1 Microlocal Analysis

Theorem 1.1 (Schwartz Kernel Theorem [Taylor, 2011, Chapter 4.6, p. 345])**.** *Let M, N be compact manifold and*

$$T : C^\infty(M) \to \mathcal{D}'(N)$$

be a continuous linear map ($C^\infty(M)$ being given Frechet space topology and $D'(N)$ the weak topology). Define a bilinear map*

$$\begin{aligned} B : C^\infty(M) \times C^\infty(N) &\to \mathbb{C} \\ (u, v) &\mapsto B(u, v) = \langle v, Tu \rangle \,. \end{aligned}$$

Then, there exist a distribution $k \in \mathcal{D}'(M \times N)$ such that for all $(u, v) \in C^\infty(M) \times C^\infty(N)$

$$B(u, v) = \langle u \otimes v, k \rangle \,.$$

We call such k the kernel of T.

1.1 Symbols

1.2 Pseudodifferential Operators (ΨDO's)

2 Sobolev Spaces [Taylor, 2011, Chapter 4]

Definition 2.1 (Sobolev Spaces)**.** Let $p \in \mathbb{R}$ and $k \in \mathbb{N}$ be given. We define the k^{th}-order L^p-based Sobolev spaces on $\mathbb{R}^n$ as the Banach space

$$W^{p,k}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) \,|\, D^\alpha u \in L^p(\mathbb{R}^n), \ |\,\alpha\,| \leqslant k\}$$

with the norm

$$\|u\|_{W^{p,k}} = \|u\|_{L^p} + \left\|D^k u\right\|_{L^p} \,.$$

For $p = 2$, we have denote $H^k := W^{2,k}$ and note that result from Fourier analysis gives

$$H^k = \left\{u \in L^2(\mathbb{R}^n) \mid \langle\xi\rangle^k \, \hat{u} \in L^2(\mathbb{R}^n)\right\}$$

allowing us to extend the definition to each real order $s \in \mathbb{R}$,

$$H^s = \left\{u \in S'(\mathbb{R}^n) \mid \langle\xi\rangle^s \, \hat{u} \in L^2(\mathbb{R}^n)\right\}$$

where $S'(\mathbb{R}^n)$ is the space of tempered distribution on $\mathbb{R}^n$. This forms a Hilbert space with inner product given by

$$\langle u, v \rangle_{H^s} = \langle \Lambda^s u, \Lambda^s v \rangle_{L^2}$$

with $\Lambda^s : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ being the operator $\Lambda^s u = \mathcal{F}^{-1}(\langle\xi\rangle^s \, \hat{u})$.

3 Linear Elliptic equations [Taylor, 2011, Chapter 5]

Theorem 3.1 (Elliptic estimate)**.** *Let $\overline{\Omega}$ be a compact Riemanian manifold and Ω be its interior. Let $L = -\Delta + X$, where Δ is the Laplacian and X and first order differential operator with smooth* *coefficient in $\overline{\Omega}$. Then, we have the estimate*

$$\|u\|^2_{H^{k+1}} \leqslant C \left(\|Lu\|^2_{H^{k-1}} + \|u\|^2_{H^k} \right)$$

for all $u \in H^{k+1}(\Omega) \cap H^1_0(\Omega)$ and for some $C > 0$.

4 Compact and Fredholm Operators

5 Miscellaneous

Lemma 5.1 (Riez's inequality). *Let X be a normed linear space. Given a non-dense subspace (or closed proper subspace) $Y \subset X$ and any $r \in (0,1)$, then there exist $x \in X$ with $\|x\| = 1$ such that*

$$\inf_{y \in Y} \|x - y\| \geqslant r.$$

Proof.
Let $x_0 \in \overline{Y}^c$ and $R = \inf_{y \in Y} \|y - x_0\| > 0$. Given any $\epsilon > 0$ we can pick (by definition of inf) a $y_0 \in Y$ such that

$$\|y_0 - x_0\| < R + \epsilon.$$

Take $\epsilon < R\frac{1-r}{r}$ and define $x \in X$ to be

$$x = \frac{y_0 - x_0}{\|y_0 - x_0\|}.$$

Observe that $\|x\| = 1$ and

$$\begin{aligned}
\inf_{y \in Y} \|x - y\| &= \frac{1}{\|x_0 - y_0\|} \inf_{y \in Y} \|y_0 - x_0 - y\,\|x_0 - y_0\|\| \\
&= \frac{1}{\|x_0 - y_0\|} \inf_{y \in Y} \|x_0 - y\| && \text{since } \alpha y - y_0 \in Y \text{ for any scalar } \alpha \\
&\geqslant \frac{R}{R + \epsilon} \\
&\geqslant \frac{R}{R + R\frac{1-r}{r}} \\
&= r
\end{aligned}$$

as required. □

Corollary. *The closed unit ball in a Banach Space X is compact iff X is finite dimensional.*

Proof. Let X be a Banach space and $\overline{B}$ be closed unit ball.
Case 1 ($\Longleftarrow$). If X is finite dimensional, it is isometrically isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$, where, by Heine-Borel theorem, the closed unit ball is compact.
Case 2 ($\Longrightarrow$). We will prove the contrapositive. Suppose, X is infinite dimensional. Let $x_0 \in \partial\overline{B}$ be an element in the unit sphere. For each $n \in \mathbb{N}$, we will use Riez Lemma to obtain a unit vector x_n such that

$$\inf_{y \in \operatorname{span}\{x_0, \ldots, x_{n-1}\}} \|x_n - y\| \geqslant \frac{1}{2}.$$

It is clear that $\{x_n \,|\, n \in \mathbb{N}\}$ is a sequence of element in $\overline{B}$ that has no convergent subsequence. Therefore $\overline{B}$ is not compact.

□

References

[Taylor, 2011] Taylor, M. (2011). *Partial Differential Equations I*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2 edition.