SHAPE AND TOPOLOGY OPTIMIZATION WITH LAGUERRE DIAGRAMS

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Abstract. Bla bla

Contents

1. Setting

The elastic displacement u is the unique solution in $H^1_{\Gamma_D}(\Omega)^d$ to the linearized elasticity system

$$\left\{ \begin{array}{cl} -\text{div}(Ae(u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ Ae(u)n = g & \text{on } \Gamma_N, \\ Ae(u)n = 0 & \text{on } \Gamma. \end{array} \right. \tag{1.1}$$

Here, the material properties of the elastic medium are encoded in the Hooke's tensor A, which reads, for any symmetric matrix ξ:

$$A\xi = 2\mu\xi + \lambda \text{tr}(\xi)\mathrm{I},$$

where λ and μ are the Lamé coefficients of the material.

This problem rewrites, under variational form: search for $u \in H^1_{\Gamma_D}(\Omega)^d$ such that

$$\forall v \in H^1_{\Gamma_D}(\Omega)^d, \;\; a(u,v) = \ell(v), \text{ where } a(u,v) = \int_\Omega Ae(u) : e(v)\,\mathrm{d}x \text{ and } \ell(v) = \int_\Omega fv\,\mathrm{d}x.$$

2. The virtual element method

We refer to [1] for a comprehensive introduction to the virtual element method, and to [2] for a description (together with a theoretical analysis) in the context of the linearized elasticity system. A pedagogical implementation is presented in the context of the 2d Laplace equation in [4]. Our implementation in the context of the 2d linear elasticity system is based on the description in [3].

The general strategy of the Virtual Element Method is that of a standard Galerkin method: a mesh $\mathcal{T}_h$ of the considered domain Ω is introduced, which is made of polygonal elements E. A finite element space $\mathcal{W}_h$ is defined based on this mesh:

$$\mathcal{W}_h = \left\{ u \in \mathcal{C}(\overline{\Omega}), \text{ for all } E \in \mathcal{T}_h, \;\; u|_E \in \mathcal{W}(E) \right\},$$

where $\mathcal{W}(E)$ is a local finite element space for vector fields on E, endowed with a suitable local basis, which we shall define in the next sections Introducing a basis $\{\psi_i\}_{i=1,\dots,N_h}$ of $\mathcal{W}_h$, we search for an approximate solution to (1.1) as the solution to the variational problem: search for $u_h \in \mathcal{W}_h$ such that

$$\forall v_h \in \mathcal{W}_h, \quad a(u_h, v_h) = \ell(v_h).$$

This system rewrites as an $N_h \times N_h$ matrix system

$$K_h U_h = F_h,$$

where the stiffness matrix K_h and the force vector $F_h \in \mathbb{R}^{N_h}$ are defined by

$$(K_h)_{ij} = a(\psi_j, \psi_i) \text{ and } (F_h)_i = \ell(\psi_i),$$

and $U_h \in \mathbb{R}^{N_h}$ is the vector of the coefficients of the sought function u_h over the basis $\{\psi_i\}$.

To assemble this stiffness matrix, we decompose the bilinear form $a(u_h, v_h)$ as the sum of its contributions over each element $E \in \mathcal{T}_h$:

$$a(u_h, v_h) = \sum_{E \in \mathcal{T}_h} a^E(u_h, v_h), \text{ where } a^E(u, v) = \int_E Ae(u) : e(v) \, \mathrm{d}x,$$

and a similar decomposition holds as regards the right-hand side $\ell(v)$.

Hence, in order to construct the full stiffness matrix for each element $E \in \mathcal{T}_h$, we are led to calculate (or approximate) the element-wise contributions $a^E(\varphi_i, \varphi_j)$, where φ_i and φ_j run through the local basis of $\mathcal{W}(E)$. This task is the purpose of the next sections.

2.1. Notations and preliminaries

Let E be an element in the mesh $\mathcal{T}_h$, whose vertices $a_1, \dots, a_n$ are assumed to be enumerated in a counterclockwise fashion. For convenience, we sometimes denote by $a_0 = a_n$. Let us introduce some additional notation, some of which being illustrated on Fig. 1.

- We denote by $\bar{a}$ the centroid of E.
- For $i = 1, \dots, n$, we denote by e_i the edge connecting the vertices a_i and a_{i+1}, with the convention that e_0 is the edge connecting a_n with a_1.
- For $i = 1, \dots, n$, we denote by $\hat{e}_i$ the edge connecting the vertices a_{i-1} and a_{i+1}. We denote by $n_{\hat{e}_i}$ the normal vector to $\hat{e}_i$ with the orientation $n_{\hat{e}_i} \cdot n_{e_j} \geq 0$ for $j = i-1, i$. A simple calculation shows that

$$|e_{i-1}| n_{e_{i-1}} + |e_i| n_{e_i} = |\hat{e}_i| n_{\hat{e}_i}. \tag{2.1}$$

- For any function $v : E \to \mathbb{R}$, we denote by

$$\bar{v} = \frac{1}{n} \sum_{i=1}^{n} v(a_i)$$

  the average value of v over the vertices of E. The same notation holds when v is a vector field on E.
- For any function $v : E \to \mathbb{R}$, we denote by

$$\langle v \rangle = \frac{1}{|E|} \int_E v(x) \, \mathrm{d}x$$

  the average of v over E. The same notation holds when v is a vector field, or a tensor field, defined on E.

2.2. The finite element space $\mathcal{W}(E)$ attached to an element $E \in \mathcal{T}_h$

Let E be a polygonal element of the mesh $\mathcal{T}_h$. The considered finite element space at the level of E is denoted by $\mathcal{W}(E)$; it is the space of vector fields $u : E \to \mathbb{R}^2$:

- The restriction $u|_e : e \to \mathbb{R}^2$ of u to any edge e of E is an (vector-valued) affine function;
- u is such that $-\mathrm{div}(Ae(u)) = 0$ in E.

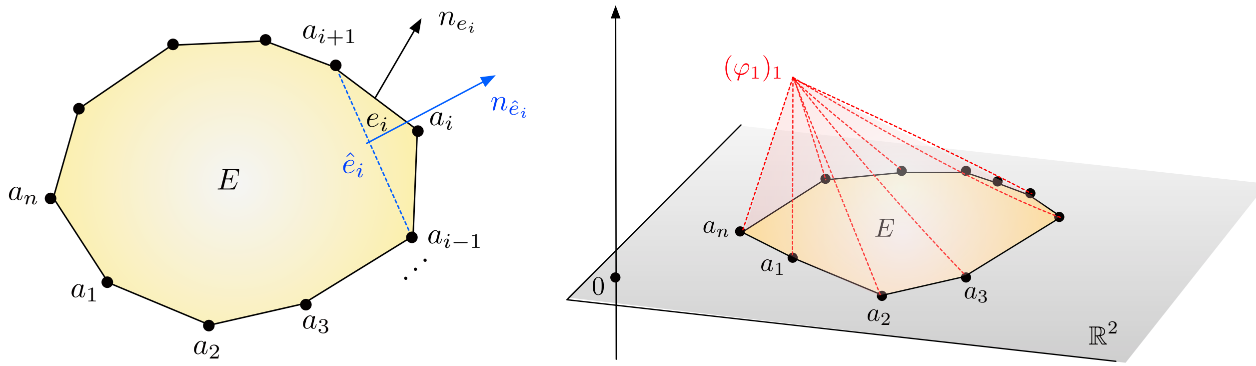

FIGURE 1. *(Left) example of a polygonal element $E \in \mathcal{T}_h$; (right) graph of the first component $(\varphi_{2i-1})_1$ of the basis function φ_{2i-1}.*

It follows from the definition that an element u in $\mathcal{W}(E)$ is uniquely defined by the values at the vertices a_i, $i = 1, \dots, n$. As a result, $\mathcal{W}(E)$ has dimension $2n$. On a different note, $\mathcal{W}(E)$ contains the space $\mathcal{P}(E)$ of affine (vector-valued) functions

$$\mathcal{P}(E) := \left\{u(x) = a + Bx,\ \ a \in \mathbb{R}^2,\ \ B \in \mathbb{R}^{2\times 2}\right\}.$$

Remark 2.1. *There is not a unique way to define $\mathcal{W}(E)$; actually, the only important point is that a function $u \in \mathcal{W}(E)$ is completely determined by its values at the vertices a_i. Nevertheless, the choice of $\mathcal{W}(E)$ is more important in 3d, where it can ease significantly the calculation some quadratures. See the appendix in* [3] *for an alternative definition of the behavior of function $u \in \mathcal{W}(E)$ inside the element E.*

We now define a basis of $\mathcal{W}(E)$ as follows: for each $i = 1, \dots, n$ the functions $\varphi_{2i-1}, \varphi_{2i} \in \mathcal{W}(E)$ are defined by the properties

$$\varphi_{2i-1}(a_i) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \varphi_{2i-1}(a_j) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } j \neq i; \tag{2.2}$$

$$\varphi_{2i}(a_i) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ and } \varphi_{2i}(a_j) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } j \neq i. \tag{2.3}$$

Let us introduce two particular subspaces of $\mathcal{W}(E)$:

- The subspace $\mathcal{W}_R(E)$ of rigid-body motions is defined by

$$\mathcal{W}_R(E) := \left\{b + M(x - \bar{a}),\ \ b \in \mathbb{R}^2,\ \ M \in \mathbb{R}^{2\times 2},\ \ M^T = -M\right\};$$

  it is spanned by the 3 vector fields

$$r_1(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad r_2(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } r_3(x) = \begin{pmatrix} -(x_2 - \bar{a}_2) \\ x_1 - \bar{a}_1 \end{pmatrix}. \tag{2.4}$$

- The subspace $\mathcal{W}_C(E)$ of vector fields with constant strain is defined by

$$\mathcal{W}_C(E) := \left\{A(x - \bar{a}),\ \ A \in \mathbb{R}^{2\times 2},\ \ M^T = M\right\}.$$

  It is spanned by the following three vector fields with constant strain $e(u)$:

$$c_1(x) = \begin{pmatrix} x_1 - \bar{a}_1 \\ 0 \end{pmatrix}, \quad c_2(x) = \begin{pmatrix} 0 \\ x_2 - \bar{a}_2 \end{pmatrix} \text{ and } c_3(x) = \begin{pmatrix} x_2 - \bar{a}_2 \\ x_1 - \bar{a}_1 \end{pmatrix}. \tag{2.5}$$

It is easily seen that the two subspaces $\mathcal{W}_R(E)$ and $\mathcal{W}_C(E)$ induce a decomposition of $\mathcal{P}(E)$:

$$\mathcal{P}(E) = \mathcal{W}_R(E) \oplus \mathcal{W}_C(E).$$

We next define projection operators from $\mathcal{W}(E)$, onto $\mathcal{W}_R(E)$ and $\mathcal{W}_C(E)$, respectively.

- For any (smooth enough) vector field $v : E \to \mathbb{R}^2$, we define $\pi_R v \in \mathcal{W}_R(E)$ by:

$$\pi_R v(x) = \bar{v} + \langle \omega(v) \rangle \begin{pmatrix} -(x_2 - \bar{a}_2) \\ x_1 - \bar{a}_1 \end{pmatrix}, \quad x \in E;$$

  where $\omega(v) := \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)$ is (half) the scalar curl of a 2d vector field v.
- For any (smooth enough) vector field $v : E \to \mathbb{R}^2$, we define $\pi_C v \in \mathcal{W}_C(E)$ by:

$$\pi_C v(x) = \langle e(v) \rangle (x - \bar{a}).$$

With these notations, the following orthogonality relation is easily verified:

$$\forall v \in \mathcal{W}_C(E),\ \ \pi_R v = 0,\ \text{ and } \forall v \in \mathcal{W}_R(E),\ \ \pi_C v = 0.$$

From these data, the orthogonal projection $\pi_{\mathcal{P}} : \mathcal{W}(E) \to \mathcal{P}(E)$ is given by

$$\pi_{\mathcal{P}} v = \pi_C v + \pi_R v.$$

Lemma 2.1. *The following energy orthogonality relation holds:*

$$\forall c \in \mathcal{W}_C(E),\ \ v \in \mathcal{W}(E), \quad a^E(c, v - \pi_C v) = 0.$$

Proof. Since the strain $e(c)$ is constant over E, one has immediately:

$$\begin{aligned} a^E(c, v - \pi_C v) &= Ae(c) : \left(\int_E (e(v) - e(\pi_C v))\, \mathrm{d}x \right) \\ &= Ae(c) : (|E| \langle (e(v) \rangle - |E| \langle e(v) \rangle \rangle\ ) \\ &= 0, \end{aligned}$$

where we have used the fact that $e(\pi_C v) = \langle e(v) \rangle$. □

Remark 2.2. *Since the vector fields in $\mathcal{W}_R(E)$ have null strain by definition, it obviously holds:*

$$\forall r \in \mathcal{W}_R(E),\ \ v \in \mathcal{W}(E), \quad a^E(r, v) = 0.$$

For any function $v \in \mathcal{W}(E)$, we then have, by a straightforward calculation:

$$\begin{aligned} a^E(v, v) &= a^E(\pi_R v + \pi_C v + (v - \pi_{\mathcal{P}} v), \pi_R v + \pi_C v + (v - \pi_{\mathcal{P}} v)) \\ &= a^E(\pi_C v + (v - \pi_{\mathcal{P}} v), \pi_C v + (v - \pi_{\mathcal{P}} v)) \\ &= a^E(\pi_C v, \pi_C v) + a^E(v - \pi_{\mathcal{P}} v, v - \pi_{\mathcal{P}} v). \end{aligned}$$

At this point, let us note that the first term in the above right-hand side can be exactly calculated, since the projections $\pi_C u$ and $\pi_C v$ have coefficients over the know basis $\{c_j\}$ that can be calculated from the values of u and v at the vertices of E, as we shall see in the following. The second term in the right-hand side is more difficult to handle: it appraises the behavior of the bilinear form $a^E(\cdot, \cdot)$ over those functions in $\mathcal{W}(E)$ that are not polynomial. The key idea is the approximate this second block by a counterpart which can be calculated just from the values of u and v at the vertices of E, so that the resulting bilinear form stays continuous and coercive.

More precisely, we approximate $a^E(\cdot, \cdot)$ by replacing the second block with a crude (still elliptic) estimate:

$$a_h^E(u, v) = a^E(\pi_C u, \pi_C v) + s^E(u - \pi_{\mathcal{P}} u, v - \pi_{\mathcal{P}} v),$$

where s^E is a bilinear, continuous and elliptic form on $\mathcal{W}(E)$.

An important property is the polynomial consistency of a_h^E with a^E, that is:

$$\begin{aligned} \forall p \in \mathcal{P},\ \ v \in \mathcal{W}(E), \quad a_h^E(p, v) &= a^E(\pi_C p, \pi_C v) \\ &= a^E(\pi_C p, v), \\ &= a^E(p, v), \end{aligned}$$

where we have used Lemma 2.1 from the first line to the second one, and Remark 2.2 from the second line to the third one. It can be proved that this polynomial consistency ensures the convergence of the Virtual Element Method, provided s^E satisfies some mild "stability conditions", see e.g. [2].

Several choices of s^E may be thought of; for instance:

$$s^E(u,v) = \alpha^E \sum_{i=1}^{n} u(a_i)\cdot v(a_i),$$

where α^E is a constant depending on the element, which is suitably chosen so a to scale as a^E. The definition and calculation of α^E is detailed in the next Section 2.3.2.

2.3. Implementation issues

Let us now carefully detail the implementation of the local stiffness matrix $K_E \in \mathbb{R}^{2n\times 2n}$ defined by

$$\forall k,l = 1,\dots,2n, \quad (K_E)_{kl} = a_h^E(\varphi_k,\varphi_l).$$

We have seen that, for all $u,v \in \mathcal{W}(E)$,

$$a_h^E(u,v) = a^E(\pi_C u, \pi_C v) + s^E(u - \pi_{\mathcal{P}} u, v - \pi_{\mathcal{P}} v).$$

Hence, K_E arises as the sum of two contributions:

$$K_E = P_E + S_E, \text{ where } P_E, S_E \in \mathbb{R}^{2n\times 2n} \text{ are defined by}$$
$$(P_E)_{kl} = a_h^E(\pi_C\varphi_k, \pi_C\varphi_l) \text{ and } (S_E)_{kl} = s^E(\varphi_k - \pi_{\mathcal{P}}\varphi_k, \varphi_l - \pi_{\mathcal{P}}\varphi_l).$$

Let W_C be the $(2n)\times 3$ matrix such that the i^{th} line of W_C represents the coordinates of the projection $\pi_C\varphi_i$ over the three basis functions c_1, c_2, c_3 defined by (2.5), namely:

$$\forall i = 1,\dots,2n, \quad \pi_C\varphi_i(x) = \sum_{j=1}^{3} (W_C)_{ij} c_j(x).$$

Likewise, let W_R be the $(2n)\times 3$ matrix such that the i^{th} line of W_R represents the coordinates of the projection $\pi_R\varphi_i$ over the three rigid-body motions r_1, r_2, r_3 defined by (2.4), namely:

$$\forall i = 1,\dots,2n, \quad \pi_R\varphi_i(x) = \sum_{j=1}^{3} (W_R)_{ij} r_j(x).$$

With these notations, the “projection” matrix P_E reads,

$$(P_E)_{ij} = a_h^E(\pi_C\varphi_i, \pi_C\varphi_j) = \sum_{k,l=1}^{3} (W_C)_{ik}(W_C)_{jl} a^E(c_k, c_l),$$

and so, in matrix form:

$$P_E = W_C D W_C^T, \text{ where } D \in \mathbb{R}^{3\times 3} \text{ is defined by } D_{ij} = a^E(c_i,c_j) = \int_E Ae(c_i) : e(c_j)\,\mathrm{d}x.$$

An elementary calculation yields:

$$D = |E| \begin{pmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & 4\mu \end{pmatrix}.$$

As far as the “stabilization” matrix S_E is concerned, we need to calculate the matrices P_C and $P_R \in \mathbb{R}^{2n\times 2n}$ of the projection mappings $\pi_C : \mathcal{W}(E) \to \mathcal{W}(E)$ and $\pi_R : \mathcal{W}(E) \to \mathcal{W}(E)$ in the basis $\{\varphi_i\}_{i=1,\dots,2n}$. To this end, let us introduce the matrix $N_C \in \mathbb{R}^{2n\times 3}$, whose j^{th} column ($j = 1,\dots,3$) contains the coordinates of c_j over the basis φ_i, that is:

$$\forall j = 1,\dots,3, \quad c_j(x) = \sum_{i=1}^{2n} (N_C)_{ij}\varphi_i(x).$$

With these notations, it holds, for $j = 1, \dots, 2n$:

$$\begin{aligned}\pi_C \varphi_j &= \sum_{k=1}^{3} (W_C)_{jk} c_k \\ &= \sum_{i=1}^{2n} \left(\sum_{k=1}^{3} (N_C)_{ik} (W_C)_{jk} \right) \varphi_i(x)\end{aligned}$$

and so, the matrix $P_C \in \mathbb{R}^{2n \times 2n}$ of the projection π_C in the basis $\{\varphi_i\}$ reads:

$$P_C = N_C W_C^T.$$

Likewise, introducing the matrix $N_R \in \mathbb{R}^{2n \times 3}$, whose j^{th} column ($j = 1, \dots, 3$) contains the coordinates of r_j over the basis φ_i, that is:

$$\forall j = 1, \dots, 3, \quad c_j(x) = \sum_{i=1}^{2n} (N_R)_{ij} \varphi_i(x),$$

we see that:

$$P_R = N_R W_R^T.$$

Finally, the matrix $P_P \in \mathbb{R}^{2n \times 2n}$ of the projection $\pi_{\mathcal{P}}$ over the space $\mathcal{P}(E)$ of affine functions reads:

$$P_P = P_R + P_C.$$

With these notations, S_E reads:

$$S_E = \alpha^E (\mathrm{I} - P_P)^T (\mathrm{I} - P_P).$$

At this point, the only remaining ingredients are the assembly of the matrices N_C, N_R, W_C, W_R, and the calculation of the coefficients α^E, which we now detail.

2.3.1. *Assembly of the matrices N_C, N_R, W_C, W_R*

For $i = 1, \dots, n$, let us introduce the vector $q_i \in \mathbb{R}^2$ defined by

$$q_i = \frac{|\hat{e}_i|}{2|E|} n_{\hat{e}_i},$$

where we recall the notation $n_{\hat{e}_i}$ introduced in (2.1).

Let us first consider the matrix $W_R \in \mathbb{R}^{2n \times 3}$ whose i^{th} line ($i = 1, \dots, 2n$) gathers the components of the projection $\pi_R \varphi_i$ of φ_i over the space of rigid-body motions, expressed in the basis $\{r_j\}_{j=1,\dots,3}$. For each vertex $i = 1, \dots, n$, we have

$$\pi_R \varphi_{2i-1}(x) = \bar{\varphi}_{2i-1} + \langle \omega(\varphi_{2i-1}) \rangle \begin{pmatrix} -(x_2 - \bar{a}_2) \\ x_1 - \bar{a}_1 \end{pmatrix},$$

where obviously, $\bar{\varphi}_{2i-1} = \frac{1}{n} r_1$ and:

$$\begin{aligned}\langle \omega(\varphi_{2i-1}) \rangle &= \frac{1}{|E|} \int_E \left(\frac{\partial (\varphi_{2i})_1}{\partial x_2} \right) \mathrm{d}x \\ &= \frac{1}{2|E|} \int_{\partial E} (\varphi_{2i-1})_1 n_2 \, \mathrm{d}\ell \\ &= \frac{1}{4|E|} (|e_{i-1}| n_{e_{i-1}} + |e_i| n_{e_i})_2 \\ &= \frac{1}{|E|} \frac{|\hat{e}_i|}{4} (n_{\widehat{e_i}})_2 \\ &= \frac{1}{2} (q_i)_2.\end{aligned}$$

Hence, we see that

$$\pi_R \varphi_{2i-1}(x) = \frac{1}{n} r_1(x) + \left(\frac{1}{2} |\hat{e}_i| (q_i)_2 \right) r_3(x).$$

A similar calculation reveals that:

$$\pi_R \varphi_{2i}(x) = \frac{1}{n} r_2(x) - \left(\frac{1}{2} |\hat{e}_i| (q_i)_1 \right) r_3(x).$$

Similar calculations lead to the formulas associated to the entries of W_C:

$$\pi_C \varphi_{2i-1}(x) = (q_i)_1 c_1(x) + \left(\frac{1}{2} (q_i)_2 \right) c_3(x),$$

and

$$\pi_C \varphi_{2i}(x) = (q_i)_2 c_2(x) + \left(\frac{1}{2} (q_i)_1 \right) c_3(x).$$

Summarizing, the matrices W_R and W_C read, in the respective bases $\{r_j\}_{j=1,\dots,3}$, $\{\varphi_i\}_{i=1,\dots,2n}$ and $\{c_j\}_{j=1,\dots,3}$, $\{\varphi_i\}_{i=1,\dots,2n}$:

$$W_R = \begin{pmatrix} \vdots & \vdots & \vdots \\ & & \\ \frac{1}{n} & 0 & \frac{1}{2}(q_i)_2 \\ 0 & \frac{1}{n} & -\frac{1}{2}(q_i)_1 \\ & & \\ \vdots & \vdots & \vdots \end{pmatrix}, \text{ and } W_C = \begin{pmatrix} \vdots & \vdots & \vdots \\ & & \\ (q_i)_1 & 0 & \frac{1}{2}(q_i)_2 \\ 0 & (q_i)_2 & \frac{1}{2}(q_i)_1 \\ & & \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

Let us now turn to the matrices N_R and N_C, which gather the components of the respective basis functions r_j and c_j in the basis $\{\varphi_i\}_{i=1,\dots,2n}$. To this end, it follows from the definition (2.2) and (2.3) of the functions φ_i that:

$$c_j(x) = \sum_{i=1}^{n} (c_j(x_i))_1 \, \varphi_{2i-1}(x) + \sum_{i=1}^{n} (c_j(x_i))_2 \, \varphi_{2i}(x).$$

Hence, a simple calculation based on the explicit form (2.5) of the basis functions c_j reveals the following expression of the matrix N_C in the bases $\{\varphi_i\}_{i=1,\dots,2n}$ and $\{c_j\}_{j=1,\dots,3}$:

$$N_C = \begin{pmatrix} \vdots & \vdots & \vdots \\ & & \\ (c_1(x_i))_1 & (c_2(x_i))_1 & (c_3(x_i))_1 \\ (c_1(x_i))_2 & (c_2(x_i))_2 & (c_3(x_i))_2 \\ & & \\ \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \vdots \\ & & \\ (x_i)_1 - \bar{a}_1 & 0 & (x_i)_2 - \bar{a}_2 \\ 0 & (x_i)_2 - \bar{a}_2 & (x_i)_1 - \bar{a}_1 \\ & & \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

In a similar fashion, we calculate the expression of the matrix N_R in the bases $\{\varphi_i\}_{i=1,\dots,2n}$ and $\{r_j\}_{j=1,\dots,3}$:

$$N_R = \begin{pmatrix} \vdots & \vdots & \vdots \\ & & \\ 1 & 0 & -((x_i)_2 - \bar{a}_2) \\ 0 & 1 & (x_i)_1 - \bar{a}_1 \\ & & \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

2.3.2. *Calculation of α^E*

As we have mentioned, the coefficient α^E is chosen so as to ensure the stability of the method. The idea from **??** consists in taking α^E such that the block $\left\{ s^E(c_k, c_l) \right\}_{k,l=1,\dots,3}$, which corresponds to the application of the approximate bilinear form s^E to the constant strain fields, scales like $\left\{a^E(c_k, c_l)\right\}_{k,l=1,\ldots,3}$, which is exactly the matrix $|E|D$. Hence, we require that

$$s^E(c_k, c_l) = \alpha^E (N_C^T N_C)_{kl} \text{ be comparable with } |E|D_{kl},$$

which entices us to choosing

$$\alpha^E = \frac{|E|\mathrm{tr}(D)}{\mathrm{tr}(N_C^T N_C)}.$$

References

[1] L. Beirão da Veiga, F. Brezzi, L. D. Marini, and A. Russo, *The hitchhiker's guide to the virtual element method*, Mathematical models and methods in applied sciences, 24 (2014), pp. 1541–1573.

[2] L. B. Da Veiga, F. Brezzi, and L. D. Marini, *Virtual elements for linear elasticity problems*, SIAM Journal on Numerical Analysis, 51 (2013), pp. 794–812.

[3] A. L. Gain, C. Talischi, and G. H. Paulino, *On the virtual element method for three-dimensional linear elasticity problems on arbitrary polyhedral meshes*, Computer Methods in Applied Mechanics and Engineering, 282 (2014), pp. 132–160.

[4] O. J. Sutton, *The virtual element method in 50 lines of matlab*, Numerical Algorithms, 75 (2017), pp. 1141–1159.