Microlocal Analysis
with Applications to Non-Elliptic Fredholm Problems

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Introduction

A linear partial differential operator of order $k \in \mathbb{N}$ in $\mathbb{R}^n$:

$$P = P(x, D_x) = \sum_{|\,\alpha\,| \leqslant k} c_\alpha(x) D_x^\alpha, \quad c_\alpha \in C_\infty^\infty(\mathbb{R}^n)$$

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Examples:

$$\Delta = D_{x_1}^2 + \cdots + D_{x_n}^2 \qquad \text{Laplace operator}$$

$$\Box = D_{x_1}^2 + \cdots + D_{x_n}^2 - D_t^2 \qquad \text{Wave operator}$$

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Weak solution and forcing:

$$\begin{aligned} u : \mathcal{S}(\mathbb{R}^n) &\to \mathbb{C} \\ \varphi &\mapsto u(\varphi) \end{aligned}$$

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \iff \varphi \in C^\infty \text{ and } \sup_x \left| x^\beta D_x^\alpha \varphi(x) \right| < \infty$$

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Fredholm theory tackles all three simultaneously!

Overview

Fredholm Operators

Definition (Fredholm operators)

A continuous linear operator $T : \mathcal{X} \to \mathcal{Y}$ between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ is Fredholm, if

- T has closed range, i.e. $T(\mathcal{X})$ is closed in $\mathcal{Y}$,
- $\ker(T) \subset \mathcal{X}$ is finite dimensional,
- $\operatorname{coker}(T) := \mathcal{Y}/T(\mathcal{X})$ is finite dimensional.

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Suppose $Tx = y$ for a given $y \in \mathcal{Y}$.

Existence a solution $x \in \mathcal{X}$ exist if and only if $y \in \operatorname{coker}(T)^{\perp}$.

Uniqueness the solution is unique if and only if $\ker(T) = 0$.

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T Fredholm
$\rightsquigarrow$ existence and uniqueness reduce to finite dimensional linear algebra.

Fredholm Estimate

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Theorem (Fredholm Estimate)

Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be Banach spaces. If

- $T : \mathcal{X} \to \mathcal{Y}$ *is continuous,*
- $\mathcal{X}$ *is compactly contained in* $\mathcal{Z}$, *i.e.* $\iota : \mathcal{X} \hookrightarrow \mathcal{Z}$ *is compact,*
- *for all* $x \in \mathcal{X}$, *there exist* $C > 0$ *such that the following estimate hold*

$$\|x\|_{\mathcal{X}} \leqslant C \left(\|Tx\|_{\mathcal{Y}} + \|x\|_{\mathcal{Z}} \right) \tag{1}$$

then T *is semi-Fredholm*

- *the image,* $T(\mathcal{X})$ *is closed, and*
- T *has finite dimensional kernel.*

Constructing a *Fredholm problem*

Fredholm Problem

Given a differential operator P, can we construct solution spaces $\mathcal{X}$ and $\mathcal{Y}$, so that

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What's the link to *regularity*? Sobolev Space!

Sobolev Space

Definition

The Sobolev space of order $k \in \mathbb{N}$ on $\mathbb{R}^n$, $H^k(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ is defined by

$$u \in H^k(\mathbb{R}^n) \iff D^\alpha u \in L^2(\mathbb{R}^n) \text{ whenever } |\,\alpha\,| \leqslant k$$
$$\iff \langle\xi\rangle^k \mathcal{F}u(\xi) \in L^2(\mathbb{R}^n).$$

$$\langle\xi\rangle := \left(1 + |\,\xi\,|^2\right)^{1/2} = \left(1 + |\,\xi_1\,|^2 + \cdots + |\,\xi_n\,|^2\right)^{1/2}$$

Hilbert space structure keeps track of (global) regularity data of u.

$$\|u\|_{H^k} = \underbrace{\|u\|_{L^2}}_{\text{global decay}} + \underbrace{\sum_{|\,\alpha\,| \leqslant k} \|D^\alpha u\|_{L^2}}_{k \text{ times differentiable}}$$

Sobolev Space on Closed Manifold

Let M be a smooth closed n-manifold (compact without boundary), $s \in \mathbb{R}$, $u \in (C^\infty(M))'$, then

$$u \in H^s(M) \iff (\chi \cdot u) \circ \Phi^{-1} \in H^s(U)$$

for any chart $\Phi : \widetilde{U} \to U \subset \mathbb{R}^n$ and smooth bump function $\chi \in C^\infty(M)$ compactly supported in the chart domain $\widetilde{U}$.

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Henceforth, M is either $\mathbb{R}^n$ or a closed n-manifold.

General Strategy

Existence, uniqueness, regularity $\rightsquigarrow$

We want to prove

$$P : H^s(M) \to H^{s'}(M)$$

is semi-Fredholm by proving

$$\|u\|_{H^s} \leqslant C \left(\|Pu\|_{H^{s'}} + \|u\|_{H^N} \right)$$

for some Sobolev orders $s, s', N \in \mathbb{R}$. Semi-Fredholm for compact M only!

For **elliptic operators**:

$$H^{s+m} \to H^s$$

For **non-elliptic operators**:

$$H^{s+m} \to H^{s+1}$$

“Elliptic operators are Fredholm”

How do we get such an estimate?

Theorem (Elliptic regularity)

Let P be an order $m \in \mathbb{R}$ elliptic differential operator on an n-manifold, M. Suppose we know a priori that $u \in H^N(M)$ for some $N \in \mathbb{R}$. Then, for any $s \in \mathbb{R}$

$$(f =)Pu \in H^s(M) \implies u \in H^{s+m}(M)$$

and u satisfies the estimates: $\exists C > 0$

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Fourier transform + integration by parts $\implies \mathcal{F}D_x = \xi\mathcal{F}$

$$(\Delta + 1)\,u = \mathcal{F}^{-1}\mathcal{F}(\Delta + 1)u = \mathcal{F}^{-1}(1 + |\,\xi\,|^2)\mathcal{F}u$$

We call $\left(1 + |\,\xi\,|^2\right)$ is the **symbol** for $(\Delta + 1)$.

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We call $\left(1 + |\,\xi\,|^2\right)$ is the **symbol** for $(\Delta + 1)$.
We expect an inverse ...

$$(\Delta + 1)^{-1}\, (\Delta + 1)\, u(x) = \mathcal{F}^{-1}(1 + |\,\xi\,|^2)^{-1}(1 + |\,\xi\,|^2)\mathcal{F} u = u$$

Pseudodifferential operators

Question: What is $(\Delta + 1)^{-1}$? Answer: **pseudo**differential operator.

$$P(x, D_x)u = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} p(x, \xi) u(y) \,\mathrm{d}y \,\mathrm{d}\xi$$

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Definition

A smooth function $p(x,\xi) \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ is a symbol of order $m \in \mathbb{R}$, i.e. $p \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$, if

$$\left| D_x^\alpha D_\xi^\beta p(x,\xi) \right| \leqslant C_{\alpha,\beta,\gamma} \left\langle \xi \right\rangle^{m-|\beta|}, \quad C_{\alpha,\beta} > 0$$

for any multi-index $\alpha, \beta \in \mathbb{N}^n$.

Pseudodifferential operators

If $P \in \Psi^m_\infty(M)$ for some $m \in \mathbb{R}$, with symbol $p(x,\xi)$, then it is **elliptic** if

$$|\,p(x,\xi)\,| \geqslant \epsilon\,\langle\xi\rangle^m \qquad \text{in } |\,\xi\,| > 1/\epsilon \text{ for some } \epsilon > 0$$

Ellipticity is a property of the principal symbol.

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Lemma

1. *$P : H^s(M) \to H^{s-m}(M)$ is continuous for any $s \in \mathbb{R}$.*
2. *If P is* **elliptic** *then there exist parametrix $Q \in \Psi^{-m}_\infty(M)$ such that*

$$QP - 1 : H^s(M) \to H^{s'}(M)$$

is continuous for any $s, s' \in \mathbb{R}$.

Proof of Elliptic Regularity

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$$\|u\|_{H^{s+m}} \leqslant \underbrace{\|QPu\|_{H^{s+m}}}_{\leqslant C\|Pu\|_{H^s}} + \underbrace{\|(QP-1)\,u\|_{H^{s+m}}}_{\leqslant C\|u\|_{H^N}}$$

using continuity $Q : H^s \to H^{s+m}$ and $(QP - 1) : H^N \to H^{s+m}$.

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using continuity $Q : H^s \to H^{s+m}$ and $(QP-1) : H^N \to H^{s+m}$.
We get

$$\|u\|_{H^{s+m}} \leqslant C\,\|Pu\|_{H^s} + C\,\|u\|_{H^N}\,.$$

Non-elliptic Fredholm problem

Theorem (Main theorem)

There exist a perturbation Q of the wave operator $\Box$ on $\mathbb{T}^{1+n}$ such that the following estimate holds for any $s, N \in \mathbb{R}$

$$\|u\|_{H^{s+2}} \leqslant C\left(\|(\Box - iQ)u\|_{H^{s+1}} + \|u\|_{H^N}\right)$$

for some $C > 0$.

$$\mathbb{T}^{1+n} := \mathbb{S}^1_t \times \underbrace{\mathbb{S}^1_{x_1} \times \mathbb{S}^1_{x_2} \times \cdots \times \mathbb{S}^1_{x_n}}_{n}$$

$$\Box := \partial_t^2 - \sum_{j=1}^{n} \partial_{x_j}^2, \quad p(t, x, \tau, \xi) = |\,\xi\,|^2 - \tau^2$$

Microlocal Viewpoint

Global elliptic

$$\iff \left|\,|\,\xi\,|^2 - \tau^2\,\right| \geqslant \epsilon\,\langle(\tau,\xi)\rangle^2 \text{ whenever } |\,(\tau,\xi)\,| > 1/\epsilon.$$

Microlocal Viewpoint

Microlocal elliptic at a point $(t_0, x_0, \tau_0, \xi_0) \in T^*\mathbb{T}^{1+n} \setminus 0$
$\iff \Big| |\xi|^2 - \tau^2 \Big| \geqslant \epsilon \langle (\tau, \xi) \rangle^2$ in

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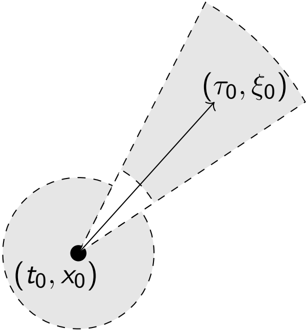

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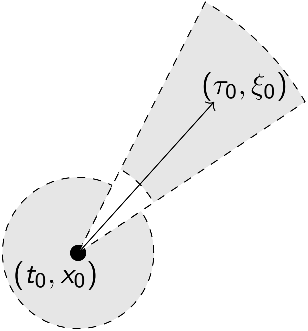


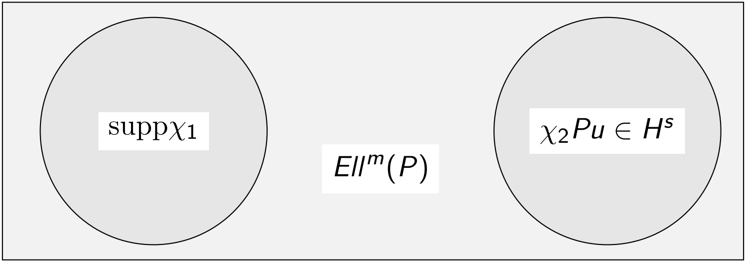
$Ell^2 = \{$points in phase space where p is elliptic$\} \setminus 0$
$\Sigma^2 = Ell^m(\Box)^c \setminus 0$.
For $\Box$: Elliptic $\iff$ not vanishing (outside of $(\tau, \xi) = 0$).

Two Major Ingredients

Theorem (Microlocal elliptic regularity)

Let $P \in \Psi^m_\infty(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. If for some $\chi_2 \in \Psi^0_\infty(\mathbb{R}^n)$, $\chi_2 Pu \in H^s(\mathbb{R}^n)$, then for any other $\chi_1 \in \Psi^0_\infty(\mathbb{R}^n)$ such that $\mathrm{WF}'(\chi_1) \subset Ell^m(P) \cap Ell^0(\chi_2)$ we have $\chi_1 u \in H^{s+m}(\mathbb{R}^n)$ and it satisfies the estimate: $\forall N \in \mathbb{R}$, $\exists C > 0$

$$\|\chi_1 u\|_{H^{s+m}} \leqslant C \left(\|\chi_2 Pu\|_{H^s} + \|u\|_{H^N}\right).$$

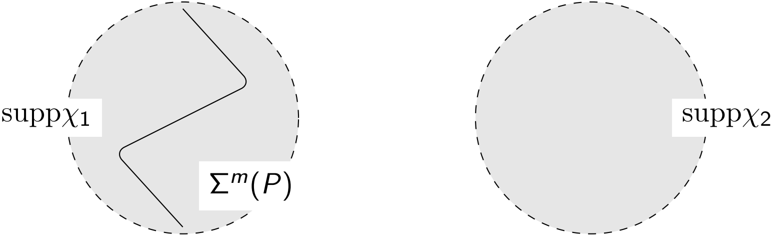


Two Major Ingredients

Theorem (Propagation of singularities)

Let $P \in \Psi^m_\infty(\mathbb{R}^n)$ is a properly supported pseudodifferential operator with polyhomogeneous principal $\sigma_m(P) = p - iq$ with real p, q. If we have $\chi_1, \chi_2, \chi_3 \in \Psi^0_\infty(\mathbb{R}^n)$ and $q \geqslant 0$ on $\mathrm{WF}'(\chi_3)$ and every $(x, \xi) \in WF'(P)$ is in the integral curve of H_p originating from $Ell^0(\chi_2)$, then for all $s, N \in \mathbb{R}$ and $u \in C^\infty(\mathbb{R}^n)$, there exist $C > 0$ such that

$$\|\chi_1 u\|_{H^{s+m}} \leqslant C \left(\|\chi_2 u\|_{H^s} + \|\chi_3 P u\|_{H^{s+1}} + \|u\|_{H^N} \right).$$



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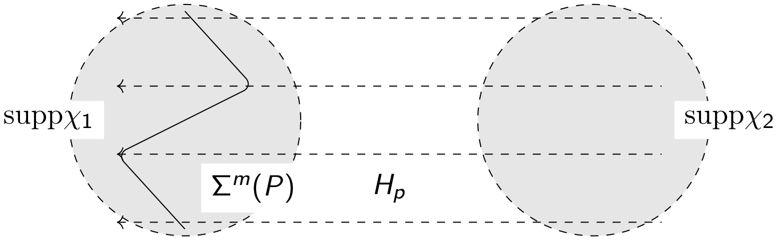
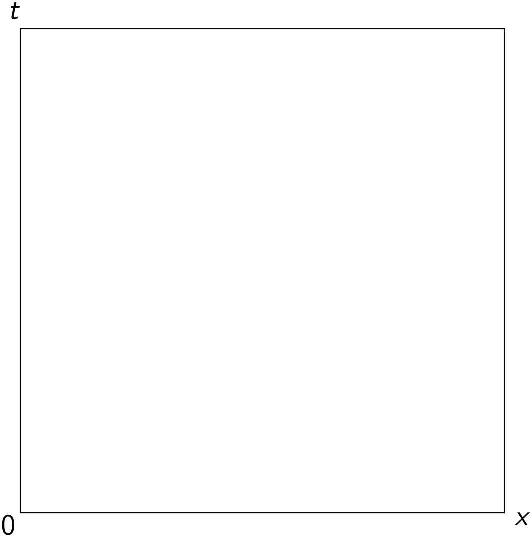


Constructions

Main idea : Create larger elliptic set to absorp singuarity! $Q = \chi(t)\partial_t^2$.



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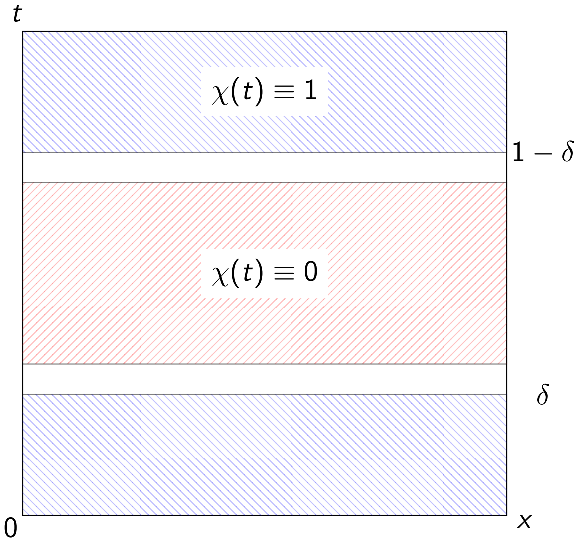

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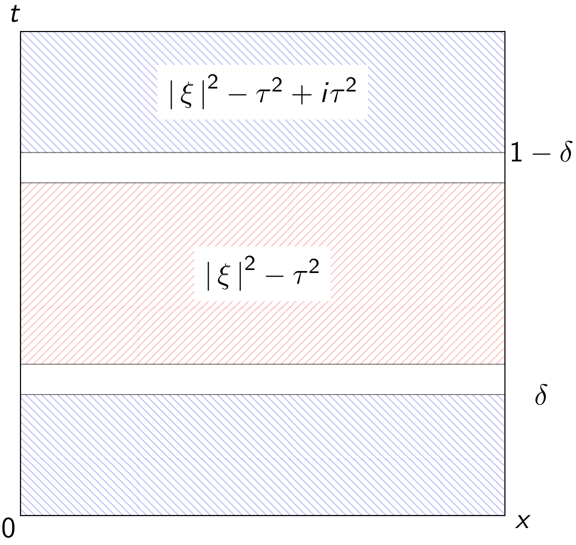
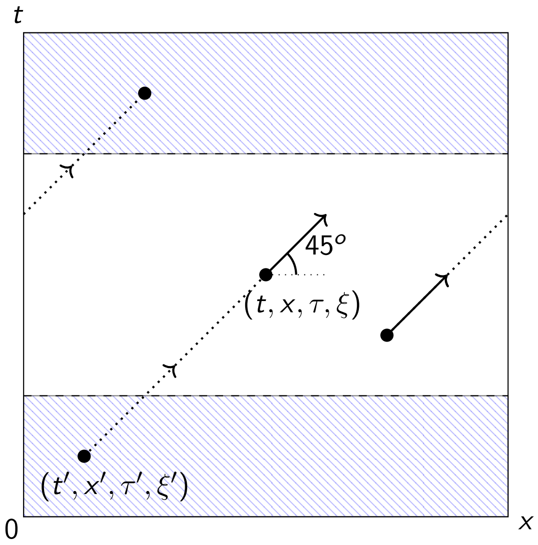


Hamilton flow of $\sigma_2(\Box)$

Hamiltonian flow: $\exp(sH_p)(t,x,\tau,\xi) = (t+s\tau, x+s\xi, \tau, \xi)$



Conclusion

Propagation of singularity $\implies$

$$\|u\|_{H^{s+2}} \leqslant C \underbrace{\|\chi(t)u\|_{H^s}}_{\text{elliptic region!}} + C\,\|(\Box - iQ)u\|_{H^{s+1}} + C\,\|u\|_{H^N}$$

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$$\|u\|_{H^{s+2}} \leqslant C'\,\|\Box - iQ\|_{H^{s-2}} + C\,\|(\Box - iQ)u\|_{H^{s+1}} + C'\,\|u\|_{H^N}$$

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$$\|u\|_{H^{s+2}} \leqslant C'\|\Box - iQ\|_{H^{s-2}} + C\|(\Box - iQ)u\|_{H^{s+1}} + C'\|u\|_{H^N}$$

$$\|u\|_{H^{s+2}} \leqslant C''\left(\|(\Box - iQ)u\|_{H^{s+1}} + \|u\|_{H^N}\right)$$

Conclusion

$$\|u\|_{H^{s+2}} \leqslant C'' \left(\|(\Box - iQ)u\|_{H^{s+1}} + \|u\|_{H^N} \right)$$

Which suggest the Hilbert space domain that we want is

$$\mathcal{X}^s = \left\{ u \in H^s \ : \ (\Box - iQ)u \in H^{s-1} \right\}.$$

And

$$\Box - iQ : \mathcal{X}^{s+2} \to H^{s+1}$$

is Fredholm for any $s \in \mathbb{R}$.

The End

Thank you!