

Microlocal Analysis Seminar

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1 Reminder: definitions and notations

1.1 Symbols

We shall here list the definition of the space of symbols of order $m \in \mathbb{N}$ in Euclidean space \mathbb{R}^n that one encounters in the literature. The main motivation is based on the property of linear differential operators of order $m \in \mathbb{N}$ with smooth coefficient that, after Fourier transform gives the polynomial of ξ with smooth coefficient

$$P(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha.$$

Definition 1.1. The **space of symbols of order m** , denoted $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$, is the space of smooth functions $a \in C^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ such that for all multi-index $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} \langle \xi \rangle^{m - |\beta|}$$

uniformly on $\mathbb{R}^p \times \mathbb{R}^n$. We can also defined the space of symbol, $S_\infty^m(\Omega; \mathbb{R}^n)$ on a set with non-empty interior $\Omega \subset \mathbb{R}^p$, $\Omega \subset \overline{\text{Int}(\Omega)}$ such that the bound above is satisfied uniformly in $(x, \xi) \in \text{Int}(\Omega) \times \mathbb{R}^n$. The subscript ∞ refers the uniform boundedness condition in x . Together with the family of seminorms (indexed by $N \in \mathbb{N}$)

$$\|a\|_{N, m} = \sup_{(x, \xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \max_{|\alpha| + |\beta| \leq N} \frac{D_x^\alpha D_\xi^\beta a(x, \xi)}{\langle \xi \rangle^{m - |\beta|}} \quad (1)$$

gives a Frechet topology to $S_\infty^m(\Omega; \mathbb{R}^n)$.

Furthermore, we define the total symbol space as

$$S_\infty^\infty(\Omega; \mathbb{R}^n) = \bigcup_{m \in \mathbb{R}} S_\infty^m(\Omega; \mathbb{R}^n)$$

and the residual *residual* space of the filtration as

$$S_\infty^{-\infty}(\Omega; \mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} S_\infty^m(\Omega; \mathbb{R}^n).$$

Note: In defining pseudodifferential operators, we shall focus on the case where $p = 2n$, i.e. $a(x, y, \xi) \in S_\infty^m(\mathbb{R}_{x, y}^{2n}; \mathbb{R}^n)$.

1.2 Quantisation

Pseudodifferential operators are defined using symbols. The main gadget is the following oscillatory integral:

$$S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n) \ni a \mapsto I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) d\xi \quad (2)$$

with action on Schwartz functions $u \in S(\mathbb{R}^n)$ given by

$$I(a)(u) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) u(y) dy d\xi. \quad (3)$$

The integral 3 above might be divergent unless $m < -n$, but it can be interpreted as a tempered distribution, i.e. a linear function on $S(\mathbb{R}^n)$, with action

$$S(\mathbb{R}^n) \ni v \mapsto I(a)(u)(v) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) u(y) v(x) dy d\xi dx \in \mathbb{C}. \quad (4)$$

The process of turning the symbol a into an operator $S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ is known as the quantisation procedure. The goal of this talk is the following:

Goal :

To establish that the procedure above is well-defined, so that for each $a \in S_{\infty}^m(\mathbb{R}^{2n}; \mathbb{R}^n)$

$$I(a) : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$$

$$u \mapsto I(a)(u) : S(\mathbb{R}^n) \rightarrow \mathbb{C}$$

$$v \mapsto I(a)(uv) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) u(y) v(x) \, dy \, d\xi \, dx$$

is a continuous linear map between Frechet spaces.

Remark. Given $a \in S_{\infty}^m(\mathbb{R}^{2n}; \mathbb{R}^n)$, we sometimes write $A = Op(a) = I(a)$ for the operator $S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ defined by quantising the symbol a . Also, once the procedure above is proven to be well-defined, we will, with abuse of notation, identify the integral

$$I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) \, d\xi \in S'(\mathbb{R}^n \times \mathbb{R}^n)$$

to be the *Schwartz Kernel* of the operator $I(a) : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$.

2 Properties of Symbols

In this section, we shall establish the following summarising theorem:

Theorem 2.1 (Summary). *Given $m \in \mathbb{R}$, $p, n \in \mathbb{N}$, then*

1. $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ is a Frechet space, hence completely metrisable.
2. $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ is a graded commutative $*$ -algebra over \mathbb{C} with continuous inclusion

$$\iota : S_\infty^m(\mathbb{R}^p; \mathbb{R}^n) \rightarrow S_\infty^{m'}(\mathbb{R}^p; \mathbb{R}^n)$$

for all $m \leq m'$.

3. $S_\infty^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$ is dense in $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ in the topology of $S_\infty^{m+\epsilon}(\mathbb{R}^p; \mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$.

Exercise : Show that symbol spaces are Frechet spaces. That is, show that the family of seminorms in 1 separates points and that if a sequence is Cauchy in each seminorm, then there exist a unique symbol where the sequence converges in each seminorm.

2.1 $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ is a graded commutative $*$ -algebra with continuous inclusion

We first prove continuous inclusion of lower order into higher order symbol space.

Proposition 2.2. *Let $p, n \in \mathbb{N}$ be given and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\text{Int}(\Omega)}$. If $m, m' \in \mathbb{R}$ such that $m \leq m'$, then $S_\infty^m(\Omega; \mathbb{R}^n) \subset S_\infty^{m'}(\Omega; \mathbb{R}^n)$. Furthermore, the inclusion map*

$$\iota : S_\infty^m(\Omega; \mathbb{R}^n) \rightarrow S_\infty^{m'}(\Omega; \mathbb{R}^n)$$

is continuous.

Proof. Let the real numbers $m \leq m'$ be given. We note that for any $\xi \in \mathbb{R}^n$

$$\langle \xi \rangle^m \leq 1 \cdot \langle \xi \rangle^{m'}$$

and thus if $a \in S_\infty^m(\Omega; \mathbb{R}^n)$, we have that $\forall \alpha \in \mathbb{N}^p, \forall \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x, \xi) \right| \leq C \langle \xi \rangle^{m-|\beta|} \leq C \langle \xi \rangle^{m'-|\beta|}$$

which show that $a \in S_\infty^{m'}(\Omega; \mathbb{R}^n)$ as well.

To show that ι is a continuous inclusion, it suffices to show that

$$\|\iota(a)\|_{N, m'} \leq C \|a\|_{N, m}$$

for any $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ and $N \in \mathbb{N}$. Indeed, this bound holds since

$$\frac{D_x^\alpha D_\xi^\beta a(x, \xi)}{\langle \xi \rangle^{m'-|\beta|}} \leq \frac{D_x^\alpha D_\xi^\beta a(x, \xi)}{\langle \xi \rangle^{m-|\beta|}}.$$

□

Proposition 2.3. *Let $p, n \in \mathbb{N}$ be given. Let $\Omega \subset \mathbb{R}^p$ be such that $\Omega \subset \overline{\text{Int}(\Omega)}$. Then, for any $m, m' \in \mathbb{R}$, we have*

$$S_{\infty}^m(\Omega; \mathbb{R}^n) \cdot S_{\infty}^{m'}(\Omega; \mathbb{R}^n) = S_{\infty}^{m+m'}(\Omega; \mathbb{R}^n)$$

Proof. Let $a \in S_{\infty}^m(\Omega; \mathbb{R}^n)$ and $b \in S_{\infty}^{m'}(\Omega; \mathbb{R}^n)$ be given. By (general) Leibniz formula, we have that for all multi-index α, β ,

$$\begin{aligned} \sup_{(x, \xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{\left| D_x^{\alpha} D_{\xi}^{\beta} a(x, \xi) b(x, \xi) \right|}{\langle \xi \rangle^{(m+m')-|\beta|}} &\leq \sum_{\mu \leq \alpha, \gamma \leq \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} \sup_{(x, \xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{\left| D_x^{\mu} D_{\xi}^{\gamma} a(x, \xi) \right| \left| D_x^{\alpha-\mu} D_{\xi}^{\beta-\gamma} b(x, \xi) \right|}{\langle \xi \rangle^{(m+m')-|\beta|}} \\ &\leq \sum_{\mu \leq \alpha, \gamma \leq \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} C \sup_{\xi \in \mathbb{R}^n} \frac{\langle \xi \rangle^{m-|\gamma|} \langle \xi \rangle^{m'-|\beta-\gamma|}}{\langle \xi \rangle^{(m+m')-|\beta|}} \\ &= \sum_{\mu \leq \alpha, \gamma \leq \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} C \sup_{\xi \in \mathbb{R}^n} \langle \xi \rangle^{|\beta|-(|\beta-\gamma|+|\gamma|)} \\ &\leq \sum_{\mu \leq \alpha, \gamma \leq \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} C \\ &< \infty \end{aligned}$$

where we have use the property of multi-index that $|\beta| = |\beta - \mu| + |\mu|$. We have thus shown that $S_{\infty}^m(\Omega; \mathbb{R}^n) \cdot S_{\infty}^{m'}(\Omega; \mathbb{R}^n) \subset S_{\infty}^{m+m'}(\Omega; \mathbb{R}^n)$

For the reverse inclusion, let $c \in S_{\infty}^{m+m'}(\Omega; \mathbb{R}^n)$ be given. Define

$$\begin{aligned} a &: (x, \xi) \mapsto \langle \xi \rangle^m \\ b &: (x, \xi) \mapsto \frac{c(x, \xi)}{a(x, \xi)} \end{aligned}$$

and observe that

- $a \in S_{\infty}^m(\Omega; \mathbb{R}^n)$. It is clear that a is smooth in both x and ξ . It is independent of x and thus any x derivative gives 0. We need only to check that for all $\beta \in \mathbb{N}^n$,

$$\left| D_{\xi}^{\beta} \langle \xi \rangle^m \right| \leq C \langle \xi \rangle^{m-|\beta|}$$

which can be proven by induction on n and β . We shall only prove the base case where $n = 1$ and $\beta = 1$. We have

$$\left| D_{\xi} \langle \xi \rangle^m \right| = \left| \partial_{\xi} (1 + \xi^2)^{m/2} \right| = \left| m \xi \langle \xi \rangle^{m-2} \right| = \left| m \frac{\xi}{\langle \xi \rangle} \right| \langle \xi \rangle^{m-1} \leq |m| \langle \xi \rangle^{m-1}$$

where we have used the fact that $|\xi| \leq \langle \xi \rangle$ for all ξ .

- $b \in S_{\infty}^{m'}(\Omega; \mathbb{R}^n)$. We note first that $\langle \xi \rangle^m \neq 0$ for all $\xi \in \mathbb{R}^n$ and thus b is well-defined. Since division by $\langle \xi \rangle^m$ does not affect any of the x derivative, we only need to show that for any $\beta \in \mathbb{N}^n$, we have

$$\left| D_{\xi}^{\beta} b(x, \xi) \right| \leq C \langle \xi \rangle^{m+m'-|\beta|}$$

for some constant $C > 0$ uniformly in ξ . Indeed, observe that by the Leibinz formula

$$\begin{aligned}
\left| D_\xi^\beta b(x, \xi) \right| &\leq \sum_{\mu \leq \beta} \binom{\beta}{\mu} \left| D_\xi^\mu c(x, \xi) \right| \left| D^{\beta-\mu} \langle \xi \rangle^{-m} \right| \\
&\leq C \sum_{\mu \leq \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m+m'-|\mu|} \langle \xi \rangle^{-m-|\beta-\mu|} \\
&\leq C \sum_{\mu \leq \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m'-(|\mu|+|\beta-\mu|)} \\
&= C \sum_{\mu \leq \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m'-|\beta|} \\
&= C 2^\beta \langle \xi \rangle^{m'-|\beta|}
\end{aligned}$$

where we have use the definition of c and applied the result proven for a with $m \mapsto -m$. Thus, $b \in S_\infty^{m'}(\Omega; \mathbb{R}^n)$.

It is clear that $a \cdot b = c$ and we have therefore shown that $S_\infty^{m+m'}(\Omega; \mathbb{R}^n) \subset S_\infty^m(\Omega; \mathbb{R}^n) \cdot S_\infty^{m'}(\Omega; \mathbb{R}^n)$. \square

The results above, together with the easily proven fact $a^*(x, \xi) := \overline{a(x, \xi)} \in S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n) \iff a \in S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n)$, gives the desired algebraic structure for $S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n)$

2.2 Density of residual space, $S_\infty^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$

Next, we have a rather technical density result : the residual space, $S_\infty^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$, is dense in $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$, but only with the topology of $S_\infty^{m+\epsilon}(\mathbb{R}^p; \mathbb{R}^n)$. The main strategy in this proof is to approximate any symbol with the very same symbol but cut off by a compactly supported function. As such, the main reason we **cannot** have density of $S_\infty^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$ in $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ is the same reason to the fact that Schwartz functions are not dense in the space of smooth bounded functions, in particular, $1 \in S_\infty^0(\Omega; \mathbb{R}^n)$ is not in the closure of $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$.

Lemma 2.4. *Given any $m \in \mathbb{R}$, $n, p \in \mathbb{N}$ and $a \in S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$, there exist a sequence in $S_\infty^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$ that is bounded in $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ and converges to a in the topology of $S_\infty^{m+\epsilon}(\mathbb{R}^p; \mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$.*

Proof. Let $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ and $\epsilon \in \mathbb{R}_{>0}$ be given. Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be a non-negative smooth cut-off function, i.e. $\chi \geq 0$ and satisfies

$$\chi(\xi) = \begin{cases} 1 & , |\xi| < 1 \\ 0 & , |\xi| > 2. \end{cases}$$

Then, for each $k \in \mathbb{N}$, we define

$$a_k(x, \xi) = \chi\left(\frac{\xi}{k}\right) a(x, \xi).$$

Now, given arbitrary $N, k \in \mathbb{N}$, observe that

$$|a_k| \leq C \langle \xi \rangle^{-N}$$

since a_k is compactly supported in ξ (as χ is compactly supported). Furthermore, by Leibinz formula and the symbol estimates on $a \in S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$, we have

$$\begin{aligned}
\left| D_x^\alpha D_\xi^\beta a_k(x, \xi) \right| &\leq \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} \left(D_\xi^\mu \chi \right) \left(\frac{\xi}{k} \right) \left| D_x^\alpha D_\xi^{\beta-\mu} a(x, \xi) \right| \\
&\leq C \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} \left(D_\xi^\mu \chi \right) \left(\frac{\xi}{k} \right) \langle \xi \rangle^{m-|\beta-\mu|}.
\end{aligned}$$

Since χ and all its derivatives are compactly supported, each term above is bounded in ξ and thus a_k is bounded in $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ and that

$$\left| D_x^\alpha D_\xi^\beta a_k(x, \xi) \right| \leq C' \langle \xi \rangle^{-N}$$

which allow us to conclude that $a_k \in S_\infty^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$.

It remains to show that $\lim_{k \rightarrow \infty} a_k = a$ in $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$. In the first symbol norm, we observe that, using the symbol estimate for a

$$\begin{aligned} \|a_k - a\|_{0, m+\epsilon} &= \sup_{(x, \xi) \in \mathbb{R}^p \times \mathbb{R}^n} \frac{|a_k(x, \xi)|}{\langle \xi \rangle^{m+\epsilon}} \\ &= \sup_{(x, \xi) \in \mathbb{R}^p \times \mathbb{R}^n} \frac{|(1 - \chi(\xi/k))| |a(x, \xi)|}{\langle \xi \rangle^{m+\epsilon}} \\ &\leq \|a\|_{0, m} \sup_{\xi \in \mathbb{R}^n} \frac{|(1 - \chi(\xi/k))|}{\langle \xi \rangle^\epsilon} \\ &\leq \|a\|_{0, m} \langle k \rangle^{-\epsilon} \\ &\rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$, since $|1 - \chi(\xi/k)|$ is 0 in the region $|\xi| \leq k$ and bounded by 1 otherwise. We remark upon the necessity of the extra decay given by $\langle \xi \rangle^{-\epsilon}$ factor. For other symbol norms we shall again use Leibinz formula:

$$\begin{aligned} \sup_{(x, \xi) \in \mathbb{R}^p \times \mathbb{R}^n} \frac{|D_x^\alpha D_\xi^\beta a_k(x, \xi)|}{\langle \xi \rangle^{m+\epsilon-|\beta|}} &\leq \sup_{(x, \xi) \in \mathbb{R}^p \times \mathbb{R}^n} \frac{1}{\langle \xi \rangle^{m+\epsilon-|\beta|}} \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu(1 - \chi)) \left(\frac{\xi}{k} \right) |D_x^\alpha D_\xi^{\beta-\mu} a(x, \xi)| \\ &\leq \sup_{(x, \xi) \in \mathbb{R}^p \times \mathbb{R}^n} \frac{C}{\langle \xi \rangle^{m+\epsilon-|\beta|}} \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu(1 - \chi)) \left(\frac{\xi}{k} \right) \langle \xi \rangle^{m-|\beta-\mu|} \\ &= C \sup_{\xi \in \mathbb{R}^n} \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu(1 - \chi)) \left(\frac{\xi}{k} \right) \langle \xi \rangle^{-\epsilon-|\mu|} \\ &\leq C' k^{-\epsilon} \\ &\rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$ by the same argument as before. Thus, we have proven that $a_k \rightarrow a$ as $k \rightarrow \infty$ in $S_\infty^{m+\epsilon}(\mathbb{R}^p; \mathbb{R}^n)$. \square

3 Quantisation

3.1 Continuity of $I(a) : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$

We first note that, if $m < -n$ (write $m = -n - \epsilon$ for some $\epsilon > 0$), the oscillatory integral 3, is absolutely convergent and defines a continuous linear operator

$$I : S_{\infty}^{-n-\epsilon}(\mathbb{R}^{2n}; \mathbb{R}^n) \rightarrow S'(\mathbb{R}^{2n})$$

$$a \mapsto I(a) : S(\mathbb{R}^{2n}) \ni \varphi \mapsto I(a)(\varphi) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) \varphi(x, y) d\xi dx dy.$$

The map above is clearly linear. Continuity comes from the bound given by the following computation: $\forall M \in \mathbb{N}, \forall a \in S_{\infty}^{-n-\epsilon}(\mathbb{R}^{2n}; \mathbb{R}^n), \forall \varphi \in S(\mathbb{R}^n)$

$$\begin{aligned} |I(a)(\varphi)| &\leq \frac{1}{(2\pi)^n} \int |a(x, y, \xi) \varphi(x, y)| d\xi dx dy \\ &\leq \frac{\|a\|_{0, -n-\epsilon}}{(2\pi)^n} \int \langle \xi \rangle^{-n-\epsilon} \langle (x, y) \rangle^{-M} \langle (x, y) \rangle^M |\varphi(x, y)| d\xi dx dy \\ &\leq \frac{\|a\|_{0, -n-\epsilon} \|\varphi\|_M}{(2\pi)^n} \int \langle \xi \rangle^{-n-\epsilon} \langle (x, y) \rangle^{-M} d\xi dx dy \end{aligned}$$

for any $M \in \mathbb{N}$, where

$$\|\varphi\|_M := \sum_{|\alpha| \leq M} \sup_{(x, y) \in \mathbb{R}^{2n}} \langle (x, y) \rangle^M |D_{x, y}^{\alpha} \varphi(x, y)| \quad (5)$$

is the Schwartz seminorm on $S(\mathbb{R}^{2n})$. If we choose $M \geq 2n + 1$, the x, y integrals are convergent and since $m = -n - \epsilon < -n$, the ξ integral converges as well, hence we have

$$|I(a)(\varphi)| \leq C \|a\|_{0, m} \|\varphi\|_M$$

which implies continuity.

The proposition below extend this result to general $m \in \mathbb{R}$.

Proposition 3.1. *The continuous linear map*

$$I : S_{\infty}^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n) \rightarrow S'(\mathbb{R}^{2n})$$

extends uniquely to a linear map

$$I : S_{\infty}^m(\mathbb{R}^{2n}; \mathbb{R}^n) \rightarrow S'(\mathbb{R}^{2n})$$

which is continuous as linear map from $S_{\infty}^{m'}(\mathbb{R}^{2n}; \mathbb{R}^n)$ to $S'(\mathbb{R}^{2n})$ for arbitrary $m \in \mathbb{R}$ and $m' > m$.

Proof. Let $m, m' \in \mathbb{R}, n \in \mathbb{N}$ with $m < m'$ be given. For any $a \in S_{\infty}^m(\mathbb{R}^{2n}; \mathbb{R}^n)$, the density of $S_{\infty}^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ in $S_{\infty}^m(\mathbb{R}^{2n}; \mathbb{R}^n)$ with the topology of $S_{\infty}^{m'}(\mathbb{R}^{2n}; \mathbb{R}^n)$ means that there exist a sequence $a_k \in S_{\infty}^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ so that $a_k \rightarrow a \in S_{\infty}^{m'}(\mathbb{R}^{2n}; \mathbb{R}^n)$. Together with the completeness of $S'(\mathbb{R}^{2n})$ (being a continuous linear map into \mathbb{C} which is complete), we have unique continuous linear extension of $I : S_{\infty}^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n) \rightarrow S'(\mathbb{R}^{2n})$ to $S_{\infty}^m(\mathbb{R}^{2n}; \mathbb{R}^n)$ given by

$$I(a) := \lim_{k \rightarrow \infty} I(a_k)$$

which is continuous in the $S_{\infty}^{m'}(\mathbb{R}^{2n}; \mathbb{R}^n)$ topology. Therefore, it is enough to show that for any $a \in S_{\infty}^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ and $\varphi \in S(\mathbb{R}^{2n})$, there exist $N, M \in \mathbb{N}$, such that

$$|I(a)(\varphi)| \leq C \|a\|_{N, m'} \|\varphi\|_M.$$

Let a, φ as above be given. Note that

$$e^{i(x-y)\xi} = \langle \xi \rangle^{-2q} (1 + \xi \cdot D_x)^q e^{i(x-y)\xi} = \langle \xi \rangle^{-2q} (1 - \xi \cdot D_y)^q e^{i(x-y)\xi}.$$

Thus, using integration by parts, for any $q \in \mathbb{N}$,

$$\begin{aligned} I(a)(\varphi) &= \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) \varphi(x, y) \, d\xi \, dx \, dy \\ &= \frac{1}{(2\pi)^n} \int \langle \xi \rangle^{-4q} (1 - \xi \cdot D_y)^q (1 + \xi \cdot D_x)^q e^{i(x-y)\xi} a(x, y, \xi) \varphi(x, y) \, d\xi \, dx \, dy \\ &= \frac{1}{(2\pi)^n} \int \langle \xi \rangle^{-4q} e^{i(x-y)\xi} (1 - \xi \cdot D_y)^q (1 + \xi \cdot D_x)^q [a(x, y, \xi) \varphi(x, y)] \, d\xi \, dx \, dy \\ &= \frac{1}{(2\pi)^n} \int \langle \xi \rangle^{-4q} e^{i(x-y)\xi} \left(\sum_{|\gamma| \leq 2q} a_\gamma(x, y, \xi) D_{x,y}^\gamma \varphi(x, y) \right) \, d\xi \, dx \, dy \end{aligned}$$

where

$$a_\gamma(x, y, \xi) = \sum_{|\mu|, |\nu| \leq q} C_{\mu\nu} \xi^{\mu+\nu} D_x^\mu D_y^\nu a(x, y, \xi)$$

for some combinatorial constants $C_{\mu\nu}$. Now, using the symbol estimate for $a \in S_\infty^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$, and that $|\mu| + |\nu| \leq 2q$

$$\begin{aligned} |a_\gamma(x, y, \xi)| &\leq \sum_{|\mu|, |\nu| \leq q} C_{\mu\nu} |\xi|^{\mu+\nu} |D_x^\mu D_y^\nu a(x, y, \xi)| \\ &= \sum_{|\mu|, |\nu| \leq q} C_{\mu\nu} |\xi|^{\mu+\nu} \langle \xi \rangle^{m'} \frac{|D_x^\mu D_y^\nu a(x, y, \xi)|}{\langle \xi \rangle^{m'}} \\ &\leq \|a\|_{2q, m'} \langle \xi \rangle^{m'} \sum_{|\mu|, |\nu| \leq q} C_{\mu\nu} |\xi|^{\mu+\nu} \\ &\leq \|a\|_{2q, m'} \langle \xi \rangle^{m'} \sum_{|\mu|, |\nu| \leq q} C_{\mu\nu} \langle \xi \rangle^{\mu+\nu} \\ &\leq \|a\|_{2q, m'} \langle \xi \rangle^{m'} \langle \xi \rangle^{2q} \sum_{|\mu|, |\nu| \leq q} C_{\mu\nu} \\ &\leq C_q \|a\|_{2q, m'} \langle \xi \rangle^{m'+2q} \end{aligned}$$

and since $|\gamma| \leq 2q$,

$$\begin{aligned} |D_{x,y}^\gamma \varphi(x, y)| &= \langle (x, y) \rangle^{-2q-2n-1} \langle (x, y) \rangle^{2q+2n+1} |D_{x,y}^\gamma \varphi(x, y)| \\ &\leq \langle (x, y) \rangle^{-2q-2n-1} \sum_{|\alpha| \leq 2q+2n+1} \sup_{(x,y) \in \mathbb{R}^{2n}} \langle (x, y) \rangle^{2q+2n+1} |D_{x,y}^\alpha \varphi(x, y)| \\ &\leq \langle (x, y) \rangle^{-2q-2n-1} \|\varphi\|_{2q+2n+1}. \end{aligned}$$

Bring together both bounds, we have

$$\begin{aligned} |I(a)(\varphi)| &\leq \frac{1}{(2\pi)^n} \int \langle \xi \rangle^{-4q} \left(\sum_{|\gamma| \leq 2q} |a_\gamma(x, y, \xi) D_{x,y}^\gamma \varphi(x, y)| \right) \, d\xi \, dx \, dy \\ &\leq C' \|a\|_{2q, m'} \|\varphi\|_{2q+2n+1} \int \langle \xi \rangle^{-4q} \langle \xi \rangle^{m'+2q} \langle (x, y) \rangle^{-2q-2n-1} \, d\xi \, dx \, dy \\ &= C' \|a\|_{2q, m'} \|\varphi\|_{2q+2n+1} \int \langle \xi \rangle^{m'-2q} \langle (x, y) \rangle^{-2q-2n-1} \, d\xi \, dx \, dy \end{aligned}$$

Thus, as long as $m' - 2q < -n$, i.e. $q > \max\left(\frac{m'+n}{2}, 0\right)$, the integral above converges. Finally, set $N = 2q$, $M = 2q + 2n + 1$, we have

$$|I(a)(\varphi)| \leq C \|a\|_{N,m'} \|\varphi\|_M$$

as required. \square

By the Schwartz Kernel theorem, each $a \in S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n)$ defines a continuous linear operator

$$I(a) : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n).$$

We can now define the space of m -order pseudo-differential operators as the space

$$\Psi_\infty^m(\mathbb{R}^n) := \{A = I(a) \mid a \in S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n)\}$$

with the total space $\Psi_\infty^\infty(\mathbb{R}^n) := \cup_{m \in \mathbb{R}} \Psi_\infty^m(\mathbb{R}^n)$ and the residual space $\Psi_\infty^{-\infty}(\mathbb{R}^n) := \cap_m \Psi_\infty^m(\mathbb{R}^n)$ defined similarly.

3.2 Composition theorem

In this section we shall prove that, just like symbol spaces, $\Psi_\infty^\infty(\mathbb{R}^n)$ forms a graded $*$ -algebra. The difference being, this time, the algebra is *non-commutative*. That is, we shall show that following theorem holds.

Theorem 3.2 (Summary). *Given $n \in \mathbb{N}$, $\Psi_\infty^\infty(\mathbb{R}^n)$ is a graded $*$ -algebra over \mathbb{C} with continuous inclusion*

$$\iota : \Psi_\infty^m(\mathbb{R}^n) \rightarrow \Psi_\infty^{m'}(\mathbb{R}^n)$$

for any $m \leq m'$.

We shall prove this theorem by first accumulate several technical lemmas, of which the most important is the reduction lemma that allow us remove the dependence of either x or y in the symbol $a(x, y, \xi) \in S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n)$.

3.2.1 Asymptotic Summation

Suppose we are given a sequence of symbols with decreasing order, $a_j \in S_\infty^{m-j}(\mathbb{R}^p; \mathbb{R}^n)$, $j \in \mathbb{N}$, we know that $a_j(x, \xi)$ has ever higher rate of decay for large $|\xi|$ with increasing j . However, the series $\sum_{j \in \mathbb{N}} a_j(x, \xi)$ need not converge. However, we have the following notion of asymptotic convergence.

Definition 3.3 (Asymptotic summation). A sequence of symbols with decreasing order, $a_j \in S_\infty^{m-j}(\mathbb{R}^p; \mathbb{R}^n)$, $j \in \mathbb{N}$ is **asymptotically summable** if there exist $a \in S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ such that for all $N \in \mathbb{N}$,

$$a - \sum_{j=0}^{N-1} a_j \in S_\infty^{m-N}(\mathbb{R}^p; \mathbb{R}^n).$$

We write

$$a \sim \sum_{j \in \mathbb{N}} a_j.$$

Lemma 3.4. *Every sequence of symbols with decreasing order is asymptotically summable. Furthermore, the sum is unique up to an additive term in $S_{\infty}^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$.*

Sketch. Let $a_j \in S_{\infty}^{m-j}(\mathbb{R}^p; \mathbb{R}^n)$, $j \in \mathbb{N}$ be given. For uniqueness, suppose $a, a' \in S_{\infty}^m(\mathbb{R}^p; \mathbb{R}^n)$ are both asymptotic sums of the sequence. We need to show that $a - a' \in S_{\infty}^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$. Indeed, for any $N \in \mathbb{N}$,

$$a - a' = \left(a - \sum_{j=0}^{N-1} a_j \right) - \left(a' - \sum_{j=0}^{N-1} a_j \right) \in S_{\infty}^{m-N}(\mathbb{R}^p; \mathbb{R}^n)$$

since both terms on the right are elements of $S_{\infty}^{m-N}(\mathbb{R}^p; \mathbb{R}^n)$. Thus,

$$a - a' \in \bigcap_{n \in \mathbb{N}} S_{\infty}^{m-N}(\mathbb{R}^p; \mathbb{R}^n) = S_{\infty}^{-\infty}(\mathbb{R}^p; \mathbb{R}^n).$$

For existence, we construct $a \in S_{\infty}^m(\mathbb{R}^p; \mathbb{R}^n)$ by Borel's method []. Let $\chi \in C_c^{\infty}(\mathbb{R}^p)$ be a bump function and define

$$a = \sum_{j \in \mathbb{N}} (1 - \chi)(\epsilon_j \xi) a_j(x, \xi)$$

where $\mathbb{R}_{>0} \ni \epsilon_j \rightarrow 0$ is a strictly monotonic decreasing sequence that converges to 0. We note that the sequence is a finite sum for any input (x, ξ) and hence define a smooth function. It remains to show that, for some choice of ϵ_j with sufficiently rapid decay,

$$\sum_{j \geq N} (1 - \chi)(\epsilon_j \xi) a_j(x, \xi)$$

converges in $S_{\infty}^{m-N}(\mathbb{R}^p; \mathbb{R}^n)$ for any $N \in \mathbb{N}$.

Note: This is again an exercise in using symbol seminorms and Leibniz formula. □

3.2.2 Reduction

We will now show that $\Psi_{\infty}^m(\mathbb{R}^n)$ is exactly the range of $I : S_{\infty}^m(\mathbb{R}^{2n}; \mathbb{R}^n) \rightarrow S'(\mathbb{R}^{2n})$ restricted to $S_{\infty}^m(\mathbb{R}^n; \mathbb{R}^n) \subset S_{\infty}^m(\mathbb{R}^{2n}; \mathbb{R}^n)$.

Definition 3.5. Let

$$\pi_L : \mathbb{R}_{x,y,\xi}^{3n} \rightarrow \mathbb{R}_{x,\xi}^{2n}$$

be the projection map $(x, y, \xi) \mapsto (x, \xi)$. We define the **left quantisation map** as

$$q_L := I \circ \pi_L^* : S_{\infty}^m(\mathbb{R}^n; \mathbb{R}^n) \rightarrow \Psi_{\infty}^m(\mathbb{R}^n)$$

with elements $a_L \in S_{\infty}^m(\mathbb{R}^n; \mathbb{R}^n)$ known as the **left reduced symbols**.

Lemma 3.6 (Reduction). *For any $a(x, y, \xi) \in S_{\infty}^m(\mathbb{R}_{x,y,\xi}^{2n}; \mathbb{R}_{\xi}^n)$ there exist unique $a_L(x, \xi) \in S_{\infty}^m(\mathbb{R}^n; \mathbb{R}^n)$ such that $I(a) = q_L(a_L) = I(a_L \circ \pi_L)$. Furthermore, with $\iota : \mathbb{R}^{2n} \ni (x, \xi) \mapsto (x, x, \xi) \in \mathbb{R}^{3n}$ being the diagonal inclusion map, we have*

$$a_L(x, \xi) \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} \iota^* D_y^{\alpha} D_{\xi}^{\alpha} a(x, y, \xi). \quad (6)$$

Sketch. Note that

$$D_{\xi}^{\alpha} e^{i(x-y)\xi} = (x-y)^{\alpha} e^{i(x-y)\xi} \implies I((x-y)^{\alpha} a) = I((-1)^{|\alpha|} D_{\xi}^{\alpha} a)$$

where we have extended the identity that is true using integration by parts in $S_{\infty}^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ to general $S_{\infty}^m(\mathbb{R}^{2n}; \mathbb{R}^n)$ using the density result of symbol space. Now, if we Taylor expand a around the diagonal $x = y$, we get

$$a(x, y, \xi) = \sum_{|\alpha| \leq N-1} \frac{(-i)^{|\alpha|}}{\alpha!} (x - y)^{\alpha} D_y^{\alpha} a(x, x, \xi) + r_N(x, y, \xi)$$

where

$$r_N(x, y, \xi) = \sum_{|\alpha|=N} \frac{(-i)^{|\alpha|}}{\alpha!} (x - y)^{\alpha} \int_0^1 (1-t)^{N-1} D_y^{\alpha} a(x, (1-t)x + ty, \xi) dt$$

for any $N \in \mathbb{N}$. Applying the identity above give us

$$\begin{aligned} I(a) &= \sum_{j=0}^{N-1} A_j + R_N \\ A_j &= I \left(\sum_{|\alpha|=j} \frac{i^{|\alpha|}}{\alpha!} D_y^{\alpha} D_{\xi}^{\alpha} a(x, x, \xi) \right) \in \Psi_{\infty}^{m-j}(\mathbb{R}^n) \\ R_N &\in \Psi_{\infty}^{m-N}(\mathbb{R}^n) \end{aligned}$$

Asymptotic summation lemma give us

$$b(x, \xi) \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} D_y^{\alpha} D_{\xi}^{\alpha} a(x, x, \xi) \in S_{\infty}^m(\mathbb{R}^n; \mathbb{R}^n)$$

so that $I(a) - I(b) \in \Psi_{\infty}^{-\infty}(\mathbb{R}^n)$. It remains to show that $A \in \Psi_{\infty}^{-\infty}(\mathbb{R}^n) \iff A = I(c), c \in S_{\infty}^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)$. \square

3.2.3 Composition theorem

Theorem 3.7 (Composition). *Let $A \in \Psi_{\infty}^m(\mathbb{R}^n)$, $B \in \Psi_{\infty}^{m'}(\mathbb{R}^n)$ for some $m, m' \in \mathbb{R}$. Then,*

1. $A^* \in \Psi_{\infty}^m(\mathbb{R}^n)$.
2. $A \circ B \in \Psi_{\infty}^{m+m'}(\mathbb{R}^n)$.

Sketch. Let $A \in \Psi_{\infty}^m(\mathbb{R}^n)$, $B \in \Psi_{\infty}^{m'}(\mathbb{R}^n)$ for some $m, m' \in \mathbb{R}$ be given. Since $A : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n)$ (??), we have the adjoint operator $A^* : S'(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ defined by

$$A^* u(\varphi) = u(\overline{A\varphi}), \quad u \in S'(\mathbb{R}^n), \varphi \in S(\mathbb{R}^n).$$

We check that $A^* u$ is indeed an element of $S'(\mathbb{R}^n)$ since it is the composition of the maps $u \in S'(\mathbb{R}^n)$ and $S(\mathbb{R}^n) \ni \varphi \mapsto \overline{A\varphi}$ which are both continuous. Let $a \in S_{\infty}^m(\mathbb{R}^{2n}; \mathbb{R}^n)$ be such that $A = I(a)$. Observe that,

$$\begin{aligned} \langle Au, \varphi \rangle_{L^2} &= \int Au(x) \overline{\varphi(x)} dx \\ &= \int u(y) \overline{\int e^{i(x-y)\xi} \overline{a(x, y, \xi)} \varphi(x) dx d\xi} dy \\ &= \int u(y) \overline{I(b)\varphi(y)} dy \\ &= \langle u, A^* \varphi \rangle_{L^2} \end{aligned}$$

where $b(x, y, \xi) = \overline{a(y, x, \xi)} \in S_{\infty}^m(\mathbb{R}^{2n}; \mathbb{R}^n)$. Thus, $A^* \in \Psi_{\infty}^m(\mathbb{R}^n)$.

For composition, applying the reduction lemma twice to get $a_L \in S_\infty^m(\mathbb{R}^n; \mathbb{R}^n)$ and $b_L \in S_\infty^{m'}(\mathbb{R}^n; \mathbb{R}^n)$ so that

$$\begin{aligned} A\varphi(x) &= \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, \xi) \varphi(y) \, dy \, d\xi \\ B^* \varphi(x) &= \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} \overline{b(x, \xi)} \varphi(y) \, dy \, d\xi \end{aligned}$$

which shows that

$$AB\varphi(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, \xi) b(y, \xi) \varphi(y) \, dy \, d\xi$$

and thus $AB = I(a(x, \xi)b(y, \xi))$. Since $a(x, \xi)b(y, \xi) \in S_\infty^{m+m'}(\mathbb{R}^{2n}; \mathbb{R}^n)$, we have the result $AB \in \Psi_\infty^{m+m'}(\mathbb{R}^n)$ as required. □

4 Appendix: Functional Analysis

Theorem 4.1 (Continuous Linear extension). *Let $T \in \mathcal{L}(V, W)$ be a continuous linear map between normed vector spaces V and W with W completely metrisable. Then, there exist unique extension $\tilde{T} \in \mathcal{L}(\tilde{V}, W)$ of T , i.e. $\tilde{T}|_V = T$ where \tilde{V} is the completion of V .*

Theorem 4.2. *Let normed vector spaces V, W be given. If W is complete, then $\mathcal{L}(V, W)$ is complete.*

Theorem 4.3 (Schwartz Kernel Theorem [?, Chapter 4.6, p. 345]). *Let M, N be compact manifold and*

$$T : C^\infty(M) \rightarrow \mathcal{D}'(N)$$

be a continuous linear map ($C^\infty(M)$ being given Frechet space topology and $\mathcal{D}'(N)$ the weak topology). Define a bilinear map*

$$\begin{aligned} B : C^\infty(M) \times C^\infty(N) &\rightarrow \mathbb{C} \\ (u, v) &\mapsto B(u, v) = \langle v, Tu \rangle. \end{aligned}$$

Then, there exist a distribution $k \in \mathcal{D}'(M \times N)$ such that for all $(u, v) \in C^\infty(M) \times C^\infty(N)$

$$B(u, v) = \langle u \otimes v, k \rangle.$$

We call such k the kernel of T .

Definition 4.4 (Frechet space).