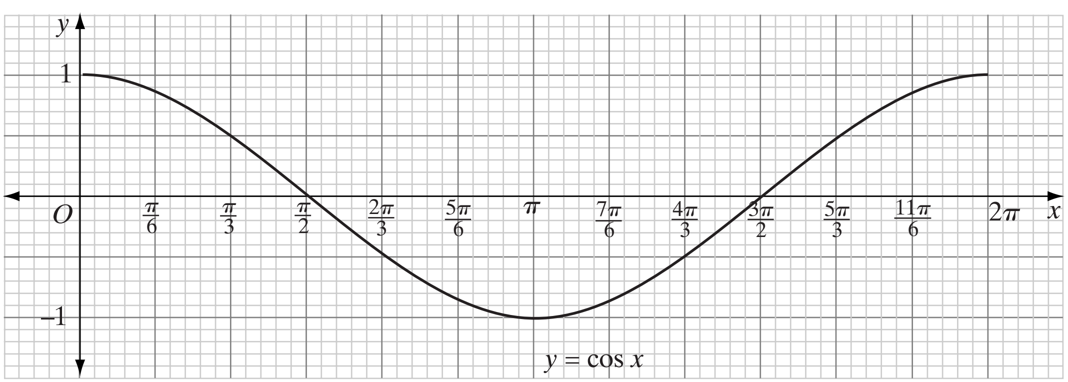
and that $\frac{\pi}{6}$ is the measure of the reference angle for angles with measures of $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$, ... We can round the rational approximation of $\cos \frac{\pi}{6}$ to two decimal places, 0.87.

We also know that

$$\cos \frac{\pi}{3} = \frac{1}{2} = 0.5$$

and that $\frac{\pi}{3}$ is the measure of the reference angle for angle with measures of $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$, ...

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos x$	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0	0.5	0.87	1



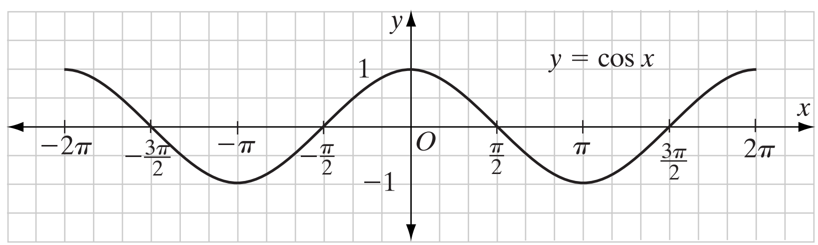
On the graph, we plot the points whose coordinates are given in the table. Through these points, we draw a smooth curve. Note how x and y change.

- As x increases from 0 to $\frac{\pi}{2}$, y decreases from 1 to 0.
- As x increases from $\frac{\pi}{2}$ to π, y decreases from 0 to -1.

- As x increases from π to $\frac{3\pi}{2}$, y increases from -1 to 0.
- As x increases from $\frac{3\pi}{2}$ to 2π, y increases from 0 to 1.

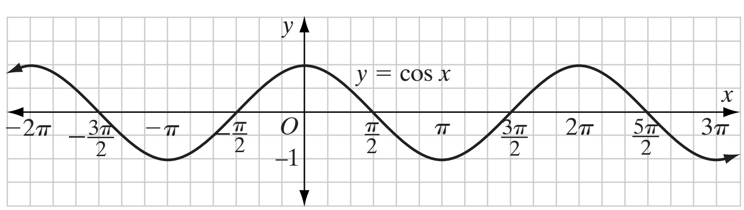
When we plot a larger subset of the domain of the cosine function, this pattern is repeated. For example, add to the points given above the point whose x-coordinates are in the interval $-2\pi \leq x \leq 0$.

x	-2π	$-\frac{11\pi}{6}$	$-\frac{5\pi}{3}$	$-\frac{3\pi}{2}$	$-\frac{4\pi}{3}$	$-\frac{7\pi}{6}$	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0
$\cos x$	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1



Each time we change the value of the x-coordinates by a multiple of 2π, the basic cosine curve is repeated. Each portion of the graph in an interval of 2π is one cycle of the cosine function.

The graph of the function $y = \cos x$ is its own image under the translation $T_{2\pi,0}$. The function $y = \cos x$ is a periodic function with a period of 2π because for every x in the domain of the cosine function, $\cos x = \cos(x + 2\pi)$.



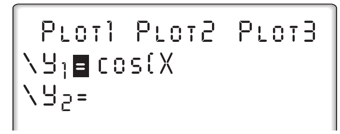
► **The period of the cosine function $y = \cos x$ is 2π.**

Each cycle of the cosine curve can be separated into four quarters. In the first quarter, the cosine curve decreases from the maximum value of the function to 0. In the second quarter, it decreases from 0 to the minimum value. In the third quarter, it increases from the minimum value to 0, and in the fourth quarter, it increases from 0 to the maximum value.

A graphing calculator will display the graph of the cosine function.

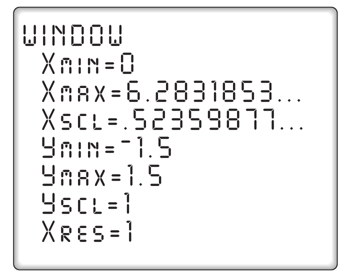
STEP 1. Put the calculator in radian mode.

STEP 2. Enter the equation for the cosine function.



ENTER: **Y=** **COS** **X,T,Θ,*n*** **ENTER**

STEP 3. To display one cycle of the curve, let the window include values from 0 to 2π for x and values slightly smaller than -1 and larger than 1 for y. Use the following viewing window: $Xmin = 0$, $Xmax = 2\pi$, $Xscl = \frac{\pi}{6}$, $Ymin = -1.5$, $Ymax = 1.5$.



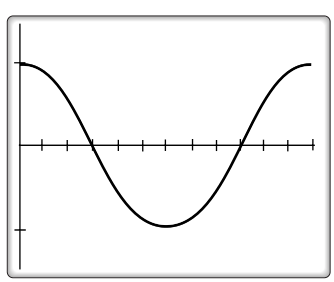
ENTER: **WINDOW** 0 **ENTER** 2 **2nd** **π** **ENTER**

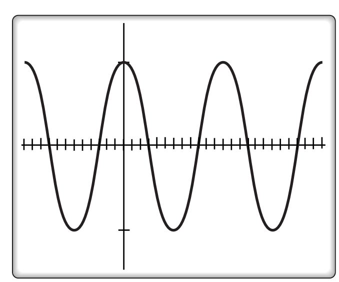
2nd **π** **÷** 6 **ENTER** -1.5 **ENTER** 1.5 **ENTER**

STEP 4. Finally, graph the sin curve by pressing **GRAPH**. To display more than one cycle of the curve, change *Xmin* or *Xmax* of the window.

ENTER: **WINDOW** -2 **2nd** **π** **ENTER** 4 **2nd** **π**

ENTER **GRAPH**

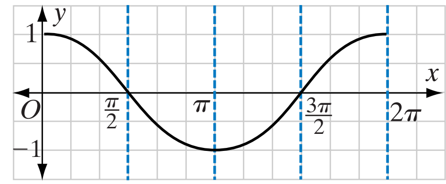




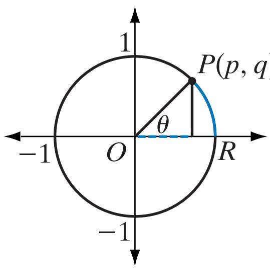
EXAMPLE 1

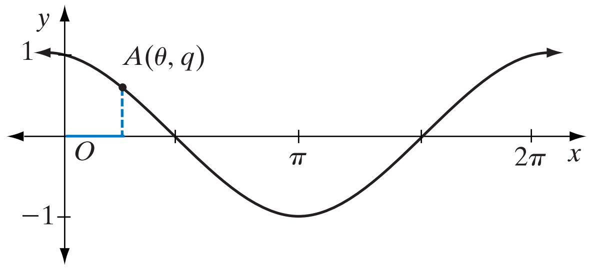
For what values of x in the interval $0 \leq x \leq 2\pi$ does $y = \cos x$ have a maximum value and for what values of x does it have a minimum value?

Solution The graph shows that $y = \cos x$ has a maximum value, 1, at $x = 0$ and at $x = 2\pi$ and has a minimum value, -1, at $x = \pi$.



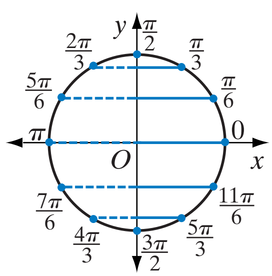
The Graph of the Cosine Function and the Unit Circle

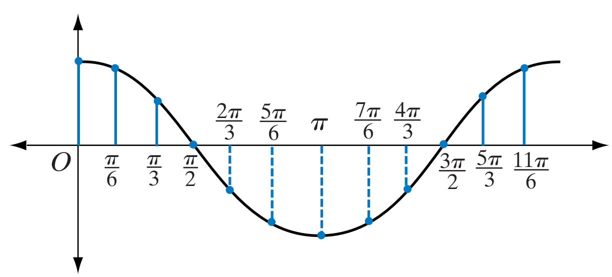




If $\angle ROP$ is an angle in standard position with measure θ and $P(p, q)$ is a point on the unit circle, then $(p, q) = (\cos\theta, \sin\theta)$ and $A(\theta, p)$ is a point on the graph of $y = \cos x$. Note that the x-coordinate of A on the graph of $y = \cos x$ is θ, the length of $\overset{\frown}{RP}$.

Compare the graph of the unit circle and the graph of $y = \cos x$ in the figures below for different values of θ.





Exercises

Writing About Mathematics

1. Is the graph of $y = \cos x$ its own image under a reflection in the y-axis? Justify your answer.

2. Is the graph of $y = \cos x$ its own image under the translation $T_{-2\pi,0}$? Justify your answer.

Developing Skills

3. Sketch the graph of $y = \cos x$ in the interval $0 \leq x \leq 4\pi$.
   a. In the interval $0 \leq x \leq 4\pi$, for what values of x is the graph of $y = \cos x$ increasing?
   b. In the interval $0 \leq x \leq 4\pi$, for what values of x is the graph of $y = \cos x$ decreasing?
   c. How many cycles of the graph of $y = \cos x$ are in the interval $0 \leq x \leq 4\pi$?
4. What is the maximum value of y on the graph of $y = \cos x$?
5. What is the minimum value of y on the graph of $y = \cos x$?
6. What is the period of the cosine function?
7. Is the cosine function one-to-one? Justify your answer.

Applying Skills

8. A function f is **even** if and only if $f(x) = f(-x)$ for all x in the domain of the function. Note that a function is even if it is symmetric with respect to the y-axis. In other words, a function is even if it is its own image under a reflection about the y-axis.
   a. Draw a unit circle and any first-quadrant angle ROP in standard position, with point P on the unit circle. Let $m\angle ROP = \theta$.
   b. On the same set of axes, draw an angle in standard position with measure $-\theta$. What is the relationship between θ and $-\theta$? Between $\cos \theta$ and $\cos(-\theta)$?
   c. Repeat steps **a** and **b** for second-, third-, and fourth-quadrant angles. Does $\cos \theta = \cos(-\theta)$ for second-, third-, and fourth-quadrant angles? Justify your answer.
   d. Does $\cos \theta = \cos(-\theta)$ for quadrantal angles? Explain.
   e. Do parts **a–d** show that $y = \cos x$ is an even function? Justify your answer.
9. A wheelchair user brings along a 6-foot long portable ramp to get into a van. For safety and ease of wheeling, the ramp should make a 5- to 10-degree angle with the ground.
   a. Let θ represent the measure of the angle that the ramp makes with the ground *in radians*. Express, as a function of θ, the distance d between the foot of ramp and the base of the van on which the ramp sits.
   b. What is the domain of the function from part **a**?
   c. Graph the function from part **a** using the domain found in part **b**.
   d. What is the smallest safe distance from the foot of the ramp to the base of the van?
10. In later courses, you will learn that the cosine function can be written as the sum of an infinite sequence. In particular, for x in radians, the cosine function can be approximated by the finite series:

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

   a. Graph $Y_1 = \cos x$ and $Y_2 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ on the graphing calculator. For what values of x does Y_2 seem to be a good approximation for Y_1?
   b. The next term of the cosine approximation is $-\frac{x^6}{6!}$. Repeat part **a** using Y_1 and $Y_3 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$. For what values of x does Y_3 seem to be a good approximation for Y_1?

c. Use Y_2 and Y_3 to find approximations to the cosine function values below. Which function gives a better approximation? Is this what you expected? Explain.

(1) $\cos -\frac{\pi}{6}$ (2) $\cos -\frac{\pi}{4}$ (3) $\cos -\pi$

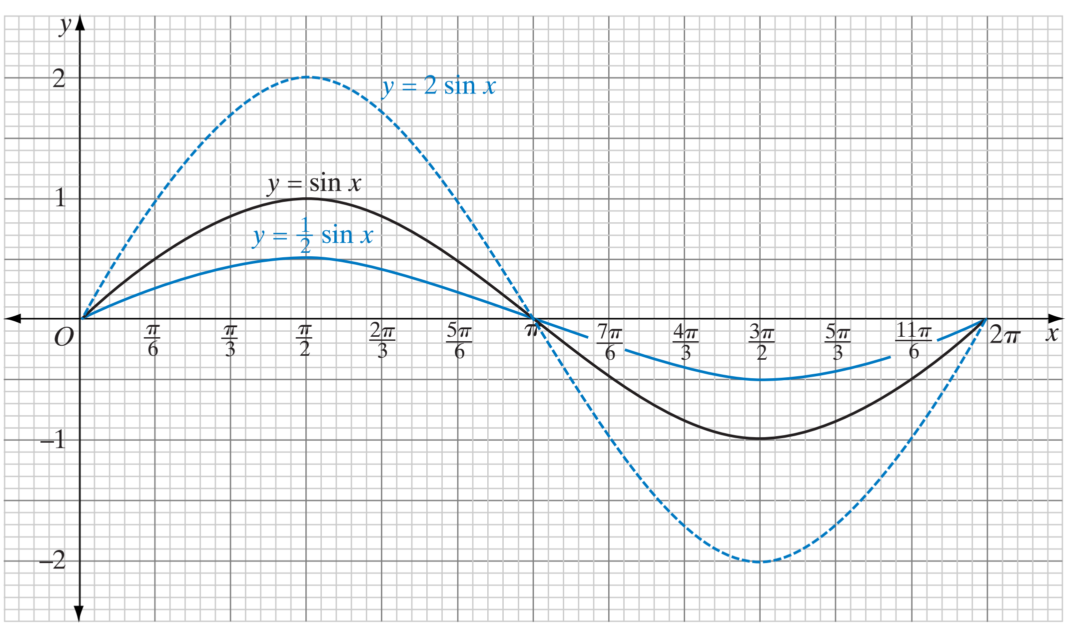
11-3 AMPLITUDE, PERIOD, AND PHASE SHIFT

In Chapter 4 we saw that the functions $af(x)$, $f(ax)$, and $f(x) + a$ are transformations of the function $f(x)$. Each of these transformations can be applied to the sine function and the cosine function.

Amplitude

How do the functions $y = 2 \sin x$ and $y = \frac{1}{2} \sin x$ compare with the function $y = \sin x$? We will make a table of values and sketch the curves. In the following table, approximate values of irrational values of the sine function are used.

x	**0**	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
sin x	0	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0
2 sin x	0	1	1.73	2	1.73	1	0	−1	−1.73	−2	−1.73	−1	0
$\frac{1}{2}$ **sin x**	0	0.25	0.43	0.5	0.43	0.25	0	−0.25	−0.43	−0.5	−0.43	−0.25	0



When we use these values to sketch the curves $y = \sin x$, $y = 2 \sin x$, and $y = \frac{1}{2} \sin x$, we see that $y = 2 \sin x$ is the function $y = \sin x$ stretched in the vertical direction and $y = \frac{1}{2} \sin x$ is the function $y = \sin x$ compressed in the vertical direction as expected.

- For $y = \sin x$, the maximum function value is 1 and the minimum function value is -1.
- For $y = 2 \sin x$, the maximum function value is 2 and the minimum function value is -2.
- For $y = \frac{1}{2} \sin x$, the maximum function value is $\frac{1}{2}$ and the minimum function value is $-\frac{1}{2}$.

In general:

► **For the function $y = a \sin x$, the maximum function value is $|a|$ and the minimum function value is $-|a|$.**

This is also true for the function $y = a \cos x$.

► **For the function $y = a \cos x$, the maximum function value is $|a|$ and the minimum function value is $-|a|$.**

The **amplitude** of a periodic function is the absolute value of one-half the difference between the maximum and minimum y-values.

- For $y = \sin x$, the amplitude is $\left|\frac{1 - (-1)}{2}\right| = 1$.
- For $y = 2 \sin x$, the amplitude is $\left|\frac{2 - (-2)}{2}\right| = 2$.
- For $y = \frac{1}{2} \sin x$, the amplitude is $\left|\frac{\frac{1}{2} - \left(-\frac{1}{2}\right)}{2}\right| = \frac{1}{2}$.

In general:

► **For $y = a \sin x$ and $y = a \cos x$, the amplitude is $\left|\frac{a - (-a)}{2}\right| = |a|$.**

EXAMPLE 1

For the function $y = 3 \cos x$:

a. What are the maximum and minimum values of the function?

b. What is the range of the function?

c. What is the amplitude of the function?

Solution The range of the function $y = \cos x$ is $-1 \leq y \leq 1$.

The function $y = 3 \cos x$ is the function $y = \cos x$ stretched by a factor of 3 in the vertical direction.

When $x = 0$, $\cos 0 = 1$, the maximum value, and $y = 3 \cos 0 = 3(1) = 3$.

When $x = \pi$, $\cos \pi = -1$, the minimum value, and $y = 3 \cos \pi = 3(-1) = -3$.

a. The maximum value of the function is 3 and the minimum value is -3. *Answer*

b. The range of $y = 3 \cos x$ is $-3 \leq y \leq 3$. *Answer*

c. The amplitude of the function is $\left|\frac{3 - (-3)}{3}\right| = 3$. *Answer*

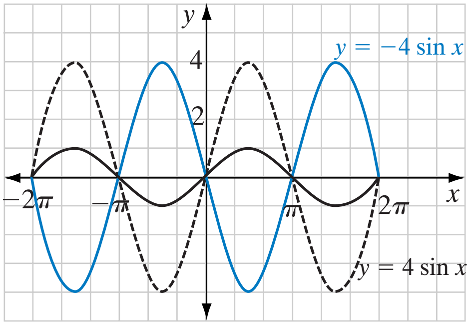
EXAMPLE 2

Describe the relationship between the graph of $y = -4 \sin x$ and the graph of $y = \sin x$ and sketch the graphs.

Solution

- The function $g(x) = 4f(x)$ is the function $f(x)$ stretched by the factor 4 in the vertical direction.
- The function $-g(x)$ is the function $g(x)$ reflected in the x-axis.

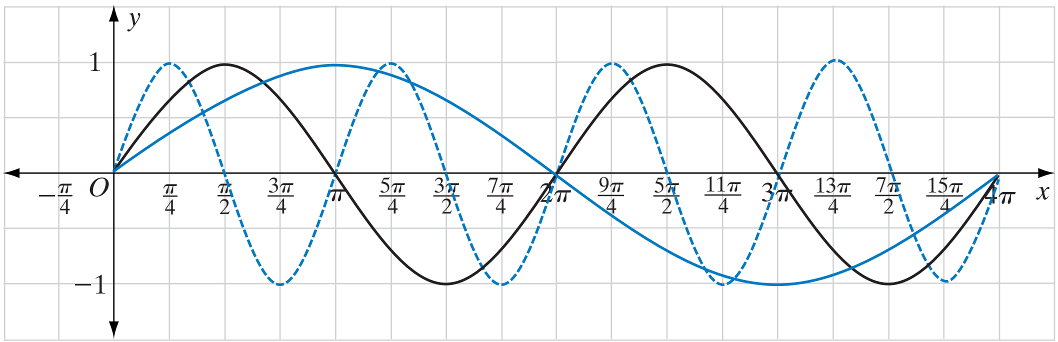
  Apply these rules to the sine function.
- The function $y = -4 \sin x$ is the function $y = \sin x$ stretched by the factor 4 in the vertical direction and reflected in the x-axis.
  - The amplitude of $y = -4 \sin x = \frac{4 - (-4)}{2} = 4$.



Period

The function $g(x) = f(ax)$ is the function $f(x)$ stretched or compressed by a factor of a in the horizontal direction. Compare the graphs of $y = \sin x$, $y = \sin 2x$, and $y = \sin \frac{1}{2}x$. Consider the maximum, zero, and minimum values of y for one cycle of the graph of $y = \sin x$.

	zero	maximum	zero	minimum	zero
y = sin x	$x = 0$	$x = \frac{\pi}{2}$	$x = \pi$	$x = \frac{3\pi}{2}$	$x = 2\pi$
y = sin 2x	$2x = 0$ $x = 0$	$2x = \frac{\pi}{2}$ $x = \frac{\pi}{4}$	$2x = \pi$ $x = \frac{\pi}{2}$	$2x = \frac{3\pi}{2}$ $x = \frac{3\pi}{4}$	$2x = 2\pi$ $x = \pi$
y = sin ½x	$\frac{1}{2}x = 0$ $x = 0$	$\frac{1}{2}x = \frac{\pi}{2}$ $x = \pi$	$\frac{1}{2}x = \pi$ $x = 2\pi$	$\frac{1}{2}x = \frac{3\pi}{2}$ $x = 3\pi$	$\frac{1}{2}x = 2\pi$ $x = 4\pi$



The graph shows the functions $y = \sin x$ (—), $y = \sin 2x$ (---), and $y = \sin \frac{1}{2}x$ (—) in the interval $0 \leq x \leq 4\pi$. The graph of $y = \sin 2x$ is the graph of $y = \sin x$ compressed by the factor $\frac{1}{2}$ in the horizontal direction. The graph of $y = \sin \frac{1}{2}x$ is the graph of $y = \sin x$ stretched by the factor 2 in the horizontal direction.

- For $y = \sin x$, there is one complete cycle in the interval $0 \leq x \leq 2\pi$.
- For $y = \sin 2x$, there is one complete cycle in the interval $0 \leq x \leq \pi$.
- For $y = \sin \frac{1}{2}x$, there is one complete cycle in the interval $0 \leq x \leq 4\pi$.

The difference between the x-coordinates of the endpoints of the interval for one cycle of the graph is the period of the graph.

- The period of $y = \sin x$ is 2π.
- The period of $y = \sin 2x$ is π.
- The period of $y = \sin \frac{1}{2}x$ is 4π.

In general:

► **The period of $y = \sin bx$ and $y = \cos bx$ is $\left|\frac{2\pi}{b}\right|$.**

EXAMPLE 3

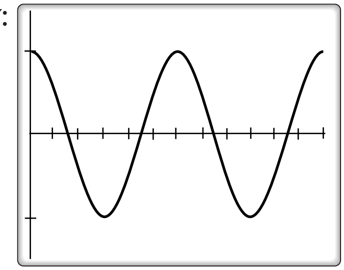
a. Use a calculator to sketch the graph of $y = \cos 2x$ in the interval $0 \leq x \leq 2\pi$.

b. What is the period of $y = \cos 2x$?

Solution **a.** With the calculator in radian mode:

ENTER: `Y=` `COS` 2 `X,T,Θ,n` `ENTER` `WINDOW` 0 `ENTER` 2 `2nd` `π` `ENTER` `2nd` `π` `÷` 6 `ENTER` −1.5 `ENTER` 1.5 `ENTER` `GRAPH`

DISPLAY:

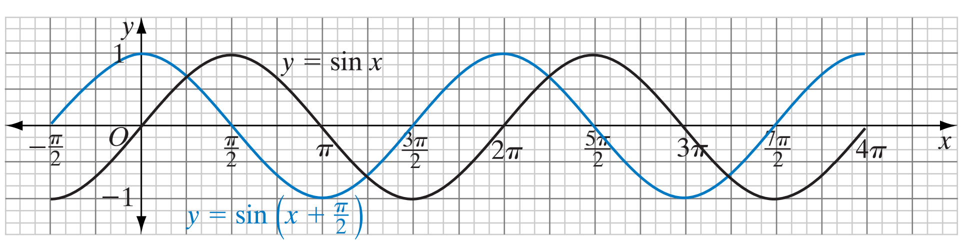


b. The graph shows that there are two cycles of the function in the 2π interval. The period of the graph is $\frac{2\pi}{2} = \pi$. *Answer*

Phase Shift

The graph of $f(x + c)$ is the graph of $f(x)$ moved $|c|$ units to the right when c is negative or $|c|$ units to the left when c is positive. The horizontal translation of a trigonometric function is called a **phase shift**. Compare the graph of $y = \sin x$ and the graph of $y = \sin\left(x + \frac{\pi}{2}\right)$.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
sin x	0	1	0	−1	0	1	0	−1	0
$x + \frac{\pi}{2}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π	$\frac{9\pi}{2}$
$\sin\left(x + \frac{\pi}{2}\right)$	1	0	−1	0	1	0	−1	0	1



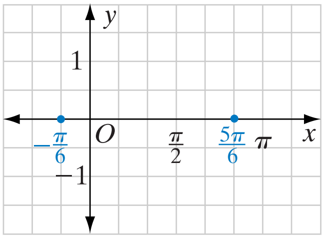
The graph of $y = \sin\left(x + \frac{\pi}{2}\right)$ is the graph of $y = \sin x$ moved $\frac{\pi}{2}$ units to the left.

EXAMPLE 4

Sketch the graph of $y = 3 \sin 2\left(x + \frac{\pi}{6}\right)$.

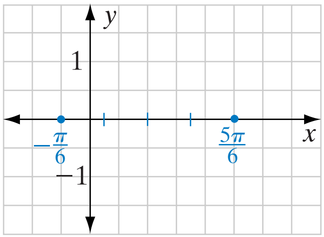
Solution *How to Proceed*

(1) The phase shift is $-\frac{\pi}{6}$. Locate a point on the x-axis that is $\frac{\pi}{6}$ to the left of the origin. This point, $\left(-\frac{\pi}{6}, 0\right)$, is the starting point of one cycle of the sine curve:



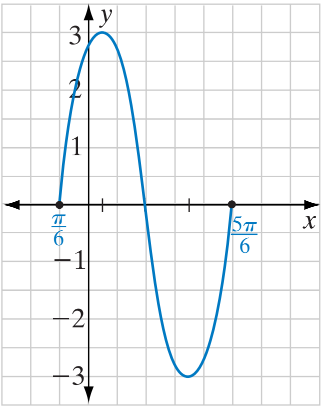
(2) The period is $\frac{2\pi}{2} = \pi$. Locate a point π units to the right of the point in step 1. This point, $\left(\frac{5\pi}{6}, 0\right)$, is the upper endpoint of one cycle of the curve:

(3) Divide the interval for one cycle into four parts of equal length along the x-axis:



(4) The amplitude of the curve is 3:

(5) The y-coordinates of the sine curve increase from 0 to the maximum in the first quarter of the cycle, decrease from the maximum to 0 in the second quarter, decrease from 0 to the minimum in the third quarter, and increase from the minimum to 0 in the fourth quarter. Sketch the curve:



SUMMARY

For the graphs of $y = a \sin b(x + c)$ and $y = a \cos b(x + c)$:

1. The amplitude is $|a|$.
   - The maximum value of the function is $|a|$ and the minimum value is $-|a|$.
2. The period of the function is $\left|\frac{2\pi}{b}\right|$.
   - There are $|b|$ cycles in a 2π interval.

3. The phase shift is $-c$.

- The graph of $y = \sin(x + c)$ or $y = \cos(x + c)$ is the graph of $y = \sin x$ or $y = \cos x$ shifted c units in the horizontal direction.
  - If c is positive, the graph is shifted $|c|$ units to the left.
  - If c is negative, the graph is shifted $|c|$ units to the right.

4. The graph of $y = -a \sin b(x + c)$ or $y = -a \cos b(x + c)$ is a reflection in the x-axis of $y = a \sin b(x + c)$ or $y = a \cos b(x + c)$.

5. The domain of the function is the set of real numbers.

6. The range of the function is the interval $[-a, a]$.

Exercises

Writing About Mathematics

1. Is the graph of $y = \sin 2\left(x + \frac{\pi}{2}\right)$ the same as the graph of $y = \sin 2\left(x - \frac{\pi}{2}\right)$? Justify your answer.

2. Is the graph of $y = \cos 2\left(x + \frac{\pi}{4}\right)$ the same as the graph of $y = \cos\left(2x + \frac{\pi}{4}\right)$? Justify your answer.

Developing Skills

In 3–10, find the amplitude of each function.

3. $y = \sin x$ **4.** $y = 2 \cos x$ **5.** $y = 5 \cos x$ **6.** $y = 3 \sin x$

7. $y = \frac{3}{4} \sin x$ **8.** $y = \frac{1}{2} \cos x$ **9.** $y = 0.6 \cos x$ **10.** $y = \frac{1}{8} \sin x$

In 11–18, find the period of each function.

11. $y = \sin x$ **12.** $y = \cos x$ **13.** $y = \cos 3x$ **14.** $y = \sin 2x$

15. $y = \cos \frac{1}{2}x$ **16.** $y = \sin \frac{1}{3}x$ **17.** $y = \sin 1.5x$ **18.** $y = \cos 0.75x$

In 19–26, find the phase shift of each function.

19. $y = \cos\left(x + \frac{\pi}{2}\right)$ **20.** $y = \cos\left(x - \frac{\pi}{2}\right)$ **21.** $y = \sin\left(x + \frac{\pi}{3}\right)$

22. $y = \sin\left(x - \frac{\pi}{4}\right)$ **23.** $y = \cos\left(x - \frac{\pi}{6}\right)$ **24.** $y = \sin 2\left(x + \frac{3\pi}{4}\right)$

25. $y = \sin 2(x + \pi)$ **26.** $y = \cos(2x - \pi)$

In 27–38, sketch one cycle of each function.

27. $y = \sin x$

28. $y = \cos x$

29. $y = \sin 2x$

30. $y = \sin \frac{1}{2}x$

31. $y = \cos 3x$

32. $y = 3 \cos x$

33. $y = 4 \sin 3x$

34. $y = \frac{1}{2} \cos \frac{1}{3}x$

35. $y = -\sin 2x$

36. $y = -\cos \frac{1}{2}x$

37. $y = \sin \left(x + \frac{\pi}{2}\right)$

38. $y = \frac{1}{2} \cos \left(x - \frac{\pi}{4}\right)$

Applying Skills

39. Show that the graph of $y = \sin x$ is the graph of $y = \cos\left(x - \frac{\pi}{2}\right)$.

40. Electromagnetic radiation emitted by a radio signal can be described by the formula

$$e = 0.014 \cos (2\pi ft)$$

where e is in volts, the frequency f is in kilohertz (kHz), and t is time.

a. Graph two cycles of e for $f = 10$ kHz.

b. What is the voltage when $t = 2$ seconds?

41. As stated in the Chapter Opener, sound can be thought of as vibrating air. Simple sounds can be modeled by a function $\mathrm{h}(t)$ of the form

$$\mathrm{h}(t) = \sin (2\pi ft)$$

where the frequency f is in kilohertz (kHz) and t is time.

a. The frequency of "middle C" is approximately 0.261 kHz. Graph two cycles of $\mathrm{h}(t)$ for middle C.

b. The frequency of C_3, or the C note that is one octave lower than middle C, is approximately 0.130 kHz. On the same set of axes, graph two cycles of $\mathrm{h}(t)$ for C_3.

c. Based on the graphs from parts **a** and **b**, the periods of each function appear to be related in what way?

42. In 2008, the temperature (in Fahrenheit) of a city can be modeled by:

$$\mathrm{f}(t) = 71.3 + 12.1 \sin (0.5t - 1.4)$$

where t represents the number of months that have passed since the first of the year. (For example, $t = 4$ represents the temperature at May 1.)

a. Graph $\mathrm{f}(t)$ in the interval $[0, 11]$.

b. Is it reasonable to extend this model to the year 2009? Explain.

Hands-On Activity

1. Sketch the graph of $y = 2 \sin x$ in the interval $[0, 2\pi]$.

2. Sketch the graph of the image of $y = 2 \sin x$ under the translation $T_{0,3}$.

3. Write an equation of the graph drawn in step 2.

4. What are the maximum and minimum values of y for the image of $y = 2 \sin x$ under the translation $T_{0,3}$?

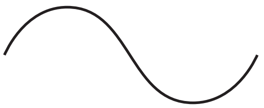
5. The amplitude of the image of $y = 2 \sin x$ under the translation $T_{0,3}$ is 2. How can the maximum and minimum values be used to find the amplitude?

6. Repeat steps 2 through 5 for the translation $T_{0,-4}$.

11-4 WRITING THE EQUATION OF A SINE OR COSINE GRAPH

Each of the graphs that we have studied in this chapter have had an equation of the form $y = a \sin b(x + c)$ or $y = a \cos b(x + c)$. Each of these graphs has a maximum value that is the amplitude, $|a|$, and a minimum, $-|a|$. The positive number b is the number of cycles in an interval of 2π, and $\frac{2\pi}{b}$ is the period or the length of one cycle. For $y = a \sin b(x + c)$, a basic cycle begins at $y = 0$, and for $y = a \cos b(x + c)$, a basic cycle begins at $y = a$. The basic cycle of the graph closest to the origin is contained in the interval $-c \leq x \leq -c + \frac{2\pi}{b}$ and the phase shift is $-c$. Using these values, we can write an equation of the graph.

$y = a \sin (bx + c)$



1. Identify the maximum and minimum values of y for the function. Find a.

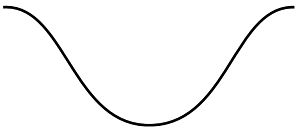
$$a = \frac{\text{maximum} - \text{mininum}}{2}$$

2. Identify one basic cycle of the sine graph that begins at $y = 0$, increases to the maximum value, decreases to 0, continues to decrease to the minimum value, and then increases to 0. Determine the x-coordinates of the endpoints of this cycle. Write in interval notation the domain of one cycle, $x_0 \leq x \leq x_1$ or $[x_0, x_1]$.

3. The period or the length of one cycle is $\frac{2\pi}{b} = x_1 - x_0$. Find b using this formula.

4. The value of c is the opposite of the lower endpoint of the interval of the basic cycle: $c = -x_0$.

$y = a \cos (bx + c)$



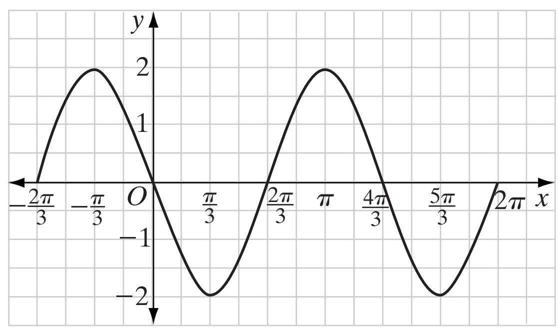
1. Identify the maximum and minimum values of y for the function. Find a.

$$a = \frac{\text{maximum} - \text{mininum}}{2}$$

2. Identify one basic cycle of the graph that begins at the maximum value, decreases to 0, continues to decrease to the minimum value, increases to 0, and then increases to the maximum value. Find the x-coordinates of the endpoints of this cycle. Write in interval notation the domain of one cycle, $x_0 \leq x \leq x_1$ or $[x_0, x_1]$.
3. The period or the length of one cycle is $\frac{2\pi}{b} = x_1 - x_0$. Find b using this formula.
4. The value of c is the opposite of the lower endpoint of the interval of the basic cycle: $c = -x_0$.

EXAMPLE 1

Write an equation of the graph below in the form $y = a \cos bx$.



Solution (1) The maximum y value is 2 and the minimum y value is -2.

$$a = \frac{2 - (-2)}{2} = 2$$

(2) There is one cycle of the curve in the interval $-\frac{\pi}{3} \leq x \leq \pi$.

(3) The period is the difference between the endpoints of the interval for one cycle. The period is $\pi - \left(-\frac{\pi}{3}\right)$ or $\frac{4\pi}{3}$. Therefore:

$$\frac{2\pi}{b} = \frac{4\pi}{3}$$
$$4\pi b = 6\pi$$
$$b = \frac{3}{2}$$

(4) The value of c is $-\left(-\frac{\pi}{3}\right)$ or $\frac{\pi}{3}$.

The phase shift is $-\frac{\pi}{3}$.

The equation of the curve is $y = 2 \cos \frac{3}{2}\left(x + \frac{\pi}{3}\right)$. *Answer*

EXAMPLE 2

Write the equation of the graph from Example 1 as a sine function.

Solution (1) The maximum y value is 2 and the minimum y value is -2.

$$a = \frac{2 - (-2)}{2} = 2$$

(2) There is one cycle of the curve in the interval $-\frac{2\pi}{3} \leq x \leq \frac{2\pi}{3}$.

(3) The period is the difference between the endpoints of the interval for one cycle. The period is $\frac{2\pi}{3} - \left(-\frac{2\pi}{3}\right)$ or $\frac{4\pi}{3}$. Therefore:

$$\frac{2\pi}{b} = \frac{4\pi}{3}$$
$$4\pi b = 6\pi$$
$$b = \frac{3}{2}$$

(4) The value of c is $-\left(-\frac{2\pi}{3}\right)$ or $\frac{2\pi}{3}$.

The phase shift is $-\frac{2\pi}{3}$.

The equation of the curve is $y = 2\sin\frac{3}{2}\left(x + \frac{2\pi}{3}\right)$. *Answer*

Note: The equations of the sine function and the cosine function differ only in the phase shift.

To determine the phase shift, we usually choose the basic cycle with its lower endpoint closest to zero. For this curve, the interval for the sine function, $\left[\frac{2\pi}{3}, 2\pi\right]$, with the lower endpoint $\frac{2\pi}{3}$ could also have been chosen. The equation could also have been written as $y = 2\sin\frac{3}{2}\left(x - \frac{2\pi}{3}\right)$.

Exercises

Writing About Mathematics

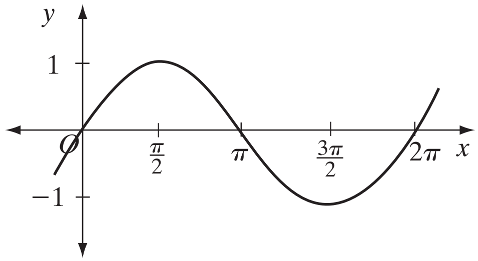
1. Tyler said that one cycle of a cosine curve has a maximum value at $\left(\frac{\pi}{4}, 5\right)$ and a minimum value at $\left(\frac{5\pi}{4}, -5\right)$. The equation of the curve is $y = 5\cos\left(2x - \frac{\pi}{2}\right)$. Do you agree with Tyler? Explain why or why not.

2. Is the graph of $y = \sin 2(x + \pi)$ the same as the graph of $y = \sin 2x$? Explain why or why not.

Developing Skills

In 3–14, for each of the following, write the equation of the graph as: **a.** a sine function **b.** a cosine function. In each case, choose the function with the smallest absolute value of the phase shift.

3.



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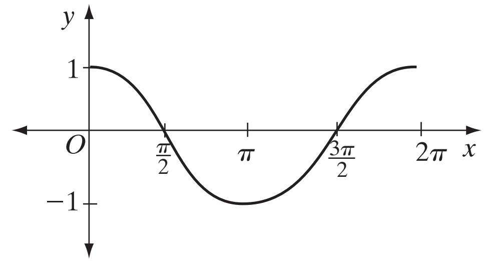

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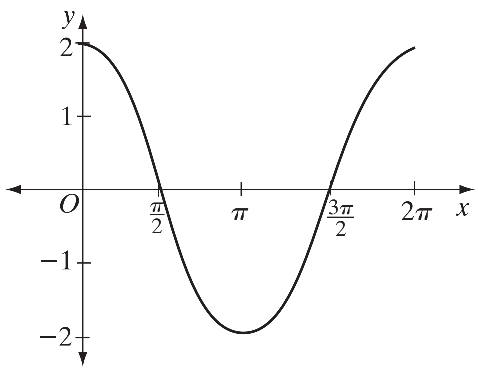

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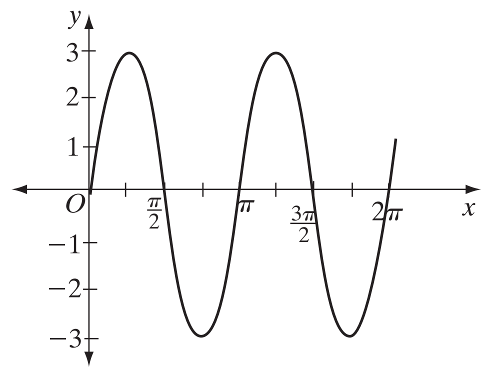

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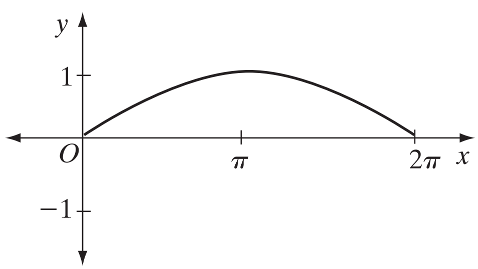

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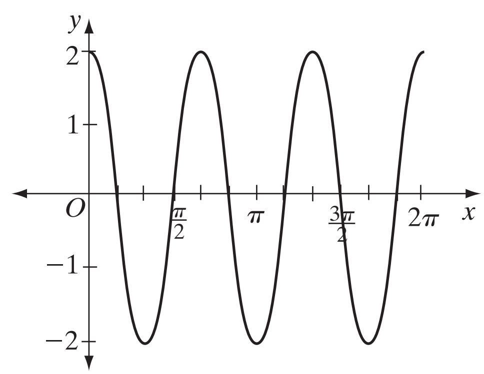

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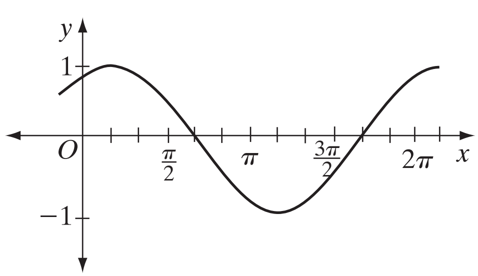

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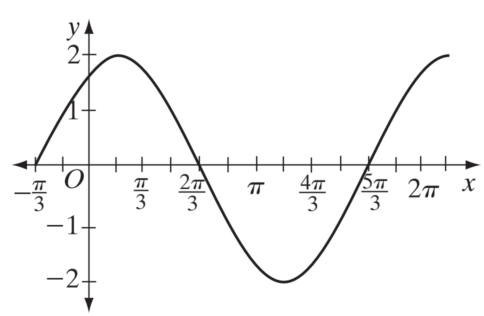

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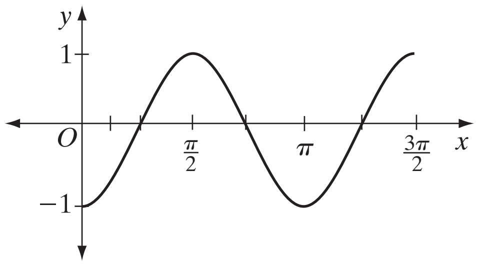

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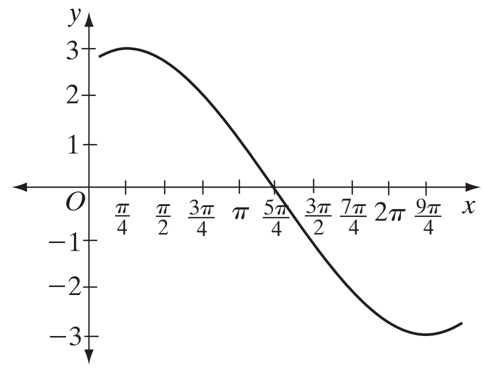

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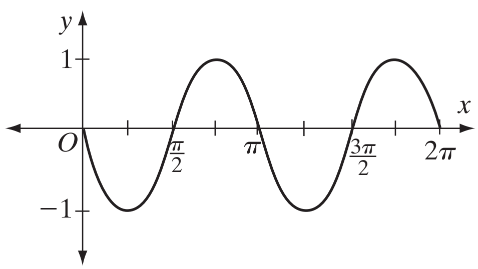

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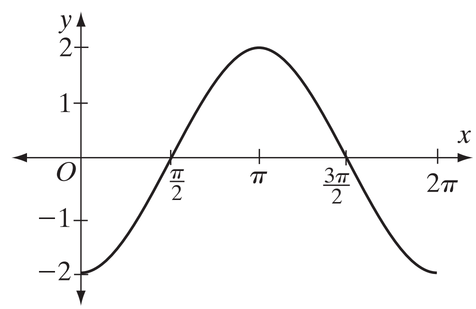
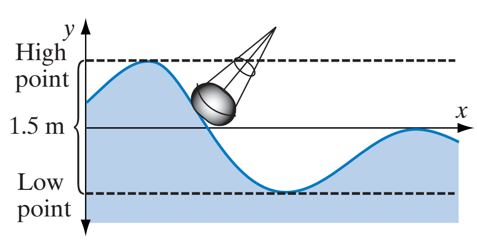


Applying Skills

15. Motion that can be described by a sine or cosine function is called **simple harmonic motion**. During the day, a buoy in the ocean oscillates in simple harmonic motion. The **frequency** of the oscillation is equal to the reciprocal of the period. The distance between its high point and its low point is 1.5 meters. It takes the buoy 5 seconds to move between its low point and its high point, or 10 seconds for one complete oscillation from high point to high point. Let h(t) represent the height of the buoy as a function of time t.



a. What is the amplitude of h(t)?

b. What is the period of h(t)?

c. What is the frequency of h(t)?

d. If h(0) represents the maximum height of the buoy, write an expression for h(t).

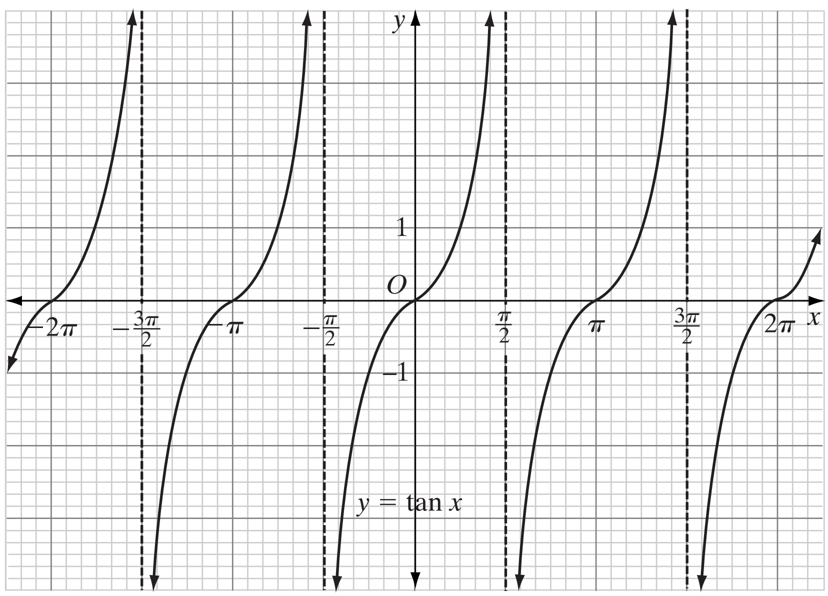
e. Is there a value of t for which h(t) = 1.5 meters? Explain.

11-5 GRAPH OF THE TANGENT FUNCTION

We can use the table shown below to draw the graph of $y = \tan x$. The values of x are given at intervals of $\frac{\pi}{6}$ from -2π to 2π. The values of $\tan x$ are the approximate decimal values displayed by a calculator, rounded to two decimal places. No value is listed for those values of x for which $\tan x$ is undefined.

x	-2π	$-\frac{11\pi}{6}$	$-\frac{5\pi}{3}$	$-\frac{3\pi}{2}$	$-\frac{4\pi}{3}$	$-\frac{7\pi}{6}$	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$
tan x	0	0.58	1.73	—	−1.73	−0.58	0	0.58	1.73	—	−1.73	−0.58

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
tan x	0	0.58	1.73	—	−1.73	−0.58	0	0.58	1.73	—	−1.73	−0.58	0



The graph of the tangent function is a curve that increases through negative values of $\tan x$ to 0 and then continues to increase through positive values. At odd multiples of $\frac{\pi}{2}$, the graph is discontinuous and then repeats the same pattern. Since there is one complete cycle of the curve in the interval from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$, the period of the curve is $\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$. The curve is its own image under the transformation $T_{\pi,0}$.

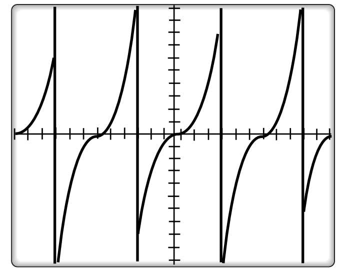
The graph shows a vertical line at $x = \frac{\pi}{2}$ and at every value of x that is an odd multiple of $\frac{\pi}{2}$. These lines are vertical asymptotes. As x approaches $\frac{\pi}{2}$ or any odd multiple of $\frac{\pi}{2}$ from the left, y increases; that is, y approaches infinity. As x

approaches $\frac{\pi}{2}$ or any odd multiple of $\frac{\pi}{2}$ from the right, y decreases; that is, y approaches negative infinity.

Compare the graph shown above with the graph displayed on a graphing calculator.

ENTER: **Y=** **TAN** **X,T,Θ,n** **ENTER**
WINDOW -2 **2nd** **π**
ENTER 2 **2nd** **π** **ENTER**
2nd **π** **÷** 6 **ENTER**
-10 **ENTER** 10 **ENTER** **GRAPH**

DISPLAY:



When in connected mode, the calculator displays a line connecting the points on each side of the values of x for which $\tan x$ is undefined. The lines that appear to be vertical lines at $-\frac{3\pi}{2}$, $-\frac{\pi}{2}$, $\frac{\pi}{2}$, and $\frac{3\pi}{2}$ are not part of the graph. On the table given by the graphing calculator, the y-values associated with these x-values are given as ERROR.

ENTER: **2nd** **TBLSET** -3 **2nd**
π **÷** 2 **ENTER** **2nd**
π **÷** 6 **ENTER** **2nd**
TABLE

DISPLAY:

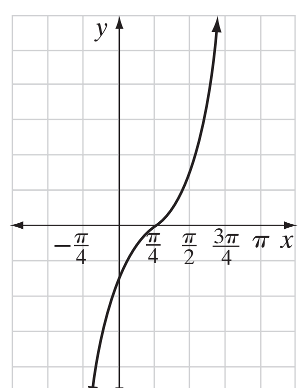
X	Y1	
-4.712	ERROR	
-4.189	-1.732	
-3.665	-.5774	
-3.142	0	
-2.618	.57735	
-2.094	1.7321	
-1.571	ERROR	

X=-4.71238898038

EXAMPLE 1

Sketch one cycle of the graph of $y = \tan\left(x - \frac{\pi}{4}\right)$.

Solution The graph of $y = \tan\left(x - \frac{\pi}{4}\right)$ is the graph of $y = \tan x$ with a phase shift of $\frac{\pi}{4}$. Since there is one cycle of $y = \tan x$ in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$, there will be one cycle of $y = \tan\left(x + \frac{\pi}{4}\right)$ in the interval $-\frac{\pi}{2} + \frac{\pi}{4} < x < \frac{\pi}{2} + \frac{\pi}{4}$, that is, in the interval $-\frac{\pi}{4} < x < \frac{3\pi}{4}$.



SUMMARY

For the graph of $y = \tan x$:

1. There is no amplitude.
   - The graph has no maximum or minimum values.
2. The period of the function is π.
3. The domain of the function is $\left\{x : x \neq \frac{\pi}{2} + n\pi \text{ for } n \text{ an integer}\right\}$.
   - For integral values of n, the graph has vertical asymptotes at $x = \frac{\pi}{2} + n\pi$.
4. The range of the function is {Real numbers} or $(-\infty, \infty)$.

Exercises

Writing About Mathematics

1. List at least three ways in which the graph of the tangent function differs from the graph of the sine function and the cosine function.

2. Does $y = \tan x$ have a maximum and a minimum value? Justify your answer.

Developing Skills

3. Sketch the graph of $y = \tan x$ from $x = -\frac{3\pi}{2}$ to $x = \frac{3\pi}{2}$.

a. What is the period of $y = \tan x$?

b. What is the domain of $y = \tan x$?

c. What is the range of $y = \tan x$?

4. a. Sketch the graph of $y = \tan x$ from $x = 0$ to $x = 2\pi$.

b. On the same set of axes, sketch the graph of $y = \cos x$ from $x = 0$ to $x = 2\pi$.

c. For how many pairs of values does $\tan x = \cos x$ in the interval $[0, 2\pi]$?

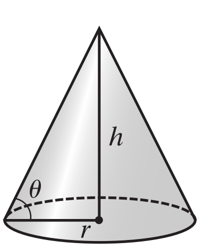
5. a. Sketch the graph of $y = \tan x$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.

b. Sketch the graph of $y = \tan(-x)$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.

c. Sketch the graph of $y = -\tan x$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.

d. How does the graph of $y = \tan(-x)$ compare with the graph of $y = -\tan x$?

Applying Skills



6. The volume of a cone is given by the formula

$$V = \frac{1}{3}Bh$$

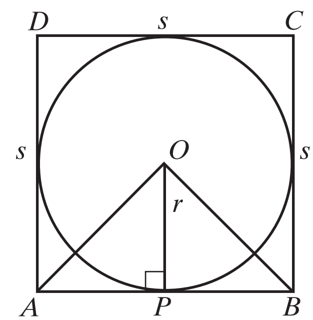
where B is the area of the base and h is the height of the cone.

a. Find a formula for h in terms of r and θ when r is the radius of the base and θ is the measure of the angle that the side of the cone makes with the base.

b. Use part **a** to write a formula for the volume of the cone in terms of r and θ.

7. Recall from your geometry course that a polygon is *circumscribed* about a circle if each side of the polygon is tangent to the circle. Since each side is tangent to the circle, the radius of the circle is perpendicular to each side at the point of tangency. We will use the tangent function to examine the formula for the perimeter of a circumscribed regular polygon.

a. Let square $ABCD$ be circumscribed about circle O. A radius of the circle, $\overline{OP}$, is perpendicular to $\overline{AB}$ at P.



(1) In radians, what is the measure of $\angle AOB$?

(2) Let $m\angle AOP = \theta$. If θ is equal to one-half the measure of $\angle AOB$, find θ.

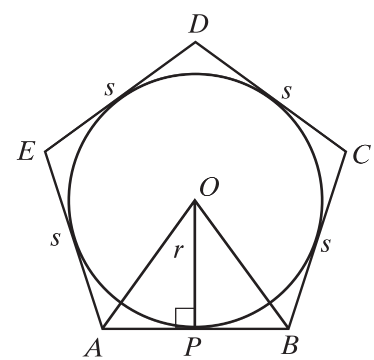
(3) Write an expression for AP in terms of $\tan\theta$ and r, the radius of the circle.

(4) Write an expression for $AB = s$ in terms of $\tan\theta$ and r.

(5) Use part (4) to write an expression for the perimeter in terms of r and the number of sides, n.

b. Let regular pentagon $ABCDE$ be circumscribed about circle O. Repeat part **a** using pentagon $ABCDE$.

c. Do you see a pattern in the formulas for the perimeter of the square and of the pentagon? If so, make a conjecture for the formula for the perimeter of a circumscribed regular polygon in terms of the radius r and the number of sides n.



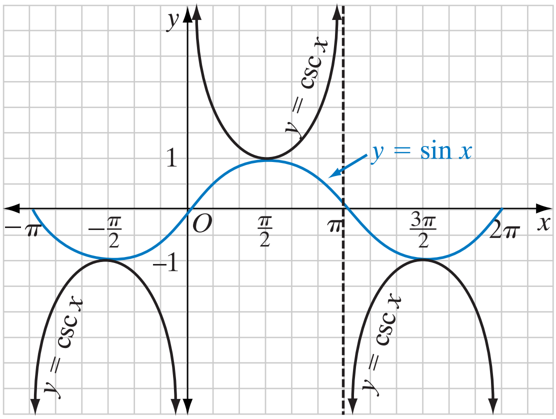
11-6 GRAPHS OF THE RECIPROCAL FUNCTIONS

The Cosecant Function

The cosecant function is defined in terms of the sine function: $\csc x = \frac{1}{\sin x}$. To graph the cosecant function, we can use the reciprocals of the sine function values. The reciprocal of 1 is 1 and the reciprocal of a positive number less than 1 is a number greater than 1. The reciprocal of -1 is -1 and the reciprocal of a negative number greater than -1 is a number less than -1. The reciprocal of 0 is undefined. Reciprocal values of the sine function exist for $-1 \leq \sin x < 0$, and for $0 < \sin x \leq 1$. Therefore:

$$-\infty < \csc x \leq -1 \qquad 1 \leq \csc x < \infty$$

(Recall that the symbol ∞ is called infinity and is used to indicate that a set of numbers has no upper bound. The symbol $-\infty$ indicates that the set of numbers has no lower bound.) For values of x that are multiples of π, $\sin x = 0$ and $\csc x$ is undefined. For integral values of n, the vertical lines on the graph at $x = n\pi$ are asymptotes.

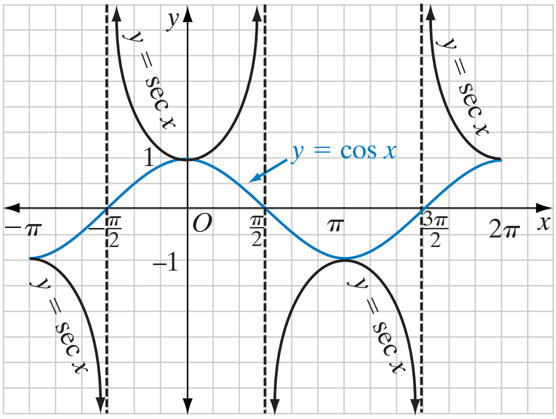


The Secant Function

The secant function is defined in terms of the cosine function: $\sec x = \frac{1}{\cos x}$. To graph the secant function, we can use the reciprocals of the cosine function values. Reciprocal values of the cosine function exist for $-1 \leq \cos x < 0$, and for $0 < \cos x \leq 1$. Therefore:

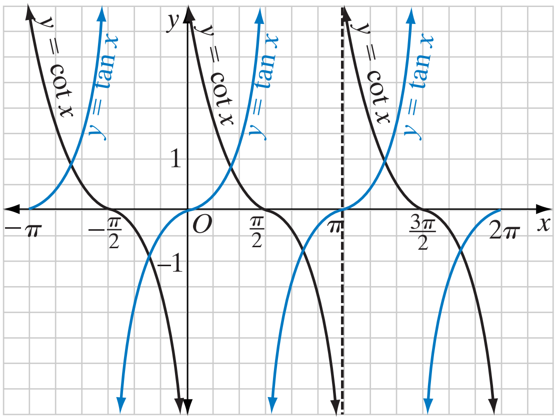
$$-\infty < \sec x \leq -1 \qquad\qquad 1 \leq \sec x < \infty$$

For values of x that are odd multiples of $\frac{\pi}{2}$, $\cos x = 0$ and $\sec x$ is undefined. For integral values of n, the vertical lines on the graph at $x = \frac{\pi}{2} + n\pi$ are asymptotes.



The Cotangent Function

The cotangent function is defined in terms of the tangent function: $\cot x = \frac{1}{\tan x}$. To graph the cotangent function, we can to use the reciprocals of the tangent function values. For values of x that are multiples of π, $\tan x = 0$ and $\cot x$ is undefined. For values of x for which $\tan x$ is undefined, $\cot x = 0$. For integral values of n, the vertical lines on the graph at $x = n\pi$ are asymptotes.



EXAMPLE 1

Use a calculator to sketch the graphs of $y = \sin x$ and $y = \csc x$ for $-2\pi < x < 2\pi$.

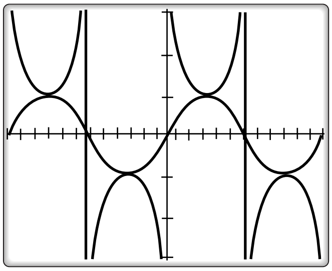
Solution There is no key for the cosecant function on the TI graphing calculator. Therefore, the function must be entered in terms of the sine function.

ENTER: Y= SIN X,T,Θ,*n*)

ENTER Y= 1 ÷ SIN

X,T,Θ,*n*) GRAPH

DISPLAY:



SUMMARY

	$y = \csc x$	$y = \sec x$	$y = \cot x$
Amplitude:	none	none	none
Maximum:	$+\infty$	$+\infty$	$+\infty$
Minimum:	$-\infty$	$-\infty$	$-\infty$
Period:	2π	2π	π
Domain:	$\{x : x \neq n\pi\}$	$\left\{x : x \neq \frac{\pi}{2} + n\pi\right\}$	$\{x : x \neq n\pi\}$
Range:	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, \infty)$

Exercises

Writing About Mathematics

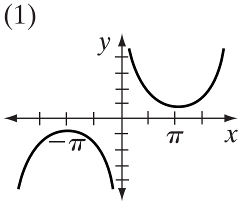
1. If $\tan x$ increases for all values of x for which it is defined, explain why $\cot x$ decreases for all values of x for which it is defined.

2. In the interval $0 \leq x \leq \pi$, $\cos x$ decreases. Describe the change in $\sec x$ in the same interval.

Developing Skills

In 3–10, match each graph with its function.

(1)



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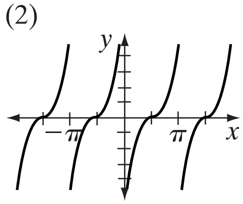

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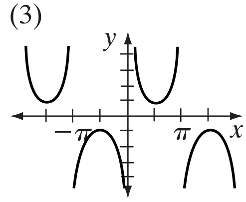

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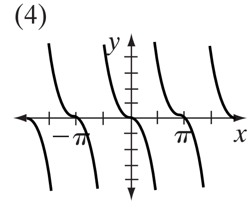

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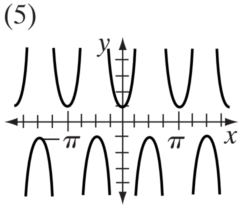

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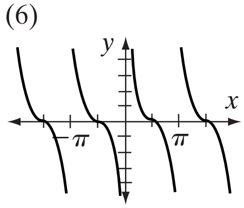

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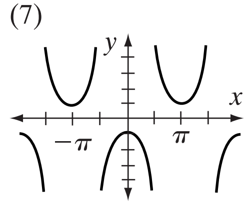

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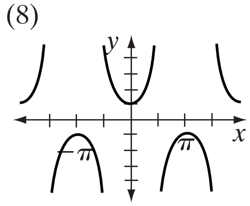


3. $y = \csc x$ **4.** $y = \sec x$ **5.** $y = \cot x$ **6.** $y = -\sec x$

7. $y = \csc \frac{x}{2}$ **8.** $y = \sec 2x$ **9.** $y = -\cot x$ **10.** $y = \cot\left(x + \frac{\pi}{2}\right)$

11. a. Sketch the graphs of $y = \sin x$ and $y = \csc x$ for $-2\pi \leq x \leq 2\pi$.

b. Name four values of x in the interval $-2\pi \leq x \leq 2\pi$ for which $\sin x = \csc x$.

12. **a.** Sketch the graphs of $y = \cos x$ and $y = \sec x$ for $-2\pi \leq x \leq 2\pi$.

b. Name four values of x in the interval $-2\pi \leq x \leq 2\pi$ for which $\cos x = \sec x$.

13. **a.** Sketch the graphs of $y = \tan x$ and $y = \cot x$ for $-\pi \leq x \leq \pi$.

b. Name four values of x in the interval $-\pi \leq x \leq \pi$ for which $\tan x = \cot x$.

14. List two values of x in the interval $-2\pi \leq x \leq 2\pi$ for which $\sec x$ is undefined.

15. List two values of x in the interval $-2\pi \leq x \leq 2\pi$ for which $\csc x$ is undefined.

16. List two values of x in the interval $-2\pi \leq x \leq 2\pi$ for which $\tan x$ is undefined.

17. List two values of x in the interval $-2\pi \leq x \leq 2\pi$ for which $\cot x$ is undefined.

18. The graphs of which two trigonometric functions have an asymptote at $x = 0$?

19. The graphs of which two trigonometric functions have an asymptote at $x = \frac{\pi}{2}$?

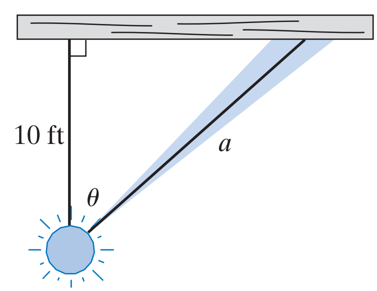
20. Using the graphs of each function, determine whether each function is even, odd, or neither.

a. $y = \tan x$ **b.** $y = \csc x$

c. $y = \sec x$ **d.** $y = \cot x$

Applying Skills

21. A rotating strobe light casts its light on the ceiling of the community center as shown in the figure. The light is located 10 feet from the ceiling.



a. Express a, the distance from the light to its projection on the ceiling, as a function of θ and a reciprocal trigonometric function.

b. Complete the following table, listing each value to the nearest tenth.

θ	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$
a				

c. The given values of θ in part **b** increase in equal increments. For these values of θ, do the distances from the light to its projection on the ceiling increase in equal increments? Explain.

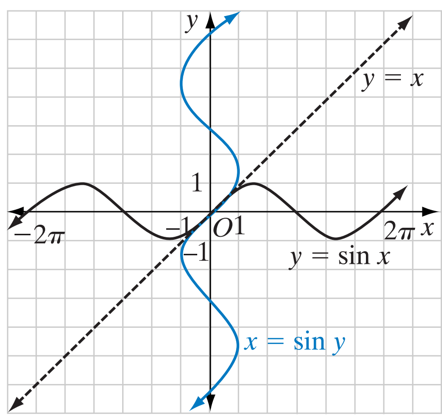
d. The maximum value of θ, in radians, before the light stops shining on the ceiling is $\frac{4\pi}{9}$. To the nearest tenth, how wide is the ceiling?

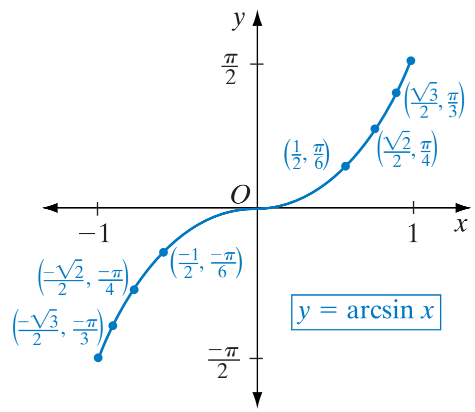
11-7 GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

The trigonometric functions are not one-to-one functions. By restricting the domain of each function, we were able to write inverse functions in Chapter 10.

Inverse of the Sine Function

The graph of $y = \sin x$ is shown below on the left. When the graph of $y = \sin x$ is reflected in the line $y = x$, the image, $x = \sin y$ or $y = \arcsin x$, is a relation that is not a function. The interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ includes all of the values of $\sin x$ from -1 to 1.





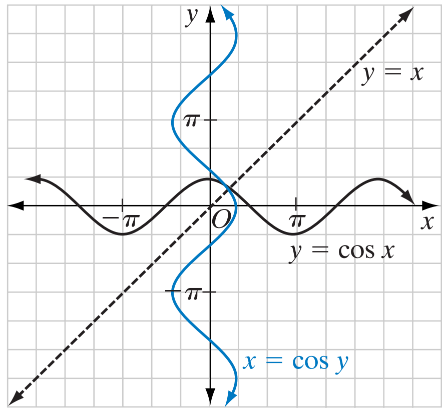
If we restrict the domain of the sine function to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, that subset of the sine function is a one-to-one function and has an inverse function. When we reflect that subset over the line $y = x$, the image is the function $y = \arcsin x$ or $y = \sin^{-1} x$.

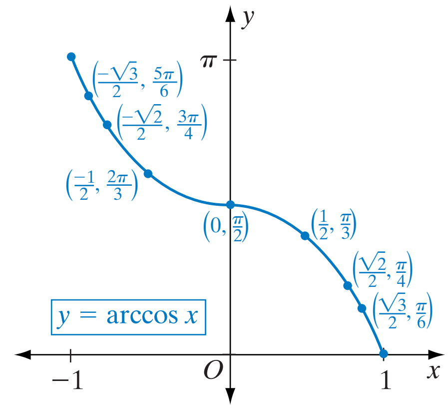
Sine Function with a Restricted Domain
$y = \sin x$
Domain $= \left\{x : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right\}$
Range $= \{y : -1 \leq y \leq 1\}$

Inverse Sine Function
$y = \arcsin x$
Domain $= \{x : -1 \leq x \leq 1\}$
Range $= \left\{y : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right\}$

Inverse of the Cosine Function

The graph of $y = \cos x$ is shown on the top of page 469. When the graph of $y = \cos x$ is reflected in the line $y = x$, the image, $x = \cos y$ or $y = \arccos x$, is a relation that is not a function. The interval $0 \leq x \leq \pi$ includes all of the values of $\cos x$ from -1 to 1.





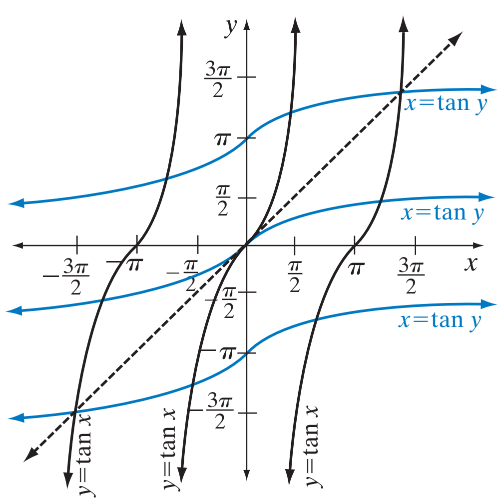
If we restrict the domain of the cosine function to $0 \leq x \leq \pi$, that subset of the cosine function is a one-to-one function and has an inverse function. When we reflect that subset over the line $y = x$, the image is the function $y = \arccos x$ or $y = \cos^{-1} x$.

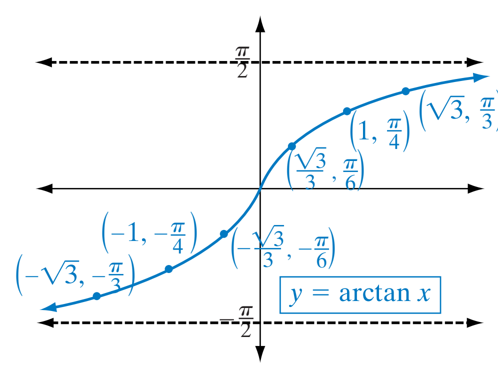
Cosine Function with a Restricted Domain
$y = \cos x$
Domain $= \{x : 0 \leq x \leq \pi\}$
Range $= \{y : -1 \leq y \leq 1\}$

Inverse Cosine Function
$y = \arccos x$
Domain $= \{x : -1 \leq x \leq 1\}$
Range $= \{y : 0 \leq y \leq \pi\}$

Inverse of the Tangent Function

The graph of $y = \tan x$ is shown below on the left. When the graph of $y = \tan y$ is reflected in the line $y = x$, the image, $x = \tan y$ or $y = \arctan x$, is a relation that is not a function. The interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ includes all of the values of tan x from $-\infty$ to ∞.





If we restrict the domain of the tangent function to $-\frac{\pi}{2} < x < \frac{\pi}{2}$, that subset of the tangent function is a one-to-one function and has an inverse function. When we reflect that subset over the line $y = x$, the image is the function $y = \arctan x$ or $y = \tan^{-1} x$.

Tangent Function with a Restricted Domain	Inverse Tangent Function
$y = \tan x$	$y = \arctan x$
Domain $= \left\{x : -\frac{\pi}{2} < x < \frac{\pi}{2}\right\}$	Domain $= \{x : x \text{ is a real number}\}$
Range $= \{y : y \text{ is a real number}\}$	Range $= \left\{y : -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}$

If we know that $\sin x = 0.5738$, a calculator will give the value of x in the restricted domain. We can write $x = \arcsin 0.5770$. On the calculator, the SIN⁻¹ key is used for the arcsin function.

To write the value of x in degrees, change the calculator to degree mode.

ENTER: MODE ▼ ▼ ▶ ENTER CLEAR 2nd SIN⁻¹ 0.5738) ENTER

DISPLAY:
```
SIN⁻¹(0.5738)
          35.01563871
```

In degrees, $\sin 35° \approx 0.5738$. This is the value of x for the restricted domain of the sine function. There is also a second quadrant angle whose reference angle is 35°. That angle is $180° - 35°$ or 145°. These two measures, 35° and 145°, and any measures that differ from one of these by a complete rotation, 360°, are a value of x. Therefore, if $\sin x = 0.5738$,

$$x = 35 + 360n \quad \text{or} \quad x = 145 + 360n$$

for all integral values of n.

EXAMPLE 1

Find, in radians, *all* values of θ such that $\tan \theta = -1.200$ in the interval $0 \leq \theta \leq 2\pi$. Express the answer to four decimal places.

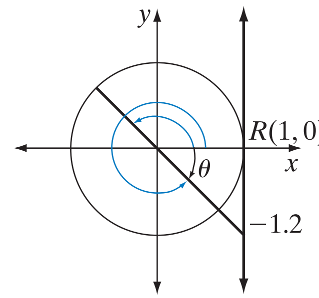
Solution If $\tan \theta = -1.200$, $\theta = \arctan - 1.200$.

Set the calculator to radian mode. Then use the TAN⁻¹ key.

ENTER: 2nd TAN⁻¹ −1.200) ENTER

DISPLAY:
```
TAN⁻¹(⁻1.200)
         ⁻.8760580506
```

To four decimal places, $\theta = -0.8761$. This is the value of θ in the restricted domain of the tangent function. An angle of -0.8761 radians is a fourth-quadrant angle. The fourth-quadrant angle with the same terminal side and a measure between 0 and 2π is $-0.8761 + 2\pi \approx 5.4070$. The tangent function values are negative in the second and fourth quadrants. There is a second-quadrant angle with the same tangent value. The first-quadrant reference angle has a measure of 0.8761 radians. The second quadrant angle with a reference angle of 0.8761 radians has a measure of $\pi - 0.8761 \approx 2.2655$.



Answer $x \approx 2.2655$ and $x \approx 5.4070$

Exercises

Writing About Mathematics

1. Show that if $\arcsin -\frac{1}{2} = x$, then the measure of the reference angle for x is 30°.

2. Is $\arctan 1 = 220°$ a true statement? Justify your answer.

Developing Skills

In 3–14, find each exact value in degrees.

3. $y = \arcsin \frac{1}{2}$
4. $y = \arccos \frac{1}{2}$
5. $y = \arctan 1$
6. $y = \arctan \sqrt{3}$
7. $y = \arcsin (-1)$
8. $y = \arccos 0$
9. $y = \arcsin \left(-\frac{\sqrt{3}}{2}\right)$
10. $y = \arccos \left(-\frac{\sqrt{2}}{2}\right)$
11. $y = \arctan (-1)$
12. $y = \arcsin \left(-\frac{\sqrt{2}}{2}\right)$
13. $y = \arctan 0$
14. $y = \arccos (-1)$

In 15–26, find each exact value in radians, expressing each answer in terms of π.

15. $y = \arcsin 1$
16. $y = \arccos 1$
17. $y = \arctan 1$
18. $y = \arcsin \left(\frac{\sqrt{3}}{2}\right)$
19. $y = \arcsin \left(-\frac{\sqrt{3}}{2}\right)$
20. $y = \arccos \frac{1}{2}$
21. $y = \arccos \left(-\frac{1}{2}\right)$
22. $y = \arctan \sqrt{3}$
23. $y = \arctan \left(-\sqrt{3}\right)$
24. $y = \arctan 0$
25. $y = \arccos 0$
26. $y = \arcsin 0$

In 27–32, for each of the given inverse trigonometric function values, find the exact function value.

27. sin (arccos 1) **28.** cos (arcsin 1) **29.** tan (arctan 1)

30. $\sin\left(\arccos -\frac{\sqrt{3}}{2}\right)$ **31.** sin (arctan −1) **32.** $\cos\left(\arccos -\frac{1}{2}\right)$

33. **a.** On the same set of axes, sketch the graph of $y = \arcsin x$ and of its inverse function.

b. What are the domain and range of each of the functions graphed in part **a**?

34. **a.** On the same set of axes, sketch the graph of $y = \arccos x$ and of its inverse function.

b. What are the domain and range of each of the functions graphed in part **a**?

35. **a.** On the same set of axes, sketch the graph of $y = \arctan x$ and of its inverse function.

b. What are the domain and range of each of the functions graphed in part **a**?

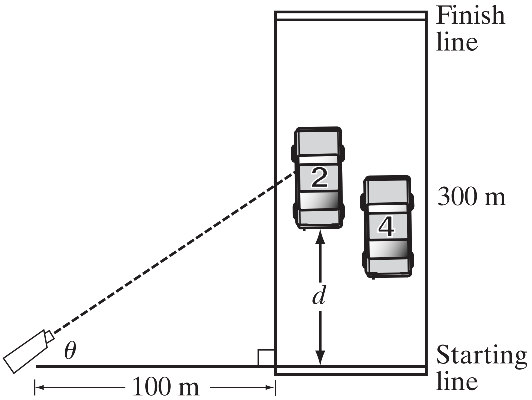
Applying Skills

36. A television camera 100 meters from the starting line is filming a car race, as shown in the figure. The camera will follow car number 2.

a. Express θ as a function of d, the distance of the car to the starting line.

b. Find θ when the car is 50 meters from the starting line.

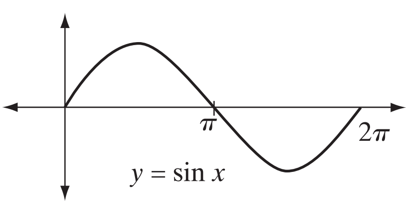
c. If the finish line is 300 meters away from the starting line, what is the maximum value of θ to the nearest minute?

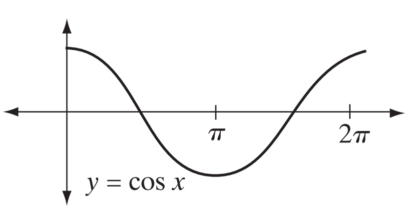


11-8 SKETCHING TRIGONOMETRIC GRAPHS

One cycle of the basic sine curve and of the basic cosine curve are shown to the right. In the previous sections, we have seen how the values of a, b, and c change these curves without changing the fundamental shape of a cycle of the graph. For $y = a \sin b(x + c)$ and for $y = a \cos b(x + c)$:

- $|a|$ = amplitude
- $|b|$ = number of cycles in a 2π interval
- $\frac{2\pi}{|b|}$ = period of the graph
- $-c$ = phase shift





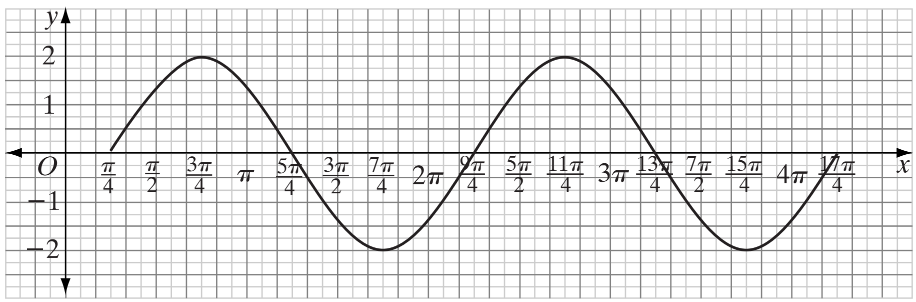
EXAMPLE 1

a. Sketch two cycles of the graph of $y = 2\sin\left(x - \frac{\pi}{4}\right)$ without using a calculator.

b. On the same set of axes, sketch one cycle of the graph of $y = \cos\frac{1}{2}x$ without using a calculator.

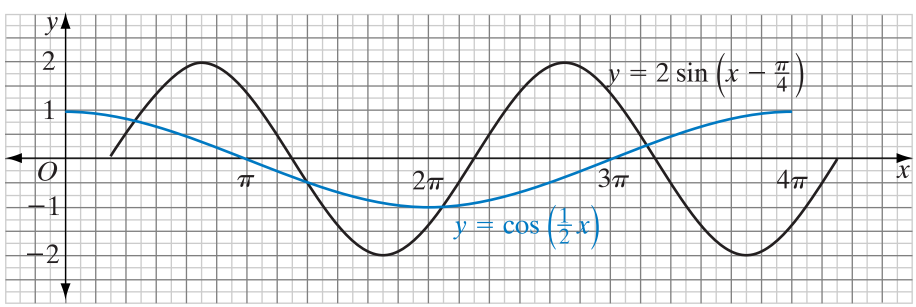
c. In the interval $0 \leq x \leq 4\pi$, how many points do the two curves have in common?

Solution



a. For the function $y = 2\sin\left(x - \frac{\pi}{4}\right)$, $a = 2$, $b = 1$, and $c = -\frac{\pi}{4}$. Therefore, one cycle begins at $x = \frac{\pi}{4}$. There is one complete cycle in the 2π interval, that is from $\frac{\pi}{4}$ to $\frac{9\pi}{4}$. Divide this interval into four equal intervals and sketch one cycle of the sine curve with a maximum of 2 and a minimum of -2.

There will be a second cycle in the interval from $\frac{9\pi}{4}$ to $\frac{17\pi}{4}$. Divide this interval into four equal intervals and sketch one cycle of the sine curve with a maximum of 2 and a minimum of -2.



b. For the function $y = \cos\frac{1}{2}x$, $a = 1$, $b = \frac{1}{2}$, and $c = 0$. Therefore, one cycle begins at the origin, $x = 0$. There is one-half of a complete cycle in the 2π interval, so the interval for one cycle is $\frac{2\pi}{\frac{1}{2}} = 4\pi$. Divide the interval from 0 to 4π into four equal intervals and sketch one cycle of the cosine curve with a maximum of 1 and a minimum of -1.

c. The curves have four points in common. *Answer*

Exercises

Writing About Mathematics

1. Calvin said that the graph of $y = \tan\left(x - \frac{\pi}{4}\right)$ has asymptotes at $x = \frac{3\pi}{4} + n\pi$ for all integral values of n. Do you agree with Calvin? Explain why or why not.

2. Is the graph of $y = \sin\left(2x - \frac{\pi}{4}\right)$ the graph of $y = \sin 2x$ moved $\frac{\pi}{4}$ units to the right? Explain why or why not.

Developing Skills

In 3–14, sketch one cycle of the graph.

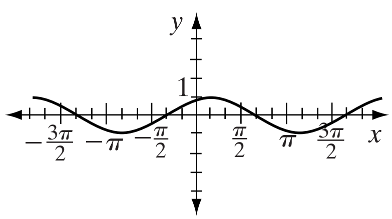
3. $y = 2\sin x$ **4.** $y = 3\sin 2x$ **5.** $y = \cos 3x$ **6.** $y = 2\sin \frac{1}{2}x$

7. $y = 4\cos 2x$ **8.** $y = 3\sin\left(x - \frac{\pi}{3}\right)$ **9.** $y = \cos 2\left(x + \frac{\pi}{6}\right)$ **10.** $y = 4\sin(x - \pi)$

11. $y = \tan x$ **12.** $y = \tan\left(x - \frac{\pi}{2}\right)$ **13.** $y = -2\sin x$ **14.** $y = -\cos x$

In 15–20, for each of the following, write the equation of the graph as: **a.** a sine function **b.** a cosine function. In each case, choose the function with the smallest absolute value of the phase shift.

15.



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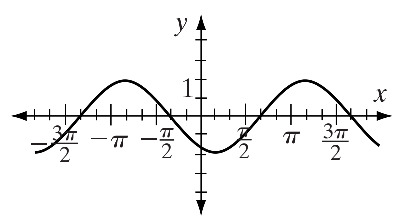

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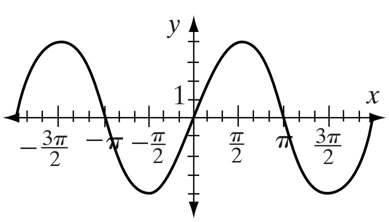

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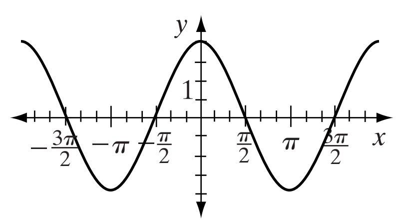

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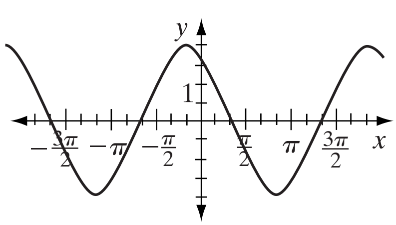

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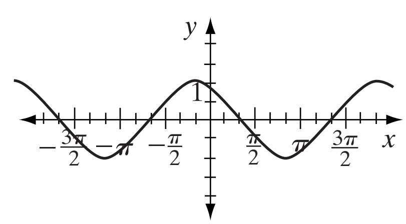


21. a. On the same set of axes, sketch the graphs of $y = 2\sin x$ and $y = \cos x$ in the interval $0 \leq x \leq 2\pi$.

b. How many points do the graphs of $y = 2\sin x$ and $y = \cos x$ have in common in the interval $0 \leq x \leq 2\pi$?

22. a. On the same set of axes, sketch the graphs of $y = \tan x$ and $y = \cos\left(x + \frac{\pi}{2}\right)$ in the interval $-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$.

b. How many points do the graphs of $y = \tan x$ and $y = \cos\left(x + \frac{\pi}{2}\right)$ have in common in the interval $-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$?

23. a. On the same set of axes, sketch the graphs of $y = \sin 3x$ and $y = 2 \cos 2x$ in the interval $0 \leq x \leq 2\pi$.

b. How many points do the graphs of $y = \sin 3x$ and $y = 2 \cos 2x$ have in common in the interval $0 \leq x \leq 2\pi$?

CHAPTER SUMMARY

	Domain (n an integer)	Range	Period	$\left(0, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \pi\right)$	$\left(\pi, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, 2\pi\right)$
$y = \sin x$	All real numbers	$[-1, 1]$	2π	increase	decrease	decrease	increase
$y = \cos x$	All real numbers	$[-1, 1]$	2π	decrease	decrease	increase	increase
$y = \tan x$	$x \neq \frac{\pi}{2} + n\pi$	All real numbers	π	increase	increase	increase	increase
$y = \csc x$	$x \neq n\pi$	$(-\infty, -1] \cup [1, \infty)$	2π	decrease	increase	increase	decrease
$y = \sec x$	$x \neq \frac{\pi}{2} + n\pi$	$(-\infty, -1] \cup [1, \infty)$	2π	increase	increase	decrease	decrease
$y = \cot x$	$x \neq n\pi$	All real numbers	π	decrease	decrease	decrease	decrease

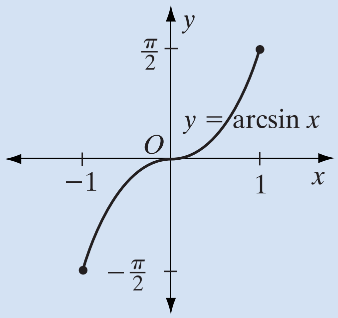
For $y = a \sin b(x + c)$ or $y = a \cos b(x + c)$:

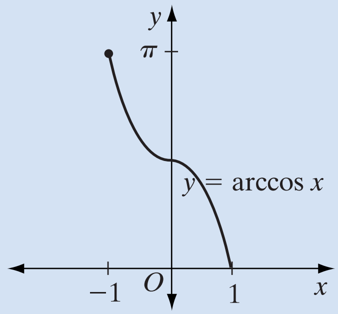
- The amplitude, $|a|$, is the maximum value of the function, and $-|a|$ is the minimum value.
- The frequency $|b|$, is the number of cycles in the 2π interval, and the period, $\frac{2\pi}{|b|}$, is the length of the interval for one cycle.
- The phase shift is $-c$. If c is positive, the graph is shifted $|c|$ units to the left. If c is negative, the graph is shifted $|c|$ units to the right and the frequency, $\frac{|b|}{2\pi}$, is the reciprocal of the period.

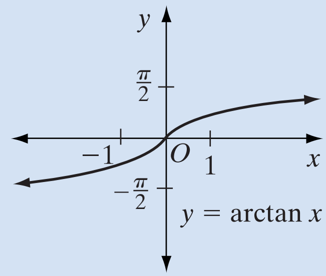
The graphs of the trigonometric functions are periodic curves. Each graph of $y = a \sin b(x + c)$ or of $y = a \cos b(x + c)$ is its own image under the translation $(x, y) \rightarrow \left(x, y + \frac{2\pi}{|b|}\right)$. The graph of $y = \tan x$ is its own image under the translation $(x, y) \rightarrow (x, y + \pi)$.

When the domain of a trigonometric function is restricted to a subset for which the function is one-to-one, the function has an inverse function.

Function	Restricted Domain	Inverse Function
$y = \sin x$	$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$	$y = \arcsin x$ or $y = \sin^{-1} x$
$y = \cos x$	$0 \leq x \leq \pi$	$y = \arccos x$ or $y = \cos^{-1} x$
$y = \tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	$y = \arctan x$ or $y = \tan^{-1} x$







VOCABULARY

11-1 Cycle • Periodic function • Period • Odd function

11-2 Even function

11-3 Amplitude • Phase shift

11-4 Simple harmonic motion • Frequency

REVIEW EXERISES

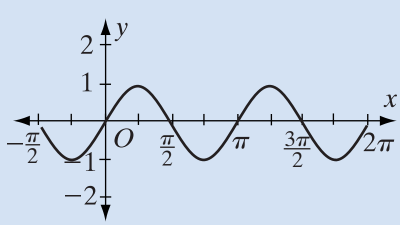
In 1–6, for each function, state: **a.** the amplitude **b.** the period **c.** the frequency **d.** the domain **e.** the range. **f.** Sketch one cycle of the graph.

1. $y = 2 \sin 3x$ **2.** $y = 3 \cos \frac{1}{2}x$ **3.** $y = \tan x$

4. $y = \cos 2\left(x - \frac{\pi}{3}\right)$ **5.** $y = \sin (x + \pi)$ **6.** $y = -2 \cos x$

In 7–10, for each graph, write the equation in the form: **a.** $y = a \sin b(x + c)$ **b.** $y = a \cos b(x + c)$. In each case, choose one cycle with its lower endpoint closest to zero to find the phase shift.

7.



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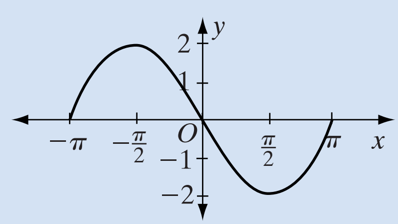

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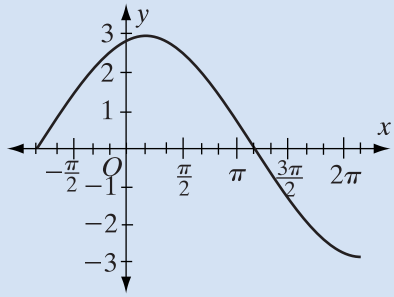

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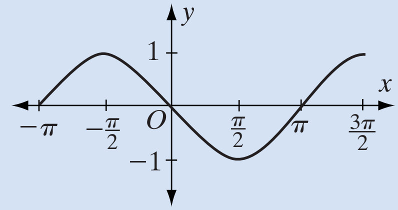
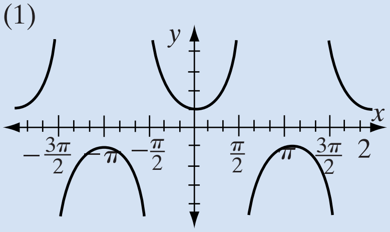


In 11–14, match each graph with its function.

(1)



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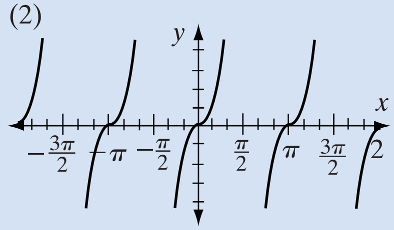

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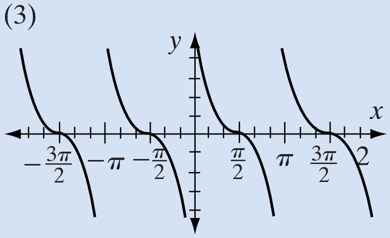

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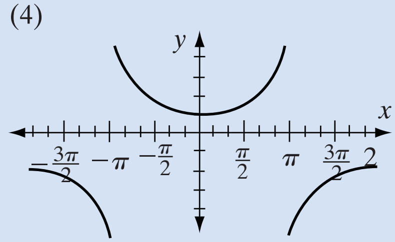


11. $y = \sec \frac{x}{2}$

12. $y = \csc\left(x + \frac{\pi}{2}\right)$

13. $y = -\cot\left(x + \frac{\pi}{2}\right)$

14. $y = -\tan\left(x - \frac{\pi}{2}\right)$

In 15–20, find, in radians, the exact value of y for each trigonometric function.

15. $y = \arcsin \frac{1}{2}$

16. $y = \sin^{-1} 1$

17. $y = \arctan \frac{\sqrt{3}}{3}$

18. $y = \arccos\left(-\frac{\sqrt{3}}{2}\right)$

19. $y = \cos^{-1} \frac{\sqrt{2}}{2}$

20. $y = \arctan\left(-\frac{\sqrt{3}}{3}\right)$

21. a. What is the restricted domain for which $y = \sin x$ is a one-to-one function?

b. What is the domain of the function $y = \arcsin x$?

c. What is the range of the function $y = \arcsin x$?

d. Sketch the graph of the function $y = \arcsin x$.

22. **a.** Sketch the graph of $y = \cos x$ in the interval $-2\pi \leq x \leq 2\pi$.

b. On the same set of axes, sketch the graph of $y = \csc x$.

c. How many points do $y = \cos x$ and $y = \csc x$ have in common?

23. What are the equations of the asymptotes of the graph of $y = \tan x$ in the interval $-2\pi < x < 2\pi$?

24. **a.** On the same set of axes, sketch the graph of $y = 2\cos x$ and $y = \sin \frac{1}{2}x$ in the interval $-\pi \leq x \leq \pi$.

b. From the graph, determine the number of values of x for which $\cos x = \sin \frac{1}{2}x$ in the interval $-\pi \leq x \leq \pi$.

25. Is the domain of $y = \csc x$ the same as the domain of $y = \sin x$? Explain why or why not.

26. The function $p(t) = 85 + 25 \sin (2\pi t)$ approximates the blood pressure of Mr. Avocado while at rest where $p(t)$ is in milligrams of mercury (mmHg) and t is in seconds.

a. Graph $p(t)$ in the interval $[0, 3]$.

b. Find the period of $p(t)$.

c. Find the amplitude of $p(t)$.

d. The higher value of the blood pressure is called the *systolic pressure*. Find Mr. Avocado's systolic pressure.

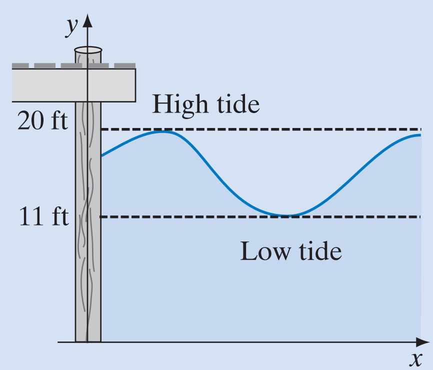
e. The lower value of the blood pressure is called the *diastolic pressure.* Find Mr. Avocado's diastolic pressure.

27. The water at a fishing pier is 11 feet deep at low tide and 20 feet deep at high tide. On a given day, low tide is at 6 A.M. and high tide is at 1 P.M. Let $h(t)$ represent the height of the tide as a function of time t.

a. What is the amplitude of $h(t)$?

b. What is the period of $h(t)$?

c. If $h(0)$ represents the height of the tide at 6 A.M., write an expression for $h(t)$.



Exploration

Natural phenomena often occur in a cyclic pattern that can be modeled by a sine or cosine function. For example, the time from sunrise to sunset for any given latitude is a maximum at the beginning of summer and a minimum at the beginning of winter. If we plot this difference at weekly intervals for a year, beginning with the first day of summer, the curve will closely resemble a cosine curve after the translation $T_{0,d}$. (A translation $T_{0,d}$ moves the graph of a function d units in the vertical direction.) The equation of the cosine curve can then be written as $y = a \cos b(x + c) + d$.

STEP 1. Research in the library or on the Internet to find the time of sunrise and sunset at weekly intervals. Let the week of June 21 be week 0 and the week of June 21 for the next year be week 52. Round the time from sunrise to sunset to the nearest quarter hour. For example, let 14 hours 12 minutes be $14\frac{1}{4}$ hours and 8 hours 35 minutes be $8\frac{1}{2}$ hours.

STEP 2. Plot the data.

STEP 3. What is the amplitude that most closely approximates the data?

STEP 4. What is the period that most closely approximates the data?

STEP 5. Let d equal the average of the maximum and minimum values of the data or:

$$d = \frac{\text{maximum} + \text{minimum}}{2}$$

Find an approximate value for d.

STEP 6. Write a cosine function of the form $y = a \cos b(x + c) + d$ that approximates the data.

CUMULATIVE REVIEW CHAPTERS 1–11

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. The sum $\sqrt{-25} + \sqrt{-9}$ is equal to
(1) $\sqrt{-34}$ (2) $-8i$ (3) $8i$ (4) $34i$

2. The solution set of $|2x + 2| - 4 = 0$ is
(1) $\varnothing$ (2) $\{1\}$ (3) $\{1, -1\}$ (4) $\{1, -3\}$

3. In radians, 225° is equivalent to
(1) $\frac{\pi}{4}$ (2) $\frac{3\pi}{4}$ (3) $\frac{5\pi}{4}$ (4) $\frac{7\pi}{4}$

4. Which of the following is a geometric sequence?
(1) 1, 2, 4, 7, 11, . . .
(2) 1, 2, 3, 4, 5, . . .
(3) 1, 1, 2, 3, 5, . . .
(4) 1, 2, 4, 8, 16, . . .

5. Which of the following functions is one-to-one?
(1) $f(x) = 2x^2$
(2) $f(x) = 2^x$
(3) $f(x) = |2x|$
(4) $f(x) = 2 \tan x$

6. When written with a rational denominator, $\frac{3}{2 - \sqrt{2}}$ is equal to

(1) $\frac{3(2 + \sqrt{2})}{2}$ (3) $3(1 + \sqrt{2})$

(2) $\frac{3(2 - \sqrt{2})}{2}$ (4) $\frac{2 + \sqrt{2}}{2}$

7. The solution set of $2x^2 + 5x - 3 = 0$ is

(1) $\left\{\frac{1}{2}, 3\right\}$ (2) $\left\{\frac{1}{2}, -3\right\}$ (3) $\left\{-\frac{1}{2}, 3\right\}$ (4) $\left\{-\frac{1}{2}, -3\right\}$

8. If $g(x) = x^2$ and $f(x) = 2x + 1$, then $g(f(x))$ equals

(1) $(2x + 1)^2$ (3) $(2x + 1)(x^2)$

(2) $2x^2 + 1$ (4) $x^2 + 2x + 1$

9. If $\theta = \arcsin\left(-\frac{\sqrt{3}}{2}\right)$, then in radians, θ is equal to

(1) $-\frac{\pi}{3}$ (2) $-\frac{\pi}{6}$ (3) $\frac{\pi}{3}$ (4) $\frac{2\pi}{3}$

10. The coordinates of the center of the circle $(x + 3)^2 + (y - 2)^2 = 9$ are

(1) $(3, -2)$ (3) $(-3, 2)$

(2) $(3, 2)$ (4) $(-3, -2)$

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Solve for x and graph the solution set on the number line:

$$|2x - 5| < 7$$

12. Find, to the nearest degree, all values of θ in the interval $0 \leq \theta \leq 360$ for which $\tan \theta = -1.54$.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Express the roots of $x^3 - 8x^2 + 25x = 0$ in simplest form.

14. Find the value of $\sum_{n=0}^{5} 3(2)^{n-1}$.

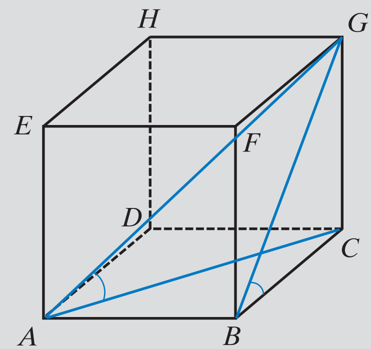
Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. *Given:* $ABCDEFGH$ is a cube with sides of length 1.

a. Find the exact measure of $\angle GBC$, the angle formed by diagonal $\overline{BG}$ and side $\overline{BC}$.

b. Find, to the nearest degree, the measure of $\angle GAC$, the angle formed by diagonals $\overline{AG}$ and $\overline{AC}$.



16. Find the solution set of the following system of equations algebraically.

$$y = 2x^2 - 3x - 5$$
$$2x - y = 7$$