Martingales for Quantitative Finance Interviews

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1 Introduction

We begin by defining terms.

A *stochastic process* is a collection of random variables that represent the evolution of a system of random variables over time. There are many kinds of stochastic process that are relevant to quantitative finance. The two kinds that are most likely to appear in a quantitative finance interview are *Markov processes* and *Martingale processes*.

A Markov process satisfies the *Markov property*, namely, that given the present state of the process, all of the past states and all of the future states are independent. Consider a discrete-time stochastic process, X, such that $X = (X_t, t \in T)$, where $T = \mathbb{N}$ is a totally ordered set corresponding to time:

$$\mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = \mathbb{P}(X_n = x_n | X_{n-1}) \tag{1}$$

This leads to the common observation that a Markov process is *memoryless*, since the future of the process is completely independent of its past.

A Martingale process satisfies the *Martingale property*, namely, that all future states of the process have a conditional expected value equal to the current state of the process. Consider, again, a discrete-time stochastic process, Y, such that $Y = (Y_t, t \in T)$, where $T = \mathbb{N}$ is a totally ordered set corresponding to time:

$$\mathbb{E}(Y_{n+1} | Y_n, \ldots, Y_0) = Y_n \tag{2}$$

One should note that a Martingale process does not have to be a Markov process, and a Markov process does not have to be a Martingale process. The rest of this discussion will focus around Martingale processes.

In order to explain Martingales further, we will consider a classic problem known as *Gambler's Ruin* as an example. In this problem, a gambler begins with an initial fortune of i dollars. On each successive game, the gambler either wins one dollar with probability p or loses one dollar with probability $1 - p$. The gambler continues to play game after game until he either accumulates N dollars or loses all of his money. We identify some useful variables and terms below:

- Label the games using T, where $T = \mathbb{N}$ is a totally ordered set.
- Let the outcome of game $t \in T$ be the random variable X_t, such that X_t has allowed values of ± 1.
- As stated above, $\mathbb{P}(X_t = 1) = p$ and $\mathbb{P}(X_t = -1) = 1 - p$
- We describe the gambler's fortune after n games as the time-series $S_n = S_0 + \sum_{t=1}^{n} X_t$, where $S_0 = i$ is the gambler's initial fortune.

We will consider two forms of the Gambler's Ruin problem: one in which the game is *fair* ($p = 0.5$) and one in which the game is *unfair* ($p \neq 0.5$).

The next sections will describe properties of Martingales in more detail. Section 2 will introduce the concept of stopping times. Section 3 will introduce the *optional stopping theorem*. Section 4 will introduce Wald's first equality. Finally, Sections 5 and 6 will apply these concepts and theorems to the Gambler's Ruin example described above.

2 Stopping times

3 Optional stopping theorem

4 Wald's equality

5 Fair Gambler's Ruin

Let's examine the fair ($p = 0.5$) Gambler's Ruin problem described in Section 1. Recall that:

- The gambler begins with a fortune of $S_0 = i$ dollars.
- The gambler plays a series of games, and in each of these games he can either win a dollar with probability $p = 0.5$ or lose a dollar with probability $1 - p = 0.5$.
- The gambler's winnings in game t are X_t.
- The gamber's fortune after n games is $S_n = S_0 + \sum_{t=1}^{n} X_t$.
- The gambler will stop when he runs out of money ($S_N = 0$) or obtains N dollars ($S_N = N$).

First, let's show that $\{S_n\}$ is a martingale:

$$\mathbb{E}[S_n] = S_{n-1} + p + (1-p) \cdot (-1) \tag{3}$$
$$\mathbb{E}[S_n] = S_{n-1} + 2p - 1 \tag{4}$$
$$\mathbb{E}[S_n] = S_{n-1} \tag{5}$$

So $\{S_n\}$ is a martingale.

We would like to evaluate the probability that the gambler accumulates N dollars and does not go bankrupt (p'). We can evaluate that probability as follows:

$$\mathbb{E}[S_n] = p' \cdot N + (1-p') \cdot (0) \tag{6}$$
$$\mathbb{E}[S_n] = p' \cdot N \tag{7}$$

The fact that $\{S_n\}$ is a martingale allows us to evaluate this probability using the optional stopping theorem:

$$\mathbb{E}[S_n] = \mathbb{E}[S_0] \tag{8}$$
$$\mathbb{E}[S_n] = i \tag{9}$$

Combining this information, we find:

$$\mathbb{E}[S_n] = p' \cdot N = i \tag{10}$$
$$p' = \frac{i}{N} \tag{11}$$

This makes some intuitive sense. The larger the gambler's initial fortune (i), the less likely he is to go bankrupt. Similar, the larger his goal (N), the less likely he is to achieve it.

We would also like to find the expected number of games (τ) it will take the gambler to either go bankrupt ($S_\tau = 0$) or reach his goal ($S_\tau = N$). To do this, we must prove that $\{S_n^2 - n\}$ is also a martingale:

$$\mathbb{E}[S_{n+1}^2 - (n+1)] = \frac{1}{2}\left(S_n + 1\right)^2 + \frac{1}{2}\left(S_n - 1\right)^2 - (n+1) \tag{12}$$
$$\mathbb{E}[S_{n+1}^2 - (n+1)] = \frac{1}{2}\left(2S_n^2 + 2\right) - (n+1) \tag{13}$$
$$\mathbb{E}[S_{n+1}^2 - (n+1)] = S_n^2 - n \tag{14}$$

So $\{S_n^2 - n\}$ is also a martingale.

Since $\{S_n^2 - n\}$ is a martingale, we will treat τ as a stopping time. Let's evaluate the expectation value of $S_n^2 - n$ at the stopping time, τ:

$$\mathbb{E}[S_\tau^2 - \tau] = \mathbb{E}[S_\tau^2] - \mathbb{E}[\tau] \tag{15}$$
$$\mathbb{E}[S_\tau^2 - \tau] = \mathbb{E}[p' \cdot \left(N^2\right) + (1 - p') \cdot (0)] - \mathbb{E}[\tau] \tag{16}$$
$$\mathbb{E}[S_\tau^2 - \tau] = p' \cdot \left(N^2\right) - \mathbb{E}[\tau] \tag{17}$$

We have already solved for p', and we plug in the value we derived:

$$\mathbb{E}[S_\tau^2 - \tau] = \left(\frac{i}{N}\right) \cdot \left(N^2\right) - \mathbb{E}[\tau] \tag{18}$$
$$\mathbb{E}[S_\tau^2 - \tau] = iN - \mathbb{E}[\tau] \tag{19}$$

Furthermore, since $\{S_n^2 - n\}$ is a martingale, we can also make use of the optional stopping theorem:

$$\mathbb{E}[S_\tau^2 - \tau] = \mathbb{E}[S_0^2 - 0] \tag{20}$$
$$\mathbb{E}[S_\tau^2 - \tau] = i^2 \tag{21}$$

Combining this information, we find:

$$\mathbb{E}[S_\tau^2 - \tau] = iN - \mathbb{E}[\tau] = i^2 \tag{22}$$
$$\mathbb{E}[\tau] = i \cdot (N - i) \tag{23}$$

So the expected number of games the gambler would have to play to either win or lose is equal to the product of the number of dollars he would have to lose to go bankrupt (i) and the number of dollars he would have to win to reach his goal ($N - i$).

6 Unfair Gambler's Ruin

Now let's examine the unfair ($p \neq 0.5$) Gambler's Ruin problem described in Section 1. Recall that:

- The gambler begins with a fortune of $S_0 = i$ dollars.
- The gambler plays a series of games, and in each of these games he can either win a dollar with probability $p \neq 0.5$ or lose a dollar with probability $1 - p \neq 0.5$.
- The gambler's winnings in game t are X_t.
- The gamber's fortune after n games is $S_n = S_0 + \sum_{t=1}^{n} X_t$.
- The gambler will stop when he runs out of money ($S_N = 0$) or obtains N dollars ($S_N = N$).

Unlike the fair Gambler's Ruin, $\{S_n\}$ is not a martingale. We will have to use a different expression. In this case, we define a new variable, λ, and a new function $f(S_n)$:

$$\lambda = \frac{1 - p}{p} \tag{24}$$
$$f(S_n) = \lambda^{S_n} \tag{25}$$

We can show that $\{f(S_n)\}$, however, *is* a martingale:

$$\mathbb{E}[f(S_n)] = \mathbb{E}[\lambda^{S_n}] \tag{26}$$
$$\mathbb{E}[f(S_n)] = p \cdot \lambda^{S_{n-1}+1} + (1-p) \cdot \lambda^{S_{n-1}-1} \tag{27}$$
$$\mathbb{E}[f(S_n)] = p \cdot \left(\frac{1-p}{p}\right) \cdot \lambda^{S_{n-1}} + (1-p) \cdot \left(\frac{p}{1-p}\right) \cdot \lambda^{S_{n-1}} \tag{28}$$
$$\mathbb{E}[f(S_n)] = (1-p) \cdot \lambda^{S_{n-1}} + p \cdot \lambda^{S_{n-1}} \tag{29}$$
$$\mathbb{E}[f(S_n)] = \lambda^{S_{n-1}} \tag{30}$$
$$\mathbb{E}[f(S_n)] = f(S_{n-1}) \tag{31}$$

So $\{f(S_n)\}$ is a martingale.

Once again, we would like to evaluate the probability that the gambler accumulates N dollars and does not go bankrupt (p'). We can evaluate that probability as follows:

$$\mathbb{E}[f(S_n)] = p' \cdot \lambda^N + (1-p') \cdot \lambda^0 \tag{32}$$
$$\mathbb{E}[f(S_n)] = p' \cdot \lambda^N + (1-p') \tag{33}$$
$$\mathbb{E}[f(S_n)] = p' \cdot \left(\lambda^N - 1\right) + 1 \tag{34}$$

The fact that $\{f(S_n)\}$ is a martingale allows us to evaluate this probability using the optional stopping theorem:

$$\mathbb{E}[f(S_n)] = \mathbb{E}[f(S_0)] \tag{35}$$
$$\mathbb{E}[f(S_n)] = \lambda^i \tag{36}$$

Combining this information, we find:

$$\mathbb{E}[f(S_n)] = p' \cdot \left(\lambda^N - 1\right) + 1 = \lambda^i \tag{37}$$
$$p' = \frac{1-\lambda^i}{1-\lambda^N} \tag{38}$$

We would also like to find the expected number of games (τ) it will take the gambler to either go bankrupt ($S_\tau = 0$) or reach his goal ($S_\tau = N$). To do this, we will use Wald's equality. Note that Wald's equality works for any stopping condition on any random series, not just martingales:

$$\mathbb{E}[S_\tau] = i + \mathbb{E}[X] \cdot \mathbb{E}[\tau] \tag{39}$$
$$\mathbb{E}[\tau] = \frac{1}{\mathbb{E}[X]} \left(\mathbb{E}[S_\tau] - i\right) \tag{40}$$

We evaluate $\mathbb{E}[X]$:

$$\mathbb{E}[X] = p - (1-p) \tag{41}$$
$$\mathbb{E}[X] = p \cdot (1-\lambda) \tag{42}$$
$$\mathbb{E}[X] = \frac{1-\lambda}{1+\lambda} \tag{43}$$

We evaluate $\mathbb{E}[S_\tau]$:

$$\mathbb{E}[S_\tau] = p' \cdot N \tag{44}$$
$$\mathbb{E}[S_\tau] = \left(\frac{1-\lambda^i}{1-\lambda^N}\right) \cdot N \tag{45}$$

We can now evaluate $\mathbb{E}[\tau]$:

$$\mathbb{E}[\tau] = \frac{1}{\mathbb{E}[X]} \left(\mathbb{E}[S_\tau] - i\right) \tag{46}$$
$$\mathbb{E}[\tau] = \left(\frac{1+\lambda}{1-\lambda}\right) \cdot \left[\left(\frac{1-\lambda^i}{1-\lambda^N}\right) \cdot N - i\right] \tag{47}$$