

1 Microlocal Analysis

Theorem 1.1 (Schwartz Kernel Theorem [Taylor, 2011, Chapter 4.6, p. 345]). *Let M, N be compact manifold and*

$$T : C^\infty(M) \rightarrow \mathcal{D}'(N)$$

be a continuous linear map ($C^\infty(M)$ being given Frechet space topology and $\mathcal{D}'(N)$ the weak topology). Define a bilinear map*

$$\begin{aligned} B : C^\infty(M) \times C^\infty(N) &\rightarrow \mathbb{C} \\ (u, v) &\mapsto B(u, v) = \langle v, Tu \rangle. \end{aligned}$$

Then, there exist a distribution $k \in \mathcal{D}'(M \times N)$ such that for all $(u, v) \in C^\infty(M) \times C^\infty(N)$

$$B(u, v) = \langle u \otimes v, k \rangle.$$

We call such k the kernel of T .

1.1 Symbols

1.2 Pseudodifferential Operators (Ψ DO's)

2 Sobolev Spaces [Taylor, 2011, Chapter 4]

Definition 2.1 (Sobolev Spaces). Let $p \in \mathbb{R}$ and $k \in \mathbb{N}$ be given. We define the k^{th} -order L^p -based Sobolev spaces on \mathbb{R}^n as the Banach space

$$W^{p,k}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) \mid D^\alpha u \in L^p(\mathbb{R}^n), |\alpha| \leq k\}$$

with the norm

$$\|u\|_{W^{p,k}} = \|u\|_{L^p} + \|D^k u\|_{L^p}.$$

For $p = 2$, we have denote $H^k := W^{2,k}$ and note that result from Fourier analysis gives

$$H^k = \left\{ u \in L^2(\mathbb{R}^n) \mid \langle \xi \rangle^k \hat{u} \in L^2(\mathbb{R}^n) \right\}$$

allowing us to extend the definition to each real order $s \in \mathbb{R}$,

$$H^s = \left\{ u \in S'(\mathbb{R}^n) \mid \langle \xi \rangle^s \hat{u} \in L^2(\mathbb{R}^n) \right\}$$

where $S'(\mathbb{R}^n)$ is the space of tempered distribution on \mathbb{R}^n . This forms a Hilbert space with inner product given by

$$\langle u, v \rangle_{H^s} = \langle \Lambda^s u, \Lambda^s v \rangle_{L^2}$$

with $\Lambda^s : S'(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ being the operator $\Lambda^s u = \mathcal{F}^{-1}(\langle \xi \rangle^s \hat{u})$.

3 Linear Elliptic equations [Taylor, 2011, Chapter 5]

Theorem 3.1 (Elliptic estimate). *Let $\bar{\Omega}$ be a compact Riemannian manifold and Ω be its interior. Let $L = -\Delta + X$, where Δ is the Laplacian and X and first order differential operator with smooth*

coefficient in $\overline{\Omega}$. Then, we have the estimate

$$\|u\|_{H^{k+1}}^2 \leq C \left(\|Lu\|_{H^{k-1}}^2 + \|u\|_{H^k}^2 \right)$$

for all $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ and for some $C > 0$.

References

[Taylor, 2011] Taylor, M. (2011). *Partial Differential Equations I*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2 edition.