Fourier Analysis and Distributions

Notes from Taylors PDE Vol 1 Chapter 3

1 Introduction

Historically, Fourier analysis was the first to give formulas to various linear PDE with constant coefficient, in particular the three classical PDE:

1. Laplace equation: $\Delta u = f$
2. Heat equation : $\partial_t u - \Delta u = f$
3. Wave equation : $\partial_t^2 u - \Delta u = f$.

We shall introduce the Fourier transform first on the space of (real or complex-valued) functions on $\mathbb{T}^n$, i.e. the Fourier series associated to periodic functions in n variables which provide basic results for harmonic functions in the plane (functions with $\Delta u = 0$) and their connection to holomorphic functions. We will then define the Fourier transform on $\mathbb{R}^n$ and its inversion formula. We shall also introduce the notion of "distributions", a generalisation of functions on $\mathbb{R}^n$, which is the natural space to which the solutions to the classical PDE above belongs.

2 Fourier Series

We shall focus on the torus $\mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n \cong S^1 \times \cdots \times S^1$ and (real or complex valued) integrable functions on the space, i.e. $L^1(\mathbb{T}^n; \mathbb{C})$. Given such a function $f : \mathbb{T}^n \to \mathbb{C}$, we define its Fourier series $\hat{f} : \mathbb{Z}^n \to \mathbb{C}$ to be

$$\mathcal{F}f(k) := \hat{f}(k) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(\theta) e^{-ik\cdot\theta}\, d\theta. \tag{1}$$

We shall note that the transformation is a continuous linear map between integrable functions on $\mathbb{T}^n$ to the space of bounded functions on $\mathbb{Z}^n$, that is

$$\mathcal{F} : L^1(\mathbb{T}^n) \to l^\infty(\mathbb{Z}^n).$$

Furthermore, if f is smooth ($f \in C^\infty(\mathbb{T}^n)$) then integration by part gives,

$$k^\alpha \hat{f}(k) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} (D^\alpha f)(\theta) e^{-ik\cdot\theta}\, d\theta \tag{2}$$

where

$$\begin{aligned} \alpha &= (\alpha_1, \alpha_2, \dots, \alpha_n) \\ k^\alpha &= k_1^{\alpha_1} \dots k_n^{\alpha_n} \\ D^\alpha &= D_1^{\alpha_1} \dots D_n^{\alpha_n} \\ D_j &= -i\, \partial_{\theta_j} \end{aligned}$$

This allow conversion of linear differential operators into algebraic expression in the frequency domain. Multiplying (2) by $\langle k \rangle^n := 1/(\sqrt{1 - |k|^2})^n$ and observe that

A consequence of this is that for all smooth functions, f on $\mathbb{T}^n$ and for all $N \in \mathbb{N}$,

$$\sup_{k\in\mathbb{Z}^n} \langle k \rangle^N |\hat{f}(k)|$$