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BASIC PRINCIPLES OF VIRTUAL ELEMENT METHODS

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We present, on the simplest possible case, what we consider as the very basic features of the (brand new) virtual element method. As the readers will easily recognize, the virtual element method could easily be regarded as the ultimate evolution of the mimetic finite differences approach. However, in their last step they became so close to the traditional

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finite elements that we decided to use a different perspective and a different name. Now the virtual element spaces are just like the usual finite element spaces with the addition of suitable *non-polynomial* functions. This is far from being a new idea. See for instance the very early approach of E. Wachspress [*A Rational Finite Element Basic* (Academic Press, 1975)] or the more recent overview of T.-P. Fries and T. Belytschko [The extended/generalized finite element method: An overview of the method and its applications, *Int. J. Numer. Methods Engrg.* **84** (2010) 253–304]. The novelty here is to take the spaces and the degrees of freedom in such a way that the elementary stiffness matrix can be computed without actually computing these non-polynomial functions, but just using the degrees of freedom. In doing that we can easily deal with complicated element geometries and/or higher-order continuity conditions (like C^1, C^2, etc.). The idea is quite general, and could be applied to a number of different situations and problems. Here however we want to be as clear as possible, and to present the simplest possible case that still gives the flavor of the whole idea.

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AMS Subject Classification: 65N30, 65N12, 65G99, 76R99

1. Introduction

Since their very beginning, mimetic finite differences (MFD) (see for instance Refs. 28, 24, 17, 25 and 26), and in particular their mathematical framework and setting, have been evolving from the original finite difference/finite volumes point of view towards more finite element-like presentations and analyses.[13,15,8,14]

In their last presentation they could be considered as a form of approximations either by means of cochains, or by finite element methods (FEMs) in which only the degrees of freedom are used (since the trial functions are not available in the interior of the elements).[31,19,1,21] This however allowed them to mimic (together with several fundamental physical laws, as their name suggests) most types of finite element spaces of lowest order (from the traditional Lagrange FEM, to the more sophisticated ones used for mixed formulations) on rather general element geometries (see for instance Refs. 18, 2, 11, 3, 10 and the references therein).

In the past years some attempts have been made to introduce higher-order MFD, making use of more general degrees of freedom[7,23,5] such as moments on faces, edges, and elements. A mimetic discretization method with arbitrary polynomial order was presented recently in Ref. 6. But the non-existence of trial and test functions inside the elements (or even inside the faces) was still making the presentation rather cumbersome.

Very recently it became clear that life would be much simpler if the degrees of freedom (typical unknowns for MFD) were attached to trial/test functions inside the elements, although not necessarily polynomials. This of course makes the method now closer to other attempts to generalize finite elements on polygons, like the PFEM (polygonal finite element methods, see for instance Refs. 29 and 30) or the extended FEMs (see Ref. 22 and the references therein). However, these methods rely on the (quite interesting) idea of adding particular shape functions that could take care of the singularities of the exact solution, and provide a better accuracy.

Here instead we want to do something simpler, preserving *the polynomial accuracy that one has on simplexes* while working on polyhedra. This will also allow us to cope with more general continuity requirements, such as $H(\text{curl})$-conformity, $H(\text{div})$-conformity, or, at the other end, C^r continuity with $r \geq 1$.

At this point it seemed more convenient to have a new, different name for this last evolution, and we chose *Virtual Element Methods*. The new approach, being much easier to deal with, and to elaborate, soon opened the way to a number of extensions of classical FEMs.

However we felt that, when proposing a new approach to the scientific community, it would be more convenient to start with a very basic paper, presenting the heart of the novelty on the simplest possible case. This is such a paper: the virtual element method is presented here on the two-dimensional Laplace equation. We preserve the generality of the shape of the elements in the decomposition of the computational domain, and the generality in the degree k of accuracy that we require to the method. All other generalizations are put aside, and left to other papers to follow.

What we consider as *the core idea* of the virtual element method can be summarized as follows:

- The trial and test functions (that here coincide) contain, on each element, all the polynomials of degree $\leq k$, plus other functions that, in general, will not be polynomials.
- When computing, on each element, the local stiffness matrix (or rather the local stiffness bilinear form) we take particular care of the cases where one of the two entries is a polynomial of degree $\leq k$. The degrees of freedom are carefully chosen in order to allow us to compute the *exact* result using only the degrees of freedom of the *other* entry that in general will not be a polynomial.
- We can show that for the remaining part of the local stiffness bilinear form (when a non-polynomial encounters another non-polynomial) we only need to produce a result with *right order of magnitude and stability properties.*

In a sense, instead of using, in a more traditional way, *nearly exact* entries in the local stiffness bilinear form (as with the use of numerical integration formulae) we have *exact* values when one of the two entries is a polynomial, and only *right order of magnitude and stability properties* in the other cases.

Note that the three properties above bring us quite close to the *patch test* used by engineers, as they imply, roughly speaking, that the method should be able to give the exact solution whenever this is a global polynomial of degree $\leq k$.

Our goal in this paper is to make each of these three ingredients as clear as possible, avoiding all non-indispensable complications, and keeping the paper as short, as simple and as clear as possible.

Throughout the paper, we will follow the usual notation for Sobolev spaces and norms (see, e.g. Ref. 20). In particular, for an open bounded domain $\mathcal{D}$, we will use $|\cdot|_{s,\mathcal{D}}$ and $\|\cdot\|_{s,\mathcal{D}}$ to denote seminorm and norm, respectively, in the Sobolev space

$H^s(\mathcal{D})$, while $(\cdot,\cdot)_{0,\mathcal{D}}$ will denote the $L^2(\mathcal{D})$ inner product. Often the subscript will be omitted when $\mathcal{D}$ is the computational domain Ω. For k a non-negative integer, $\mathbb{P}_k(\mathcal{D})$ will denote the space of polynomials of degree $\leq k$ on $\mathcal{D}$. Conventionally, $\mathbb{P}_{-1}(\mathcal{D}) = \{0\}$. Moreover, $P_k^{\mathcal{D}}$ will denote the usual $L^2(\mathcal{D})$-orthogonal projection onto $\mathbb{P}_k(\mathcal{D})$. Finally, C will be a generic constant independent of the decomposition that could change from one occurrence to the other.

The layout of the paper is the following. In Sec. 2 we present the model problem, and in Sec. 3 the abstract framework for the virtual element method, including the required fundamental assumptions and the ensuing convergence estimates. In Sec. 4 we address the actual construction of the method, in terms of the virtual discrete space V_h, the bilinear form a_h and the loading term f_h. In Sec. 5 we draw some conclusions.

2. The Continuous Problem

We consider the problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma = \partial\Omega, \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain and $f \in L^2(\Omega)$. The variational formulation reads

$$\begin{cases} \text{find } u \in V := H_0^1(\Omega) \quad \text{such that} \\ a(u,v) = (f,v) \quad \forall\, v \in V, \end{cases} \tag{2.2}$$

with $(\cdot,\cdot) =$ scalar product in L^2, $a(u,v) = (\nabla u, \nabla v)$, $|v|_1^2 = a(v,v)$. It is clear that Poincaré inequality and the boundary conditions imply that the seminorm $|\cdot|_1$ is actually *a norm* on $H_0^1(\Omega)$, equivalent to the usual $H^1(\Omega)$-norm. It is also well known that problem (2.2) has a unique solution, since

$$a(u,v) \leq M|u|_1|v|_1, \quad a(v,v) \geq \alpha|v|_1^2 \quad \forall\, u, v \in V, \tag{2.3}$$

with $\alpha = M = 1$ in our simplified case.

3. The Discrete Problem: Abstract Framework

Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into elements K, and let $\mathcal{E}_h$ be the set of edges e of $\mathcal{T}_h$. As usual, h will also denote the maximum of the diameters of the elements in $\mathcal{T}_h$. For the moment we just assume that:

A0.1. For every h, the decomposition $\mathcal{T}_h$ is made of a finite number of *simple polygons* (meaning open simply connected sets whose boundary is a non-intersecting line made of a finite number of straight line segments — not very far away from what every kid would draw).

The bilinear form $a(\cdot,\cdot)$ and the norm $|\cdot|_1$ can obviously be split as

$$a(u,v) = \sum_{K\in\mathcal{T}_h} a^K(u,v) \quad \forall\, u,\, v \in V, \quad |v|_1 = \left(\sum_{K\in\mathcal{T}_h} |v|_{1,K}^2\right)^{1/2} \quad \forall\, v \in V. \tag{3.1}$$

Since in what follows we shall also deal with functions belonging to the space $H^1(\mathcal{T}_h) := \prod_{K\in\mathcal{T}_h} H^1(K)$, we need to define a broken H^1-seminorm:

$$|v|_{h,1} := \left(\sum_{K\in\mathcal{T}_h} |\nabla v|^2_{0,K} \right)^{1/2}. \tag{3.2}$$

Note that, for discontinuous functions, this is *really* a seminorm and not a norm: for instance, $|c_h|_{h,1} \equiv 0$ for every piecewise constant function c_h.

Additional conditions on the decompositions will be introduced in Assumption A0.2 in Sec. 4.2 and in Assumption A0.3 in Sec. 4.6.

A1. We assume to have, for each h,

- a space $V_h \subset V$;
- a symmetric bilinear form a_h from $V_h \times V_h$ to $\mathbb{R}$ which can be split as

$$a_h(u_h, v_h) = \sum_{K\in\mathcal{T}_h} a_h^K(u_h, v_h) \quad \forall\, u_h,\, v_h \in V_h, \tag{3.3}$$

  where $a_h^K(\cdot,\cdot)$ is a bilinear form on $V_{h|K} \times V_{h|K}$;
- an element $f_h \in V_h'$.

We will do this in such a way that the discrete problem:

$$\begin{cases} \text{find } u_h \in V_h \quad \text{such that} \\ a_h(u_h, v_h) = \langle f_h, v_h\rangle \quad \forall\, v_h \in V_h, \end{cases} \tag{3.4}$$

has a unique solution u_h, and good approximation properties hold. Namely, if $k \geq 1$ is the target degree of accuracy, and the solution u of (2.2) is smooth enough, we want to have

$$|u - u_h|_1 \leq C h^k |u|_{k+1,\Omega}. \tag{3.5}$$

3.1. *An abstract convergence theorem*

Together with **A1** we further assume the following crucial properties.

A2. There exists an integer $k \geq 1$ (that will be our order of accuracy) such that for all h, and for all K in $\mathcal{T}_h$, we have $\mathbb{P}_k(K) \subset V_{h|K}$ and

- k-*Consistency*: For all $p \in \mathbb{P}_k(K)$ and for all $v_h \in V_{h|K}$,

$$a_h^K(p, v_h) = a^K(p, v_h). \tag{3.6}$$

- *Stability*: There exist two positive constants α_* and α^*, independent of h and of K, such that

$$\forall\, v_h \in V_{h|K}, \quad \alpha_*\, a^K(v_h, v_h) \leq a_h^K(v_h, v_h) \leq \alpha^*\, a^K(v_h, v_h). \tag{3.7}$$

We notice that the symmetry of a_h, property (3.7) and the definiton of a^K easily imply the continuity of a_h:

$$\begin{aligned} a_h^K(u,v) &\leq (a_h^K(u,u))^{1/2}(a_h^K(v,v))^{1/2} \leq \alpha^*(a^K(u,u))^{1/2}(a^K(v,v))^{1/2} \\ &= \alpha^*|u|_{1,K}|v|_{1,K} \quad \text{for all } u,v \in V_{h|K}. \end{aligned} \tag{3.8}$$

We have the following convergence theorem.

Theorem 3.1. *Under Assumptions* **A1–A2**, *the discrete problem: Find* $u_h \in V_h$ *such that*

$$a_h(u_h, v_h) = \langle f_h, v_h \rangle \quad \forall\, v_h \in V_h, \tag{3.9}$$

has a unique solution u_h. *Moreover, for every approximation* $u_I \in V_h$ *of* u *and for every approximation* u_π *of* u *that is piecewise in* $\mathbb{P}_k$, *we have*

$$|u - u_h|_1 \leq C(|u - u_I|_1 + |u - u_\pi|_{h,1} + \mathfrak{F}_h),$$

where C *is a constant depending only on* α_* *and* α^*, *and, for any* h, $\mathfrak{F}_h$ $(\equiv \|f - f_h\|_{V_h'})$ *is the smallest constant such that*

$$(f, v) - \langle f_h, v \rangle \leq \mathfrak{F}_h |v|_1 \quad \forall\, v \in V_h. \tag{3.10}$$

Proof. Existence and uniqueness of the solution of (3.9) is a consequence of (3.7) and (2.3). Next, setting $\delta_h := u_h - u_I$ we have

$$\begin{aligned} \alpha_*|\delta_h|_1^2 &= \alpha_* a(\delta_h,\delta_h) \leq a_h(\delta_h,\delta_h) \\ &= a_h(u_h,\delta_h) - a_h(u_I,\delta_h) \quad \text{(use (3.9) and (3.3))} \\ &= \langle f_h,\delta_h\rangle - \sum_K a_h^K(u_I,\delta_h) \quad \text{(use } \pm u_\pi) \\ &= \langle f_h,\delta_h\rangle - \sum_K (a_h^K(u_I-u_\pi,\delta_h) + a_h^K(u_\pi,\delta_h)) \quad \text{(use (3.6))} \\ &= \langle f_h,\delta_h\rangle - \sum_K (a_h^K(u_I-u_\pi,\delta_h) + a^K(u_\pi,\delta_h)) \quad \text{(use } \pm u \text{ and (3.1))} \\ &= \langle f_h,\delta_h\rangle - \sum_K (a_h^K(u_I-u_\pi,\delta_h) + a^K(u_\pi-u,\delta_h)) - a(u,\delta_h) \quad \text{(use (2.2))} \\ &= \langle f_h,\delta_h\rangle - \sum_K (a_h^K(u_I-u_\pi,\delta_h) + a^K(u_\pi-u,\delta_h)) - (f,\delta_h) \\ &= \langle f_h,\delta_h\rangle - (f,\delta_h) - \sum_K (a_h^K(u_I-u_\pi,\delta_h) + a^K(u_\pi-u,\delta_h)). \end{aligned} \tag{3.11}$$

Now use (3.10), (3.8) and the continuity of each a^K in (3.11) to obtain

$$|\delta_h|_1^2 \leq C(\mathfrak{F}_h + |u_I - u_\pi|_{h,1} + |u - u_\pi|_{h,1})|\delta_h|_1 \tag{3.12}$$

for some constant C depending only on α_* and α^*. Then the result follows easily by the triangle inequality. $\square$

4. Discretization

4.1. *Degrees of freedom on each element*

We consider now a *simple polygon* K with n edges and

$$\mathbf{x}_K = \text{barycenter of } K, \quad h_K = \text{diameter of } K.$$

We define for $k \geq 1$

$$\mathbb{B}_k(\partial K) := \{v \in C^0(\partial K) : v_{|e} \in \mathbb{P}_k(e) \, \forall \, e \subset \partial K\}. \tag{4.1}$$

It is not difficult to see that $\mathbb{B}_k(\partial K)$ is a linear space of dimension $n+n(k-1) = nk$. Indeed, a continuous function on ∂K which is a polynomial of degree $\leq k$ on each edge is uniquely determined by its values at the vertices (n conditions) plus, for $k > 1$, by its values at $k-1$ additional points on each edge (hence $n+n(k-1) = nk$ conditions in total).

We then consider for $k \geq 1$ the finite-dimensional space $V^{K,k}$ defined as

$$V^{K,k} = \{v \in H^1(K) : v_{|\partial K} \in \mathbb{B}_k(\partial K), \Delta v_{|K} \in \mathbb{P}_{k-2}(K)\}. \tag{4.2}$$

Recall that $\mathbb{P}_{-1}(K) = \{0\}$.

For $k = 1$ this is made of functions that are *linear* on each edge, completely determined by their value at the n vertices. Inside, $V^{K,1}$ is made of *harmonic functions* and its total dimension is equal to n.

For $k = 2$ we have, on the boundary, functions that are polynomials of degree ≤ 2 on each edge: they can be identified (always on ∂K), by the values at the vertices and at the midpoint of each edge, and the dimension of their boundary values is $2n$. Inside, according to the definition (4.2), their Laplacian is constant. For every constant c and for every boundary value $g \in \mathbb{B}_2(\partial K)$ we can find a unique function $v \in H^1(K)$ such that $\Delta v = c$ in K and $v = g$ on ∂K. Hence the dimension of $V^{K,2}$ is equal to $2n + 1$.

More generally, for every given $q_{k-2} \in \mathbb{P}_{k-2}(K)$ and for every $g \in \mathbb{B}_k(\partial K)$ there is a unique function $v \in H^1(K)$ such that $\Delta v = q_{k-2}$ in K and $v = g$ on ∂K. Hence the dimension of $V^{K,k}$ is given by

$$N^K := \dim V^{K,k} = nk + k(k-1)/2, \tag{4.3}$$

where the last term corresponds to the dimension of polynomials of degree $\leq k-2$ in two dimensions.

In $V^{K,k}$ we can choose the following *degrees of freedom*:

- $\mathcal{V}^{K,k}$: The values of v_h at the *vertices*.
- $\mathcal{E}^{K,k}$: For $k > 1$, the values of v_h at $k-1$ uniformly spaced points on each edge e.
- $\mathcal{P}^{K,k}$: For $k > 1$, the moments $\frac{1}{|K|}\int_K m(\mathbf{x}) v_h(\mathbf{x}) d\mathbf{x} \; \forall \, m \in \mathcal{M}_{k-2}(K)$,

where we have denoted by $\mathcal{M}_{k-2}$ the set of $(k^2 - k)/2$ monomials

$$\mathcal{M}_{k-2} = \left\{ \left(\frac{\mathbf{x} - \mathbf{x}_K}{h_K} \right)^{\mathbf{s}}, |\mathbf{s}| \leq k-2 \right\}, \tag{4.4}$$

and for a multi-index $\mathbf{s}$, we denoted $|\mathbf{s}| := s_1 + s_2$ and $\mathbf{x}^{\mathbf{s}} := x_1^{s_1} x_2^{s_2}$. Note that $\mathcal{M}_{k-2}$ is a basis for $\mathbb{P}_{k-2}(K)$; the interest of this choice will be clear later on.

It is not difficult to check that the dimension N^K of $V^{K,k}$, computed in (4.3), equals the total number of degrees of freedom $\mathcal{V}^{K,k}$ plus $\mathcal{E}^{K,k}$ plus $\mathcal{P}^{K,k}$.

Remark 4.1. We note that the degrees of freedom $\mathcal{V}^{K,k}$ plus $\mathcal{E}^{K,k}$ uniquely determine a polynomial of degree $\leq k$ on each edge of K, that is, $\mathcal{V}^{K,k}$ plus $\mathcal{E}^{K,k}$ are equivalent to prescribe v_h on ∂K. On the other hand, the degrees of freedom $\mathcal{P}^{K,k}$ are equivalent to prescribe $P^K_{k-2} v_h$ in K. We recall that P^K_{k-2} is the projection operator, in the $L^2(K)$-norm, onto the space $\mathbb{P}_{k-2}(K)$.

For the space $V^{K,k}$ and the degrees of freedom $\mathcal{V}^{K,k}$ plus $\mathcal{E}^{K,k}$ plus $\mathcal{P}^{K,k}$ we have the following *unisolvence result*.

Proposition 4.1. *Let K be a simple polygon with n edges, and let the space $V^{K,k}$ be defined as in* (4.2)*. The degrees of freedom $\mathcal{V}^{K,k}$ plus $\mathcal{E}^{K,k}$ plus $\mathcal{P}^{K,k}$ are unisolvent for $V^{K,k}$.*

Proof. According to Remark 4.1, to prove the proposition it is enough to see that a function $v_h \in V^{K,k}$, such that

$$v_h = 0 \quad \text{on } \partial K \quad \forall\, K \in \mathcal{T}_h, \tag{4.5}$$

and

$$P^K_{k-2} v_h = 0 \quad \text{in } K \quad \forall\, K \in \mathcal{T}_h, \tag{4.6}$$

is actually identically zero in K. In order to prove this, we show that $\Delta v_h = 0$ in K (that joined with (4.5) gives $v_h \equiv 0$). To this end, we first solve, for every $q \in \mathbb{P}_{k-2}(K)$, the following auxiliary problem: *Find $w \in H^1_0(K)$ such that*

$$a^K(w, v) = (q, v)_{0,K} \quad \forall\, v \in H^1_0(K), \tag{4.7}$$

which could also be written as

$$-\Delta w = q \quad \text{in } K, \quad w = 0 \quad \text{on } \partial K, \quad \text{or else } w = -\Delta^{-1}_{0,K}(q). \tag{4.8}$$

Next, we consider the map R, from $\mathbb{P}_{k-2}(K)$ into itself, defined by

$$Rq := P^K_{k-2}(-\Delta^{-1}_{0,K}(q)) \equiv P^K_{k-2} w. \tag{4.9}$$

We claim that R, with this definition, is an isomorphism. Indeed, from (4.9), the definition of P^K_{k-2}, and (4.7) we have, for every $q \in \mathbb{P}_{k-2}(K)$,

$$(R(q), q)_{0,K} = (P^K_{k-2}(-\Delta^{-1}_{0,K}(q)), q)_{0,K} = (P^K_{k-2} w, q)_{0,K} = (w, q)_{0,K} = a^K(w, w).$$

Since w is in $H^1_0(K)$ we have then that

$$\{R(q) = 0\} \Leftrightarrow \{a^K(w, w) = 0\} \Leftrightarrow \{w = 0\} \Leftrightarrow \{q = 0\}. \tag{4.10}$$

We notice that, if $v_h = 0$ on ∂K, then

$$P^K_{k-2} v_h = P^K_{k-2}(-\Delta^{-1}_{0,K}(-\Delta v_h)) = R(-\Delta v_h).$$

Hence, $P^K_{k-2} v_h = 0 \Rightarrow R(-\Delta v_h) = 0 \Rightarrow -\Delta v_h = 0$, and the proof is concluded. $\square$

Remark 4.2. We point out that the Laplace operator Δ appearing in definition (4.2) is the most natural choice, but it could be replaced by other second-order elliptic operators. More generally, we could just require that the space $V^{K,k}$: has dimension N^K, is made of functions that are polynomials of degree $\leq k$ on each edge, contains $\mathbb{P}_k$, and is such that the degrees of freedom $\mathcal{V}^{K,k}$, $\mathcal{E}^{K,k}$, and $\mathcal{P}^{K,k}$ are unisolvent.

4.2. *Projection error*

The assumption **A0.1** that we made so far would not be enough to ensure, given a smooth function w on K, the existence of a local approximation $w_\pi \in \mathbb{P}_k(K)$ with optimal approximation properties. In order to have it we might add, for instance, the following assumption.

A0.2. We assume that there exists a $\gamma > 0$ such that, for all h, each element K in $\mathcal{T}_h$ is star-shaped with respect to a ball of radius $\geq \gamma h_K$, where h_K is the diameter of K.

According to the classical Scott–Dupont theory (see, e.g. Ref. 9) we have then the following result.

Proposition 4.2. *Assume that assumption* **A0.2** *is satisfied. Then there exists a constant* C, *depending only on* k *and* γ, *such that for every* s *with* $1 \leq s \leq k+1$ *and for every* $w \in H^s(K)$ *there exists a* $w_\pi \in \mathbb{P}_k(K)$ *such that*

$$\|w - w_\pi\|_{0,K} + h_K |w - w_\pi|_{1,K} \leq C h_K^s |w|_{s,K}. \tag{4.11}$$

Remark 4.3. Always following Ref. 9 we note that we could take the weaker assumption that (roughly speaking) every K is the union of a finite (and uniformly bounded) number of star-shaped domains, each satisfying **A0.2**.

4.3. *Construction of V_h*

We can now use what we learned on individual polygons in order to design a virtual element space on the whole Ω. For every decomposition $\mathcal{T}_h$ of Ω into simple polygons K and for every $k \geq 1$ we define

$$V_h = \{v \in V : v_{|\partial K} \in \mathbb{B}_k(\partial K) \text{ and } \Delta v_{|K} \in \mathbb{P}_{k-2}(K)\, \forall\, K \in \mathcal{T}_h\}. \tag{4.12}$$

Arguing as we did in the case of a single polygon (but remembering that on $\partial\Omega$ we set homogeneous Dirichlet boundary conditions (meaning, $u = 0$)), we can easily see that the dimension of the whole space V_h is given by

$$N^{\text{tot}} \equiv \dim V_h = N_V + N_E(k-1) + N_P \frac{k(k-1)}{2}, \tag{4.13}$$

where N_V, N_E and N_P are, respectively, the total number of *internal vertices*, *internal edges* and *elements* (*polygons*) in $\mathcal{T}_h$.

In agreement with the local choice of the degrees of freedom, in V_h we choose the following *degrees of freedom*:

- $\mathcal{V}$: The values of v_h at the *internal vertices.*
- $\mathcal{E}$: For $k > 1$, the values of v_h at $k-1$ uniformly spaced points on each *internal edge e.*
- $\mathcal{P}$: For $k > 1$, the moments $\frac{1}{|K|}\int_K q(\mathbf{x})v_h(\mathbf{x})d\mathbf{x}\ \forall q \in \mathcal{M}_{k-2}(K)$ in each *element K.*

We explicitly recall once more that the request $V_h \subset V$ implies $v_h = 0$ on the nodes and on the edges belonging to the boundary $\partial\Omega$.

It is not difficult to check that, here too, the dimension N^{tot} of V_h, computed in (4.13), equals the total number of degrees of freedom $\mathcal{V}$ plus $\mathcal{E}$ plus $\mathcal{P}$. Proposition 4.1 will now easily imply that the global degrees of freedom are unisolvent for the global space V_h. Exactly as it happens for the usual finite element spaces.

4.4. *Interpolation error*

For each element $K \in \mathcal{T}_h$ we denote by χ_i, $i = 1, \dots, N^K$, the operator that to each smooth enough function φ associates the ith local degree of freedom $\chi_i(\varphi)$. It follows easily from the above construction that for every smooth enough w there exists a unique element w_I of $V^{K,k}$ such that

$$\chi_i(w - w_I) = 0, \quad i = 1, \dots, N^K. \tag{4.14}$$

More generally, always following for instance Ref. 9 it is not difficult to see that the following result holds.

Proposition 4.3. *Assume that assumption* **A0.2** *is satisfied. Then there exists a constant C, depending only on k and γ, such that for every s with $2 \le s \le k+1$, for every h, for all $K \in \mathcal{T}_h$ and for every $w \in H^s(K)$ there exists a $w_I \in V^{K,k}$ such that*

$$\|w - w_I\|_{0,K} + h_K|w - w_I|_{1,K} \le C h_K^s |w|_{s,K}. \tag{4.15}$$

Remark 4.4. As in Remark 4.3 we could replace in **A0.2** "star-shaped domain" with "union of a finite number of star-shaped domains". We will not insist on these struggles for generality.

4.5. *Construction of a_h*

First of all, we observe that the local degrees of freedom allow us to compute exactly $a^K(p, v)$ for any $p \in \mathbb{P}_k(K)$ and for any $v \in V^{K,k}$. Indeed,

$$a^K(p, v) = \int_K \nabla p \cdot \nabla v dx = -\int_K \Delta p v dx + \int_{\partial K} \frac{\partial p}{\partial n} v ds. \tag{4.16}$$

Since $\Delta p \in \mathbb{P}_{k-2}(K)$, $\frac{\partial p}{\partial n} \in \mathbb{P}_{k-1}(e)$ and $v \in \mathbb{P}_k(e)$ for all $e \subset \partial K$, the last two integrals can be computed exactly without knowing v in the interior of K.

Remark 4.5. In general, one can use on each edge e the values at the vertices and at $k-1$ internal points, not necessarily uniformly spaced, to reconstruct v_h (which is a polynomial of degree k on e). In particular, we point out that, using as degrees of freedom the $k-1$ internal Gauss–Lobatto points on each edge one can compute the boundary integrals in (4.16) exactly using only the given degrees of freedom (and without reconstructing v_h on each edge). This is indeed what was done in Ref. 6 in the spirit of MFD.

At this point we are left to show how to construct a (computable!) a_h that satisfies (3.6) and (3.7). For any $K \in \mathcal{T}_h$ and for any sufficiently regular function φ we set

$$\overline{\varphi} := \frac{1}{n}\sum_{i=1}^{n} \varphi(V_i), \quad V_i = \text{vertices of } K. \tag{4.17}$$

Next, we define the operator $\Pi_k^K : V^{K,k} \to \mathbb{P}_k(K) \subset V^{K,k}$ as the solution of

$$\begin{cases} a^K(\Pi_k^K v, q) = a^K(v,q) \quad \forall\, q \in \mathbb{P}_k(K), \\ \overline{\Pi_k^K v} = \overline{v}, \end{cases} \tag{4.18}$$

for all $v \in V^{K,k}$. We notice that (4.18) clearly implies

$$\Pi_k^K q = q \quad \forall\, q \in \mathbb{P}_k(K), \tag{4.19}$$

since the first equation will tell us that q and $\Pi_k^K q$ have the same gradient, and the second equation takes care of the constant part.

At this point, choosing $a_h^K(u,v) = a^K(\Pi_k^K u, \Pi_k^K v)$ would ensure property (3.6), but (3.7) in general would not be verified. We need to add a term able to ensure (3.7). Let then $S^K(u,v)$ be any symmetric positive definite bilinear form to be chosen to verify

$$c_0 a^K(v,v) \le S^K(v,v) \le c_1 a^K(v,v) \quad \forall\, v \in V^{K,k} \quad \text{with } \Pi_k^K v = 0 \tag{4.20}$$

for some positive constants c_0, c_1 independent of K and h_K. Then set

$$a_h^K(u,v) = a^K(\Pi_k^K u, \Pi_k^K v) + S^K(u - \Pi_k^K u, v - \Pi_k^K v) \quad \forall\, u,v \in V^{K,k}. \tag{4.21}$$

We point out that definition (4.21) imitates the following identity:

$$a^K(u,v) = a^K(\Pi_k^K u, \Pi_k^K v) + a^K(u - \Pi_k^K u, v - \Pi_k^K v) \quad \forall\, u,v \in V^{K,k}, \tag{4.22}$$

which for $u = v$ is nothing else but the Pythagoras theorem.

Theorem 4.1. *The bilinear form* (4.21) *satisfies the consistency property* (3.6) *and the stability property* (3.7).

Proof. Property (3.6) follows immediately from (4.19) and (4.18): for p in $\mathbb{P}_k(K)$ (4.19) implies $S^K(p - \Pi_k^K p, v - \Pi_k^K v) = 0$. Hence, for all $v \in V^{K,k}$ it holds

$$a_h^K(p, v) = a^K(\Pi_k^K p, \Pi_k^K v) = a^K(p, v). \tag{4.23}$$

Property (3.7) follows from (4.20) and (4.22): for all $v \in V^{K,k}$

$$\begin{aligned} a_h^K(v, v) &\le a^K(\Pi_k^K v, \Pi_k^K v) + c_1 a^K(v - \Pi_k^K v, v - \Pi_k^K v) \\ &\le \max\{1, c_1\}(a^K(\Pi_k^K v, \Pi_k^K v) + a^K(v - \Pi_k^K v, v - \Pi_k^K v)) = \alpha^* a^K(v, v). \end{aligned}$$

Similarly, for all $v \in V^{K,k}$,

$$\begin{aligned} a_h^K(v, v) &\ge \min\{1, c_0\}(a^K(\Pi_k^K v, \Pi_k^K v) + a^K(v - \Pi_k^K v, v - \Pi_k^K v)) \\ &= \alpha_* a^K(v, v). \end{aligned}$$

□

4.6. *Choice of S^K*

In general, the choice of the bilinear form S^K would depend on the problem and on the degrees of freedom. From (4.20) it is clear that S^K must scale like $a^K(\cdot,\cdot)$ on the kernel of Π_k^K. Choosing then the canonical basis $\varphi_1, \ldots, \varphi_{N^K}$ as

$$\chi_i(\varphi_j) = \delta_{ij}, \quad i, j = 1, 2, \ldots, N^K, \tag{4.24}$$

the local stiffness matrix is given by

$$a_h^K(\varphi_i, \varphi_j) = a^K(\Pi_k^K \varphi_i, \Pi_k^K \varphi_j) + S^K(\varphi_i - \Pi_k^K \varphi_i, \varphi_j - \Pi_k^K \varphi_j). \tag{4.25}$$

In our case it is easy to check that, on a "reasonable" polygon, $a^K(\varphi_i, \varphi_i) \simeq 1$. Note that this holds true for all $i = 1, 2, \ldots, N^K$ since we defined the local degrees of freedom suitably, and this explains the choice of $\mathcal{M}_{k-2}$ instead of the usual $\mathbb{P}_{k-2}$ in the definition of the internal degree of freedom.

However, several types of misbehavior can occur for awkwardly-shaped polygons, in particular if two or more vertices tend to coalesce, although, in our numerical experiments, the method appears to be quite robust in this respect. Hence it would be wiser to introduce a further, and last requirement for the elements of our decompositions.

A0.3. We assume that there exists a $\gamma > 0$ such that for all h and for each element K in $\mathcal{T}_h$ the distance between any two vertices of K is $\ge \gamma h_K$.

If assumption **A0.3** is verified, then indeed we will have $a^K(\varphi_i, \varphi_i) \simeq 1$ for all i. As a consequence it will be sufficient to take the simple choice

$$S^K(\varphi_i - \Pi_k^K \varphi_i, \varphi_j - \Pi_k^K \varphi_j) = \sum_{r=1}^{N^K} \chi_r(\varphi_i - \Pi_k^K \varphi_i)\chi_r(\varphi_j - \Pi_k^K \varphi_j)$$

in order to satisfy (4.20).

Remark 4.6. The present method uses the same degrees of freedom as the mimetic discretization method for linear elliptic problems introduced in Ref. 6. The stabilizing part (here depending on S^K) will in general be different, but this is not so important. In our opinion the major novelty here is in the construction of the method, and in particular the construction of a subspace V_h of V that puts the method back in the framework of conforming Galerkin methods. Hence the present methodology is much easier to describe, and much better suited for the extension to other problems.

Remark 4.7. The construction (4.21) of the local stiffness matrix is not really too far away from the classical constructions done in MFD. Indeed, looking closer, one might recognize that the first part, that is $a^K(\Pi_k^K u, \Pi_k^K v)$, is what is commonly used in MFD to take care of the *polynomial consistency*, while the second part $S^K(u - \Pi_k^K u, v - \Pi_k^K v)$ is the one usually added to recover *stability*. Note however that, in particular, here the construction does not require to choose a set of linearly independent vectors among the ones that are locally orthogonal to polynomials, as done, from the very beginning of these constructions, in Ref. 15.

4.7. *Construction of the right-hand side*

We consider first the case $k \geq 2$, and define f_h on each element K as the $L^2(K)$-projection of f onto the space $\mathbb{P}_{k-2}$, that is,

$$f_h = P_{k-2}^K f \quad \text{on each } K \in \mathcal{T}_h.$$

Consequently, the associated right-hand side

$$\langle f_h, v_h \rangle = \sum_{K \in \mathcal{T}_h} \int_K f_h v_h \mathrm{d}x \equiv \sum_{K \in \mathcal{T}_h} \int_K (P_{k-2}^K f) v_h \mathrm{d}x = \sum_{K \in \mathcal{T}_h} \int_K f (P_{k-2}^K v_h) \mathrm{d}x$$

can be exactly computed using the degrees of freedom for V_h that represent the internal moments. Then, standard L^2-orthogonality and approximation estimates on star-shaped domains yield

$$\begin{aligned} \langle f_h, v_h \rangle - (f, v_h) &= \sum_{K \in \mathcal{T}_h} \int_K (P_{k-2}^K f - f)(v_h - P_0^K v_h) \mathrm{d}x \\ &\leq C \sum_{K \in \mathcal{T}_h} h_K^{k-1} |f|_{k-1,K} h_K |v_h|_{1,K} \\ &\leq C h^k \left(\sum_{K \in \mathcal{T}_h} |f|_{k-1,K}^2 \right)^{1/2} |v_h|_1, \end{aligned} \tag{4.26}$$

and thus, recalling (3.10),

$$\mathfrak{F}_h \leq C h^k \left(\sum_{K \in \mathcal{T}_h} |f|_{k-1,K}^2 \right)^{1/2}. \tag{4.27}$$

For $k = 1$ we approximate f by a piecewise constant, and define

$$\langle f_h, v_h \rangle = \sum_{K \in \mathcal{T}_h} \int_K P_0^K f \, \overline{v}_h \mathrm{d}x = \sum_{K \in \mathcal{T}_h} |K| P_0^K f \, \overline{v}_h, \tag{4.28}$$

with $\overline{v}_h$ defined as in (4.17). We have

$$\begin{aligned} \langle f_h, v_h \rangle - (f, v_h) &= \sum_{K \in \mathcal{T}_h} \int_K ((P_0^K f - f)\overline{v}_h + f(\overline{v}_h - v_h))\mathrm{d}x \\ &\leq C \sum_{K \in \mathcal{T}_h} (h_K |f|_{1,K} \|v_h\|_{0,K} + \|f\|_{0,K} h_K |v_h|_{1,K}) \\ &\leq Ch \left(\sum_{K \in \mathcal{T}_h} |f|_{1,K}^2 \right)^{1/2} |v_h|_1. \end{aligned}$$

Thus,

$$\mathfrak{F}_h \leq Ch \left(\sum_{K \in \mathcal{T}_h} |f|_{1,K}^2 \right)^{1/2}. \tag{4.29}$$

Remark 4.8. Optimal order error estimates in the L^2-norm can be easily derived with the usual duality argument techniques. However, we remark that in the case $k = 1, 2$ a more accurate approximation of the right-hand side is needed. This is the reason why we decided not to include the L^2-analysis in the present paper. More details can be found in Ref. 4.

5. Conclusions

We have presented the virtual element method in the simplest possible case to show the essential features of the method. The virtual element method approach maintains, as MFD, the capability to reproduce several physical laws exactly, and to deal with complicated element geometries. At the same time, it shares the advantages of finite element-like formulations. Virtual elements have already been applied to other problems, such as linear elasticity[4] and plate bending problems.[16] The method seems to be particularly suited to deal with higher-order continuity requirements, and it allows to design easily C^1-approximations, as shown in Ref. 16. We believe that the new methodology might open the way to new perspectives in a number of different directions.

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References

1. J. E. Aarnes, S. Krogstad and K.-A. Lie, Multiscale mixed/mimetic methods on corner-point grids, *Comput. Geosci.* **12** (2008) 297–315. Special Issue on Multiscale Methods.
2. L. Beirão da Veiga, A residual based error estimator for the mimetic finite difference method, *Numer. Math.* **108** (2008) 387–406.
3. L. Beirão da Veiga, A mimetic finite difference method for linear elasticity, *M2AN*: *Math. Model. Numer. Anal.* **44** (2010) 231–250.
4. L. Beirão da Veiga, F. Brezzi and L. D. Marini, Virtual elements for linear elasticity problems, submitted for publication.
5. L. Beirão da Veiga, K. Lipnikov and G. Manzini, Convergence analysis of the high-order mimetic finite difference method, *Numer. Math.* **113** (2009) 325–356.
6. L. Beirão da Veiga, K. Lipnikov and G. Manzini, Arbitrary-order nodal mimetic discretizations of elliptic problems on polygonal meshes, *SIAM J. Numer. Anal.* **49** (2011) 1737–1760.
7. L. Beirão da Veiga and G. Manzini, A higher-order formulation of the mimetic finite difference method, *SIAM J. Sci. Comput.* **31** (2008) 732–760.
8. P. Bochev and J. M. Hyman, Principle of mimetic discretizations of differential operators, in *Compatible Discretizations. Proc. IMA Hot Topics Workshop on Compatible Discretizations*, eds. D. Arnold, P. Bochev, R. Lehoucq, R. Nicolaides and M. Shashkov, IMA Vol. 142 (Springer-Verlag, 2006).
9. S. C. Brenner and R. L. Scott, *The Mathematical Theory of Finite Element Methods*, Texts in Applied Mathematics, Vol. 15 (Springer-Verlag, 2008).
10. F. Brezzi and A. Buffa, Innovative mimetic discretizations for electromagnetic problems, *J. Comput. Appl. Math.* **234** (2010) 1980–1987.
11. F. Brezzi, A. Buffa and K. Lipnikov, Mimetic finite differences for elliptic problems, *M2AN*: *Math. Model. Numer. Anal.* **43** (2009) 277–295.
12. F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods* (Springer-Verlag, 1991).
13. F. Brezzi, K. Lipnikov and M. Shashkov, Convergence of mimetic finite difference method for diffusion problems on polyhedral meshes, *SIAM J. Numer. Anal.* **43** (2005) 1872–1896.
14. F. Brezzi, K. Lipnikov, M. Shashkov and V. Simoncini, A new discretization methodology for diffusion problems on generalized polyhedral meshes, *Comput. Methods Appl. Mech. Engrg.* **196** (2007) 3682–3692.
15. F. Brezzi, K. Lipnikov and V. Simoncini, A family of mimetic finite difference methods on polygonal and polyhedral meshes, *Math. Models Methods Appl. Sci.* **15** (2005) 1533–1553.
16. F. Brezzi and L. D. Marini, Virtual elements for plate bending problems, submitted for publication.
17. J. C. Campbell and M. J. Shashkov, A tensor artificial viscosity using a mimetic finite difference algorithm, *J. Comput. Phys.* **172** (2001) 739–765.
18. A. Cangiani and G. Manzini, Flux reconstruction and pressure post-processing in mimetic finite difference methods, *Comput. Methods Appl. Mech. Engrg.* **197** (2008) 933–945.
19. A. Cangiani, G. Manzini and A. Russo, Convergence analysis of the mimetic finite difference method for elliptic problems, *SIAM J. Numer. Anal.* **47** (2009) 2612–2637.
20. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems* (North-Holland, 1978).
21. J. Droniou, R. Eymard, T. Gallouët and R. Herbin, A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods, *Math. Models Methods Appl. Sci.* **20** (2010) 265–295.

22. T.-P. Fries and T. Belytschko, The extended/generalized finite element method: An overview of the method and its applications, *Int. J. Numer. Methods Engrg.* **84** (2010) 253–304.
23. V. Gyrya and K. Lipnikov, High-order mimetic finite difference method for diffusion problems on polygonal meshes, *J. Comput. Phys.* **227** (2008) 8841–8854.
24. J. M. Hyman and M. Shashkov, The orthogonal decomposition theorems for mimetic finite difference methods, *SIAM J. Numer. Anal.* **36** (1999) 788–818.
25. J. Hyman and M. Shashkov, Mimetic discretisations of Maxwell's equations and the equations of magnetic diffusion, *Progr. Electromag. Res.* **32** (2001) 89–121.
26. Yu. Kuznetsov and S. Repin, New mixed finite element method on polygonal and polyhedral meshes, *Russian J. Numer. Anal. Math. Modelling* **18** (2003) 261–278.
27. S. Rjasanow and S. Weißer, Higher order BEM-based FEM on polygonal meshes, preprint Nr. 297, Fachrichtung 6.1 - Mathematik, Saarbrücken (2011).
28. M. Shashkov and S. Steinberg, Solving diffusion equations with rough coefficients in rough grids, *J. Comput. Phys.* **129** (1996) 383–405.
29. A. Tabarraei and N. Sukumar, Conforming polygonal finite elements, *Int. J. Numer. Methods Engrg.* **61** (2004) 2045–2066.
30. A. Tabarraei and N. Sukumar, Extended finite element method on polygonal and quadtree meshes, *Comput. Methods Appl. Mech. Engrg.* **197** (2007) 425–438.
31. K. A. Trapp, Inner products in covolume and mimetic methods, *M2AN*: *Math. Model. Numer. Anal.* **42** (2008) 941–959.
32. E. Wachspress, *A Rational Finite Element Basis* (Academic Press, 1975).