1 More General Consumption Processes

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2 Setup

Assume now that:

$$c_t dt = \int_{-\infty}^{t} f(t-s) dQ_s dt$$

(this is a generalization of $f(t-s) = \phi$, the random walk case from BPP).

We have

$$\begin{aligned}
\Delta \bar{c}_T &= \int_{T-1}^{T} \int_{-\infty}^{t} f(t-s) dQ_s dt - \int_{T-2}^{T-1} \int_{-\infty}^{t} f(t-s) dQ_s dt \\
&= \int_{T-1}^{T} \Big(\int_{-\infty}^{T-2} f(t-s) dQ_s + \int_{T-2}^{T-1} f(t-s) dQ_s + \int_{T-1}^{t} f(t-s) dQ_s \Big) dt \\
&\quad - \int_{T-2}^{T-1} \Big(\int_{-\infty}^{T-2} f(t-s) dQ_s + \int_{T-1}^{t} f(t-s) dQ_s \Big) dt \\
&= \int_{-\infty}^{T-2} \Big(\int_{T-1}^{T} f(t-s) dt - \int_{T-2}^{T-1} f(t-s) dt \Big) dQ_s \\
&\quad + \int_{T-2}^{T-1} \Big(\int_{T-1}^{T} f(t-s) dt - \int_{s}^{T-1} f(t-s) dt \Big) dQ_s \\
&\quad + \int_{T-1}^{T} \Big(\int_{s}^{T} f(t-s) dt \Big) dQ_s
\end{aligned}$$

3 Examples

3.1 Example 1: Exponential Decay

The first example will assume exponential decay of consumption, with households eventually using up all their money

$$f(t-s) = \phi e^{-\phi(t-s)}$$

This implies an annual MPC of

$$\begin{aligned}\text{MPC} &= \int_0^1 \phi e^{-\phi t} dt \\ &= \Big[- e^{-\phi t} \Big]_0^1 \\ &= 1 - e^{-\phi}\end{aligned}$$

Similarly quarterly MPC is $1 - e^{-0.25\phi}$. We have:

$$\begin{aligned}\Delta \bar{c}_T = &\int_{-\infty}^{T-2} \Big(e^{-\phi(T-s)}(e^{\phi} - 1) - e^{-\phi(T-1-s)}(e^{\phi} - 1) \Big) dQ_s \\ &+ \int_{T-2}^{T-1} \Big(e^{-\phi(T-s)}(e^{\phi} - 1) - 1 + e^{-\phi(T-1-s)} \Big) dQ_s \\ &+ \int_{T-1}^{T} \Big(1 - e^{-\phi(T-s)} \Big) dQ_s\end{aligned}$$

$$\begin{aligned}\Delta \bar{c}_T = &\int_{-\infty}^{T-2} \Big(- e^{-\phi(T-s)}(e^{\phi} - 1)^2 \Big) dQ_s \\ &+ \int_{T-2}^{T-1} \Big(e^{-\phi(T-s)}(2e^{\phi} - 1) - 1 \Big) dQ_s \\ &+ \int_{T-1}^{T} \Big(1 - e^{-\phi(T-s)} \Big) dQ_s\end{aligned}$$

3.2 Example 2: Splurge, then constant

This example has two parameters and will be identified by both $cov(\Delta \bar{c}_t, \Delta \bar{y}_{t+1})$ and $cov(\Delta \bar{c}_t, \Delta \bar{y}_{t-1})$. It is equivalent to an initial 'splurge' followed by a fixed increase in consumption. It can be mapped to a model with durable expenditure where there is an initial large sum spent on durable goods. Formally:

$$f(t-s) = \psi_1 \delta_0(t-s) + \psi_2 \mathbb{1}_{t-s>0}$$

where δ_0 is the dirac delta function.

$$\begin{aligned}\Delta \bar{c}_T = \psi_1 &\Big(\int_{T-1}^{T} dQ_s - \int_{T-2}^{T-1} dQ_s \Big) \\ &+ \psi_2 \Big(\int_{T-2}^{T-1} (s - (T-2)) dQ_s + \int_{T-1}^{T} (T-s) dQ_s \Big)\end{aligned}$$

$$\begin{aligned}\Delta \bar{c}_T = &\int_{T-1}^{T} \big(\psi_1 + \psi_2 (T-s) \big) dQ_s \\ &+ \int_{T-2}^{T-1} \big(\psi_2 (s - (T-2)) - \psi_1 \big) dQ_s \qquad (1)\end{aligned}$$

4 Persistence in Transitory Shock

4.1 Most Closely Parallel to MA(1)

BPP add MA(1) persistence in the transitory shock. The most closely related continuous time version (that doesn't seem to relate to reality, but might work in practice), is to assume a transitory payment of 1 followed by another payment of θ exactly one year later. That is:

$$y_t dt = \Big(\int_0^t dP_s\Big)dt + dQ_t + \theta dQ_{t-1}$$

So that:

$$\begin{aligned}\Delta \bar{y}_T = &\Big(\int_{T-1}^{T}(T-s)dP_s + \int_{T-2}^{T-1}(s-(T-2))dP_s\Big) \\ &+ \Big(\int_{T-1}^{T} dQ_s - (1-\theta)\int_{T-2}^{T-1} dQ_s - \theta\int_{T-3}^{T-2} dQ_s\Big) \qquad (2)\end{aligned}$$

4.2 Combining persistence with splurge

Here I calculate the moments related to a model of equation 1 extended to for permanent shocks, combined with the income process 2.

$$
\begin{aligned}
cov(\Delta\bar{c}_T, \Delta\bar{y}_{T-1}) &= \int_{T-2}^{T-1} (T-1-s)(\phi_2(s-(T-2)) - \phi_1) dP_s dP_s \\
&\quad + \int_{T-2}^{T-1} \big(\psi_2(s-(T-2)) - \psi_1\big) dQ_s dQ_s \\
&= \frac{1}{6}\phi_2\sigma_P^2 - \frac{1}{2}\phi_1\sigma_P^2 + \frac{1}{2}\psi_2\sigma_Q^2 - \psi_1\sigma_Q^2 \\
cov(\Delta\bar{c}_T, \Delta\bar{y}_T) &= \int_{T-1}^{T} (\phi_1 + \phi_2(T-s))(T-s) dP_s dP_s + \int_{T-2}^{T-1} (\phi_2(s-(T-2) - \phi_1)(s-(T-2)) dP_s dP_s \\
&\quad \int_{T-1}^{T} (\psi_1 + \psi_2(T-s)) dQ_s dQ_s - (1-\theta) \int_{T-2}^{T-1} (\psi_2(s-(T-2)) - \psi_1) dQ_s dQ_s \\
&= \frac{1}{2}\phi_1\sigma_P^2 + \frac{1}{3}\phi_2\sigma_P^2 + \frac{1}{3}\phi_2\sigma_P^2 - \frac{1}{2}\phi_1\sigma_P^2 \\
&\quad + \psi_1\sigma_Q^2 + \frac{1}{2}\psi_2\sigma_Q^2 - (1-\theta)\Big(\frac{1}{2}\psi_2 - \psi_1\Big)\sigma_Q^2 \\
&= \frac{2}{3}\phi_2\sigma_P^2 + (2-\theta)\psi_1\sigma_Q^2 + \frac{1}{2}\theta\psi_2\sigma_Q^2 \\
cov(\Delta\bar{c}_T, \Delta\bar{y}_{T+1}) &= \int_{T-1}^{T} (\phi_1 + \phi_2(T-s))(s-(T-1)) dP_s dP_s \\
&\quad - (1-\theta) \int_{T-1}^{T} (\psi_1 + \psi_2(T-s)) dQ_s dQ_s - \theta \int_{T-2}^{T-1} (\psi_2(s-(T-2)) - \psi_1) dQ_s dQ_s \\
&= \frac{1}{2}\phi_1\sigma_P^2 + \frac{1}{6}\phi_2\sigma_P^2 - (1-\theta)(\psi_1 + \frac{1}{2}\psi_2)\sigma_Q^2 - \theta(\frac{1}{2}\psi_2 - \psi_1)\sigma_Q^2 \\
&= \frac{1}{2}\phi_1\sigma_P^2 + \frac{1}{6}\phi_2\sigma_P^2 - (1-2\theta)\psi_1\sigma_Q^2 - \frac{1}{2}\psi_2\sigma_Q^2 \\
cov(\Delta\bar{c}_T, \Delta\bar{y}_{T+2}) &= -\theta \int_{T-1}^{T} (\psi_1 + \psi_2(T-s)) dQ_s dQ_s \\
&= -\theta(\psi_1 + \frac{1}{2}\psi_2)\sigma_Q^2
\end{aligned}
$$

4.3 Transitory Shock that Lasts for Time τ

Now suppose the transitory shock is not instantaneous but instead spread out over a period of time τ. That is instantaneous income is now:

$$
y_t dt = \Big(\int_0^t dP_s\Big)dt + \Big(\int_{t-\tau}^{t} \frac{1}{\tau} dQ_s\Big)dt
$$