Solving heterogeneous agent models in discrete time with many idiosyncratic states by perturbation methods*

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Abstract

This paper describes a method for solving heterogeneous agent models with aggregate risk and many idiosyncratic states formulated in discrete time. It extends the method proposed by Reiter (2009) and complements recent work by Ahn et al. (2017) on how to solve such models in continuous time. We suggest first solving for the stationary equilibrium of the model without aggregate risk. We then write the functionals that describe the recursive equilibrium as sparse expansions around their stationary equilibrium counterparts. Finally we use the perturbation method of Schmitt-Grohé and Uribe (2004) to approximate the aggregate dynamics of the model.

Keywords: Numerical Methods, Heterogeneous Agent Models, Linearization, Incomplete Markets

JEL-Codes: C63, E32

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1 Introduction

Models of heterogeneous agents have become widespread in macroeconomics, at least since Krusell and Smith (1997, 1998) developed the first widely applicable algorithm to solve them in an environment of aggregate risk. Yet, their use has been limited initially by the computational resources needed to solve these models. Over the last decade, substantial progress has been made in developing algorithms that can solve these models more efficiently. One of the most popular and powerful of these methods was originally developed by Reiter (2002, 2009). This method extends perturbation methods to heterogeneous agent environments, i.e., it builds on the methods often used to solve dynamic stochastic general equilibrium models with a representative agent (see, e.g., Schmitt-Grohé and Uribe, 2004). Our paper restates this procedure and additionally shows how the necessary dimensionality reduction of the heterogeneous agent model can be achieved in a new, intuitive way.

The extension of perturbation methods to heterogeneous agent models relies on writing the model in the form of a non-linear difference equation that is function-valued instead of vector-valued (as in representative agent models). This equation is then (linearly) approximated around the stationary equilibrium of the heterogeneous agent model without aggregate risk. The (at least) two functionals that enter the difference equation are the distribution of agents over idiosyncratic states (e.g., the wealth distribution) and the function (value or policy function) that describes the optimal individual behavior. These functionals can be seen as replacements of the aggregate capital accumulation and consumption Euler equation in representative agent models. These replacements allow us to maintain all non-linearity with respect to microeconomic shocks—yet obtaining a model that is linear in aggregate variables.

While all of this is straightforward in theory, the key practical issue is how to approximate the functionals involved because they need to be replaced by finite-dimensional objects for the actual computation of the model's dynamics. In particular, when the individual planning problem is rich insofar as it has many idiosyncratic states, this issue is severe. The curse of dimensionality implies that it is hard to come up with a small enough finite-dimensional representation of the distribution function and the value/policy function without having any *a priori* knowledge of their shape.

However, the solution of the stationary equilibrium provides us with such knowledge. Therefore, we propose a dimensionality reduction step *after* the stationary equilibrium of the economy (i.e., without aggregate risk) has been determined, but *before* perturbing the system. This dimensionality reduction is adaptive and takes into account the shape

of the distribution and value function in the stationary equilibrium. As a result, it does not constrain the computation of the stationary equilibrium such that standard methods can be applied here.

In detail, we suggest using sparse expansions of value and distribution functions around their non-sparse stationary equilibrium counterparts. First, we write the value function in the stationary equilibrium as a sum of a full set of basis functions and determine the coefficients on these. We then allow only those coefficients of the basis functions to vary outside the stationary equilibrium that are significantly different from zero in the stationary equilibrium. A particularly numerically efficient way to do this replacement is through the use of the discrete cosine transformation as is done in lossy image compression techniques. Second, we approximate high-dimensional distribution functions by histograms of their marginals and the copula of the stationary equilibrium. This second dimensionality reduction picks up the idea of Krusell and Smith (1997, 1998) that not all moments of the wealth-income distribution are equally important for price formation and therefore relevant for the equilibrium dynamics. The assumption of a fixed copula implies that the rank correlation among, say, wealth in various kinds of assets and income is time constant without imposing any restriction on changes in the shape of the marginal distributions. Here we show, both for an incomplete markets model with one asset and for a model with two assets, that the fixed copula assumption has virtually no impact on the model dynamics but substantially speeds up the computation. The largest share of the computation time falls on the calculation of the stationary equilibrium followed by the calculation of the derivatives of the non-linear difference equation. However, both can be sped up by parallelization. At any rate, the models we consider can all be solved on a standard desktop computer in a matter of seconds or minutes using our algorithm.

By reducing the dimensionality after the solution of the stationary equilibrium but before linearizing, our method differs from existing proposals. The original proposal by Reiter (2002) was to represent distribution functions by histograms without any dimensionality reduction and to write value functions (or other functionals describing the dynamic planning problem) as finite-dimensional parametric objects—for example, by using splines. However, when the individual planning problem is rich insofar as it has many idiosyncratic states, this procedure can become inaccurate and in many cases even infeasible to solve numerically. The first idea to tackle this issue was to be as sparse as possible in the parametric approximation of functions when solving for the stationary equilibrium (see, e.g., Reiter, 2009), e.g., through sparse grid methods in the dynamic planning problem (see, e.g., Bungartz and Griebel, 2004; Krueger and Kubler, 2004) and

by using mixtures of parametric distributions as proposed by Winberry (2016). In other words, these methods rely on achieving dimensionality reduction *ex ante*, before solving for the stationary equilibrium, and hence impose a numerical constraint on this solution.

An alternative attack, also suggested by Reiter (2009), is to use singular value decomposition for dimensionality reduction of the Jacobian of the system *after* linearizing the difference equation but before solving it. Ahn et al. (2017) develop this approach further in that they write the planning problem in continuous time and suggest using automatic differentiation in order to obtain a sparse Jacobian. This helps both with the memory requirements, precision, and with the computing time for both the singular value decomposition and the solving of the difference equation itself. In addition, they suggest perturbing the deviations of value and distribution functions from their stationary equilibrium counterparts instead of perturbing the functions themselves. This allows for different parametric classes for deviations and stationary equilibriums functionals. As a result, it decouples the number of perturbed parameters from the number of parameters used in the approximation of the functions in the stationary equilibrium (which can potentially be richer). Our approach shares the latter aspect with the approach of Ahn et al. (2017). Compared to their method, ours has the advantage of avoiding the calculation of a very large Jacobian because the dimensionality is reduced before this step. Thus, it can be applied to models formulated in discrete time, where the Jacobian would otherwise be too non-sparse to be efficiently stored in a PC's memory.

The remainder of the paper is organized as follows: Section 2 defines the generic model we aim to solve with our method and lays out the solution method itself. Section 3 provides the economic model of two application examples: first, a standard incomplete markets model with just a single asset, capital, as in Krusell and Smith (1998); second, an extensions of that model, in which households have to choose between assets of different liquidity. They can hold a liquid nominal asset or illiquid capital. We add a nominal rigidity to this model, such that it is of the New-Keynesian flavor. For the first model variant, we can compare our solution to the original Krusell and Smith (1998) algorithm and to the standard Reiter (2009) approach. Our method is equally precise as Reiter's standard approach but faster. It is both faster and more precise than the Krusell and Smith algorithm in our example. The second model variant is too rich in terms of aggregate states to be solved by Reiter's standard approach. Here we only show that simulating the model along the lines of Den Haan (2010)'s test proves the method to be accurate. Section 5 concludes. Example codes are provided as an online appendix.

2 Method

We consider a generic economy with a continuum of heterogeneous agents and aggregate risk. We first define the objects we need to work with. Thereafter, we define a stationary equilibrium and a sequential equilibrium (with recursive individual planning) for this economy. Then, we describe how the sequential equilibrium can be solved for locally and how a reduction of the state space can be achieved. Finally, we give an overview of the suggested algorithm.

2.1 Prerequisites and notation

Let $S_t \in \mathbb{R}^n$ denote the aggregate states in this economy other than the distribution of agents over their idiosyncratic states $s_{it} \in \mathbb{R}^m$ for individual i at time t. In a representative agent model these S_t would be the only state variables. With heterogeneous agents, the distribution *function* μ_t of agents over s_{it} is also part of the aggregate states of the economy but for notational purposes shall not be included in S_t.

Both S_t and s_{it} shall be partitioned in an exogenous stochastic and an endogenous deterministic component

$$S_t = \begin{bmatrix} S_t^0 \\ S_t^1 \end{bmatrix}, \quad s_{it} = \begin{bmatrix} s_{it}^0 \\ s_{it}^1 \end{bmatrix}, \tag{1}$$

with length n_0, n_1, m_0, and m_1, respectively.

With the stochastic elements in S_t and s_{it}, agents in the economy face both aggregate and idiosyncratic risk. We denote the stochastic elements of the aggregate and idiosyncratic state space by S_t^0 and s_{it}^0, respectively. We assume that all stochastic variables follow a stationary Markov chain, such that

$$S_{t+1}^0 = H^0(S_t^0) + \varepsilon_{t+1}^S, \quad s_{it+1}^0 = h^0(s_{it}^0) + \epsilon_{it+1}^s, \tag{2}$$

and the innovations ($\varepsilon_{t+1} = [\varepsilon_{t+1}^S, \epsilon_{it+1}^s]$) have variances $\omega_S \Omega_S$ and $\sigma_s \Sigma_s$ for the aggregate and idiosyncratic variables, respectively.

The remaining idiosyncratic state variables s_{it}^1 are chosen by households in order to maximize their utility. This choice shall be described by the generic planning problem

$$\nu(s_{it}, S_t, \mu_t) \quad = \quad \max_{s_{it+1}^1 \in \Gamma(s_{it}, P_t)} u(s_{it}, s_{it+1}^1) + \beta \mathbb{E}\nu(s_{it+1}, S_{t+1}, \mu_{t+1}), \tag{3}$$

where Γ is a budget set and $P_t = P(S_t, \mu_t)$ is a pricing kernel. Prices may result from market clearing in the sense introduced below, but may also be directly determined by

the aggregate state or the distribution, such as, e.g., interest rates set by the central bank or the wage rate as a function of the aggregate amount of capital. The further aggregate states move for simplicity according to some given law of motion $S^1_{t+1} = H^1(S_t, \mu_t)$.[1] Note that this does not preclude prices from depending also on choices for state variables S_{t+1} made at time t because we can write these as functions of states in t.

It will come in handy later to simplify notation for the Bellman equation by observing that, from the individual's point of view, aggregates and distributions only matter through prices. These, in turn, we can summarize by adding a time index t to the value functions. Dropping the time indexes to the *idiosyncratic* states and using $'$ to denote next period variables, we can write the recursive planning problem of the individual household as:

$$\nu(s_i, t) = \max_{s_i^{1\prime} \in \Gamma(s_i, P(t))} u(s_i, s_i^{1\prime}) + \beta \mathbb{E}\nu(s_i', t+1), \tag{4}$$

where the time index here stands for conditioning the individual planning problem and the pricing kernel on all state variables of time t. Individual policy functions h^1_t can be defined accordingly.

To close the model, we need a description of market clearing. We define an excess demand function $\Phi(h^1_t(s), d\mu_t, P_t, S_t)$ that maps the idiosyncratic policies, the distribution, prices and aggregate states into a real vector. Typically, we have as many prices as idiosyncratic endogenous states, given that we assumed an exogenous law of motion for aggregate states, i.e, $\Phi(h^1_t(s), d\mu_t, P_t, S_t) \in \mathbb{R}^{m_1}$.

For example, in an economy as in Krusell and Smith (1998), i.e., with capital and aggregate productivity risk, Φ is given by the difference between the marginal product of capital and the rate of return on capital. In a bond economy with only IOUs, in contrast, we would have $\Phi = \int h^1_t(s) d\mu_t$, and in an economy with government bonds this would be $\Phi = \int h^1_t(s) d\mu_t - B_t$, where B_t is the amount of government bonds issued and circulating in t.

2.2 Stationary equilibrium and approximate solution

Since the method developed by Reiter (2009) approximates the aggregate dynamics around the stationary equilibrium, we first consider an economy without aggregate risk, i.e., where $\omega_S = 0$. For such an economy, prices, distributions, and hence value functions do not change over time, and we can define a stationary equilibrium generically as follows.

[1] The law of motion H can be the outcome of some other aggregate planning problem as well. Importantly it is neither stochastic nor influenced by a single individual decision.

Definition 1. *A **stationary equilibrium** is a value function $\bar{\nu}$, a distribution function $\bar{\mu}$, a policy function $\bar{h}^1(s)$, and prices $\bar{P}$ such that*

1. *The individual policy is the maximizer of the Bellman equation (3) given prices $\bar{P}$.*
2. *The value function solves the Bellman equation (3) given the individual policy $\bar{h}^1$.*
3. *Markets clear, i.e., $\Phi(\bar{h}^1, \bar{\mu}, \bar{P}) = 0$.*
4. *The distribution $\bar{\mu}$ is the stationary distribution of the Markov chain induced by $\bar{h}(s, \epsilon^s) := \begin{bmatrix} h^0(s) + \epsilon^s \\ \bar{h}^1(s) \end{bmatrix}$.*

To solve for the equilibrium it is necessary to approximate the model. Typically, the model is solved for a (full tensor) grid of points in $\mathbb{R}^m$ replacing the functionals by some parametric approximation. A common approach is, for example, to replace the value functions by splines with the nodes of the spline being equal to the grid points. When first-order conditions are sufficient and the problem is differentiable, we can replace the Bellman equation with an Euler equation to describe the planning problem. Since the techniques to find the equilibrium value functions are standard, we refer only to the literature here (see, e.g., Carroll, 2006; Hintermaier and Koeniger, 2010).

Similarly, the distribution is often approximated by a step function (the density being replaced by a point mass, i.e., a histogram) on the grid or by a piecewise linear function (the density function being a step function). Since policy functions map potentially into non-grid points, a standard technique is to introduce some trembling to the policy function such that policies fall on neighboring grid points with such probabilities that the off-grid policy equals the expected value of the tremble.

Under these assumptions, the dynamics of the wealth distribution can be described by the histogram $d\mu$ and a transition matrix $\Pi_{\bar{h}}$ induced by the policy function $\bar{h}$. In the stationary economy

$$d\bar{\mu} = d\bar{\mu}\Pi_{\bar{h}} \tag{5}$$

needs to hold. This is the discrete time analogue to the Kolmogorov forward / Fokker-Planck equation in continuous time systems. For a given transition probability matrix, i.e., for a given policy function, the stationary distribution can then be calculated efficiently by determining the left unit eigenvector of $\Pi_{\bar{h}}$. Similarly, if we assume that the value function is replaced by a linear interpolant, we obtain the result that the solution to the Bellman equation is given by a finite vector of values, with a slight abuse of

notation also denoted by $\bar{\nu}$, which needs to satisfy

$$\bar{\nu} = u_{\bar{h}} + \beta \Pi_{\bar{h}} \bar{\nu}, \tag{6}$$

where $u_{\bar{h}}$ is the period payoff under the optimal policy:[2]

$$\bar{h}(s) = \arg \max_{s^{1\prime} \in \Gamma\left(s, \bar{P}\right)} u(s, s^{1\prime}) + \beta \mathbb{E} \bar{\nu}(s'). \tag{7}$$

In the following, we assume that the stationary equilibrium is solved for in this way on a full tensor grid, because these methods are readily available, easy to implement, and their application is in most cases not constrained by memory availability even on desktop computers. However, the method laid out below extends readily to the case where the stationary equilibrium is solved for by sparse grid methods; see Bungartz and Griebel (2004) or Krueger and Kubler (2004).

2.3 Sequential equilibrium with recursive individual planning

If there is uncertainty regarding the aggregate states, value functions written as functions of idiosyncratic states are no longer time constant in equilibrium. The same holds true for the distribution functions. Instead, if the model is stationary, value functions and distributions will converge to a sequence that fulfills the following equilibrium conditions.[3]

Definition 2. *A **sequential competitive equilibrium with recursive individual planning** is a sequence of value functions ν_t, a sequence of distribution functions μ_t, a sequence of policy functions $h_t^1(s)$, a sequence of aggregate states S_t, and a sequence of prices P_t such that at each point in time t:*

1. *The individual policy is the maximizer of the the Bellman equation (3) given the prices P_t.*

2. *The value function solves the Bellman equation (3) given the individual policy h_t^1 and the expected continuation value ν_{t+1}.*

[2] If first-order conditions are sufficient such that, say, a standard consumption Euler equation holds, we can also work with

$$\bar{u}'_{\bar{h}} = \beta(1+r)\Pi_{\bar{h}} \bar{u}'_{\bar{h}},$$

instead of (6), where $\bar{u}'_{\bar{h}}$ is the marginal utility of consumption under the optimal policy.

[3] Note that we write the problem still in recursive form from a household's point of view for notational convenience.

3. *Markets clear, i.e.,* $\Phi(h_t^1, \mu_t, P_t, S_t) = 0$.

4. *The distribution* μ_{t+1} *is induced by* $h_t(s, \epsilon^s) := \begin{bmatrix} h^0(s) + \epsilon^s \\ h_t^1(s) \end{bmatrix}$ *and the distribution* μ_t.

5. *The sequence of aggregate states is induced by* $S_{t+1} = H(S_t) + \begin{bmatrix} \varepsilon_{t+1}^S \\ 0 \end{bmatrix}$

Again, we need to approximate the functions involved in the model in a suitable way to solve the model. For that purpose, we replace the distribution function by a step function and add trembles to the policy. Finally, we write the value function as a linear interpolant. This implies that the discrete time Fokker-Planck equation (5) takes the form

$$d\mu_{t+1} = d\mu_t \Pi_{h_t}, \tag{8}$$

which makes clear its forward equation character. Further note that due to the continuum-of-agents assumption, there is no randomness in the transition other than through aggregate states and therefore shocks changing h_t. The Bellman equation (6) now takes the form

$$\nu_t = u_{h_t} + \beta \Pi_{h_t} \nu_{t+1}, \tag{9}$$

where u_{h_t} is the period payoff under the optimal policy:

$$h_t(s) = \arg \max_{s^{1\prime} \in \Gamma(s, P_t)} u(s, s^{1\prime}) + \beta \mathbb{E} \nu_{t+1}(s'). \tag{10}$$

Combining these equilibrium conditions, we can summarize the sequential equilibrium conditions by the non-linear difference equation given by

$$F\left(d\mu_t, S_t, d\mu_{t+1}, S_{t+1}, \nu_t, P_t, \nu_{t+1}, P_{t+1}, \varepsilon_{t+1}\right) = \begin{bmatrix} d\mu_{t+1} - d\mu_t \Pi_{h_t} \\ S_{t+1}^0 - H^0(S_t) + \varepsilon_{t+1} \\ S_{t+1}^1 - H^1(S_t, d\mu_t) \\ \nu_t - (u_{h_t} + \beta \Pi_{h_t} \nu_{t+1}) \\ \Phi(h_t^1, d\mu_t, P_t, S_t) \\ \varepsilon_{t+1} \end{bmatrix} \tag{11}$$

s.t.

$$h_t(s) = \arg \max_{s^{1\prime} \in \Gamma(s, P_t)} u(s, s^{1\prime}) + \beta \mathbb{E} \nu_{t+1}(s'). \tag{12}$$

A sequential equilibrium now fulfills

$$\mathbb{E}_t F\left(d\mu_t, S_t, d\mu_{t+1}, S_{t+1}, \nu_t, P_t, \nu_{t+1}, P_{t+1}, \varepsilon_{t+1}\right) = 0. \tag{13}$$

For notational simplicity, it is useful to define , $X_t := \begin{bmatrix} S_t & d\mu_t \end{bmatrix}'$ as the states of this system. The control variables, $Y_t := \begin{bmatrix} \nu_t & P_t \end{bmatrix}'$ are prices and value functions; to be more precise, their values at the nodes. Again if working with first-order conditions, value functions might be replaced with marginal utilities.

2.4 Approximating the sequential equilibrium around the stationary equilibrium

There are various ways to solve the non-linear difference equation, $\mathbb{E}_t F = 0$, by perturbation methods. Here we follow Schmitt-Grohé and Uribe (2004), who show how to solve the system (11) by first- and second-order perturbation. These methods can be readily applied here as well, choosing the stationary equilibrium solution as the point around which to perturb the system, as in Reiter (2009).

For expositional purposes, we focus on first-order perturbation here. This means that it is necessary to calculate the Jacobi matrix of the system (dropping ε_{t+1}), $J = \begin{bmatrix} F_X & F_{X'} & F_Y & F_{Y'} \end{bmatrix}$, and solve the linearized difference equation relating its solution to the generalized eigenvalue problem

$$\underbrace{\begin{bmatrix} F_X & F_Y \end{bmatrix}}_{A:=} Z\Lambda = \underbrace{-\begin{bmatrix} F_{X'} & F_{Y'} \end{bmatrix}}_{B:=} Z, \tag{14}$$

with Z being the matrix of eigenvectors and Λ the diagonal matrix of eigenvalues. Splitting the eigenvalues such that Λ_1 contains the eigenvalues in the unit circle, we can write $\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$ and $Z = \begin{bmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \end{bmatrix}$. If a local equilibrium exists and is unique, the number of eigenvalues in the unit-circle is equal to the number of state variables and the linearized law of motion for state variables is given by $H := Z_{11}\Lambda_1 Z_{11}^{-1}$, while states map to controls through $G := Z_{12}Z_{11}^{-1}$. For details we refer to Schmitt-Grohé and Uribe (2004). The fact that the distribution function over idiosyncratic states is part of the aggregate state vector and that the value functions (or marginal utilities) are part of the aggregate vector of controls does not change the solution in principle.

In practice, however, solving the generalized eigenvalue problem (or equivalently making a qz-decomposition of A, B) becomes easily numerically infeasible or at least

very imprecise in practice because the number of aggregate state variables (and controls) becomes very large, and thus A and B are large matrices. If the idiosyncratic state-space is high dimensional, both value functions and distribution functions are objects hard to approximate. A simple tensor grid to describe the value function or histogram has easily a large number points, even if it has a small number of points in each dimension of heterogeneity among households. Consider, for example, a household planning problem with two assets and idiosyncratic income. Even if we use only 9 points for the income grid and 50 points for each of the two asset grids, then both $d\mu$ and ν are vectors with a length of 22,500 entries, and with this resolution, the precision is at the lower bound of what one would like to have.

This creates various numerical problems. First, one needs to calculate many derivatives numerically. In our example, both A and B would be more than $45,000 \times 45,000$ entries large. While this calculating of the Jacobian is time consuming, the numerical complexity is only quadratic in the number of grid points. On top, modern automatic differentiation can speed this up. Notwithstanding, the matrix to be stored remains large; each has more than 7GB in our example if stored as a full double precision matrix.[4] Second, the qz-decomposition and the calculation of generalized eigenvalues become imprecise and numerically complex (cubic in the number of grid points).

The literature has suggested ways to deal with the issue. First, Reiter (2009) suggested replacing the value function with splines in order to decrease the number of nodes needed to describe the value function. Building on this suggestion, Winberry (2016) suggests using parametric families for the distribution functions to reduce the number of parameters that describe the distributions at each point in time. A downside of these two approaches is that they might impose tight restrictions on the value function and distribution in the stationary equilibrium. What is more, they no longer allow us to represent the Bellman equation and the distribution dynamics by conveniently linear systems. For this reason, Ahn et al. (2017) suggest working in continuous time, which increases the sparsity of the Jacobians. Then they suggest, following the original paper by Reiter (2009), to use singular-value decomposition of the Jacobians to project the state space of the model into a lower dimensional space without losing much of the dynamics of the system. Similar to what we suggest next, Ahn et al. (2017) linearize around the stationary equilibrium value and distribution functions without imposing any a priori restriction on the functional forms.

[4]Clearly, many of the derivatives are (close to) zero and thus storing the matrices as sparse matrices further helps. In fact, this is one of the main advantages of writing the model in continuous time, because then Π_h is very sparse and it is easy to see how this translates into a very sparse Jacobian; see Ahn et al. (2017).

2.5 State-space reduction: Fixed copula, compressed value function

We suggest reducing the dimensionality of the dynamic system before calculating the Jacobian. This allows us to solve the model in discrete time, where the Jacobian of the full system is much less sparse than in continuous time. We achieve this by assuming the copula of the distribution function to be constant at its stationary equilibrium value, $\bar{C}$:

$$\hat{\mu}_t(s) \quad = \quad \bar{C}\left\{\mu_{1t}(s), \ldots, \mu_{mt}(s)\right\}. \tag{15}$$

The idea here follows the general insight by Krusell and Smith (1998) that not all moments of the cross-sectional distribution μ have a strong impact on the distribution of prices economic agents need to forecast. Reiter (2009) proposes reducing the dimensionality of the state space by projecting the histogram of the joint distribution on a lower dimensional object that is perturbed instead. The projection can be done in such a way that some moments of the distribution are ensured to be precise. Yet, using this approach, the distribution function will in general not maintain the shape it has in the stationary equilibrium. With our method by contrast, it maintains its shape. In addition, the method can be expected to be locally exact if the rank-correlation structure has no significant impact on the equilibrium prices or is relatively constant; see Bayer et al. (2015) or Luetticke (2018) for examples.

Regarding the number of controls, we can achieve dimensionality reduction of the control space by writing the node values (on the tensor grid) of the value functions as some form of sparse (polynomial) expansions around their stationary equilibrium values:[5]

$$\hat{\nu}_t(s) \quad = \quad \bar{\nu}(s) + g_\nu(s; \theta_t^\nu) \tag{16}$$

We use the sparse expansions only to calculate the values for the nodes and maintain the assumption that the value function is generated through (linear) interpolation everywhere else to avoid oscillating behavior of the value function elsewhere.

One particularly useful way to construct the sparse expansions g^ν is through (inverse) discrete cosine transformations of the stationary equilibrium value function. With a slight abuse of notation, let $\bar{\nu}$ be the array of the value function values at the nodes of

[5] We use the term sparse polynomial of order k to describe a polynomial of the form

$$p(x_1, \ldots, x_m; \theta) = \sum_{i_1=0}^{k} \sum_{i_2=0}^{k-i_1} \cdots \sum_{i_m=0}^{k-i_1-i_2\cdots-i_{m-1}} \theta(i_1, i_2, \ldots, i_m)\phi_{i_1}(x_1)\phi_{i_2}(x_2) \cdot_{\ldots} \cdot \phi_{i_m}(x_m)$$

with basis functions $\phi_j(x)$. One can further reduce the degree of the polynomial by reducing the highest order quicker than one for one in the degree along other dimensions.

the full tensor grid in the stationary equilibrium. Then let $\Theta = dct(\bar{\nu})$ be its discrete cosine transform (using the index numbers as function arguments). By shrinking all but the largest elements (the squared sum of which is a fraction x of the squared Euclidean norm of Θ) to zero, one can compress the value function (akin to image compression like jpeg) without losing too much information. In other words: let $\mathcal{I}$ be the indexes of the largest elements of Θ, then define $\tilde{\Theta}(\theta|\mathcal{I}) = \begin{cases} \theta(i) & \forall i \in \mathcal{I} \\ 0 & \text{else} \end{cases}$, where the vector θ is sparse. Then the inverse discrete cosine transformation $idct(\tilde{\Theta})$ is roughly equal to $\bar{\nu}$. This allows us to define the sparse expansion as the inverse discrete cosine transform of the array $\tilde{\Theta}(\theta|\mathcal{I})$ where the non-zero entries correspond to those coefficients θ retained in the compression step, $g^{\nu}(\theta) := idct(\tilde{\Theta}(\theta|\mathcal{I}))$. The advantage of the discrete cosine transformation over other basis functions is that it can be calculated without inverting large matrices. Importantly, since only the *difference* to the stationary equilibrium value function is perturbed, the approximation puts no restriction on the stationary equilibrium. What is more, it does not require prior knowledge of the shape of the value function itself but the basis functions that are important are chosen from the shape of the value function of the stationary equilibrium.[6]

Under this approach, the dynamic system F replaces value functions and distributions by the parameters $\theta_t^{\nu}, d\mu_{1t}, \ldots, d\mu_{mt}$, where the $d\mu$-terms are the histograms of the marginal distributions. Since the system has more equations than unknowns now, we need to reduce the dimensionality of F. This can either be done by projecting the differences back to a lower dimensional space. For example, for the distribution functions this can be done by comparing only the marginal distributions. For the value functions, one can focus on the coefficients of the discrete cosine transformation of the error terms on the value functions at all nodes $\nu_t - (u_{h_t} + \beta \Pi_{h_t} \nu_{t+1})$ that correspond to the retained coefficients θ.

One advantage of reducing the state space *before* calculating the Jacobian of the difference equation through fixing the copula and "compressing" the value function instead of reducing it *after* calculating the Jacobian (as in Reiter, 2009; Ahn et al., 2017) is that it reduces substantially the time needed for calculating derivatives and avoids the potentially large memory requirements to store them that arise in discrete time models. In addition, it avoids the singular-value decomposition altogether.

[6]Since the dimensionality of the polynomial is implicitly chosen by the researcher, we suggest checking robustness of results, in particular the dynamics of aggregates, with respect to the maximum order of the polynomial or the number of coefficients retained in the compression step. Yet, for the state-space reduction based on singular-value decompositions of the Jacobian as in Ahn et al. (2017), one also needs to decide on the minimal singular value that is retained.

Its disadvantage is that it precludes studying shocks that directly alter the copula of the wealth-income distribution, such as targeted transfers. In addition, it is not guaranteed that the coefficients of the expansion around the stationary equilibrium value function that are shrunk to zero are unimportant for the shape of the value function outside the stationary equilibrium. They are only arguably unimportant in the stationary equilibrium. Yet, whether the latter leads to low-quality approximations can be checked through simulating the model along the lines of the tests suggested by Den Haan (2010).

2.6 The algorithm in a nutshell

To give a practical guide on the implementation, we finally provide a summary of the proposed algorithm. Concrete implementations can differ in particular in how the dynamic programming problem is solved. In particular, we provide the algorithm here on the basis of value function iteration, for simplicity and generality. In practice, another recursive method such as an endogenous grid method might well be preferable.

For our algorithm, define grids $\mathfrak{s}^j = \{s^j_0 \dots s^j_{m_j}\}$ for each $j = 0 \dots J$ of the idiosyncratic state variables, with m_j being the number of grid points used for variable j (note that different from the section before, here we explicitly split up the endogenous state variables in their J-dimensions next to one dimension for the exogenous idiosyncratic state variable). Let $\mathcal{S}$ be the tensor product (mesh) of these grids, and $\mathcal{IS}$ be the corresponding tensor product (mesh) of the indexes. This mesh has in total $m = \prod_{j=0}^{J} n_j$ grid points. The exogenous stochastic state(s) s^0 evolves on the grid $\mathfrak{s}^0$, which together with the transition matrix Π_0 defines a discrete Markov chain for this state variable.

We define $\mathcal{V}$ as the $J+1$-dimensional array that stores the values of a value function at each point of the mesh $\mathcal{S}$. We define $\hat{\nu}[(s^0, s^1 \dots s^J)|\Pi_0\mathcal{V}]$ as the linear interpolant defined by the mesh $\mathcal{S}$ and node values $\Pi_0\mathcal{V}$ evaluated at $(s^0, s^1 \dots s^J)$, where $\Pi_0\mathcal{V}$ is the matrix product of Π_0 and $\mathcal{V}$ reshaped accordingly. With $d\mu \in \mathbb{R}^{m_0 \times m_1 \times \dots m_J}$ we denote the histogram of the distribution of agents over all states $s \in \mathcal{S}$ in array form; $\vec{d\mu}$ is the same, but vectorized (stacked). Let S be the (exogenous) aggregate state of the economy with S^* its steady-state value.

Prerequisits 1.

1. *Define for a given price system P a mapping $T(\mathcal{V}|P) : \mathbb{R}^m \to \mathbb{R}^m$ such that*

$$\forall s = (s^0 \dots s^J) \in \mathcal{S} : T(\mathcal{V}|P)(s) := \max_{(s^{1\prime} \dots s^{J\prime}) \in \Gamma(s,P)} u(s, s^{1\prime} \dots s^{J\prime}) + \hat{\nu}[(s^0, s^{1\prime} \dots s^{J\prime})|\Pi_0\mathcal{V}].$$

In words, this mapping is one iteration of the value function. Define $h(\mathcal{V}|P) : \mathbb{R}^N \to \mathbb{R}^{m/m_0}$ *as the corresponding policy function (the* $\arg\max$*).*

2. *Define a mapping* $\Pi = \Pi(\mathcal{V}|P) : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ *such that*

$$\forall x = (x^0 \dots x^J), y = (y^0 \dots y^J) \in \mathcal{IS} : \Pi(\mathcal{V}|P)(x,y) = \Pi_0(x^0, y^0) \prod_{j=1}^{J} \Pi_j(x,y),$$

where Π_j *are the coefficients to represent the policy* $h(\mathcal{V}|P)(x) = (h^1(x) \dots h^J(x))$ *as convex combinations of the nearest neighbors on the index mesh* $\mathcal{IS}$*, i.e.,*

$$\Pi_j(x,y) = \begin{cases} 0 & \text{if } h^j \notin [s^j_{y^j-1}, s^j_{y^j+1}] \\ 1 - \frac{h^j - s^j_{y^j}}{s^j_{y^j+1} - s^j_{y^j}} & \text{if } s^j_{y^j+1} \geq h^j > s^j_{y^j} \\ \frac{h^j - s^j_{y^j-1}}{s^j_{y^j} - s^j_{y^j-1}} & \text{if } s^j_{y^j} \geq h^j \geq s^j_{y^j-1} \end{cases}. \tag{17}$$

3. *The discrete cosine transformation of an array* A *along a dimension* j *is given by pre-multiplying a transformation matrix* C_j *to array* A *along the* j*-dimension. This is done by permuting the array such that dimension* j *becomes the first one and reshaping the array to matrix form. The result of this matrix multiplication has to be reshaped back to its array form, permuting the now first dimension back to the j-th position. The inverse is defined analogously through pre-multiplication of* C_j^{-1}*. The matrix* C_j *is constructed as*

$$C_j(k,l) = \sqrt{2/m_j} \cos\left(\pi \frac{(l-1/2)(k-1)}{m_j}\right) \forall k,l = 1 \dots m_j \tag{18}$$

Algorithm 1.

1. ***Finding the stationary equilibrium***
   (a) *For a given price system* P *iterate* $T^{(n)} = \underbrace{T(T(\dots T(\mathcal{V}^{(0)}|P)|P)|P)}_{n \text{ times}}$ *until convergence to obtain an equilibrium value function* $\mathcal{V}^*$ *as the limit* $n \to \infty$*.*
   (b) *Calculate the equilibrium distribution by solving* $\vec{d\mu}^*(P) = \vec{d\mu}^*(P).\Pi(\mathcal{V}^*|P)$
   (c) *Calculate excess demand* $\mathcal{E}$ *as a function* $\mathcal{E}[d\mu^*(P), h(\mathcal{V}^*|P), S^*]$*.*
   (d) *Search over prices, repeating (a) to (c) until* $\mathcal{E}(P^*) = 0$*.* P^* *is the price system in the stationary equilibrium.*

2. ***Dimensionality reduction***

   (a) *Define the joint distribution function* $\mu^*(s) = \sum_{x \leq s} d\mu^*(x)$. *Define* $\mu^{*j} \in [0,1]^{m_j}, j = 0 \ldots J$ *as the* J *vectors of the marginal distributions corresponding to the* m_j *points on the* $\mathfrak{s}_j$*-grids. Generate the fixed copula* $\mathcal{C}^*(\mu^0, \ldots, \mu^J | \mu^*) : [0,1]^{J+1} \to [0,1]$ *as a linear interpolant of* μ^* *on the tensor product* $\otimes_{j=0}^{J} \mu^{*j}$.

   (b) *Calculate the discrete cosine transformation of* $\mathcal{V}^*$ *along all* $J+1$ *dimensions iteratively. This yields coefficients* Θ^*. *Find the minimal index set* $\mathcal{I}^*$, *such that* $\frac{\sum_{i \in \mathcal{I}} \vec{\Theta}^*(i)^2}{\sum_i \vec{\Theta}^*(i)^2} > 1 - \epsilon$ *(by sorting the coefficients and retaining only the largest ones). This set has typically substantially fewer elements,* $\#\mathcal{I}^*$, *than the full tensor grid has, which is* m.

   (c) *Define a vector* $\theta \in \mathbb{R}^{\#\mathcal{I}^*}$ *that now is shorter than* $\Theta \in \mathbb{R}^m$. *This vector is used to assign values to those coefficients of the discrete cosine transformation of* $\mathcal{V}$ *which were found to be different from zero and hence important. In other words, it assigns a value to each coefficient in the index set* $\mathcal{I}^*$, *such that we obtain the full set of coefficients,* $\tilde{\Theta}(\theta | \Theta^*, \mathcal{I}^*) \in \mathbb{R}^m$, *which is given by*

$$\tilde{\Theta}(i) = \begin{cases} \Theta^*(i) & \text{if } i \notin \mathcal{I}^* \\ \theta(i) & \text{if } i \in \mathcal{I}^* \end{cases}$$

   and the mapping of this array $\tilde{\Theta}$ *to the value function values* $\tilde{\mathcal{V}}(\theta)$ *is obtained through an inverse cosine transformation.*

3. ***Linearization***

   (a) *Define the following objects:*

   - *the difference between the value function implied from one backward iteration based on its value at time* $t+1$ *and the value function for time* t *as implied by* θ_t. *We apply the discrete cosine transformation to the value functions and evaluate on all points in* $\mathcal{S}$

$$\Delta_1(\theta_t, \theta_{t+1}, P_t) := \tilde{\Theta}(\theta_t) - dct\left\{T\left[\tilde{\mathcal{V}}(\theta_{t+1}) | P_t\right]\right\} \in \mathbb{R}^m.$$

     The shorter vector Δ_1^* *selects out of* Δ_1 *only those elements that correspond to the index set* $\mathcal{I}^*$.

   - *the difference between the distribution for time* $t+1$ *implied by* (μ_{t+1}^j) *(for all* $j = 0 \ldots J$*) and the copula* $\mathcal{C}^*$ *and the distribution obtained from iter-*

ating forward once the distribution implied by $(\mu_t^j)_{j=0...J}$ *and the copula* $\mathcal{C}^*$

$$\Delta_2[(\mu_t{}^j)_{j=0...J}, (\mu_{t+1}^j)_{j=0...J}, P_t, \theta_{t+1}] := \\ d\mathcal{C}^*[(\mu_t{}^j)_{j=0...J}]\Pi[\tilde{\mathcal{V}}(\theta_{t+1})|P_t] - d\mathcal{C}^*[(\mu_{t+1}{}^j)_{j=0...J}] \in \mathbb{R}^m,$$

where $d\mathcal{C}^*$ *denotes the histogram obtained from the copula evaluated on all points of the tensor product spanned by* $(\mu^j)_{j=0...J}$. Δ_2^* *is the corresponding difference of the marginal distributions.*

- *the excess demand function* $\mathcal{E} = \mathcal{E}\left\{d\mathcal{C}^*[(\mu_t{}^j)_{j=0...J}], h[\tilde{\mathcal{V}}(\theta_{t+1})|P_t], S_t, S_{t+1}\right\}$.

(b) Use these differences to define a function

$$F((\mu_t{}^j)_{j=0...J}, (\mu_{t+1}^j)_{j=0...J}, S_t, S_{t+1}, \theta_t, \theta_{t+1}, P_t, P_{t+1}|\mathcal{C}^*, \mathcal{V}^*, \mathcal{I}^*, \Theta^*)$$

that describes the economy as a system of non-linear difference equations

$$F = \begin{bmatrix} \Delta_1^*(\theta_t, \theta_{t+1}, P_t) \\ \Delta_2^*[(\mu_t{}^j)_{j=0...J}, (\mu_{t+1}^j)_{j=0...J}, P_t, \theta_{t+1}] \\ S_{t+1} - H(S_t) \\ \mathcal{E}\left\{d\mathcal{C}^*[(\mu_t{}^j)_{j=0...J}], h[\tilde{\mathcal{V}}(\theta_{t+1})|P_t], S_t, S_{t+1}\right\} \end{bmatrix} \tag{19}$$

(c) Calculate the Jacobian of F. Define A,B as defined in the text before and as in Schmitt-Grohé and Uribe (2004).

(d) Calculate the qz decomposition and solve for the linearized dynamics using the algorithm provided by Schmitt-Grohé and Uribe (2004).

3 Examples

In the following, we discuss two examples to illustrate our modification of Reiter's method to solve general equilibrium models with heterogeneous agents and aggregate risk. Both examples share the same model of consumption-savings choice in which households face uninsurable income risk and use assets to self-insure. We then specify two variants of the model: one without nominal frictions and only one asset, i.e., the setup of Krusell and Smith (1998); second, a setup with two assets of different liquidity and a nominal rigidity. The first example can be solved using the original Krusell and Smith algorithm and the Reiter algorithm without state-space reduction. For the second example, state-

space reduction is necessary to render the computation feasible. Details on the numerical precision of the various algorithms are provided in Section 4.

3.1 Household sector

There is a continuum of ex-ante identical households of measure one, indexed by i. Households are infinitely lived, have time-separable preferences with time-discount factor β, and derive felicity from consumption c_{it} and leisure. Households have Greenwood-Hercowitz-Huffman (GHH) preferences, and maximize the discounted sum of felicity:

$$E_0 \max_{\{c_{it}, n_{it}, \Delta k_{it}\}} \sum_{t=0}^{\infty} \beta^t u \left[c_{it} - G(h_{it}, n_{it}) \right].$$

The assumption of GHH preferences simplifies the numerical analysis of the stationary equilibrium substantially but is not necessary for our implementation of Reiter's method. The maximization is subject to the budget constraints described further below. The felicity function u exhibits a constant relative risk aversion (CRRA) with risk aversion parameter $\xi > 0$,

$$u(x_{it}) = \frac{1}{1-\xi} x_{it}^{1-\xi},$$

where $x_{it} = c_{it} - G(h_{it}, n_{it})$ is household i's composite demand for goods consumption c_{it} and leisure and G measures the disutility from work. Goods consumption bundles varieties j of differentiated goods according to a Dixit-Stiglitz aggregator:

$$c_{it} = \left(\int c_{ijt}^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}}.$$

Each of these differentiated goods is offered at price p_{jt}, so that for the aggregate price level, $P_t = \left(\int p_{jt}^{1-\eta} dj \right)^{\frac{1}{1-\eta}}$, the demand for each of the varieties is given by

$$c_{ijt} = \left(\frac{p_{jt}}{P_t} \right)^{-\eta} c_{it}.$$

The disutility of work, $G(h_{it}, n_{it})$, determines a household's labor supply given the aggregate wage rate, w_t, and a labor income tax, τ, through the first-order condition:

$$\frac{\partial G(h_{it}, n_{it})}{\partial n_{it}} = (1-\tau) w_t h_{it}.$$

Assuming that G has a constant elasticity w.r.t. n, $\frac{\partial G(h_{it}, n_{it})}{\partial n_{it}} = (1+\gamma) \frac{G(h_{it}, n_{it})}{n_{it}}$ with

$\gamma > 0$, we can simplify the expression for the composite consumption good x_{it} making use of the first-order condition (3.1):

$$x_{it} = c_{it} - G(h_{it}, n_{it}) = c_{it} - \frac{(1-\tau)w_t h_{it} n_{it}}{1+\gamma}.$$

When the Frisch elasticity of labor supply is constant, the disutility of labor is always a constant fraction of labor income. Therefore, in both the budget constraint of the household and its felicity function only after-tax income enters and neither hours worked nor productivity appears separately.

This implies that we can assume $G(h_{it}, n_{it}) = h_{it}\frac{n_{it}^{1+\gamma}}{1+\gamma}$ without further loss of generality as long as we treat the empirical distribution of income as a calibration target. This functional form simplifies the household problem as h_{it} drops out from the first-order condition and all households supply the same number of hours $n_{it} = N(w_t)$. Total effective labor input, $\int n_{it} h_{it} di$, is hence also equal to $N(w_t)$ because $\int h_{it} di = 1$.

A household's labor income $w_t h_{it} n_{it}$ is composed of the aggregate wage rate, w_t, the household's hours worked, n_{it}, and its idiosyncratic labor productivity, h_{it}. Productivity evolves according to a log-AR(1) process and a fixed probability of transition to a high income state in which $h_{it} = 0$ but households receive a share of pure rents, i.e., they become entrepreneurs:

$$h_{it} = \begin{cases} \exp\left(\rho_h \log h_{it-1} + \epsilon_{it}^h\right) & \text{with probability } 1-\zeta \text{ if } h_{it-1} \neq 0, \\ 1 & \text{with probability } \iota \text{ if } h_{it-1} = 0, \\ 0 & \text{otherwise,} \end{cases}$$

with shocks to productivity ϵ_{it}^h being normally distributed.

With probability ζ households become entrepreneurs ($h = 0$). With probability ι an entrepreneur returns to the labor force with median productivity. An entrepreneurial household obtains a fixed share of the pure rents, Π_t, in the economy (from monopolistic competition and creation of capital). We assume that the claim to the pure rent cannot be traded as an asset. The idea here is that a household that becomes an entrepreneur develops a variety only she can produce out of intermediate goods and it loses this capacity (because her variety is replaced by another household's drastic innovation) when returning to the labor force.

3.2 Price setting

These entrepreneur households, i.e., the final-goods producers, differentiate the intermediate good and set prices. We assume price adjustment costs à la Rotemberg (1982). For tractability, we assume that the actual price setting is delegated to a mass-zero group of households (managers) that are risk neutral and compensated by a share in profits. They do not participate in any asset market. Under this assumption, managers maximize the present value of real profits given the demand for good j,

$$y_{jt} = (p_{jt}/P_t)^{-\eta} Y_t,$$

and quadratic costs of price adjustment, i.e., they maximize:

$$E_0 \sum_{t=0}^{\infty} \beta^t Y_t \left\{ \left(\frac{p_{jt}}{P_t} - MC_t \right) \left(\frac{p_{jt}}{P_t} \right)^{-\eta} - \frac{\eta}{2\kappa} \left(\log \frac{p_{jt}}{p_{jt-1}} \right)^2 \right\},$$

with a time-constant discount factor. From the corresponding first-order condition for price setting, it is straightforward to derive the Phillips curve:

$$\log(\pi_t) = \beta E_t \left[\log(\pi_{t+1}) \tfrac{Y_{t+1}}{Y_t} \right] + \kappa \left(MC_t - \tfrac{\eta-1}{\eta} \right), \tag{20}$$

where π_t is the gross inflation rate, $\pi_t := \frac{P_t}{P_{t-1}}$, and MC_t is the real marginal costs. The price adjustment then creates real costs $\frac{\eta}{2\kappa} Y_t \log(\pi_t)^2$.

Since managers are a mass-zero group in the economy, their consumption does not show up in any resource constraint and all profits – net of price adjustment costs – go to the entrepreneur households (whose $h = 0$). In the case of the two-asset economy, these households also obtain profit income from adjusting the aggregate capital stock. They can transform I_t consumption goods into ΔK_{t+1} new capital goods (and back) according to the transformation function:

$$I_t = \tfrac{\phi}{2} (\Delta K_{t+1}/K_t)^2 K_t + \Delta K_{t+1}.$$

Since they are facing perfect competition in this market, entrepreneurs will adjust the stock of capital until the following first-order condition holds:

$$q_t = 1 + \phi \Delta K_{t+1}/K_t,$$

where q_t is the price of capital.[7]

3.3 Intermediate-goods producers

Intermediate goods are produced with a constant returns to scale production function:

$$Y_t = S_t N_t^{\alpha} K_t^{(1-\alpha)},$$

where $K_t = E(k_{it})$ is the aggregate capital supply, $N_t = E(h)[(1-\tau)w_t]^{\frac{1}{\gamma}}$ is the aggregate labor supply, and S_t is total factor productivity.

Let MC_t be the relative price at which the intermediate good is sold to entrepreneurs. The intermediate-good producer maximizes profits,

$$MC_t Y_t - w_t N_t - (r_t + \delta)K_t = MC_t S_t N_t^{\alpha} K_t^{(1-\alpha)} - w_t N_t - (r_t + \delta)K_t,$$

but it operates in perfectly competitive markets, such that the real wage and the user costs of capital are given by the marginal products of labor and capital:

$$w_t = \alpha S_t MC_t \left(K_t/N_t\right)^{1-\alpha}, \qquad r_t + \delta = (1-\alpha) S_t MC_t \left(N_t/K_t\right)^{\alpha}.$$

3.4 Model variants

To close the model, we still need to define which assets households can trade. As stated before, we consider two model variants. First, a variant of the original Krusell and Smith (1998) economy where only capital is traded, which is a perfectly liquid asset. This variant serves to benchmark our solution strategy against other discrete time methods. Second, we use the economy as in Bayer et al. (2015) and Luetticke (2018) with a liquid nominal asset and illiquid capital. This economy cannot be solved without state-space reduction and serves as an application example for those cases.

3.4.1 A neoclassical economy with one asset: The Krusell-Smith setup

Our model nests the Krusell and Smith (1998) economy (except for the different functional form of labor supply). In that economy, households save only in capital that is perfectly liquid. There are no entrepreneurs ($\zeta = 0$), competition is perfect, and price adjustment is costless ($\eta, \kappa \to \infty, \frac{\eta}{\kappa} \to 0$). In addition there is no capital adjustment cost, $\phi = 0$, such that $q_t = 1$. Taxes τ are zero, too.

[7] We assume for simplicity that all depreciation is replaced immediately through maintenance investment which transforms consumption goods into replacement investment one-for-one.

Therefore, households optimize subject to this budget constraint:

$$\begin{aligned} c_{it} + k_{it+1} &= k_{it}(1 + r_t) + (1 - \tau)(w_t h_{it} N_t), \\ k_{it+1} &\geq 0, \end{aligned}$$

where r_t is the real return on capital.

Substituting the expression $c_{it} = x_{it} + \frac{(1-\tau)w_t h_{it} N_t}{1+\gamma}$ for consumption, we obtain:

$$\begin{aligned} x_{it} + k_{it+1} &= k_{it}(1 + r_t) + (1 - \tau)\left(\tfrac{\gamma}{1+\gamma} w_t h_{it} N_t\right), \\ k_{it+1} &\geq 0. \end{aligned}$$

With this setup, one Bellman equation characterizes the dynamic planning problem of a household:

$$V(k, h; \Theta, S) = \max_{k'} u[x(k, k', h)] + \beta V(k', h'; \Theta', S'),$$

where Θ is the wealth-income distribution and S is aggregate productivity as the only other state variable and capital and labor market clearing are the only equilibrium conditions (there is classical dichotomy and the nominal side is not determined):

$$w_t = \alpha S_t MC_t \left(K_t/N_t\right)^{1-\alpha}, \qquad r_t + \delta = (1 - \alpha) S_t MC_t \left(N_t/K_t\right)^{\alpha}.$$

3.4.2 New-Keynesian variant with liquid and illiquid assets

The second model variant introduces a nominal rigidity, such that the Phillips curve (20) is not vertical, and a nominal bond that pays R_t and makes capital illiquid, such that the two assets are not close substitutes. Illiquidity is modeled as follows: only a randomly selected fraction of households, ν, participates in the market for capital each period and can thus actively sell or buy capital. All other households obtain dividends, but may only adjust their holdings of nominal bonds. Holdings of bonds have to be above an exogenous debt limit $\underline{B}$, and holdings of capital have to be non-negative.

Therefore, households optimize subject to their budget constraint:

$$\begin{aligned} c_{it} + b_{it+1} + q_t k_{it+1} &= b_{it} \tfrac{R(b_{it}, R_t^b)}{\pi_t} + (q_t + r_t) k_{it} + (1 - \tau)(w_t h_{it} N_t + \mathbb{I}_{h_{it}=0} \Pi_t), \\ k_{it+1} &\geq 0, b_{it+1} \geq \underline{B}, \end{aligned}$$

where b_{it} is real bond holdings, $\underline{B}$ is an exogenous borrowing constraint, k_{it} is the amount

of illiquid assets, q_t is the price of these assets, r_t is their dividend, $\pi_t = \frac{P_t - P_{t-1}}{P_{t-1}}$ is realized inflation, and R is the nominal interest rate on bonds, which depends on the portfolio position of the household and the central bank's interest rate R_t^b, which is set one period before. All households that do not participate in the capital market ($k_{it+1} = k_{it}$) still obtain dividends and can adjust their bond holdings. Depreciated capital has to be replaced for maintenance, such that the dividend, r_t, is the net return on capital.

We assume that there is a wasted intermediation cost, $\overline{R}$, when households resort to unsecured borrowing and specify:

$$R(b_{it}, R_t^b) = \begin{cases} R_t^b & \text{if } b_{it} \geq 0 \\ R_t^b + \overline{R} & \text{if } b_{it} < 0. \end{cases}$$

This assumption creates a mass of households with zero unsecured credit but with the possibility to borrow, though at a penalty rate.

Substituting the expression $c_{it} = x_{it} + \frac{(1-\tau) w_t h_{it} N_t}{1+\gamma}$ for consumption, we obtain:

$$\begin{aligned} x_{it} + b_{it+1} + q_t k_{it+1} = b_{it} \tfrac{R(b_{it}, R_t^b)}{\pi_t} + (q_t + r_t) k_{it} + (1-\tau) \left(\tfrac{\gamma}{1+\gamma} w_t h_{it} N_t + \mathbb{I}_{h_{it}=0} \Pi_t \right), \\ k_{it+1} \geq 0, \quad b_{it+1} \geq \underline{B}. \end{aligned}$$

With this setup, two Bellman equations characterize the dynamic planning problem of a household: V_a in the case where the household can adjust its capital holdings and V_n otherwise:

$$\begin{aligned} V_a(b, k, h; \Theta, R^b, s) = & \max_{k', b'_a} u[x(b, b'_a, k, k', h)] + \beta[\nu EV^a(b'_a, k', h'; \Theta', R^{b'}, s') \\ & + (1-\nu) EV^n(b'_a, k', h'; \Theta', R^{b'}, s')] \\ V_n(b, k, h; \Theta, R^b, s) = & \max_{b'_n} u[x(b, b'_n, k, k, h)] + \beta[\nu EV^a(b'_n, k, h'; \Theta', R^{b'}, s') \\ & + (1-\nu) EV^n(b'_n, k, h'; \Theta', R^{b'}, s')] \end{aligned}$$

Since we allow for a nominal rigidity, the equilibrium is only determined when a monetary and a fiscal policy are specified. Monetary policy controls the nominal interest rate on liquid assets, while fiscal policy determines the amount of government bonds by controlling fiscal deficits through the adjustment of expenditures. We assume that the monetary and fiscal authorities operate independently and their behavior is described by simple rules.

We assume that monetary policy sets the nominal interest rate on bonds following a

Taylor (1993)-type rule with interest rate smoothing:

$$\frac{R^b_{t+1}}{\bar{R}^b} = \left(\frac{R^b_t}{\bar{R}^b}\right)^{\rho_R} \left(\frac{\pi_t}{\bar{\pi}}\right)^{(1-\rho_R)\theta_\pi}.$$

The coefficient $\bar{R}^b \geq 0$ determines the nominal interest rate in the steady state. The coefficient $\theta_\pi \geq 0$ governs the extent to which the central bank attempts to stabilize inflation around its steady-state value: the larger θ_π the stronger is the reaction of the central bank to deviations from the inflation target. When $\theta_\pi \to \infty$, inflation is perfectly stabilized at its steady-state value. $\rho_R \geq 0$ captures interest rate smoothing.

We assume that the government issues bonds according to the rule (c.f. Woodford, 1995):

$$\frac{B_{t+1}}{\bar{B}} = \left(\frac{B_t R^b_t/\pi_t}{\bar{B}\bar{R}^b/\bar{\pi}}\right)^{\rho_B} \left(\frac{\pi_t}{\bar{\pi}}\right)^{-\gamma_\pi} \left(\frac{\mathcal{T}_t}{\bar{\mathcal{T}}}\right)^{-\gamma_\mathcal{T}},$$

using tax revenues $\mathcal{T}_t = \tau(w_t N_t + \Pi_t)$ to finance government consumption, G_t, and interest on debt. In other words, the government seeks to stabilize debt in the long run and output in the short run. The coefficient ρ_B captures whether and how fast the government seeks to repay its outstanding obligations $B_t R^b_t/\pi_t$. For $\rho_B < 1$ the government actively stabilizes real government debt, and for $\rho_B = 1$ the government rolls over all outstanding debt including interest. The coefficients $\gamma_\pi, \gamma_\mathcal{T}$ capture the cyclicality of debt issuance: for $\gamma_\pi = \gamma_\mathcal{T} = 0$, new debt does not actively react to tax revenues and inflation, but only to the value of outstanding debt. For $\gamma_\pi > 0 > \gamma_\mathcal{T}$, debt is countercyclical; for $\gamma_\pi < 0 < \gamma_\mathcal{T}$ it is procyclical.

In equilibrium, we need both factor markets to clear, such that

$$w_t = \alpha MC_t S_t \left(K_t/N_t\right)^{1-\alpha}, \qquad r_t + \delta = (1-\alpha) S_t MC_t \left(N_t/K_t\right)^{\alpha},$$

and we also need asset markets to clear. This requires first

$$B_{t+1} = B^d(\Theta_t; R^b_t, s_t; q_t, \pi_t) := E\left[\nu b^*_a + (1-\nu) b^*_n\right], \tag{21}$$

where b^*_a, b^*_n are bond demand functions of adjusters and non-adjusters. They are functions in the states $(b, k, h; R^b_t, s_t)$, of current prices q_t, π_t, and of expectations of future prices. Expectations in the right-hand-side expression are taken w.r.t. the distribution $\Theta_t(b, k, h)$. Equilibrium requires the total *net* amount of bonds the household sector demands, B^d, to equal the supply of government bonds. In gross terms there are more liquid assets in circulation as some households borrow up to $\underline{B}$.

Second, the asset market for capital has to clear. This requires that

$$\begin{aligned} q_t &= 1 + \phi \frac{K_{t+1} - K_t}{K_t}, \\ K_{t+1} &= K^d(\Theta_t; R_t^b, s_t; q_t, \pi_t) := E[\nu k^* + (1-\nu)k]. \end{aligned} \tag{22}$$

Again expectations are taken w.r.t. the distribution $\Theta_t(b, k, h)$.

4 Numerical Performance

In the following we first demonstrate the performance and accuracy of our method by comparing it to the Krusell and Smith (1998) algorithm for the standard Krusell and Smith (K-S) model, as described in Section 3.4.1. We then show the scalability of our method by solving heterogeneous agent New-Keynesian (HANK) models with higher dimensional heterogeneity, providing accuracy measures for the variant described in section 3.4.2. All codes are available on the authors' websites.

4.1 Comparison to Krusell & Smith (1998)

To compare the performance and accuracy of our method, we solve Krusell and Smith's (1998) model with the standard parameterization of the JEDC comparison project (except for the GHH preferences) (c.f. Den Haan et al., 2010).[8] A period in the model is a quarter, the discount factor is $\beta = 0.99$, the coefficient of relative risk aversion is $\xi = 1$, the Frisch elasticity is $\gamma = 1$, and the rate of depreciation equals 2.5% per quarter.[9] Idiosyncratic and aggregate productivity risk both follow two-state Markov chains. We solve the household problem on 100 grid points for idiosyncratic capital. The grid for the aggregate capital stock has 3 points for the Krusell-Smith algorithm and covers the unconditional ± 3 STD interval from the linearized solution.

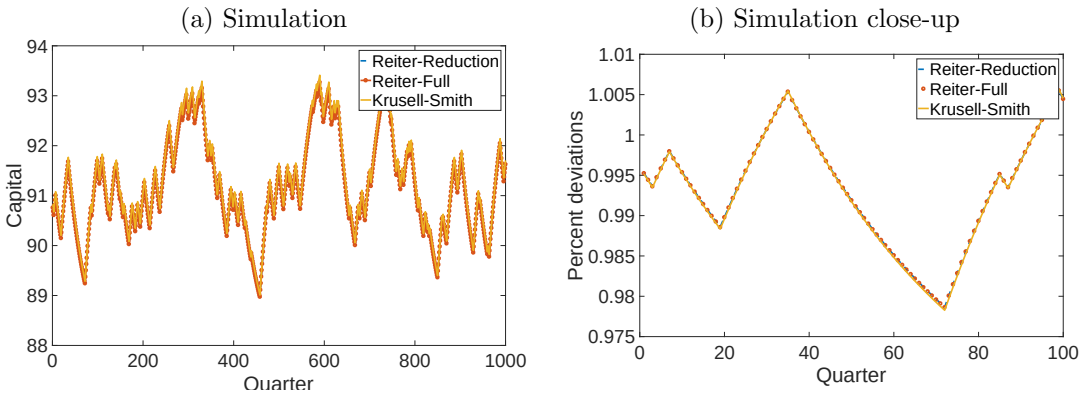
Figure 1 shows simulations of the K-S model for three different solution methods: (1) Reiter's method with state-space reduction via the fixed copula assumption and value function compression (73 coefficients of the discrete cosine transformation conserve 99.99% of the energy), (2) Reiter's method with a full value function and histogram on the tensor product of the income and capital grid, and (3) the original Krusell and Smith algorithm.[10] The response of aggregate capital to TFP shocks is virtually the same in all three simulations. Table 1 confirms this. The mean absolute error between the time

[8] Setting $\eta \to \infty$ and $\kappa \to \infty$, i.e., no markups and flexible prices, yields the standard neoclassical incomplete markets model but with GHH preferences.

[9] See Appendix A Table 5 for the calibration.

[10] The simulations start from the steady state without aggregate risk, which is the same for all three methods.

Figure 1: Simulations of Krusell & Smith model



Notes: Both panels show simulations of the Krusell & Smith (1998) model with TFP shocks solved with (1) the Reiter method with our proposed state-space reduction, (2) the original Reiter method without state-space reduction, (3) the original Krusell & Smith algorithm.

Table 1: Simulation errors relative to Krusell & Smith algorithm

	Absolute difference (in %) of capital stocks K_t between simulations		
	Reiter-Reduction vs. K-S	Reiter-Full vs. K-S	Reiter-Reduction vs. Reiter-Full
Mean	0.0844	0.0843	0.0029
Max	0.1618	0.1558	0.0073

Notes: Differences in percent between simulations of aggregate capital for the Krusell and Smith (1998) model solved with (1) the Reiter method with our proposed state-space reduction, (2) the original Reiter method without state-space reduction, (3) the original Krusell & Smith algorithm. The first two columns show the performance of (1) and (2) relative to (3), and the last column shows the performance of (1) relative to (2) for 1,000 periods. The draws for the productivity process are kept constant across solution methods.

series from the two linearization methods and the K-S algorithm is 0.08%. What is more, the linearization methods with and without state and control space reduction yield basically the same simulation for the aggregate stock of capital with a maximum absolute error of 0.01%.

To further evaluate the accuracy of our solution method, we use the error metric

suggested by Den Haan (2010), which compares the simulation from the linearized solution of the model to a simulation in which we solve for the equilibrium interest rate every period and track the full histogram over time. The mean absolute error is 0.03% and the maximum error is 0.05%; see Table 2. The K-S algorithm, which is the most accurate algorithm in Den Haan et al. (2010), yields a substantially larger maximum error of 0.56%.

Table 2: Den Haan errors

	Absolute error (in %) for capital K_t		
	Reiter-Reduction	Reiter-Full	K-S
Mean	0.0312	0.0311	0.2159
Max	0.0535	0.0505	0.5603

Notes: Differences in percent between the simulation of the linearized solutions of the model and simulations in which we solve for the intratemporal equilibrium prices in every period and track the full histogram over time for $t = \{1, ..., 1000\}$; see Den Haan (2010).

Finally, Table 3 shows the run times of all three methods and the steady state separately. The Reiter method with state and control space reduction only takes 0.38 second. This makes it more than 130 times faster than the Krusell and Smith algorithm. Without reduction, the run time increases by 100%. Even when the time to compute the stationary equilibrium is taken into account, the linearization methods are 8 times faster than the Krusell-Smith algorithm. The main advantage of linearization with state and control space reduction, however, lies in its capacity to solve models with many idiosyncratic states fast and precisely as the next example shows.

Table 3: Run time for Krusell & Smith model

	Stationary equilibrium	Krusell & Smith	Reiter-Reduction	Reiter-Full
in seconds	6.28	49.85	0.43	0.91

Notes: Run time in seconds on a Dell laptop with an Intel i7-7500U CPU @ 2.70GHz 4. Model calibration and number of grid points as in Den Haan et al. (2010). Code in Matlab.

Table 4: Run times and accuracy for two-asset model

	Running times*	
	Stationary equilibrium	Reiter-Reduction
In seconds	388.14	80.38

	Absolute error (in %)**	
	For capital K_t	For bonds B_t
Mean	0.0314	0.1513
Max	0.1279	1.0612

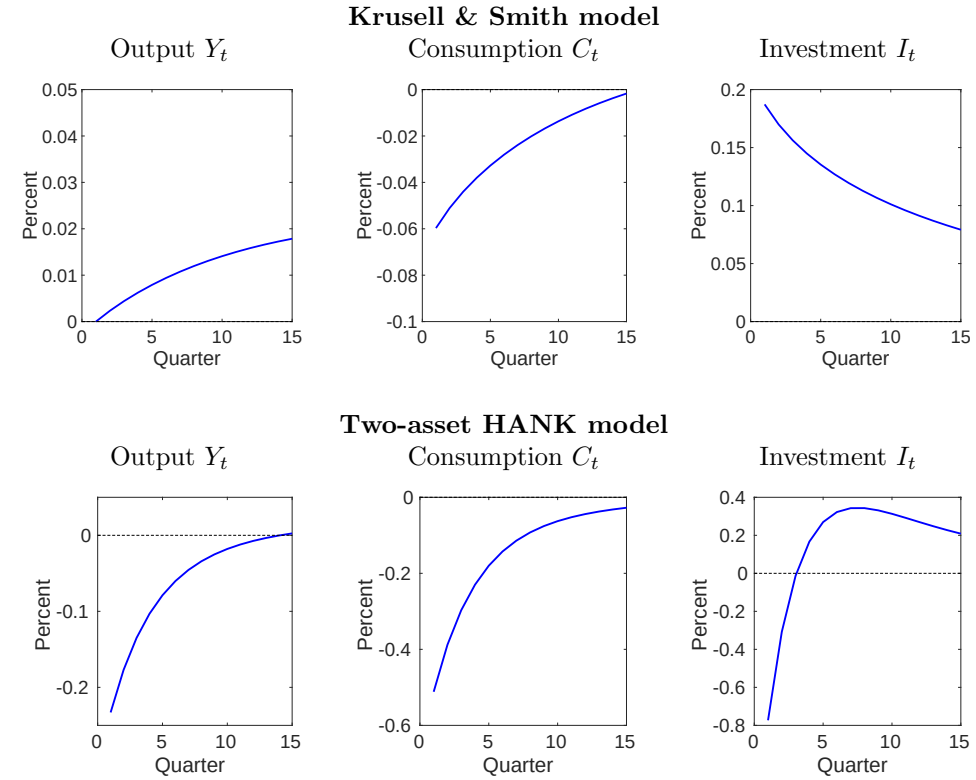
* On a Dell laptop with an Intel i7-7500U CPU @ 2.70GHz, 4 cores. Code in Matlab.
** Differences in percent between the simulation of the linearized solution of the model and a simulation in which we solve for the intratemporal equilibrium prices in every period and track the full histogram over time for $t = \{1, ..., 1.000\}$; see Den Haan (2010).

4.2 Two-asset model

The true advantage of the state and control space reduction through fixing the copula and compressing the value function lies in breaking the curse of dimensionality and, thus, making it possible to solve models with high dimensional heterogeneity. In the following, we provide accuracy statistics and computational time for our model with a portfolio choice between liquid and illiquid assets as set out in Section 3, in particular 3.4.2. This model features heterogeneity with respect to three dimensions: (1) liquid asset holdings, (2) illiquid asset holdings, and (3) idiosyncratic productivity. We solve the household problem on 100 grid points for both asset choices and 4 grid points for productivity. With 40,000 states and 80,000 controls (for the two value functions), it is infeasible to solve for the aggregate dynamics of the model on the full histogram. The copula approximation reduces the number of states to 204. Maintaining only the coefficients of the discrete cosine of the value functions with the cumulative highest 99.999% energy reduces the number of controls to 635. This all together makes it possible to solve the model on a laptop computer in, as the top panel of Table 4 shows, only 80 seconds (plus an additional 6 minutes for the stationary equilibrium).

We first solve the model for the same calibration as the Krusell and Smith model in the previous section.[11] Table 4 shows the error metric suggested by Den Haan (2010) for the capital stock implied by the two-asset model in response to TFP shocks. The maximum absolute error is 0.13% and the mean absolute error is 0.03%, which are comparable to the errors in Table 2 for the single-asset model. The errors for equilibrium bonds are slightly larger.

Figure 2: Aggregate response to idiosyncratic uncertainty shock



Notes: Impulse responses to a 54% increase in uncertainty (measured by STD) of idiosyncratic income tomorrow.

Being able to solve the two-asset model is important, because it generates "wealthy hand-to-mouth" households (Kaplan and Violante, 2014) and implies different investment behavior (see Bayer et al. (2015) and Luetticke (2018)). Figure 2 shows the effect of higher uncertainty about idiosyncratic productivity in the Krusell and Smith model

[11] Appendix A Table 6 provides the full calibration.

and the two-asset HANK model. Consumption falls in both models as households increase their precautionary savings in response to higher uncertainty. In the Krusell and Smith model, higher savings translate one-for-one into capital, which leads to an economic expansion. In the two-asset model, in contrast, households prefer to hold more liquid portfolios. They sell illiquid capital to save more in liquid assets. Higher uncertainty therefore causes a simultaneous fall in consumption, investment, and output. The recessionary effect is further amplified through sticky prices, which makes the economy demand-driven in the short run. See Bayer et al. (2015) for a more detailed discussion of the portfolio rebalancing channel of uncertainty.[12]

5 Conclusion

In this paper, we have proposed an extension of Reiter's method to solve heterogeneous agent models with aggregate risk by perturbation. The proposed method relies on reducing the state space after solving for the stationary equilibrium but before linearizing the non-linear difference equation that characterizes the equilibrium dynamics. The state-space reduction is achieved by "lossy compression" of the value functions, which are control variables of the system, and by approximating the dynamics of the multi-dimensional distribution of individual characteristics by a distribution with a fixed copula and varying marginals. Both steps effectively break the curse of dimensionality and allow us to efficiently and precisely solve for the equilibrium dynamics of heterogeneous agent economies as we have shown in two examples.

Breaking the curse of dimensionality is essential because it allows us to analyze business cycle models with rich heterogeneity. Examples that go beyond what we show here are models where aging adds another dimension to the household problem or where a richer household portfolio needs to be modeled, e.g. when households own liquid asset, own houses, and write mortgages at the same time. To all these setups, the proposed method lends itself well to solve for equilibrium dynamics.

References

Ahn, S., Kaplan, G., Moll, B., Winberry, T., and Wolf, C. (2017). When inequality matters for macro and macro matters for inequality. *NBER Macroeconomics Annual*, Volume 32.

[12] Appendix A Table 7 summarizes the calibration.

Bayer, C., Luetticke, R., Pham-Dao, L., and Tjaden, V. (2015). Precautionary savings, illiquid assets, and the aggregate consequences of shocks to household income risk. *CEPR Discussion Paper No. DP10849*.

Bungartz, H.-J. and Griebel, M. (2004). Sparse grids. *Acta Numerica*, 13:147269.

Carroll, C. (2006). The method of endogenous gridpoints for solving dynamic stochastic optimization problems. *Economics Letters*, 91(3):312–320.

Den Haan, W. J. (2010). Assessing the accuracy of the aggregate law of motion in models with heterogeneous agents. *Journal of Economic Dynamics and Control*, 34(1):79–99.

Den Haan, W. J., Judd, K. L., and Juillard, M. (2010). Computational suite of models with heterogeneous agents: Incomplete markets and aggregate uncertainty. *Journal of Economic Dynamics and Control*, 34(1):1–3.

Hintermaier, T. and Koeniger, W. (2010). The method of endogenous gridpoints with occasionally binding constraints among endogenous variables. *Journal of Economic Dynamics and Control*, 34(10):2074–2088.

Kaplan, G. and Violante, G. L. (2014). A model of the consumption response to fiscal stimulus payments. *Econometrica*, 82:1199–1239.

Krueger, D. and Kubler, F. (2004). Computing equilibrium in OLG models with stochastic production. *Journal of Economic Dynamics and Control*, 28(7):1411–1436.

Krusell, P. and Smith, A. A. (1997). Income and wealth heterogeneity, portfolio choice, and equilibrium asset returns. *Macroeconomic Dynamics*, 1(02):387–422.

Krusell, P. and Smith, A. A. (1998). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5):867–896.

Luetticke, R. (2018). Transmission of monetary policy with heterogeneity in household portfolios. *CFM Discussion Paper Series*, (CFM-DP2018-19).

Reiter, M. (2002). Recursive computation of heterogeneous agent models. *mimeo, Universitat Pompeu Fabra*.

Reiter, M. (2009). Solving heterogeneous-agent models by projection and perturbation. *Journal of Economic Dynamics and Control*, 33(3):649–665.

Rotemberg, J. J. (1982). Sticky prices in the United States. *Journal of Political Economy*, pages 1187–1211.

Schmitt-Grohé, S. and Uribe, M. (2004). Solving dynamic general equilibrium models using a second-order approximation to the policy function. *Journal of Economic Dynamics and Control*, 28(4):755–775.

Taylor, J. B. (1993). Discretion versus policy rules in practice. In *Carnegie-Rochester Conference Series on Public Policy*, volume 39, pages 195–214. Elsevier.

Winberry, T. (2016). A toolbox for solving and estimating heterogeneous agent macro models. *mimeo, Chicago Booth*.

Woodford, M. (1995). Price-level determinacy without control of a monetary aggregate. *Carnegie-Rochester Conference Series on Public Policy*, 43(Supplement C):1 – 46.

A Calibrations

Table 5: Parameters of the Krusell & Smith model

Parameter	Value	Description	Target
Households			
β	0.99	Discount factor	Den Haan et al. (2010)
ξ	1	Relative risk aversion	Den Haan et al. (2010)
γ	1	Inv. Frisch elasticity	Standard value
ρ_h	0.98	Persistence of productivity	Standard value
σ_h	0.06	STD of innovations	Standard value
Intermediate Goods			
α	64%	Share of labor	Den Haan et al. (2010)
δ	2.5%	Depreciation rate	Den Haan et al. (2010)
ρ_Z	0.75	Persistence of productivity	Den Haan et al. (2010)
σ_Z	0.07	STD of innovations	Den Haan et al. (2010)

Notes: All values are reported for the quarterly frequency of the model.

Table 6: Parameters of the two-asset HANK model for Table 4

Parameter	Value	Description	Target
Households			
β	0.99	Discount factor	Den Haan et al. (2010)
ν	6.5%	Participation frequency	Luetticke (2018)
ξ	1	Relative risk aversion	Den Haan et al. (2010)
γ	1	Inv. Frisch elasticity	Standard value
$\overline{R}$	12.5%	Borrowing penalty	Bayer et al. (2015)
ρ_h	0.98	Persistence of productivity	Standard value
σ_h	0.06	STD of innovations	Standard value
Intermediate Goods			
α	67%	Share of labor	Den Haan et al. (2010)
δ	2.5%	Depreciation rate	Den Haan et al. (2010)
ρ_Z	0.75	Persistence of productivity	Den Haan et al. (2010)
σ_Z	0.07	STD of innovations	Den Haan et al. (2010)
Final Goods			
κ	∞	Price stickiness	0 quarters
η	20	Elasticity of substitution	5% markup
Capital Goods			
ϕ	0	Capital adjustment costs	Den Haan et al. (2010)
Fiscal Policy			
τ	0.3	Tax rate	$G/Y = 20\%$
ρ_B	0.86	Autocorrelation of debt	Bayer et al. (2015)
γ_π	0	Reaction to inflation	
$\gamma_\mathcal{T}$	0	Reaction to taxes	
Monetary Policy			
Π	1	Inflation	0% p.a.
R^B	1.0025	Nominal interest rate	1% p.a.
θ_π	1.25	Reaction to inflation	Standard value
ρ_R	0.8	Interest rate smoothing	Standard value

Notes: All values are reported for the quarterly frequency of the model.

Table 7: Parameters of the two-asset HANK model for Figure 2

Parameter	Value	Description	Source
Households			
β	0.98	Discount factor	Bayer et al. (2015)
ν	6.5%	Participation frequency	Luetticke (2018)
ξ	4	Relative risk aversion	Bayer et al. (2015)
γ	1	Inv. Frisch elasticity	Bayer et al. (2015)
$\overline{R}$	11%	Borrowing penalty	Bayer et al. (2015)
ρ_h	0.98	Persistence of productivity	Bayer et al. (2015)
σ_h	0.06	STD of innovations	Bayer et al. (2015)
ρ_S	0.84	Persistence of uncertainty	Bayer et al. (2015)
σ_S	0.54	STD of uncertainty shocks	Bayer et al. (2015)
Intermediate Goods			
α	70%	Share of labor	Income share labor of 66%
δ	1.35%	Depreciation rate	NIPA: Fixed assets
Final Goods			
κ	0.09	Price stickiness	4 quarters
η	20	Elasticity of substitution	5% markup
Capital Goods			
ϕ	11.4	Capital adjustment costs	Bayer et al. (2015)
Fiscal Policy			
τ	0.3	Tax rate	$G/Y = 20\%$
ρ_B	0.86	Autocorrelation of debt	Bayer et al. (2015)
γ_π	1.5	Reaction to inflation	Bayer et al. (2015)
$\gamma_\mathcal{T}$	0.5075	Reaction to taxes	Bayer et al. (2015)
Monetary Policy			
Π	1	Inflation	0% p.a.
R^B	1.0062	Nominal interest rate	2.5% p.a.
θ_π	1.25	Reaction to inflation	Standard value
ρ_R	0.8	Interest rate smoothing	Standard value

Notes: All values are reported for the quarterly frequency of the model.