Presentation notes

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1 Speech script

1.1 Slide 1: Introducing partial differential operator

Thanks for the introduction. I promise I will clarify what I meant by non-elliptic Fredholm problem in the next few minutes.

First, here's something familiar. A linear partial differential operator in $\mathbb{R}^n$ (or in local coordinate on manifold) of order k, which we will denote as P throughout, is just a sum of partial derivatives with smooth bounded coefficient. Here we have used the multi-index notation, which is just n-tuples of powers that adds up to less than k, because it's of order k and D_x^α is just a notation for the multiplication with each term raise to their corresponding power. Because of how important Fourier transform is to us, we will use multiply the usual partial derivative by $-i$ just so we have the nicer looking identity, which says that Fourier transform of the mix-partial derivatives is the same as doing fourier transform first and then multiply by the monomial with the same multi-index.

That's a lot of notation to absorb. Here's some concrete examples. In fact, they are each representative of two very important classes of differential operator. The Laplace operator which is in the class of elliptic operator and the wave operator which is hyperbolic.

1.2 Slide 2: Introducing PDE

Another familiar thing. This is an order k linear partial differential equation. P the operator and f the forcing term are the givens and u the solution. Since we are taking derivative, we might expect u and f to be smooth functions, but a more effective way of studying pde is to allow weak solution and forcing called tempered distributions. These are linear functional that takes in rapidly decaying smooth functions and spits out a complex scalar. Rapid decay is just a short way of saying that the phi's are smooth functions and their derivatives decay faster than any polynomial.

Given a PDE, we can and we will 3 all important questions

1.3 Slide 3: Introducing the main questions in PDE

One, existence: For which f can we find solution u?
Two, the uniqueness question: if we found one, is it the only one?
Three, regularity: How does the regularity of the forcing affect regularity of the solution? For instance, does smooth f beget smooth u?

In this talk, we will employ Fredholm theory from functional analysis to tackle all three question in one swoop.
It's my job now to explain how.

1.4 Slide 4: Overview

First I will define Fredholm operators and explain what it has to do with existence, uniqueness and regularity?
After explaining the why, we get into the how.
I will prove things in general for a class of operator call the elliptic operators. I will define that later.
For non-elliptic ones, the situation is more complicated. It's only recently that we have the technology to tackle them. I will prove a particular example in this class too.

1.5 Fredholm Operators

Let's dive into it. Fredholm operators. Given Banach spaces X and Y and a continuous linear map T between them. We say T is Fredholm if its image in Y is a closed set and both it's kernel and cokernel are finite dimensional.

Suppose you're given a y and you want to solve Tx = y. Linear algebra says, a solution exist if and only if y is perpendicular to the cokernel. and the solution is unique if and only if the kernel is trivial.

If we know T is Fredholm, we have then reduce existence and uniqueness to finite dimensional linear algebra!

1.6 Slide 5: Fredholm estimate

In analysis, we usually translates topological or algebraic statements into estimates. I am sure most of you are familiar with checking continuity of linear operator is the same as checking an inequality. Continuity have continuity estimates, Fredholm has Fredholm estimate too.
Again, T is continuous linear between X and Y. If the size of every element in the domain X is control by the size of its image under T in Y plus a small term, then T is semi-Fredholm. That is it has closed image and finite dimensional kernel. We think of the last term being small because we require that X be compactly contained in Z. The only thing missing for semi-Fredholm operator is finite dimensional cokernel. Notice that using Banach isomorphism theorem, if we manage to prove the same estimate for the dual of T, then T will be fully Fredholm.

1.7 Slide 6: Fredholm problem

That's quite enough functional analysis. Let's go back to PDE. An operator can't be Fredholm in isolation, we need to specify its domain and codomain!

So when we say "contructing a Fredholm problem for a differential operator P", we mean, find solution space X and Y so that P acts as a Fredholm map between them.

If you're following, you will notice that I owe you something: I have said something about existence and uniquess, but I owe you regularity. The strategy here is to built some measures of regularity into X and Y. Happily, this has been done for us in the form of Sobolev space.

1.8 Slide 7: Sobolev space

Sobolev space are space of distributions with global regularity and decay data built into its Hilbert space structure. A distribution is in the kth orde Sobolev space if and only if and weak derivative of order at most k is square integrable. Fourier analysis tells us that this is equivalent to saying that its fourier transform decay fast enough that even when we try to grow it by a kth order polynomial, it is still square integrable. This weighting condition allow us to generalise from integer order k to non-integer order s.
Note that we can give it a Hilbert inner product that induce a norm of the this form. The first term measure global decay of the distribution itself. The rest measure decay of the derivative upto order k.

1.9 Slide 8: Sobolec space on closed manifold

Spoiler alert, we will be working on the torus later. For that and other reasons we would really like to have Sobolev space on closed manifold, that is smooth compact manifold without boundary, like the torus. These are just distribution on the manifold that locally in a coordinate patch looks behaves like an element of the Sobolev space of $\mathbb{R}^n$.
For the rest of the talk, M will either be $\mathbb{R}^n$ or closed manifold.

1.10 Slide 9: General strategy

I have told you why we care if an operator is Fredholm, I haven't tell you how. Hopefully this is not a surprise by now, we do that by proving estimates on sobolev spaces, something that looks a lot like Fredholm estimate.

More precisely, we want to control the regularity of distribution by the regularity of its forcing term Pu (remember this is just f). We need to handle this annoying last term, which is usually taken to be highly negative so that higher order sobolev space s is compactly contained in it. This is only possible on closed manifold M. For non-compact one like $\mathbb{R}^n$ we need extra conditions which we will not discuss today.

Looking ahead: For the nice class of elliptic operators, we can do this for any s, provided regularity of the forcing term s' is of lower order. How much lower depends on the order m of the differential operator.

For non-elliptic one, we need a bit more, we need the forcing term to have one order higher regularity to the estimate to work.

1.11 Slide 10: Elliptic operators

Let's get into the hows of proving estimates now. There is a common refrain in PDE that says: "Elliptic operators are Fredholm". That's because of the following elliptic regularity theorem:
Suppose P is an order m elliptic differential operator. I will owe you for now the definition of ellipticity. If we know before hand that a potential solution u is has at least regularity N (think negative a million). If the equation tell us that the forcing term has regularity s, we immediately know that any solution have m order better regularity. Furthermore, the regularity is control by that of its forcing term plus its a priori estimate.

The only thing you need get out of this now is: This is precisely the Fredholm estimate we needed!

We will now go about proving elliptic regularity

1.12 Slide 11: Elliptic operator

First some intuition about what is elliptic operator. They are operators that generalise the Laplace operator. To avoid smoothness issue later, we will discuss del +1 here. Again, recall that fourier transform and integration by parts says that we make differential operators into multiplicative ones. Using this fact, the action of del +1 on distribution u is the same as: first taking the fourier transform, multiply by this second order polynomial and then take inverse transform. We call that polynomial the symbol for del +1

If we stare at that long enough we might say, hey what if we have another operator that multiply by the inverse of that symbol? Well ok, suppose we can, and boldly denote it as del +1 inverse, we can check that that will indeed invert the del +1 operator!

1.13 Slide 12: Pseudodifferential operator

But the question is, what is this del + 1 inverse operator? The answer is its our first example of pseudodifferential operator! What are these things. We have seen that using fourier transform the usual differential operator acts by integrating its input u againts wave amplitude amplitude p(x, xi) which are polynomial in xi. Pseudodifferential operator generalise this by allow a more general class of amplitudes called symbols. These are just smooth functions that satisfies bounds that you would expect if they are polynomials. That is, an mth order symbol is bounded in x but we allow m order growth in xi, but if you differentiate in xi the growth order have to be reduced accordingly as well.

An mth order pseudodifferential operator is then just an operator that act on distribution by integrating against amplitude given by an mth order symbol.

Admittedly we are suppressing a lot of convergence and functional analytic issue here where it is fully developed in the research. Also, these definitions can be shown to be diffeomorphism invariant and hence generalisable to closed manifold. In that case, we say $x \in M$ is the space variable of the base space M and ξ is the dual variable in that belong to the cotangent space T^*M.

1.14 Slide 13: Properties of Pseudos

Sometimes solution comes hand in hand with generalisation. The following lemma about pseudodifferential operators will allow us to prove elliptic regularity with suprising ease:
If P is a pseudo of order m on some manifold, then one, it is a continuous map between the s order sobolev space to m order lower sobolev space. two, if P is elliptic... ah now I better pay you back the definition of ellipticity by now, here it is ... P is elliptic if it has symbol is not only bounded above by order m polynomial, but is bounded below by m order polynomial as well. Actually, we only need that to be true away from some compact set around the origin in fibre of the cotangent space.
so if P is elliptic, then we can construct something call the parametrix Q of P. Which is something very close to an inverse. More precisely, Q is an order -m pseudo and the difference of QP with the identity is continuous between any two sobolev space!

1.15 Slide 14: Proof of elliptic regularity

We are now set to prove elliptic regularity.

So we are given P an elliptic pseudo of order m and we know we have a parametrix Q. We are also given u in some H^N. First, we add and substract QPu to rewrite u as follow. Recall that our goal is to bound the H^{s+m} norm, triangle inequality give us that the norm s + m norm of u is bounded by the norm of QPu and the norm of the second term. But from before, we have continuity of Q and and QP -1. Continuity translate to inequality and we have the desired elliptic regularity estimate!

On closed manifold, we can now conclude P is semi fredholm between s + m to s. The theory of pseudodifferential operator also tell us that the dual of elliptic operator is also elliptic. So we can run the same argument and show that P is indeed fully fredholm.

1.16 Slide 15: Non-elliptic Fredholm Problem

We will now move on to the more complicated and perhaps more exciting recent progress in non-elliptic fredholm problem. We will illustrate this by proving the following result:
This is a result about the Fredholm problem of the wave operator on the n +1 torus, we distinguish one extra dimension for time.
The wave operator in local coordinates looks like this and it's a second order differential operator. If it were elliptic we would expect it to be fredholm from Hs+2 to Hs, but it is not elliptic, so need to make compromises. One, we need to perturb it a little by some operator Q. Two, we need to assume more regularity on the forcing, that is the codomain is Hs+1 instead of Hs. Three, we need to restrict to a subspace X of Hs+2.
The claim is that, there is a perturbation Q, a subspace X such that the perturb operator box -iQ act as a fredholm map.
In what follows, p is the symbol for the wave operator. It is a function of the cotangent space of the torus, with t, x being the base variable and tau xi the dual variable. Happily, our symbol is independent of t and x. For now.

1.17 Slide 16: Microlocal viewpoint

To tackle this problem we need to graduate ourselves to the microlocal viewpoint. Recall that p will be globally elliptic if it is bounded below for large dual variables everywhere. But our symbol is zero on what we call the light cone, that is when xi and tau have the same magnitude. So it is not globally elliptic.
But there is a notion of microlocal ellipticity. Microlocal because we are not only locallising in the base manifold, but also in the cotangent fibre. Our symbol is microlocally elliptic at the point t0 x0 in the base

space and tau0 and xi0 in the fibre if the same bound holds, not everywhere, but in some neighbourhood of the base point and in some span of direction near tau0 xi0. So our symbol is microlocally elliptic whenever we stay away from the light cone.
We call the set set of points in cotangent space where p is elliptic the elliptic set of the wave operator and where it is not, we call it the characteristic set. Since we only care about the direction in the fibre, we will remove the zero section from our discussion.

1.18 Slide 17: Microlocal elliptic regularity

Now, we need two more ingredients to prove our theorem. The first one is the microlocal version of elliptic regularity. This theorem says that as long as you stay in the elliptic set of your operator, you get the very same elliptic estimate. The "Staying within" part can be achieve by multiplying all your terms by cut-offs functions that is zero outside of the elliptic set.
So, if we know something about the regularity of the image of u under P somewhere in the elliptic set, we can conclude m order better regularity anywhere else in the elliptic set.
Side note: for the microlocal analyst among us, I have forgo defining wavefront sets. What I have said is a weaker version that suits our purpose.

1.19 Slide 18: Propagation of singularities

The second and perhaps the most important ingredient is the propagation of singularity estimate. The previous theorem tells us how to handle thing in the elliptic set and this one will tell us how to handle the characteristic set.

So here's the characteristic set, it happens to be a codimension one submanifold. We want to know the regularity of u in this neighbourhood so we cut-off using a bump function chi1. Suppressing a little sympletic geometry, the horizontal lines represents the hamilton flow of the principal symbol. For our purpose, this is translates to following the light rays. The theorem says that if every point in the characteristic set is hit by a light ray originating from the support of another bump function, all the while staying in the elliptic region of another bump function, then if we know u is Hs here, we can conclude u is m order better here. In a sense we are propagating regularity information along the hamilton flow within elliptic sets.
But compare to just ellipticity we have a lousier bound. Inregion that we desire information, the regularity is bounded by the region we propagate from plus the m - 1 order lower norm of the image under P, as oppose to m order lower. And then there is always the residue.

1.20 Slide 19: Constructions

We are now equipped with everything we need to prove fredholm estimate. The main idea is the following. Knowing that we can propagate regularity along light rays, we will perturb the wave operator Q judicially to introduce a big enough elliptic region. We do this by a bump function in the time dimension.
Here's the fundamental domain of our torus. Time runs vertically and all the space dimension is collapsed to one dimension running horizontally in our picture.
Our elliptic region will be where the bump function is identically 1. So in the time variable the graph will look like this. In the red region, the pertubation is zero, so we regain our wave operator with principal symbol like that. We know it is non-elliptic on the light cone. In the blue region, however, our principal symbol becomes. For it to be non-elliptic we need it to be zero. So both the real and imaginary part needs to vanish. That means fisrt that tau is zero, which means that xi is also zero. But that means that we are on the zero section which if you remember is excluded from our discussion.

1.21 Slide 20: Hamilton flow

Remember that the hamilton flow of xi2 - tau2 is given by the light ray which are just 45 degree rays originating from the base point you choose. The important thing to realise here is that every point in the middle where we have characteristic set, comes from a light ray that originates in the gray region where we have full ellipticity.

1.22 Slide 21: Conclusion

Finally, we are in this happy scenario where every point in the elliptic region propagates to hit every point in our characteristic set. So, propagation of singularity applies. Recall this is the estimate provided. Everything needs to be in the elliptic set of chi3. But we are on compact manifold, might as well take it to be the identity. We can also take chi1 to be the identity since our elliptic region propagates to hit characteristic point everywhere. chi2 needs to be supported in the elliptic region, so just take chi(t) we define before. m is 2 since the wave operator is order 2 operator. We get the estimate. Not quite like a Fredholm estimate yet. We need to get rid of this extra term, happily it's in the elliptic region, microlocal elliptic estimate applies and give Absorbing smaller terms into higher term paying the price of larger contants we get our desired Fredholm estimate.
We are almost there. We just need to specify the domain. The estimate suggest that we need a priori better regularity or the forcing term. So, we just define a subspace where that is achieved. It is easy to show that this is again a Hilbert space. Finally, conclude that box -iQ is Fredholm between said subspace and Hs+1.

Abstract

In the analysis of partial differential equations, we are interested in the existence, uniqueness and regularity of the solutions. Given a linear differential operator, one typically asks the following question: can we construct Banach spaces of a priori (weak) solutions between which the given operator acts as a Fredholm map? The Fredholm property will then reduce the question of existence and uniqueness to finite dimensional linear algebra. For differential operators, such problems are frequently solved with Sobolev spaces where regularity data of their elements are built into their Hilbert space structure, thus providing the link to regularity.

Traditionally, Fredholm problems are associated with elliptic operators. However, there has been recent progress in non-elliptic setting using microlocal analysis. In this talk, we will introduce the central machineries of microlocal analysis, in particular, the propagation of singularity estimate and use it to construct a non-elliptic Fredholm problem for a perturbation of the wave operator on the torus.

A Introduction

First, the setup and some notations. A linear differential operator on $\mathbb{R}^n$ of order $k \in \mathbb{N}$, looks like

$$P = \sum_{|\alpha| \leqslant k} c_\alpha(x) D_x^\alpha = \sum_{\alpha : \alpha_1 + \dots \alpha_n \leqslant k} c_\alpha(x) \left(-i\partial_{x_1}\right)^{\alpha_1} \left(-i\partial_{x_2}\right)^{\alpha_2} \dots \left(-i\partial_{x_n}\right)^{\alpha_n}$$

explain multi-index and D. A partial differential equation of P will look like

$$Pu = f$$

for some functions $u, f : \mathbb{R}^n \to \mathbb{C}$. Since we are taking upto k derivatives, it seems reasonable to demand that $u \in C^k(\mathbb{R}^n)$, but turns out that's too restrictive and, in fact, there is a consistent way of talking solutions that need not even be continuous. We call these weak solutions or *distributions*. These are linear functionals

$$u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$$

e.g. the Dirac delta distribution $\delta(\varphi) = \varphi(0)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We are mainly interested in tempered distributions because these are distribution where Fourier transform works and Fourier transform is central to microlocal analysis.
Given a PDE, there are three immediate question that we are interested in:

1. Does a solution u exists?

2. If it does, is it unique?

3. How regular is the solution? Is it a weak solution? Is it continuous? Is it differentiable? smooth? rapidly decaying?

Functional analysis, or of particular concern for us, the theory of Fredholm (differential) operators provides us with a way of tackling all three question simultaneously. In this this talk, we will

- see how being Freholm is link to existence, uniqueness and regularity.
- briefly see why elliptic operators, which are essentially invertible operators, are Fredholm. mention that this is a typical and traditional case of study
- There are non-elliptic Fredholm operators too! We will get our hands dirty and actually prove one.

There are a couple of missing information in these statements which we will rectify along the way. Also along the way, we will introduce some relevant machineries from microlocal analysis name drop pseudodifferential operators here?? useful for such problems. refer to thesis for more complete development of microlocal analysis and pseudodifferential calculus.
All the statements above needs some qualifying like compact manfold and domains codomainsn for Fredholms maps. This will be part of the content.

B Fredholm Operators

So, we recall Fredholm maps. and an excuse to introduce spaces and notations for later.

Definition B.1 (Fredholm operators)**.** *A continuous linear operator $T : \mathcal{X} \to \mathcal{Y}$ between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ is Fredholm, if*

- *T has closed range, i.e. $T(\mathcal{X})$ is closed in $\mathcal{Y}$,*
- *$\ker(T) \subset \mathcal{X}$ is finite dimensional,*
- *$\operatorname{coker}(T) := \mathcal{Y}/T(\mathcal{X})$ is finite dimensional.*

If we are given $y \in \mathcal{Y}$ and we want to solve for x in $Tx = y$, then recall that for linear maps the "size" of the kernel is a measure of non-uniqueness of solution x and the cokernel is the amount of obstructions to solvability. More precisely,

- solvable if and only if $y \notin \operatorname{coker}(T)$.
- unique if and only if $\ker(T) = 0$.

The finite dimension property of the Fredholm maps thus reduces questions of existence and uniqueness of solutions to

$$Pu = f$$

to checking a finite number of conditions. And here we encounter our first piece of missing information: we can talk about a differential operator being Fredholm only when we have specify its domain and codomain, i.e. where f lives and where we expect u to belong. So, the question becomes

Given a P, can we construct $\mathcal{X}, \mathcal{Y}$ so that $P : \mathcal{X} \to \mathcal{Y}$ is Fredholm?

For differential operators, we will want $\mathcal{X}, \mathcal{Y}$ to be spaces of distributions. This is known as "constructing a Fredholm problem" for the operator P. In PDE, this is frequently done with Sobolev spaces.

Definition B.2 (Sobolev Spaces)**.** *Given $n, k \in \mathbb{N}$, the Sobolev space of order k on $\mathbb{R}^n$ is*

$$H^k(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : D^\alpha u \in L^2(\mathbb{R}^n), \quad |\alpha| \leqslant k \right\}$$

where $D^\alpha u$ is the distributional derivative of u. Result from Fourier analysis shows that

$$u \in H^k(\mathbb{R}^n) \iff \left(1 + |\xi|^2\right)^{k/2} \mathcal{F}u(\xi) \in L^2(\mathbb{R}^n)$$

allowing using to generalise to arbitrary real order $s \in \mathbb{R}$, giving

$$H^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \,:\, \left(1 + |\,\xi\,|^2\right)^{s/2} \mathcal{F}u \in L^2(\mathbb{R}^n) \right\}.$$

We can give a Hilber space structure on $H^s(\mathbb{R}^n)$, by

$$\langle u, v \rangle_{H^s} = \langle \Lambda^s u, \Lambda^s v \rangle_{L^2}.$$

where

$$\Lambda^s u = \mathcal{F}^{-1}(\langle \xi \rangle^s \, \hat{u})$$

is a topological isormophism $\Lambda^s : \mathcal{S}' \to \mathcal{S}'$.

Sobolev spaces are important in PDE because they measures (global) regularity and decay.

$$\|u\|_{H^k} = \|u\|_{L^2} + \sum_{|\,\alpha\,| \leqslant k} \|D^\alpha u\|_{L^2}$$

First term sufficient decay and later terms boundedness in derivatives. Connect with Fourier analysis, intuition: Need arbitrarily high frequency wave to approximate jump discontinuity.
So, if we can construction Fredholm problem of the operator P of the form

$$P : H^s \to H^{s'}$$

for some s, s' dependend on P, we will then have information about regularity of the solutions of $Pu = f$ as well.
In practise, instead of topological statements, we translates properties like continuity and Fredholm into estimates. For continuity of an operator $T : \mathcal{X} \to \mathcal{Y}$, we have the familiar boundedness condition $\|Tx\|_{\mathcal{Y}} \leqslant C\,\|x\|_{\mathcal{X}}$ for all $u \in \mathcal{X}$. For Fredholm, we will need the following theorem.

Theorem B.1. *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be Banach spaces. If*

- *$T : \mathcal{X} \to \mathcal{Y}$ is continuous,*
- *$\mathcal{X}$ is compactly contained in $\mathcal{Z}$, i.e. the injection $\iota : \mathcal{X} \hookrightarrow \mathcal{Z}$ is compact,*
- *for all $u \in \mathcal{X}$, there exist $C > 0$ such that the estimate*

$$\|x\|_{\mathcal{X}} \leqslant C\left(\|Tx\|_{\mathcal{Y}} + \|u\|_{\mathcal{Z}}\right)$$

  holds,

then the image, $T(\mathcal{X})$ is closed, and T has finite dimensional kernel.

For differential operators between Sobolev spaces, $P : H^s \to H^{s'}$, the estimate we want is

$$\|u\|_{H^s} \leqslant C\,(\|Pu\|_{H^{s'}} + \|u\|_{H^N}).$$

The idea is that we can take N to be highly negative, then, at least on compact manifold M (such as the torus!), we have $H^s(M) \Subset H^N M$. For non-compact manifold or Euclidean $\mathbb{R}^n$ we will need weighted sobolev spaces and decay conditions for the operator. Need to mention the dual map.

C Pseudodifferential operators and the elliptic ones

We have now relate Fredholm operators with existence, uniqueness and, importantly, regularity of its solutions and has reduced Fredholm conditions to estimates. The problem now is: How do we get those estimates? We will approach this problem from the perspective of microlocal analysis, in particular, we will now introduce

- Pseudodifferential operators. Which generalise linear differential operator P.
- The idea of microlocalisation. That is to say, we not only keep track of behaviours of solutions and operators in the base space $\mathbb{R}^n$ (or some manifold as base space), we will keep track of directional data as well, i.e. we will work in the cotangent bundlel $T^*\mathbb{R}^n \cong \mathbb{R}^n_x \times \mathbb{R}^n_\xi$.

segue to introduce the more general pseudos here? or perhaps introduce them earlier? Will need it later anyway.

C.1 Pseudodifferential operators

It is a distinctive property of Fourier transform $\mathcal{F}$ that we can turn the action of operators of the form

$$P = \sum_{|\alpha| \leqslant k} c_\alpha(x) D_x^\alpha.$$

to algebraic operation in the dual space. If $u \in \mathcal{S}(\mathbb{R}^n)$, we have $Pu = \mathcal{F}^{-1}\mathcal{F}Pu = \mathcal{F}^{-1}p(x,\xi)\mathcal{F}u$ or explicitly,

$$Pu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} p(x,\xi) u(y)\, \mathrm{d}y\, \mathrm{d}\xi$$

where

$$p(x,\xi) = \sum_{|\alpha| \leqslant k} c_\alpha(x) \xi^\alpha$$

is the *symbol* of P.
We generalise by introducing a larger class of symbols that behave asymptotically like polynomials. also, introduce incoming variable y.

Definition C.1. *The space $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ of order m is the space of smooth functions $a \in C^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ such that for all multi-index $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^n$*

$$\left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}$$

uniformly on $\mathbb{R}^p \times \mathbb{R}^n$. Together with the family of seminorm (indexed by $N \in \mathbb{N}$)

$$\|a\|_{N,m} = \sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \max_{|\alpha|+|\beta| \leqslant N} \frac{\left| D_x^\alpha D_\xi^\beta a(x,\xi) \right|}{\langle \xi \rangle^{m-|\beta|}}$$

gives a Frechet topology to $S^m_\infty(\Omega;\mathbb{R}^n)$.

We turn symbols of order m in to pseudodifferential operators via quantisation procedures in analogy of turning classical observables like momentum, which are smooth functions to quantum observables which are self-adjoint operators

$$Op(a)u := \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} a(x,y,\xi) u(y)\, \mathrm{d}\xi$$

We denote the space of order m pseudodifferential operators as the image of $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ under Op, i.e.

$$\Psi^m_\infty(\mathbb{R}^n) := Op\left(S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)\right).$$

there are a lot of techinical details being suppressed here. Mention some. For the purpose of this talk, any explicitly written down operator is simply the usual differential operator (of constant coefficient even!) We will see example presently.

C.2 Elliptic operators

Example Δ or $\Delta + 1$. How to invert them. One reason to introduce pseudos is that it is closed under taking elliptic parametrix.

Definition C.2. *Given $p, n \in \mathbb{N}$ and $m \in \mathbb{R}$, an order m symbol $a \in S^m_\infty(\mathbb{R}^p; \mathbb{R}^n)$ is (globally)* ***elliptic*** *if there exist $\epsilon \in \mathbb{R}_{>0}$ such that*

$$\inf_{|\,\xi\,| \geqslant 1/\epsilon} |\, a(x, \xi)\,| \geqslant \epsilon \,\langle \xi \rangle^m .$$

The importance of elliptic symbol is that they are invertible modulo $S^{-\infty}_\infty(\mathbb{R}^p; \mathbb{R}^n)$ as shown in the next lemma.

Lemma C.1. *Let $p, n \in \mathbb{N}$, $m \in \mathbb{R}$ be given and let $a \in S^m_\infty(\mathbb{R}^p; \mathbb{R}^n)$ be an elliptic symbol of order m. Then there exist a symbol $b \in S^{-m}_\infty(\mathbb{R}^p; \mathbb{R}^n)$ such that*

$$a \cdot b - 1 \in S^{-\infty}_\infty(\mathbb{R}^p; \mathbb{R}^n).$$

The following theorem characterise globally elliptic pseudodifferential operators.

Theorem C.2. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ be a pseudodifferential operator. Then, the following are equivalent*

1. *A is an elliptic pseudodifferential operator.*
2. *$\sigma_L(A) \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$ is an elliptic symbol.*
3. *$\exists b \in S^{-m}_\infty(\mathbb{R}^n; \mathbb{R}^n)$, s.t. $\sigma_L(A) \cdot b - 1 \in S^{-\infty}_\infty(\mathbb{R}^n; \mathbb{R}^n)$.*
4. *the principal symbol of A is invertible in the quotient symbol space, i.e.*

$$\exists [b] \in S^{m-[1]}_\infty(\mathbb{R}^n; \mathbb{R}^n), \quad s.t. \quad \sigma_m(A) \cdot [b] = [1] \in S^{0-[1]}_\infty(\mathbb{R}^n; \mathbb{R}^n)$$

where $S^{m-[1]}_\infty(\mathbb{R}^n; \mathbb{R}^n)$ denotes the quotient space $S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)/S^{m-1}_\infty(\mathbb{R}^n; \mathbb{R}^n)$.

Proposition C.3. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ be elliptic and $u \in H^N(\mathbb{R}^n)$ for some $N \in \mathbb{R}$. Then, for any $s \in \mathbb{R}$*

$$Au \in H^s(\mathbb{R}^n) \implies u \in H^{s+m}(\mathbb{R}^n)$$

and u satisfies the estimates: $\exists C > 0$

$$\|u\|_{H^{s+m}} \leqslant C\left(\|Au\|_{H^s} + \|u\|_{H^N}\right).$$

Proof. Again, let $B \in \Psi^{-m}_\infty(\mathbb{R}^n)$ be the elliptic parametrix so that $E := BA - 1 \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. We know that $B : H^s \to H^{s+m}$ and $E : H^N \to H^{s+m}$ are bounded linear map. Using $u = BAu + Eu$, we have

$$\|u\|_{H^{s+m}} \leqslant \|BAu\|_{H^{s+m}} + \|u\|_{H^{s+m}} \leqslant C\left(\|Au\|_{H^s} + \|u\|_{H^N}\right)$$

for some $C > 0$. ∎

Based on previous discussion, we have thus establish that on compact manifold or $\mathbb{R}^n$, elliptic pseudos are Fredholm $H^s \to H^{s-m}$. A practical implication is that, if we know that f is smooth, then u is automatically smooth!

D Non-Elliptic problems

Sadly, the situation is more complicated for non-elliptic pseudos. To construct Fredholm problem for non-elliptic operators, such as the wave operator, we will need two more ingredients: microlocal elliptic estimate and propagation of singularities estimate.
picture of propagation estimate
light cone on torus

Definition D.1. *A pseudodifferential operator, $A \in \Psi^m_\infty(\mathbb{R}^n), m \in \mathbb{R}$ is* ***elliptic at a point*** $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ *if there exist $\epsilon > 0$ such that its left-reduced symbol satisfies the lower bound*

$$|\,\sigma_L(A)(x,\xi)\,| \geqslant \epsilon \,\langle \xi \rangle^m$$

in the region

$$\overline{U}_\epsilon = \left\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \,:\, |\,x - x_0\,| \leqslant \epsilon, \left|\,\widehat{\xi} - \widehat{\xi}_0\,\right| \leqslant \epsilon, |\,\xi\,| \geqslant 1/\epsilon\right\}$$

where $\widehat{\xi} = \xi/\,|\,\xi\,|$ denotes the unit vector in the direction of ξ for any non-zero $\xi \in \mathbb{R}^n$. We denote the set of all elliptic points of A as

$$Ell^m(A) = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \,:\, A \text{ is elliptic of order } m \text{ at } (x,\xi)\}$$

and its complement in $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ as

$$\begin{aligned}\Sigma^m(A) &= Ell^m(A)^c \setminus \{(x,0) \,:\, x \in \mathbb{R}^n\} \\ &= \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \,:\, A \text{ is } \textbf{not} \text{ elliptic of order } m \text{ at } (x,\xi)\}\end{aligned}$$

The notion of wavefront set, WF, is central to microlocal analysis. Roughly, given a distribution u, a point $(x,\xi) \in T^*\mathbb{R}^n$ is **not** in the the wavefront set $\mathrm{WF}(u)$ if there exist a "microlocal cut-off" $A \in \Psi^0_\infty(\mathbb{R}^n)$, such that Au is smooth. The role of A is to locallise around x in the base space and conically locallise around ξ, i.e. we care only of the direction (hence $\xi = 0$ would not be considered).

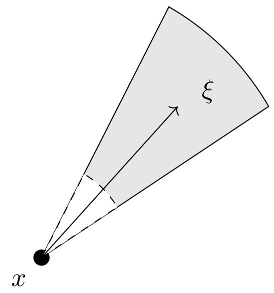

Figure 1: Microlocalisation in phase space.

Definition D.2. *The* ***wavefront set*** *of a compactly supported tempered distribution*

$$u \in C_c^{-\infty}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \,:\, \mathrm{supp}(u) \Subset \mathbb{R}^n\}$$

is given by

$$\mathrm{WF}(u) = \bigcap \left\{\Sigma^0(A) \,:\, A \in \Psi^0_\infty(\mathbb{R}^n), Au \in C^\infty(\mathbb{R}^n)\right\}.$$

For general tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$, its wavefront set is given by

$$\mathrm{WF}(u) = \bigcup_{\chi \in C_c^\infty(\mathbb{R}^n)} \mathrm{WF}(\chi u).$$

E Fredholm problem for the wave operator on torus

Definition E.1. *Let $\mathbb{S}^1 = [0,1]/(0 \sim 1)$ denote the circle and for any $k \in \mathbb{N}$ let*

$$\mathbb{T}^k := \underbrace{\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{k}$$

denote the k-dimensional torus. We shall study the totally periodic wave operator, on $M :== \mathbb{T}_t^1 \times \mathbb{T}_x^n$ given by

$$\Box := \partial_t^2 - \sum_{j=1}^{n-1} \partial_{x_j}^2 \tag{1}$$

where $(t, x_1, \ldots, x_n)$ are the local coordinates on M.

F Notes:

"propagation of singularity" is one of the most basic phenomenon in non-elliptic setting (connected microlocal energy esitmates). If $P \in \Psi^m_\infty(\mathbb{R}^n)$, with real homogeneous principal symbol

- $\mathrm{WF}^s(u)$ measures if u is microlocally in H^s. $\alpha \notin \mathrm{WF}^s(u) \iff$ there is $A \in \Psi^0_\infty(\mathbb{R}^n)$ elliptic at α and $Au \in H^s$.
- Away from $\Sigma^m(P)$, we have microlocal elliptic regularity, i.e. $\mathrm{WF}^s(u) \setminus \Sigma(P) \subset \mathrm{WF}^{s-m}(Pu)$.
- In $\Sigma^m(P) \setminus \mathrm{WF}^{s-m+1}(Pu)$, however, $\mathrm{WF}^s(u)$ is a union of maximally extended bicharacteristics. (Closed graph theorem gives estimates, estimates about both regularity and decay) (complex absorption is an option to control the initial regularity estimate on the right hand side of the inequality.)

Generalisation to Pseudodifferential operators. Keeping concepts like symbols and principal symbols, wave front set. Theorems go through. Generalisation to manifold (problem with compactness. Need decay). Wave operator are typical of hyperbolic equations.

F.1 alternative for elliptic differential operators

To illustrate, we consider constant coefficient differential operator of order m

$$P = \sum_{|\alpha| \leqslant m} c_\alpha D_x^\alpha, \quad c_\alpha \in \mathbb{C}$$

It can be shown that an m^{th} order differential operator defines a continuous (i.e. bounded) map

$$P : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$$

that is, we decrease the order of regularity by m. The symbol of P is the the polynomial

$$p(\xi) = \sum_{|\alpha| \leqslant m} c_\alpha \xi^\alpha$$

which is precisely the polynomial that makes the following square commutes i.e.

$$\mathcal{F}Pu = p(\xi)\mathcal{F}u \quad u \in \mathcal{S}(\mathbb{R}^n)$$

Question: When is the map above surjective? Answer: When it is elliptic.

Theorem F.1. *Let P be an order m differential operator. If for some $\epsilon > 0$, $p(\xi)$ above satisfies*

$$|p(\xi)| \geqslant \epsilon \left(1 + |\xi|^2\right)^{m/2} \qquad \text{whenever } |\xi| > 1/\epsilon,$$

we say $p(\xi)$ and therefore P are elliptic of order m and as a map

$$P : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$$

P is surjective.

show graph of $p(\xi)$ bounded below by polynomial outside of a initial compact set

The statement is actually a statement of the principal part of the symbol only.

Lemma F.2. *A polynomial of order m, $p(\xi) = \sum_{|\alpha| \leqslant m} c_\alpha \xi^\alpha$ is elliptic of of order m if and only if its leading part is never zero except perhaps at $\xi = 0$, i.e.*

$$p_m(\xi) := \sum_{|\alpha| = m} c_\alpha \xi^\alpha \neq 0$$

for any $\xi \neq 0$.