1 Exponentially Decaying Income Process

A shock to transitory income decays exponentially according to the function:

$$f(t) = \frac{\Omega}{1-e^{-\Omega}} e^{-\Omega t}$$

The constant in front of the exponential is so that the income in the first year following a unit shock will be equal to one.

The flow of income at a point in time s is therefore:

$$y(t) = \frac{\Omega}{1-e^{-\Omega}} \int_{-\infty}^{t} e^{-\Omega(t-s)} dQ_s$$

Observed income over the year T is the integral of the income flow over that year:

$$\begin{aligned}
y_T^{obs} &= \frac{\Omega}{1-e^{-\Omega}} \int_{T-1}^{T} \int_{-\infty}^{t} e^{-\Omega(t-s)} dQ_s dt \\
&= \frac{\Omega}{1-e^{-\Omega}} \left[\int_{T-1}^{T} \int_{-\infty}^{T-1} e^{-\Omega(t-s)} dQ_s dt + \int_{T-1}^{T} \int_{T-1}^{t} e^{-\Omega(t-s)} dQ_s dt \right]
\end{aligned}$$

Swapping the order of the integrals gives:

$$\begin{aligned}
y_T^{obs} &= \frac{\Omega}{1-e^{-\Omega}} \left[\int_{-\infty}^{T-1} \int_{T-1}^{T} e^{-\Omega(t-s)} dt dQ_s + \int_{T-1}^{T} \int_{s}^{T} e^{-\Omega(t-s)} dt dQ_s \right] \\
&= \frac{1}{1-e^{-\Omega}} \left[\int_{-\infty}^{T-1} (e^{-\Omega(T-1-s)} - e^{-\Omega(T-s)}) dQ_s + \int_{T-1}^{T} (1-e^{-\Omega(T-s)}) dQ_s \right] \\
&= \frac{1}{1-e^{-\Omega}} \int_{T-1}^{T} (1-e^{-\Omega(T-s)}) dQ_s + \int_{-\infty}^{T-1} e^{-\Omega(T-1-s)} dQ_s
\end{aligned}$$

Now take the first difference:

$$\begin{aligned}
\Delta y_T^{obs} &= \frac{1}{1-e^{-\Omega}} \int_{T-1}^{T} (1-e^{-\Omega(T-s)}) dQ_s \\
&\quad + \int_{T-2}^{T-1} \left(e^{-\Omega(T-1-s)} - \frac{1}{1-e^{-\Omega}} (1-e^{-\Omega(T-1-s)}) \right) dQ_s \\
&\quad - \int_{-\infty}^{T-2} e^{-\Omega(T-2-s)} (1-e^{-\Omega}) dQ_s \\
&= \frac{1}{1-e^{-\Omega}} \int_{T-1}^{T} (1-e^{-\Omega(T-s)}) dQ_s \\
&\quad + \frac{1}{1-e^{-\Omega}} \int_{T-2}^{T-1} \left((2-e^{-\Omega}) e^{-\Omega(T-1-s)} - 1 \right) dQ_s \\
&\quad - (1-e^{-\Omega}) \int_{-\infty}^{T-2} e^{-\Omega(T-2-s)} dQ_s
\end{aligned}$$

Calculate covariances - first the variance:

$$\begin{aligned}
\mathrm{Var}(\Delta y_T^{obs}) &= \frac{1}{(1-e^{-\Omega})^2}\int_{T-1}^{T}(1-2e^{-\Omega(T-s)}+e^{-2\Omega(T-s)})ds \\
&\quad + \frac{1}{(1-e^{-\Omega})^2}\int_{T-2}^{T-1}\left((2-e^{-\Omega})^2e^{-2\Omega(T-1-s)}-2(2-e^{-\Omega})e^{-\Omega(T-1-s)}+1\right)ds \\
&\quad + (1-e^{-\Omega})^2\int_{-\infty}^{T-2}e^{-2\Omega(T-2-s)}ds \\
&= \frac{1}{(1-e^{-\Omega})^2}\left(1-\frac{2}{\Omega}(1-e^{-\Omega})+\frac{1}{2\Omega}(1-e^{-2\Omega})\right) \\
&\quad + \frac{1}{(1-e^{-\Omega})^2}\left((2-e^{-\Omega})^2\frac{1}{2\Omega}(1-e^{-2\Omega})-2(2-e^{-\Omega})\frac{1}{\Omega}(1-e^{-\Omega})+1\right) \\
&\quad + \frac{1}{2\Omega}(1-e^{-\Omega})^2 \\
&= \frac{1}{(1-e^{-\Omega})^2}\left(2+\left((2-e^{-\Omega})^2+1\right)\frac{1}{2\Omega}(1-e^{-2\Omega})-(3-e^{-\Omega})\frac{2}{\Omega}(1-e^{-\Omega})\right) \\
&\quad + \frac{1}{2\Omega}(1-e^{-\Omega})^2 \\
&= \frac{1}{(1-e^{-\Omega})^2}\left(2-\frac{1}{2\Omega}\left(7-12e^{-\Omega}+8e^{-2\Omega}-4e^{-3\Omega}+e^{-4\Omega}\right)\right) \\
&\quad + \frac{1}{2\Omega}(1-e^{-\Omega})^2 \\
&= \frac{1}{(1-e^{-\Omega})^2}\left(2-\frac{1}{\Omega}\left(3-4e^{-\Omega}+e^{-2\Omega}\right)\right) \\
&= \frac{2}{(1-e^{-\Omega})^2}-\frac{3-e^{-\Omega}}{\Omega(1-e^{-\Omega})}
\end{aligned}$$

Next calculate covariance with one lag:

$$\begin{aligned}
\mathrm{Cov}(\Delta y_T^{obs},\Delta y_{T-1}^{obs}) &= \frac{1}{(1-e^{-\Omega})^2}\int_{T-2}^{T-1}(1-e^{-\Omega(T-1-s)})\left((2-e^{-\Omega})e^{-\Omega(T-1-s)}-1\right)ds \\
&\quad - \int_{T-3}^{T-2}\left((2-e^{-\Omega})e^{-\Omega(T-2-s)}-1\right)e^{-\Omega(T-2-s)}ds \\
&\quad + (1-e^{-\Omega})^2\int_{-\infty}^{T-3}e^{-\Omega(T-3-s)}e^{-\Omega(T-2-s)}dQ_s \\
&= \frac{1}{2\Omega}(2-e^{-\Omega})-\frac{1}{(1-e^{-\Omega})^2}(1-\frac{1-e^{-\Omega}}{\Omega}) \\
&\quad - \frac{1-e^{-2\Omega}}{2\Omega}\left(2-e^{-\Omega}\right)+\frac{1}{\Omega}(1-e^{-\Omega}) \\
&\quad + \frac{1}{2\Omega}e^{-\Omega}(1-e^{-\Omega})^2 \\
&= \frac{1}{2\Omega}(2-e^{-\Omega})-\frac{1}{(1-e^{-\Omega})^2}(1-\frac{1-e^{-\Omega}}{\Omega})
\end{aligned}$$

And the covariance with $M \geq 2$ lags:

$$\begin{aligned}
\mathrm{Cov}(\Delta y_T^{obs}, \Delta y_{T-M}^{obs}) = &- \int_{T-M-1}^{T-M} (1 - e^{-\Omega(T-M-s)}) e^{-\Omega(T-2-s)} ds \\
&- \int_{T-M-2}^{T-M-1} \left((2 - e^{-\Omega}) e^{-\Omega(T-M-1-s)} - 1\right) e^{-\Omega(T-2-s)} ds \\
&+ (1 - e^{-\Omega})^2 \int_{-\infty}^{T-M-2} e^{-\Omega(T-M-2-s)} e^{-\Omega(T-2-s)} ds \\
= &- \frac{1}{\Omega}(1 - e^{-\Omega}) e^{-\Omega(M-2)} + \frac{1}{2\Omega}(1 - e^{-2\Omega}) e^{-\Omega(M-2)} \\
&- (2 - e^{-\Omega}) e^{-\Omega(M-1)} \frac{1}{2\Omega}(1 - e^{-2\Omega}) + \frac{1}{\Omega} e^{-\Omega(M-1)} (1 - e^{-\Omega}) \\
&+ (1 - e^{-\Omega})^2 \frac{1}{2\Omega} e^{-\Omega M}
\end{aligned}$$

Note the variance of y_T^{obs} is not equal to one (as in the discrete time case). For comparison I calculate it here:

$$\begin{aligned}
\mathrm{Var}(y_T^{obs}) = &\frac{1}{(1 - e^{-\Omega})^2} \int_{T-1}^{T} (1 - 2e^{-\Omega(T-s)} + e^{-2\Omega(T-s)}) ds \\
&+ \int_{-\infty}^{T-1} e^{-2\Omega(T-1-s)} ds \\
= &\frac{1}{(1 - e^{-\Omega})^2} \left(1 - \frac{2}{\Omega}(1 - e^{-\Omega}) + \frac{1}{2\Omega}(1 - e^{-2\Omega})\right) + \frac{1}{2\Omega}
\end{aligned}$$

2 Exponentially Decaying Consumption Process

Consumption responds to a transitory income shock according to the function:

$$g(t) = \frac{\psi\theta}{1 - e^{-\theta}} e^{-\theta t}$$

The flow of consumption is observed at the end of each calendar year:

$$c_T^{obs} = \frac{\psi\theta}{1 - e^{-\theta}} \int_{-\infty}^{T} e^{-\theta(T-s)} dQ_s$$

Now take the first difference

$$\begin{aligned}
\Delta c_T^{obs} &= \frac{\psi\theta}{1 - e^{-\theta}} \left[\int_{T-1}^{T} e^{-\theta(T-s)} dQ_s + \int_{-\infty}^{T-1} e^{-\theta(T-s)} - e^{-\theta(T-1-s)} dQ_s\right] \\
&= \frac{\psi\theta}{1 - e^{-\theta}} \int_{T-1}^{T} e^{-\theta(T-s)} dQ_s - \psi\theta \int_{-\infty}^{T-1} e^{-\theta(T-1-s)} dQ_s
\end{aligned}$$

Calculate covariances:

$$
\begin{aligned}
\mathrm{Var}(\Delta c_T^{obs}) &= \frac{\psi^2\theta^2}{(1-e^{-\theta})^2}\int_{T-1}^{T} e^{-2\theta(T-s)}ds + \psi^2\theta^2\int_{-\infty}^{T-1} e^{-2\theta(T-1-s)}ds \\
&= \frac{\psi^2\theta}{2}\left(1+\frac{1-e^{-2\theta}}{(1-e^{-\theta})^2}\right) \\
&= \frac{\psi^2\theta}{1-e^{-\theta}}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}(\Delta c_T^{obs}, \Delta c_{T-M}^{obs}) &= \frac{-\psi^2\theta^2}{(1-e^{-\theta})}\int_{T-M-1}^{T-M} e^{-\theta M}e^{-\theta(2(T-M-s)-1)}ds \\
&\quad + \psi^2\theta^2\int_{-\infty}^{T-M-1} e^{-\theta M}e^{-2\theta(T-M-1-s)}ds \\
&= \frac{\psi^2\theta}{2}e^{-\theta(M-1)}\left[\frac{e^{-2\theta}-1}{1-e^{-\theta}}+e^{-\theta}\right] \\
&= \frac{-\psi^2\theta}{2}e^{-\theta(M-1)}
\end{aligned}
$$

3 Covariance of Income and Consumption

$$
\begin{aligned}
\mathrm{Cov}(\Delta c_T^{obs}, \Delta y_T^{obs}) &= \frac{\psi\theta}{(1-e^{-\Omega})(1-e^{-\theta})}\int_{T-1}^{T}(1-e^{-\Omega(T-s)})e^{-\theta(T-s)}ds \\
&\quad - \frac{\psi\theta}{1-e^{-\Omega}}\int_{T-2}^{T-1}\left((2-e^{-\Omega})e^{-\Omega(T-1-s)}-1\right)e^{-\theta(T-1-s)}ds \\
&\quad + \psi\theta(1-e^{-\Omega})\int_{-\infty}^{T-2} e^{-\Omega(T-2-s)}e^{-\theta(T-1-s)}ds \\
&= \frac{\psi\theta}{(1-e^{-\Omega})(1-e^{-\theta})}\left[\frac{1}{\theta}(1-e^{-\theta})-\frac{1}{\Omega+\theta}(1-e^{-(\Omega+\theta)})\right] \\
&\quad - \frac{\psi\theta}{1-e^{-\Omega}}\left[(2-e^{-\Omega})\frac{1}{\Omega+\theta}(1-e^{-(\Omega+\theta)})-\frac{1}{\theta}(1-e^{-\theta})\right] \\
&\quad + \psi\theta(1-e^{-\Omega})e^{-\theta}\frac{1}{\Omega+\theta}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}(\Delta c_T^{obs}, \Delta y_{T+1}^{obs}) &= \frac{\psi\theta}{(1-e^{-\Omega})(1-e^{-\theta})} \int_{T-1}^{T} \left((2-e^{-\Omega})e^{-\Omega(T-s)} - 1\right) e^{-\theta(T-s)} ds \\
&\quad + \psi\theta(1-e^{-\Omega}) \int_{-\infty}^{T-1} e^{-\Omega(T-1-s)} e^{-\theta(T-1-s)} ds \\
&= \frac{\psi\theta}{(1-e^{-\Omega})(1-e^{-\theta})} \left[(2-e^{-\Omega})\frac{1}{\Omega+\theta}(1-e^{-(\Omega+\theta)}) - \frac{1}{\theta}(1-e^{-\theta})\right] \\
&\quad + \psi\theta(1-e^{-\Omega})\frac{1}{\Omega+\theta}
\end{aligned}
$$

For $M \geq 2$:

$$
\begin{aligned}
\mathrm{Cov}(\Delta c_T^{obs}, \Delta y_{T+M}^{obs}) &= -\psi\theta\frac{1-e^{-\Omega}}{1-e^{-\theta}} e^{-\Omega(M-2)} \int_{T-1}^{T} e^{-\Omega(T-s)} e^{-\theta(T-s)} ds \\
&\quad + \psi\theta(1-e^{-\Omega})e^{-\Omega(M-1)} \int_{-\infty}^{T-1} e^{-\Omega(T-1-s)} e^{-\theta(T-1-s)} ds \\
&= -\psi\theta\frac{1-e^{-\Omega}}{1-e^{-\theta}} e^{-\Omega(M-2)} \frac{1}{\Omega+\theta}(1-e^{-(\Omega+\theta)}) \\
&\quad + \psi\theta(1-e^{-\Omega})e^{-\Omega(M-1)} \frac{1}{\Omega+\theta}
\end{aligned}
$$

For $M \geq 1$

$$
\begin{aligned}
\mathrm{Cov}(\Delta c_{T+M}^{obs}, \Delta y_T^{obs}) &= -\frac{\psi\theta}{1-e^{-\Omega}} e^{-\theta(M-1)} \int_{T-1}^{T} (1-e^{-\Omega(T-s)}) e^{-\theta(T-s)} ds \\
&\quad + -\frac{\psi\theta}{1-e^{-\Omega}} e^{-\theta M} \int_{T-2}^{T-1} \left((2-e^{-\Omega})e^{-\Omega(T-1-s)} - 1\right) e^{-\theta(T-1-s)} ds \\
&\quad + \psi\theta(1-e^{-\Omega})e^{-\theta(M+1)} \int_{-\infty}^{T-2} e^{-\Omega(T-2-s)} e^{-\theta(T-2-s)} ds \\
&= -\frac{\psi\theta}{1-e^{-\Omega}} e^{-\theta(M-1)} \left[\frac{1}{\theta}(1-e^{-\theta}) - \frac{1}{\Omega+\theta}(1-e^{-(\Omega+\theta)})\right] \\
&\quad + -\frac{\psi\theta}{1-e^{-\Omega}} e^{-\theta M} \left[(2-e^{-\Omega})\frac{1}{\Omega+\theta}(1-e^{-(\Omega+\theta)}) - \frac{1}{\theta}(1-e^{-\theta})\right] \\
&\quad + \psi\theta(1-e^{-\Omega})e^{-\theta(M+1)} \frac{1}{\Omega+\theta}
\end{aligned}
$$