# 1 Defnition in Continuous Time

The permanent component of wages follows a martingale $P_{i,t}$ for $i \in \{1,2\}$
The transitory component of wages consists of a shocks that persist for a period of time $a$, so that at time $t$ the transitory component of wages is $\frac{1}{a}\int_{t-a}^{t} dQ_{i,t}$.

# 2 Mapping Observable Changes in Continuous Time

In discrete time:

$$\begin{aligned}
\Delta c_t &\approx \kappa_{c,u_1}\Delta u_{1,t} + \kappa_{c,u_2}\Delta u_{2,t} + \kappa_{c,v_1} v_{1,t} + \kappa_{c,v_2} v_{2,t} \\
\Delta h_{1,t} &\approx \kappa_{h_1,u_1}\Delta u_{1,t} + \kappa_{h_1,u_2}\Delta u_{2,t} + \kappa_{h_1,v_1} v_{1,t} + \kappa_{h_1,v_2} v_{2,t} \\
\Delta h_{2,t} &\approx \kappa_{h_2,u_1}\Delta u_{1,t} + \kappa_{h_2,u_2}\Delta u_{2,t} + \kappa_{h_2,v_1} v_{1,t} + \kappa_{h_2,v_2} v_{2,t}
\end{aligned}$$

In continuous time replace $\Delta u_{i,t}$ with $\frac{1}{a}\left(\int_{t-a}^{t} dQ_{i,s} - \int_{t-2-a}^{t-2} dQ_{i,s}\right)$
and replace $v_{i,t}$ with $\int_{t-2}^{t} dP_{i,t}$. Which translates to the continuous time model above as:

$$\begin{aligned}
\Delta c_t \approx{}& \kappa_{c,u_1}\frac{1}{a}\left(\int_{t-a}^{t} dQ_{1,s} - \int_{t-2-a}^{t-2} dQ_{1,s}\right) + \kappa_{c,u_2}\frac{1}{a}\left(\int_{t-a}^{t} dQ_{2,s} - \int_{t-2-a}^{t-2} dQ_{2,s}\right) \\
&+ \kappa_{c,v_1}\int_{t-2}^{t} dP_{1,t} + \kappa_{c,v_2}\int_{t-2}^{t} dP_{2,t} \\
\Delta h_{1,t} \approx{}& \kappa_{h_1,u_1}\frac{1}{a}\left(\int_{t-a}^{t} dQ_{1,s} - \int_{t-2-a}^{t-2} dQ_{1,s}\right) + \kappa_{h_1,u_2}\frac{1}{a}\left(\int_{t-a}^{t} dQ_{2,s} - \int_{t-2-a}^{t-2} dQ_{2,s}\right) \\
&+ \kappa_{h_1,v_1}\int_{t-2}^{t} dP_{1,t} + \kappa_{h_1,v_2}\int_{t-2}^{t} dP_{2,t} \\
\Delta h_{2,t} \approx{}& \kappa_{h_2,u_1}\frac{1}{a}\left(\int_{t-a}^{t} dQ_{1,s} - \int_{t-2-a}^{t-2} dQ_{1,s}\right) + \kappa_{h_2,u_2}\frac{1}{a}\left(\int_{t-a}^{t} dQ_{2,s} - \int_{t-2-a}^{t-2} dQ_{2,s}\right) \\
&+ \kappa_{h_2,v_1}\int_{t-2}^{t} dP_{1,t} + \kappa_{h_2,v_2}\int_{t-2}^{t} dP_{2,t}
\end{aligned}$$

However, we do not observe this snapshot at time $t$ (except for consumption), instead we observe:

$$\begin{aligned}
\Delta c_T^{obs} &= \Delta c_T \\
\Delta h_{i,T}^{obs} &= \int_{T-1}^{T} \Delta h_{i,t} dt
\end{aligned}$$

Define for transitory shocks:

$$
\begin{aligned}
\Delta\mathcal{Q}_{i,T} &= \frac{1}{a}\left(\int_{T-a}^{T} dQ_{i,s} - \int_{T-2-a}^{T-2} dQ_{i,s}\right)\\
\overline{\Delta\mathcal{Q}}_{i,T} &= \int_{T-1}^{T} \Delta\mathcal{Q}_{i,s} ds\\
&= \int_{T-1-a}^{T-1} \frac{s-(T-1-a)}{a} dQ_{i,s} + \int_{T-1}^{T-a} dQ_{i,s} + \int_{T-a}^{T} \frac{T-s}{a} dQ_{i,s}\\
&\quad - \int_{T-3-a}^{T-3} \frac{s-(T-3-a)}{a} dQ_{i,s} - \int_{T-3}^{T-2-a} dQ_{i,s} - \int_{T-2-a}^{T-2} \frac{T-2-s}{a} dQ_{i,s}
\end{aligned}
$$

and for permanent shocks:

$$
\begin{aligned}
\Delta\mathcal{P}_{i,T} &= \int_{T-2}^{T} dP_{i,s}\\
\overline{\Delta\mathcal{P}}_{i,T} &= \int_{T-1}^{T} \Delta\mathcal{P}_{i,s} ds\\
&= \int_{T-3}^{T-2} (s-(T-3)) dP_{i,s} + \int_{T-2}^{T-1} dP_{i,s} + \int_{T-1}^{T} (T-s) dP_{i,s}
\end{aligned}
$$

then

$$
\begin{aligned}
\Delta c_{i,T}^{obs} &= \Delta c_{i,T}\\
&\approx \kappa_{c,u_1}\Delta\mathcal{Q}_{1,T} + \kappa_{c,u_2}\Delta\mathcal{Q}_{2,T}\\
&\quad + \kappa_{c,v_1}\Delta\mathcal{P}_{1,T} + \kappa_{c,v_2}\Delta\mathcal{P}_{2,T}
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta h_{i,T}^{obs} &= \int_{T-1}^{T} \Delta h_{i,t} dt\\
&\approx \kappa_{h_i,u_1}\overline{\Delta\mathcal{Q}}_{1,T} + \kappa_{h_i,u_2}\overline{\Delta\mathcal{Q}}_{2,T}\\
&\quad + \kappa_{h_i,v_1}\overline{\Delta\mathcal{P}}_{1,T} + \kappa_{h_i,v_2}\overline{\Delta\mathcal{P}}_{2,T}
\end{aligned}
$$

furthermore

$$
\begin{aligned}
\Delta w_{i,T}^{obs} &= \int_{T-1}^{T} \Delta w_{i,t} dt\\
&\approx \overline{\Delta\mathcal{Q}}_{i,T} + \overline{\Delta\mathcal{P}}_{i,T}
\end{aligned}
$$

Now calculate variances for transitory shocks (defining $\sigma^2_{Q_{i,T}}$ as the variance of transitory wage shocks in year $T$ for male or female, and $\sigma_{Q_{ij,T}}$ as the covariance of transitory wage shocks between men and women ($i \neq j$).

$$
\begin{aligned}
\mathrm{Var}(\Delta\mathcal{Q}_{i,T}) &= \frac{1}{a}\left(\sigma^2_{Q_{i,T}} + \sigma^2_{Q_{i,T-2}}\right) \\
\mathrm{Cov}(\Delta\mathcal{Q}_{i,T}, \Delta\mathcal{Q}_{i,T-2}) &= -\frac{1}{a}\sigma^2_{Q_{i,T-2}} \\
\mathrm{Cov}(\Delta\mathcal{Q}_{i,T}, \Delta\mathcal{Q}_{j,T}) &= \frac{1}{a}\left(\sigma_{Q_{ij,T}} + \sigma_{Q_{ij,T-2}}\right) \\
\mathrm{Cov}(\Delta\mathcal{Q}_{i,T}, \Delta\mathcal{Q}_{j,T-2}) &= -\frac{1}{a}\sigma_{Q_{ij,T-2}} \\
\mathrm{Var}(\overline{\Delta\mathcal{Q}}_{i,T}) &= \frac{1}{3}a\sigma^2_{Q_{i,T-1}} + (1-\frac{2}{3}a)\sigma^2_{Q_{i,T}} + \frac{1}{3}a\sigma^2_{Q_{i,T-3}} + (1-\frac{2}{3}a)\sigma^2_{Q_{i,T-2}} \\
\mathrm{Cov}(\overline{\Delta\mathcal{Q}}_{i,T}, \overline{\Delta\mathcal{Q}}_{i,T-2}) &= -\frac{1}{3}a\sigma^2_{Q_{i,T-3}} - (1-\frac{2}{3}a)\sigma^2_{Q_{i,T-2}} \\
\mathrm{Cov}(\overline{\Delta\mathcal{Q}}_{i,T}, \overline{\Delta\mathcal{Q}}_{j,T}) &= \frac{1}{3}a\sigma_{Q_{ij,T-1}} + (1-\frac{2}{3}a)\sigma_{Q_{ij,T}} + \frac{1}{3}a\sigma_{Q_{ij,T-3}} + (1-\frac{2}{3}a)\sigma_{Q_{ij,T-4}} \\
\mathrm{Cov}(\overline{\Delta\mathcal{Q}}_{i,T}, \overline{\Delta\mathcal{Q}}_{j,T-2}) &= -\frac{1}{3}a\sigma_{Q_{ij,T-3}} - (1-\frac{2}{3}a)\sigma_{Q_{ij,T-2}}
\end{aligned}
$$

and (NEED TO ADD IN MALE FEMALE COVARIANCE, same as above)

$$
\begin{aligned}
\mathrm{Cov}(\Delta\mathcal{Q}_{i,T}, \overline{\Delta\mathcal{Q}}_{i,T}) &= \frac{1}{2}(\sigma^2_{Q_{i,T}} + \sigma^2_{Q_{i,T-2}}) \\
\mathrm{Cov}(\overline{\Delta\mathcal{Q}}_{i,T}, \Delta\mathcal{Q}_{i,T-2}) &= -\frac{1}{2}\sigma^2_{Q_{i,T-2}} \\
\mathrm{Cov}(\Delta\mathcal{Q}_{i,T}, \overline{\Delta\mathcal{Q}}_{i,T-2}) &= -\frac{1}{2}\sigma^2_{Q_{i,T-2}}
\end{aligned}
$$

and for permanent shocks:

$$
\begin{aligned}
\mathrm{Var}(\Delta\mathcal{P}_{i,T}) &= \sigma^2_{P_{i,T}} + \sigma^2_{P_{i,T-1}} \\
\mathrm{Cov}(\Delta\mathcal{P}_{i,T}, \Delta\mathcal{P}_{i,T-2}) &= 0 \\
\mathrm{Var}(\overline{\Delta\mathcal{P}}_{i,T}) &= \frac{1}{3}\sigma^2_{P_{i,T}} + \sigma^2_{P_{i,T-1}} + \frac{1}{3}\sigma^2_{P_{i,T-2}} \\
\mathrm{Cov}(\overline{\Delta\mathcal{P}}_{i,T}, \overline{\Delta\mathcal{P}}_{i,T-2}) &= \frac{1}{6}\sigma^2_{P_{i,T-2}}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Cov}(\Delta\mathcal{P}_{i,T}, \overline{\Delta\mathcal{P}}_{i,T}) &= \frac{1}{2}\sigma^2_{P_{i,T}} + \sigma^2_{P_{i,T-1}} \\
\mathrm{Cov}(\overline{\Delta\mathcal{P}}_{i,T}, \Delta\mathcal{P}_{i,T-2}) &= \frac{1}{2}\sigma^2_{P_{i,T-2}} \\
\mathrm{Cov}(\Delta\mathcal{P}_{i,T}, \overline{\Delta\mathcal{P}}_{i,T-2}) &= \frac{1}{2}\sigma^2_{P_{i,T-2}}
\end{aligned}
$$