We shall follow the presentation given in [?].

1 Motivation for Pseudodifferential operators

- Solving PDEs via Fourier transform. For example, in Euclidean space, $\mathbb{R}^n$, constant coefficient linear PDE
$$P(D)u = \sum_{|\alpha| \leqslant n} c_\alpha D^\alpha u = f, \quad c_\alpha \in \mathbb{R}$$
where $P \in \mathbb{R}[x]$, can solved by applying Fourier transform which gives a solution of the form
$$u(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} f(y) \frac{1}{P(\xi)} \mathrm{d}y \, \mathrm{d}\xi$$
due to the observation that
$$\mathcal{F}P(D)u = P(\xi)\mathcal{F}u.$$
Moreover, for linear differential operators with smooth coefficients
$$P(x, D) : u \mapsto \sum_{|\alpha| \leqslant n} a_\alpha D^\alpha u, \quad a_\alpha \in C^\infty(\mathbb{R}^n)$$
we have
$$P(x, D)u = \frac{1}{(2\pi)^n} \int P(x, \xi) e^{i(x-y)\xi} u(y) \mathrm{d}y \, \mathrm{d}\xi \, .$$
We would like to generalise the above so that $P(x, \xi)$ are smooth functions satisfying certain uniform bounds, called *symbols*, instead of just polynomials in ξ. This gives us a class of operators, called pseudodifferential operators, that acts as
$$A_a u(x) = \frac{1}{(2\pi)^n} \int a(x, \xi) e^{i(x-y)\xi} u(y) \mathrm{d}y \, \mathrm{d}\xi$$
for each symbol a.
- There isn't enough differential operators with smooth coefficient in the sense that elliptic differential operators are not, in general, invertible in this class. For example, the operator
$$u \mapsto (\Delta + 1)u$$
has inverse that acts as (using construction via Fourier transform shown above)
$$(\Delta + 1)^{-1} f = \frac{1}{(2\pi)^n} \int \frac{1}{1 + |\xi|^2} e^{i(x-y)\xi} f(y) \mathrm{d}y \, \mathrm{d}\xi$$
which is a pseudodifferential operator with symbol $a(x, \xi) = (1 + |\xi|^2)^{-1}$.
- Motivation from quantum mechanics. The notion of "quantisation" in quantum mechanics can be formalised as the map that sends a symbol a (a smooth function that represent determistic observable in classical mechanics) to its corresponding pseudodifferential operator (i.e. the corresponding quantum observable)
$$A_a : \psi \mapsto \frac{1}{(2\pi)^n} \int \frac{1}{1 + |\xi|^2} e^{i(x-y)\xi} \psi(y) \mathrm{d}y \, \mathrm{d}\xi$$
that acts on the wavefunction ψ.
- Used in the formulation and proof of Atiyah-Singer Index theorem.

We shall define, on Euclidean space, the space of symbols, $S^m(\mathbb{R}^{2n}_{x,y}; \mathbb{R}^n_\xi)$ and the corresponding space of pseudodifferential operators, $\Psi^m(\mathbb{R}^n)$ which acts on distributions via the Schwartz kernel given by the oscilliatory integral
$$I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) \mathrm{d}\xi \, .$$
We note that we have introduced an extra variable y which will help in explicating the properties of pseudodifferential operators. However, the extra variable does not change the essence of the theory.

2 Symbols

We shall here list the definition of the space of symbols of order $m \in \mathbb{N}$ in Euclidean space $\mathbb{R}^n$ that one encounter in the literature. The main motivation is again based on the property of linear differential operators of order $m \in \mathbb{N}$ with smooth coefficient that, after Fourier transform gives the polynomial of ξ with smooth coefficient

$$P(x,\xi) = \sum |\,\alpha\,| \leqslant m a_\alpha(x)\xi^\alpha.$$

It has the property that

$$\left| D_x^\alpha D_\xi^\beta P(x,\xi) \right| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-|\,\beta\,|}$$

i.e. $P(x,\xi)$ is smooth and decreases in order as $\xi \to \infty$ with successive ξ-derivative.

Definition 2.1. The space $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ of order m is the space of smooth functions $a \in C^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ such that for all multi-index $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-|\,\beta\,|}$$

uniformly on $\mathbb{R}^p \times \mathbb{R}^n$. We can also defined the space of symbol, $S^m_\infty(\Omega;\mathbb{R}^n)$ on a set with non-empty interior $\Omega \subset \mathbb{R}^p$, $\Omega \subset \overline{\mathrm{Int}(\Omega)}$ such that the bound above is satisfied uniformly in $(x,\xi) \in \mathrm{Int}(\Omega) \times \mathbb{R}^n$. The subscript ∞ refers the uniform boundedness condition. Together with the family of seminorm (indexed by $N \in \mathbb{N}$)

$$\|a\|_{N,m} = \sup_{(x,\xi)\in\mathrm{Int}(\Omega)\times\mathbb{R}^n} \max_{|\,\alpha\,|+|\,\beta\,|\leqslant N} \frac{D_x^\alpha D_\xi^\beta a(x,\xi)}{\langle \xi \rangle^{m-|\,\beta\,|}}$$

gives a Frechet topology to $S^m_\infty(\Omega;\mathbb{R}^n)$.

Note: In defining pseudodifferential operators, we shall focus on the case where $p = 2n$.

Definition 2.2. A **symbol** of type $S^{m,l_1,l_2}_{\delta,\delta'}$ where $m, l_1, l_2 \in \mathbb{R}$ and $\delta, \delta' \in [0, 1/2)$ is an element of $C^\infty(\mathbb{R}^n_x;\mathbb{R}^n_y;\mathbb{R}^n_\xi)$ satisfying

$$\frac{\left| D_x^\alpha D_y^\beta D_\xi^\gamma a(x,y,\xi) \right|}{\langle \xi \rangle^{m-|\,\gamma\,|} \langle x \rangle^{l_1-|\,\alpha\,|} \langle y \rangle^{l_2-|\,\beta\,|} \langle \xi \rangle^{\delta|\,(\alpha,\beta,\gamma)\,|} \langle x,y \rangle^{\delta'|\,(\alpha,\beta,\gamma)\,|}} \leqslant C_{\alpha,\beta,\gamma}$$

uniformly in $\mathbb{R}^{3n}$. Taking the supremum over $\mathbb{R}^{3n}$, we get a family of seminorms, indexed by $N \in \mathbb{N}$ defined by

$$\|a\|_{S^{m,l_1,l_2}_{\delta,\delta'},N} := \sum_{|\,(\alpha,\beta,\gamma)\,|\leqslant N} \inf C_{\alpha,\beta,\gamma}$$

which gives $S^{m,l_1,l_2}_{\delta,\delta'}$ a Frechet topology.

Definition 2.3. A (Kohn-Nirenberg) **symbol** of order $m \in \mathbb{R}$ on $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}_{x,\xi}$ is a smooth function $a = a(x,\xi)$ satisfying

$$\forall \alpha,\beta \in \mathbb{N}^n, \exists C \in \mathbb{R}_{\geqslant 0} \ : \left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \leqslant C \langle \xi \rangle^{m-|\,\beta\,|}$$

uniformly in x. The **space of symbol of order** m on $T^*\mathbb{R}^n$

Definition 2.4. Let $n \in \mathbb{N}$ be given. An **order function** $g \in C^\infty(\mathbb{R}^n; \mathbb{R}_{\geqslant 0})$ is a *non-negative* function satisfying

$$\forall \alpha \in N^n \exists C \in \mathbb{R}_{\geqslant 0} \ : \partial^\alpha g \leqslant Cg$$

uniformly on $\mathbb{R}^n$, i.e. $\partial^\alpha g = O(g)$ uniformly on $\mathbb{R}^n$.

Given an order function g, a **symbol** of order g is a smooth function $a = a(x,\xi) \in C^\infty(T^*\mathbb{R}^n)$ satisfying

$$\forall \alpha, \beta \in \mathbb{N}^n \ : \left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \leqslant Cg(\xi)$$

uniformly in x.

2.1 Properties of Symbols

Proposition 2.5. *Let $p, n \in \mathbb{N}$ be given and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$. If $m, m' \in \mathbb{R}$ such that $m \leqslant m'$, then $S^m_\infty(\Omega; \mathbb{R}^n) \subset S^{m'}_\infty(\mathbb{R}^)$. Furthermore, the inclusion map*

$$\iota : S^m_\infty(\Omega; \mathbb{R}^n) \to S^{m'}_\infty(\Omega; \mathbb{R}^n)$$

is continuous.

Proof. Let the real numbers $m \leqslant m'$ be given. We note that for any $\xi \in \mathbb{R}^n$

$$\langle \xi \rangle^m \leqslant 1 \cdot \langle \xi \rangle^{m'}$$

and thus if $a \in S^m_\infty(\Omega; \mathbb{R}^n)$, we have that $\forall \alpha \in \mathbb{N}^p, \forall \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \leqslant C \langle \xi \rangle^{m - |\beta|} \leqslant C \langle \xi \rangle^{m' - |\beta|}$$

which show that $a \in S^{m'}_\infty(\Omega; \mathbb{R}^n)$ as well.

To show that ι is a continuous inclusion, it suffices to show that

$$\|\iota(a)\|_{N,m'} \leqslant C \|a\|_{N,m}$$

for any $a \in S^m_\infty(\Omega; \mathbb{R}^n)$ and $N \in \mathbb{N}$. Indeed, his bound holds since

$$\frac{D_x^\alpha D_\xi^\beta a(x,\xi)}{\langle \xi \rangle^{m' - |\beta|}} \leqslant \frac{D_x^\alpha D_\xi^\beta a(x,\xi)}{\langle \xi \rangle^{m - |\beta|}}.$$

□

This inclusion property allow us to consider $S^m_\infty(\Omega; \mathbb{R}^n)$ as the filtration of the space

$$S^\infty_\infty(\Omega; \mathbb{R}^n) = \bigcup_{m \in \mathbb{R}} S^m_\infty(\Omega; \mathbb{R}^n)$$

and we shall denote the *residual* space of the filtration as

$$S^{-\infty}_\infty(\Omega; \mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} S^m_\infty(\Omega; \mathbb{R}^n).$$

We have a rather technical result of the density of the residual space in $S^m_\infty(\Omega; \mathbb{R}^n)$.

Lemma 2.6. *Given any $m \in \mathbb{R}$ and $a \in S^m_\infty(\Omega;\mathbb{R}^n)$, there exist a sequence in $S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$ such that bounded in $S^m_\infty(\Omega;\mathbb{R}^n)$ and converges to a in the topology of $S^{m+\epsilon}_\infty(\Omega;\mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$. In other words, for any $m \in \mathbb{R}$ and $\epsilon > 0$, $S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$ is dense in $S^m_\infty(\Omega;\mathbb{R}^n)$ with the topology of $S^{m+\epsilon}_\infty(\Omega;\mathbb{R}^n)$.*

Proposition 2.7. *Let $p, n \in \mathbb{N}$ be given. Let $\Omega \subset \mathbb{R}^p$ be such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$. Then, for any $m, m' \in \mathbb{R}$, we have*

$$S^m_\infty(\Omega;\mathbb{R}^n) \cdot S^{m'}_\infty(\Omega;\mathbb{R}^n) = S^{m+m'}_\infty(\Omega;\mathbb{R}^n)$$

Proof. Let $a \in S^m_\infty(\Omega;\mathbb{R}^n)$ and $b \in S^{m'}_\infty(\Omega;\mathbb{R}^n)$ be given. By (general) Leibinz formula, we have that for all multi-index α, β,

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}^n} \frac{\left| D^\alpha_x D^\beta_\xi a(x,\xi) b(x,\xi) \right|}{\langle\xi\rangle^{(m+m')-|\,\beta\,|}} &\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} \sup_{\xi\in\mathbb{R}^n} \frac{\left| D^\mu_x D^\gamma_\xi a(x,\xi)\right| \left| D^{\alpha-\mu}_x D^{\beta-\gamma}_\xi b(x,\xi) \right|}{\langle\xi\rangle^{(m+m')-|\,\beta\,|}} \\
&\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \sup_{\xi\in\mathbb{R}^n} \frac{\langle\xi\rangle^{m-|\,\gamma\,|}\langle\xi\rangle^{m'-|\,\beta-\gamma\,|}}{\langle\xi\rangle^{(m+m')-|\,\beta\,|}} \\
&= \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{|\,\beta\,|-(|\,\beta-\gamma\,|+|\,\gamma\,|)} \\
&\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \\
&< \infty
\end{aligned}$$

where we have use the property of multi-index that $|\,\beta\,| = |\,\beta-\mu\,| + |\,\mu\,|$. We have thus shown that $S^m_\infty(\Omega;\mathbb{R}^n) \cdot S^{m'}_\infty(\Omega;\mathbb{R}^n) \subset S^{m+m'}_\infty(\Omega;\mathbb{R}^n)$

For the reverse inclusion, let $c \in S^{m+m'}_\infty(\Omega;\mathbb{R}^n)$ be given. Define

$$\begin{aligned}
a &: (x,\xi) \mapsto \langle\xi\rangle^m \\
b &: (x,\xi) \mapsto \frac{c(x,\xi)}{a(x,\xi))}
\end{aligned}$$

and observe that

- $a \in S^m_\infty(\Omega;\mathbb{R}^n)$. It is clear that a is smooth in both x and ξ. It is independent of x and thus any x derivative gives 0. We need only to check that for all $\beta \in \mathbb{N}^n$,

  $$\left| D^\beta_\xi \langle\xi\rangle^m \right| \leqslant C \langle\xi\rangle^{m-|\,\beta\,|}$$

  which can be proven by induction on n and β. We shall only prove the base case where $n = 1$ and $\beta = 1$. We have

  $$|\, D_\xi \langle\xi\rangle^m \,| = \left| \partial_\xi (1+\xi^2)^{m/2} \right| = \left| m\xi \langle\xi\rangle^{m-2} \right| = \left| m\frac{\xi}{\langle\xi\rangle} \right| \langle\xi\rangle^{m-1} \leqslant |\, m\,| \langle\xi\rangle^{m-1}$$

  where we have used the fact that $|\,\xi\,| \leqslant \langle\xi\rangle$ for all ξ.

- $b \in S^{m'}_\infty(\Omega;\mathbb{R}^n)$. We note first that $\langle\xi\rangle^m \neq 0$ for all $\xi \in \mathbb{R}^n$ and thus b is well-defined. Since division by $\langle\xi\rangle^m$ does not affect any of the x derivative, we only need to show that for any $\beta \in \mathbb{N}^n$, we have

  $$\left| D^\beta_\xi b(x,\xi) \right| \leqslant C \langle\xi\rangle^{m+m'-|\,\beta\,|}$$

for some constant $C > 0$ uniformly in ξ. Indeed, observe that by the Leibinz formula

$$\begin{aligned}\left| D_\xi^\beta b(x,\xi) \right| &\leqslant \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \left| D_\xi^\mu c(x,\xi) \right| \left| D^{\beta-\mu} \langle \xi \rangle^{-m} \right| \\ &\leqslant C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m+m'-|\mu|} \langle \xi \rangle^{-m-|\beta-\mu|} \\ &\leqslant C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m'-(|\mu|+|\beta-\mu|)} \\ &= C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m'-|\beta|} \\ &= C 2^\beta \langle \xi \rangle^{m'-|\beta|}\end{aligned}$$

where we have use the definition of c and applied the result proven for a with $m \mapsto -m$. Thus, $b \in S_\infty^{m'}(\Omega; \mathbb{R}^n)$.

It is clear that $a \cdot b = c$ and we have therefore shown that $S_\infty^{m+m'}(\Omega; \mathbb{R}^n) \subset S_\infty^m(\Omega; \mathbb{R}^n) \cdot S_\infty^{m'}(\Omega; \mathbb{R}^n)$.

□

A sumarising theorem:

Theorem 2.8. *Given $p, n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$. Let*

$$S_\infty^\infty(\Omega; \mathbb{R}^n) = \bigcup_{m \in \mathbb{R}} S_\infty^m(\Omega; \mathbb{R}^n).$$

Then $S_\infty^\infty(\Omega; \mathbb{R}^n)$ is a graded algebra over $\mathbb{R}$ with continuous inclusion $S_\infty^m(\Omega; \mathbb{R}^n) \to S_\infty^{m'}(\Omega; \mathbb{R}^n)$ for all $m \leqslant m'$.

2.2 Ellipticity of symbols

Definition 2.9. Given $p, n \in \mathbb{N}$, $m \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$, an order m symbol $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ is (globally) **elliptic** if there exist $\epsilon \in \mathbb{R}_{>0}$ such that

$$\inf_{|\xi| \geqslant 1/\epsilon} |a(x,\xi)| \geqslant \epsilon \langle \xi \rangle^m.$$

The importance of elliptic symbol is that they are invertible modulo $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$.

Lemma 2.10. *Given $p, n \in \mathbb{N}$, $m \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$. Let $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ be an elliptic symbol of order m. Then there exist a symbol $b \in S_\infty^{-m}(\Omega; \mathbb{R}^n)$ such that*

$$a \cdot b - 1 \in S_\infty^{-\infty}(\Omega; \mathbb{R}^n).$$

3 Pseudodifferential Operators (ΨDO's)

4 Appendix

4.1 Stationary phase lemma

In the study of pseudodifferential operators, we often encounter integral of highly oscillatory functions of the form

$$I(h) = \int_{\mathbb{R}} a(x) e^{i\varphi(x)/h} dx$$

where $a \in C_c^\infty(\mathbb{R})$, $\varphi \in C^\infty(\mathbb{R})$ and we are interested in the asymptotic behaviour as $h \to 0$. We note that if φ is linear (or constant), i.e. $\varphi(x) = \alpha x + \beta$, $\alpha, \beta \in \mathbb{R}$, then,

$$|\, I(h) \,| = \left| \int_{\mathbb{R}} a(x) e^{i(\alpha x + \beta)/h} dx \right| = \left| e^{i\beta/h} \right| \left| \int_{\mathbb{R}} a(x) e^{i\alpha x/h} dx \right| = \left| \int_{\mathbb{R}} a(x) e^{i\alpha x/h} dx \right| \to 0$$

as $h \to 0$ by Riemann-Lebesgue lemma. That is to say, as the length scale of the oscillation tends to zero, the values of the integrand achieve perfect cancellation. In general, if $\varphi'(x) \neq 0$, we expect $e^{i\varphi(x)/h}$ to oscillate at length scale of order h and thus as $h \to 0$,

5 Miscenllaneous

Theorem 5.1 (Schwartz Kernel Theorem [Taylor, 2011, Chapter 4.6, p. 345])**.** *Let M, N be compact manifold and*

$$T : C^\infty(M) \to \mathcal{D}'(N)$$

be a continuous linear map ($C^\infty(M)$ being given Frechet space topology and $D'(N)$ the weak topology). Define a bilinear map*

$$\begin{aligned} B : C^\infty(M) \times C^\infty(N) &\to \mathbb{C} \\ (u, v) &\mapsto B(u, v) = \langle v, Tu \rangle \,. \end{aligned}$$

Then, there exist a distribution $k \in \mathcal{D}'(M \times N)$ such that for all $(u, v) \in C^\infty(M) \times C^\infty(N)$

$$B(u, v) = \langle u \otimes v, k \rangle \,.$$

We call such k the kernel of T.

References

[Taylor, 2011] Taylor, M. (2011). *Partial Differential Equations I*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2 edition.