1 Polyhomogeneity of Riemann Map for polygonal region

Theorem 1.1 (Riemann Mapping Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a simply connected region which is not the whole plane and $z_0 \in \Omega$. There exists a unique one-to-one analytic function $f : \Omega \to D$, with $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$ being the open unit disk, such that $f(z_0) = 0$ and $f'(z_0) > 0$.*

It can also be shown that if the boundary, $\partial\Omega$ of the region is a Jordan Curve, the Riemann Map can be extended to an analytic one-to-one function on $\overline{\Omega}$ onto the closed unit disk, i.e. $f : \overline{\Omega} \to \overline{D}$. When extended, map $f : \Omega \to D$, simply by virtue of being a topological map (i.e. homeomorphism), will map boundary to boundary.

1.1 Riemann Map for polygonal region

In this section we shall exhibit an explicit formula for the (inverse of) Riemann map for a polygonal region $\Omega \subset \mathbb{C}$. An n-gon can be specified by an ordered sequence of n distinct complex numbers $(z_k)_{1 \leqslant k \leqslant n}$. We shall let $(\alpha_k \, \pi)_{1 \leqslant k \leqslant n}$ denote the interior angles at z_k, and $(\beta_k \, \pi)$ the corresponding exterior angles. Since the (extended) Riemann Map will map boundary to boundary, the points z_k will be mapped to $w_k \in S^1 \subset \overline{D}$. With these notations in place, we shall give the following formula for the conformal of Ω to D.

Theorem 1.2 (Schwarz-Christoffel Formula)**.** *The function $z = F(w)$ which map D, the open unit disk, conformally onto an n-gon defined by $(z_k)_{1 \leqslant k \leqslant n}$ with exterior angles $(\beta_k \, \pi)_{1 \leqslant k \leqslant n}$ is given by*

$$F(w) = C \int_0^w \prod_{k=1}^{n} (\eta - w_k)^{-\beta_k} \, d\eta + C' \tag{1}$$

for some $C, C' \in \mathbb{C}$, with $z_k = \lim_{w \to w_k} F(w)$.

1.2 Polyhomegeity

In order to understand the behaviour of the conformal map as we approach a corner of the polygon, we shall seek asymptotic expansion of the map F in terms of r, the distance from a particular $w_l \in \{w_1, w_2, \ldots, w_n\}$. To this end, we shall apply the following theorem repeatedly.

Theorem 1.3 (Generalised Binomial Theorem)**.** *For $x, \alpha \in \mathbb{C}$,*

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

where

$$\binom{\alpha}{k} = \frac{(\alpha)_k}{k!} = \frac{(\alpha)(\alpha-1)\ldots(\alpha-k+1)}{k!}.$$

The series is absolutely convergent for any $\alpha \in \mathbb{C}$ if $|x| < 1$.

Let $\alpha = w_l - re^{i\theta}$, where $r \in \mathbb{R}_{>0}$ and $\theta \in [-\pi, \pi]$ are such that α stays in the open unit disk, i.e. $|\alpha| < 1$. Let $I(w)$ denote the integral in the Schwarz-Christoffel Formula. Observe that

$$\begin{aligned}
I(\alpha) &= \int_0^\alpha \prod_{k=1}^{n} (\eta - w_k)^{-\beta_k} \, d\eta \\
&= \alpha \int_0^1 \prod_{k=1}^{n} (\alpha t - w_k)^{-\beta_k} \, dt \qquad \text{using a radial path } t \mapsto \alpha t \\
&= \alpha \left(\prod_{k=1}^{n} (-w_k)^{-\beta_k} \right) \int_0^1 \prod_{k=1}^{n} \left(1 - \frac{\alpha t}{w_k}\right)^{-\beta_k} dt \\
&= \alpha \left(\prod_{k=1}^{n} (-w_k)^{-\beta_k} \right) \int_0^1 \prod_{k=1}^{n} \sum_{j_k=0}^{\infty} \binom{-\beta_k}{j_k} \left(-\frac{\alpha t}{w_k}\right)^{j_k} dt
\end{aligned}$$

using the Generalised Binomial theorem, with

$$\binom{-\beta_k}{j_k} = \frac{(-\beta_k)_{j_k}}{j_k!} = \frac{(-\beta_k)(-\beta_k-1)\dots(-\beta_k-j_k+1)}{j_k!}.$$

We note that $\left|-\frac{\alpha t}{w_k}\right| < 1$, and thus the series is absolutely convergent. Taking finite product of absolutely convergent series, we get

$$I(\alpha) = \alpha\left(\prod_{k=1}^{n}(-w_k)^{-\beta_k}\right)\int_0^1 \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_n=0}^{\infty}\frac{(-\beta_1)_{j_1}(-\beta_2)_{j_2}\dots(-\beta_n)_{j_n}}{j_1!j_2!\dots j_n!}w_1^{-j_1}w_2^{-j_2}\dots w_n^{-j_n}(-\alpha t)^{j_1+j_2+\cdots+j_n}\,dt$$

The absolute value of each summand of the series in the integrand is dominated by the corresponding summand with $t=1$. Since, the resulting series with $t=1$ is again absolutely convergent, the series is uniformly convergent (by Weierstrass M-test). Therefore, we can Interchanging sums and integral and get

$$\begin{aligned}
I(\alpha) &= \alpha\left(\prod_{k=1}^{n}(-w_k)^{-\beta_k}\right)\sum_{(j_1,\dots,j_n)\in\mathbb{N}^n}\frac{(-\beta_1)_{j_1}\dots(-\beta_n)_{j_n}}{j_1!\dots j_n!}w_1^{-j_1}\dots w_n^{-j_n}(-\alpha)^{j_1+j_2+\cdots+j_n}\int_0^1 t^{j_1+\cdots+j_n}\,dt\\
&= \alpha\left(\prod_{k=1}^{n}(-w_k)^{-\beta_k}\right)\sum_{(j_1,\dots,j_n)\in\mathbb{N}^n}\frac{(-\beta_1)_{j_1}\dots(-\beta_n)_{j_n}}{j_1!\dots j_n!}\frac{w_1^{-j_1}\dots w_n^{-j_n}}{j_1+j_2+\cdots+j_n+1}(-\alpha)^{j_1+j_2+\cdots+j_n}\\
&= \sum_{(j_1,\dots,j_n)\in\mathbb{N}^n}\beta_{j_1j_2\dots j_n}\alpha^{j_1+\cdots+j_n+1}
\end{aligned}$$

where

$$\beta_{j_1\dots j_n} = \left(\prod_{k=1}^{n}(-w_k)^{-\beta_k}\right)\frac{(-\beta_1)_{j_1}\dots(-\beta_n)_{j_n}}{j_1!\dots j_n!}\frac{w_1^{-j_1}\dots w_n^{-j_n}}{j_1+j_2+\cdots+j_n+1}(-1)^{j_1+j_2+\cdots+j_n}.$$

The series is absolutely convergent for $|\alpha|<1$. We can reenumerate $\mathbb{N}^n$ and rearrange the sum to be

$$\begin{aligned}
I(\alpha) &= \sum_{m=0}^{\infty}\beta_m\alpha^{m+1}\\
\beta_m &= \sum_{j_1+\dots j_n=m}\beta_{j_1\dots j_n}
\end{aligned}$$

which give us

$$\begin{aligned}
I(\alpha) &= \sum_{m=0}^{\infty}\beta_m\left(w_l - re^{i\theta}\right)^{m+1}\\
&= \sum_{m=0}^{\infty}\beta_m w_l^{m+1}\left(1-\frac{re^{i\theta}}{w_l}\right)^{m+1}\\
&= \sum_{m=0}^{\infty}\beta_m w_l^{m+1}\sum_{s=0}^{m+1}\binom{m+1}{s}\left(-\frac{re^{i\theta}}{w_l}\right)^{s}\\
&= \sum_{k=0}^{\infty}\gamma_k\, r^k
\end{aligned}$$

with

$$\gamma_k = \sum_{m=0}^{\infty}\beta_m w_l^{m+1}\binom{m+1}{k}\left(-\frac{e^{i\theta}}{w_l}\right)^{k}.$$

We have thus exhibit I and therefore F as a convergent power series of r. [1]

[1] This is perhaps expected since we have proven that the Riemann map extended to an analytic function on $\overline{\Omega}$.