# Microlocal Analysis
## with Applications to Non-Elliptic Fredholm Problems

Edmund Lau
Supervised by: Dr Jesse Gell-Redman

The University of Melbourne

*elau1@student.unimelb.edu.au*

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# Introduction

A linear partial differential operator of order $k \in \mathbb{N}$ in $\mathbb{R}^n$ :

$$P = P(x, D_x) = \sum_{|\alpha| \leqslant k} c_\alpha(x) D_x^\alpha, \quad c_\alpha \in C_\infty^\infty(\mathbb{R}^n)$$

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where

$$\begin{aligned} \alpha &= (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n && \text{multi-index} \\ |\alpha| &= \alpha_1 + \alpha_2 + \cdots + \alpha_n && \text{order of multi-index} \\ D_x^\alpha &= (-i\partial_{x_1})^{\alpha_1} (-i\partial_{x_2})^{\alpha_2} \dots (-i\partial_{x_n})^{\alpha_n} \\ D_{x_i} &= -i\partial_{x_i} \implies \mathcal{F} D_x^\alpha = \xi^\alpha \mathcal{F} \end{aligned}$$

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Examples:

$$\begin{aligned} \Delta &= D_{x_1}^2 + \cdots + D_{x_n}^2 && \text{Laplace operator} \\ \Box u &= D_{x_1}^2 + \cdots + D_{x_n}^2 - D_t^2 && \text{Wave operator} \end{aligned}$$

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Weak solution and forcing:

$$\begin{aligned} u : \mathcal{S}(\mathbb{R}^n) &\to \mathbb{C} \\ \varphi &\mapsto u(\varphi) \end{aligned}$$

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \iff \varphi \in C^\infty \text{ and } \sup_x \left| x^\beta D_x^\alpha \varphi(x) \right| < \infty$$

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**Fredholm** theory tackles all three simultaneously!

# Overview


1. Introduction
2. Fredholm Operators and Regularity
3. “Elliptic operators are Fredholm”
4. A Non-elliptic Fredholm problem

# Fredholm Operators

## Definition (Fredholm operators)

A continuous linear operator $T : \mathcal{X} \to \mathcal{Y}$ between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ is Fredholm, if

- $T$ has closed range, i.e. $T(\mathcal{X})$ is closed in $\mathcal{Y}$,
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$T$ Fredholm
$\rightsquigarrow$ existence and uniqueness reduce to finite dimensional linear algebra.

# Fredholm Estimate

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**Theorem (Fredholm Estimate)**

*Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be Banach spaces. If*

- $T : \mathcal{X} \to \mathcal{Y}$ *is continuous,*
- $\mathcal{X}$ *is compactly contained in* $\mathcal{Z}$, *i.e.* $\iota : \mathcal{X} \hookrightarrow \mathcal{Z}$ *is compact,*
- *for all* $x \in \mathcal{X}$, *there exist* $C > 0$ *such that the following estimate hold*

$$\|x\|_{\mathcal{X}} \leqslant C \left( \|Tx\|_{\mathcal{Y}} + \|x\|_{\mathcal{Z}} \right) \tag{1}$$

*then* $T$ *is semi-Fredholm*

- *the image,* $T(\mathcal{X})$ *is closed, and*
- $T$ *has finite dimensional kernel.*

# Constructing a *Fredholm problem*

What's a Fredholm differential operator? . . . what's $\mathcal{X}$ and $\mathcal{Y}$?

**Fredholm Problem**

Given a differential operator $P$, can we construct solution spaces $\mathcal{X}$ and $\mathcal{Y}$, so that

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What's the link to *regularity*? Sobolev Space!

# Sobolev Space

## Definition

The Sobolev space of order $k \in \mathbb{N}$ on $\mathbb{R}^n$, $H^k(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ is defined by

$$u \in H^k(\mathbb{R}^n) \iff D^\alpha u \in L^2(\mathbb{R}^n) \text{ whenever } |\alpha| \leqslant k$$
$$\iff \langle \xi \rangle^k \mathcal{F}u(\xi) \in L^2(\mathbb{R}^n).$$

$$\langle \xi \rangle := \left(1 + |\xi|^2\right)^{1/2} = \left(1 + |\xi_1|^2 + \cdots + |\xi_n|^2\right)^{1/2}$$

Hilbert space structure that keeps track of (global) regularity data of $u$.

$$\|u\|_{H^k} = \underbrace{\|u\|_{L^2}}_{\text{global decay}} + \underbrace{\sum_{|\alpha| \leqslant k} \|D^\alpha u\|_{L^2}}_{k \text{ times differentiable}}$$

## Sobolev Space on Closed Manifold

Let $M$ be a smooth closed $n$-manifold (compact without boundary), $s \in \mathbb{R}$, $u \in (C^\infty(M))'$, then

$$u \in H^s(M) \iff (\chi \cdot u) \circ \Phi^{-1} \in H^s(U)$$

for any chart $\Phi : \widetilde{U} \to U \subset \mathbb{R}^n$ and smooth bump function $\chi \in C^\infty(M)$ compactly supported in the chart domain $\widetilde{U}$.

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Henceforth, $M$ is either $\mathbb{R}^n$ or a closed $n$-manifold.

# General Strategy

Existence, uniqueness, regularity $\rightsquigarrow$

For what $s, s' \in \mathbb{R}$ can we prove

$$\|u\|_{H^s} \leqslant C \left(\|Pu\|_{H^{s'}} + \|u\|_{H^N}\right).$$

so that $P : H^s(M) \to H^{s'}(M)$ is (semi-) Fredholm?

For **elliptic operators**:

$$\text{any } s + m \text{ and } s' = s.$$

For **non-elliptic operators**:

$$\text{any } s + m \text{ and } s' = s + 1.$$

Only certain subsets of Sobolev spaces allowed.

# “Elliptic operators are Fredholm”

How do we get such an estimate?

**Theorem (Elliptic regularity)**

*Let $P$ be an order $m \in \mathbb{R}$ elliptic differential operator on an $n$-manifold, $M$. Suppose we know a priori that $u \in H^N(M)$ for some $N \in \mathbb{R}$. Then, for any $s \in \mathbb{R}$*

$$(f =)Pu \in H^s(M) \implies u \in H^{s+m}(M)$$

*and $u$ satisfies the estimates: $\exists C > 0$*

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$$(\Delta + 1)\, u = \mathcal{F}^{-1}\mathcal{F}(\Delta + 1)u = \mathcal{F}^{-1}(1 + |\,\xi\,|^2)\mathcal{F}u$$

We call $\left(1 + |\,\xi\,|^2\right)$ is the **symbol** for $(\Delta + 1)$.

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We call $\left(1 + |\, \xi\, |^2\right)$ is the **symbol** for $(\Delta + 1)$.
We expect an inverse ...

$$(\Delta + 1)^{-1}\, (\Delta + 1)\, u(x) = \mathcal{F}^{-1}(1 + |\, \xi\, |^2)^{-1}(1 + |\, \xi\, |^2) \mathcal{F} u = u$$

# Pseudodifferential operator

Question: What is $(\Delta + 1)^{-1}$? Answer: **pseudo**differential operator.

$$P(x, D_x)u = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} p(x, \xi) u(y) \, \mathrm{d}y \, \mathrm{d}\xi$$

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## Definition

A smooth function $p(x,\xi) \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ is a symbol of order $m \in \mathbb{R}$, i.e. $p \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$, if

$$\left| D_x^\alpha D_\xi^\beta p(x,\xi) \right| \leqslant C_{\alpha,\beta,\gamma} \langle \xi \rangle^{m - |\beta|}, \quad C_{\alpha,\beta} > 0$$

for any multi-index $\alpha, \beta \in \mathbb{N}^n$.
A pseudodifferential operator, $P \in \Psi^m_\infty(\mathbb{R}^n)$ of order $m$ with (left reduced) symbol $p \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$ has action on $u \in \mathcal{S}'(\mathbb{R}^n)$ given by the integral above.

# Pseudodifferential operators

## Lemma

*If $P \in \Psi^m_\infty(M)$ for some $m \in \mathbb{R}$,*

1. $P : H^s(M) \to H^{s-m}(M)$ *is continuous for any* $s \in \mathbb{R}$.
2. *If $P$ is* **elliptic** *of order $m$, i.e. its symbol $p$ satisfies*

$$|\, p(x,\xi)\,| \geqslant \epsilon \langle\xi\rangle^m \qquad \text{in } |\,\xi\,| > \epsilon \text{ for some } \epsilon > 0$$

*then there exist parametrix $Q \in \Psi^{-m}_\infty(M)$ such that*

$$QP - 1 : H^s(M) \to H^{s'}(M)$$

*is continuous for any* $s, s' \in \mathbb{R}$.

# Proof of Elliptic Regularity

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using continuity $Q : H^{s+m} \to H^s$ and $(QP - 1) : H^{s+m} \to H^N$.

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We get

$$\|u\|_{H^{s+m}} \leqslant C\,\|Pu\|_{H^s} + C\,\|u\|_{H^N}\,.$$

# Non-elliptic Fredholm problem

**Theorem (Main theorem)**

*There exist a perturbation $Q$ of the wave operator $\Box$ on $\mathbb{T}^{1+n}$ and a subspace $\mathcal{X}^{s+2} \subset H^{s+2}(\mathbb{T}^{1+n})$ for each $s \in \mathbb{R}$, such that the operator:*

$$(\Box - iQ) : \mathcal{X}^{s+2} \to H^{s+1}(\mathbb{T}^n)$$

*is Fredholm.*

$$\mathbb{T}^{1+n} := \mathbb{S}^1_t \times \underbrace{\mathbb{S}^1_{x_1} \times \mathbb{S}^1_{x_2} \times \cdots \times \mathbb{S}^1_{x_n}}_{n}$$

$$\Box := \partial_t^2 - \sum_{j=1}^{n-1} \partial_{x_j}^2, \quad p(t, x, \tau, \xi) = |\,\xi\,|^2 - \tau^2$$

# Microlocal Viewpoint

Global ellipticity

$$\iff \left| \, |\, \xi \,|^2 - \tau^2 \, \right| \geqslant \epsilon \, \langle (\tau, \xi) \rangle^2 \text{ whenever } |\, (\tau, \xi) \,| > 1/\epsilon.$$

# Microlocal Viewpoint

Microlocal ellipticity at a point $(t_0, x_0, \tau_0, \xi_0) \in T^*\mathbb{T}^{1+n} \setminus 0$
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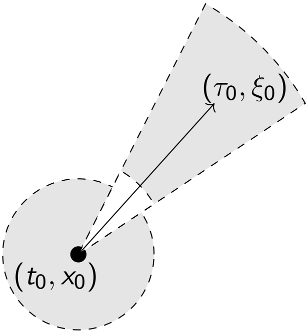

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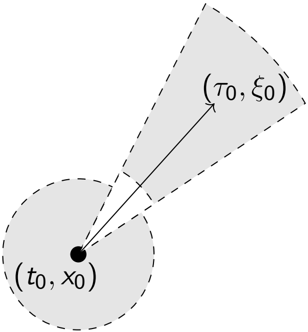


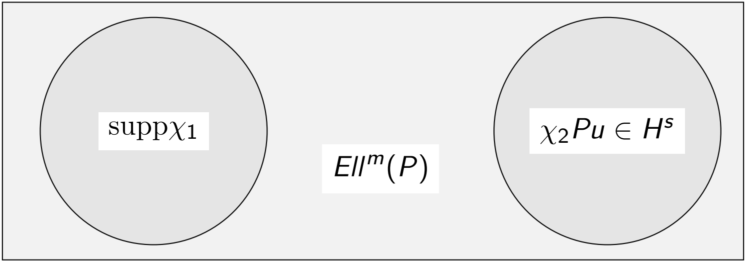

$Ell^2 = \{\text{points in phase space where } p \text{ is elliptic}\} \setminus 0$
$\Sigma^2 = Ell^m(\Box)^c \setminus 0.$
Note: For differential operators: Elliptic $\iff$ principal symbol is non-zero (outside of zero section).

# Two Major Ingredients

**Theorem (Microlocal elliptic regularity)**

*Let $P \in \Psi^m_\infty(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. If for some $\chi_2 \in \Psi^0_\infty(\mathbb{R}^n)$, $\chi_2 Pu \in H^s(\mathbb{R}^n)$, then for any other $\chi_1 \in \Psi^0_\infty(\mathbb{R}^n)$ such that $\mathrm{WF}'(\chi_1) \subset Ell^m(P) \cap Ell^0(\chi_2)$ we have $\chi_1 u \in H^{s+m}(\mathbb{R}^n)$ and it satisfies the estimate: $\forall N \in \mathbb{R}$, $\exists C > 0$*

$$\|\chi_1 u\|_{H^{s+m}} \leqslant C\left(\|\chi_2 Pu\|_{H^s} + \|u\|_{H^N}\right).$$

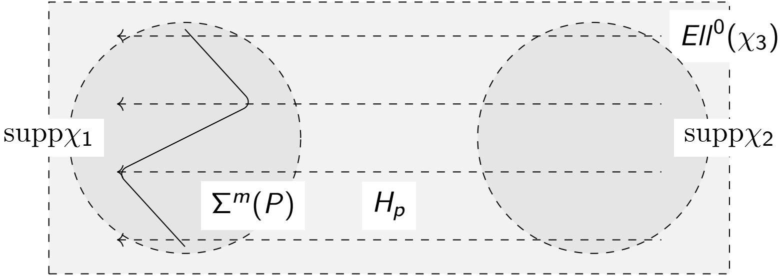


# Two Major Ingredients

## Theorem (Propagation of singularities)

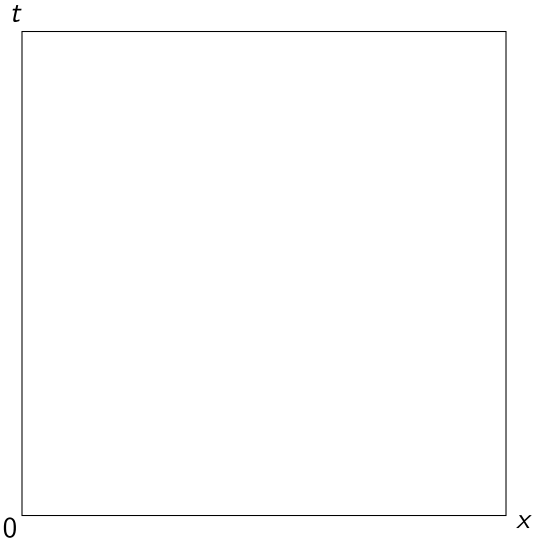
*Let $P \in \Psi^m_\infty(\mathbb{R}^n)$ is a properly supported pseudodifferential operator with polyhomogeneous principal $\sigma_m(P) = p - iq$ with real $p, q$. If we have $\chi_1, \chi_2, \chi_3 \in \Psi^0_\infty(\mathbb{R}^n)$ and $q \geqslant 0$ on $\mathrm{WF}'(\chi_3)$ and every $(x, \xi) \in WF'(P)$ is in the integral curve of $H_p$ originating from $Ell^0(\chi_2)$, then for all $s, N \in \mathbb{R}$ and $u \in C^\infty(\mathbb{R}^n)$, there exist $C > 0$ such that*

$$\|\chi_1 u\|_{H^{s+m}} \leqslant C\left(\|\chi_2 u\|_{H^s} + \|\chi_3 Pu\|_{H^{s+1}} + \|u\|_{H^N}\right).$$



# Constructions

Main idea : Create enough elliptic region! $\Box - iQ = \Box - i\chi(t)\partial_t^2$.



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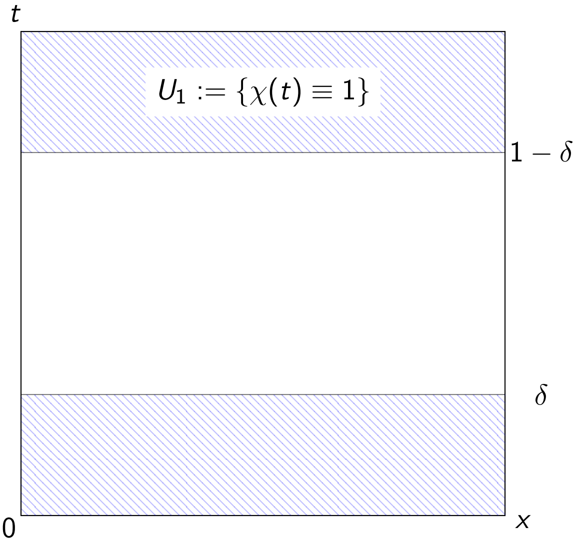

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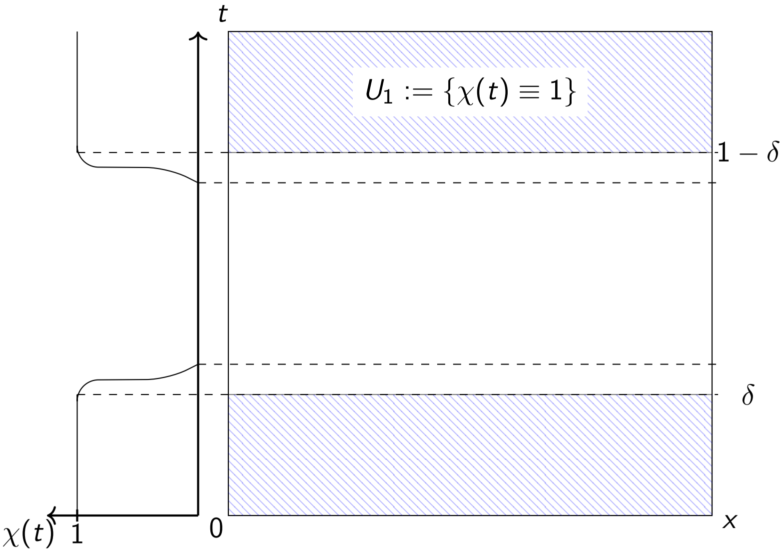

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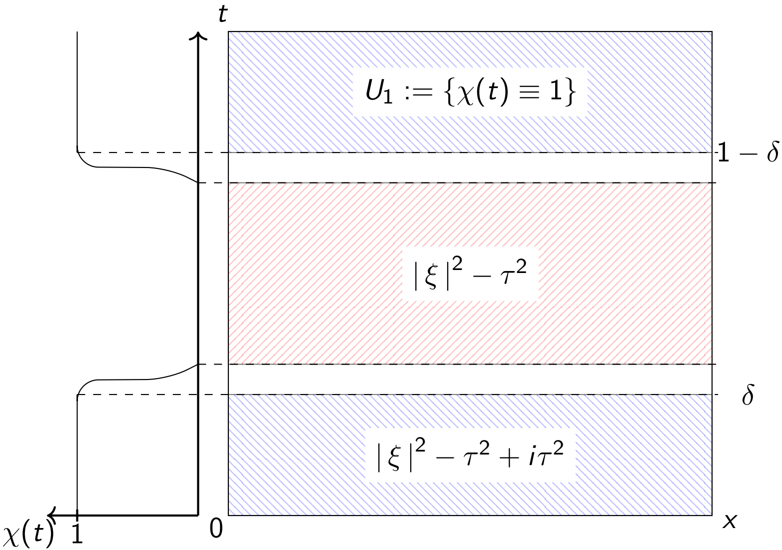
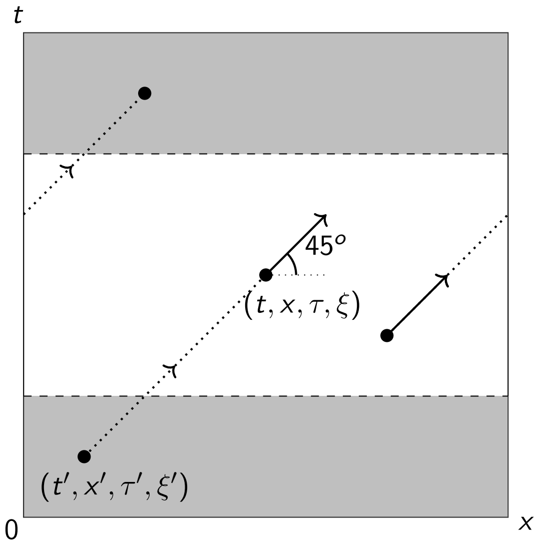


# Hamilton flow of $\sigma_2(\Box)$

Hamiltonian flow: $\exp(sH_p)(t,x,\tau,\xi) = (t+s\tau, x+s\xi, \tau, \xi)$



# Constructions

- (principal) symbol of the form $p - iq$, $p = \sigma_2(\Box)$, $q \geqslant 0$. ($\checkmark$)
- Elliptic region that propagates to hit every point in $\Sigma^2(\Box - iQ)$. ($\checkmark$)

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$$\|u\|_{H^{s+2}} \leqslant C'\|\Box - iQ\|_{H^{s-2}} + C\|(\Box - iQ)u\|_{H^{s+1}} + C'\|u\|_{H^N}$$

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$$\begin{aligned}
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\|u\|_{H^{s+2}} &\leqslant C \underbrace{\|\chi(t) u\|_{H^s}}_{\text{elliptic region!}} + C \|(\Box - iQ)u\|_{H^{s+1}} + C \|u\|_{H^N} \\
\|u\|_{H^{s+2}} &\leqslant C' \|\Box - iQ\|_{H^{s-2}} + C \|(\Box - iQ)u\|_{H^{s+1}} + C' \|u\|_{H^N} \\
\|u\|_{H^{s+2}} &\leqslant C'' \left(\|(\Box - iQ)u\|_{H^{s+1}} + \|u\|_{H^N}\right)
\end{aligned}$$

Almost there!

$$\|u\|_{H^{s+2}} \leqslant C'' \left(\|(\Box - iQ)u\|_{H^{s+1}} + \|u\|_{H^N}\right)$$

Which suggest the Hilbert space domain that we want is

$$\mathcal{X}^s = \left\{u \in H^s \,:\, (\Box - iQ)u \in H^{s-1}\right\}.$$

And

$$\Box - iQ : \mathcal{X}^{s+2} \to H^{s+1}$$

is (semi-) Fredholm for any $s \in \mathbb{R}$.

Thank you!

Questions are welcomed!