uflvector_solution

April 6, 2022

1 Problem

We want to use quadratic finite elements over

$$\Omega = [-\frac{1}{2}, \frac{1}{2}]^2$$

to solve the vector valued PDE:

$$\begin{aligned} -\Delta u_1 + u_1 &= f_1 \ , \qquad \nabla u_1 \cdot n = 0 \ , \\ -\Delta u_2 + u_2 &= f_2 \ , \qquad \nabla u_2 \cdot n = g \ . \end{aligned}$$

where f_1, f_2, g are chosen so that

$$u_1 = \sin(\pi x) + \sin(\pi y) \ , \qquad u_2 = \sin(4\pi x \cdot x) \ .$$

1.1 Necessary imports

[1]:
```
import numpy as np

from dune.grid import cartesianDomain
from dune.alugrid import aluConformGrid as leafGridView

from ufl import SpatialCoordinate, sin, sqrt, dot
from ufl import as_vector
from ufl import TrialFunction, TestFunction
from ufl import grad, div, inner, dx, pi
from ufl import FacetNormal, ds

from dune.fem.space import lagrange
from dune.fem.scheme import galerkin
from dune.fem.function import uflFunction
from dune.fem.plotting import plotPointData as plot

from dune.fem.function import integrate
```

1.2 Setup of grid and space

We want to setup a quadratic finite element space V_h over $\mathcal{T}_h$.

[2]:
```
domain = cartesianDomain([-0.5,-0.5],[0.5,0.5],[4,4])
view = leafGridView(domain)
space = lagrange(view, order=2, dimRange=2)

x,n = SpatialCoordinate(space), FacetNormal(space)
u,v = TrialFunction(space), TestFunction(space)
```

```
Created parallel ALUGrid<2,2,simplex,conforming> from input stream.
```

```
GridParameterBlock: Parameter 'refinementedge' not specified, defaulting to
'ARBITRARY'.
WARNING (ignored): Could not open file 'alugrid.cfg', using default values 0 <
[balance] < 1.2, partitioning method 'ALUGRID_SpaceFillingCurve(9)'.

You are using DUNE-ALUGrid, please don't forget to cite the paper:
Alkaemper, Dedner, Kloefkorn, Nolte. The DUNE-ALUGrid Module, 2016.
```

The exact solution is given by $u = (u_1, u_2)^T$.

[3]:
```
exact = as_vector( [sin(pi*x[0]) + sin(pi*x[1]),
                    sin(4*pi*dot(x,x))] )
```

The weak form is given by

$$a(u, v) = \int_\Omega \nabla u \nabla v + uv \, dx$$

$$b(v) = \int_\Omega (\Delta u_1 + u_1) v_1 \, dx + \int_\Omega (\Delta u_2 + u_2) v_2 \, dx + \int_\Omega (\nabla u_2 \cdot n) v_2 \, ds$$

[4]:
```
a  = inner( grad(u), grad(v) ) * dx + dot(u,v) * dx
b  = (-div(grad(exact[0])) + exact[0]) * v[0] * dx
b += (-div(grad(exact[1])) + exact[1]) * v[1] * dx
b += dot(grad(exact[1]),n) * v[1] * ds
```

By default the scheme uses a *gmres* solver but the given bilinear form is symmetric so we can use *cg* instead:

[5]:
```
scheme = galerkin(a==b, solver="cg")
```

Create $u_h \in V_h$.

[6]:
```
uh = space.interpolate([0,0], name="solution")
```

Refine the grid a few times and on each level compute the solution and compute the errors

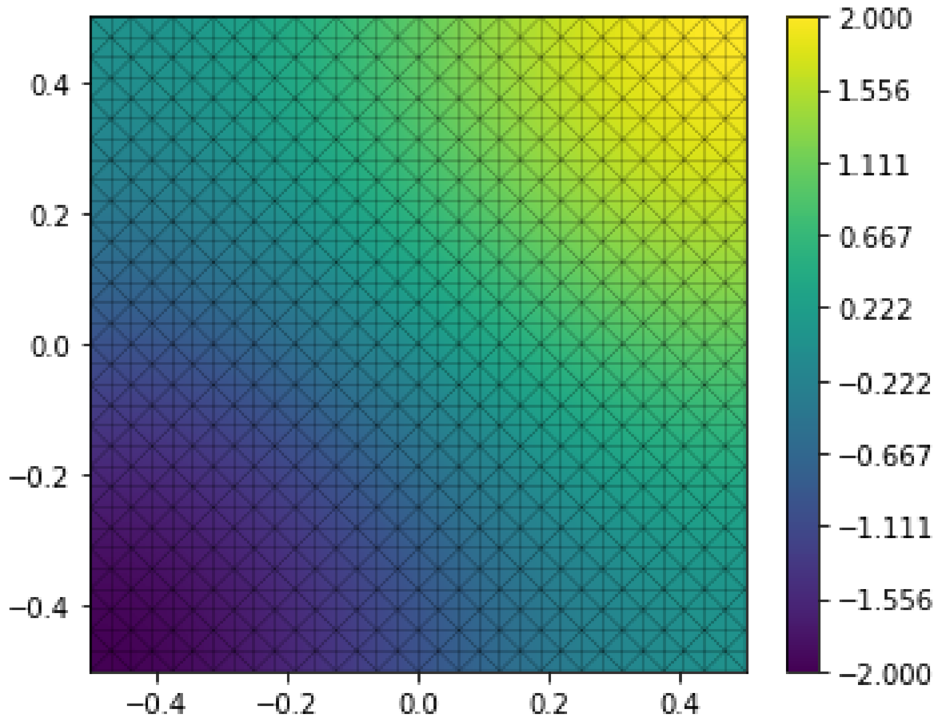
$$\|e_h\|_0^2 := \int_\Omega |e_h|^2 \;, \qquad \|e_h\|_1^2 := \int_\Omega |\nabla e_h|^2 \;.$$

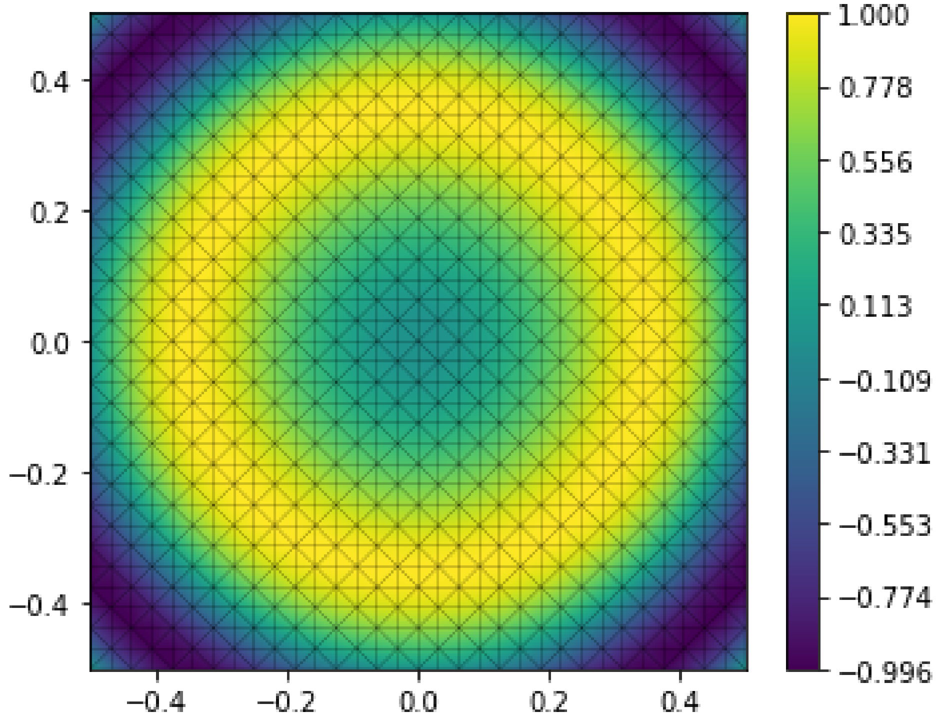
[7]:
```
for l in range(3):
    view.hierarchicalGrid.globalRefine(2)
    scheme.solve(target=uh)
    error = uh-exact
    h1Error = as_vector( [inner(error,error),
                          inner(grad(error),grad(error))] )

    print(view.size(0), space.size,
                np.sqrt( integrate(view,h1Error,order=5) )
          )
```

```
128 578 [0.01040168 0.60037702]
512 2178 [0.00181493 0.21224711]
2048 8450 [0.00024122 0.05405574]
```

[8]:
```
plot(uh[0])
plot(uh[1])
```





0.4
0.2
0.0
−0.2
−0.4
−0.4
−0.2
0.0
0.2
0.4
1.000
0.778
0.556
0.335
0.113
−0.109
−0.331
−0.553
−0.774
−0.996