Problem Set 7 (Solution)
Due 24. November, 14:15

Exercise 7.1: An Aiyagari model with endogenous labor supply

Consider a closed economy with ex-ante identical households that - conditional on the states (a_t, y_t) - solve the same recursive consumer problem in every period t

$$V(a_t, y_t) = u(c_t) - v(h_t) + \beta \mathrm{E}_t \left[V(a_{t+1}, y_{t+1}) \right],$$

subject to

$$\begin{aligned} c_t + a_{t+1} &= (1 + r_t)a_t + h_t y_t w_t \\ y_{t+1} &\sim \Gamma(y_t), \quad y_{t+1} \in Y = \{\bar{y}_1, ..., \bar{y}_N\}, \, N < \infty \\ c_t &\geq 0, \, h_t \geq 0, \, a_{t+1} \geq -b, \end{aligned}$$

where c_t is individual consumption, a_t asset holdings, h_t labor supply, and y_t can be interpreted as an idiosyncratic productivity shock that follows a stochastic Markov process with transition probability matrix, $\Gamma(y_t)$, and realizations drawn from the finite-valued set Y. The parameters are restricted to be, $0 < \beta < 1$ (subjective discount factor) and $b \geq 0$ (borrowing constraint), and the efficiency wage $w_t > 0$ as well as the interest rate $r_t > 0$ are determined in the competitive production sector. What makes households ex-post heterogeneous is that each of them draws an individual-specific productivity realization y_{t+1} from the same distribution, and since financial markets are incomplete (there is only a risk-free asset available, but no state-contingent assets) agents with different shock histories will be different ex post.

(a) Derive the household's optimality conditions with respect to consumption, labor supply, and future asset holdings, taking as given factor prices, w_t and r_t, and the transition probabilities, $\Gamma(y_t)$. (Hint: you can ignore the positivity constraints on consumption and labor supply, the functional forms of utility will make sure later this is satisfied. But you have to incorporate the borrowing constraint on the asset holdings.)

Solution:

The optimality conditions fore every period t are

$$\begin{aligned} 0 &= u'(c_t) - \lambda_t \\ 0 &= -v(h_t) + \lambda_t y_t w_t \\ 0 &= \beta \mathrm{E}_t \left[\frac{\partial V(a_{t+1}, y_{t+1})}{\partial a_{t+1}} \right] - \lambda_t + \mu_t, \quad \mu_t(a_{t+1} + b) = 0, \, \mu_t \geq 0, \\ \frac{\partial V(a_t, y_t)}{\partial a_t} &= \lambda_t (1 + r_t) \\ 0 &= (1 + r_t)a_t + h_t y_t w_t - c_t - a_{t+1}. \end{aligned}$$

Eliminating the Lagrange multiplier, λ_t, and the Envelope condition yields the set of equations

$$\begin{aligned} v'(h_t) &= u'(c_t)y_t w_t \\ u'(c_t) &= \beta \mathrm{E}_t\left[u'(c_{t+1})(1+r_t)\right] + \mu_t \\ \mu_t(a_{t+1}+b) &= 0,\ \mu_t \geq 0 \\ a_{t+1} &= (1+r_t)a_t + h_t y_t w_t - c_t, \end{aligned}$$

that characterizes the optimality of the household's choice.

(b) Let the inverse function of the marginal disutility of labor be denoted by $(v')^{-1}(\cdot)$. Characterize the optimal labor supply, as a function of the optimal consumption and the productivity realization, $\mathcal{H}(c_t, y_t)$.
Solution:

The intratemporal optimality condition implies

$$\mathcal{H}(c_t, y_t) \equiv (v')^{-1}\left(u'(c_t)y_t w_t\right).$$

(c) Guess a decision rule (subscripts indicate functions that depend on this guess) for the optimal consumption as a function of the future states,

$$c_0(a_{t+1}, y_{t+1}).$$

Let the future asset level be consistent with a non-binding borrowing constraint, $a_{t+1} \geq -b$ and the shock realizations be drawn from the finite valued set, $Y = \{\bar{y}_1, ..., \bar{y}_N\}$ with transition probabilities defined by

$$\pi_{n,m} \equiv \pi(y_{t+1} = \bar{y}_n | y_t = \bar{y}_m), \quad \forall n, m \in \{1, ..., N\},$$

such that

$$\Gamma(y_t) = \begin{bmatrix} \pi_{1,1} & \cdots & \pi_{1,m} & \cdots & \pi_{1,N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \pi_{n,1} & \cdots & \pi_{n,m} & \cdots & \pi_{n,N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \pi_{N,1} & \cdots & \pi_{N,m} & \cdots & \pi_{1,N} \end{bmatrix}.$$

Denote the inverse marginal utility function of consumption by $(u')^{-1}(\cdot)$. Derive the function, $\mathcal{C}_0(a_{t+1}, y_t)$, that measures the current consumption level as a function of the future state, a_{t+1}, the current state, y_t, and the guess, $c_0(a_{t+1}, y_{t+1})$ if the the borrowing constraint is not binding, $\mu_t = 0$.
Solution:

Given the guess, the Euler equation characterizes the current consumption level

$$\begin{aligned}\mathcal{C}_0(a_{t+1}, y_t) &= (u')^{-1}\left(\beta(1+r_{t+1})\mathrm{E}_t u'\left(c_0(a_{t+1}, y_{t+1})\right)\right) \\ &= (u')^{-1}\left(\beta(1+r_{t+1})\sum_{y_{t+1}\in Y}\pi(y_{t+1}|y_t)u'\left(c_0(a_{t+1}, y_{t+1})\right)\right).\end{aligned}$$

(d) Now, find the current asset level $\mathcal{A}_0(a_{t+1}, y_t)$ that is consistent with the guess of the consumption decision rule. Or, on other words, $\mathcal{A}_0(a_{t+1}, y_t)$ defines the current assets holdings for a household with labor productivity, y_t, who has chosen to save a_{t+1} for tomorrow).

Solution:

From the budget constraint we can derive the current asset level that is consistent with the guess,

$$\mathcal{A}_0(a_{t+1}, y_t) = \frac{1}{1+r_t}\left[\mathcal{C}_0(a_{t+1}, y_t) + a_{t+1} - \mathcal{H}\left(\mathcal{C}_0(a_{t+1}, y_t), y_t\right) y_t w_t\right].$$

Note that $\mathcal{A}_0(a_{t+1}, y_t)$ defines the current assets holdings for a household with labor productivity, y_t, who has chosen to save a_{t+1} for tomorrow.

(e) The pair of functions, $\langle \mathcal{C}_0(a_{t+1}, y_t), y_t), \mathcal{A}_0(a_{t+1}, y_t)\rangle$ implicitly defines a relationship between current consumption, c_t, and the current asset level and productivity realization, (a_t, y_t), given that the borrowing constraint does not bind, $\mu_t = 0$. Thus, the guess on the consumption function can be updated by simply "interpolating" $c_0\left(\mathcal{A}_0(a_{t+1}, y_t), y_t\right)$ on (a_{t+1}, y_{t+1}) to yield

$$c_1\left(a_{t+1}, y_{t+1}\right), \quad a_{t+1} \geq \mathcal{A}_0(-b, y_t)$$

For future asset levels a_{t+1} that are smaller than the lowest current asset level that is exactly consistent with a binding borrowing constraint in the future

$$a_{t+1} < \mathcal{A}_0(-b, y_t),$$

the procedure is different: for those asset levels, update the consumption function that is consistent with a binding borrowing constraint, $a_{t+1} = -b$. Consider the following explicit form for the marginal utility functions (this parametrization is also valid for the upcoming questions)

$$\begin{aligned}u'(c) &= c^{-\gamma}, \quad \gamma = 3/2 \\ v'(h) &= h^{1/\varphi}, \quad \varphi = 2/3.\end{aligned}$$

Derive the updated consumption function for the case $a_{t+1} < \mathcal{A}_0(-b, y_t)$.

Solution:

For a_{t+1} that are below the stated threshold, update the consumption function according to the budget constraint

$$c_1\left(a_t, y_t\right) = (1+r_t)a_t + \mathcal{H}(c_1\left(a_t, y_t\right), y_t)y_t w_t + b,$$

to find the consumption consistent with a binding borrowing constraint, $a_{t+1} = -b$. Then simply iterate one period forward to yield $c_1(a_{t+1}, y_{t+1})$. Note that under the given parameter restriction

$$\mathcal{H}(c, y) = \left(c^{-3/2} y w\right)^{2/3},$$

such that the consumption function can be expressed as the quadratic equation

$$\begin{aligned} 0 &= -c + (1+r)a + \left(c^{-3/2} y w\right)^{2/3} y w + b \\ 0 &= c^2 - [(1+r)a + b]c - (yw)^{1+2/3}. \end{aligned}$$

The two solutions are

$$c = \frac{[(1+r)a + b] \pm \sqrt{[(1+r)a+b]^2 + 4(yw)^{1+2/3}}}{2},$$

where the lower root can be ruled out because it violates the non-negativity constraint of consumption. Thus, the updated consumption function is given by

$$c_1(a, y) = \frac{[(1+r)a + b] + \sqrt{[(1+r)a+b]^2 + 4(yw)^{1+2/3}}}{2}.$$

(f) From here we aim to solve for the stationary equilibrium of this incomplete markets economy with Matlab. We concentrate on stationary equilibria in the sense that we require the interest rate (and therefore also the labor wage, $w_t = w$) to be constant $r_t = r$ and consistent with capital market clearing.

Set $\beta = 97/100$, $Y \equiv \{\bar{y}_1, \bar{y}_2\} = \{95/100, 105/100\}$, and let the state transition probabilities be symmetric,

$$\rho \equiv \pi(\bar{y}_1 | \bar{y}_1) = \pi(\bar{y}_2 | \bar{y}_2) = 9/10.$$

For the discretization of the future asset level, a_{t+1}, consider $M = 250$ equally spaced grid points on the grid

$$A = \{\bar{a}_1, \ldots, \bar{a}_M\}, \quad \bar{a}_1 < \cdots < \bar{a}_M,\ \bar{a}_1 = -b,\ \bar{a}_M = 45,$$

and set the borrowing constraint to $b = 0$. Start with the guess that agents consume their asset income plus labor income with unit labor supply,

$$c_0(a_{t+1}, y_{t+1}) = r a_{t+1} + y_t w.$$

For the moment set the stationary prices to $r_0 = (1/\beta - 1) - 10^{-4}$ and $w_0 = 1$, we will solve for the exact equilibrium values in the next step. Assuming that a_{t+1} and y_{t+1} can only take values on the corresponding grids (that's what is usually meant by discretization), A and Y, respectively, this can be expressed as

$$c_0(A, Y) = r_0 A + Y w_0.$$

Use the results from the previous subquestions to write a program that updates this consumption function iteratively until the following convergence criterion is satisfied,

$$\left|\left|\frac{c_k(A, Y) - c_{k-1}(A, Y)}{c_{k-1}(A, Y)}\right|\right|_2 \leq 10^{-6},$$

where k denotes the current iteration and $||\cdot||_2$ denotes the 2-norm. Note that the endogenous values for the current asset level, $\mathcal{A}_k(A, Y)$, will in general be off the grid points of A, thus you will have to use an interpolation routine (in Matlab you could use interp1) to update

$$c_k(a_{t+1}, y_{t+1}) = c_k(A, Y),$$

given that you can compute

$$c_k(a_t, y_t) = c_k(\mathcal{A}_k(A, Y), Y).$$

Solution:

See code, PS7main.m.

(g) Extend your program and simulate the optimal consumption, savings, and labor supply of $I = 10^4$ individuals over $T = 10^4$ periods for the given guess on the stationary prices, r_0 and w_0. The guess on the interest rate r_0 will in general not be consistent with market clearing in the stationary equilibrium. Thus, let us update the stationary prices according to marginal pricing in a Cobb-Douglas production sector, $F(K, L) = K^\alpha L^{1-\alpha}$,

$$r_1 = \alpha \left(\frac{K_1}{L_1}\right)^{\alpha - 1} - \delta, \quad w_1 = (1 - \alpha) \left(\frac{\alpha}{r_1 + \delta}\right)^{\alpha/(1-\alpha)}.$$

where K_1 and L_1 denote the average capital and labor supply over the last 100 simulation periods,

$$K_1 = \frac{1}{100 \times I} \sum_{t=T-100}^{T} \sum_{i=1}^{I} a_t^i(r_0), \quad L_1 = \frac{1}{100 \times I} \sum_{t=T-100}^{T} I^{-1} \sum_{i=1}^{I} y_t^i h_t^i(r_0).$$

Set $\alpha = 1/3$ and $\delta = 5/100$, and update the interest rate iteratively until convergence. The stationary equilibrium is very sensitive to the interest rate, so you should limit the search for the equilibrium interest rate to the interval

$$[(1/\beta - 1) - 10^{-12}, (1/\beta - 1) - 10^{-4}].$$

A good way to organize the program is to write a function (you could call it stationary_equilibrium.m) that - for a given interest rate - (i) solves for the optimal decision rules of the agents, (2) then simulates I indviduals over T periods, (3) returns the residual relative to the market clearing interest rate. You can then use Matlab's rootfinding routine fzero to search for the interest rate that makes the residual of the above function equal to zero (which is the stationary equilibrium interest rate).

Solution:

See code, PS7main.m.

(h) Compared to the transition probability of $\rho = 9/10$ that we considered so far, how does the stationary wealth distribution change if the productivity shocks are instead i.i.d. over time, $\rho = 1/2$?

Solution:

See code, PS7main.m. In general, there is less probability mass in the tails of the distribution (less agent's are effectively borrowing constraint) and the ergodic set is more narrow. Also, the average asset level is lower such that the stationary interest rate is higher.