Useful Series for Quantitative Finance Interviews

Edmund A. Berry

March 12, 2015

1 Introduction

How to evaluate some useful series.

2 Useful series

Series are frequently useful when evaluating expected values or variances of outcomes of Bernoulli experiments. For example, say you have a fair coin ($p(\text{heads}) = p = 0.5$). How many times do you expect to have to flip your coin before "heads" comes up?

The outcome of observing $N-1$ tails outcomes before finally seeing a heads outcome on toss N has a well-defined probability, $P(N)$:

$$P(N) = (1-p)^{N-1}p \tag{1}$$

This means that the expected value of the number of coin flips is also well-defined:

$$\mathbb{E}[N] = \sum_{n=1}^{\infty} n \cdot P(n) \tag{2}$$

$$\mathbb{E}[N] = \sum_{n=1}^{\infty} n \cdot (1-p)^{n-1}p \tag{3}$$

Let's assume that the coin is fair, so that $p = (1-p) = 0.5$:

$$\mathbb{E}[N] = \sum_{n=1}^{\infty} n \cdot p^n \tag{4}$$

We evaluate this series by trying to establish a recurrence relation:

$$p \cdot \mathbb{E}[N] = \sum_{n=1}^{\infty} n \cdot p^{n+1} \tag{5}$$

$$p \cdot \mathbb{E}[N] = \sum_{n=2}^{\infty} (n-1) \cdot p^n \tag{6}$$

$$\mathbb{E}[N] - p \cdot \mathbb{E}[N] = \left(\sum_{n=1}^{\infty} n \cdot p^n\right) - \left(\sum_{n=2}^{\infty} (n-1) \cdot p^n\right) \tag{7}$$

$$\mathbb{E}[N] \cdot (1-p) = \left(p + \sum_{n=2}^{\infty} n \cdot p^n\right) - \left(\sum_{n=2}^{\infty} (n-1) \cdot p^n\right) \tag{8}$$

$$\mathbb{E}[N] \cdot (1-p) = p + \sum_{n=2}^{\infty} p^n \tag{9}$$

$$\mathbb{E}[N] \cdot (1-p) = \sum_{n=1}^{\infty} p^n \tag{10}$$

This last series is simply an infinite geometric series with first term $a_0 = p$, which we know to be $\frac{p}{1-p}$.

$$\mathbb{E}[N] \cdot (1-p) = \frac{p}{1-p} \tag{11}$$

$$\mathbb{E}[N] = \frac{p}{(1-p)^2} \tag{12}$$

To summarize:

$$\boxed{\sum_{n=1}^{\infty} n \cdot p^n = \frac{p}{(1-p)^2}} \tag{13}$$

Now that we know the expected number of coin flips it will take for use to observe a head, we might want to know the variance:

$$\text{Var}[N] = \mathbb{E}[N^2] - \mathbb{E}[N]^2 \tag{14}$$

We already know $\mathbb{E}[N]^2 = \frac{p^2}{(1-p)^4}$, so we just need to evaluate $\mathbb{E}[N^2]$. As before, this is well-defined:

$$\mathbb{E}[N^2] = \sum_{n=1}^{\infty} n^2 \cdot p^n \tag{15}$$

We take a similar approach to the one we took last time. Namely, we multiply both sides of the above equation by p in an effort to establish a recurrence relation.

$$p \cdot \mathbb{E}[N^2] = \sum_{n=1}^{\infty} n^2 \cdot p^{n+1} \tag{16}$$

$$p \cdot \mathbb{E}[N^2] = \sum_{n=2}^{\infty} (n-1)^2 \cdot p^n \tag{17}$$

$$\mathbb{E}[N^2] - p \cdot \mathbb{E}[N^2] = \left(\sum_{n=1}^{\infty} n^2 \cdot p^n\right) - \left(\sum_{n=2}^{\infty} (n-1)^2 \cdot p^n\right) \tag{18}$$

$$\mathbb{E}[N^2] \cdot (1-p) = \left(p + \sum_{n=2}^{\infty} n^2 \cdot p^n\right) - \left(\sum_{n=2}^{\infty} (n-1)^2 \cdot p^n\right) \tag{19}$$

$$\mathbb{E}[N^2] \cdot (1-p) = p + \sum_{n=2}^{\infty} \left(n^2 - (n-1)^2\right) \cdot p^n \tag{20}$$

$$\mathbb{E}[N^2] \cdot (1-p) = p + \sum_{n=2}^{\infty} (2n-1) \cdot p^n \tag{21}$$

$$\mathbb{E}[N^2] \cdot (1-p) = p + 2\left(\sum_{n=2}^{\infty} n \cdot p^n\right) - \left(\sum_{n=2}^{\infty} p^n\right) \tag{22}$$

We know how to evaluate the first series using Equation 13, and the second series is just an infinite geometric series that starts with p^2.

$$\mathbb{E}[N^2] \cdot (1-p) = p + 2\left(\frac{p}{(1-p)^2} - p\right) - \left(\frac{p^2}{1-p}\right) \tag{23}$$

$$\mathbb{E}[N^2] \cdot (1-p) = \frac{p \cdot (1-p)^2}{(1-p)^2} + 2\left(\frac{p}{(1-p)^2} - \frac{p \cdot (1-p)^2}{(1-p)^2}\right) - \left(\frac{p^2 \cdot (1-p)}{(1-p)^2}\right) \tag{24}$$

$$\mathbb{E}[N^2] \cdot (1-p) = \frac{p \cdot (1-p)^2}{(1-p)^2} + 2\left(\frac{p}{(1-p)^2} - \frac{p \cdot (1-p)^2}{(1-p)^2}\right) - \left(\frac{p^2 \cdot (1-p)}{(1-p)^2}\right) \tag{25}$$

$$\mathbb{E}[N^2] \cdot (1-p) = \frac{p \cdot (p+1)}{(1-p)^2} \tag{26}$$

$$\mathbb{E}[N^2] = \frac{p \cdot (p+1)}{(1-p)^3} \tag{27}$$

$$\tag{28}$$

To summarize:

$$\boxed{\sum_{n=1}^{\infty} n^2 \cdot p^n = \frac{p \cdot (p+1)}{(1-p)^3}} \tag{29}$$

This allows us to evaluate the variance:

$$\mathrm{Var}[N] = \mathbb{E}[N^2] - \mathbb{E}[N]^2 \tag{30}$$

$$\mathrm{Var}[N] = \frac{p \cdot (p+1)}{(1-p)^3} - \frac{p^2}{(1-p)^4} \tag{31}$$

$$\mathrm{Var}[N] = \frac{p \cdot (p+1) \cdot (1-p)}{(1-p)^4} - \frac{p^2}{(1-p)^4} \tag{32}$$

$$\tag{33}$$