

We shall follow the presentation given in [?].

## 1 Motivation for Pseudodifferential operators

- Solving PDEs via Fourier transform. For example, in Euclidean space,  $\mathbb{R}^n$ , constant coefficient linear PDE

$$P(D)u = \sum_{|\alpha| \leq n} c_\alpha D^\alpha u = f, \quad c_\alpha \in \mathbb{R}$$

where  $P \in \mathbb{R}[x]$ , can be solved by applying Fourier transform which gives a solution of the form

$$u(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y) \cdot \xi} f(y) \frac{1}{P(\xi)} dy d\xi$$

due to the observation that

$$\mathcal{F}P(D)u = P(\xi)\mathcal{F}u.$$

Moreover, for linear differential operators with smooth coefficients

$$P(x, D) : u \mapsto \sum_{|\alpha| \leq n} a_\alpha D^\alpha u, \quad a_\alpha \in C^\infty(\mathbb{R}^n)$$

we have

$$P(x, D)u = \frac{1}{(2\pi)^n} \int P(x, \xi) e^{i(x-y)\xi} u(y) dy d\xi.$$

We would like to generalise the above so that  $P(x, \xi)$  are smooth functions satisfying certain uniform bounds, called *symbols*, instead of just polynomials in  $\xi$ . This gives us a class of operators, called pseudodifferential operators, that acts as

$$A_a u(x) = \frac{1}{(2\pi)^n} \int a(x, \xi) e^{i(x-y)\xi} u(y) dy d\xi$$

for each symbol  $a$ .

- There isn't enough differential operators with smooth coefficient in the sense that elliptic differential operators are not, in general, invertible in this class. For example, the operator

$$u \mapsto (\Delta + 1)u$$

has inverse that acts as (using construction via Fourier transform shown above)

$$(\Delta + 1)^{-1} f = \frac{1}{(2\pi)^n} \int \frac{1}{1 + |\xi|^2} e^{i(x-y)\xi} f(y) dy d\xi$$

which is a pseudodifferential operator with symbol  $a(x, \xi) = (1 + |\xi|^2)^{-1}$ .

- Motivation from quantum mechanics. The notion of “quantisation” in quantum mechanics can be formalised as the map that sends a symbol  $a$  (a smooth function that represent deterministic observable in classical mechanics) to its corresponding pseudodifferential operator (i.e. the corresponding quantum observable)

$$A_a : \psi \mapsto \frac{1}{(2\pi)^n} \int \frac{1}{1 + |\xi|^2} e^{i(x-y)\xi} \psi(y) dy d\xi$$

that acts on the wavefunction  $\psi$ .

- Used in the formulation and proof of Atiyah-Singer Index theorem.

We shall define, on Euclidean space, the space of symbols,  $S^m(\mathbb{R}_{x,y}^{2n}; \mathbb{R}_\xi^n)$  and the corresponding space of pseudodifferential operators,  $\Psi^m(\mathbb{R}^n)$  which acts on distributions via the Schwartz kernel given by the oscillatory integral

$$I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) d\xi.$$

We note that we have introduced an extra variable  $y$  which will help in explicating the properties of pseudodifferential operators. However, the extra variable does not change the essence of the theory.

## 2 Symbols

We shall here list the definition of the space of symbols of order  $m \in \mathbb{N}$  in Euclidean space  $\mathbb{R}^n$  that one encounter in the literature. The main motivation is again based on the property of linear differential operators of order  $m \in \mathbb{N}$  with smooth coefficient that, after Fourier transform gives the polynomial of  $\xi$  with smooth coefficient

$$P(x, \xi) = \sum |\alpha| \leq m a_\alpha(x) \xi^\alpha.$$

It has the property that

$$\left| D_x^\alpha D_\xi^\beta P(x, \xi) \right| \leq C_{\alpha, \beta} \langle \xi \rangle^{m - |\beta|}$$

i.e.  $P(x, \xi)$  is smooth and decreases in order as  $\xi \rightarrow \infty$  with successive  $\xi$ -derivative.

**Definition 2.1.** The space  $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$  of order  $m$  is the space of smooth functions  $a \in C^\infty(\mathbb{R}^p \times \mathbb{R}^n)$  such that for all multi-index  $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} \langle \xi \rangle^{m - |\beta|}$$

uniformly on  $\mathbb{R}^p \times \mathbb{R}^n$ . We can also defined the space of symbol,  $S_\infty^m(\Omega; \mathbb{R}^n)$  on a set with non-empty interior  $\Omega \subset \mathbb{R}^p, \Omega \subset \text{Int}(\Omega)$  such that the bound above is satisfied uniformly in  $(x, \xi) \in \text{Int}(\Omega) \times \mathbb{R}^n$ . The subscript  $\infty$  refers the uniform boundedness condition. Together with the family of seminorm (indexed by  $N \in \mathbb{N}$ )

$$\|a\|_{N, m} = \sup_{(x, \xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \max_{|\alpha| + |\beta| \leq N} \frac{D_x^\alpha D_\xi^\beta a(x, \xi)}{\langle \xi \rangle^{m - |\beta|}}$$

gives a Frechet topology to  $S_\infty^m(\Omega; \mathbb{R}^n)$ .

Note: In defining pseudodifferential operators, we shall focus on the case where  $p = 2n$ .

**Definition 2.2.** A **symbol** of type  $S_{\delta, \delta'}^{m, l_1, l_2}$  where  $m, l_1, l_2 \in \mathbb{R}$  and  $\delta, \delta' \in [0, 1/2)$  is an element of  $C^\infty(\mathbb{R}_x^n; \mathbb{R}_y^n; \mathbb{R}_\xi^n)$  satisfying

$$\frac{\left| D_x^\alpha D_y^\beta D_\xi^\gamma a(x, y, \xi) \right|}{\langle \xi \rangle^{m - |\gamma|} \langle x \rangle^{l_1 - |\alpha|} \langle y \rangle^{l_2 - |\beta|} \langle \xi \rangle^{\delta |(\alpha, \beta, \gamma)|} \langle x, y \rangle^{\delta' |(\alpha, \beta, \gamma)|}} \leq C_{\alpha, \beta, \gamma}$$

uniformly in  $\mathbb{R}^{3n}$ . Taking the supremum over  $\mathbb{R}^{3n}$ , we get a family of seminorms, indexed by  $N \in \mathbb{N}$  defined by

$$\|a\|_{S_{\delta, \delta'}^{m, l_1, l_2}, N} := \sum_{|(\alpha, \beta, \gamma)| \leq N} \inf C_{\alpha, \beta, \gamma}$$

which gives  $S_{\delta, \delta'}^{m, l_1, l_2}$  a Frechet topology.

**Definition 2.3.** A (Kohn-Nirenberg) **symbol** of order  $m \in \mathbb{R}$  on  $T^*\mathbb{R}^n \cong \mathbb{R}_{x, \xi}^{2n}$  is a smooth function  $a = a(x, \xi)$  satisfying

$$\forall \alpha, \beta \in \mathbb{N}^n, \exists C \in \mathbb{R}_{\geq 0} : \left| D_x^\alpha D_\xi^\beta a(x, \xi) \right| \leq C \langle \xi \rangle^{m - |\beta|}$$

uniformly in  $x$ . The **space of symbol of order  $m$**  on  $T^*\mathbb{R}^n$

**Definition 2.4.** Let  $n \in \mathbb{N}$  be given. An **order function**  $g \in C^\infty(\mathbb{R}^n; \mathbb{R}_{\geq 0})$  is a *non-negative* function satisfying

$$\forall \alpha \in \mathbb{N}^n \exists C \in \mathbb{R}_{\geq 0} : \partial^\alpha g \leq Cg$$

uniformly on  $\mathbb{R}^n$ , i.e.  $\partial^\alpha g = O(g)$  uniformly on  $\mathbb{R}^n$ .

Given an order function  $g$ , a **symbol** of order  $g$  is a smooth function  $a = a(x, \xi) \in C^\infty(T^*\mathbb{R}^n)$  satisfying

$$\forall \alpha, \beta \in \mathbb{N}^n : \left| D_x^\alpha D_\xi^\beta a(x, \xi) \right| \leq Cg(\xi)$$

uniformly in  $x$ .

## 2.1 Properties of Symbols

**Proposition 2.5.** Let  $p, n \in \mathbb{N}$  be given and  $\Omega \subset \mathbb{R}^p$  such that  $\Omega \subset \overline{\text{Int}(\Omega)}$ . If  $m, m' \in \mathbb{R}$  such that  $m \leq m'$ , then  $S_\infty^m(\Omega; \mathbb{R}^n) \subset S_\infty^{m'}(\Omega; \mathbb{R}^n)$ . Furthermore, the inclusion map

$$\iota : S_\infty^m(\Omega; \mathbb{R}^n) \rightarrow S_\infty^{m'}(\Omega; \mathbb{R}^n)$$

is continuous.

*Proof.* Let the real numbers  $m \leq m'$  be given. We note that for any  $\xi \in \mathbb{R}^n$

$$\langle \xi \rangle^m \leq 1 \cdot \langle \xi \rangle^{m'}$$

and thus if  $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ , we have that  $\forall \alpha \in \mathbb{N}^p, \forall \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x, \xi) \right| \leq C \langle \xi \rangle^{m-|\beta|} \leq C \langle \xi \rangle^{m'-|\beta|}$$

which show that  $a \in S_\infty^{m'}(\Omega; \mathbb{R}^n)$  as well.

To show that  $\iota$  is a continuous inclusion, it suffices to show that

$$\|\iota(a)\|_{N, m'} \leq C \|a\|_{N, m}$$

for any  $a \in S_\infty^m(\Omega; \mathbb{R}^n)$  and  $N \in \mathbb{N}$ . Indeed, this bound holds since

$$\frac{D_x^\alpha D_\xi^\beta a(x, \xi)}{\langle \xi \rangle^{m'-|\beta|}} \leq \frac{D_x^\alpha D_\xi^\beta a(x, \xi)}{\langle \xi \rangle^{m-|\beta|}}.$$

□

This inclusion property allows us to consider  $S_\infty^m(\Omega; \mathbb{R}^n)$  as the filtration of the space

$$S_\infty(\Omega; \mathbb{R}^n) = \bigcup_{m \in \mathbb{R}} S_\infty^m(\Omega; \mathbb{R}^n)$$

and we shall denote the *residual* space of the filtration as

$$S_\infty^{-\infty}(\Omega; \mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} S_\infty^m(\Omega; \mathbb{R}^n).$$

We have a rather technical result of the density of the residual space in  $S_\infty^m(\Omega; \mathbb{R}^n)$ .

**Lemma 2.6.** *Given any  $m \in \mathbb{R}$  and  $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ , there exist a sequence in  $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$  such that bounded in  $S_\infty^m(\Omega; \mathbb{R}^n)$  and converges to  $a$  in the topology of  $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$  for any  $\epsilon \in \mathbb{R}_{>0}$ . In other words, for any  $m \in \mathbb{R}$  and  $\epsilon > 0$ ,  $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$  is dense in  $S_\infty^m(\Omega; \mathbb{R}^n)$  with the topology of  $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$ .*

*Proof.* The main strategy in this proof is to approximate any symbol with the very same symbol but cut off by a compactly supported function. As such, the main reason we can't have density of  $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$  in  $S_\infty^m(\Omega; \mathbb{R}^n)$  is the same reason to the fact that Schwartz functions are not dense in the space of smooth bounded functions, in particular,  $1 \in S_\infty^0(\Omega; \mathbb{R}^n)$  is not in the closure of  $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$ .

Now, let  $a \in S_\infty^m(\Omega; \mathbb{R}^n)$  and  $\epsilon \in \mathbb{R}_{>0}$  be given. Take any smooth cut off functions supported in the unit ball, i.e. take  $\phi \in C_c^\infty(\mathbb{R}^n)$  such that  $0 \leq \phi \leq 1$  and  $\phi(\xi) = 1$  if  $|\xi| < 1$  and  $\phi(\xi) = 0$  if  $|\xi| > 2$ . We define for each  $k \in \mathbb{N}$

$$a_k(x, \xi) = \phi\left(\frac{\xi}{k}\right) a(x, \xi)$$

and we check the following

1.  $a_k \in S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$  for all  $k \in \mathbb{N}$ ;
2.  $a_k$  are bounded in  $S_\infty^m(\Omega; \mathbb{R}^n)$  for all  $k \in \mathbb{N}$ ;
3.  $a_k \rightarrow a$  as  $k \rightarrow \infty$  in  $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$ .

Given arbitrary  $N, k \in \mathbb{N}$ , observe that

$$|a_k| \leq C \langle \xi \rangle^{-N}$$

since  $a_k$  is compactly supported in  $\xi$  (as  $\phi$  is compactly supported) and by Leibniz formula and symbol estimates on  $a \in S_\infty^m(\Omega; \mathbb{R}^n)$

$$\left| D_x^\alpha D_\xi^\beta a_k(x, \xi) \right| \leq \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu \phi) \left( \frac{\xi}{k} \right) \left| D_x^\alpha D_\xi^{\beta-\mu} a(x, \xi) \right| \leq C \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu \phi) \left( \frac{\xi}{k} \right) \langle \xi \rangle^{m-|\beta-\mu|}.$$

Since  $\phi$  and all its derivatives are compactly supported, each term above is bounded in  $\xi$  and thus  $a_k$  is bounded in  $S_\infty^m(\Omega; \mathbb{R}^n)$  and that

$$\left| D_x^\alpha D_\xi^\beta a_k(x, \xi) \right| \leq C' \langle \xi \rangle^{-N}$$

which allow us to conclude that  $a_k \in S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$ .

It remains to show that  $\lim_{k \rightarrow \infty} a_k = a$  in  $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$ . In the first symbol norm, we observe that, using the symbol estimate for  $a$

$$\begin{aligned} \|a_k - a\|_{0, m+\epsilon} &= \sup_{(x, \xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{|a_k(x, \xi)|}{\langle \xi \rangle^{m+\epsilon}} \\ &= \sup_{(x, \xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{|(1 - \phi(\xi/k))| |a(x, \xi)|}{\langle \xi \rangle^{m+\epsilon}} \\ &\leq \|a\|_{0, m} \sup_{\xi \in \mathbb{R}^n} \frac{|(1 - \phi(\xi/k))|}{\langle \xi \rangle^\epsilon} \\ &\leq \|a\|_{0, m} \langle k \rangle^{-\epsilon} \\ &\rightarrow 0 \end{aligned}$$

as  $k \rightarrow \infty$ , since  $|1 - \phi(\xi/k)|$  is 0 in the region  $|\xi| \leq k$  and bounded by 1 otherwise. We remark upon the necessity of the extra decay given by  $\langle \xi \rangle^{-\epsilon}$  factor. For other symbol norm we shall again use Leibniz

formula to obtain

$$\begin{aligned}
\sup_{(x,\xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{|D_x^\alpha D_\xi^\beta a_k(x, \xi)|}{\langle \xi \rangle^{m+\epsilon-|\beta|}} &\leq \sup_{(x,\xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{1}{\langle \xi \rangle^{m+\epsilon-|\beta|}} \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu(1-\phi)) \left(\frac{\xi}{k}\right) |D_x^\alpha D_\xi^{\beta-\mu} a(x, \xi)| \\
&\leq \sup_{(x,\xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{C}{\langle \xi \rangle^{m+\epsilon-|\beta|}} \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu(1-\phi)) \left(\frac{\xi}{k}\right) \langle \xi \rangle^{m-|\beta-\mu|} \\
&= C \sup_{\xi \in \mathbb{R}^n} \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu(1-\phi)) \left(\frac{\xi}{k}\right) \langle \xi \rangle^{-\epsilon-|\mu|} \\
&\leq C' k^{-\epsilon} \\
&\rightarrow 0
\end{aligned}$$

as  $k \rightarrow \infty$  by the same argument as before. Thus, we have proven that  $a_k \rightarrow a$  as  $k \rightarrow \infty$  in  $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$ .  $\square$

**Proposition 2.7.** *Let  $p, n \in \mathbb{N}$  be given. Let  $\Omega \subset \mathbb{R}^p$  be such that  $\Omega \subset \overline{\text{Int}(\Omega)}$ . Then, for any  $m, m' \in \mathbb{R}$ , we have*

$$S_\infty^m(\Omega; \mathbb{R}^n) \cdot S_\infty^{m'}(\Omega; \mathbb{R}^n) = S_\infty^{m+m'}(\Omega; \mathbb{R}^n)$$

*Proof.* Let  $a \in S_\infty^m(\Omega; \mathbb{R}^n)$  and  $b \in S_\infty^{m'}(\Omega; \mathbb{R}^n)$  be given. By (general) Leibinz formula, we have that for all multi-index  $\alpha, \beta$ ,

$$\begin{aligned}
\sup_{(x,\xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{|D_x^\alpha D_\xi^\beta a(x, \xi) b(x, \xi)|}{\langle \xi \rangle^{(m+m')-|\beta|}} &\leq \sum_{\mu \leq \alpha, \gamma \leq \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} \sup_{(x,\xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{|D_x^\mu D_\xi^\gamma a(x, \xi)|}{\langle \xi \rangle^{m-|\mu|}} \frac{|D_x^{\alpha-\mu} D_\xi^{\beta-\gamma} b(x, \xi)|}{\langle \xi \rangle^{(m+m')-|\beta|}} \\
&\leq \sum_{\mu \leq \alpha, \gamma \leq \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} C \sup_{\xi \in \mathbb{R}^n} \frac{\langle \xi \rangle^{m-|\mu|} \langle \xi \rangle^{m'-|\beta-\gamma|}}{\langle \xi \rangle^{(m+m')-|\beta|}} \\
&= \sum_{\mu \leq \alpha, \gamma \leq \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} C \sup_{\xi \in \mathbb{R}^n} \langle \xi \rangle^{|\beta|-(|\beta-\gamma|+|\mu|)} \\
&\leq \sum_{\mu \leq \alpha, \gamma \leq \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} C \\
&< \infty
\end{aligned}$$

where we have use the property of multi-index that  $|\beta| = |\beta - \mu| + |\mu|$ . We have thus shown that  $S_\infty^m(\Omega; \mathbb{R}^n) \cdot S_\infty^{m'}(\Omega; \mathbb{R}^n) \subset S_\infty^{m+m'}(\Omega; \mathbb{R}^n)$

For the reverse inclusion, let  $c \in S_\infty^{m+m'}(\Omega; \mathbb{R}^n)$  be given. Define

$$\begin{aligned}
a &: (x, \xi) \mapsto \langle \xi \rangle^m \\
b &: (x, \xi) \mapsto \frac{c(x, \xi)}{a(x, \xi)}
\end{aligned}$$

and observe that

- $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ . It is clear that  $a$  is smooth in both  $x$  and  $\xi$ . It is independent of  $x$  and thus any  $x$  derivative gives 0. We need only to check that for all  $\beta \in \mathbb{N}^n$ ,

$$|D_\xi^\beta \langle \xi \rangle^m| \leq C \langle \xi \rangle^{m-|\beta|}$$

which can be proven by induction on  $n$  and  $\beta$ . We shall only prove the base case where  $n = 1$  and  $\beta = 1$ . We have

$$|D_\xi \langle \xi \rangle^m| = \left| \partial_\xi (1 + \xi^2)^{m/2} \right| = \left| m \xi \langle \xi \rangle^{m-2} \right| = \left| m \frac{\xi}{\langle \xi \rangle} \right| \langle \xi \rangle^{m-1} \leq |m| \langle \xi \rangle^{m-1}$$

where we have used the fact that  $|\xi| \leq \langle \xi \rangle$  for all  $\xi$ .

- $b \in S_{\infty}^{m'}(\Omega; \mathbb{R}^n)$ . We note first that  $\langle \xi \rangle^m \neq 0$  for all  $\xi \in \mathbb{R}^n$  and thus  $b$  is well-defined. Since division by  $\langle \xi \rangle^m$  does not affect any of the  $x$  derivative, we only need to show that for any  $\beta \in \mathbb{N}^n$ , we have

$$\left| D_{\xi}^{\beta} b(x, \xi) \right| \leq C \langle \xi \rangle^{m+m'-|\beta|}$$

for some constant  $C > 0$  uniformly in  $\xi$ . Indeed, observe that by the Leibinz formula

$$\begin{aligned} \left| D_{\xi}^{\beta} b(x, \xi) \right| &\leq \sum_{\mu \leq \beta} \binom{\beta}{\mu} \left| D_{\xi}^{\mu} c(x, \xi) \right| \left| D^{\beta-\mu} \langle \xi \rangle^{-m} \right| \\ &\leq C \sum_{\mu \leq \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m+m'-|\mu|} \langle \xi \rangle^{-m-|\beta-\mu|} \\ &\leq C \sum_{\mu \leq \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m'-(|\mu|+|\beta-\mu|)} \\ &= C \sum_{\mu \leq \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m'-|\beta|} \\ &= C 2^{\beta} \langle \xi \rangle^{m'-|\beta|} \end{aligned}$$

where we have use the definition of  $c$  and applied the result proven for  $a$  with  $m \mapsto -m$ . Thus,  $b \in S_{\infty}^{m'}(\Omega; \mathbb{R}^n)$ .

It is clear that  $a \cdot b = c$  and we have therefore shown that  $S_{\infty}^{m+m'}(\Omega; \mathbb{R}^n) \subset S_{\infty}^m(\Omega; \mathbb{R}^n) \cdot S_{\infty}^{m'}(\Omega; \mathbb{R}^n)$ .  $\square$

A summarising theorem:

**Theorem 2.8.** *Given  $p, n \in \mathbb{N}$  and  $\Omega \subset \mathbb{R}^p$  such that  $\Omega \subset \overline{\text{Int}(\Omega)}$ . Let*

$$S_{\infty}^{\infty}(\Omega; \mathbb{R}^n) = \bigcup_{m \in \mathbb{R}} S_{\infty}^m(\Omega; \mathbb{R}^n).$$

*Then  $S_{\infty}^{\infty}(\Omega; \mathbb{R}^n)$  is a graded algebra over  $\mathbb{R}$  with continuous inclusion  $S_{\infty}^m(\Omega; \mathbb{R}^n) \rightarrow S_{\infty}^{m'}(\Omega; \mathbb{R}^n)$  for all  $m \leq m'$ .*

## 2.2 Ellipticity of symbols

**Definition 2.9.** Given  $p, n \in \mathbb{N}$ ,  $m \in \mathbb{R}$  and  $\Omega \subset \mathbb{R}^p$  such that  $\Omega \subset \overline{\text{Int}(\Omega)}$ , an order  $m$  symbol  $a \in S_{\infty}^m(\Omega; \mathbb{R}^n)$  is (globally) **elliptic** if there exist  $\epsilon \in \mathbb{R}_{>0}$  such that

$$\inf_{|\xi| \geq 1/\epsilon} |a(x, \xi)| \geq \epsilon \langle \xi \rangle^m.$$

The importance of elliptic symbol is that they are invertible modulo  $S_{\infty}^{-\infty}(\Omega; \mathbb{R}^n)$ .

**Lemma 2.10.** *Given  $p, n \in \mathbb{N}$ ,  $m \in \mathbb{R}$  and  $\Omega \subset \mathbb{R}^p$  such that  $\Omega \subset \overline{\text{Int}(\Omega)}$ . Let  $a \in S_{\infty}^m(\Omega; \mathbb{R}^n)$  be an elliptic symbol of order  $m$ . Then there exist a symbol  $b \in S_{\infty}^{-m}(\Omega; \mathbb{R}^n)$  such that*

$$a \cdot b - 1 \in S_{\infty}^{-\infty}(\Omega; \mathbb{R}^n).$$

*Proof.* We shall follow the general strategy of inverting the symbol outside of a compact set. Let  $\phi \in C_c^{\infty}(\mathbb{R}^n)$  be a smooth cut off function, i.e  $0 \leq \phi \leq 1$  and  $\phi(\xi) = 1$  for  $|\xi| < 1$  and  $\phi(\xi) = 0$  for  $|\xi| > 2$ .

Let  $a \in S_{\infty}^m(\Omega; \mathbb{R}^n)$  be an elliptic symbol, that is, for any fixed  $\epsilon \in \mathbb{R}_{>0}$ , we have

$$|a(x, \xi)| \geq \epsilon \langle \xi \rangle^m$$

for any  $|\xi| \geq 1/\epsilon$ . Thus, we can define

$$b(x, \xi) = \begin{cases} \frac{1-\phi(\epsilon\xi/2)}{a(x, \xi)} & |\xi| \geq 1/\epsilon \\ 0 & |\xi| < 1/\epsilon. \end{cases}$$

We check:

**$b$  is well-defined and smooth.**

We note that  $|a(x, \xi)| > 0$  whenever  $|\xi| \geq 1/\epsilon$  and therefore  $b$  is well defined in that region. For smoothness, we note first that  $b$  is smooth in the regions  $|\xi| > 1/\epsilon$  and  $|\xi| < 1/\epsilon$ . Set  $\delta = 1/(2\epsilon)$ . In the region where  $1/\epsilon - \delta < |\xi| < 1/\epsilon + \delta$ , we have  $|\epsilon\xi/2| < 1/\epsilon$  and therefore  $b(x, \xi) \equiv 0$  in this region and is thus smooth. Since the we have covered  $\Omega \times \mathbb{R}^n$  by the three chart domain above,  $b$  is smooth by the (smooth) gluing lemma.

**$b$  is a symbol of order  $-m$ .**

We can prove by induction that in the region  $|\xi| \geq 1/\epsilon$

$$D_x^\alpha D_\xi^\beta b = a^{-1-|\alpha|-|\beta|} G_{\alpha\beta}$$

for all multi-index  $\alpha, \beta$ , where  $G_{\alpha\beta}$  is a symbol of order  $(|\alpha| + |\beta|)m - |\beta|$ . Therefore, using the ellipticity estimate for  $a$ , we get

$$\begin{aligned} \|b\|_{k, -m} &= \sup_{(x, \xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{|D_x^\alpha D_\xi^\beta b(x, \xi)|}{\langle \xi \rangle^{-m-k}} \\ &= \sup_{|\xi| \geq 1/\epsilon} \left| a^{-1-|\alpha|-|\beta|} G_{\alpha\beta} \right| \langle \xi \rangle^{m+k} \\ &\leq \frac{\|G_{\alpha\beta}\|_{0, (|\alpha|+|\beta|)m-|\beta|}}{\epsilon} \sup_{|\xi| \geq 1/\epsilon^{1+|\alpha|+|\beta|}} \langle \xi \rangle^{-m(1+|\alpha|+|\beta|)} \langle \xi \rangle^{m+k} \\ &< \infty \end{aligned}$$

as required.

**$b$  is an inverse of  $a$  modulo  $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$ .**

The main observation is that the set where  $b$  fails to be the multiplicative inverse of  $a$  is a compact set (in  $\xi$ ) and thus  $a \cdot b - 1$  is in fact a compactly supported smooth function of  $\xi$  which is a subset of  $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$ .

Explicitly, for any  $N \in \mathbb{N}$

$$\sup_{(x, \xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{|D_x^\alpha D_\xi^\beta (a \cdot b - 1)|}{\langle \xi \rangle^{-N}} \leq \sup_{|\xi| \leq 1/\epsilon} \langle \xi \rangle^N \left| D_x^\alpha D_\xi^\beta (\phi(\xi\epsilon/2)) \right| < \infty.$$

□

### 3 Pseudodifferential Operators ( $\Psi$ DO's)

As mentioned in section ??, we wanted to generalise the action of differential operators

$$P(x, D)u = \frac{1}{(2\pi)^n} \int P(x, \xi) e^{i(x-y)\xi} u(y) dy d\xi$$

where  $P$  is an  $m^{th}$  order polynomial in  $\xi$  with  $C^\infty$  coefficient, to the actions of symbols  $a \in S_\infty^m(\mathbb{R}^n; \mathbb{R}^n)$

$$A_a u = \frac{1}{(2\pi)^n} \int a(x, \xi) e^{i(x-y)\xi} u(y) dy d\xi$$

or  $a \in S_\infty^m(\mathbb{R}_{(x,y)}^{2n}; \mathbb{R}^n)$  with action

$$A_a u = \frac{1}{(2\pi)^n} \int a(x, y, \xi) e^{i(x-y)\xi} u(y) dy d\xi.$$

One of the result we will prove is that action of  $a(x, y, \xi)$  as in the later case can always be reduced to the action of some other  $a(x, \xi)$  as in the former case.

Here we shall introduce a slightly more general symbol space,  $\langle x - y \rangle^w S_\infty^m(\Omega; \mathbb{R}^n)$ , to allow for polynomial growth perpendicular to the diagonal.

**Definition 3.1.** Given  $m, w \in \mathbb{R}$ , a  $w$ -**weighted symbol space of order  $m$** ,  $\langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n)$  is given by

$$a \in \langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n) \iff a(x, y, \xi) = \langle x - y \rangle^w \tilde{a}(x, y, \xi), \tilde{a} \in S_\infty^m(\Omega; \mathbb{R}^n)$$

or equivalently,  $a \in \langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n)$  if and only if for all multi-index  $\alpha, \beta, \gamma$ ,

$$\left| D_x^\alpha D_y^\beta D_\xi^\gamma a(x, y, \xi) \right| \leq C \langle x - y \rangle^w \langle \xi \rangle^{m-|\gamma|}.$$

We shall show that the elements  $a \in \langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n)$  acts on  $S(\mathbb{R}^n)$  via the Schwartz kernel

$$I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) d\xi.$$

**Proposition 3.2.** Let  $n \in \mathbb{N}$  and  $m, w \in \mathbb{R}$  with  $m < -n$ , then the map

$$\begin{aligned} I : \langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n) &\rightarrow (1 + |x|^2 + |y|^2) C_\infty^0(\mathbb{R}^{2n}) \\ a &\mapsto I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) d\xi \end{aligned}$$

extends by continuity to

$$I : \langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n) \rightarrow S'(\mathbb{R}^{2n})$$

in the topology of  $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$  for any  $\epsilon \in \mathbb{R}_{>0}$ .

*Proof.* □

## 4 Microlocalisation

Roughly, the support of a distribution in  $\mathbb{R}^n$  consist of points  $x \in \mathbb{R}^n$  where the distribution is non-zero after any smooth cut-offs near  $x$ .

**Definition 4.1.** The **support of a tempered distribution**  $u \in S'(\mathbb{R}^n)$  is given by the set

$$\text{supp}(u) = \{x \in \mathbb{R}^n \mid \exists \phi \in S(\mathbb{R}^n), \phi(x) \neq 0, \phi u = 0\}^c.$$

And the singular support of a distribution is the set of points where the distribution fails to behave like an element of  $S(\mathbb{R}^n)$ .

**Definition 4.2.** The **singular support of a tempered distribution**  $u \in S'(\mathbb{R}^n)$  is given by the set

$$\text{singsupp}(u) = \{x \in \mathbb{R}^n \mid \exists \phi \in S(\mathbb{R}^n), \phi(x) \neq 0, \phi(u) \in S(\mathbb{R}^n)\}^c$$

Note that if the singular support of a tempered distribution is empty, it is then an element of  $C^\infty(\mathbb{R}^n)$ . The support of an operator is given by the support of its Schwartz kernel.



**Definition 4.3.** The **support of a continuous linear operator**  $A : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$  is given by

$$\text{supp}(A) = \text{supp}(K_A) \subset \mathbb{R}^n \times \mathbb{R}^n$$

where  $K_A \in S'(\mathbb{R}^n \times \mathbb{R}^n)$  is the Schwartz kernel of  $A$ .

We note from the above that supports or singular supports are complement of open sets, therefore they are closed. We have the following result relating the support of a smooth function after the action of a continuous linear operator.

**Proposition 4.4** (Calculus of support). *Let  $A : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n)$  be a continuous linear operator and  $\phi \in C_c^\infty(\mathbb{R}^n)$ , then*

$$\text{supp}(A\phi) \subset \text{supp}(A) \circ \text{supp}(\phi) := \{x \in \mathbb{R}^n \mid \exists y \in \text{supp}(\phi), (x, y) \in \text{supp}(A)\}.$$

*Proof.* We shall show the contrapositive statement:

$$x \notin \text{supp}(A) \circ \text{supp}(\phi) \implies x \notin \text{supp}(A\phi).$$

Suppose  $x \notin \text{supp}(A) \circ \text{supp}(\phi)$ . Observe that

$$\text{supp}(A) \circ \text{supp}(\phi) = \pi_x(\pi_y^{-1}(\text{supp}(\phi)) \cap \text{supp}(A))$$

where  $\pi_{x,y} : \mathbb{R}^2 \rightarrow \mathbb{R}$  are the projection map to the respective coordinates. Since  $\text{supp}(A)$  is closed and  $\text{supp}(\phi)$  is compact, we have that  $\text{supp}(A) \circ \text{supp}(\phi)$  is closed and thus  $x$  belongs to an open set. We can therefore choose a smooth cutt-off function  $\chi \in C_c^\infty(\mathbb{R}^n)$  supported at  $x$  ( $\chi(x) \neq 0$ ) but away from  $\text{supp}(A) \circ \text{supp}(\phi)$ . Thus,

$$\text{supp}(A) \cap (\text{supp}(\chi) \times \text{supp}(\phi)) = \emptyset$$

and hence  $\chi(x)K_A(x, y)\phi(y) = 0 \implies \chi A\phi = 0$ , as required.  $\square$

## 4.1 Pseudolocality

We shall show two results on the singular support of pseudodifferential operators. The first result is that the singular support of any  $\Phi$ DO is contained within the diagonal, i.e. they are smooth away from  $x = y$ . The second result is the pseudolocality result that says that action  $\Psi$ DO's do not increase singular support of distributions.

**Proposition 4.5.** *Let  $A \in \Psi_\infty^m(\mathbb{R}^n)$  for some  $m \in \mathbb{R}$ , then*

$$\text{singsupp}(A) \subset \{(x, y) \in \mathbb{R}^{2n} \mid x = y\}.$$

*Proof.* We shall prove this theorem for elements of  $\Psi_\infty^{-\infty}(\mathbb{R}^n)$  and then extend by continuity to all orders. Let  $A \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$  with symbol  $a \in S_\infty^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ . Its singular support is given by the singular support of the kernel. Since all derivatives of  $a$  are  $O(\langle \xi \rangle^{-\infty})$ , the oscillatory integral below representing the kernel is absolutely convergent and, using integration by parts, we get

$$\begin{aligned} I(a) &= \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) d\xi \\ &= \frac{1}{(2\pi)^n} \int \frac{1}{(x-y)^\alpha} (-D_\xi^\alpha) \left( e^{i(x-y)\xi} \right) a(x, y, \xi) d\xi \\ &= \frac{1}{(x-y)^\alpha} \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} (-D_\xi^\alpha) a(x, y, \xi) d\xi \\ &= \frac{1}{(x-y)^\alpha} I((-D_\xi^\alpha) a) \end{aligned}$$

which is true for all multi-index  $\alpha$  of any order. Since all  $x, y$ -derivatives of  $a$  are uniformly bounded by  $\langle \xi \rangle^{-N}$  for any  $N \in \mathbb{N}$ , we can differentiate under the integral sign to get the equation

$$\begin{aligned} D_x^\beta D_y^\gamma (x-y)^\alpha I(a) &= \frac{1}{(2\pi)^n} \int D_x^\beta D_y^\gamma e^{i(x-y)\xi} (-D_\xi^\alpha) a(x, y, \xi) d\xi \\ &= \frac{1}{(2\pi)^n} \int (-1)^{|\beta|} \xi^{\beta+\gamma} e^{i(x-y)\xi} (-D_\xi^\alpha) a(x, y, \xi) d\xi \end{aligned}$$

where the last integral gives a smooth function, thus showing that  $(x-y)^\alpha I(a)$  is smooth for all  $\alpha$ , and hence  $I(a)$  is smooth away from  $x = y$ .

Now, for a general  $A \in \Psi_\infty^m(\mathbb{R}^n)$ ,  $m \in \mathbb{R}$ , we shall use the density of  $S_\infty^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n) \subset S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n)$  and that  $I$  extends by continuity to a map  $I : S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n) \rightarrow S'(\mathbb{R}^{2n})$  in the topology  $S_\infty^{m+\epsilon}(\mathbb{R}^{2n}; \mathbb{R}^n)$  for any  $\epsilon > 0$  ??.

□

**Proposition 4.6.** *Let  $A \in \Psi_\infty^m(\mathbb{R}^n)$  for some  $m \in \mathbb{R}$  and  $u \in C^{-\infty}(\mathbb{R}^n)$ , then*

$$\text{singsupp}(Au) \subset \text{singsupp}(u).$$

*We call operators that satisfies the above property pseudolocal*

*Proof.* Again we shall prove the contrapositive statement that

$$x \notin \text{singsupp}(u) \implies x \notin \text{singsupp}(Au)$$

Let  $u \in S'(\mathbb{R}^n)$  be compactly supported and  $x_0 \notin \text{singsupp}(u)$ . We can choose  $\chi \in S(\mathbb{R}^n)$ , (normalised) so that  $\chi \equiv 1$  in a neighbourhood of  $x_0$  and that  $\chi u \in S(\mathbb{R}^n)$ . Observe that

$$Au = A(\chi u + (1 - \chi)u) = A(\chi u) + A(1 - \chi)u.$$

Since  $\chi x u \in S(\mathbb{R}^n) \implies A\chi u \in S(\mathbb{R}^n)$  [?], we have that

$$\text{singsupp}(Au) = \text{singsupp}(A(1 - \chi)u).$$

Furthermore, we know that  $x_0 \notin \text{supp}((1 - \chi)u)$ . Now, we shall further cut-off near  $x_0$  by choosing a  $\phi \in S(\mathbb{R}^n)$  compactly supported away from  $\text{supp}(1 - \chi)$  and  $\phi \equiv 1$  near  $x_0$ , i.e.

$$\text{supp}(1 - \chi) \cap \text{supp}\phi = \emptyset.$$

We now have an operator  $\phi A(1 - \chi)$  with kernel

$$\phi(x) K_A(x, y) (1 - \phi(y))$$

that vanishes (in particular smooth) on the diagonal. Since we have shown that the singular support of a pseudodifferential operator have to be contained in the diagonal, we conclude that  $\phi A(1 - \chi)$  is a smoothing operator, and thus  $\phi A(1 - \chi)u \in C^\infty(\mathbb{R}^n)$  as required. .

□

## 4.2 Elliptic, Characteristic, Wavefront sets

We will now define *ellipticity at a point* in phase space which allow up to define various microlocal contructions that focus on localised (conically in phase space) behaviour  $\Psi$ DO's and distributions.

**Definition 4.7.** A pseudodifferential operator,  $A \in \Psi_\infty^m(\mathbb{R}^n)$ ,  $m \in \mathbb{R}$  is **elliptic at a point**  $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$  if there exist  $\epsilon > 0$  such that its left-reduced symbol satisfies the lower bound

$$|\sigma_L(A)(x, \xi)| \geq \epsilon \langle \xi \rangle^m$$

in the region

$$\bar{U}_\epsilon = \left\{ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid |x - x_0| \leq \epsilon, \left| \hat{\xi} - \hat{\xi}_0 \right| \leq \epsilon, |\xi| \geq 1/\epsilon \right\}$$

where  $\widehat{\xi} = \xi/|\xi|$  for any non-zero  $\xi \in \mathbb{R}^n$ . We denote the set of all elliptic points of  $A$  as

$$Ell^m(A) = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid A \text{ is elliptic of order } m \text{ at } (x, \xi)\}$$

and its complement in  $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$  as

$$\begin{aligned} \Sigma^m(A) &= Ell^m(A)^c \setminus \{(x, 0) \mid x \in \mathbb{R}^n\} \\ &= \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid A \text{ is **not** elliptic of order } m \text{ at } (x, \xi)\} \end{aligned}$$

**Lemma 4.8.** *Let  $A \in \Psi_\infty^m(\mathbb{R}^n)$ ,  $m \in \mathbb{R}$ .*

1. *If  $\sigma_m(A)(x, \xi)$  is homogeneous of degree  $m$  in  $\xi$ , then*

$$Ell^m(A) = \{(x_0, \xi_0) \mid \xi_0 \neq 0, \sigma_m(A)(x_0, \xi_0) \neq 0\}.$$

2.  *$Ell^m(A)$  is open in  $\mathbb{R}^n \times \mathbb{R}^n$ .*

3.  *$Ell^m(A)$  is conic in  $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ , in the sense that*

$$(x_0, \xi_0) \in Ell^m(A) \implies (x_0, t\xi_0) \in Ell^m(A), \forall t \in \mathbb{R}_{>0}.$$

4.  *$\Sigma^m(A)$  is closed conic.*

5. *if  $B \in \Psi^{m'}(\mathbb{R}^n)$ , then*

$$\Sigma^{m+m'}(A \circ B) = \Sigma^m(A) \cup \Sigma^{m'}(B).$$

*Proof.* Let  $A \in \Psi_\infty^m(\mathbb{R}^n)$ ,  $m \in \mathbb{R}$  be given.

1. Suppose the principal symbol  $\sigma_m(A)(x, \xi)$  is homogeneous of order  $m$  in  $\xi$ . We need to show that

$$(x_0, \xi_0) \in Ell^m(A) \iff \xi_0 \neq 0, \sigma_m(A)(x_0, \xi_0) \neq 0.$$

If  $\xi_0 = 0$ ,  $(x_0, \xi_0) \notin Ell_\infty^m$  by definition of ellipticity. If  $\sigma_m(x_0, \xi_0) = 0$ , by homogeneity, we have

$$\sigma_m(x_0, t\xi_0) = t^m \sigma_m(x_0, \xi_0) = t^m \cdot 0 = 0$$

for all  $t \in \mathbb{R}_{>0}$ . By definition of principal symbol, we can write the left symbol of  $A$  as

$$\sigma_L(A) = \sigma_m(A) + a$$

where  $a \in S_\infty^{m-1}(\mathbb{R}^n; \mathbb{R}^n)$ . Now, observe that for any  $\epsilon > 0$ , the set

$$\overline{U}_\epsilon = \left\{ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid |x - x_0| \leq \epsilon, \left| \widehat{\xi} - \widehat{\xi}_0 \right| \leq \epsilon, |\xi| \geq 1/\epsilon \right\}$$

contains the (open) half-line starting at  $\widehat{\xi}_0/\epsilon$ , i.e. the set  $\{(x_0, t\xi_0/(|\xi_0|\epsilon)) \mid t > 0\}$ . However, by the symbol estimate of  $a$ ,

$$\begin{aligned} \left| \sigma_L(A) \left( x_0, \frac{t\xi_0}{|\xi_0|\epsilon} \right) \right| &\leq \left( \frac{t}{\epsilon|\xi_0|} \right)^m |\sigma_m(x_0, \xi_0)| + \left| a \left( x_0, \frac{t\xi_0}{|\xi_0|\epsilon} \right) \right| \\ &= 0 + \left| a \left( x_0, \frac{t\xi_0}{|\xi_0|\epsilon} \right) \right| \\ &\leq C \left\langle \frac{t\xi_0}{|\xi_0|\epsilon} \right\rangle^{m-1} \\ &= C \langle t/\epsilon \rangle^{m-1} \end{aligned}$$

and therefore

$$\begin{aligned}
\inf_{(x,\xi) \in \overline{U}_\epsilon} \frac{|\sigma_L(A)(x,\xi)|}{\langle \xi \rangle^m} &\leq \inf_{t>0} \frac{\left| \sigma_L(A) \left( x_0, \frac{t\xi_0}{|\xi_0| \epsilon} \right) \right|}{\langle t/\epsilon \rangle^m} \\
&\leq \inf_{t>0} \frac{C \langle t/\epsilon \rangle^{m-1}}{\langle t/\epsilon \rangle^m} \\
&= C \inf_{t>0} \langle t/\epsilon \rangle^{-1} \\
&= 0
\end{aligned}$$

which means that  $(x_0, \xi_0) \notin Ell^m(A)$ .

Conversely, if  $\sigma_m(A)(x_0, \xi_0) \neq 0$ , by continuity and homogeneity,  $\sigma_m(A)$ , is non-zero in a (closed) conic neighbourhood, i.e. there exist  $\epsilon > 0$  such that  $\sigma_m(A) \neq 0$  in

$$\overline{U}_\epsilon = \left\{ (x, \xi) \mid |x - x_0| \leq \epsilon, \left| \widehat{\xi} - \widehat{\xi}_0 \right| \leq \epsilon, |\xi| \geq 1/\epsilon \right\}.$$

Again, writing the left symbol as a sum of the principal symbol and a lower order term, we observe that in  $\overline{U}_\epsilon$ ,

$$\begin{aligned}
\frac{|\sigma_L(A)(x, \xi)|}{\langle \xi \rangle^m} &\geq \frac{|\sigma_m(A)(x, \xi)| - |a(x, \xi)|}{\langle \xi \rangle^m} \\
&= \left| \frac{|\xi|^m}{\langle \xi \rangle^m} \sigma_m(A)(x, \widehat{\xi}) - \frac{|a(x, \xi)|}{\langle \xi \rangle^m} \right|
\end{aligned}$$

By the symbol estimate of  $a$ , the second term is tending to 0 which the first term is bounded below by  $C = \inf_{(x,\xi) \in \overline{U}_\epsilon} |\sigma_m(A)(x, \xi)| > 0$ . Therefore, choosing a smaller  $\epsilon$  if necessary, we have  $|a(x, \xi)| / \langle \xi \rangle^m < C$  and thus

$$\inf_{(x,\xi) \in \overline{U}_\epsilon} \frac{|\sigma_L(A)(x, \xi)|}{\langle \xi \rangle^m} \geq C' \geq \epsilon.$$

and therefore  $(x_0, \xi_0) \in Ell^m(A)$ .

2. We note first that if the principal symbol is homogeneous of degree  $m$ , the previous result applies and continuity of the principal symbol guarantee that the elliptic set is open (if  $\sigma_m(A)$  is non-zero at a point, it is non-zero in an open neighbourhood of the point).

For the general case, suppose  $(x_0, \xi_0) \in Ell^m(A)$ . We therefore have for some  $\epsilon > 0$ ,

$$|\sigma_L(A)(x, \xi)| \geq \epsilon \langle \xi \rangle^m$$

in the region

$$\overline{U}_\epsilon(x_0, \xi_0) = \left\{ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid |x - x_0| \leq \epsilon, \left| \widehat{\xi} - \widehat{\xi}_0 \right| \leq \epsilon, |\xi| \geq 1/\epsilon \right\}.$$

It suffice to show that there is an open neighbourhood of  $(x_0, \xi_0)$  where  $A$  remains elliptic. We can take desired open neighbourhood to be

$$V = \left\{ (x', \xi') \mid \xi' \neq 0, |x' - x_0| < \epsilon/2, \left| \widehat{\xi'} - \widehat{\xi}_0 \right| < \epsilon/2 \right\}.$$

Then, we can check that for every  $(x', \xi') \in V$ ,  $A$  satisfies the elliptic estimate in  $\overline{U}_{\epsilon/2}(x', \xi')$ . Indeed, if  $(x, \xi) \in \overline{U}_{\epsilon/2}(x', \xi')$ , then

$$\begin{aligned}
|x - x_0| &\leq |x - x'| + |x' - x_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\
\left| \widehat{\xi} - \widehat{\xi}_0 \right| &\leq \left| \widehat{\xi} - \widehat{\xi'} \right| + \left| \widehat{\xi'} - \widehat{\xi}_0 \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\
|\xi| &\geq 2/\epsilon \geq 1/\epsilon
\end{aligned}$$

which shows that  $\overline{U}_{\epsilon/2}(x', \xi') \subset \overline{U}_\epsilon(x_0, \xi_0)$ . Therefore,

$$\inf_{(x, \xi) \in \overline{U}_{\epsilon/2}(x', \xi')} \frac{|\sigma_L(A)(x, \xi)|}{\langle \xi \rangle^m} \geq \inf_{(x, \xi) \in \overline{U}_\epsilon(x_0, \xi_0)} \frac{|\sigma_L(A)(x, \xi)|}{\langle \xi \rangle^m} \geq \epsilon \geq \epsilon/2$$

as required.

3. Again, this result is immediate if the principal symbol is homogeneous in  $\xi$ . In general, this result come from the observation that only  $\widehat{\xi} = \xi/|\xi|$  appears in  $\overline{U}_\epsilon$  in the definition of  $Ell^m(A)$ , i.e. only the *direction* in the dual variable is important.

Explicitly, let  $(x_0, \xi_0) \in Ell^m(A)$  and  $t \in \mathbb{R}_{>0}$ . Clearly  $t\xi_0 \neq 0$ . And note that

$$\overline{U}_\epsilon(x_0, \xi_0) = \overline{U}_\epsilon(x_0, t\xi_0)$$

since  $\widehat{\xi} = t\widehat{\xi}$ .

4.  $\Sigma^m(A) = Ell^m(A)^c$  where  $Ell^m(A)$  is open and conic. Since complement of conic set is conic, and complement of open is closed, we conclude that  $\Sigma^m(A)$  is closed conic.
5. If both principal symbols are homoeogenous of degree  $m, m'$  respectively, we can applied the result above and by symbol calculus, we have

$$\begin{aligned} Ell^{m+m'}(A \circ B) &= \{(x, \xi) \mid \xi \neq 0, \sigma_{m+m'}(A \circ B) = \sigma_m(A)\sigma_{m'}(B) \neq 0\} \\ &= \{(x, \xi) \mid \xi \neq 0, \sigma_m(A) \neq 0\} \cap \{(x, \xi) \mid \xi \neq 0, \sigma_{m'}(B) \neq 0\} \\ &= Ell^m(A) \cap Ell^{m'}(B). \end{aligned}$$

Taking complement give the desired result.

In general,

□

**Definition 4.9.** The **wavefront set** of a compactly supported tempered distribution

$$u \in C_c^{-\infty}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) \mid \text{supp}(u) \Subset \mathbb{R}^n\}$$

is given by

$$\text{WF}(u) = \bigcap \{Ell^0(A) \mid A \in \Psi_\infty^0(\mathbb{R}^n), Au \in C^\infty(\mathbb{R}^n)\}.$$

For general tempered distribution  $u \in S'(\mathbb{R}^n)$ , its wavefront set is given by

$$\text{WF}(u) = \bigcup_{\chi \in C_c^\infty(\mathbb{R}^n)} \text{WF}(\chi u).$$

**Proposition 4.10.** For compactly supported tempered distribution,  $u \in C_c^{-\infty}(\mathbb{R}^n)$ ,

$$\pi(\text{WF}(u)) = \text{singsupp}(u).$$

where  $\pi(x, y) = x$  is the projection map.

*Proof.* To show  $\pi(\text{WF}(u)) \subset \text{singsupp}(u)$ , we observe that, by definition of singular support,

$$x_0 \notin \text{singsupp}(u) \implies \exists \phi \in S(\mathbb{R}^n), \phi(x_0) \neq 0, \phi u \in S(\mathbb{R}^n).$$

But since multiplication by  $\phi$  gives an operator in  $\Psi_\infty^0(\mathbb{R}^n)$  which is elliptic at  $(x_0, \xi)$  for any  $\xi \neq 0$  ( $\phi$  is its own principal symbol which happens to be homogeneous and non-zero for any  $(x_0, \xi), \xi \neq 0$ ).

Therefore,  $x_0 \notin \pi(\text{WF}(u))$ .

Conversely, if  $x_0 \notin \pi(\text{WF}(u))$ , then for all  $\xi \neq 0$ , there exist  $A_\xi \in \Psi_\infty^0(\mathbb{R}^n)$  such that  $A_\xi$  is elliptic at  $(x_0, \xi)$  and  $A_\xi u \in C^\infty(\mathbb{R}^n)$ . Since elliptic set  $\text{Ell}^0(A_\xi)$  is open and conic, we know that there exist  $\epsilon = \epsilon(\xi)$  such that  $A_\xi$  is elliptic in the open conic set

$$V_\xi = \left\{ (x', \xi') \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \mid |x' - x_0| < \epsilon, \left| \widehat{\xi'} - \widehat{\xi} \right| < \epsilon \right\}.$$

Compactness of the sphere (note that  $\xi' \mapsto \widehat{\xi'}$  is an embedding of  $\mathbb{R}^n \setminus \{0\}$  into  $S^n$ ) allow us to cover  $\{x_0\} \times (\mathbb{R}^n \setminus \{0\})$  with finite number of  $V_{\xi_j}, j = 1, \dots, N$  with corresponding operators  $A_{\xi_j}$ .

Now, consider the operator

$$A = \sum_{j=1}^N A_{\xi_j}^* A_{\xi_j}.$$

□

**Definition 4.11.** Let  $a \in S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$  for some  $m \in \mathbb{R}$ ,  $p, n \in \mathbb{N}$  be a symbol. We say  $a$  is of order  $-\infty$  at a point  $(x_0, \xi_0) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\}$  (write  $a = O(\langle \xi \rangle^{-\infty})$ ) if there exist  $\epsilon \in \mathbb{R}_{>0}$  such that for all  $M \in \mathbb{R}$ , there is a constant  $C_M > 0$  such that

$$|a(x, \xi)| \leq C_M \langle \xi \rangle^{-M}$$

in the neighbourhood of  $(x_0, \xi_0)$  given by

$$\overline{U}_{(x_0, \xi_0)} = \left\{ (x, \xi) \in \mathbb{R}^p \times \mathbb{R}^n \mid |x - x_0| \leq \epsilon, \left| \widehat{\xi} - \widehat{\xi_0} \right| \leq \epsilon \right\}.$$

We define the cone support of the symbol  $a$  to be all the points in phase space that where it fails to be  $O(\langle \xi \rangle^{-\infty})$ .

$$\text{conesupp}(a) = \left\{ (x, \xi) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\} \mid a = O(\langle \xi \rangle^{-\infty}) \text{ at } (x, \xi) \right\}^c.$$

**Lemma 4.12.** Let  $a \in S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ , then

1.  $\text{conesupp}(a)$  is a closed conic set in  $\mathbb{R}^p \times \mathbb{R}^n$ .
2. If  $a = O(\langle \xi \rangle^{-\infty})$  at  $(x_0, \xi_0) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\}$ , then so is  $D_x^\alpha D_\xi^\beta a(x, \xi)$  for any multi-index  $\alpha, \beta$

With the lemma above, we can in fact define the cone support of a symbol to be the smallest closed conic subset of the phase space (with  $\xi \neq 0$ ) such that, in the complement,  $a$  and all its derivatives are of order  $-\infty$ .

**Definition 4.13.** Let  $A \in \Psi_\infty^m(\mathbb{R}^n)$ ,  $m \in \mathbb{R}$  be pseudodifferential operator. We define the **essential support**,  $\text{WF}'(A)$ , of  $A$  to be the cone support of its left symbol, i.e.

$$\text{WF}'(A) = \text{conesupp}(\sigma_L(A)) \subset \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$$

Using symbol calculus we can prove the following result.

**Lemma 4.14.** Let  $A \in \Psi_\infty^m(\mathbb{R}^n)$ ,  $B \in \Psi_\infty^{m'}(\mathbb{R}^n)$  be pseudifferential operators. Then

1.  $\text{WF}'(A) = \text{conesupp}(\sigma_R(A))$ .
2.  $\text{WF}'(A \circ B) \subset \text{WF}'(A) \cap \text{WF}'(B)$ .

$$3. \text{WF}'(A + B) = \text{WF}'(A) \cup \text{WF}'(B).$$

With the concept of essential support we can define the notion of *microlocal elliptic parametrix* which can be thought of as local inverse at an elliptic point of  $\Psi\text{DO}$ 's.

**Proposition 4.15.** *Let  $A \in \Psi_\infty^m(\mathbb{R}^n)$  and  $z \notin \Sigma^m(A)$ . Then there exist a (two-sided) microlocal parametrix  $B \in \Psi^{-m}(\mathbb{R}^n)$  such that*

$$z \notin \text{WF}'(1 - AB) \text{ and } z \notin \text{WF}'(1 - BA).$$

**Proposition 4.16.** *Pseudodifferential operators are microlocal in the following sense: Let  $A \in \Psi_\infty^m(\mathbb{R}^n)$  and  $u \in C_c^{-\infty}(\mathbb{R}^n)$ , then*

$$\text{WF}(Au) \subset \text{WF}(u). \quad (1)$$

*In fact, we have*

$$\text{WF}(Au) \subset \text{WF}'(A) \cap \text{WF}(u).$$

A partial converse to the above is given by the following proposition.

**Proposition 4.17.** *Let  $A \in \Psi_\infty^m(\mathbb{R}^n)$  and  $u \in C_c^{-\infty}(\mathbb{R}^n)$ , then*

$$\text{WF}(u) \subset \text{WF}(Au) \cup \Sigma^m(A).$$

## 5 Appendix

### 5.1 Stationary phase lemma

In the study of pseudodifferential operators, we often encounter integral of highly oscillatory functions of the form

$$I(h) = \int_{\mathbb{R}} a(x) e^{i\varphi(x)/h} dx$$

where  $a \in C_c^\infty(\mathbb{R})$ ,  $\varphi \in C^\infty(\mathbb{R})$  and we are interested in the asymptotic behaviour as  $h \rightarrow 0$ . We note that if  $\varphi$  is linear (or constant), i.e.  $\varphi(x) = \alpha x + \beta$ ,  $\alpha, \beta \in \mathbb{R}$ , then,

$$|I(h)| = \left| \int_{\mathbb{R}} a(x) e^{i(\alpha x + \beta)/h} dx \right| = \left| e^{i\beta/h} \right| \left| \int_{\mathbb{R}} a(x) e^{i\alpha x/h} dx \right| = \left| \int_{\mathbb{R}} a(x) e^{i\alpha x/h} dx \right| \rightarrow 0$$

as  $h \rightarrow 0$  by Riemann-Lebesgue lemma. That is to say, as the length scale of the oscillation tends to zero, the values of the integrand achieve perfect cancellation. In general, if  $\varphi'(x) \neq 0$ , we expect  $e^{i\varphi(x)/h}$  to oscillate at length scale of order  $h$  and thus as  $h \rightarrow 0$ ,

**Theorem 5.1** (Schwartz Kernel Theorem [Taylor, 2011, Chapter 4.6, p. 345]). *Let  $M, N$  be compact manifold and*

$$T : C^\infty(M) \rightarrow \mathcal{D}'(N)$$

*be a continuous linear map ( $C^\infty(M)$  being given Frechet space topology and  $\mathcal{D}'(N)$  the weak\* topology). Define a bilinear map*

$$B : C^\infty(M) \times C^\infty(N) \rightarrow \mathbb{C} \\ (u, v) \mapsto B(u, v) = \langle v, Tu \rangle.$$

Then, there exist a distribution  $k \in \mathcal{D}'(M \times N)$  such that for all  $(u, v) \in C^\infty(M) \times C^\infty(N)$

$$B(u, v) = \langle u \otimes v, k \rangle.$$

We call such  $k$  the kernel of  $T$ .

## References

[Taylor, 2011] Taylor, M. (2011). *Partial Differential Equations I*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2 edition.