

1 Exponentially Decaying Income Process

A shock to transitory income decays exponentially according to the function:

$$f(t) = \frac{\Omega}{1 - e^{-\Omega}} e^{-\Omega t}$$

The constant in front of the exponential is so that the income in the first year following a unit shock will be equal to one.

The flow of income at a point in time s is therefore:

$$y(t) = \frac{\Omega}{1 - e^{-\Omega}} \int_{-\infty}^t e^{-\Omega(t-s)} dQ_s$$

Observed income over the year T is the integral of the income flow over that year:

$$\begin{aligned} y_T^{obs} &= \frac{\Omega}{1 - e^{-\Omega}} \int_{T-1}^T \int_{-\infty}^t e^{-\Omega(t-s)} dQ_s dt \\ &= \frac{\Omega}{1 - e^{-\Omega}} \left[\int_{T-1}^T \int_{-\infty}^{T-1} e^{-\Omega(t-s)} dQ_s dt + \int_{T-1}^T \int_{T-1}^t e^{-\Omega(t-s)} dQ_s dt \right] \end{aligned}$$

Swapping the order of the integrals gives:

$$\begin{aligned} y_T^{obs} &= \frac{\Omega}{1 - e^{-\Omega}} \left[\int_{-\infty}^{T-1} \int_{T-1}^T e^{-\Omega(t-s)} dt dQ_s + \int_{T-1}^T \int_s^T e^{-\Omega(t-s)} dt dQ_s \right] \\ &= \frac{1}{1 - e^{-\Omega}} \left[\int_{-\infty}^{T-1} (e^{-\Omega(T-1-s)} - e^{-\Omega(T-s)}) dQ_s + \int_{T-1}^T (1 - e^{-\Omega(T-s)}) dQ_s \right] \\ &= \frac{1}{1 - e^{-\Omega}} \int_{T-1}^T (1 - e^{-\Omega(T-s)}) dQ_s + \int_{-\infty}^{T-1} e^{-\Omega(T-1-s)} dQ_s \end{aligned}$$

Now take the first difference:

$$\begin{aligned} \Delta y_T^{obs} &= \frac{1}{1 - e^{-\Omega}} \int_{T-1}^T (1 - e^{-\Omega(T-s)}) dQ_s \\ &\quad + \int_{T-2}^{T-1} \left(e^{-\Omega(T-1-s)} - \frac{1}{1 - e^{-\Omega}} (1 - e^{-\Omega(T-1-s)}) \right) dQ_s \\ &\quad - \int_{-\infty}^{T-2} e^{-\Omega(T-2-s)} (1 - e^{-\Omega}) dQ_s \\ &= \frac{1}{1 - e^{-\Omega}} \int_{T-1}^T (1 - e^{-\Omega(T-s)}) dQ_s \\ &\quad + \frac{1}{1 - e^{-\Omega}} \int_{T-2}^{T-1} ((2 - e^{-\Omega}) e^{-\Omega(T-1-s)} - 1) dQ_s \\ &\quad - (1 - e^{-\Omega}) \int_{-\infty}^{T-2} e^{-\Omega(T-2-s)} dQ_s \end{aligned}$$

Calculate covariances - first the variance:

$$\begin{aligned}
\text{Var}(\Delta y_T^{obs}) &= \frac{1}{(1 - e^{-\Omega})^2} \int_{T-1}^T (1 - 2e^{-\Omega(T-s)} + e^{-2\Omega(T-s)}) ds \\
&\quad + \frac{1}{(1 - e^{-\Omega})^2} \int_{T-2}^{T-1} ((2 - e^{-\Omega})^2 e^{-2\Omega(T-1-s)} - 2(2 - e^{-\Omega})e^{-\Omega(T-1-s)} + 1) ds \\
&\quad + (1 - e^{-\Omega})^2 \int_{-\infty}^{T-2} e^{-2\Omega(T-2-s)} ds \\
&= \frac{1}{(1 - e^{-\Omega})^2} \left(1 - \frac{2}{\Omega}(1 - e^{-\Omega}) + \frac{1}{2\Omega}(1 - e^{-2\Omega}) \right) \\
&\quad + \frac{1}{(1 - e^{-\Omega})^2} \left((2 - e^{-\Omega})^2 \frac{1}{2\Omega}(1 - e^{-2\Omega}) - 2(2 - e^{-\Omega}) \frac{1}{\Omega}(1 - e^{-\Omega}) + 1 \right) \\
&\quad + \frac{1}{2\Omega}(1 - e^{-\Omega})^2 \\
&= \frac{1}{(1 - e^{-\Omega})^2} \left(2 + ((2 - e^{-\Omega})^2 + 1) \frac{1}{2\Omega}(1 - e^{-2\Omega}) - (3 - e^{-\Omega}) \frac{2}{\Omega}(1 - e^{-\Omega}) \right) \\
&\quad + \frac{1}{2\Omega}(1 - e^{-\Omega})^2 \\
&= \frac{1}{(1 - e^{-\Omega})^2} \left(2 - \frac{1}{2\Omega} (7 - 12e^{-\Omega} + 8e^{-2\Omega} - 4e^{-3\Omega} + e^{-4\Omega}) \right) \\
&\quad + \frac{1}{2\Omega}(1 - e^{-\Omega})^2 \\
&= \frac{1}{(1 - e^{-\Omega})^2} \left(2 - \frac{1}{\Omega} (3 - 4e^{-\Omega} + e^{-2\Omega}) \right) \\
&= \frac{2}{(1 - e^{-\Omega})^2} - \frac{3 - e^{-\Omega}}{\Omega(1 - e^{-\Omega})}
\end{aligned}$$

Next calculate covariance with one lag:

$$\begin{aligned}
\text{Cov}(\Delta y_T^{obs}, \Delta y_{T-1}^{obs}) &= \frac{1}{(1 - e^{-\Omega})^2} \int_{T-2}^{T-1} (1 - e^{-\Omega(T-1-s)}) ((2 - e^{-\Omega})e^{-\Omega(T-1-s)} - 1) ds \\
&\quad - \int_{T-3}^{T-2} ((2 - e^{-\Omega})e^{-\Omega(T-2-s)} - 1) e^{-\Omega(T-2-s)} ds \\
&\quad + (1 - e^{-\Omega})^2 \int_{-\infty}^{T-3} e^{-\Omega(T-3-s)} e^{-\Omega(T-2-s)} dQ_s \\
&= \frac{1}{2\Omega}(2 - e^{-\Omega}) - \frac{1}{(1 - e^{-\Omega})^2} \left(1 - \frac{1 - e^{-\Omega}}{\Omega} \right) \\
&\quad - \frac{1 - e^{-2\Omega}}{2\Omega} (2 - e^{-\Omega}) + \frac{1}{\Omega}(1 - e^{-\Omega}) \\
&\quad + \frac{1}{2\Omega} e^{-\Omega}(1 - e^{-\Omega})^2 \\
&= \frac{1}{2\Omega}(2 - e^{-\Omega}) - \frac{1}{(1 - e^{-\Omega})^2} \left(1 - \frac{1 - e^{-\Omega}}{\Omega} \right)
\end{aligned}$$

And the covariance with $M \geq 2$ lags:

$$\begin{aligned}
\text{Cov}(\Delta y_T^{obs}, \Delta y_{T-M}^{obs}) &= - \int_{T-M-1}^{T-M} (1 - e^{-\Omega(T-M-s)}) e^{-\Omega(T-2-s)} ds \\
&\quad - \int_{T-M-2}^{T-M-1} ((2 - e^{-\Omega}) e^{-\Omega(T-M-1-s)} - 1) e^{-\Omega(T-2-s)} ds \\
&\quad + (1 - e^{-\Omega})^2 \int_{-\infty}^{T-M-2} e^{-\Omega(T-M-2-s)} e^{-\Omega(T-2-s)} ds \\
&= -\frac{1}{\Omega} (1 - e^{-\Omega}) e^{-\Omega(M-2)} + \frac{1}{2\Omega} (1 - e^{-2\Omega}) e^{-\Omega(M-2)} \\
&\quad - (2 - e^{-\Omega}) e^{-\Omega(M-1)} \frac{1}{2\Omega} (1 - e^{-2\Omega}) + \frac{1}{\Omega} e^{-\Omega(M-1)} (1 - e^{-\Omega}) \\
&\quad + (1 - e^{-\Omega})^2 \frac{1}{2\Omega} e^{-\Omega M}
\end{aligned}$$

Note the variance of y_T^{obs} is not equal to one (as in the discrete time case). For comparison I calculate it here:

$$\begin{aligned}
\text{Var}(y_T^{obs}) &= \frac{1}{(1 - e^{-\Omega})^2} \int_{T-1}^T (1 - 2e^{-\Omega(T-s)} + e^{-2\Omega(T-s)}) ds \\
&\quad + \int_{-\infty}^{T-1} e^{-2\Omega(T-1-s)} ds \\
&= \frac{1}{(1 - e^{-\Omega})^2} \left(1 - \frac{2}{\Omega} (1 - e^{-\Omega}) + \frac{1}{2\Omega} (1 - e^{-2\Omega}) \right) + \frac{1}{2\Omega}
\end{aligned}$$

2 Exponentially Decaying Consumption Process

Consumption responds to a transitory income shock according to the function:

$$g(t) = \frac{\psi\theta}{1 - e^{-\theta}} e^{-\theta t}$$

The flow of consumption is observed at the end of each calendar year:

$$c_T^{obs} = \frac{\psi\theta}{1 - e^{-\theta}} \int_{-\infty}^T e^{-\theta(T-s)} dQ_s$$

Now take the first difference

$$\begin{aligned}
\Delta c_T^{obs} &= \frac{\psi\theta}{1 - e^{-\theta}} \left[\int_{T-1}^T e^{-\theta(T-s)} dQ_s + \int_{-\infty}^{T-1} e^{-\theta(T-s)} - e^{-\theta(T-1-s)} dQ_s \right] \\
&= \frac{\psi\theta}{1 - e^{-\theta}} \int_{T-1}^T e^{-\theta(T-s)} dQ_s - \psi\theta \int_{-\infty}^{T-1} e^{-\theta(T-1-s)} dQ_s
\end{aligned}$$

Calculate covariances:

$$\begin{aligned}
\text{Var}(\Delta c_T^{obs}) &= \frac{\psi^2 \theta^2}{(1 - e^{-\theta})^2} \int_{T-1}^T e^{-2\theta(T-s)} ds + \psi^2 \theta^2 \int_{-\infty}^{T-1} e^{-2\theta(T-1-s)} ds \\
&= \frac{\psi^2 \theta}{2} \left(1 + \frac{1 - e^{-2\theta}}{(1 - e^{-\theta})^2} \right) \\
&= \frac{\psi^2 \theta}{1 - e^{-\theta}}
\end{aligned}$$

$$\begin{aligned}
\text{Cov}(\Delta c_T^{obs}, \Delta c_{T-M}^{obs}) &= \frac{-\psi^2 \theta^2}{(1 - e^{-\theta})} \int_{T-M-1}^{T-M} e^{-\theta M} e^{-\theta(2(T-M-s)-1)} ds \\
&\quad + \psi^2 \theta^2 \int_{-\infty}^{T-M-1} e^{-\theta M} e^{-2\theta(T-M-1-s)} ds \\
&= \frac{\psi^2 \theta}{2} e^{-\theta(M-1)} \left[\frac{e^{-2\theta} - 1}{1 - e^{-\theta}} + e^{-\theta} \right] \\
&= \frac{-\psi^2 \theta}{2} e^{-\theta(M-1)}
\end{aligned}$$