Microlocal Analysis
with Applications to Non-Elliptic Fredholm Problems

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Introduction

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where

$$\begin{aligned}
\alpha &= (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n && \text{multi-index} \\
|\alpha| &= \alpha_1 + \alpha_2 + \cdots + \alpha_n && \text{order of multi-index} \\
c_\alpha &\in C_\infty^\infty(\mathbb{R}^n) && \text{bounded smooth functions} \\
D_{x_i} &= -i\partial_{x_i} \\
D_x^\alpha &= (-i\partial_{x_1})^{\alpha_1} (-i\partial_{x_2})^{\alpha_2} \ldots (-i\partial_{x_n})^{\alpha_n}
\end{aligned}$$

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A linear partial differential equation (PDE) :

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Smooth solution and forcing:

$$C^\infty(\mathbb{R}^n) \ni u,\, f : \mathbb{R}^n \to \mathbb{C}$$

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Example:

$$\Delta u = -\partial_{x_1}^2 u - \cdots - \partial_{x_n}^2 u = \delta \qquad \text{Laplace operator}$$

$$\Box u = \partial_t^2 u - \partial_{x_1}^2 u - \cdots - \partial_{x_n}^2 u = 0 \qquad \text{Wave operator}$$

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We will use the **Fredholm** theory to tackle all three simultaneously!

Overview

1. Introduction
2. Fredholm Operators and Regularity
3. "Elliptic operators are Fredholm"
4. A Non-elliptic Fredholm problem

Fredholm Operators

Definition (Fredholm operators)

A continuous linear operator $T : \mathcal{X} \to \mathcal{Y}$ between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ is Fredholm, if

- T has closed range, i.e. $T(\mathcal{X})$ is closed in $\mathcal{Y}$,
- $\ker(T) \subset \mathcal{X}$ is finite dimensional,
- $\operatorname{coker}(T) := \mathcal{Y}/T(\mathcal{X})$ is finite dimensional.

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Suppose $Tx = y$ for a given $y \in \mathcal{Y}$.

Existence a solution $x \in \mathcal{X}$ exist if and only if $y \notin \operatorname{coker}(T)$.

Uniqueness the solution is unique if and only if $\ker(T) = 0$.

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T Fredholm

$\implies$ existence and uniqueness is a finite dimensional linear algebraic problem.

Fredholm Estimate

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Theorem

Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be Banach spaces. If

- *$T : \mathcal{X} \to \mathcal{Y}$ is continuous,*
- *$\mathcal{X}$ is compactly contained in $\mathcal{Z}$, i.e. $\iota : \mathcal{X} \hookrightarrow \mathcal{Z}$ is compact,*
- *for all $u \in \mathcal{X}$, there exist $C > 0$ such that the following estimate hold*

$$\|x\|_{\mathcal{X}} \leqslant C \left(\|Tx\|_{\mathcal{Y}} + \|u\|_{\mathcal{Z}} \right) \tag{3}$$

then

- *the image, $T(\mathcal{X})$ is closed, and*
- *T has finite dimensional kernel.*

Constructing a *Fredholm problem*

What's a Fredholm differential operator? ... what's $\mathcal{X}$ and $\mathcal{Y}$?

Fredholm Problem

Given a differential operator P, can we construct solution spaces $\mathcal{X}$ and $\mathcal{Y}$, so that

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What's the link to *regularity*? Answer: Sobolev spaces.

Sobolev Space

Definition

The Sobolev space of order $k \in \mathbb{N}$ on $\mathbb{R}^n$, $H^k(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ is defined by

$$\begin{aligned} u \in H^k(\mathbb{R}^n) &\iff D^\alpha u \in L^2(\mathbb{R}^n) \text{ whenever } |\alpha| \leqslant k \\ &\iff \langle \xi \rangle^k \, \widehat{u}(\xi) \in L^2(\mathbb{R}^n). \end{aligned}$$

$$\langle \xi \rangle := \left(1 + |\xi|^2\right)^{1/2} = \left(1 + |\xi_1|^2 + \cdots + |\xi_n|^2\right)^{1/2}$$

$$\widehat{u}(\xi) := \mathcal{F}u(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{-ix\cdot\xi} u(x)\, \mathrm{d}x$$

Hilbert space structure that keeps track of (global) regularity data of u.

$$\|u\|_{H^k} = \underbrace{\|u\|_{L^2}}_{\text{global decay}} + \underbrace{\sum_{|\alpha| \leqslant k} \|D^\alpha u\|_{L^2}}_{k \text{ times differentiable}}$$

Sobolev Space on Compact Manifold

Let M be a smooth compact n-manifold without boundary (i.e. closed manifold), $s \in \mathbb{R}$, $u \in (C^\infty(M))'$, then

$$u \in H^s(M) \iff (\chi \cdot u) \circ \Phi^{-1} \in H^s(U)$$

for any chart $\Phi : \widetilde{U} \to U \subset \mathbb{R}^n$ and smooth bump function $\chi \in C^\infty(M)$ compactly supported in the chart domain $\widetilde{U}$.

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for any chart $\Phi : \widetilde{U} \to U \subset \mathbb{R}^n$ and smooth bump function $\chi \in C^\infty(M)$ compactly supported in the chart domain $\widetilde{U}$.

Henceforth, M is either $\mathbb{R}^n$ or a closed n-manifold.

General Strategy

Reduced question

Existence, uniqueness, regularity $\rightsquigarrow$ can we find $\mathcal{X} \subset H^s(M)$, $\mathcal{Y} \subset H^{s'}(M)$ so that

$$P : \mathcal{X} \to \mathcal{Y}$$

is Fredholm?

On closed manifold, $H^s(M) \Subset H^{s-\epsilon}(M)$ for any $\epsilon > 0$.

General Strategy

Prove that for **any** $s, N \in \mathbb{R}$, there exist $C > 0$,

$$\|u\|_{H^s} \leqslant C \left(\|Pu\|_{H^{s'}} + \|u\|_{H^N} \right).$$

"Elliptic operators are Fredholm"

How do we get such an estimate?

Theorem (Elliptic regularity)

Let P be an order $m \in \mathbb{R}$ elliptic differential operator on an n-manifold, M. Suppose we know a priori that $u \in H^N(M)$ for some $N \in \mathbb{R}$. Then, for any $s \in \mathbb{R}$

$$(f =)Pu \in H^s(\mathbb{R}^n) \implies u \in H^{s+m}(\mathbb{R}^n)$$

and u satisfies the estimates: $\exists C > 0$

$$\|u\|_{H^{s+m}} \leqslant C\left(\|Pu\|_{H^s} + \|u\|_{H^N}\right).$$

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Let P be an order $m \in \mathbb{R}$ **elliptic (pseudo-)** *differential operator on an n-manifold, M. Suppose we know a priori that $u \in H^N(M)$ for some $N \in \mathbb{R}$. Then, for any $s \in \mathbb{R}$*

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Elliptic operator

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Fourier transform + integration by parts $\implies$

$$(\Delta + 1)\, u(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} \left(1 + |\,\xi\,|^2\right) u(y)\, \mathrm{d}y\, \mathrm{d}\xi$$

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We expect the existenc of an inverse ...

$$\begin{aligned}
&(\Delta + 1)^{-1} (\Delta + 1)\, u(x) \\
&= \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} \underbrace{\left(1 + |\,\xi\,|^2\right)^{-1} \left(1 + |\,\xi\,|^2\right)}_{=1} u(y)\, \mathrm{d}y\, \mathrm{d}\xi = u(x)
\end{aligned}$$

$\left(1 + |\,\xi\,|^2\right)^{\pm 1}$ are the **symbols** for $(\Delta + 1)^{\pm 1}$.

Pseudodifferential operator

Question: What is $(\Delta + 1)^{-1}$? Answer: **pseudo**differential operator.

$$P(x, D_x)u = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} p(x,\xi) u(y) \,\mathrm{d}y \,\mathrm{d}\xi \tag{4}$$

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Definition

A smooth function $p(x,\xi) \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ is a symbol of order $m \in \mathbb{R}$, i.e. $p \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$, if

$$\left| D_x^\alpha D_\xi^\beta p(x,\xi) \right| \leqslant C_{\alpha,\beta,\gamma} \left\langle \xi \right\rangle^{m - |\beta|}, \quad C_{\alpha,\beta} > 0$$

for any multi-index $\alpha, \beta \in \mathbb{N}^n$.
A pseudodifferential operator, $P \in \Psi^m_\infty(\mathbb{R}^n)$ of order m with (left reduced) symbol $p \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$ has action on $u \in \mathcal{S}'(\mathbb{R}^n)$ given by (4).

Proof sketch of elliptic regularity

Theorem

The total spaces $\cup_m S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ *and* $\cup_m \Psi^m_\infty(\mathbb{R}^n)$ *form filtered *-algebra over* $\mathbb{C}$. *If* $P \in \Psi^m_\infty(\mathbb{R}^n)$ *for some* $m \in \mathbb{R}$,

1. $P : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ *is continuous for any* $s \in \mathbb{R}$.
2. *If* P *is* **elliptic** *of order* m*, i.e. its symbol* p *satisfies*

$$|\, p(x,\xi)\,| \geqslant \epsilon\,\langle\xi\rangle^m \qquad \text{in } |\,\xi\,| > \epsilon \text{ for some } \epsilon > 0$$

   then there exist parametrix $Q \in \Psi^{-m}_\infty(\mathbb{R}^n)$ *such that* $QP - 1 : H^s \to H^{s'}$ *is continuous for any* $s, s' \in \mathbb{R}$.

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Proof sketch (elliptic regularity).

$$u = QPu - (QP-1)\,u$$

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Proof sketch (elliptic regularity).

$$\|u\|_{H^{s+m}} \leqslant \underbrace{\|QPu\|_{H^{s+m}}}_{\leqslant C\|Pu\|_{H^s}} + \underbrace{\|(QP-1)\,u\|_{H^{s+m}}}_{\leqslant C\|u\|_{H^N}}$$

using continuity $Q : H^{s+m} \to H^s$ and $(QP-1) : H^{s+m} \to H^N$.

Non-elliptic Fredholm problem

Recent work by [?, ?] show that we can construct Fredholm problem for **non-elliptic** operators too!
We'll illustrate by sketching the proof for a pertubation of the wave operator

$$\Box := \partial_t^2 - \sum_{j=1}^{n-1} \partial_{x_j}^2$$

on the $1+n$-torus,

$$\mathbb{T}^{1+n} := \mathbb{S}^1_t \times \underbrace{\mathbb{S}^1_{x_1} \times \mathbb{S}^1_{x_2} \times \cdots \times \mathbb{S}^1_{x_n}}_{n}$$

where $\mathbb{S}^1 := [0,1]/(0 \sim 1)$ is the circle and $(t,x) = (t, x_1, \ldots, x_n)$ are the local coordinates on $\mathbb{T}^{1+n}$.

Main theorem

Theorem

There exist a perturbation Q of the wave operator $\Box$ on $\mathbb{T}^{1+n}$ and a subspace $\mathcal{X}^s \subset H^s(\mathbb{T}^{1+n})$ for each $s \in \mathbb{R}$, such that the operator:

$$(\Box - iQ) : \mathcal{X}^s \to H^{s-1}(\mathbb{T}^n)$$

is Fredholm.

Two Major Ingredients

Theorem (Microlocal elliptic regularity)

Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. If for some $Q' \in \Psi^0_\infty(\mathbb{R}^n)$, $Q'Au \in H^s(\mathbb{R}^n)$, then for any other $Q \in \Psi^0_\infty(\mathbb{R}^n)$ such that $\mathrm{WF}'(Q) \subset Ell^m(A) \cap Ell^0(Q')$ we have $Qu \in H^{s+m}(\mathbb{R}^n)$ and it satisfies the estimate: $\forall N \in \mathbb{R}$, $\exists C > 0$

$$\|Qu\|_{H^{s+m}} \leqslant C \left(\left\|Q'Au\right\|_{H^s} + \|u\|_{H^N} \right).$$

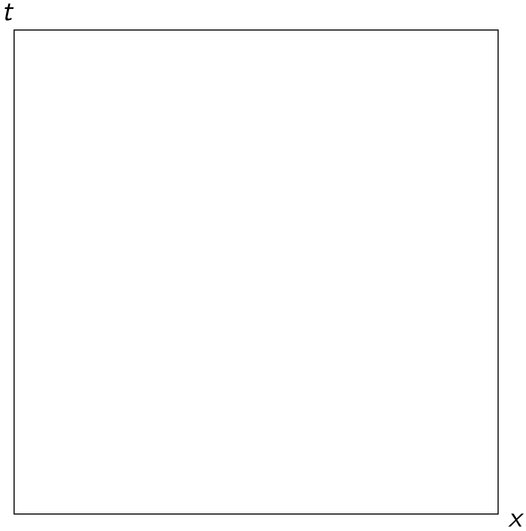
Theorem (Propagation of singularities)

Let $P \in \Psi^m_\infty(\mathbb{R}^n)$ is a properly supported pseudodifferential operator with polyhomogeneous principal $\sigma_m(P) = p - iq$ with real p, q. If we have $A, B, B' \in \Psi^0_\infty(\mathbb{R}^n)$ and $q \geqslant 0$ on $\mathrm{WF}'(B')$ and every $(x, \xi) \in WF'(A)$ is in the integral curve of H_p originating from $Ell^0(B)$, then for all $s, N \in \mathbb{R}$ and $u \in C^\infty(\mathbb{R}^n)$, there exist $C > 0$ such that

$$\|Au\|_{H^s} \leqslant C \left(\|Bu\|_{H^s} + \left\|B'Pu\right\|_{H^{s-m+1}} + \|u\|_{H^{-N}} \right).$$

Constructions

First, we define a bump functions in the time dimension $\chi(t)$ as below:



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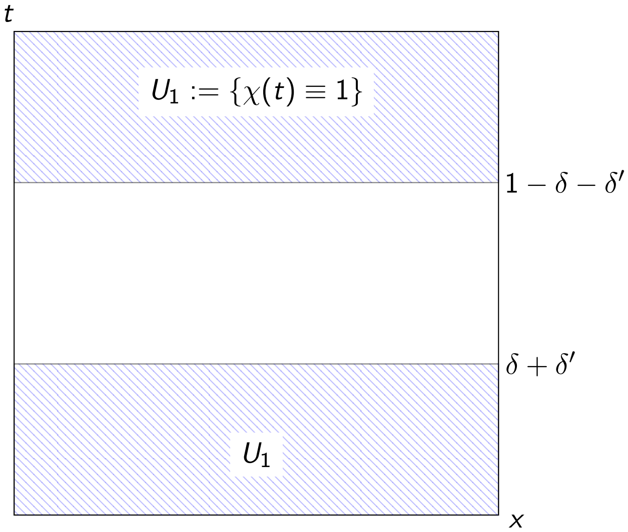

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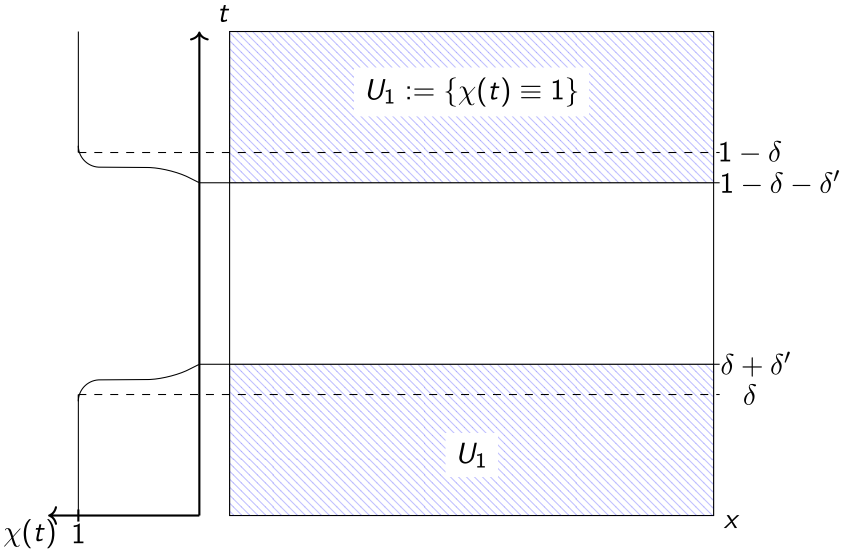

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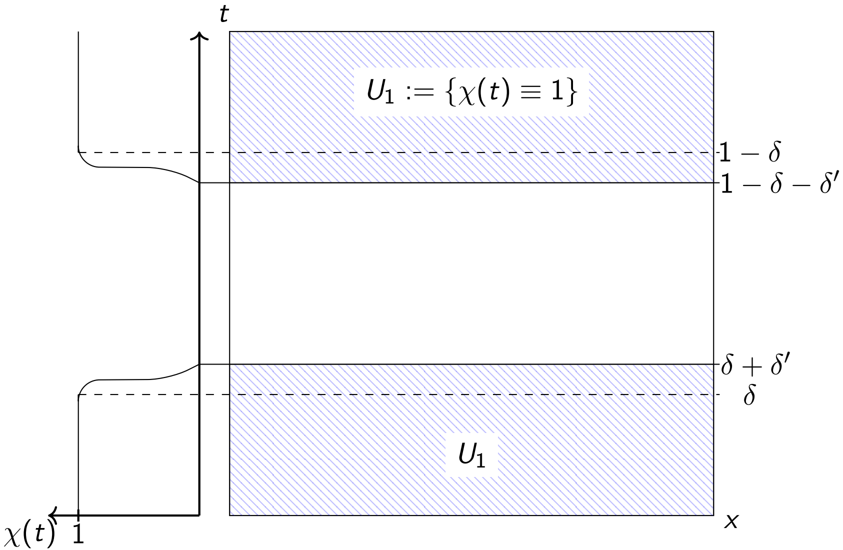


Constructions

Define in local coordinates

$$Q = \chi(t)\partial_t^2 \implies \Box - iQ = (1 - i\chi(t))\,\partial_t - \sum_{i=1}^{n} \partial_{x_i}^2$$

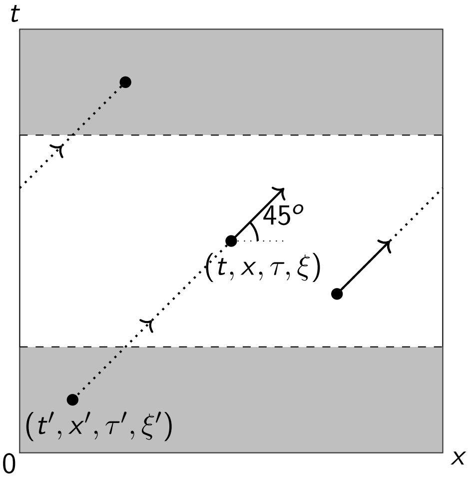
with principal symbol

$$\begin{aligned} p &:= \sigma_2(\Box) = \tau^2 - |\,\xi\,|^2 \\ q &:= \sigma_2(Q) = \chi(t)\tau^2 \implies \sigma_2(\Box - iQ) = (1 - i\chi(t))\,\tau - |\,\xi\,|^2 \,. \end{aligned}$$

Hamilton flow of $\sigma_2(\Box)$

Hamiltonian flow:

$$\exp(sH_p)(t,x,\tau,\xi) = (t+s\tau, x+s\xi, \tau, \xi)$$



Fourier analysis relates the (global) regularity of functions to their fourier transform. e.g. in the 'superposition of wave' picture, only waves with high frequency can approximate jump discontinuity, linking continuity with the decay of fourier coefficients. Microlocal analysis also keeps track of the direction of decay.

Theorem (Rank-nullity)

If $T : V \to W$ be a linear operator between finite dimensional vector space V and W, then

$$\mathrm{Ind}(T) := \dim \ker T - \dim \mathrm{coker}\, T = \dim W - \dim V.$$

Theorem (Atiyah-Singer index theorem)

Given

- X *a smooth comapct manifold,*
- E, F *smooth vector bundles over* X,
- $P : \Gamma(E) \to \Gamma(F)$ *be an elliptic differential operators between the space of sections of* E *and* F.

Then, P *is Fredholm and its Fredholm index is related to its topological index.*