1 Appendices

1.1 Allowing for Persistence in the Transitory Shock

In this section we formalize the continuous time model and calculate the relevant variance and covariances. We begin by defining permanent income. Let p_t for $t \in \mathbb{R}^+$ be a martingale process (possibly with jumps) with independent stationary increments and ν_p be such that $\mathbb{E}(e^{p_t - p_{t-1}}) = e^{\nu_p}$. Define the permanent component of income as:

$$P_t = e^{p_t - t\nu_p}$$

Note that $\mathbb{E}\Big(\frac{P_{t+s}}{P_t}\Big) = 1$ for all $s \geq 0$.

Next we define transitory income. Let q_t on $t \in \mathbb{R}^+$ also be a martingale process, independent of p_t, with independent stationary increments. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be the impulse response of income to changes in q_t. We will assume that the impulse response to a transitory shock to income is over after two years, that is $f(s) = 0$ for $s > 2$. The transitory component of income is then defined as:

$$\theta_t = e^{\int_{t-2}^{t} f(t-s)dq_s - \nu_q}$$

where $e^{\nu_q} = \mathbb{E}e^{\int_{t-2}^{t} f(t-s)dq_s}$ so that $\mathbb{E}\theta_t = 1$.

We are now in a position to talk about total income. Total income *flow* at time t is given by:

$$\begin{aligned} Y_t &= P_t\theta_t \\ &= e^{p_t - t\nu_p + \int_{t-2}^{t} f(t-s)dq_s - \nu_q} \end{aligned}$$

Observable income is the sum of income *flow* over a one year period, that is:

$$\bar{Y}_T = \int_{T-1}^{T} P_t\theta_t dt$$

We will be focused on the log of observable income growth over N years:

$$\begin{aligned} \Delta^N \log(\bar{y}_T) &= \log\Big(\int_{T-1}^{T} P_t\theta_t dt\Big) - \log\Big(\int_{T-N-1}^{T-N} P_t\theta_t dt\Big) \\ &= \log(\frac{P_{T-1}}{P_{T-N}}) + \log\left(\int_{T-1}^{T} \frac{P_t}{P_{T-1}}\theta_t dt\right) - \log\left(\int_{T-N-1}^{T-N} \frac{P_t}{P_{T-N}}\theta_t dt\right) \end{aligned} \tag{1}$$

Note that if $N \geq 3$ each of the three components of equation 1 are mutually independent because both p_t and q_t have independent increments, and θ_t is independent of q_s for $s < t-2$ and $s > t$. Defining $\mathcal{P}_{T,N}$, $\mathcal{Q}^1_{T,N}$ and $\mathcal{Q}^2_{T,N}$ to be the three parts of the sum in equation 1 respectively, we have:

$$\begin{aligned} \mathcal{P}_{T,N} &= \log(\frac{P_{T-1}}{P_{T-N}}) \\ \Rightarrow \mathrm{Var}(\mathcal{P}_{T,N}) &= (N-1)\mathrm{Var}\Big(\log(\frac{P_T}{P_{T-1}})\Big) \end{aligned}$$

$$= (N-1)\sigma_P^2$$

where σ_P^2 is defined to be $\text{Var}\Big(\log(\frac{P_T}{P_{T-1}})\Big)$, which does not depend on T because p_t has independent increments. Moving on to the components that contain a mix of both permanent and transitory income, and defining $\bar{\theta}_T = \int_{T-1}^{T} \theta_t dt$, we have

$$\begin{aligned}
\mathcal{Q}^1_{T,N} &= \log\left(\int_{T-1}^{T} \frac{P_t}{P_{T-1}}\theta_t dt\right) \\
&= \log\left(\int_{T-1}^{T} \theta_t dt + \int_{T-1}^{T} \Big(\frac{P_t}{P_{T-1}} - 1\Big)\theta_t dt\right) \\
&= \log\left(\bar{\theta}_T\right) + \log\left(1 + \int_{T-1}^{T} \Big(\frac{P_t}{P_{T-1}} - 1\Big)\frac{\theta_t}{\bar{\theta}_T} dt\right) \\
&\approx \log\left(\bar{\theta}_T\right) + \int_{T-1}^{T} \Big(\frac{P_t}{P_{T-1}} - 1\Big)\frac{\theta_t}{\bar{\theta}_T} dt
\end{aligned}$$

Where the approximation holds so long as $\frac{P_t}{P_{T-1}}$ is close to 1 for $T-1 \leq t \leq T$, that is the permanent shock does not move a lot in the course of one year **(*********NOTE: What is the exact notion of approx that I am using here. It should be some probabilistic notion, as it is clearly not true over the entire space of possibly values for P_t)**. Define:

$$\sigma_\theta^2 = \text{Var}\left(\log\left(\bar{\theta}_T\right)\right)$$

So that

$$\begin{aligned}
\text{Var}\big(\mathcal{Q}^1_{T,N}\big) &\approx \sigma_\theta^2 + \mathbb{E}\left(\int_{T-1}^{T} \Big(\frac{P_t}{P_{T-1}} - 1\Big)\frac{\theta_t}{\bar{\theta}_T} dt\right)^2 \\
&= \sigma_\theta^2 + \mathbb{E}\left(\int_{T-1}^{T}\int_{T-1}^{T} \Big(\frac{P_t}{P_{T-1}} - 1\Big)\Big(\frac{P_s}{P_{T-1}} - 1\Big)\frac{\theta_t\theta_s}{\bar{\theta}_T^2} dtds\right) \\
&= \sigma_\theta^2 + \int_{T-1}^{T}\int_{T-1}^{T} \mathbb{E}\left(\Big(\frac{P_{\min(t,s)}}{P_{T-1}}\Big)^2 \frac{P_{\max(t,s)}}{P_{\min(t,s)}} - \frac{P_t}{P_{T-1}} - \frac{P_s}{P_{T-1}} - 1\right)\mathbb{E}\left(\frac{\theta_t\theta_s}{\bar{\theta}_T^2}\right) dtds \\
&= \sigma_\theta^2 + \int_{T-1}^{T}\int_{T-1}^{T} \text{Var}\left(\frac{P_{\min(t,s)}}{P_{T-1}}\right)\mathbb{E}\left(\frac{\theta_t\theta_s}{\bar{\theta}_T^2}\right) dtds \\
&\approx \sigma_\theta^2 + \sigma_P^2 \int_{T-1}^{T}\int_{T-1}^{T} \min(t,s)\mathbb{E}\left(\frac{\theta_t\theta_s}{\bar{\theta}_T^2}\right) dtds \\
&= \sigma_\theta^2 + \sigma_P^2 \int_{T-1}^{T}\int_{T-1}^{T} \min(t,s)\mathbb{E}\left(\Big(1 + \frac{\theta_t - \bar{\theta}_T}{\bar{\theta}_T}\Big)\Big(1 + \frac{\theta_s - \bar{\theta}_T}{\bar{\theta}_T}\Big)\right) dtds
\end{aligned}$$

$$= \sigma_\theta^2 + \sigma_P^2 \int_{T-1}^{T}\int_{T-1}^{T} \min(t,s)\Bigg(1 + \mathbb{E}\Big(\hat{\theta}_{t,T}\Big) + \mathbb{E}\Big(\hat{\theta}_{s,T}\Big) + \mathbb{E}\Big(\hat{\theta}_{t,T}\hat{\theta}_{s,T}\Big)\Bigg)dtds$$

where $\hat{\theta}_{t,T} = \frac{\theta_t - \bar{\theta}_T}{\bar{\theta}_T}$. Continuing:

$$\begin{aligned}
\mathrm{Var}\big(\mathcal{Q}^1_{T,N}\big) &\approx \sigma_\theta^2 + \sigma_P^2 \int_{T-1}^{T}\int_{T-1}^{T} \min(t,s)dtds \\
&\quad + \sigma_P^2 \underbrace{\int_{T-1}^{T}\int_{T-1}^{T} \min(t,s)\Bigg(\mathbb{E}\Big(\hat{\theta}_{t,T}\Big) + \mathbb{E}\Big(\hat{\theta}_{s,T}\Big) + \mathbb{E}\Big(\hat{\theta}_{t,T}\hat{\theta}_{s,T}\Big)\Bigg)dtds}_{\approx 0} \\
&= \sigma_\theta^2 + \sigma_P^2 \int_{T-1}^{T}\Bigg(\int_{T-1}^{s} tdt + \int_{s}^{T} sdt\Bigg)ds \\
&= \sigma_\theta^2 + \frac{1}{3}\sigma_P^2
\end{aligned}$$

*********NOTE: why and I able to set the $\hat{\theta}_{t,T}$ parts to zero. I believe it it exactly true when there is no persistence in the transitory shocks, and also if the variance of θ is zero. What exact notions am I calling on here...?**

A very similar calculation shows that:

$$\mathrm{Var}\big(\mathcal{Q}^2_{T,N}\big) \approx \sigma_\theta^2 + \frac{1}{3}\sigma_P^2$$

So we get that:

$$\begin{aligned}
\mathrm{Var}\Big(\Delta^N \log(\bar{y}_T)\Big) &= \mathrm{Var}\big(\mathcal{P}_{T,N}\big) + \mathrm{Var}\big(\mathcal{Q}^1_{T,N}\big) + \mathrm{Var}\big(\mathcal{Q}^2_{T,N}\big) \\
&\approx (N-1)\sigma_P^2 + (\sigma_\theta^2 + \frac{1}{3}\sigma_P^2) + (\sigma_\theta^2 + \frac{1}{3}\sigma_P^2) \\
&= (N - \frac{1}{3})\sigma_P^2 + 2\sigma_\theta^2
\end{aligned}$$

Now we turn to consumption. Consumption responds to permanent income with elasticity ϕ, while the impulse response to a transitory shock is given by some function $g : \mathbb{R}^+ \to \mathbb{R}$ with $g(s) = 0$ for $s > 2$. Total consumption *flow* is then given by:

$$C_t = C_t^P C_t^\theta$$

where

$$\begin{aligned}
C_t^P &= e^{\phi p_t - t\nu_{pc}} \\
C_t^\theta &= e^{\int_{t-2}^{t} g(t-s)dq_s - \nu_{qc}}
\end{aligned}$$

and ν_{p_c} and ν_{q_c} are defined such that $\mathbb{E}\Big(\frac{C_t^P}{C_s^P}\Big) = \mathbb{E}(C_t^\theta) = 1$ for all $t \geq s$. Analogous to the case with log income growth over N years (equation 1) we get:

$$\Delta^N \log(\bar{c}_T) = \log(\frac{C_{T-1}^P}{C_{T-N}^P}) + \log\Bigg(\int_{T-1}^{T} \frac{C_t^P}{C_{T-1}^P} C_t^\theta dt\Bigg) - \log\Bigg(\int_{T-N-1}^{T-N} \frac{C_t^P}{C_{T-N}^P} C_t^\theta dt\Bigg) \quad (2)$$

Defining $\mathcal{C}^P_{T,N}$, $\mathcal{C}^1_{T,N}$ and $\mathcal{C}^2_{T,N}$ to be the three parts of the sum in equation 2 respectively, we have:

$$\begin{aligned}
\mathcal{C}^P_{T,N} &= \log(\frac{C^P_{T-1}}{C^P_{T-N}}) \\
&= \phi\log(\frac{P_{T-1}}{P_{T-N}}) - (N-1)(\nu_{p_c} - \phi\nu_p) \\
\Rightarrow \mathrm{Cov}(\mathcal{P}_{T,N}, \mathcal{C}^P_{T,N}) &= (N-1)\phi \mathrm{Var}\Big(\log(\frac{P_T}{P_{T-1}})\Big) \\
&= (N-1)\phi\sigma^2_P
\end{aligned}$$

and that:

$$\begin{aligned}
\mathcal{C}^1_{T,N} &= \log\left(\int_{T-1}^{T} \frac{C^P_t}{C^P_{T-1}} C^\theta_t dt\right) \\
&= \log\left(\int_{T-1}^{T} \Big(\frac{P_t}{P_{T-1}}\Big)^\phi e^{-(t-(T-1))(\nu_{p_c}-\phi\nu_p)} C^\theta_t dt\right) \\
&\approx \log\left(\bar{C^\theta_T}\right) + \int_{T-1}^{T} \Big(\Big(\frac{P_t}{P_{T-1}}\Big)^\phi e^{-(t-(T-1))(\nu_{p_c}-\phi\nu_p)} - 1\Big) \frac{C^\theta_t}{\bar{C^\theta_T}} dt
\end{aligned}$$

where the steps taken in the approximation are the same as we did in the case of income.

$$\begin{aligned}
\mathrm{Cov}\Big(\mathcal{Q}^1_{T,N}, \mathcal{C}^1_{T,N}\Big) &= \mathrm{Cov}\Big(\log\Big(\bar{\theta}_T\Big), \log\Big(\bar{C^\theta_T}\Big)\Big) \\
&\quad + \mathbb{E}\left(\int_{T-1}^{T}\int_{T-1}^{T} \Big(\frac{P_t}{P_{T-1}} - 1\Big)\Big(\Big(\frac{P_s}{P_{T-1}}\Big)^\phi e^{-(s-(T-1))(\nu_{p_c}-\phi\nu_p)} - 1\Big) \frac{\theta_t}{\bar{\theta}_T}\frac{C^\theta_s}{\bar{C^\theta_T}} dtds\right) \\
&= \mathrm{Cov}\Big(\log\Big(\bar{\theta}_T\Big), \log\Big(\bar{C^\theta_T}\Big)\Big) \\
&\quad + \mathbb{E}\left(\int_{T-1}^{T}\int_{T-1}^{T} \Big(\Big(\frac{P_{\min(t,s)}}{P_{T-1}}\Big)^{1+\phi} e^{-(\min(t,s)-(T-1))(\nu_{p_c}-\phi\nu_p)} - 1\Big) \frac{\theta_t}{\bar{\theta}_T}\frac{C^\theta_s}{\bar{C^\theta_T}} dtds\right) \\
&= \mathrm{Cov}\Big(\log\Big(\bar{\theta}_T\Big), \log\Big(\bar{C^\theta_T}\Big)\Big) \\
&\quad + \int_{T-1}^{T}\int_{T-1}^{T} \mathbb{E}\Big(\Big(\frac{P_{\min(t,s)}}{P_{T-1}}\Big)^{1+\phi} e^{-(\min(t,s)-(T-1))(\nu_{p_c}-\phi\nu_p)} - 1\Big) dtds \\
&\approx 0 \begin{cases} + \int_{T-1}^{T}\int_{T-1}^{T} \mathbb{E}\left(\Big(\Big(\frac{P_{\min(t,s)}}{P_{T-1}}\Big)^{1+\phi} e^{-(\min(t,s)-(T-1))(\nu_{p_c}-\phi\nu_p)} - 1\Big)\right. \\ \left. \qquad \times\Big(\mathbb{E}\Big(\hat{\theta}_t\Big) + \mathbb{E}\Big(\hat{C}^\theta_s\Big) + \mathbb{E}\Big(\hat{\theta}_t\hat{C}^\theta_s\Big)\Big)\right) dtds \end{cases} \\
&= \mathrm{Cov}\Big(\log\Big(\bar{\theta}_T\Big), \log\Big(\bar{C^\theta_T}\Big)\Big) \\
&\quad + \int_0^1\int_0^1 \mathbb{E}\Big(P^{1+\phi}_{\min(t,s)} e^{-\min(t,s)(\nu_{p_c}-\phi\nu_p)} - 1\Big) dtds
\end{aligned}$$

where $\hat{C}^\theta_{t,T} = \frac{C^\theta_t - \bar{C}^\theta_T}{\bar{C}^\theta_T}$. We now assume that p_t has no jumps, and is therefore a Brownian motion. With this assumption, $\nu_p = \frac{1}{2}\sigma_P^2$ and $\nu_{p_c} = \frac{1}{2}\phi^2\sigma_P^2$ and $\mathbb{E}(P_t^{1+\phi}) = e^{\frac{1}{2}t(1+\phi)^2\sigma_P^2 - \frac{1}{2}t(1+\phi)\sigma_P^2}$ so that:

$$
\begin{aligned}
\mathrm{Cov}\Big(\mathcal{Q}^1_{T,N}, \mathcal{C}^1_{T,N}\Big) &= \mathrm{Cov}\Big(\log\Big(\bar{\theta}_T\Big), \log\Big(\bar{C^\theta_T}\Big)\Big) \\
&\quad + \int_0^1 \int_0^1 \Big(e^{\frac{1}{2}\min(s,t)\sigma_P^2((1+\phi)^2-(1+\phi)-\phi^2+\phi)} - 1\Big) dtds \\
&= \mathrm{Cov}\Big(\log\Big(\bar{\theta}_T\Big), \log\Big(\bar{C^\theta_T}\Big)\Big) \\
&\quad + \int_0^1 \int_0^1 \Big(e^{\min(s,t)\phi\sigma_P^2} - 1\Big) dtds \\
&\approx \mathrm{Cov}\Big(\log\Big(\bar{\theta}_T\Big), \log\Big(\bar{C^\theta_T}\Big)\Big) + \phi\sigma_P^2 \int_0^1 \int_0^1 \min(s,t) dtds \\
&= \mathrm{Cov}\Big(\log\Big(\bar{\theta}_T\Big), \log\Big(\bar{C^\theta_T}\Big)\Big) + \frac{1}{3}\phi\sigma_P^2
\end{aligned}
$$

Similarly

$$
\mathrm{Cov}\Big(\mathcal{Q}^2_{T,N}, \mathcal{C}^2_{T,N}\Big) \approx \mathrm{Cov}\Big(\log\Big(\bar{\theta}_T\Big), \log\Big(\bar{C^\theta_T}\Big)\Big) + \frac{1}{3}\phi\sigma_P^2
$$

So that the covariance of income growth with consumption growth over N years is:

$$
\mathrm{Cov}\Big(\Delta^N \log(\bar{y}_T), \Delta^N \log(\bar{c}_T)\Big) = (N - \frac{1}{3})\phi\sigma_P^2 + 2\mathrm{Cov}(\tilde{y}, \tilde{c})
$$

where $\tilde{y} = \log\Big(\bar{\theta}_T\Big)$ and $\tilde{c} = \log\Big(\bar{C^\theta_T}\Big)$