Greedy Method

(最美的臉型與五官未必能組合成最美的臉孔)

There exist some problems (usually, optimization problems) whose solutions consist of n components ($x_1, x_2, \ldots, x_n$).

Solutions of these problems each involve n decisions, each for determining one component, i.e.,

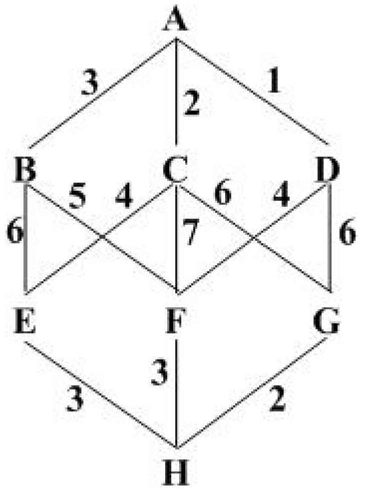
For $i \leftarrow 1$ to n

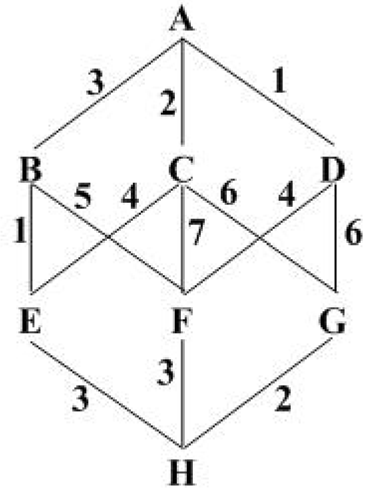
determine x_i (find locally optimal x_i).

Since x_i is locally optimal, ($x_1, x_2, \ldots, x_n$) is not necessarily globally optimal.

The greedy method is often used in designing approximation algorithms.

Ex. **Find a shortest path from *A* to *H* by the greedy method.**





x_1=AD, x_2=DF, x_3=FH

optimal solution: A-D-F-H

x_1=AD, x_2=DF, x_3=FH

optimal solution: A-B-E-H

Ex. The Knapsack Problem.

We are given n objects and a knapsack. Object i has a weight $w_i > 0$ and the knapsack has a capacity M. If a fraction x_i $(0 \le x_i \le 1)$ of object i is placed into the knapsack, then a profit of $p_i x_i > 0$ is earned. The objective is to obtain a filling of the knapsack that maximizes the total profit earned.

Mathematically, the problem may be formulated as follows.

$$\text{Max} \quad \sum_{i=1}^{n} p_i x_i \qquad (1)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} w_i x_i \le M \qquad (2)$$

$$0 \le x_i \le 1 \qquad (3)$$

A feasible solution is any set $(x_1, x_2, \ldots, x_n)$ satisfying (2) and (3). An optimal solution is a feasible solution that maximizes (1).

Greedy method 1 : largest profit first.

Objects are selected in nonincreasing order of p_i,

i. e., $p_{s1} \geq p_{s2} \geq \ldots \geq p_{sn}$.

Greedy method 2 : smallest weight first.

Objects are selected in nondecreasing order of w_i,

i. e., $w_{s1} \leq w_{s2} \leq \ldots \leq w_{sn}$.

Greedy method 3 : maximal profit per unit of capacity first.

Objects are selected in nonincreasing order of p_i/w_i,

i.e., $p_{s1}/w_{s1} \geq p_{s2}/w_{s2} \geq \ldots \geq p_{sn}/w_{sn}$.

For example, consider the following problem instance.

$$\begin{array}{ll} \text{Max} & 25x_1 + 24x_2 + 15x_3 \\ \text{s.t.} & 18x_1 + 15x_2 + 10x_3 \le 20 \\ & 0 \le x_1, x_2, x_3 \le 1 \end{array}$$

	$s1$	$s2$	$s3$	x_1	x_2	x_3	total profit
Method 1	1	2	3	1	2/15	0	28.2
Method 2	3	2	1	0	2/3	1	31
Method 3	2	3	1	0	1	1/2	31.5

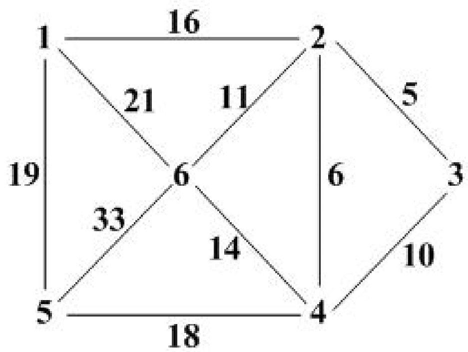
The time complexities of the above three greedy methods are all $O(n\log n)$.

Theorem. **Greedy method 3 always generates an optimal solution to the knapsack problem.**

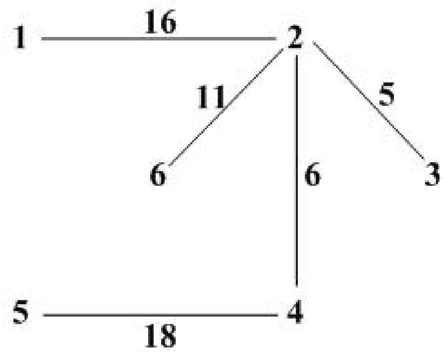
Proof. **Refer to *Computer Algorithms*, by Horowitz, Sahni, and Rajasekaran, p. 201.**

Ex. The Minimum Spanning Tree Problem.

Let *G* = (*V*, *E*) be a weighted connected undirected graph. A spanning subgraph *G'*=(*V*, *E'*) (*E'* ⊆ *E*) of *G* is a *spanning tree* of *G* iff *G'* is a tree. *G'* is called a *minimum spanning tree* of *G* if it has the smallest total weight of *E'* among all spanning trees of *G*.



$G = (V, E)$



the minimum spanning tree of *G*

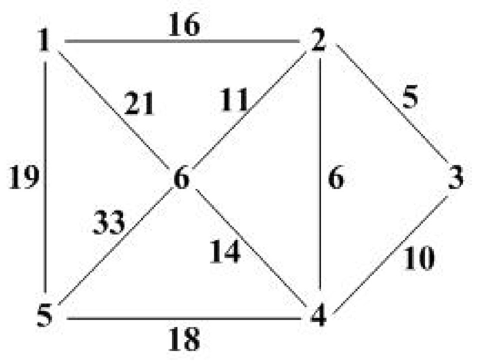
Greedy method 1. Prim's algorithm.

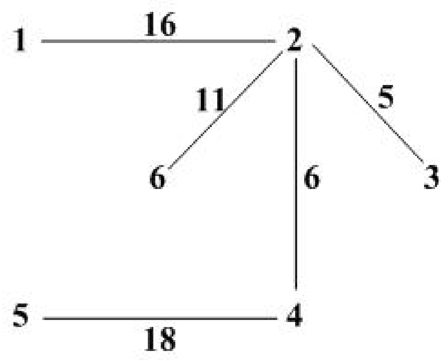
Step 1. Arbitrarily choose a vertex from V.

Step 2. Select the edge with minimum weight among all edges that connect the chosen vertices with those unchosen ones.

Step 3. Repeat step 2 until a spanning tree is formed.

For example, if we start at vertex 1, then the edges are selected in the sequence of (1, 2), (2, 3), (2, 4), (2, 6), (4, 5).





Theorem. The spanning tree produced by Prim's algorithm is minimum.

Proof. Refer to *Graph Algorithms*, by Even, p. 24.

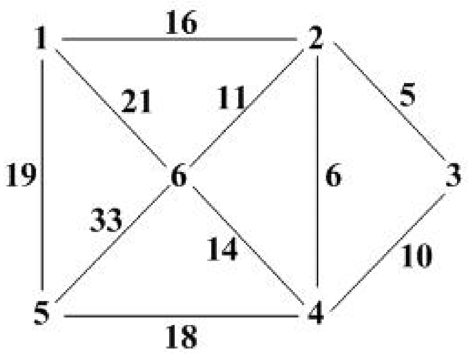
Prim's algorithm takes $O(|V|^2)$ time.

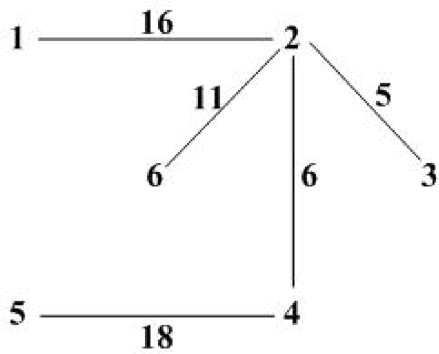
Greedy method 2. Kruskal's algorithm.

Step 1. Sort edges in nondecreasing order of weights.

Step 2. Select feasible edges (not form a cycle), one at a time, from the sorted list of edges.

For example, for the same example, feasible edges are selected in the sequence of (2, 3), (2, 4), (2, 6), (1, 2), (4, 5).



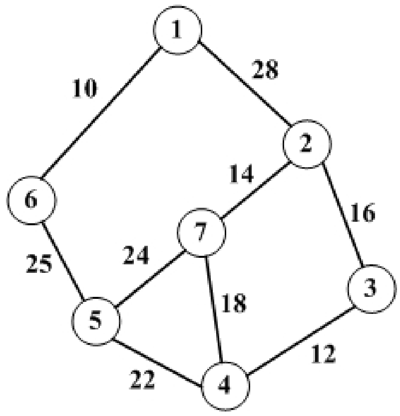


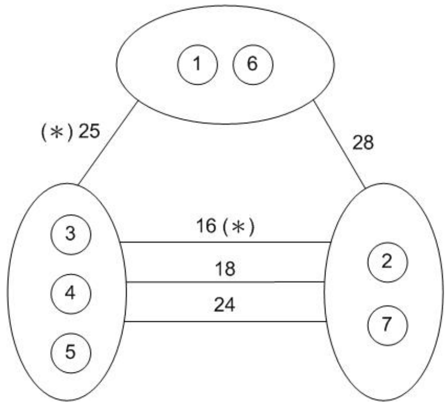
Theorem. **The spanning tree produced by Kruskal's algorithm is minimum.**

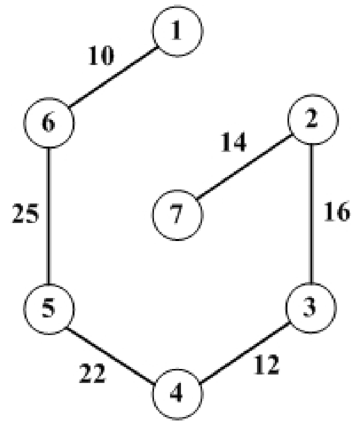
Proof. **Refer to *Computer Algorithms*, by Horowitz, Sahni, and Rajasekaran, p. 225.**

Kruskal's algorithm takes $O(|E|\cdot\log|E|)$ time.

Greedy method 3. Sollin's algorithm





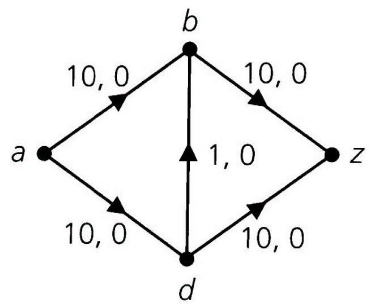


Sollin's algorithm can be implemented in $O(|E| \cdot \log\log|V|)$ time.

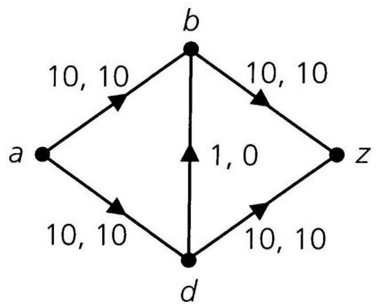
Refer to: A. C. Yao, "An $O(|E| \cdot \log\log|V|)$ algorithm for finding minimum spanning trees," *Info. Proc. Lett.*, vol. 4, no. 1, pp. 21-23, 1975.

An example of inferior greedy algorithms.

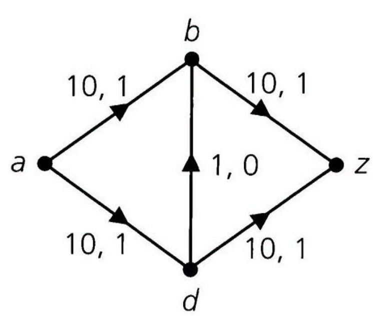
Ford & Fulkerson's maximum-flow minimum-cut algorithm:



(*a*, *b*, *z*) and (*a*, *d*, *z*) are selected as augmenting paths.



(*a*, *d*, *b*, *z*) and (*a*, *b*, *d*, *z*) are selected, in the sequence, as augmenting paths.



The worst-case time complexity is exponential.