Inverted Pendulum on a Cart

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This document outlines the control of an inverted pendulum on a cart. This is a common problem used in control theory as it is a simple yet under-actuated system. Demonstrated will be the system, the derivation of its equations of motion, how this is linearised and represented in state-space, the application of a LQR-based control, and its performance.

1 System Description

The system depicted in Fig. 1 shows a pendulum connected to a cart.

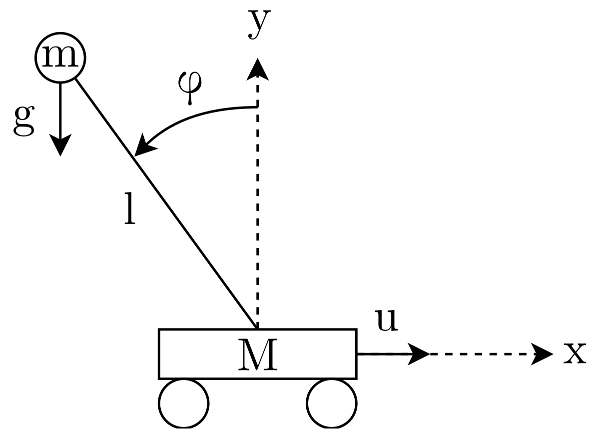

Figure 1: Model of system

The connection of the pendulum to the cart is such that the pendulum may fall to either side along a single axis. This same axis is the one in which the cart is able to move. The objective is to move the cart such that the pendulum remains balanced i.e. $\varphi = 0$.

2 Derivation of Equations of Motion

The equations of motion of the system were derived using Lagrange's equations. To do this, the position of the pendulum's mass was described in x and y coordinates.

$$x_p = x - l\sin\varphi\,, \tag{1}$$
$$y_p = l\cos\varphi\,. \tag{2}$$

As velocity is required in the kinetic energy portion of the Lagrange equations, the derivative of these coordinates with respect to time was required.

$$\dot{x}_p = \dot{x} - l\dot{\varphi}\cos\varphi\,, \tag{3}$$
$$\dot{y}_p = -l\dot{\varphi}\sin\varphi\,. \tag{4}$$

Now, the potential V and kinetic T energies could be defined:

$$V = mgl\cos\varphi\,, \tag{5}$$
$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left((\dot{x}_p)^2 + (\dot{y}_p)^2\right)\,, \tag{6}$$
$$= \frac{1}{2}(M+m)\,\dot{x}^2 + \frac{1}{2}ml^2\dot{\varphi}^2 - m\dot{x}l\dot{\varphi}\cos\varphi\,. \tag{7}$$

Lagrange's equations are set up as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i\,, \tag{8}$$
$$L = T - V\,. \tag{9}$$

This was applied to the obtained equations for potential and kinetic energy to give the following equations:

$$L = \frac{1}{2}(M+m)\,\dot{x}^2 + \frac{1}{2}ml^2\dot{\varphi}^2 - m\dot{x}l\dot{\varphi}\cos\varphi - mgl\cos\varphi\,, \tag{10}$$
$$x : (M+m)\,\ddot{x} - ml\ddot{\varphi}\cos\varphi + ml\dot{\varphi}^2\sin\varphi - 0 = u\,, \tag{11}$$
$$\varphi : ml^2\ddot{\varphi} - m\ddot{x}l\cos\varphi + m\dot{x}l\dot{\varphi}\sin\varphi - m\dot{x}l\dot{\varphi}\sin\varphi - mgl\sin\varphi = 0\,. \tag{12}$$

Which gave rise to the equations of motions:

$$(M+m)\,\ddot{x} - ml\ddot{\varphi}\cos\varphi + ml\dot{\varphi}^2\sin\varphi = u\,, \tag{13}$$
$$l\ddot{\varphi} - \ddot{x}\cos\varphi - g\sin\varphi = 0\,. \tag{14}$$

3 Linearisation of the Equations of Motion

Small angle approximations may be made, as the objective is to keep $\varphi = 0$. This gives rise to the following simplifications:

$$\sin\varphi \approx \varphi\,, \tag{15}$$
$$\cos\varphi \approx 1\,, \tag{16}$$
$$\varphi^2 \approx 0\,. \tag{17}$$

Applying this theorem to the equations of motion linearises the system to the following equations:

$$(M+m)\,\ddot{x} - ml\ddot{\varphi} = u\,, \tag{18}$$
$$l\ddot{\varphi} - \ddot{x} - g\varphi = 0\,. \tag{19}$$

4 State-space Model Representation

To obtain a state-space model for the system, Eq. 18 & 19 must explicitly state the highest-order term using lower order terms. This is done by substituting the equations into each other.

First, both equations are rearranged in terms of the highest-order state:

$$\ddot{x} = \frac{1}{M+m}\left(u + ml\ddot{\varphi}\right) , \tag{20}$$

$$\ddot{\varphi} = \frac{1}{l}\left(\ddot{x} + g\varphi\right) . \tag{21}$$

Next, Eq. 21 is substituted into Eq. 20 to obtain the expression for $\ddot{x}$:

$$\ddot{x} = \frac{1}{M+m}\left(u + m\ddot{x} + mg\varphi\right) , \tag{22}$$

$$\ddot{x}\left(1 - \frac{m}{M+m}\right) = \frac{1}{M+m}\left(u + mg\varphi\right) , \tag{23}$$

$$\ddot{x} = \frac{1}{M}\left(u + mg\varphi\right) . \tag{24}$$

Then, Eq. 24 is substituted into Eq. 21 to obtain the expression for $\ddot{\varphi}$:

$$\ddot{\varphi} = \frac{1}{l}\left(\frac{u}{M} + \frac{mg\varphi}{M} + g\varphi\right) , \tag{25}$$

$$= \frac{1}{lM}\left[u + \left(m+M\right)g\varphi\right] . \tag{26}$$

With Eq. 24 & 26, the state-space representation can be obtained:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\varphi} \\ \ddot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(m+M)g}{lM} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \varphi \\ \dot{\varphi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{lM} \end{bmatrix} u \tag{27}$$

5 LQR control with Integral Action

The objective for the controller will be to keep the pendulum upright while being able to set the desired position of the cart. The applied controller will be a LQR and as this is to be simulated, the system will need to be discretised.

Before this, however, the measurement equation is set to measure the position of the cart and the angle of the pendulum. Therefore:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \varphi \\ \dot{\varphi} \end{bmatrix} \tag{28}$$

The parameters for the system were chosen to be as follows:

Table 1: Parameters used when simulating the system

Parameter	Value	Unit
Mass of cart (M)	1	kg
Mass at end of pendulum (m)	0.5	kg
Length of pendulum (l)	1	m
Acceleration due to gravity (g)	9.81	ms^{-2}

The system was discretised with a step size of 0.001 s, as this would be fast enough for the system and simulations could still be executed quick enough.

As integral action is used to set the position of the cart, the system had to be augmented to accommodate this feature. Then, the discrete-time LQR gains were calculated for this augmented system. The discretised system was augmented as follows:

$$A_{d,\text{ aug}} = \begin{bmatrix} A_d & \mathbf{0} \\ -T_s C_d[1,:] & \mathbf{1} \end{bmatrix}, \tag{29}$$

$$B_{d,\text{ aug}} = \begin{bmatrix} B_d \\ \mathbf{1} \end{bmatrix}. \tag{30}$$

Note, that only the first row of C_d is used, as the reference to be set only concerns the first output which is the position of the cart.

Using this augmented system, the discrete-time LQR system was calculated using the following weights:

$$Q = \begin{bmatrix} 200 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 200 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1000 \end{bmatrix}, \tag{31}$$

$$R = 1\,. \tag{32}$$

The integral gain is the negative of the final gain from this calculation, the rest are gains that are multiplied with the states.

6 Simulation Implementation

In MATLAB, the loop for the simulation was implemented as follows:

```
% Initialize
x = [-1;0;0;0];
xi = -1.48;      % Observed when stable at -1m
u = 0;
k = 1;

% Generate reference square wave
ref = squarewave(time,4);

% Simulation loop
for t = time
        % System Dynamics
        x = Ad*x+Bd*u;
        y = Cd*x+Dd*u;

        % Optional noise to states
        x_n = [x(1);x(2);x(3) + randn*0.1;x(4)];

        % Control Law
        u = -Kx*x_n+Ki*xi;

        % Error calculation
        e = ref(k)-x_n(1);
        % Integral state
        xi = xi + e*ts;

        k = k + 1;
end
```

It is assumed, that the states are all known for the calculation of the control law. However, this may not always be the case realistically. If it weren't, a Kalman filter may be used to infer the states based on the horizontal and angular positions. Another addition to the loop is the optional noise. This may be altered to vary the calculation of the control law based on noise to the states.

7 Simulation results

The simulation was run for a 20 s time period, where the reference changed 4 times between 1 and -1. The result can be seen in Fig. 2.

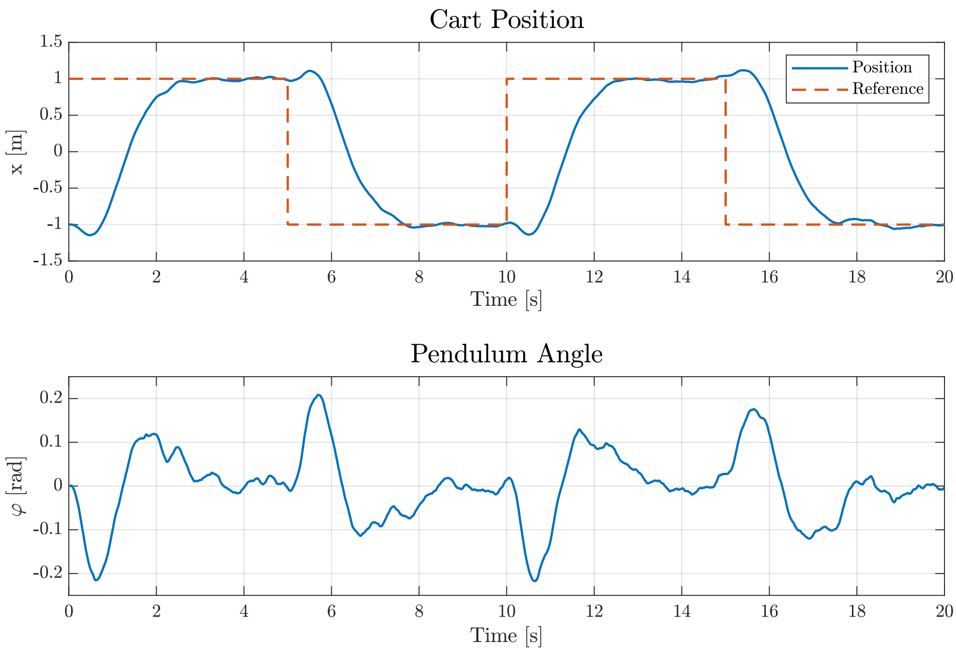

Figure 2: Result of the control of the cart to balance the pendulum

The noise to the pendulum angle measurement means that the calculation of the control action wasn't always as optimal as it could have been. Nevertheless, the pendulum could remain balanced as this variation wasn't significant enough to destabilize the control.

It can be seen that in order for the cart to move to its next position, it has to move in the opposite direction to cause the pendulum to fall into the desired direction. This allows the pendulum to remain balanced while the cart is moving.