Probability Theory and Lattice Models

- **Expectation**
    - E[XY] != E[X]E[Y] **unless** correlation is 0
        - Cov(X,Y) = E[XY] – E[X]E[Y] = rho * sx * sy
- **Variance**
    - V[aX+b]
        - b constant doesn't affect fluctuation, only moves central location up/down, a multiplies the effect of the variance so a^2
    - V[X+Y] = a^2V[X] + b^2V[Y] + 2abCOV(X,Y)
- **Mode** is differentiate one time set to 0 and solve aka max
- **Skewness**
    - 3rd central moment / sd ^ 3
    - Skewness of normal distribution = 0
    - Positive skewness -> a lot of large number
    - Negative skewness -> a lot of small numbers (more common in returns distribution, goes up by abit falls by alot)
- **Kurtosis**
    - 4th central moment / sd ^ 4
    - Kurtosis normal distribution = 3, Excess kurtosis = kurtosis - 3
    - +ve -> **narrow peak**, **wider tail**, if move will move larger than normal, probability of extreme case is higher
- **MGF**

$$M_X(\theta) = E[e^{\theta X}] = 1 + \theta E[X] + \frac{\theta^2 E[X^2]}{2!} + \frac{\theta^3 E[X^3]}{3!} + \frac{\theta^4 E[X^4]}{4!}$$

- Differentiate MGF and set $\theta = 0$, one time get mean, second time get $E[X^2]$
- MGF normal Distribution, X ~ N (μ, σ^2)
    - $M_X(\theta) = E[e^{\theta X}] = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2} / \int_{-\infty}^{\infty} e^{\theta X}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$
- MGF standard normal, Z ~ N (0, 1)
    - $M_Z(\theta) = e^{\frac{\theta^2}{2}}$
- MGF lognormal where Y = e^X, ln(Y) ~ N (μ, σ^2)
    - E[Y] = $E[e^X] = e^{\mu + \frac{1}{2}\sigma^2}$, $\theta = 1$

- **Leibnitz Rule** – differentiate an integral wrt x
    - $\int_x^{x^2}\frac{\sin(xt)}{t}dt$
    - $V(x) = x^2,\ u(x) = x, f(x,t) = \frac{\sin(xt)}{t}$
    - $f(x, v(x)) * v'(x) - f(x, u(x)) * u'(x) + \frac{df(x,t)}{dx}$
    - $\frac{\sin(x^3)}{x^2}(2x) - \frac{\sin(x^2)}{x}(1) + \int_x^{x^2}\frac{t\cos(xt)}{t}dt$

Binomial Trees and Risk neutral

- **Binomial Trees (2 State Model)**
    - No arbitrage, value of portfolio has to be worth the same
        - Δ * 110 (up) – 1 * 10 (payoff) = Δ * 90 (down) – 1 * 0 (payoff)
        - Δ = 0.5
        - Portfolio value at T1 = 100 * 0.5 - C_0
    - Risk neutral probability
        - E[S_1] = 110p* + 90(1 – p*) = S_0
        - E[C_1] = payoff x p*, use this if know current option price
- **Binomial Trees (3 State Model)**
    - Take extreme states, calculate hedge amount Δ
        - Δ * down –payoff < Δ * Current value - C_0 < Δ * up –payoff
    - Risk neutral probability
        - P1 + P2 + P3 = 1
        - 0 < P1 < 1
- **Radon-Nikodym Derivative**
    - $\frac{dQ}{dP}$, $E^Q[S_2] = E^P\left[S_2 * \frac{dQ}{dP}\right]$ = all possible S_2* P prob * $\frac{dQ}{dP}$
    - Trick: (previous prob * 1st up) + (previous prob * 1st down) / 2
- **Non-Zero Interest Rates**
    - If inequality is not fulfilled, arbitrage occurs:
        - where <Sd, borrow money and buy stock today
        - >Su short the stock and put in risk free then redeem the riskfree
        - Su, Sd decided by ourselves in the binomial model, how much you want the stock to go up and down
    - P* = $\frac{(1+r)-d}{u-d}$, discounting = $\frac{1}{(1+r)^n}$, down = $\frac{1}{u}$
    - 121, 1331, 14641, 15101051, 1615201561
- **European and American options (put = K – S)**
    - EU
        - Calculate payoff at end
        - Discounting to t0 * [p* (payoff) + (1 - p*)(payoff)]
    - American Puts (Puts more expensive than European, calls same unless dividends)
        - Calculate payoff
        - Max{discounting * [p* (payoff) + (1 - p*)(payoff)], (strike – price)+}
        - Repeat for all time steps and possible payoffs
- **Law of iterated expectation**
    - $E_1[E_2[S_3]]$, given S_2 = x, what is Expectation of S_3, calculate from inside out
- **Discounted stock price is a martingale**
    - $E_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] = E_n\left[\frac{S_n}{(1+r)^{n+1}}\frac{S_{n+1}}{S_n}\right] = \frac{S_n}{(1+r)^n}\frac{1}{(1+r)}E_n\left[\frac{S_{n+1}}{S_n}\right] = \frac{S_n}{(1+r)^n}\frac{p\,X\,u + q\,X\,d}{(1+r)} = \frac{S_n}{(1+r)^n}$, $E_n\left[\frac{S_{n+1}}{S_n}\right] = \frac{p\,X\,u\,X\,S_n + q\,X\,d\,X\,S_n}{S_n}$

Brownian Motion and Martingale

- $W_0 = 0, W_t \sim N(0,t), W_t - W_s \sim N(0, t-s)$, continuous and no jumps, stationary and independent increments, increments follow normal distribution
- $E[W_t^4] = E[t^2X^4] = 3t^2$
- $W_t - W_s \sim N(0, t-s) = W_t - W_s \sim \sqrt{t-s}X$
- **Properties**
    - $E[W_t] = 0,\ W_t \sim N(0,t)$
    - $E[W_t^2] = t$, variance = t
    - $E[W_t - W_s] = 0$
    - $V[W_t - W_s] = t - s$
    - $Cov(W_s, W_t) = s, s < t, Cov(W_s, W_t) = Cov(W_s, W_t - W_s + W_s) = Cov(W_s, W_t - W_s) + Cov(W_s, W_s) = 0 + s$, correlation = cov / sdsd
- **Up and down Brownian probabilities**
    - $P(W_2 < 0|W_1 > 0)$
        - $P(W_2 < W_1|W_1 > 0)XP(|W_2 - W_1| > |W_1 - W_0||W_1 > 0)$
        - Step down (0.5) AND magnitude must be big enough (0.5) = 1/4
    - $P(W_2 > 0|W_1 > 0)$
        - $P(W_2 > W_1|W_1 > 0)XP(W_2 < W_1, |W_2 - W_1| < |W_1 - W_0||W_1 > 0)$
        - ½ + ½*½ = ¼
    - $P(W_1 X W_2 > 0) = P(W_1 > 0, W_2 > 0) + P(W_1 < 0, W_2 < 0)$
        - $P(W_2 > 0|W_1 > 0)\,P(W_1 > 0) + P(W_2 < 0|W_1 < 0)\,P(W_1 < 0)$
        - ¾ * ½ + ¾ * ½ = ¾
    - $P(W_2 < 0 \cap W_1 < 0)$
        - $P(W_2 < 0|W_1 < 0)\,P(W_1 < 0)$
        - ¾ * ½ = 3/8
- **Markovian**
    - $E_s[W_t] = E_s[W_t - W_s + W_s] = E_s[W_t - W_s] + E_s[W_s] = W_s$
- **Brownian Martingale**
    - Show that $E_s[brownian\ function\ at\ time\ t] = brownian\ function\ at\ time\ s$
    - Use $W_t = (W_t - W_s) + W_s$
    - $E[W_t^{odd}] = 0$, 1 3 15 105

Stochastic integrals and Ito formula

- $S_t = S_0 + \mu t + \sigma W_t$
- $E[S_t] = S_0 + \mu t$
- $V[S_t] = \sigma^2 t$
- $dS_t = \mu dt + \sigma dW_t$
- **Properties of stochastic integrals**
    - $\int_0^t f(u, W_u)dW_u$
    - E[I] = 0
    - $E[I^2] = E\left[\int_0^t f(u, W_u)dW_u{}^2\right] = E\left[\int_0^t f(u, W_u)^2 du\right]$
    - ito isometry: $E\left[\int_0^t X_t dW_t{}^2\right] = E\left[\int_0^t X_t{}^2 dt\right] = \int_0^t E[X_t{}^2]dt$
- **ito's formula**
    - $dY_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2$
    - $dY_t = g_t(t, X_t)dt + g_x(t, X_t)dX_t + \frac{1}{2}g_{xx}(t, X_t)(dX_t)^2$
    - $\int_0^T W_u dW_u, let X_t = W_t^2 \mid \int_0^T dW_u^2, let X_t = W_t^2$
- **Product/Quotient Rules**
    - $d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$
    - $d\left(X_t, \frac{1}{Y_t}\right) = \frac{1}{Y_t}dX_t - \frac{X_t}{Y_t^2}dY_t + \frac{X_t}{Y_t^3}(dY_t)^2 - \frac{1}{Y_t^2}dX_t dY_t$
    - $dX_t = f_w(\ )dW_t + f_z(\ )dZ_t + \frac{1}{2}[f_{ww}(\ )(dW_t)^2 + 2f_{wz}(\ )dW_t dZ_t + f_{zz}(\ )(dZ_t)^2]$ dt ρ dt dt
      2. (a) Let W_t denote a standard Brownian motion. Derive the stochastic differential equations
        - For Wt^2/Zt

Stochastic Differential Equations

- **SDE and martingale**
    - As long as got time component aka coefficient of dt != 0 then is not martingale
- $\phi: PDF, \Phi: CDF, \Phi(\infty) = 1$
- **Bachelier Model**
    - Stock price process follows normal distribution, lack of lower bound 0, can go negative, only used for interest rates
    - $S_t = S_0 + \sigma W_t, W_t \sim N(0,T)$
    - $\boldsymbol{SDE: dS_t = \sigma dW_t}$
    - $S_t \sim N(S_0, \sigma^2 T)$
    - **VALUE OF CALL**
    - $V_c = e^{-rT}E[(S_t - K)^+] = e^{-rT}E[(S_0 + \sigma W_t - K)^+]$
        - $\frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[(S_0 + \sigma\sqrt{T}x - K)^+]e^{-\frac{x^2}{2}}dx$, Since normal distribution
        - Inequality to satisfy, $S_0 + \sigma\sqrt{T}x - K > 0 => x > \frac{K - S_0}{\sigma\sqrt{T}} = x^*$
        - $\frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}(S_0 + \sigma\sqrt{T}x - K)e^{-\frac{x^2}{2}}dx$, (for cash or nothing (…) is 1, for asset or nothing (…) is S_t)
        - $\frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}(S_0 - K)e^{-\frac{x^2}{2}}dx + \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}(\sigma\sqrt{T}x)e^{-\frac{x^2}{2}}dx$
        - $(S_0 - K)e^{-rT}[\Phi(\infty) - \Phi(x^*)] - \frac{e^{-rT}\sigma\sqrt{T}}{\sqrt{2\pi}}[e^{-\frac{x^2}{2}}](\infty, x^*)$
        - $e^{-rT}\left[(S_0 - K)\Phi(-x^*) + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{(-x^*)^2}{2}}\right]$
        - $e^{-rT}\left[(S_0 - K)\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right)\right]$
        -
    - **VALUE OF PUT**
    - $V_p = e^{-rT}E[(K - S_t)^+ = e^{-rT}E[(K - S_0 - \sigma W_t)^+]$
        - $\frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[(K - S_0 - \sigma\sqrt{T}x)^+]e^{-\frac{x^2}{2}}dx$, Since normal distribution
        - Inequality to satisfy, $K - S_0 - \sigma\sqrt{T}x > 0 => x < \frac{K - S_0}{\sigma\sqrt{T}} = x^*$
        - $\frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{x^*}(K - S_0 - \sigma\sqrt{T}x)e^{-\frac{x^2}{2}}dx$
        - $\frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{x^*}(K - S_0)e^{-\frac{x^2}{2}}dx - \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{x^*}(\sigma\sqrt{T}x)e^{-\frac{x^2}{2}}dx$
        - $(K - S_0)e^{-rT}[\Phi(x^*) - \Phi(-\infty)] + \frac{e^{-rT}\sigma\sqrt{T}}{\sqrt{2\pi}}[e^{-\frac{x^2}{2}}](x^*, -\infty)$
        - $e^{-rT}\left[(K - S_0)\Phi(x^*) + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{x^{*2}}{2}}\right]$
        - $e^{-rT}\left[(K - S_0)\Phi\left(\frac{K - S_0}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{K - S_0}{\sigma\sqrt{T}}\right)\right]$
        - For ATM options, $S_0 = K, V_c = e^{-rT}\sigma\sqrt{\frac{T}{2\pi}}$
- **Black Scholes**
    - $S_t = S_0\exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_t\right], W_t \sim N(0,T)$
    - $\frac{S_t}{S_0} = \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma N(0,T)\right]$
    - $\log\left(\frac{S_t}{S_0}\right) = \left(r - \frac{\sigma^2}{2}\right)T + \sigma N(0,T) \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$
    - **SDE:** $\boldsymbol{dS_t = rS_t dt + \sigma S_t dW_t}$
        - Solve SDE by letting $X_t = log(S_t)$
    - **VALUE OF CALL**
        - $V_c = e^{-rT}E[(S_t - K)^+] = \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x} - K)^+ e^{-\frac{x^2}{2}}dx$
        - Inequality **to get payoff!**, $S_T - K > 0 => x > \frac{\log\left(\frac{K}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^*$
        - $\frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}(S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x} - K)e^{-\frac{x^2}{2}}dx$
        - $\frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}(S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x}e^{-\frac{x^2}{2}}dx - \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}Ke^{-\frac{x^2}{2}}dx$
        - $\frac{S_0 e^{-\frac{\sigma^2}{2}T}}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^{-\frac{x^2 - 2\sigma\sqrt{T}x}{2}}dx - \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^{-\frac{x^2}{2}}dx$
        - $\frac{S_0 e^{-\frac{\sigma^2}{2}T}}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^{-\frac{x^2 - 2\sigma\sqrt{T}x + \sigma^2 T - \sigma^2 T}{2}}dx - \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^{-\frac{x^2}{2}}dx$
        - $\frac{S_0 \cancel{e^{-\frac{\sigma^2}{2}T}}}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^{-\frac{(x - \sigma\sqrt{T})^2}{2}}\cancel{e^{\frac{\sigma^2 T}{2}}}dx - \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^{-\frac{x^2}{2}}dx$
        - $\frac{S_0}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^{-\frac{(x - \sigma\sqrt{T})^2}{2}}dx - \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^{-\frac{x^2}{2}}dx$
        - $S_0[\Phi(\infty) - \Phi(x^* + \sigma\sqrt{T})] - Ke^{-rT}\Phi(-x^*)$
        - $S_0[\Phi(-x^* + \sigma\sqrt{T})] - Ke^{-rT}\Phi(-x^*)$
        - $S_0\Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$
    - **VALUE OF PUT**
        - $V_p = e^{-rT}E[(K - S_t)^+ = \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(K - S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x})^+ e^{-\frac{x^2}{2}}dx$
        - Inequality to satisfy, $K - S_T > 0 => x > \frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^*$
        - $\frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{x^*}(K - S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x})e^{-\frac{x^2}{2}}dx$
        - $-\frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{x^*}(S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x}e^{-\frac{x^2}{2}}dx + \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{x^*}Ke^{-\frac{x^2}{2}}dx$
        - $-\frac{S_0 e^{-\frac{\sigma^2}{2}T}}{\sqrt{2\pi}}\int_{-\infty}^{x^*}e^{-\frac{x^2 - 2\sigma\sqrt{T}x}{2}}dx + \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{x^*}e^{-\frac{x^2}{2}}dx$
        - $-\frac{S_0 e^{-\frac{\sigma^2}{2}T}}{\sqrt{2\pi}}\int_{-\infty}^{x^*}e^{-\frac{x^2 - 2\sigma\sqrt{T}x + \sigma^2 T - \sigma^2 T}{2}}dx + \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{x^*}e^{-\frac{x^2}{2}}dx$
        - $-\frac{S_0 \cancel{e^{-\frac{\sigma^2}{2}T}}}{\sqrt{2\pi}}\int_{-\infty}^{x^*}e^{-\frac{(x - \sigma\sqrt{T})^2}{2}}\cancel{e^{\frac{\sigma^2 T}{2}}}dx + \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{x^*}e^{-\frac{x^2}{2}}dx$

    - $-\frac{S_0}{\sqrt{2\pi}}\int_{-\infty}^{x^*} e^{-\frac{(x-\sigma\sqrt{T})^2}{2}}dx + \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{x^*} e^{-\frac{x^2}{2}}dx$
    - $Ke^{-rT}\Phi(-x^*) - S_0\left[\Phi\left(x^* + \sigma\sqrt{T}\right) - \Phi(\infty)\right]$
    - $Ke^{-rT}\Phi(-\mathrm{x}^*) - S_0\left[\Phi\left(-\mathrm{x}^* - \sigma\sqrt{T}\right)\right]$
    - $Ke^{-rT}\Phi\left(-\frac{\log\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - S_0\Phi\left(-\frac{\log\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$

- **Black Model / Black76**
  - r is not included because when pricing the forward price, the drift would have been taken into account already
  - if not $F_t = S_t e^{r(T-t)}$
    - SDE: $dF_t = (\mu - r)F_t dt + \sigma F_t dW_t$
  - $\boldsymbol{SDE: dF_t = \sigma F_0 dW_t}$
    - Solve SDE by letting $X_t = log(F_t)$
    - $F_t = F_0 e^{\left[\left(-\frac{\sigma^2}{2}\right)T + \sigma W_T\right]}$
  - $e^{-rT}\left[F_0\Phi\left(-\frac{\log\left(\frac{F_0}{K}\right)+\left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - K\Phi\left(-\frac{\log\left(\frac{F_0}{K}\right)-\left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right]$
- **Mean Reverting Process – Vasicek Model**
  - $\boldsymbol{SDE: dr_t = k(\theta - r_t)dt + \sigma W_t}$
  - $k = mean\ reversion\ speed, \theta = long - run\ average$
    - $(\theta - r_t) = 0, flat$
    - $(\theta - r_t) > 0, drift\ up$
    - $(\theta - r_t) < 0, drift\ down$
  - SOLVE SDE: $X_t = e^{kt}r_t$
    - $de^{kt}r_t = ke^{kt}r_t dt + e^{kt}dr_t$
    - $de^{kt}r_t = k\theta e^{kt}dt + \sigma e^{kt}dW_t$
    - $\int_0^t de^{ku}r_u = \int_0^t k\theta e^{ku}du + \int_0^t \sigma e^{ku}dW_u$
    - $e^{kt}r_t - e^{k0}r_0 = \theta(e^{kt} - e^0) + \sigma\int_0^t e^{ku}dW_u, divide\ by\ e^{kt}$
    - $r_t = r_0e^{-kt} + \theta(1 - e^{-kt}) + \sigma\int_0^t e^{k(u-t)}dW_u$
  - $E[r_t] = r_0e^{-kt} + \theta(1 - e^{-kt})$
  - $V[r_t] = E\left[\left(\sigma\int_0^t e^{k(u-t)}dW_u\right)^2\right] = E[\sigma^2\int_0^t e^{2k(u-t)}du] = \frac{\sigma^2}{2k}(1 - e^{-2kt})$
  - $r_t \sim N\left(r_0e^{-kt} + \theta(1 - e^{-kt}), \frac{\sigma^2}{2k}(1 - e^{-2kt})\right)$

Valuation Framework and Stochastic Volatility Models

- Displaced-Diffusion Model
  - $dF_t = \sigma F_0 dW_t$
  - $d(F_t + \alpha) = \sigma(F_0 + \alpha)dW_t$
  - $dF_t = \sigma(F_0 + \alpha)dW_t$
  - $dF_t = \sigma[\beta F_t + (1 - \beta)F_0]dW_t, where\ 1st\ term\ is\ lognormal, 2nd\ is\ Normal$
  - Allows us to adjust how log normal of a distribution we want by adjusting beta, higher beta more log normal because Ft follows lognormal
    - Solve SDE: $X_t = \log[\beta F_t + (1 - \beta)F_0]$
    - $f'(F_t) = \frac{\beta}{[\beta F_t + (1-\beta)F_0]}, f''(F_t) = -\frac{\beta^2}{[\beta F_t + (1-\beta)F_0]^2}$
    - $\int_0^T dX_t = \int_0^T \beta\sigma dW_t - \int_0^t \frac{1}{2}\beta^2\sigma^2 dt$
    - $X_t - X_0 = \beta\sigma W_T - \frac{1}{2}\beta^2\sigma^2 T,$ $\boldsymbol{simplify\ denominator\ 1st}$!
    - $F_t = \frac{F_0}{\beta}\exp\left[\beta\sigma W_T - \frac{1}{2}\beta^2\sigma^2 T\right] - \frac{1-\beta}{\beta}F_0$
  - Comparing with black model: $F_t = F_0 e^{\left[\left(-\frac{\sigma^2}{2}\right)T + \sigma W_T\right]}$
    - To turn black model to displaced diffusion,
      - Black(F, K, σ, T)
      - DD($\frac{F_0}{\beta}, K + \frac{1-\beta}{\beta}F_0, \sigma\beta$, T)

Equivalent Martingale Measure

- Girsanov
  - If W_t is a standard Brownian motion under P then it becomes Brownian motion with drift coefficient $-k$ under Q ($W_t^* = W_t + kt$)
    - $E^Q[W_t^*] = 0$
    - $E^Q\left[e^{\sigma W_t^*}\right] = e^{\frac{1}{2}\sigma^2 t}$
    - $E^Q\left[e^{\sigma W_{t+s}^* - W_s^*}|s\right] = e^{\frac{1}{2}\sigma^2 t}$
  - Let W_t denote a P Brownian motion:
    - $E^Q[W_t] = E^Q[W_t^* - kt] = -kt$
    - $E^p[W_t^*] = E^p[W_t + kt] = kt$
  - SDE: P: $dX_t = \mu X_t dt + \sigma X_t dW_t$
    - To change measure so that drift coefficient of X_t is v instead of μ
    - $dX_t = vX_t dt - vX_t dt + \mu X_t dt + \sigma X_t dW_t$
    - $dX_t = vX_t dt + \sigma X_t\left(\frac{\mu - v}{\sigma}dt + dW_t\right), let\ k = \left(\frac{\mu - v}{\sigma}\right)$
    - $dW_t^* = dW_t + \left(\frac{\mu - v}{\sigma}\right)dt$
    - $dX_t = vX_t dt + \sigma X_t dW_t^*$, where W_t^* is a Q Brownian motion
  - By Girsanov's Theorem, under the Radon-Nikodym derivative
    - $\frac{dQ}{dP} = e^{-kW_t - \frac{1}{2}k^2 T}\ where\ k = \frac{\mu - r}{\sigma}$
    - there exists a probability measure ˜P which is equivalent to P and ˜Wt is a ˜P-Brownian motion, and we have
  - under risk neutral probability measure, stock price follows the SDE: $dS_t = rS_t dt + \sigma dW_t^*$
    - $\frac{S_t}{B_t}$ is a martingale, applying ito formula, $X_t = \frac{S_t}{B_t}$
    - $fb = -\frac{s}{b^2}, fs = \frac{1}{b}, fss = 0, dX_t = -\frac{S_t}{B_t^2}dB_t + \frac{1}{B_t}dS_t$
    - $dX_t = (\mu - r)X_t dt + \sigma X_t dW_t$, since this process is a martingale, rewrite to $dX_t = \sigma X_t\left(\frac{(\mu - r)}{\sigma}dt + dW_t\right) = \sigma X_t dW_t^B$
    - Sub into S_t, $dS_t = \mu S_t dt + \sigma S_t\left(dW_t^* - \frac{(\mu - r)}{\sigma}dt\right)$
- to determine the probability of $\{S_t > K\}$, follow blackscholes and set payoff to 1
- as MM need to always convert to risk neutral measure before valuing

Static Replication

- $V_0 = e^{-rT}h(F) + \int_0^F h''(K)P(K)dK + \int_F^\infty h''(K)C(K)dK$
- h(F) is the payoff, F is forward price, S_0e^{rt}
- integration by parts
  - $\int_0^F uv'd = [uv]limits - \int_0^F u'vd$

Random notes

- Black Scholes Assumptions: underlying is lognormal with constant mean and variance, risk free rate is constant no dividend paid during life of option, short selling is allowed, no risk free arbitrage, trading is possible in continuous time, no transaction cost/tax/trading limits
- lognormal distribution (mode < median < mode). In particular, the median of a lognormal distribution is always below its mean. Since the median is always lying below the mean, it follows that, more often than not, the realised value of a lognormal random variable falls below its expected value.
- When strike is equal to forward value of stock, call and put option are of the same value. But if calls have unlimited upside, why is it priced the same? It is more likely for the call option to expire OTM compared to the put option. This balances out the effect of the call option having no theoretical upperbound. Due to the skewness of lognormal, the moneyness is being taken away from the put option, resulting in C(K+10) > P(K-10)

Valuation Framework and Stochastic Volatility Models

- based on option price can use black Scholes to solve for IV
- volatility smile reflects demand, higher demand => higher option price => higher IV, more demand for downside protection so puts have higher IV than calls
  - commodity is opp, calls higher IV than puts, fear of price surge so need hedge
- Displaced Diffusion (deterministic volatility)
  - Allows us to adjust how log normal of a distribution we want by adjusting beta, higher beta more log normal because Ft follows lognormal
- SABR (stochastic volatility)
  - Alpha (real positive number) changes the level
  - Rho [-1,1] changes the skewness of the returns distribution, negative rho => high volatility when price falls, price of otm puts increases while price of otm calls decreases
  - Nu (real positive number) is the vol of vol, increasing nu increases kurtosis of returns, price of otm calls and puts increases if nu is 0 then volatility becomes deterministic
  - Beta [0,1] is the shape of the returns, beta 0 is normal, 1 is lognormal