Expected Utility Theory

- **Non-Satiation**
  - Wealth = Utility
  - 1. Strictly increasing utility function (Non-satiation, investor always prefer more wealth)
    - Strictly increasing marginal utility (differentiate utility function), slope positive, marginal utility $U'(W) > 0$
  - 2. Marginal utility should decrease with wealth (the richer you are the less additional marginal utility received with every unit increase in wealth) => concave function
    - X which is between 2 points of the concave function > the linear function between the 2 points
    - First derivative of marginal utility must be <= 0
      - Second derivative of utility $U''(W) <= 0$
      - Slope of the slope needs to be gentler
- **Risk Aversion**
  - In a fair lottery, a risk averse investor will be unwilling to accept the fair lottery because current utility >= E[utility of wealth + lottery]
- **Risk Premium**
  - How much is an investor willing to pay to avoid a gamble
  - (Utility – risk premium) = E[utility of wealth + lottery]
  - Taylor polynomial holds if gamble is small relative to existing wealth
  - Investors willing to pay affected by
    - Variance => riskiness
    - $-U''(W)/U'(W)$ = > coefficient of absolute risk aversion (relative risk aversion just * W), willingness to pay in dollar terms
      - Numerator is negative because risk averse, $U''(W)$ is negative
      - Premium will be positive because the minus cancels outs
  - Relative risk aversion can be argued to be a constant because proportional of wealth willing to gamble is the same when wealth changes (willing to pay 1% stays the same as wealth increases)
- **Power utility**
  - Gamma < 1 means utility function negative
  - When gamma = 1
    - $u'(w) = 1/w$
      - $u(w) = \ln(w)$
- **Normal Returns**
  - To calculate utility of random return, start with baseline mu, expected return
    - Taylor series R = mu + (R – mu)
  - Note for this must consider all terms because returns is not a small amount
    - Doesn't apply to quadratic utility because 3^{rd} derivative and so on = 0
      - Not realistic, implies satiation
    - Assume normal distribution
      - Symmetric so all the odd moments are out
      - Kurtosis can ignore all the even moments because they will be functions of variance
      - Stable under addition => add 2 normal distribution still normal
        - BUT unlimited liability because normal distribution is unbounded
    - Opposite is lognormal which has limited liability but not stable under addition, need to **multiply**
- **Indifference curve**
  - indifference curve cannot intersect
    - else it means that 2 portfolios have the same expected returns
  - higher risk aversion steeper convex curve

Efficient Frontier

- **Investment environment**
  - Positive definite: $z'Vz > 0$
    - (1 x n) x (n x n) x (n x 1) > 0
  - Assume no 2 assets are identical
    - Correlation coefficient of 2 assets != 1 or -1
  - Cannot have 2 assets replicate 1 asset
    - $R1 = wR2 + (1-w)R3$
- **Portfolio Weights**
  - Weights can be +ve or -ve
  - Sum of weights = 1, you can only invest your entire wealth
  - Expected return = $w'R$
  - Variance = $w'Vw > 0$
- **Asset Allocation**
  - Fix return find lowest risk
    - Constraints:
      - $w'R = Rp$, mean return of portfolio = Rp
      - $w'e = 1$, weights sum to 1
    - Minimize $w'Vw$
    - Take first derivative = Vw = 0, Vw is a n x 1 matrix
    - Add the 2 constraints to the objective function to minimize
    - Pre multiply R' (1 x n) to w* (n x 1) results is scaler 1 x 1 (order matters for matrix multiplication)
    - Investors only interested in the top halve (efficient frontier), maximize return for given amount of risk
      - For every portfolio on the bottom halve there is a corresponding portfolio in the upper halve
    - Minimum variance frontier consists of portfolio with the lowest risk for that given return (entire curve)
    - If x axis is sd then can superimpose indifference curve until it is tangent to the frontier, this gives the optimal portfolio for risk adverse investors
  - Fix risk find maximum return
  - **Affine Combination** (2 Fund Separation Theorem)
    - If choose 2 portfolios on the minimum variance frontier with sum of weights of each portfolio = 1
    - Then it is also a frontier portfolio
    - More convenient since investor can generate efficient frontier and construct optimal portfolio without knowing R and V for individual assets
  - **Orthogonal Frontier portfolio**
    - If choose portfolio in the efficient front and set covariance = 0, then p2 would lie in the inefficient frontier
    - Because p1 – Rmv > 0
    - If get line tangent to p1 then the corresponding Rp at y intercept = return of p2 which is orthogonal to p1
  - Indifference curve will be on the CAL, it will always be a combination of tangency portfolio and riskless bond
    - Weight between rf and tangency, some money in riskless
    - Else borrow money at riskless
- **Constant Absolute Risk Aversion**
  - If return follow normal distribution, then e ^ return follow lognormal distribution
  - Since exp is increasing and concave, just need too maximize exponents
    - Since exponent contains values not dependent on weights, only need to focus on maximizing the weights
      - Leads to a parabola
  - W* is optimal portfolio that maximizes utility
  - If investor have same absolute coefficient of absolute risk aversion then they will all be willing to invest in a risky asset, initial wealth is irrelevant
  - Cannot use Constant Relative Risk Aversion because result is completely avoid risky portfolio and only invest in riskless asset because CRRA need to avoid investments that have unlimited downside

CAPM

- If all investors agree to hold a portfolio somewhere on the CAL and can borrow and lend at riskfree rate then the tangency portfolio become market portfolio and the CAL line becomes the CML
- Create negative beta by shorting positive beta assets
- Portfolios that are below Rf always have negative beta because Rm – Rf is negative
- SML applies to any asset or portfolio, CML is only combination of portfolio and riskless bond
- Slope of SML = treynor ratio – risk premium to beta, all investments have the same treynor ratio
- Risk factor part 2, sub Bi = COV(Ri, Rm) / COV(Rm, Rm) then will = 0
- CAPM assumes pooling of systematic risk, ignoring industry risk etc
- All portfolio along the horizontal line of the efficient frontier have the same beta => point on the efficient frontier is the one with the lowest idiosyncratic risk
- OLS is reducing VAR(epsilon), variance of idiosyncratic risk
- If alpha, the intercept, is non zero and significant => pricing error for that asset/passive portfolio meaning CAPM is over/under predicting the returns, the asset is over/under priced, see SML
- Zero beta portfolio
  - if there is no riskfree rate then investors will choose some point on the efficient frontier as the 'market' portfolio
  - if take tangent to this 'market' portfolio, the point on the frontier that lies on the same line as the y intercept is the zero beta portfolio because
    - zero beta that is orthogonal to market portfolio has a covariance of 0 with the market portfolio
    - hence if covariance is 0 then beta is 0
- Big v small cap
  - If use std which includes market and size risk vs beta which is only market risk, shows that CAPM doesn't account for the size risk
    - Else the difference in the sd and beta for small cap and big cap should be the same because both std and beta would capture market risk only
    - There exists extra 'systematic' risk that is not related to market
    - This risk only affects small cap and not big cap
    - Hence size risk

Linear Factor Models Performance Measurement

- **SMB** (market cap)
  - Positive small cap, Negative big cap
    - If not significant then mid cap
    - Small cap tend to outperform big cap
- **HML** (book-market, price-book)
  - Positive value, Negative growth
    - If not significant then blended
    - Value then to outperform growth
- If model is correct then intercept coefficient should be 0 for any individual asset / passive portfolio else it is pricing error relative to FF model
- Downside beta
  - B*
    - Passive: Investors loss averse
    - Active: timing of market
- Sharpe Ratio
  - Better for diversified portfolio
  - Any investment with higher idiosyncratic risk becomes less useful
  - Cannot compare investment and portfolio, unless same idiosyncratic risk which will end up being comparing portfolio with portfolio
  - Only if returns follow normal distribution
- Jensens alpha
  - Active portfolio the alpha is the ability to identify abnormal returns due to fund manager's ability to identify over/underpriced assets
- Sortino
  - Can distinguish between asymmetric returns distribution with same variance but different skewness
    - If same variance sharpe same
    - But sortino is better as left skewed (many small/negative returns but once in a while have a big winner) distribution is more favorable than right skew (more positive returns but once in a while portfolio lose all money)

Efficient frontier revisited

- Efficient frontier difficult to estimate mean returns and estimate got standard error of sd/rt(n)
- Black litterman's indirect method can calculate the estimated risk premiums using CARA, market portfolio and covariance of n tradable risky asset
- **FOR EXAM CONSTRUCT P AND Q**
- Coskewness
  - Change in skewness of market when allocating more weight into the asset
  - Bigger gamma, more positively skewed the investment become which is desirable
    - Willing to pay higher price, accept lower mean return for pi2 < 0

Stochastic Discount Factor

- Smaller the delta the more impatient, (discount rate), usually set at 0.99
- Choose initial consumption and portfolio weights that maximize utility
- If U'' < 0 aka strictly concave, **the only required condition**, then => U' is decreasing function so expected marginal utility weighted return decreases as you increase allocation to asset i
  - So in order to equalize marginal utility, reduce i increase other asset weight until equal
- Adjust choices until marginal utility from one unit of consumption at T0 = discounted expected marginal utility of T1 if you invest at T0 instead
- IMRS: investor willingness to transfer between investment and consumption
- Consumption CAPM
  - If numerator is negative then return premium is large
    - Negative Covariance is undesirable, investors aren't willing to pay as much for the asset
    - Low return when marginal utility is high (when existing consumption is low)
    - Positive correlation when return is high and consumption is low (negative beta, counter cyclical)

Consumption	Marginal Consumption	Return	Cov	Remarks
+	-	+	- (undesirable)	Well off when getting more return
-	+	-	- (undesirable)	If poor and really want to receive more consumption, return on asset is low

- HJ bound
  - Lower bound of volatility of pricing kernal
  - Equity premium puzzle
    - Cannot generate equity premium is large enough unless you assume that investors have unreasonably high risk aversion
    - Either super high risk aversion or the HJ bound doesn't hold
- Assumptions
  - Time separable power utility of consumption (CRRA)
  - Lognormal consumption growth
- To Solve equity premium puzzle must change assumption (keep CRRA, change lognormal consumption growth to add v to make probability distribution of consumption growth more left skewed)

Multi-Period Asset Pricing

- Marginal consumption at time t is the discounted expected marginal consumption at t+1 and return in t+1, condition on the returns distribution of t+1

- Price at time t is known so E[|t] the price can remove
- IID, independent so can split the E[XY] because COV(X,Y) = 0, identical so can combine into the product pi

Behavioural Finance

- V: dollar amount, v: excess return
- Bigger b0 more effect of prospect theory
- Lambda: degree of loss aversion, gamma: CRRA, risk aversion

State Prices

- If got more than 3 assets can just create k portfolio with linearly independent payoff
- If market is not complete X may not be invertible
- In complete market can replicate any investment with a combination of the n underlying asset
- Y (kX1) = X (KXN) * N (NX1)
- Initial price: (P1 P2) = P'X^-1, what you pay at time 0 to receive a payoff of 1 at time 1 of that specific state
    - State prices are unique and strictly positive as long as investors are non-satiated.
- Bad: final consumption low, marginal utility of final consumption high, Ms > E[M]
- Risk neutral probability expected return is rf but using physical return probability is higher because of risk premium
- More weight on bad state, less on good state to eliminate the risk premium (radon nikodym)
- Using risk neutral probability take the risk neutral probability * final payoff then discount back to today