Bond Market and Bond Risk Management

- Interest rate always annualized
- Simple / linear compounding – returns = $1 + \Delta R$, usually for < 1 year
- Discrete compounding – returns = $\left(1+\frac{R}{m}\right)^{mn}$, m = no. payments, n = number of years
- Continuous compounding – e^{Rt} (Not usually used in FI as no continuous compounding, fastest is 1 day, overnight, for convenience only)
- Discounting just take 1 /
- D(0, time) – 0 denotes today, time is maturity
- Effective annual rate – make different rate payment frequency comparable by converting to annual
  - $EAR = \left(1+\frac{r}{m}\right)^m - 1$
  - Special case – Bond Equivalent Yield (used in US) because they got semi-annual rates
    - $BEY = \left[\left(1+\frac{r}{m}\right)^{\frac{m}{2}} - 1\right] * 2$
- Day count conventions
  - If payment date falls on non-business day, then:
    - Actual, Following / Previous, Modified following / Previous (rolled to next business day except if the next business day is the next month)
- Bond instruments
  - Zero-coupon bonds – pure discount, single payment at maturity (t-bills)
  - Coupon bonds – periodic fixed coupons, longer maturity
    - US: zero-coupon bonds (bills), 2-10years (notes), >10years (bonds)
  - Floating rate – coupons not fixed but depend on benchmark interest rate
- YTM – rate that discounts bond to PV, Par yield – rate that discounts bond to 100
- Bootstrapping bond curve – use maturity, coupon and price to find the discount factor for n number of bonds with n different maturities
  - For arbitrage, only trade bonds with cashflow shorter than or up to the underpriced bond, **NOT** those with longer maturities
  - Cashflow netting (to secure riskless profits) – w * coupon + w * principal = 0
  - To get weights: cashflow netting each time period, Solve simultaneous
- Clean price – quoted in the market without **accrued interest**
- Dirty price = clean price + Accrued interest, Accrued interest = $coupon * \frac{days\ since\ last\ coupon}{days\ between\ coupon\ dates}$
- Floating rate notes – linked to variable interest rate, eliminates interest rate sensitivity, price action driven mostly by changes in market-perceived credit quality of the issuer
  - If rate goes up, more coupon CF BUT higher discount rate so bond price falls, result in stable bond price, higher coupons offset price decrease, interest rate exposure lesser
- Bond duration and convexity
  - Taylor series expansion of $B(y+\Delta y) = B(y) + \frac{dB}{dy}(\Delta y) + \frac{1}{2}\left(\frac{d^2b}{dy^2}\right)(\Delta y)^2 + \cdots$
  - To hedge bond can only use another bond because yield is not tradable
  - $modified\ Duration = \frac{Macaulay\ duration}{1+\frac{y}{m}}$
    - Macaulay uses discrete compounding $\sum_{i=1}^{n}\frac{t_i c_i}{\left(1+\frac{y}{m}\right)^{m_i t_i}}$, modified duration uses continuous compounding
  - Modified duration = $-\frac{dB}{dy}\left(\frac{1}{B}\right) = \frac{1}{B}\sum_{i=1}^{n} t_i c_i e^{-rt}$ (<u>add par, 100, at maturity to coupon</u>)
    - Time weighted average of the cashflow's NPV
    - % change in bond price = $-D\Delta y$
    - If yield goes up, $-D\Delta y$ is negative so returns drop, vice versa
    - Underestimates return when yield drops, overestimates when yield increases because doesn't account for convexity
    - Only can use for small changes in yield
  - Convexity = $\frac{1}{B}\left(\frac{d^2B}{dy^2}\right) = \frac{1}{B}\sum_{i=1}^{n} t_i^2 c_i e^{-rt}$
    - Longer maturity more convex more discounted cash flow + square time
    - Regardless if yield goes up/down, higher convexity higher return because of the yield squared
  - $change\ in\ bond\ price = -modified\ duration \Delta y B + \frac{1}{2}C(\Delta y)^2 B$
  - bond portfolio, don't divide bond price because itll be normalized for that bond => \$ duration = $-\frac{dB}{dy} = \sum_{i=1}^{n} t_i c_i e^{-rt}$, \$ convexity = $\frac{d^2B}{dy^2} = \sum_{i=1}^{n} t_i^2 c_i e^{-rt}$, additive
  - DV01 – yield change 0.01%, what is \$ change (for hedging to offset price movement)

Interest Rate and Swap Market

- Bond portfolio immunization
  - $\pi = V + B_1 + B_2$
  - $-D_\$(V) = B_1 D_1 + B_2 D_2$, bond 1 & 2 give -ve duration of portfolio so net 0 hedged
  - $-C_\$(V) = B_1 C_1 + B_2 C_2$, bond 1 & 2 give -ve convexity of portfolio so net 0 hedged
- Negative yielding bond
  - Regulatory requirement – central banks required to hold bonds to meet liquidity requirements and pledge as collateral
  - Potential capital gain – currency appreciation
  - Holding cash not optimal – If bond yield -ve, cash interest rate will be negative also
- LIBOR vs SOFR (overnight, lower credit exposure), rate mismatch, different credit exposure
- Spot LIBOR tells you how to discount future cash flow to today
- Forward Rate Agreement (FRA) – like forwards, at t0 price is 0
  - Buy – borrow at some future rate, if rate end up higher, good, borrow at lower rate
  - Sell – deposit/lend money at some rate in the future
  - Quote – Expiry X (length of depositing period + expiry, when everything ends)
    - 3 X 9, 3 month to expiry, 6 month LIBOR, starts 3 months from now ends 9 months from now
  - If think 1m spot unchanged 1m later, short 1X2 FRA as F(1,2) > spot (borrow at lower rate to lend at higher rate)
- If 3m LIBOR rate is 4% and 6m LIBOR rate is 6%, what is in-3m-for-3m forward rate F(3m, 6m)?
  - Need to be higher than 4% and 6% to average out to be ~6%
  - Its ~8% & because interest on interest, need to be slightly < 8% (compounding)
- Forward rate = $\frac{1}{day\ count\ fraction} * \left(\frac{closer\ maturity\ discount\ rate}{further\ maturity\ discount\ rate} - 1\right)$
- Forward discount factor = $\frac{\boldsymbol{further\ maturity\ discount\ rate}}{\boldsymbol{closer\ maturity\ discount\ rate}}$
- Interest rate swaps – naming convention based on fixed
  - Payer: pay fixed, Receiver: receive fixed
  - LIBOR fixed in advance, paid in arrears, fixed at T0, paid at T1
  - Par swap rate = $\frac{D(0,T_0)-D(0,T_n)}{\sum_{i=1}^{n}\Delta_{i-1}D(0,T_i)}$, $\sum_{i=1}^{n}\Delta_{i-1}D(0,T_i)\ is\ PV01\ (zero\ coupon\ bonds)$
  - **Given zero rates always convert to discount factor first**
  - Once you pass the 6m LIBOR period then that 6m LIBOR rate will be fixed
  - Payer swap value = PV float – PV fix, pay fix receive float so – fix
  - Rate increase, Payer swap value increase, more float received (fixed discount more)
- Valuing swap
  - At inception
    - Use IRS, compute missing discount factor by taking PV fixed = PV float, using linear interpolation for missing discount factor
      - PV fixed = par swap rate * payment freq * (discount factors by payment freq)
      - PV float = 1 – D(0, maturity)
    - Use par swap rate formula
  - At some time after inception
    - PV fixed = par swap rate * payment freq * (discount factors by payment freq)
    - PV float = payment freq * ($\sum$ **forward LIBOR rate** * zero rate discount factors)
      - Note for next payment date it is already fixed
      - For zero rates, interpolate, calculate discount factors
      - Calculate forward LIBOR using forward rate formula (note DF t = t of zero rate)
  - Forward swap rate
    - Use IRS, compute missing discount factor by taking PV fixed = PV float, using linear interpolation for missing discount factor
      - PV fixed = par swap rate * frequency * (discount factors by frequency)
      - PV float = 1 – D(0, maturity)
    - Forward swap rate = $\frac{D(0,T_{start})-D(0,T_{start+tenor})}{\sum_{i=1}^{n}\Delta_{i-1}D(0,T_{tenor})}$

Multicurve Framework and OIS Discounting

- Secured SOFR open to bigger pool of participants, Unsecured EFFR only open for banks
- FOMC can use money supply to indirectly control the rate
  - Increase money supply => more lending => rates decrease
- Overnight index futures
  - Current month expected overall average overnight rate = 100 – (weighted average of realized overnight rates + expected overnight rate for remainder of month)
  - Probability can be calculated also
- Posting collateral – borrow at LIBOR rate but receive overnight rate which is < LIBOR
- Matching collateral and payment current **<u>(please check OIS RATE is daily or !!)</u>**
  - What discount rate to use?
    - If > 1 day then is just multiply the daily OIS
    - $D(t,T) = \prod_{i=1}^{\frac{T-t}{\delta}} \frac{1}{1+\delta f(t+(i-1)\delta, t+i\delta)}$
- Matching collateral with different currency
  - $discount\ factor = D_{x,y}(t,T) = \frac{FX_{x,y}(t,T)D_y(t,T)}{FX_{x,y}(t,t)}$
- If collateralized, no credit risk, NPV higher
- OIS discount factor is different from LIBOR discount factor
  - **<u>Collateralized framework uses OIS discount factor to get forward LIBOR</u>**
  - $D_0(0,n) = \left(\frac{1}{1+\frac{1}{360}*OIS}\right)^{years*360}$
  - $2y\ IRS: IRS\ rate * 0.5 * \left(D_0(0,1) + D_0(0,2)\right) = \Delta D_0(0,1) * 1y\ spot\ LIBOR + \Delta D_0(0,2) * foward\ LIBOR(1y, 2y)$
  - Which can then be used to compute the LIBOR discount factor
- Liquidity value adjustment = OIS DF PV – LIBOR DF PV (legacy method to calculate the OIS PV)

LIBOR and Swap Market Models

- Q* is not a good numeraire when doing interest rate derivatives because the expectation contains stochastic interest rate and stochastic payoff leading to covariance which complicates computation
  - $\frac{C_i(0)}{B_0} = E^*\left[\frac{C_i(T)}{B_T}\right] \Rightarrow C_i(0) = B_0 E^*\left[\frac{\Delta_i(L_i(T_i)-K)^+}{B_0 e^{\int_t^T r_u du}}\right]$
  - $C_i(0) = E^*\left[\Delta_i(L_i(T_i)-K)^+ * \frac{D(T,T)/D(0,T)}{B_T/B_0}\right] * D(0,T)$
  - $C_i(0) = D(0,T) * E^{*T}\left[\Delta_i(L_i(T_i)-K)^+ * \frac{dQ^T}{\cancel{dQ^*}}\right]$
- Changing to Qt measure results in discount factor * expectation of payoff which is more convenient
- LIBOR market model – using zero coupon bond maturing at T_{i+1} as numeraire is better
  - **SDE:** $\boldsymbol{dL_i(t) = \sigma_i L_i(t) dW^{i+1}(t)}$, $W^{i+1}$is Brownian motion under the Q^{i+1} measure
  - $L_i(T) = L_i(0)\exp\left[\left(-\frac{\sigma_i^2}{2}\right)T + \sigma_i W^{i+1}(T)\right]$
  - Each LIBOR has own measure as numeraire follows one where LIBOR is paid (i+1)
- Pricing a caplet / floorlet (call / put on forward LIBOR)
  - Caplet: $C_i(T_{i+1}) = \Delta_i(L_i(T_i)-K)^+$
    - $\frac{C_i(0)}{D_{i+1}(0)} = E^{i+1}\left[\frac{C_i(T_{i+1})}{D_{i+1}(T_{i+1})}\right]$, $D_{i+1}(T_{i+1})$ =1 at maturity
    - $C_i(0) = D_{i+1}(0)\Delta_i E^{i+1}[(L_i(T_i)-K)^+]$
    - $D_{i+1}(0)\Delta_i \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(L_i(0)e^{\left(-\frac{\sigma_i^2}{2}\right)T+\sigma_i\sqrt{T}x} - K)^+ e^{-\frac{x^2}{2}}dx$
    - $D_{i+1}(0)\Delta_i[L_i(0)\Phi(d_1) - K\Phi(d_2)]$
    - $d_1 = \frac{\log\left(\frac{L_i(0)}{K}\right)+\left(\frac{\sigma_i^2}{2}\right)T}{\sigma_i\sqrt{T}}, d_2 = \frac{\log\left(\frac{L_i(0)}{K}\right)-\left(\frac{\sigma_i^2}{2}\right)T}{\sigma_i\sqrt{T}}$ (cash/nothing d2, asset d1)
  - Floorlet: $P_i(T_{i+1}) = \Delta_i\left(K - L_i(T_i)\right)^+$
    - $D_{i+1}(0)\Delta_i[K\Phi(-d_2) - L_i(0)\Phi(-d_1)]$
- Swap market model, PVBP, $\sum_{i=n+1}^{N}\Delta_{i-1}D_i(t)$, is the numeraire
  - Par swap rate $S_{n,N}(t) = \frac{D_n(t)-D_N(t)}{\sum_{i=n+1}^{N}\Delta_{i-1}D_i(t)}, P_{n+1,N}(t) = \sum_{i=n+1}^{N}\Delta_{i-1}D_i(t)$
  - **SDE:** $\boldsymbol{dS_{n,N}(t) = \sigma_{n,N}S_{n,N}(t)dW^{n+1,N}(t)}$
  - Note black **<u>normal</u>** is like bachelier, $S_{n,N}(T) = S_{n,N}(0) + \sigma_{n,N}S_{n,N}(0)W^{n+1,N}(T)$
    - $V_{n,N}^{payer}(0) = P_{n+1,N}(0)E^{n+1,N}\left[\left(S_{n,N}(T)-K\right)^+\right]$
    - $P_{n+1,N}(0)\frac{1}{\sqrt{2\pi}}\left[\int_{x^*}^{\infty}(S_{n,N}(0)-K)e^{-\frac{x^2}{2}}dx + \int_{x^*}^{\infty}(S_{n,N}(0)\sigma_{n,N}\sqrt{T}x)e^{-\frac{x^2}{2}}dx\right]$
    - $P_{n+1,N}(0)[(S_{n,N}(0)-K)\Phi(-x^*) + (S_{n,N}(0)\sigma_{n,N}\sqrt{T})\phi(x^*)]$
  - $S_{n,N}(T) = S_{n,N}(0)\exp\left[\left(-\frac{\sigma_{n,N}^2}{2}\right)T + \sigma_{n,N}W^{n+1,N}(T)\right]$
  - Payer swap (like call) = $P_{n+1,N}(T)\left(S_{n,N}(T)-K\right)^+$
    - Payoff = $[P_{n+1,N}(T)\left(S_{n,N}(T)-K\right)]^+ = P_{n+1,N}(T)\left(S_{n,N}(T)-K\right)^+$
    - $\frac{V_{n,N}^{payer}(0)}{P_{n+1,N}(0)} = E^{n+1,N}\left[\frac{P_{n+1,N}(T)\left(S_{n,N}(T)-K\right)^+}{P_{n+1,N}(T)}\right]$
    - $V_{n,N}^{payer}(0) = P_{n+1,N}(0)E^{n+1,N}\left[\left(S_{n,N}(T)-K\right)^+\right]$
    - $P_{n+1,N}(0)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(S_{n,N}(0)\exp\left(\left(-\frac{\sigma_{n,N}^2}{2}\right)T + \sigma_i\sqrt{T}x\right) - K\right)^+ e^{-\frac{x^2}{2}}dx$
    - $P_{n+1,N}(0)\left[S_{n,N}(0)\Phi\left(\frac{\log\left(\frac{S_{n,N}(0)}{K}\right)+\left(\frac{\sigma_{n,N}^2}{2}\right)T}{\sigma_{n,N}\sqrt{T}}\right) - K\Phi\left(\frac{\log\left(\frac{S_{n,N}(0)}{K}\right)-\left(\frac{\sigma_{n,N}^2}{2}\right)T}{\sigma_{n,N}\sqrt{T}}\right)\right]$
  - Rec swap (put) $P_{n+1,N}(T)\left(K - S_{n,N}(T)\right)^+, P_{n+1,N}(0)[K\Phi(-d_2) - S_{n,N}(0)\Phi(-d_1)]$
  - $dS_{n,N}(t) = \sigma_{n,N}S_{n,N}(t)dW^{n+1,N}(t)$, $W^{n+1,N}$ Brownian motion under $Q^{n+1,N}$ measure
  - Swaption notation: expiry x tenor (10 x 10 expire 10y later, if exercise enter 10y swap)
  - DD Model Solve **SDE:** $\boldsymbol{X_t = \log[\beta F_t + (1-\beta)F_0]}$

    $f'(F_t) = \frac{\beta}{[\beta F_t + (1-\beta)F_0]}, f''(F_t) = -\frac{\beta^2}{[\beta F_t + (1-\beta)F_0]^2}$
  - $\int_0^T dX_t = \int_0^T \beta\sigma dW_t - \int_0^t \frac{1}{2}\beta^2\sigma^2 dt$
  - $X_t - X_0 = \beta\sigma W_T - \frac{1}{2}\beta^2\sigma^2 T$, ***simplify denominator 1st***!
  - $F_t = \frac{F_0}{\beta}\exp\left[\beta\sigma W_T - \frac{1}{2}\beta^2\sigma^2 T\right] - \frac{1-\beta}{\beta}F_0$, Black(F, K, σ, T) Vs DD($\frac{F_0}{\beta}$, $K + \frac{1-\beta}{\beta}F_0$, $\sigma\beta$, T)

Constant Maturity Swap Payoffs

- IRR settled swaptions
  - Payer = $\left[IRR(S)\left(S_{n,N}(t)-K\right)\right]^+$, Receiver = $\left[IRR(S)\left(K - S_{n,N}(t)\right)\right]^+$
  - IRR(S) = $\sum_{i=1}^{(T_N-T_n)*m}\frac{\frac{1}{m}}{\left(1+\frac{S}{m}\right)^i}$, s = market swap rate, $\frac{1}{m}$ = day count fraction
  - Valuation $= D(0,T)IRR(S)black(S_{n,N}(0), K, \sigma_{n,N}, T)$, zero coupon as numeraire
  - Settled in cash based on value of payoff observed at maturity

- Constant maturity swap – pays swap rate rather than LIBOR on floating leg
  - Exposure to fixed length longer term interest rates
    - If think yield curve steepen, long 10y payer swap (makes money when curve steepen, pay fixed receive more float) BUT if curve steepens after 4 years then the 10yr IRS will become 6yr IRS, wont get maximum benefit so 10y CMS will constantly be 10yrs, make money
- Risk-Neutral density of forward swap rate
  - $V^{pay}(K) = D(t,T)E^T[IRR(S)(S-K)^+] = D(t,T)\int_K^{inf} IRR(s)(S-K)f(s)ds$
  - $\frac{dV^{pay}(K)}{dK} = -D(t,T)\int_K^{inf} IRR(s)f(s)ds, \frac{d^2V^{pay}(K)}{dK^2} = D(t,T)IRR(K)f(K)$
  - $f(K) = \frac{d^2V^{pay}(K)}{dK^2} * \frac{1}{D(t,T)IRR(K)}$
- Static replication
  - $let\ h(K) = \frac{g(K)}{IRR(K)}, h'(K) = \frac{IRR(K)g'(K) - g(K)IRR'(K)}{IRR(K)^2}, h''(K) = \frac{IRR(K)g''(K) - g(K)IRR''(K) - 2IRR'(K)g'(K)}{IRR(K)^2} + \frac{2IRR'(K)^2 g(K)}{IRR(K)^3}, g(K)\ is\ payoff$
  - $V_0 = D(0,T)E^T[g(S)] = D(0,T)\int_0^{inf} g(K)f(K)dK$
  - $D(0,T)\int_0^{inf} g(K)\frac{d^2V^{pay}(K)}{dK^2} * \frac{1}{D(0,T)IRR(K)}dK$
  - $\int_0^F h(K)\frac{d^2V^{Rev}(K)}{dK^2} * dK + \int_F^{inf} h(K)\frac{d^2V^{Pay}(K)}{dK^2} * dK$
  - $integration\ by\ parts\ twice$

$$= \left[h(K)\frac{\partial V^{rec}(K)}{\partial K}\right]_0^F - \int_0^F h'(K)\frac{\partial V^{rec}(K)}{\partial K}dK$$
$$+ \left[h(K)\frac{\partial V^{pay}(K)}{\partial K}\right]_F^\infty - \int_F^\infty h'(K)\frac{\partial V^{pay}(K)}{\partial K}dK$$
$$= h(F)\frac{\partial V^{rec}(F)}{\partial K} - \cancel{h(0)\frac{\partial V^{rec}(0)}{\partial K}}^{0} - [h'(K)V^{rec}(K)]_0^F + \int_0^F h''(K)V^{rec}(K)dK$$
$$+ \cancel{h(\infty)\frac{\partial V^{pay}(\infty)}{\partial K}}^{0} - h(F)\frac{\partial V^{pay}(F)}{\partial K} - [h'(K)V^{pay}(K)]_F^\infty + \int_F^\infty h''(K)V^{pay}(K)dK$$
$$= h(F)\frac{\partial V^{rec}(F)}{\partial K} - h'(F)V^{rec}(F) + \cancel{h'(0)V^{rec}(0)}^{0} + \int_0^F h''(K)V^{rec}(K)dK$$
$$- h(F)\frac{\partial V^{pay}(F)}{\partial K} - \cancel{h'(\infty)V^{pay}(\infty)}^{0} + h'(F)V^{pay}(F) + \int_F^\infty h''(K)V^{pay}(K)dK$$
$$= -h(F)\left[\frac{\partial V^{pay}(F)}{\partial K} - \frac{\partial V^{rec}(F)}{\partial K}\right] + h'(F)[V^{pay}(F) - V^{rec}(F)]$$
$$+ \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK$$

  - By put call parity, $[V^{pay}(F) - V^{rec}(F)] = 0$
  - $\frac{dV^{pay}(K)}{dK} - \frac{dV^{rec}(K)}{dK} = -D(0,T)IRR(S)$
  - $\boldsymbol{V_0 = D(0,T)g(F) + h'(F)[V^{pay}(F) - V^{rec}(F)] + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK}$
  - F is the IRS, g(F) = F observe rate and pay it
    - CMS caplet payoff = F – L, no receiver swap as integrate from L-inf,

$$h'(F)[V^{pay}(F) - V^{rec}(F)] = h'(F)[V^{pay}(F)] \Rightarrow h'(F)[V^{pay}(F)] + \int_L^\infty h''(K)V^{pay}(K)dK$$

    - CMS floorlet payoff = F – L, no payer swap as integrate from 0-L,

$$-h'(F)[V^{Rec}(F)] + \int_0^L h''(K)V^{pay}(K)dK$$

  - $\int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK$ => convexity correction

Change of Numeraire Theorem & Convexity Correction

- Single Currency Change of numeraire
  - Convexity correction
  - $E^N[H_T] = E^M\left[H_T * \frac{N_T/N_0}{M_T/M_0}\right] \Rightarrow E^N[H_T] = E^{\cancel{M}}\left[H_T * \frac{dQ^N}{dQ^{\cancel{M}}}\right]$, E under N measure
  - $E^N[H_T] = E^N[H_T]$
- LIBOR in arrears
  - No L0 payment, have L4, Receive same period payment 1 period earlier
  - $V^{LIA}(0) = D_i(0)E^i[L_i(T_i)]$, L_i not a martingale under Q^i as LMM follows W_t^{i+1}
  - $E^i[L_i(T_i)] = E^{i+1}\left[L_i(T_i) * \frac{dQ^i}{dQ^{i+1}}\right]$, $\frac{dQ^i}{dQ^{i+1}} = \frac{D_i(T_i)/D_i(0)}{D_{i+1}(T_{i+1})/D_{i+1}(0)} = \frac{D_i(T_i)/D_{i+1}(T_{i+1})}{D_i(0)/D_{i+1}(0)} = \frac{1+\Delta_i L_i(T_i)}{1+\Delta_i L_i(0)}$, $L_i(T_i) = \frac{1}{\Delta_i} * \frac{D_i(T_i) - D_{i+1}(T_{i+1})}{D_{i+1}(T_{i+1})}$
  - $E^i[L_i(T_i)] = \frac{1}{1+\Delta_i L_i(0)}E^{i+1}[L_i(T_i) * (1 + \Delta_i L_i(T_i))]$
  - $E^i[L_i(T_i)] = \frac{E^{i+1}[L_i(T_i)] + \Delta_i E^{i+1}[L_i(T_i)^2]}{1+\Delta_i L_i(0)}$,
    - **SDE:** $\boldsymbol{dL_i(t) = \sigma_i L_i(t)dW^{i+1}(t)}$
    - Sub $L_i(T) = L_i(0)\exp\left[\left(-\frac{\sigma_i^2}{2}\right)T + \sigma_i W^{i+1}(T)\right]$
  - $E^i[L_i(T_i)] = \frac{L_i(0) + \Delta_i L_i(0)^2 e^{\sigma_i^2 T_i}}{1+\Delta_i L_i(0)}$
  - Convexity correction always > 1, adjust LIBOR in arrears up

The LIBOR-in-arrear caplet contract can be valued as

$$V_0 = D_i(0)\mathbb{E}^i[(L_i(T) - K)^+]$$
$$= D_i(0)\mathbb{E}^{i+1}\left[\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}}(L_i(T) - K)^+\right]$$
$$= D_i(0)\mathbb{E}^{i+1}\left[\frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)}(L_i(T) - K)^+\right]$$
$$= D_{i+1}(0)\mathbb{E}^{i+1}\left[(1 + \Delta_i L_i(T)) \cdot (L_i(T) - K)^+\right]$$
$$= D_{i+1}(0)\left\{\mathbb{E}^{i+1}[(L_i(T) - K)^+] + \Delta_i\mathbb{E}^{i+1}[L_i(T)(L_i(T) - K)^+]\right\}$$
$$= D_{i+1}(0)\left[L_i(0)\Phi\left(\frac{\log\frac{L_i(0)}{K} + \frac{\sigma_i^2 T}{2}}{\sigma_i\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{L_i(0)}{K} - \frac{\sigma_i^2 T}{2}}{\sigma_i\sqrt{T}}\right)\right]$$
$$+ \Delta_i D_{i+1}(0)\left[L_i(0)^2 e^{\sigma_i^2 T}\Phi\left(-x^* + 2\sigma_i\sqrt{T}\right) - L_i(0)K\Phi\left(-x^* + \sigma_i\sqrt{T}\right)\right]$$
$$= D_{i+1}(0)\left[L_i(0)\Phi\left(\frac{\log\frac{L_i(0)}{K} + \frac{\sigma_i^2 T}{2}}{\sigma_i\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{L_i(0)}{K} - \frac{\sigma_i^2 T}{2}}{\sigma_i\sqrt{T}}\right)\right]$$
$$+ \Delta_i D_{i+1}(0)\left[L_i(0)^2 e^{\sigma_i^2 T}\Phi\left(\frac{\log\frac{L_i(0)}{K} + \frac{3\sigma_i^2 T}{2}}{\sigma_i\sqrt{T}}\right)\right.$$
$$\left. - L_i(0)K\Phi\left(\frac{\log\frac{L_i(0)}{K} + \frac{\sigma_i^2 T}{2}}{\sigma_i\sqrt{T}}\right)\right] \quad \triangleleft$$

- Cholesky 3rd Variable Coefficient: $\rho_{1,3}, \frac{\rho_{2,3} - \rho_{1,2}\rho_{1,3}}{\sqrt{1-\rho_{1,2}^2}}, \sqrt{1 - \rho_{1,3}^2 - \frac{\rho_{2,3} - \rho_{1,2}\rho_{1,3}}{\sqrt{1-\rho_{1,2}^2}}^2}$
- $FX_t = FX_0 * \frac{D_f(0,T)}{D_d(0,T)} = FX_0 * \frac{(1+domestic)^T}{(1+foreign)^T} = FX_0 e^{(r^d - r^f)T}$, if x rate > y rate, and forward > spot, long x, short forward + y @ forward * D of x, at maturity, y = \$ / D of y, x = forward, profit
- Foreign investor will see this SDE : $dX_t = (r^D - r^F)X_t dt + \sigma X_t dW_t^D$
- $let\ Y_t = \frac{1}{X_t} \Rightarrow f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 \Rightarrow d\frac{1}{X_t} = (r^F - r^D + \sigma^2)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^D$
- $To\ get\ foreign\ measure, starting\ with\ B_t^D and\ dX_t = \mu X_t dt + \sigma X_t dW_t$
  - IF given $\frac{1}{X_t}$: $let\ Y_t = \frac{B_t^D}{X_t} \Rightarrow \frac{1}{X_t}dB_t^D + B_t^D d\left(\frac{1}{X_t}\right) \Rightarrow dY_t = (r^D + \mu)Y_t dt + \sigma Y_t dW_t$
- $let\ Y_t = \frac{B_t^D}{X_t} \Rightarrow \frac{1}{X_t}dB_t^D - \frac{B_t^D}{X_t^2}dX_t + \frac{1}{2} * \frac{2B_t^D}{X_t^3}(dX_t)^2 \Rightarrow dY_t = (r^D + \sigma^2 - \mu)Y_t dt - \sigma Y_t dW_t$
- $let\ Z_t = \frac{Y_t}{B_t^F} \Rightarrow \frac{1}{B_t^F}dY_t - \frac{Y_t}{B_t^F}dB_t^F \Rightarrow dZ_t = (r^D - r^F + \sigma^2 - \mu)Z_t dt - \sigma Z_t dW_t = -\sigma Z_t\left(dW_t - \frac{(r^D - r^F + \sigma^2 - \mu)}{\sigma}dt\right) = -\sigma Z_t dW_t^F$
- $sub\ dW_t = dW_t^F + \left(\frac{(r^D - r^F + \sigma^2 - \mu)}{\sigma}dt\right)\ to\ dX_t = \mu X_t dt + \sigma X_t dW_t$
- $dX_t = (r^D - r^F + \sigma^2)X_t dt + \sigma X_t dW_t^F, solve\ sde\ for \frac{1}{X_t}, d\frac{1}{X_t} = (r^F - r^D)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^F$
- Domestic investor will see $let\ Y_t = B_t^F X_t \Rightarrow dY_t = (\mu + r^F)Y_t dt + \sigma Y_t dW_t$
- $let\ Z_t = \frac{Y_t}{B_t^D} \Rightarrow dZ_t = (r^F - r^D + \mu)Z_t dt + \sigma Z_t dW_t = -\sigma Z_t\left(dW_t - \frac{(r^F - r^D + \mu)}{\sigma}dt\right) = -\sigma Z_t dW_t^D$
- $sub\ dW_t = dW_t^D + \left(\frac{(r^F - r^D + \mu)}{\sigma}dt\right)\ to\ dX_t = \mu X_t dt + \sigma X_t dW_t$
- $dX_t = (r^D - r^F)X_t dt + \sigma X_t dW_t^D$
- Quanto (if taking foreign perspective, given domestic LIBOR then correlation need be -ve)
  - Interest paid in the wrong currency so need convexity correction
  - Payment period is correct, currency is wrong
  - $E^{i+1,D}[L_i^F(T)] = E^{i+1,F}\left[L_i^F(T) * \frac{dQ^{i+1,D}}{dQ^{i+1,F}}\right] = E^{i+1,F}\left[L_i^F(T) * \frac{D_{i+1}^D(T_{i+1})/D_{i+1}^D(0)}{X_T D_{i+1}^F(T_{i+1})/X_0 D_{i+1}^F(0)}\right]$
  - $E^{i+1,F}\left[L_i^F(T) * \frac{1/F_{T_{i+1}}}{1/F_0}\right], \frac{D}{F} = \frac{1}{F_T}$
  - $\frac{1}{F_T} = \frac{1}{F_0}\exp\left[\left(-\frac{\sigma_x^2}{2}\right)T + \sigma_x W_T^F\right], L_i(T) = L_i(0)\exp\left[\left(-\frac{\sigma_i^2}{2}\right)T + \sigma_i W_T^{i+1}\right]$
  - $E^{i+1,F}\left[L_i^F(0)\exp\left[\left(-\frac{\sigma_i^2}{2}\right)T + \sigma_i W_T^{i+1}\right] * \frac{1}{F_0}\exp\left[\left(-\frac{\sigma_x^2}{2}\right)T + \sigma_x W_T^F\right]/1/F_0\right]$
  - $L_i^F(0)\exp\left[\left(-\frac{\sigma_i^2}{2}\right)T\right]\exp\left[\left(-\frac{\sigma_x^2}{2}\right)T\right] * E^{i+1,F}[\exp[\sigma_i W_T^{i+1}] * \exp[\sigma_x W_T^F]]$
  - Cholesky decomposition: $E^{i+1,F}[\exp[\sigma_i Z_1] * \exp[\sigma_x\rho Z_1 + \sigma_x\sqrt{1-\rho^2}Z_2]]$
  - $E^{i+1,F}[\exp[(\sigma_i + \sigma_x\rho)Z_1]] * E^{i+1,F}[\exp[\sigma_x\sqrt{1-\rho^2}Z_2]]$, because independent
  - $L_i^F(T) = L_i^F(0)e^{\rho\sigma_i\sigma_x T}$
  - If correlation +ve, LIBOR up, foreign currency $\left(\frac{1}{F}\right)$ depreciate (down), I get interest but when convert back to domestic to pay client, worth lesser, need more money
  - If correlation is -ve, LIBOR up, foreign currency appreciate, hedge lesser because money converted to domestic current worth more, don't have to top up

Short Rate Models and Term Structure

- $dr_t = \mu_t + \sigma_t dW^*$
- Drift in short rate process
  - Short rate process: $r_t = r_0 + \mu t$
  - Spot curve in terms of short rate: R(t, T) = $\frac{1}{T-t}\log(D(t,T)) = \frac{1}{2}\mu(T-t) + r_t$
  - Exponential function is convex function, therefore expectation of exp function > exp of expectation of x, that's why when doing binomial trees, the expectation of discount factor > expectation of rate then discount, so rate becomes smaller and not martingale
- Volatility in short rate process
  - Short rate process: $r_t = r_0 + \sigma W_t^*$
  - $\int_t^T r_u du = r_0(T-t) + \sigma\int_t^T W_u^* dW_u = r_t(T-t) + \sigma\int_t^T W_u^* - W_t^* dW_u$
  - $\int_t^T r_u du \sim N\left(r_t(T-t), \frac{\sigma^2}{3}(T-t)^3\right)$
  - $D(t,T) = E_t^*\left[e^{-\int_t^T r_u du}\right] = e^{-r_t(T-t) + \frac{\sigma^2}{6}(T-t)^3}$, recall MGF $\theta = -1$
  - Spot curve in terms of short rate: R(t, T) = $\frac{1}{T-t}\log(D(t,T)) = r_t - \frac{\sigma^2}{6}(T-t)^2$

Ho-Lee & Hull-White Models

- $\int_0^T W_t dt = \int_0^T\int_0^t dW_u\,dt = \int_0^T\int_u^T dt dW_u = \int_0^T(T-u)dW_u = \lim_{n\to\infty}\sum_{i=1}^N(T - t_i)(W_{t,i+1}W_{t,i})$
- Ho-Lee: $dr_t = \theta(t)dt + \sigma dW_t^*, \theta(T) = -\frac{d^2}{dT^2}logD(0,T) + \sigma^2 T$, if is t,T, integral is t,s then t,T
  - $r_t = r_0 + \int_0^t\theta(s)ds + \int_0^t\sigma dW_t^*$
  - $\int_0^T r_u du = \int_0^T r_0 du + \int_0^T\int_0^u\theta(s)\,ds du + \int_0^T\int_0^u\sigma dW_s^*\,du$
  - $= \int_0^T r_0 du + \int_0^T\int_s^T\theta(s)\,duds + \int_0^T\int_s^T\sigma dudW_s^*$
  - $= r_0 T + \int_0^T\theta(s)(T-s)\,ds + \int_0^T\sigma(T-s)dW_s^*$
  - $V\left[\int_0^T r_u du\right] = \int_0^T\sigma^2(T-s)^2 ds = \frac{1}{3}\sigma^2 T^3$
  - $D(0,T) = E\left[e^{-\int_0^T r_u du}\right] = E\left[e^{-r_0 T - \int_0^T\theta(s)(T-s)ds - \int_0^T\sigma(T-s)dW_s^*}\right]$
  - $\exp\left[-r_0 T - \int_0^T\theta(s)(T-s)\,ds + \frac{1}{2} * \frac{1}{3}\sigma^2 T^3\right]$
  - $\log D(0,T) = -r_0 T - \int_0^T\theta(s)(T-s)\,ds + \frac{1}{6}\sigma^2 T^3$
  - $\frac{d}{dt}\log D(0,T) = -r_0 - \left[\cancel{\frac{\theta(T)(T-T)dT}{dT}} - \cancel{\frac{\theta(0)(T-0)d0}{dt}} + \int_0^T\theta(s)(1)\,ds\right] + \frac{1}{2}\sigma^2 T^2$
  - $\frac{d^2}{dt^2}\log D(0,T) = -\left[\frac{\theta(T)dT}{dT} - \cancel{\frac{\theta(0)d0}{dt}} + \cancel{\int_0^T 0\,ds}\right] + \sigma^2 T$
  - Reconstructing discount factor
    - $e^{-R(t,T)(T-t)} = D(t,T) = e^{A(t,T) - r_t B(t,T)}$
    - $A(t,T) = -\int_t^T\theta(s)(T-s)\,ds + \frac{1}{6}\sigma^2(T-t)^3, B(t,T) = T - t$
    - $f_t(t,x) = e^{A(t,T) - xB(t,T)}\left[\frac{dA(t,T)}{dt} - x * \frac{dB(t,T)}{dt}\right]$
    - $f_x(t,x) = e^{A(t,T) - xB(t,T)}[-B(t,T)]$
    - $f_{xx}(t,x) = e^{A(t,T) - xB(t,T)}[B(t,T)^2]$
    - $\frac{dA(t,T)}{dt} = \theta(t)(T-t) - \frac{\sigma^2(T-t)^2}{2}, \frac{dB(t,T)}{dt} = -1$
    - $dD(t,T) = f_t(t,r_t)dt + f_x(t,r_t)dr_t + \frac{1}{2}f_{xx}(t,r_t)(dr_t)^2$
    - $D(t,T)\left[\frac{dA(t,T)}{dt} - r_t * \frac{dB(t,T)}{dt}\right]dt - D(t,T)(T-t)(\theta(t)dt + \sigma dW_t^*) + \frac{1}{2}D(t,T)(T-t)^2\sigma^2 dt$
    - $D(t,T)\left[\cancel{\theta(t)(T-t)} - \cancel{\frac{\sigma^2(T-t)^2}{2}} + r_t\right]dt - D(t,T)(T-t)(\cancel{\theta(t)dt} + \sigma dW_t^*) + \cancel{\frac{1}{2}D(t,T)(T-t)^2\sigma^2 dt}$
    - $D(t,T)r_t dt - (T-t)\sigma D(t,T)dW_t^*$
  - Discount factor process fixes the start (D(0,T)) and end (1), the closer to maturity T, the smaller the effect of the stochastic coefficient D(T,t)
  - Affine I/R model, zero coupon bond = $D(t,T) = e^{A(t,T) - r_t B(t,T)}$for some deterministic functions of A & B of t &T only => zero rates are affine functions of spot rates
  - Binomial tree: $D(0,3) = E^*[D(0,1) * D(1,2) * D(2,3)]$
    - $= D(0,1)E^*[D(1,2) * D(2,3)] = D(0,1)E^*[D(1,2) * E_1^*[D(2,3)]]$
    - $D(0,1)E^*\left[D_{up}(1,2) * E_1^*[D(2,3)] + D_{down}(1,2) * E_1^*[D(2,3)]\right]$
- Hull-White: $dr_t = k(\theta(t) - r_t)dt + \sigma dW_t^*$
- $k = mean\ reversion\ speed, \theta = long-run\ average$
  - $(\theta - r_t) = 0, flat, (\theta - r_t) > 0, drift\ up, (\theta - r_t) < 0, drift\ down$
- Vasicek SOLVE SDE: $X_t = e^{kt}r_t$
  - $de^{kt}r_t = ke^{kt}r_t dt + e^{kt}dr_t$
  - $de^{kt}r_t = k\theta e^{kt}dt + \sigma e^{kt}dW_t$
  - $\int_0^t de^{ku}r_u = \int_0^t k\theta e^{ku}du + \int_0^t\sigma e^{ku}dW_u$
  - $e^{kt}r_t - e^{k0}r_0 = \theta(e^{kt} - e^0) + \sigma\int_0^t e^{ku}dW_u, divide\ by\ e^{kt}$
  - $r_t = r_0 e^{-kt} + \theta(1 - e^{-kt}) + \sigma\int_0^t e^{k(u-t)}dW_u$
  - $E[r_t] = r_0 e^{-kt} + \theta(1 - e^{-kt})$
  - $V[r_t] = E\left[\left(\sigma\int_0^t e^{k(u-t)}dW_u\right)^2\right] = E\left[\sigma^2\int_0^t e^{2k(u-t)}du\right] = \frac{\sigma^2}{2k}(1 - e^{-2kt})$
  - $\int_0^T r_t dt = \int_0^T r_0 e^{-kt}dt + \int_0^T\theta(1 - e^{-kt})dt + \int_0^T\int_0^t\sigma e^{k(u-t)}dW_u^*\,dt$
  - $\frac{r_0}{k}(1 - e^{-kT}) + \theta T - \frac{\theta}{k}(1 - e^{-kT}) + \int_0^T\int_u^T\sigma e^{k(u-t)}\,dtdW_u^*$
  - $\frac{r_0}{k}(1 - e^{-kT}) + \theta T - \frac{\theta}{k}(1 - e^{-kT}) + \int_0^T\left[-\frac{\sigma}{k}e^{k(u-t)}\right]T, u\,dW_u^*$
  - $\frac{r_0}{k}(1 - e^{-kT}) + \theta T - \frac{\theta}{k}(1 - e^{-kT}) + \frac{\sigma}{k}\int_0^T(1 - e^{k(u-T)})\,dW_u^*$
  - $E\left[\int_0^T r_t dt\right] = \frac{r_0}{k}(1 - e^{-kT}) + \theta T - \frac{\theta}{k}(1 - e^{-kT})$
  - $V\left[\int_0^T r_t dt\right] = E\left[\left(\frac{\sigma}{k}\int_0^T(1 - e^{k(u-T)})\,dW_u^*\right)^2\right] = \frac{\sigma^2}{k^2}\int_0^t(1 - e^{-2kt})^2\,du = \frac{\sigma^2}{k^2}\int_0^t(1 - 2e^{k(u-T)} + e^{2k(u-T)})\,du\right] = \frac{\sigma^2}{k^2}\left[T - \frac{2}{k}(1 - e^{-kt}) + \frac{1}{2k}(1 - e^{-2kT})\right]$
  - $D(0,T) = \exp\left[-\frac{r_0}{k}(1 - e^{-kT}) - \theta T + \frac{\theta}{k}(1 - e^{-kT}) + \frac{1}{2}\frac{\sigma^2}{k^2}\left[T - \frac{2}{k}(1 - e^{-kt}) + \frac{1}{2k}(1 - e^{-2kT})\right]\right]$