Microlocal Analysis

with an Application to the Non-elliptic Fredholm Problem of the Wave Operator on $n+1$-Torus

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Abstract

Acknowledgement

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Chapter 1

Introduction

It is well known in Fourier analysis that the rate of decay of the Fourier transform of a distribution is a measure of smoothness of the distribution. Microlocal analysis, first introduced by Sato [] and Hormander [], is the study of distributions that takes advantage of this observation by keeping track of not only the singular points in the base manifold, but also the *directions* in the cotangent bundle in which the Fourier transform has insufficient decay.

Pseudifferential operators are the central object of study in microlocal analysis. They generalise traditional differential operators with smooth coefficients. The theory of pseudodifferential operators provides insights into the study of linear partial differential equations. For instance, the theory provides techniques to prove that certain differential operators are *Fredholm* as linear maps between appropriate Sobolev spaces. The property of an operator being Fredholm reduces the question of solvability to checking a finite number of conditions and restricts the non-uniqueness of any solution to a finite dimensional space (see []). While traditionally only elliptic operators are associated with Fredholm problems, there has been recent fruitful application of results from microlocal analysis, particularly the propagation of singularities estimates **??**, to the study of non-elliptic Fredholm problems [].

Outline :

Chapter 2 : Beside fixing notations that will be used throughout the paper, this chapter serves to introduce functional analysis concepts and theorems that are central to microlocal analysis. In particular, the space of Schwartz functions, tempered distributions and their relationship with the Fourier transform are introduced. Several results concerning compact and Fredholm operators between Banach spaces will also be stated in preparation for chapter 5.

Chapter 3 : Here we will define and prove various topological and algebraic properties of pseudodifferential operators.

Chapter 4: We will then turn our attention to pseudodifferential equations, i.e. equations of the form $Au = f$ with A being a pseudodifferential operator, u

the unknown and f a given distribution. Of particular importance is when A is an *elliptic* operator which allow us to invert A up to "trivial" terms. For each operator A, we will wish to answer the question : "How does the presence of singularities in f affect the regularity of the solution u?"

Chapter 5: Here we will consider the Fredholm problem of the totally periodic wave operator on the $n+1$-torus. We will see the application of the propagation of singularities theorem to construct subspaces of the Sobolev spaces on the torus on which the wave operator acts as a Fredholm operator after some pertubation.

Chapter 2

Functional analysis background

This chapter serves to introduces concepts and theorems that are integral to the theory of microlocal analysis and its application to Fredholm problems for pseudodifferential operators.

Some notations

We will employ the following notations throught the rest of the paper. Let $\mathbb{N} = \{0, 1, \dots\}$ denote the set of natural numbers. Given $n \in \mathbb{N}$, a *multi-index* is a n-tuple of natural numbers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$. For any multi-indices $\alpha, \beta \in \mathbb{N}^n$ and any n-tuples $x, y \in \mathbb{R}^n$, we define

$$\begin{aligned} x^\beta &:= x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n} = \prod_{j=1}^n x_j^{\beta_j} \\ (x+y)^\alpha &:= \prod_{j=1}^n (x_j + y_j)^{\beta_j} \\ D_x^\alpha &:= D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \dots D_{x_n}^{\alpha_n} \end{aligned}$$

where $D_{x_j} := -i\partial_{x_j}$with $i \in \mathbb{C}$ being the imaginary unit and ∂_{x_j} the x_j-partial derivative operator. Furthermore, we define

$$\begin{aligned} &|\,\alpha\,| := \alpha_1 + \alpha_2 + \dots + \alpha_n \\ &\alpha! := \alpha_1! \alpha_2! \dots \alpha_n! \\ &\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \dots \binom{\alpha_n}{\beta_n} = \frac{\alpha!}{(\beta-\alpha)!\beta!} \\ &\alpha \leqslant \beta \iff \alpha_i \leqslant \beta_i, \quad \forall i \in \{1, \dots, n\}\,. \end{aligned}$$

We shall now state the Leibniz formula, not only to illustrate the multi-index notation, but it will also be a theorem that we shall use repeatedly.

Theorem 2.1 (Leibinz formula). *Let $f, g \in C^\infty(\mathbb{R}^n)$, then*

$$D_x^\alpha(fg) = (-i)^{|\alpha|}\partial_x^\alpha(fg) = (-i)^{|\alpha|}\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\left(\partial_x^\beta f\right)\left(\partial_x^{\alpha-\beta}g\right).$$

Next, in discussing the order of growth of smooth functions, $f = f(x)$ on $\mathbb{R}^n$ as $\|x\| \to \infty$, it is often convenient to compare f to another smooth function. Hence, we define the bracket

$$\begin{aligned} \langle\cdot\rangle : \mathbb{R}^n &\to \mathbb{R}_{\geqslant 1} \\ x \mapsto \langle x\rangle &:= \left(1 + x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2} = \left(1 + \|x\|^2\right)^{1/2}. \end{aligned}$$

The main point of this bracket is that, $\langle x\rangle$ is a smooth function asymptotically equivalent to $\|x\|$ for large x.

2.1 Schwartz functions and tempered distributions

Definition 2.1 (Schwartz space). The space of Schwartz (test) functions of rapidly decaying functions on $\mathbb{R}^n$, denoted $\mathcal{S}(\mathbb{R}^n)$, is the space of smooth functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ such that for any $\alpha \in \mathbb{N}^n$,

$$\sup_{x\in\mathbb{R}^n}\left|\langle x\rangle^k D_x^\alpha\varphi(x)\right| < \infty. \tag{2.1}$$

We can define a countable family of seminorm on $\mathcal{S}(\mathbb{R}^n)$ by

$$\|\varphi\|_k := \max_{|\alpha|\leqslant k}\sup_{x\in\mathbb{R}^n}\left|\langle x\rangle^k D_x^\alpha\varphi(x)\right| \tag{2.2}$$

for $k \in \mathbb{N}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$. This makes $\mathcal{S}(\mathbb{R}^n)$ a Frechet space with metric

$$d(\varphi,\psi) = \sum_{k\in\mathbb{N}} 2^{-k}\frac{\|\varphi-\psi\|_k}{1 + \|\varphi-\psi\|_k}$$

for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, which defines a complete metric topology on $\mathcal{S}(\mathbb{R}^n)$.

Remark 2.2.

1. We note that the space $\mathcal{S}(\mathbb{R}^n)$ is non-empty since it contains all the compactly supported smooth functions $\chi : \mathbb{R}^n \to \mathbb{C}$. In fact,

   $$C_c^\infty(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) \,:\, \exists C \in \mathbb{R}_{>0},\, |x| > C \implies u(x) = 0\} \subset \mathcal{S}(\mathbb{R}^n)$$

   is a dense inclusion.

2. $\mathcal{S}(\mathbb{R}^n)$ with pointwise multiplication and addition is a commutative algebra over $\mathbb{C}$ without identity since $1 \notin \mathcal{S}(\mathbb{R}^n)$. It is also closed under several useful elementary operations including coordinate multiplication and partial differentiation

$$\begin{aligned} x_j &: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \\ D_{x_j} &: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n). \end{aligned}$$

Definition 2.3 (Tempered distribution)**.** The space of tempered distribution is the dual space of Schwartz space. More precisely, the space of tempered distribution $\mathcal{S}'(\mathbb{R}^n)$ on $\mathbb{R}^n$ is defined by

$$\mathcal{S}'(\mathbb{R}^n) = (\mathcal{S}(\mathbb{R}^n))' = \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathbb{C})$$

where $\mathcal{L}(V, W)$ denotes the continuous linear maps between any topological vector spaces $V \to W$. Explicitly in terms of seminorms on $\mathcal{S}(\mathbb{R}^n)$, the elements $u \in \mathcal{S}'(\mathbb{R}^n)$ are linear functionals $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ satisfying: for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, there exist $k \in \mathbb{N}$ and $C \in \mathbb{R}_{>0}$ such that

$$|\, u(\varphi) \,| \leqslant C \left\| \varphi \right\|_k .$$

We usually equip $\mathcal{S}'(\mathbb{R}^n)$ with the weak-* topology, i.e. the weakest topology for which all linear maps of the form $\langle \varphi, \cdot \rangle : \mathcal{S}'(\mathbb{R}^n) \to \mathbb{C}$ for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ are continuous. Here, $\langle \cdot, \cdot \rangle : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n) \to \mathbb{C}$ is the Frechet space pairing defined by

$$\langle \varphi, u \rangle := u(\varphi).$$

A neighbourhood basis around $0 \in \mathcal{S}'(\mathbb{R}^n)$ for the topology is given by the collection of sets of the form

$$\{u \in \mathcal{S}'(\mathbb{R}^n) \;:\; |\, u(\varphi_j) \,| < \epsilon_j,\, \varphi_j \in \mathcal{S}(\mathbb{R}^n), \epsilon_j \in \mathbb{R}_{>0},\; j = 1, \ldots, N\}$$

for any $N \in \mathbb{N}$.

The following two standard results are important in the development of pseudodifferential calculus. The first of which allows us to extend results concerning continuous linear maps on $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

Lemma 2.2. *Let $\iota : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be the injection map defined by the integral pairing*

$$\iota(\varphi)(\psi) = \int \varphi(x)\psi(x)\,\mathrm{d}x \in \mathbb{C}$$

for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$. Then, the image of ι is dense in $\mathcal{S}'(\mathbb{R}^n)$ with the weak- topology.*

The second result is the celebrated Schwartz kernel theorem. To motivate this theorem, observe that any element $k \in \mathcal{S}'(\mathbb{R}^{n+m})$ defines a continuous linear *operator* of the form

$$\begin{aligned} A_k &: \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}'(\mathbb{R}^n) \\ &\varphi(x) \to A_k\varphi : \psi(y) \in \mathcal{S}(\mathbb{R}^n) \mapsto A_k\varphi(\psi) := k(\varphi(x)\psi(y)). \end{aligned}$$

The Schwartz kernel theorem states that the converse is also true.

Theorem 2.3 (Schwartz kernel theorem)**.** *Let $m, n \in \mathbb{N}$ be given. Then, $A : \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}'(\mathbb{R}^n)$ is a continous linear operator if and only if there exist unique $k \in \mathcal{S}'(\mathbb{R}^{n+m})$ such that*

$$A\varphi(\psi) = k(\varphi \cdot \psi)$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^m)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$.

We call the unique tempered distribution $k \in \mathcal{S}'(\mathbb{R}^{n+m})$ representing the operator $A : \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}'(\mathbb{R}^m)$ the Schwartz kernel of A.

2.2 Fourier transform

Fourier transform plays a crucial role in the theory of pseudodifferential operator.

Definition 2.4. The Fourier transform $\mathcal{F}f$ of a function $f \in L^1(\mathbb{R}^n)$ is defined by the integral

$$\widehat{f}(\xi) := \mathcal{F}f(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{-ix\cdot\xi} f(x)\,\mathrm{d}x \in L^\infty(\mathbb{R}^n).$$

We can verify that, on Schwartz space, the same integral operation defines a continuous linear map

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

Its L^2-adjoint is given by

$$\mathcal{F}^* f(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi} f(x)\,\mathrm{d}x, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

The Fourier inversion theorem states that $\mathcal{F}^*$ is the continuous inverse, i.e. $\mathcal{F}^* = \mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Explicitly, this means that for any $f \in \mathcal{S}(\mathbb{R}^n)$

$$f(x) = \mathcal{F}^*\mathcal{F}f(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} f(y)\,\mathrm{d}y\,\mathrm{d}\xi.$$

With the inversion formula and the definition of L^2-dual, we can easily see that

$$\langle f, g\rangle_{L^2} = \langle f, \mathcal{F}^*\mathcal{F}g\rangle_{L^2} = \langle \mathcal{F}f, \mathcal{F}g\rangle_{L^2}, \quad f, g \in \mathcal{S}(\mathbb{R}^n).$$

This allows us to uniquely extend $\mathcal{F}$, $\mathcal{F}^*$ from $\mathcal{S}(\mathbb{R}^n)$ to a unitary map

$$\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

with unitary inverse $\mathcal{F}^*$. This is also known as the Plancherel theorem [].

2.3 Sobolev Spaces

[1, Chapter 4]

Definition 2.5 (Sobolev Spaces)**.** Let $p \in \mathbb{R}$ and $n, k \in \mathbb{N}$ be given. We define the k^{th}-order L^p-based Sobolev space on $\mathbb{R}^n$ as the Banach space

$$W^{k,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) \,:\, D^\alpha u \in L^p(\mathbb{R}^n), \ |\,\alpha\,| \leqslant k\}$$

with the norm

$$\|u\|_{W^{p,k}} = \|u\|_{L^p} + \sum_{j=1}^{k} \left\|D^j u\right\|_{L^p}.$$

For $p = 2$, we have denote $H^k := W^{k,2}$ and note that result from Fourier analysis gives

$$H^k(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n) \,:\, \langle\xi\rangle^k \hat{u} \in L^2(\mathbb{R}^n)\right\}$$

allowing us to extend the definition to each real order $s \in \mathbb{R}$,

$$H^s(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n) \,:\, \langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^n)\right\} = \Lambda^{-s} L^2(\mathbb{R}^n)$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distribution on $\mathbb{R}^n$ and

$$\Lambda^s : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$$

being the operator defined by $\Lambda^s u = \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u})$. This forms a Hilbert space with inner product given by

$$\langle u, v\rangle_{H^s} = \langle \Lambda^s u, \Lambda^s v\rangle_{L^2}.$$

Remark 2.6. It is straightforward to show that the derivative operator D_{x_j} is a continuous linear operator $D_{x_j} : H^s(\mathbb{R}^n) \to H^{s-1}(\mathbb{R}^n)$ and thus by induction, for any multi-index $\alpha \in \mathbb{N}^n$, $D^\alpha : H^s(\mathbb{R}^n) \to H^{s-|\,\alpha\,|}(\mathbb{R}^n)$.

It can be proven that **??** the definition above can be extended to any smooth compact manifold M.

Definition 2.7. Let M be a smooth compact manifold and

$$\mathcal{D}'(M) = (C^\infty(M))'$$

be the space of distribution. Then, the Sobolev space $H^s(M)$ for $s \in \mathbb{R}$, is the set of distribution $u \in \mathcal{D}'(M)$ satisfying

$$(\chi u) \circ \Phi_U^{-1} \in H^s(\mathbb{R}^n)$$

for any coordinate domain $U \subset M$, chart $\Phi_U : U \to \mathbb{R}^n$ and any compactly supported smooth function $\chi \in C_c^\infty(U)$.

corollary in Hilbert space

2.4 Compact and Fredholm operators

In this section we shall restrict our attention to just maps between Banach spaces. A compact operator between Banach spaces is one where the image of all bounded sets are precompact. We shall denote the set of all compact (continuous) operators between V and W as $\mathcal{K}(V, W) \subset \mathcal{L}(V, W)$. Some useful results regarding linear maps between Banach spaces are given below.

Lemma 2.4. **??** *Let $T : X \to Y$ be a bounded linear map between Banach spaces X, Y and let $T' : Y' \to X'$ be the dual linear map. Then*

1. $\mathrm{ker} T' = T(X)^\perp = \{\omega \in Y' : \omega(Tx) = 0, \forall x \in X\}$.
2. *If T has closed range, then $T'(Y') = \mathrm{ker} T'$.*

Proof. Let $\langle x, \omega \rangle = \omega(x)$ denote the pairing between a Banach space with its dual. The dual map T' of T is characterised by $\langle Tx, \omega \rangle = \langle x, T'\omega \rangle$ for any $x \in X$, $\omega \in Y'$. Observe that

$$\begin{aligned} \omega \in T(X)^\perp &\iff \forall x \in X,\ \langle Tx, \omega \rangle = 0 \\ &\iff \forall x \in X,\ \langle x, T'\omega \rangle = 0 \\ &\iff \omega \in \mathrm{ker} T' \end{aligned}$$

which proves the first statement.

For the second statement, if $T(X)$ is a closed linear subspace of Y, then

$$\begin{aligned} \widetilde{T} : Y/\mathrm{ker} T &\to T(X) \\ [y] &\mapsto T(y) \end{aligned}$$

defines a topological isomorphism, which in turns give rise to the topological isomorphism

$$\widetilde{T}' : T(X)' \to (X/\mathrm{ker} T)'.$$

We also know that $(X/kerT)' \cong (\mathrm{ker}T)^{\perp}$ are naturally isomorphic as Banach spaces. There is also a natural projection $p : Y' \to T(X)'$. We can then express $T' : Y' \to X'$ as the composition

$$Y' \xrightarrow{p} T(X)' \xrightarrow{\widetilde{T}} (X/\mathrm{ker}T)' \xrightarrow{\sim} (\mathrm{ker}T)^{\perp}$$

which gives the desired $T'(Y') = (\mathrm{ker}T)^{\perp}$. verify proof!!

□

Theorem 2.5. *Let V, W, Y be Banach spaces, $T \in \mathcal{L}(V, W)$ and $K \in \mathcal{K}(V, Y)$. If for all $u \in V$, the estimate*

$$\|u\|_V \leqslant C\left(\|Tu\|_W + \|Ku\|_Y\right)$$

holds for some positive real constant $C \in \mathbb{R}_{>0}$, then the image, $T(V)$ is closed, and has finite dimensional kernel. Add in and prove finite dim kernel part of the statement!

Proof. Let $\{Tu_n \in T(V) : n \in \mathbb{N}, u_n \in V\}$ be a convergent sequence in $T(V)$ with limit $w \in W$. We need to show that there exist $v \in V$ such that $Tv = w$. Let $L = \mathrm{ker}T$. There are two cases

Case ($\forall n \in \mathbb{N}$, $d(u_n, L) \leqslant a < \infty$).
By definition of distance of a point to a set, for each n there exist $x_n \in L$ such that $\|u_n - x_n\| \leqslant 2a$. We can therefore define, for each n, $v_n = u_n - x_n$. Note that $\|v_n\| \leqslant 2a$ and $\lim_{n\to\infty} Tv_n = \lim_{n\to\infty} Tu_n + Tx_n = \lim_{n\to\infty} Tu_n + 0 = w$. Since the sequence v_n is bounded and K is compact, there exist a subsequence $\left\{v_{n_j}\right\}_{j\in\mathbb{N}}$ such that $Kv_{n_j} \to y_0 \in Y$. Then, applying the estimate on $v_{n_j} - v_{n_{j+k}}$, we get, as $j \to \infty$

$$\begin{aligned}
\left\|v_{n_j} - v_{n_{j+k}}\right\|_V &\leqslant C\left(\left\|Tv_{n_j} - Tv_{n_{j+k}}\right\|_W + \left\|Kv_{n_j} - Kv_{n_{j+k}}\right\|_Y\right)\\
&\to \left(\|w - w\|_W + \|y_0 - y_0\|_Y\right)\\
&= 0
\end{aligned}$$

which shows that $\left\{v_{n_j}\right\}_j$ is a Cauchy and therefore has a limit $v \in V$. Using continuity we get $w = \lim_{n\to\infty} Tv_n = \lim_{j\to\infty} Tu_{n_j} = T\lim_{j\to\infty} u_{n_j} = Tv$ as required.

Case ($d(u_n, L) \to \infty$ as $n \to \infty$).
We can assume without loss of generality that $d(u_n, L) \geqslant 1, \forall n$. For each n, there exist $x_n \in L$ such that $1 \leqslant d(u_n, L) \leqslant \|v_n\| \leqslant d(u_n, L) + 1$ where $v_n := u_n - x_n$. Define $w_n = v_n / \|v_n\|$. Since w_n is a bounded sequence (bounded by 1), there is a subsequence Kw_{n_j} that converges, with limit $y_0 \in Y$. Furthermore, $T(w_n) =$

$T(u_n - x_n)/\|u_n - x_n\| \to 0$ as $n \to \infty$ since $\|u_n - x_n\| \geqslant d(u_n, L) \to \infty$. Therefore, the estimate applied on $w_{n_j} - w_{n_{j+k}}$ gives

$$\begin{aligned}
\left\|w_{n_j} - w_{n_{j+k}}\right\|_V &\leqslant C\left(\left\|Tw_{n_j} - Tw_{n_{j+k}}\right\|_W + \left\|Kw_{n_j} - Kw_{n_{j+k}}\right\|_Y\right) \\
&\to \left(\|0-0\|_W + \|y_0 - y_0\|_Y\right) \\
&= 0
\end{aligned}$$

as $j \to \infty$, showing that $\left\{w_{n_j}\right\}_j$ is a Cauchy sequence and therefore have a limit $w \in V$. But, $Tw = \lim_{j\to\infty} Tw_{n_j} = 0 \implies w \in L$, yet

$$\begin{aligned}
d(w_n, L) &= \inf_{x\in L}\left\|\frac{v_n}{\|v_n\|} - x\right\| \\
&= \|v_n\| \inf_{x\in L}\|v_n - x\| \\
&= \|v_n\| \inf_{x\in L}\|u_n - x\| \\
&= \|v_n\|\, d(u_n, L) \\
&\geqslant 1
\end{aligned}$$

implying that, in the limit as $n \to \infty$, $d(w, L) \geqslant 1$ which is a contraction.

To show that T has finite dimensional kernel, it suffice to show that the closed unit ball in ker$T$ is compact. Let $u_n \in \ker T$, $n \in \mathbb{N}$ be any sequence in the closed unit ball of ker$T$, i.e. $\|u_n\|_V \leqslant 1$, for all $n \in \mathbb{N}$. In otherwords, $\{u_n\}_{n\in\mathbb{N}}$ is a bounded set (bounded by 1). Since K is a compact operator, the exist a subsequence indexed by n_j for which Ku_{n_j} converges in Y. Since convergent sequence is necessarily Cauchy, using the estimate provided we have

$$\begin{aligned}
\left\|u_{n_j} - u_{n_{j+k}}\right\|_V &\leqslant C\left\|Tu_{n_j} - Tu_{n_{j+k}}\right\|_W + C\left\|Ku_{n_j} - Ku_{n_{j+k}}\right\| \\
&\leqslant C\cdot 0 + C\left\|Ku_{n_j} - Ku_{n_{j+k}}\right\| \\
&\to 0
\end{aligned}$$

as $j \to \infty$ and any $k \in \mathbb{N}$. This shows that $\left\{u_{n_j}\right\}_j$ is a Cauchy sequence in the closed unit ball of ker$T$ which is complete (being a closed subspace of a complete space). Hence, the original sequence $\{u_n\}_n$ has a convergent subsequence in the closed unit ball. We have just shown that every sequence in the closed unit ball of ker$T$ has a convergent subsequence, i.e. the closed unit ball is compact as required. □

Lemma 2.6 (Riez's lemma)**.** *Let X be a normed linear space. Given a non-dense subspace (in particular, proper subspaces) $Y \subset X$ and any $r \in (0,1)$, then there exist $x \in X$ with $\|x\| = 1$ such that*

$$\inf_{y\in Y}\|x - y\| \geqslant r.$$

Proof.
Let $x_0 \in \overline{Y}^c$ and $R = \inf_{y\in Y} \|y - x_0\| > 0$. Given any $\epsilon > 0$ we can pick (by definition of inf) a $y_0 \in Y$ such that

$$\|y_0 - x_0\| < R + \epsilon.$$

Take $\epsilon < R\frac{1-r}{r}$ and define $x \in X$ to be

$$x = \frac{y_0 - x_0}{\|y_0 - x_0\|}.$$

Observe that $\|x\| = 1$ and

$$\begin{aligned}
\inf_{y\in Y} \|x - y\| &= \frac{1}{\|x_0 - y_0\|} \inf_{y\in Y} \|y_0 - x_0 - y\,\|x_0 - y_0\|\,\| \\
&= \frac{1}{\|x_0 - y_0\|} \inf_{y\in Y} \|x_0 - y\| \qquad \text{since } \alpha y - y_0 \in Y \text{ for any scalar } \alpha \\
&\geqslant \frac{R}{R + \epsilon} \\
&\geqslant \frac{R}{R + R\frac{1-r}{r}} \\
&= r
\end{aligned}$$

as required. □

Riez's lemma gives us a clear distinction between finite and infinite dimensional Banach spaces.

Corollary. *The closed unit ball in a Banach Space X is compact iff X is finite dimensional.*

Proof. Let X be a Banach space and $\overline{B}$ be closed unit ball.
Case ($\Longleftarrow$). If X is finite dimensional, it is isometrically isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$, where, by Heine-Borel theorem, the closed unit ball is compact.
Case ($\Longrightarrow$). We will prove the contrapositive. Suppose, X is infinite dimensional. Let $x_0 \in \partial\overline{B}$ be an element in the unit sphere. For each $n \in \mathbb{N}$, we will use Riez Lemma to obtain a unit vector x_n such that

$$\inf_{y\in \text{span}\{x_0,\ldots,x_{n-1}\}} \|x_n - y\| \geqslant \frac{1}{2}.$$

It is clear that $\{x_n : n \in \mathbb{N}\}$ is a sequence of element in $\overline{B}$ that has no convergent subsequence. Therefore $\overline{B}$ is not compact.

□

Chapter 3

Pseudodifferential Calculus

In this chapter we recall the basic definitions and some well known properties about pseudodifferential operators. We include proofs where appropriate but leave some to the reader. Our main sources for this material are [Vasy, Melrose, Dyatlov-Zworski]
In this chapter, we will introduce pseudodifferential operators (or ΨDOs) that generalises differential operators in Euclidean spaces, $\mathbb{R}^n$. We have shown before (**??**) that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a continuous linear isomorphism. Using integration by parts, we see that the action on $\mathcal{S}(\mathbb{R}^n)$ of a m^{th} order differential operator with rapidly decaying smooth coefficient $c_\alpha \in \mathcal{S}(\mathbb{R}^n)$

$$P(x, D) = \sum_{|\,\alpha\,| \leqslant m} c_\alpha(x) D_x^\alpha : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \tag{3.1}$$

is given by

$$P(x, D)u = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} P(x, \xi) u(y) dy \, \mathrm{d}\xi \tag{3.2}$$

where $P(x, \xi) = \sum_{|\,\alpha\,| \leqslant m} c_\alpha(x)\xi^\alpha$ is the 'characteristic polynomial'.

Pseudodifferential operators are operators with the similar actions but with $P(x, \xi)$ generalised from polynomial in ξ to *symbols* $a = a(x, y, \xi) \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$. These are smooth functions with certain decay conditions in ξ similar to those of polynomials. We allow symbols to depends on an additional variable taking the role of y in the integral (3.2) above. A m^{th} order pseudodifferential operators, $A \in \Psi^m_\infty(\mathbb{R}^n)$ is thus an operator with action

$$Au(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} a(x, y, \xi) u(y) \, \mathrm{d}y \, \mathrm{d}\xi.$$

The procedure of turning a symbol into a pseudodifferential operator is known as the quantisation procedure.

The goal of this chapter is to build a 'calculus' of pseudodifferential operators and their symbols. This will include:

- rigourously define the space of symbols $S^m_\infty(\mathbb{R}^p; \mathbb{R}^n)$.

- show that the quantisation procedure outlined above is well-defined.

Here, we will introduce pseudodifferential operators only on Euclidean spaces. However, the results we obtain will invariant under any change of variables. This allow us to define symbols and corresponding pseudodifferential operators on the cotangent bundle T^*M on any smooth manifold M (see for example **??**).

3.1 Symbols

The most important characteristic of symbols is their behaviour at infinity. In analogy to differential operator (3.1), we require $a(x, \xi)$ to be bounded in x and have polynomial decay of increasing order with successive ξ-derivative.
Historical note about hormander symbol class. Vasy's definition. Melrose generalisation etc...

Definition 3.1. The space $S^m_\infty(\mathbb{R}^p; \mathbb{R}^n)$ of order m is the space of smooth functions $a \in C^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ such that for all multi-index $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}$$

uniformly on $\mathbb{R}^p \times \mathbb{R}^n$. Together with the family of seminorm (indexed by $N \in \mathbb{N}$)

$$\|a\|_{N,m} = \sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \max_{|\alpha|+|\beta|\leqslant N} \frac{\left| D_x^\alpha D_\xi^\beta a(x,\xi) \right|}{\langle \xi \rangle^{m-|\beta|}}$$

gives a Frechet topology to $S^m_\infty(\Omega; \mathbb{R}^n)$.

Remark 3.2.

- Above, we defined symbols $a = a(x, \xi)$ as smooth functions in $\mathbb{R}^p_x \times \mathbb{R}^n_\xi$ for some $p, n \in \mathbb{N}$. When we define pseudodifferential operators, we will then take $p = 2n$, so that $a = a(x, y, \xi) \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$. The variables $x, y, \xi \in \mathbb{R}^n$ are collectively known as the phase space variables. The variables x, y are sometime known as the "space variables" and ξ is often known as the "dual", "Fourier" or "fibre" variable.

- We can also defined the space of symbol, $S^m_\infty(\Omega; \mathbb{R}^n)$ on a set with non-empty interior $\Omega \subset \mathbb{R}^p$ with $\Omega \subset \overline{\mathrm{Int}(\Omega)}$, so that the supremum in the seminorm above is taken over $(x, \xi) \in \mathrm{Int}(\Omega) \times \mathbb{R}^n$.

- The subscript ∞ in S^m_∞ refers the uniform boundedness condition in the space variable.

- There are various generalisation of the symbol spaces which result in similar pseudodifferential calculus. For instance, we could allow polynomial growth in the space variable, and extra-decay in the dual variable, resulting in the space $S^{m,l_1,l_2}_{\infty,\delta}(\mathbb{R}^{2n}_{x,y};\mathbb{R}^n_\xi)$, with element $a \in C^\infty(\mathbb{R}^{3n})$ satisfying

$$\left| D_x^\alpha D_y^\beta D_\xi^\gamma a \right| \leqslant C \langle x\rangle^{l_1} \langle y\rangle^{l_2} \langle \xi\rangle^{m-|\gamma|+\delta|(\alpha,\beta,\gamma)|}.$$

- polyhomogeneous subspace

Example 3.1. EXAMPLES!!

1. microlocal cut-off
2. $P(x,\xi)$
3. polyhomogeneous ones

3.2 Properties of Symbols

3.2.1 Symbols form graded commutative *-algebra

We shall establish the following summarising theorem.

Theorem 3.1 (Summary). *Given $m \in \mathbb{R}$, $p, n \in \mathbb{N}$, then*

1. *$S^\infty_\infty(\mathbb{R}^p;\mathbb{R}^n)$ is a graded commutative *-algebra over $\mathbb{C}$ with continuous inclusion*
$$\iota : S^m_\infty(\mathbb{R}^p;\mathbb{R}^n) \to S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n)$$
for all $m \leqslant m'$.

2. *$S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ is dense in $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ in the topology of $S^{m+\epsilon}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$.*

Refactorise this sections so that it has better narative flow.
Something about classical symbols to prepare for propagation of singularity
We first prove continuous inclusion of lower order into higher order symbol space.

Proposition 3.2. *Let $p, n \in \mathbb{N}$ be given. If $m, m' \in \mathbb{R}$ such that $m \leqslant m'$, then $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n) \subset S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n)$. Furthermore, the inclusion map*

$$\iota : S^m_\infty(\mathbb{R}^p;\mathbb{R}^n) \hookrightarrow S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n)$$

is continuous.

Proof. Let the real numbers $m \leqslant m'$ be given. We note that for any $\xi \in \mathbb{R}^n$

$$\langle \xi\rangle^m \leqslant 1 \cdot \langle \xi\rangle^{m'}$$

and thus if $a \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$, we have that $\forall \alpha \in \mathbb{N}^p, \forall \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \leqslant C \langle \xi\rangle^{m-|\beta|} \leqslant C \langle \xi\rangle^{m'-|\beta|}$$

which show that $a \in S^{m'}_{\infty}(\Omega;\mathbb{R}^n)$ as well.

To show that ι is a continuous inclusion, it suffices to show that

$$\|\iota(a)\|_{N,m'} \leqslant C\,\|a\|_{N,m}$$

for any $a \in S^m_{\infty}(\Omega;\mathbb{R}^n)$ and $N \in \mathbb{N}$. Indeed, this bound holds since

$$\frac{\left| D^{\alpha}_x D^{\beta}_{\xi} a(x,\xi) \right|}{\langle \xi \rangle^{m'-|\beta|}} \leqslant \frac{\left| D^{\alpha}_x D^{\beta}_{\xi} a(x,\xi) \right|}{\langle \xi \rangle^{m-|\beta|}}.$$

for any $x,\xi \in \mathbb{R}^p \times \mathbb{R}^n$.

□

Next, we prove the filtration property of the symbol spaces.

Proposition 3.3. *Let $p,n \in \mathbb{N}$ be given. Then, for any $m,m' \in \mathbb{R}$,*

$$S^m_{\infty}(\Omega;\mathbb{R}^n)\cdot S^{m'}_{\infty}(\Omega;\mathbb{R}^n) = S^{m+m'}_{\infty}(\Omega;\mathbb{R}^n)$$

Proof. Let $a \in S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ and $b \in S^{m'}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ be given. By Leibinz formula **??**, we have that for all multi-index $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^n$,

$$\begin{aligned}
&\sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{\left| D^{\alpha}_x D^{\beta}_{\xi} a(x,\xi) b(x,\xi) \right|}{\langle \xi \rangle^{(m+m')-|\beta|}} \\
&\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} \sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{\left| D^{\mu}_x D^{\gamma}_{\xi} a(x,\xi) \right| \left| D^{\alpha-\mu}_x D^{\beta-\gamma}_{\xi} b(x,\xi) \right|}{\langle \xi \rangle^{(m+m')-|\beta|}} \\
&\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \sup_{\xi\in\mathbb{R}^n} \frac{\langle \xi \rangle^{m-|\gamma|} \langle \xi \rangle^{m'-|\beta-\gamma|}}{\langle \xi \rangle^{(m+m')-|\beta|}} \\
&= \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \sup_{\xi\in\mathbb{R}^n} \langle \xi \rangle^{|\beta|-(|\beta-\gamma|+|\gamma|)} \\
&\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \\
&< \infty
\end{aligned}$$

where we have use the property of multi-index that $|\beta| = |\beta-\mu| + |\mu|$. We have thus shown that $S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)\cdot S^{m'}_{\infty}(\mathbb{R}^p;\mathbb{R}^n) \subset S^{m+m'}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$

For the reverse inclusion, let $c \in S^{m+m'}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ be given. Define

$$\begin{aligned}
a(x,\xi) &= \langle \xi \rangle^m \\
b(x,\xi) &= \frac{c(x,\xi)}{a(x,\xi)}
\end{aligned}$$

and observe that

- $a \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$: It is clear that a is smooth in both x and ξ. It is independent of x and thus any x derivative gives 0. We need only to check that for all $\beta \in \mathbb{N}^n$,
$$\left| D^\beta_\xi \langle\xi\rangle^m \right| \leqslant C \langle\xi\rangle^{m-|\beta|}$$
which can be proven by induction on n and β. We shall only prove the base case where $n = 1$ and $\beta = 1$. We have
$$|D_\xi \langle\xi\rangle^m| = \left|\partial_\xi(1+\xi^2)^{m/2}\right| = \left|m\xi\langle\xi\rangle^{m-2}\right| = \left|m\frac{\xi}{\langle\xi\rangle}\right|\langle\xi\rangle^{m-1} \leqslant |m|\langle\xi\rangle^{m-1}$$
where we have used the fact that $|\xi| \leqslant \langle\xi\rangle$ for all ξ.

- $b \in S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n)$: We note first that $\langle\xi\rangle^m \neq 0$ for all $\xi \in \mathbb{R}^n$ and thus b is well-defined. Since division by $\langle\xi\rangle^m$ does not affect any of the x derivative, we only need to show that for any $\beta \in \mathbb{N}^n$, we have
$$\left| D^\beta_\xi b(x,\xi) \right| \leqslant C \langle\xi\rangle^{m+m'-|\beta|}$$
for some constant $C > 0$ uniformly in ξ. Indeed, observe that by the Leibniz formula
$$\begin{aligned}\left| D^\beta_\xi b(x,\xi)\right| &\leqslant \sum_{\mu\leqslant\beta}\binom{\beta}{\mu}\left|D^\mu_\xi c(x,\xi)\right|\left|D^{\beta-\mu}\langle\xi\rangle^{-m}\right| \\ &\leqslant C\sum_{\mu\leqslant\beta}\binom{\beta}{\mu}\langle\xi\rangle^{m+m'-|\mu|}\langle\xi\rangle^{-m-|\beta-\mu|} \\ &\leqslant C\sum_{\mu\leqslant\beta}\binom{\beta}{\mu}\langle\xi\rangle^{m'-(|\mu|+|\beta-\mu|)} \\ &= C\sum_{\mu\leqslant\beta}\binom{\beta}{\mu}\langle\xi\rangle^{m'-|\beta|} \\ &= C2^\beta\langle\xi\rangle^{m'-|\beta|}\end{aligned}$$
where we have use the definition of c and applied the result proven for a with $m \mapsto -m$. Thus, $b \in S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n)$.

It is clear that $a \cdot b = c$ and we have therefore shown that $S^{m+m'}_\infty(\mathbb{R}^p;\mathbb{R}^n) \subset S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)\cdot S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n)$. □

It is clear that
$$a^*(x,\xi) := \overline{a(x,\xi)} \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \iff a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n).$$

Together with the result above, we obtain the desired algebraic structure for $S^\infty_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ as claimed in **??**.

3.2.2 Density of residual symbol space

Next, we have a rather technical density result. It states that the residual space $S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ is dense in $S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$, but only as a subspace of $S^{m+\epsilon}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$. The main strategy in this proof is to approximate any symbol with the very same symbol but cut off by a compactly supported function. As such, the main reason we **cannot** have density of $S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ in $S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ is the same reason to the fact that Schwartz functions are not dense in the space of bounded smooth functions. For example, the first Schwartz seminorm

$$\sup_{x\in\mathbb{R}^n} |\, f(x) - 1\,| \geqslant 1$$

between of any rapidly decaying function f and the constant function $1 \in S^0_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ is always bounded below by 1.

Lemma 3.4. *Given any $m \in \mathbb{R}$, $n, p \in \mathbb{N}$ and $a \in S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$, there exist a sequence in $S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ that is bounded in $S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ and converges to a in the topology of $S^{m+\epsilon}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$.*

Proof. Let $a \in S^m_{\infty}(\Omega;\mathbb{R}^n)$ and $\epsilon \in \mathbb{R}_{>0}$ be given. Let $\chi \in C^{\infty}_c(\mathbb{R}^n)$ be a non-negative smooth cut-off function, i.e. $0 \leqslant \chi \leqslant 1$ and satisfies

$$\chi(\xi) = \begin{cases} 1 & , |\,\xi\,| < 1 \\ 0 & , |\,\xi\,| > 2. \end{cases}$$

Then, for each $k \in \mathbb{N}$, we define

$$a_k(x,\xi) = \chi\left(\frac{\xi}{k}\right) a(x,\xi).$$

Now, given arbitrary $N, k \in \mathbb{N}$, observe that

$$|\, a_k\,| \leqslant C \langle\xi\rangle^{-N}$$

since a_k is compactly supported in ξ (as χ is compactly supported). Furthermore, by Leibinz formula and the symbol estimates on $a \in S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$, we have

$$\begin{aligned} \left| D^{\alpha}_x D^{\beta}_{\xi} a_k(x,\xi) \right| &\leqslant \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D^{\mu}_{\xi}\chi\right)\left(\frac{\xi}{k}\right) \left| D^{\alpha}_x D^{\beta-\mu}_{\xi} a(x,\xi) \right| \\ &\leqslant C \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D^{\mu}_{\xi}\chi\right)\left(\frac{\xi}{k}\right) \langle\xi\rangle^{m-|\,\beta-\mu\,|}. \end{aligned}$$

Since χ and all its derivatives are compactly supported, each term in the sum above is zero outside of a compact region. Hence, given any $N \in \mathbb{N}$, for a big enough $C > 0$,

$$\left| D^{\alpha}_x D^{\beta}_{\xi} a_k(x,\xi) \right| \leqslant C' \langle\xi\rangle^{-N}$$

which allow us to conclude that $a_k \in S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ and is a bounded sequence in $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$.

It remains to show that $\lim_{k\to\infty} a_k = a$ in $S^{m+\epsilon}_\infty(\Omega;\mathbb{R}^n)$. In the first symbol norm, we observe that, using the symbol estimate for a

$$\begin{aligned}
\|a_k - a\|_{0,m+\epsilon} &= \sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{|\, a_k(x,\xi)\,|}{\langle\xi\rangle^{m+\epsilon}} \\
&= \sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{|\,(1-\chi(\xi/k))\,|\,|\,a(x,\xi)\,|}{\langle\xi\rangle^{m+\epsilon}} \\
&\leqslant \|a\|_{0,m} \sup_{\xi\in\mathbb{R}^n} \frac{|\,(1-\chi(\xi/k))\,|}{\langle\xi\rangle^{\epsilon}} \\
&\leqslant \|a\|_{0,m} \langle k\rangle^{-\epsilon} \\
&\to 0
\end{aligned}$$

as $k\to\infty$, since $|\,(1-\chi(\xi/k))\,|$ is 0 in the region $|\,\xi\,| \leqslant k$ and bounded by 1 otherwise. We remark upon the necessity of the extra decay given by $\langle\xi\rangle^{-\epsilon}$ factor. For other symbol norms we shall again use Leibinz formula:

$$\begin{aligned}
&\sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{\left| D^\alpha_x D^\beta_\xi a_k(x,\xi)\right|}{\langle\xi\rangle^{m+\epsilon-|\,\beta\,|}} \\
&\leqslant \sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{1}{\langle\xi\rangle^{m+\epsilon-|\,\beta\,|}} \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D^\mu(1-\chi)\right)\left(\frac{\xi}{k}\right) \left| D^\alpha_x D^{\beta-\mu}_\xi a(x,\xi)\right| \\
&\leqslant \sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{C}{\langle\xi\rangle^{m+\epsilon-|\,\beta\,|}} \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D^\mu(1-\chi)\right)\left(\frac{\xi}{k}\right) \langle\xi\rangle^{m-|\,\beta-\mu\,|} \\
&= C \sup_{\xi\in\mathbb{R}^n} \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D^\mu(1-\chi)\right)\left(\frac{\xi}{k}\right) \langle\xi\rangle^{-\epsilon-|\,\mu\,|} \\
&\leqslant C' k^{-\epsilon} \\
&\to 0
\end{aligned}$$

as $k\to\infty$ by the same argument as before. Thus, we have proven that $a_k\to a$ as $k\to\infty$ in $S^{m+\epsilon}_\infty(\mathbb{R}^p;\mathbb{R}^n)$.

□

3.3 Pseudodifferential operators

As noted in (**??**), pseudodifferential operators are obtained via symbols via the quantisation procedure

$$S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \ni a \mapsto I(a) = \frac{1}{(2\pi)^n}\int e^{i(x-y)\xi} a(x,y,\xi)\,\mathrm{d}\xi \tag{3.3}$$

with action on Schwartz functions $u \in \mathcal{S}(\mathbb{R}^n)$ given by

$$I(a)(u) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi) u(y) \,\mathrm{d}y \,\mathrm{d}\xi. \tag{3.4}$$

Unfortunately, unless $m < -n$, we have no guarantee that the integral 3.4 above is convergent. However, it can be interpreted as a tempered distribution, with action on a Schwartz function$v \in \mathcal{S}(\mathbb{R}^n)$ given by

$$I(a)(u)(v) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi) u(y) v(x) \,\mathrm{d}y \,\mathrm{d}\xi \,\mathrm{d}x \in \mathbb{C}. \tag{3.5}$$

Hence, our immediate concern is to ensure that this quantisation procedure is well-defined. Explicitly, we want to show that for each $a \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$

$$\begin{aligned} I(a) : \mathcal{S}(\mathbb{R}^n) &\to \mathcal{S}'(\mathbb{R}^n) \\ u &\mapsto I(a)(u) \end{aligned}$$

is a continuous linear map between Frechet spaces. By theorem **??** , this is equivalent to showing that its Schwartz kernel exist.
We will first establish the case for $m < -n$ (write $m = -n - \epsilon$ for some $\epsilon > 0$). As mentioned, the oscillatory integral 3.4, is absolutely convergent and continuity comes from the bound given by the following computation: $\forall M \in \mathbb{N}$, $\forall a \in S^{-n-\epsilon}_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$, $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned} |\, I(a)(\varphi)\,| &\leqslant \frac{1}{(2\pi)^n} \int |\, a(x,y,\xi)\varphi(x,y)\,| \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}y \\ &\leqslant \frac{\|a\|_{0,-n-\epsilon}}{(2\pi)^n} \int \langle \xi \rangle^{-n-\epsilon} \langle (x,y) \rangle^{-M} \langle (x,y) \rangle^{M} \,|\, \varphi(x,y)\,| \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}y \\ &\leqslant \frac{\|a\|_{0,-n-\epsilon} \|\varphi\|_M}{(2\pi)^n} \int \langle \xi \rangle^{-n-\epsilon} \langle (x,y) \rangle^{-M} \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}y \end{aligned}$$

for any $M \in \mathbb{N}$, where Remember to make consistent choice for schwartz norm

$$\|\varphi\|_M := \max_{|\,\alpha\,| \leqslant M} \sup_{(x,y) \in \mathbb{R}^{2n}} \langle (x,y) \rangle^M \left|\, D^\alpha_{x,y} \varphi(x,y) \,\right| \tag{3.6}$$

is the Schwartz seminorm on $\mathcal{S}(\mathbb{R}^{2n})$ (see **??**). If we choose $M \geqslant 2n + 1$, the x, y integrals are convergent and since $m = -n - \epsilon < -n$, the ξ integral converges as well, hence we have

$$|\, I(a)(\varphi)\,| \leqslant C \,\|a\|_{0,m} \,\|\varphi\|_M$$

and hence $\|I(a)\|_{\mathcal{S}'} \leqslant C \,|\, a \,|_{0,m}$ which is the continuity statement for linear maps $S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^{2n})$.

The proposition below extend this result to general $m \in \mathbb{R}$.

Proposition 3.5. *The continuous linear map*

$$I : S^{-\infty}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^{2n})$$

extends uniquely to a linear map

$$I : S^{m}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^{2n})$$

which is continuous as linear map from $S^{m'}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^{2n})$ for arbitrary $m \in \mathbb{R}$ and $m' > m$.

Proof. Let $m, m' \in \mathbb{R}$, $n \in \mathbb{N}$ with $m < m'$ be given. For any $a \in S^{m}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$, the density of $S^{-\infty}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ in $S^{m}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ with the topology of $S^{m'}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ means that there exist a sequence $a_k \in S^{-\infty}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ so that $a_k \to a \in S^{m'}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$. Together with the completeness of $\mathcal{S}'(\mathbb{R}^{2n})$ (being a continuous linear map into $\mathbb{C}$ which is complete), we have unique continuous linear extension of $I : S^{-\infty}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^{2n})$ to $S^{m}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ given by

$$I(a) := \lim_{k \to \infty} I(a_k)$$

which is continuous in the $S^{m'}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ topology. Therefore, it is enough to show that for any $a \in S^{-\infty}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$, there exist $N, M \in \mathbb{N}$, such that

$$|\, I(a)(\varphi)\, | \leqslant C \left\| a \right\|_{N,m'} \left\| \varphi \right\|_{M}.$$

Let $a \in S^{-\infty}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ as above be given. It can be shown by induction that for any $N \in \mathbb{N}$,

$$\left(1 + \Delta_y\right)^N e^{i(x-y)\cdot\xi} = \langle \xi \rangle^N e^{i(x-y)\cdot\xi}$$

where $\Delta_y = -\sum_{j=1}^{n} \partial^2_{y_j}$ is the Laplacian on $\mathbb{R}^n$.
With this, we can use integratusing integration by parts to introduce extra ξ-decay in the integral. Explicitly, for any $N \in \mathbb{N}$,

$$\begin{aligned}
I(a)(\varphi) &= \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi)\varphi(x,y)\, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y \\
&= \frac{1}{(2\pi)^n} \int \langle \xi \rangle^{-2N} \left(1 + \Delta_y\right)^N a(x,y,\xi)\varphi(x,y)\, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y \\
&= \frac{1}{(2\pi)^n} \int \langle \xi \rangle^{-2N} e^{i(x-y)\xi} \left(\sum_{|\,\mu\,|+|\,\nu\,| \leqslant 2N} C_{\mu,\nu} D^{\mu}_y a(x,y,\xi) D^{\nu}_y \varphi(x,y) \right) \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y
\end{aligned}$$

where $C_{\mu,\nu}$ is a complex constant, independent of a, φ, involving only the binomial coefficient. Now, note that using the $2N^{th}$ symbol seminorm in $S^{m'}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ we have the bound

$$\begin{aligned}
\left| D_y^\mu a(x,y,\xi) \right| &= \langle \xi \rangle^{m'} \frac{\left| D_y^\mu a(x,y,\xi) \right|}{\langle \xi \rangle^{m'}} \\
&\leqslant \langle \xi \rangle^{m'} \sup_{(x,\xi)\in\mathbb{R}^{2n}\times\mathbb{R}^n} \max_{|\mu|+|\mu'|+|\mu''|\leqslant 2N} \frac{\left| D_x^{\mu'} D_y^\mu a(x,y,\xi) \right|}{\langle \xi \rangle^{m'-|\mu''|}} \\
&= \langle \xi \rangle^{m'} \|a\|_{2N,m'} .
\end{aligned}$$

And using Schwartz seminorm, we have that for any $M \in \mathbb{N}$ greater than N,

$$\begin{aligned}
\left| D_y^\nu \varphi(x,y) \right| &= \langle (x,y) \rangle^{-M} \langle (x,y) \rangle^{M} \left| D_y^\nu \varphi(x,y) \right| \\
&\leqslant \langle (x,y) \rangle^{-M} \max_{\nu\leqslant\leqslant M} \sup_{(x,y)\in\mathbb{R}^{2n}} \langle (x,y) \rangle^{M} \left| D_y^\nu \varphi(x,y) \right| \\
&\leqslant \langle (x,y) \rangle^{-M} \|\varphi\|_M .
\end{aligned}$$

Bring together both bounds, we have for all positive integers $M > N$,

$$\begin{aligned}
| I(a)(\varphi) | &\leqslant \frac{1}{(2\pi)^n} \sum_{|\mu|+|\nu|\leqslant 2N} C_{\mu,\nu} \int \langle \xi \rangle^{-2N} \left| D_y^\mu a(x,y,\xi) D_y^\nu \varphi(x,y) \right| \,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant C' \|a\|_{2N,m'} \|\varphi\|_M \int \langle \xi \rangle^{m'-2N} \langle (x,y) \rangle^{-M} \,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Thus, as long as $m'-2N < -n$, i.e. $N > \max\left(\frac{m'+n}{2}, 0\right)$, the integral above converges and there exist $C > 0$ independent of a, φ such that

$$| I(a)(\varphi) | \leqslant C \|a\|_{2N,m'} \|\varphi\|_M$$

which makes $I(a)$ a continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$.

□

By the Schwartz Kernel theorem ([]), each $a \in S_\infty^m(\mathbb{R}^{2n};\mathbb{R}^n)$ defines a continuous linear operator

$$I(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$$

which is the desired result.
We can now define the space of m-order pseudodifferential operators to be the image of $S_\infty^m(\mathbb{R}^{2n};\mathbb{R}^n)$ under I.

Definition 3.3.

$$\Psi_\infty^m(\mathbb{R}^n) := \left\{ A = I(a) \;:\; a \in S_\infty^m(\mathbb{R}^{2n};\mathbb{R}^n) \right\}$$

with the total space $\Psi_\infty^\infty(\mathbb{R}^n) := \cup_{m\in\mathbb{R}} \Psi_\infty^m(\mathbb{R}^n)$ and the residual space $\Psi_\infty^{-\infty}(\mathbb{R}^n) := \cap_m \Psi_\infty^m(\mathbb{R}^n)$ are defined similarly to that of symbol spaces.

Now, we make the observation that

$$D_{x_j} e^{i(x-y)\cdot\xi} = i\xi_j(-i)e^{i(x-y)\cdot\xi} = \xi_j e^{i(x-y)\cdot\xi}$$
$$D_{\xi_j} e^{i(x-y)\cdot\xi} = i(x_j - y_j)(-i)e^{i(x-y)\cdot\xi} = (x_j - y_j)e^{i(x-y)\cdot\xi}$$

and thus, by induction and Lebinz formula,

$$D_x^\alpha e^{i(x-y)\cdot\xi} = \xi^\alpha e^{i(x-y)\cdot\xi}$$
$$x^\beta e^{i(x-y)\cdot\xi} = (y - D_\xi)^\beta e^{i(x-y)\cdot\xi} = \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} y^\mu D_\xi^{\beta-\mu} e^{i(x-y)\cdot\xi}.$$

Together with integration by parts, this shows that polynomial multiplication and derivative operations $x^\beta D_x^\alpha$ on $I(a)u, a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n), u \in \mathcal{S}(\mathbb{R}^n)$ can be transform into operations involving only y and ξ variables, namely the integration variables in $I(a)u$. This suggest the following sharper result.

Proposition 3.6. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ with Schwartz kernel $I(a)$, $a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$, then,*

$$A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is a continuous linear map.

Proof. From the proof of (**??**), for sufficiently large $N \in \mathbb{N}$, we have that

$$I(a)\varphi(x) = \sum_{|\mu|+|\nu|\leqslant 2N} C_{\mu,\nu} \int \langle\xi\rangle^{-2N} e^{i(x-y)\xi} D_y^\mu a(x,y,\xi) D_y^\nu \varphi(y)\,\mathrm{d}\xi\,\mathrm{d}y$$

is an absolutely convergent integral for any $a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Thus, we can differentiate under the integral sign and apply (**??**) to get

$$\begin{aligned} D_x^\alpha I(a)\varphi(x) &= \sum_{|\mu|+|\nu|\leqslant 2N} C_{\mu,\nu} \int \langle\xi\rangle^{-2N} D_x^\alpha \left(e^{i(x-y)\xi} D_y^\mu a(x,y,\xi)\right) D_y^\nu \varphi(y)\,\mathrm{d}\xi\,\mathrm{d}y \\ &= \sum_{\substack{\gamma\leqslant\alpha \\ |\mu|+|\nu|\leqslant 2N}} C_{\mu,\nu,\gamma} \int \langle\xi\rangle^{-2N} \xi^{\alpha-\gamma} e^{i(x-y)\xi} D_x^\gamma D_y^\mu a(x,y,\xi) D_y^\nu \varphi(y)\,\mathrm{d}\xi\,\mathrm{d}y. \end{aligned}$$

for any multi-index $\alpha \in \mathbb{N}^n$. Similarly, for multiplication by x^β, $\beta \in \mathbb{N}^n$, we can use (**??**). That, together with integration by parts in ξ gives

$$\begin{aligned} &x^\beta D_x^\alpha I(a)\varphi(x) \\ &= \sum_{\substack{\gamma\leqslant\alpha,\,\lambda\leqslant\beta \\ |\mu|+|\nu|\leqslant 2N}} C_{\mu,\nu,\gamma,\lambda} \int \langle\xi\rangle^{-2N} \xi^{\alpha-\gamma} y^\lambda \left(D_\xi^{\beta-\lambda} e^{i(x-y)\xi}\right) D_x^\gamma D_y^\mu a(x,y,\xi) D_y^\nu \varphi(y)\,\mathrm{d}\xi\,\mathrm{d}y. \end{aligned}$$

Thus, similar to the proof of (**??**),

$$
\begin{aligned}
&\left| x^\beta D_x^\alpha I(a)\varphi(x) \right| \\
&\leqslant \sum_{\substack{\gamma\leqslant\alpha,\,\lambda\leqslant\beta \\ |\,\mu\,|+|\,\nu\,|\leqslant 2N}} C_{\mu,\nu,\gamma,\lambda} \int \langle\xi\rangle^{-2N} \left|\,\xi^{\alpha-\gamma}\,\right| \left|\,y^\lambda\,\right| \left|\,D_x^\gamma D_y^\mu a(x,y,\xi)\,\right| \left|\,D_y^\nu \varphi(y)\,\right| \,\mathrm{d}\xi\,\mathrm{d}y \\
&\leqslant \|a\|_{N,m} \|\varphi\|_M \sum_{\substack{\gamma\leqslant\alpha,\,\lambda\leqslant\beta \\ |\,\mu\,|+|\,\nu\,|\leqslant 2N}} C_{\mu,\nu,\gamma,\lambda} \int \langle\xi\rangle^{-2N+|\,\alpha\,|+m} \langle y\rangle^{|\,\beta\,|-M} \,\mathrm{d}\xi\,\mathrm{d}y \\
&\leqslant C_{\alpha,\beta} \|a\|_{N,m} \|\varphi\|_M
\end{aligned}
$$

where $N, M \in \mathbb{N}$ are chosen so that

$$
\begin{aligned}
N &> \frac{m+|\,\alpha\,|+n}{2} \\
M &> n + |\,\beta\,|\,.
\end{aligned}
$$

Hence, the (equivalent) Schwartz seminorm of $I(a)(\varphi)$ is bounded and hence $I(a)(\varphi) \in \mathcal{S}(\mathbb{R}^n)$ as required.

□

3.3.1 Adjoint

Now, we have shown that every pseudodifferential operator $A \in \Psi_\infty^m(\mathbb{R}^n)$ is an operator

$$A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).$$

Therefore, it has a Frechet space adjoint

$$A^\dagger : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$$

defined by

$$A^\dagger u(\varphi) = u(A\varphi)$$

for all $u \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Lemma 3.7. *Given the map*

$$
\begin{aligned}
T : \mathbb{R}^{2n} \times \mathbb{R} &\to \mathbb{R}^{2n} \times \mathbb{R} \\
(x,y,\xi) &\mapsto (y,x,-\xi)
\end{aligned}
$$

and a symbol $aS_\infty^m(\mathbb{R}^{2n};\mathbb{R}^n)$, adjoint of the operator A with Schwartz kernel $I(a)$ is uniquely given by the operator whose symbol correspond to the pullback of a under T, i.e.

$$I(a)^\dagger = I(T^*a) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$$

Proof. Let $\iota : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be the continuous inclusion of Schwartz function into the space of tempered distribution given by $\iota(u)\varphi = \int u(x)\varphi(x)dx$ for all $u, \varphi \in \mathcal{S}(\mathbb{R}^n)$.

Suppose first $a \in S^{-\infty}_{\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$. We know from (??) that $I(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, thus $\iota \circ I(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a continuous function given by

$$\begin{aligned}
\iota(I(a)u)\varphi &= \int I(a)(u)(x)\varphi(x)dx \\
&= \frac{1}{(2\pi)^n} \int\int u(y)e^{i(x-y)\cdot\xi} a(x,y,\xi)\varphi(x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{(2\pi)^n} \int\int u(y)e^{i(y-x)\cdot\xi} a(x,y,-\xi)\varphi(x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y \\
&= \int u(y) I(T^*a)\varphi(y)\,\mathrm{d}y \\
&= \iota(u)(I(T^*a)\varphi)
\end{aligned}$$

forall $u, \varphi \in \mathcal{S}(\mathbb{R}^n)$. Using the density of the residual space in $S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$ with the topology of $S^{m'}_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$, $m' > m$, the identity above holds for any $a \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$. Finally, by the weak-* density of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$, $\iota \circ I(a)$ has a unique continuous linear extension to a map $S : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$S(u)(\varphi) = u(I(a)\varphi)$$

for any tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$. The last expression is also given by the adjoint $I(a)^\dagger(u)\varphi$, i.e. $\iota \circ I(a)$ extends continuously and uniquely to its adjoint. Therefore, together with the result above, we have

$$I(a)^\dagger = I(T^*a)$$

as required. □

Remark 3.4. Since $I(T^*a)$ is a composition of continuous map and that $T^*T^*a = a$, we can conclude that any symbol $a \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$ defines a continuous function

$$I(a) = I(T^*a)^\dagger : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

A similar conclusion can be made if we instead use the L^2-based pairing, i.e. the inner product on the Hilbert space $L^2(\mathbb{R}^n)$,

$$\langle f, g\rangle := \int f(x)\overline{g(x)}\,\mathrm{d}x.$$

The corresponding Hilbert space adjoint, T^* of an operator T is then defined by

$$\int Tf(x)\overline{g(x)}\,\mathrm{d}x = \int f(x)\overline{T^*g(x)}\,\mathrm{d}x$$

Lemma 3.8. *Given the transposition map*

$$\begin{aligned} F : \mathbb{R}^{2n} \times \mathbb{R} &\to \mathbb{R}^{2n} \times \mathbb{R} \\ (x, y, \xi) &\mapsto (y, x, \xi) \end{aligned}$$

and a symbol $aS^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$*,* L^2*-adjoint of the operator* A *with Schwartz kernel* $I(a)$ *is uniquely given by the operator whose symbol correspond to the complex conjugate of the pullback of* a *under* F*, i.e.*

$$I(a)^* = I(\overline{F^* a})$$

Proof. The proof is similar to that of (**??**) with the computation replaced by

$$\begin{aligned} \int I(a)u(x)\overline{\varphi(x)}\, \mathrm{d}x &= \frac{1}{(2\pi)^n} \int u(y) \overline{\int e^{i(y-x)\cdot\xi}\overline{a(x, y, \xi)}\varphi(x)\, \mathrm{d}x\, \mathrm{d}\xi}\, \mathrm{d}y \\ &= \int u(y)\overline{I(\overline{F^* a})\varphi(y)}\, \mathrm{d}y \end{aligned}$$

□

3.3.2 Composition theorem

In this section we shall prove that, just like symbol spaces, $\Psi^\infty_\infty(\mathbb{R}^n)$ forms a graded *-algebra. The difference being, this time, the algebra is *non-commutative.*

Theorem 3.9 (Summary). *Given* $n \in \mathbb{N}$*,* $\Psi^\infty_\infty(\mathbb{R}^n)$ *is a graded *-algebra over* $\mathbb{C}$ *with continuous inclusion*

$$\iota : \Psi^m_\infty(\mathbb{R}^n) \to \Psi^{m'}_\infty(\mathbb{R}^n)$$

for any $m \leqslant m'$*.*

We shall prove this theorem by first accumulate several technical lemmas. Among them, the most important and useful result is the reduction lemma **??**, which arise from the observation that for any symbol $a = a(x, y, \xi) \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$, there exist a unique symbol $a_L = a_L(x, \xi) \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$ without y dependence, that quantise to the same operator, i.e. $I(a) = I(a_L)$. In fact, for any $t \in [0, 1]$, there is a unique $a_t = a_t((1-t)x + ty, \xi)$ that quantise to the same operator. This shows that $I : S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n) \to \Psi^m_\infty(\mathbb{R}^n)$ is highly non-injective. The reduction lemma allow us to construct an injective quantisation procedure

$$q_L : S^m_\infty(\mathbb{R}^n; \mathbb{R}^n) \to \Psi^m_\infty(\mathbb{R}^n).$$

3.3.3 Asymptotic Summation

Suppose we are given a sequence of symbols with decreasing order, $a_j \in S^{m-j}_\infty(\mathbb{R}^p; \mathbb{R}^n)$, $j \in \mathbb{N}$, we know that $a_j(x, \xi)$ has ever higher rate of decay for large $|\,\xi\,|$ with increasing j. However, the series $\sum_{j\in\mathbb{N}} a_j(x, \xi)$ need not converge. However, we have the following notion of asymptotic convergence.

Definition 3.5 (Asymptotic summation)**.** A sequence of symbols with decreasing order, $a_j \in S^{m-j}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$, $j \in \mathbb{N}$ is **asymptotically summable** if there exist $a \in S^{m}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ such that for all $N \in \mathbb{N}$,

$$a - \sum_{j=0}^{N-1} a_j \in S^{m-N}_{\infty}(\mathbb{R}^p;\mathbb{R}^n).$$

We write

$$a \sim \sum_{j\in\mathbb{N}} a_j.$$

Lemma 3.10. *Every sequence of symbols with decreasing order is asymptotically summable. Furthermore, the sum is unique up to an additive term in* $S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$*.*

Proof sketch. Let $a_j \in S^{m-j}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$, $j \in \mathbb{N}$ be given. For uniqueness, suppose $a, a' \in S^{m}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ are both asymptotic sums of the sequence. We need to show that $a - a' \in S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$. Indeed, for any $N \in \mathbb{N}$,

$$a - a' = \left(a - \sum_{j=0}^{N-1} a_j\right) - \left(a' - \sum_{j=0}^{N-1} a_j\right) \in S^{m-N}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$$

since both terms on the right are elements of $S^{m-N}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$. Thus,

$$a - a' \in \cap_{n\in\mathbb{N}} S^{m-N}_{\infty}(\mathbb{R}^p;\mathbb{R}^n) = S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n).$$

For existence, we construct $a \in S^{m}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ by Borel's method []. Let $\chi \in C^{\infty}_{c}(\mathbb{R}^p)$ be a bump function, that is $0 \leqslant \chi \leqslant 1$ and $\chi|_{\|x\|\leqslant 1} \equiv 1$ and $\chi|_{\|x\|>2} \equiv 0$. We then define

$$a = \sum_{j\in\mathbb{N}} (1-\chi)(\epsilon_j \xi) a_j(x,\xi)$$

where $\mathbb{R}_{>0} \ni \epsilon_j \to 0$ is a strictly monotonic decreasing sequence that converges to 0. We note that the sequence is a finite sum for any input (x,ξ) and hence define a smooth function. It remains to show that, for some choice of ϵ_j with sufficiently rapid decay,

$$\sum_{j\geqslant N} (1-\chi)(\epsilon_j \xi) a_j(x,\xi)$$

converges in $S^{m-N}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ for any $N \in \mathbb{N}$.
The result will then follow by applying Leibinz formula on the expression above and notice that with the choice of

$$\epsilon_j = 2^{-j}$$

the sum converges in all the symbol seminorm converges to a.

□

3.3.4 Reduction

We will now show that $\Psi^m_\infty(\mathbb{R}^n)$ is exactly the range of $I : S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^{2n})$ restricted to $S^m_\infty(\mathbb{R}^n;\mathbb{R}^n) \subset S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$.

Definition 3.6. Let

$$\pi_L : \mathbb{R}^{3n}_{x,y,\xi} \to \mathbb{R}^{2n}_{x,\xi}$$

be the projection map $(x,y,\xi) \mapsto (x,\xi)$. We define the **left quantisation map** as

$$q_L := I \circ \pi_L^* : S^m_\infty(\mathbb{R}^n;\mathbb{R}^n) \to \Psi^m_\infty(\mathbb{R}^n)$$

with elements $a_L \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ known as the **left reduced symbols**.

Remark 3.7. Replacing the projection map with projection onto the second variable instead will give us *right reduced symbols*. The results below apply to both left and right reduced symbols. Indeed, there is a family of quantisation corresponding to the projections $(x,y,\xi) \mapsto ((1-t)x+ty,\xi)$ for each $t \in [0,1]$. Left and right quantisation are special cases corresponding to $t = 0, 1$ respectively.

Lemma 3.11 (Reduction)**.** *For any $a(x,y,\xi) \in S^m_\infty(\mathbb{R}^{2n}_{x,y};\mathbb{R}^n_\xi)$ there exist unique $a_L(x,\xi) \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that $I(a) = q_L(a_L) = I(a_L \circ \pi_L)$. Furthermore, with $\iota : \mathbb{R}^{2n} \ni (x,\xi) \mapsto (x,x,\xi) \in \mathbb{R}^{3n}$ being the diagonal inclusion map, we have*

$$a_L(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \iota^* D^\alpha_y D^\alpha_\xi a(x,y,\xi). \tag{3.7}$$

Sketch. Note that

$$D^\alpha_\xi e^{i(x-y)\xi} = (x-y)^\alpha e^{i(x-y)\cdot\xi} \implies I((x-y)^\alpha a) = I((-1)^{|\alpha|D^\alpha_\xi a})$$

where we have extended the identity that is true using integration by parts in $S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ to general $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ using the density result of symbol space. Now, if we Taylor expand a around the diagonal $x = y$, we get

$$a(x,y,\xi) = \sum_{|\alpha| \leqslant N-1} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha D^\alpha_y a(x,x,\xi) + r_N(x,y,\xi)$$

where

$$r_N(x,y,\xi) = \sum_{|\alpha|=N} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha \int_0^1 (1-t)^{N-1} D^\alpha_y a(x,(1-t)x+ty,\xi)\,\mathrm{d}t$$

for any $N \in \mathbb{N}$. Applying the identity above give us

$$I(a) = \sum_{j=0}^{N-1} A_j + R_N$$

$$A_j = I\left(\sum_{|\alpha|=j} \frac{i^{|\alpha|}}{\alpha!} D_y^\alpha D_\xi^\alpha a(x,x,\xi)\right) \in \Psi_\infty^{m-j}(\mathbb{R}^n)$$

$$R_N \in \Psi_\infty^{m-N}(\mathbb{R}^n)$$

Asymptotic summation lemma give us

$$b(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_y^\alpha D_\xi^\alpha a(x,x,\xi) \in S_\infty^m(\mathbb{R}^n;\mathbb{R}^n)$$

so that $I(a) - I(b) \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$. We have thus reduce the problem to showing that the elements in $\Psi_\infty^{-\infty}(\mathbb{R}^n)$ has left reduced symbol. This we will do prove in the next proposition.

□

about smoothing operators

Proposition 3.12. *A pseudodifferential operator $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is an element of $\Psi_\infty^{-\infty}(\mathbb{R}^n)$ if and only if its Schwartz kernel, K, is smooth and satisfies the estimate*

$$\left| D^\beta D^\alpha K(x,y) \right| \leqslant C_{N,\alpha,\beta} \langle x-y \rangle^{-N}$$

for any $N \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}^n$.

proof sketch. prooofff!!! Suppose $A \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$, then for some $a \in S_\infty^{-\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$, $I(a) = A$, with the Schwartz kernel $I(a) = I(a)(x,y)$ represented by an absolutely convergent integral. The Schwartz kernel can be shown to be smooth and satisfies the precribed esitmates using integration by parts,

$$(x-y)^\alpha D_x^\beta D_y^\gamma I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} (-D_\xi)^\alpha (D_x + i\xi)^\beta (D_y - i\xi)^\gamma a(x,y,\xi)\, \mathrm{d}\xi$$

which again defines an absolutely and uniformly convergent integral.

Conversely, given a smooth Schwartz kernel $K(x,y)$, we define $g(x,z) = K(x,x-z)$. That K satisfies the prescribed estimate means that g is a Schwartz function in z and thus have a Fourier transform in z. We then define a left reduced symbol

$$b(x,\xi) := \frac{1}{(2\pi)^{n/2}} \int e^{-iz\cdot\xi} g(x,z) dz$$

which is a symbol in $S_\infty^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$. Taking inverse Fourier transform shows that

$$I(b)(x,y) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} b(x,\xi)\, \mathrm{d}y\, \mathrm{d}\xi = g(x,x-y) = K(x,y)$$

which means $I(b) = A$ as required.

□

Theorem 3.13 (Composition). *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$ for some $m, m' \in \mathbb{R}$. Then,*

1. $A^* \in \Psi^m_\infty(\mathbb{R}^n)$.
2. $A \circ B \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$.

proof sketch. Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$ for some $m, m' \in \mathbb{R}$ be given. Since $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ (**??**), we have the adjoint operator $A^* : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ defined by

$$A^*u(\varphi) = u(\overline{A\varphi}), \quad u \in \mathcal{S}'(\mathbb{R}^n), \varphi \in \mathcal{S}(\mathbb{R}^n).$$

We check that A^*u is indeed an element of $\mathcal{S}'(\mathbb{R}^n)$ since it is the composition of the maps $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n) \ni \varphi \mapsto \overline{A\varphi}$ which are both continuous. Let $a \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$ be such that $A = I(a)$. Observe that,

$$\begin{aligned}
\langle Au, \varphi \rangle_{L^2} &= \int Au(x)\overline{\varphi(x)}\,\mathrm{d}x \\
&= \int u(y)\overline{\int e^{i(x-y)\cdot\xi}\overline{a(x,y,\xi)}\varphi(x)\,\mathrm{d}x\,\mathrm{d}\xi}\,\mathrm{d}y \\
&= \int u(y)\overline{I(b)\varphi(y)}\,\mathrm{d}y \\
&= \langle u, A^*\varphi \rangle_{L^2}
\end{aligned}$$

where $b(x,y,\xi) = \overline{a(y,x,\xi)} \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$. Thus, $A^* \in \Psi^m_\infty(\mathbb{R}^n)$.

For composition, applying the reduction lemma twice to get $a_L \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$ and $b_L \in S^{m'}_\infty(\mathbb{R}^n; \mathbb{R}^n)$ so that

$$\begin{aligned}
A\varphi(x) &= \frac{1}{(2\pi)^n}\int e^{i(x-y)\cdot\xi}a(x,\xi)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi \\
B^*\varphi(x) &= \frac{1}{(2\pi)^n}\int e^{i(x-y)\cdot\xi}\overline{b(x,\xi)}\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi
\end{aligned}$$

which shows that

$$AB\varphi(x) = \frac{1}{(2\pi)^n}\int e^{i(x-y)\cdot\xi}a(x,\xi)b(y,\xi)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi$$

and thus $AB = I(a(x,\xi)b(y,\xi))$. Since $a(x,\xi)b(y,\xi) \in S^{m+m'}_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$, we have the result $AB \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$ as required.

□

3.4 Principal symbol

The existence and uniqueness of the left or right reduced symbol a_L, a_R of any pseudodifferential operator $A \in \Psi^m_\infty(\mathbb{R}^n)$ shows that the left and right quantisation

$$q_L, q_R : S^m_\infty(\mathbb{R}^n; \mathbb{R}^n) \to \Psi^m_\infty(\mathbb{R}^n)$$

are in fact topological isomorphisms. Therefore, we can define their inverse

$$\sigma_L, \sigma_R : \Psi^m_\infty(\mathbb{R}^n) \to S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$$

which are called the left, resp. right *full symbol map*. As a corollary of the reduction lemma, we can show that for any $a \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$

$$\sigma_L(q_R(a))(x, \xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_x D^\alpha_\xi a(x, \xi). \tag{3.8}$$

Notice that, except for the first term (which is $a(x, y)$ itself, each term in the sum above are at least one order lower. More precisely

$$D^\alpha_x D^\alpha_\xi a(x, \xi) \in S^{m-|\alpha|}_\infty(\mathbb{R}^n; \mathbb{R}^n).$$

In other words, we have

$$\sigma_L(A) - \sigma_R(A) \in S^{m-1}_\infty(\mathbb{R}^n; \mathbb{R}^n)$$

for any $A \in \Psi^m_\infty(\mathbb{R}^n)$. This suggest that we can in fact isolate the leading order behaviour.

Definition 3.8 (Principal symbol)**.** The principal symbol map of order $m \in \mathbb{R}$ is the map

$$\sigma_m : \Psi^m_\infty(\mathbb{R}^n) \to S^{m-[1]}_\infty(\mathbb{R}^n; \mathbb{R}^n)$$

defined by

$$\sigma_m(A) = [\sigma_L(A)] = [\sigma_R(A)]$$

where $[a]$ is the equivalence class of a in the quotient symbol space

$$S^{m-[1]}_\infty(\mathbb{R}^n; \mathbb{R}^n) := S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)/S^{m-1}_\infty(\mathbb{R}^n; \mathbb{R}^n)$$

With the principal symbol map, it is clear that we can form the following short exact sequence.

Lemma 3.14.

$$0 \to \Psi^{m-1}_\infty(\mathbb{R}^n) \hookrightarrow \Psi^m_\infty(\mathbb{R}^n) \to S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)/S^{m-1}_\infty(\mathbb{R}^{2n}; \mathbb{R}^n) \to 0$$

Further exploitation of the asymptotic expansion 3.8 allow us to prove the following central result concerning principal symbols of pseudodifferential operators. The first result translate roughly to the statement that pseudodifferential operators commutes to leading order.

Proposition 3.15. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ and $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$, then*

$$\sigma_{m+m'}(A \circ B) = \sigma_m(A)\sigma_{m'}(B).$$

proof sketch. Using the proof of the composition theorem, we know that $A \circ B = I(a(x,\xi)b(y,\xi))$ has full symbol,

$$a(x,\xi)b(y,\xi) \in S^{m+m'}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$$

with $a, b \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that $I(a) = A$, $I(b) = B$. The algebraic property of the quotient space $S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)/S^{m-1}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ then give the desired result

$$\sigma_{m+m'}(I(a(x,\xi)b(y,\xi))) = [ab] = [a][b] = \sigma_m(A)\sigma_{m'}(B).$$

□

The second result goes one order further in the asymptotic expansion 3.8, where we encounter the sympletic structure on $T^*\mathbb{R}^n$. Recall that the Poisson bracket on $T^*\mathbb{R}^n \cong \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ is given by

$$\{a, b\} := \sum_{j=1}^n \partial_{\xi_j} a \partial_{x_j} b - \partial_{x_j} a \partial_{\xi_j} b = H_a b$$

where H_a is the Hamilton vector field associated with a. We then have the following result.

Proposition 3.16. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ and $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$, then $[A, B] := AB - BA \in \Psi^{m+m'-1}_\infty(\mathbb{R}^n)$ and*

$$\sigma_{m+m'-1}([A, B]) = -i\{\sigma_m(A), \sigma_{m'}(B)\}.$$

3.5 L^2 and Sobolev boundedness

We shall also note here that, using Hormander's square root construction, we can prove the following boundedness result of pseudodifferential operators acting on L^2 and on Sobolev spaces. (for proof see)

Proposition 3.17. *Let $A \in \Psi^0_\infty(\mathbb{R}^n)$. Then $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator. Let $a \in \sigma_0(A)$ be a representative of the principal symbol, then for any $u \in L^2(\mathbb{R}^n)$,*

$$\|Au\|_{L^2} \leqslant C \|u\|_{L^2} + |\langle Eu, u\rangle_{L^2}|$$

for some $C > 0$ and $E \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$.

Proposition 3.18. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, then for any $s \in \mathbb{R}$, $A : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ is a bounded linear map.*

Lemma 3.19 (Square root construction)**.**

Proposition 3.20. $A : H^{s,r}(\mathbb{R}^n) \to H^{s-m,r}(\mathbb{R}^n)$

Chapter 4

Ellipticity and Microlocalisation

Having defined and developed the calculus of pseudodifferential operators, we now turn to matter concerning solving (*pseudo*)differential equations. Typically, we have equation of the form

$$Au = f, \quad A \in \Psi^m_\infty(\mathbb{R}^n),\, u \in \mathcal{S}'(\mathbb{R}^n), f \in H^s(\mathbb{R}^n). \tag{4.1}$$

The goal is to study how regularity and singularity of Au (which is given as f), affects that of the solution, u if it exist.

4.1 Pseudodifferential operators are pseudolocal

We shall show two results on the singular support of pseudodifferential operators. The first result is that the singular support of any pseudodifferential operator is contained within the diagonal, i.e. they are smooth away from $x = y$. The second result is the pseudolocality result that says that action pseudodifferential operator do not increase singular support of distributions. In terms of 4.1, we can say that the solution u is singular at all the points where f is singular.

4.1.1 Support and singular support

First, we need the definition of support and singular supports of operators and distributions. Roughly, the support of a distribution in $\mathbb{R}^n$ consist of points $x \in \mathbb{R}^n$ where the distribution is non-zero after any smooth cut-offs near x.

Definition 4.1. The **support of a tempered distribution** $u \in \mathcal{S}'(\mathbb{R}^n)$ is given by the set

$$\operatorname{supp}(u) = \{x \in \mathbb{R}^n \,:\, \exists \phi \in \mathcal{S}(\mathbb{R}^n), \phi(x) \neq 0, \phi u = 0\}^c\,.$$

And the singular support of a distribution is the set of points where the distribution fails to behave like an element of $\mathcal{S}(\mathbb{R}^n)$.

Definition 4.2. The **singular support of a tempered distribution** $u \in \mathcal{S}'(\mathbb{R}^n)$ is given by the set

$$\operatorname{singsupp}(u) = \{x \in \mathbb{R}^n : \exists \phi \in S(R^n), \phi(x) \neq 0, \phi(u) \in \mathcal{S}(\mathbb{R}^n)\}^c$$

Note that if the singular support of a tempered distribution is empty, it is then an element of $C^\infty(\mathbb{R}^n)$ The support of an operator is given by the support of its Schwartz kernel.

Definition 4.3. The **support of a continuous linear operator** $A : S(R^n) \to \mathcal{S}'(\mathbb{R}^n)$ is given by

$$\operatorname{supp}(A) = \operatorname{supp}(K_A) \subset \mathbb{R}^n \times \mathbb{R}^n$$

where $K_A \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ is the Schwartz kernel of A.

We note from the above that supports or singular supports are complement of open sets, therefore they are closed.
Now, we are ready to show that pseudodifferential operators are smooth away from the diagonal.

Proposition 4.1. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ for some $m \in \mathbb{R}$, then*

$$\operatorname{singsupp}(A) \subset \left\{(x, y) \in \mathbb{R}^{2n} : x = y\right\}.$$

Proof. Using lemma **??**, it suffices to prove this theorem for elements of $\Psi^{-\infty}_\infty(\mathbb{R}^n)$ and then extend by continuity to all orders.

Let $A \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$ with symbol $a \in S^{-\infty}_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$. Its singular support is given by the singular support of the kernel. Since all derivatives of a are $O(\langle\xi\rangle^{-\infty})$, the oscillatory integral below representing the kernel is absolutely convergent and, using integration by parts, we get

$$\begin{aligned}
I(a) &= \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} a(x, y, \xi)\, \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int \frac{1}{(x-y)^\alpha} (-D^\alpha_\xi) \left(e^{i(x-y)\cdot\xi}\right) a(x, y, \xi)\, \mathrm{d}\xi \\
&= \frac{1}{(x-y)^\alpha} \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} (-D^\alpha_\xi) a(x, y, \xi)\, \mathrm{d}\xi \\
&= \frac{1}{(x-y)^\alpha} I((-D^\alpha_\xi)a)
\end{aligned}$$

which is true for all multi-index α of any order. Since all x, y-derivatives of a are uniformly bounded by $\langle\xi\rangle^{-N}$ for any $N \in \mathbb{N}$, we can differentiate under the integral sign to get the equation

$$\begin{aligned} D_x^\beta D_y^\gamma (x-y)^\alpha I(a) &= \frac{1}{(2\pi)^n} \int D_x^\beta D_y^\gamma e^{i(x-y)\cdot\xi} (-D_\xi^\alpha) a(x,y,\xi)\, \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n} \int (-1)^{|\beta|} \xi^{\beta+\gamma} e^{i(x-y)\cdot\xi} (-D_\xi^\alpha) a(x,y,\xi)\, \mathrm{d}\xi \end{aligned}$$

where the last integral gives a smooth function, thus showing that $(x-y)^\alpha I(a)$ is smooth for all α, and hence $I(a)$ is smooth away from $x = y$.

□

Proposition 4.2. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ for some $m \in \mathbb{R}$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ be compactly supported tempered distribution, then*

$$\operatorname{singsupp}(Au) \subset \operatorname{singsupp}(u).$$

We call operators that satisfies the above property pseudolocal operators.

Proof. Again we shall prove the contrapositive statement:

$$x \notin \operatorname{singsupp}(u) \implies x \notin \operatorname{singsupp}(Au)$$

Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be compactly supported and $x_0 \notin \operatorname{singsupp}(u)$. We can choose $\chi \in \mathcal{S}(\mathbb{R}^n)$, (normalised) so that $\chi \equiv 1$ in a neighbourhood of x_0 and that $\chi u \in \mathcal{S}(\mathbb{R}^n)$. Observe that

$$Au = A(\chi u + (1-\chi)u) = A(\chi u) + A(1-\chi)u.$$

Since $\chi u \in \mathcal{S}(\mathbb{R}^n) \implies A\chi u \in \mathcal{S}(\mathbb{R}^n)$ [**?**], we have that

$$\operatorname{singsupp}(Au) = \operatorname{singsupp}(A(1-\chi)u).$$

Furthermore, we know that $x_0 \notin \operatorname{supp}((1-\chi)u)$. Now, we shall further cut-off near x_0 by choosing a $\phi \in \mathcal{S}(\mathbb{R}^n)$ compactly supported away from $\operatorname{supp}(1-\chi)$ and $\phi \equiv 1$ near x_0, i.e.

$$\operatorname{supp}(1-\chi) \cap \operatorname{supp}\phi = \emptyset.$$

We now have an operator $\phi A(1-\chi)$ with kernel

$$\phi(x) K_A(x,y)(1-\phi(y))$$

that vanishes (in particular smooth) on the diagonal. Since we have shown that the singular support of a pseudodifferential operator have to be contained in the diagonal, we conclude that $\phi A(1-\chi)$ is a smoothing operator, and thus $\phi A(1-\chi)u \in C^\infty(\mathbb{R}^n)$ as required.

□

4.2 Global ellipticity

There is a subset of operators for which we can conclude much more than just pseudolocality, namely the set of operators that are *elliptic*, i.e. element of the algebra that are invertible upto additive elements in $\Psi_\infty^{-\infty}(\mathbb{R}^n)$.

Definition 4.4. A pseudodifferential operator $A \in \Psi_\infty^m(\mathbb{R}^n)$ is elliptic if there exist $B \in \Psi_\infty^{-m}(\mathbb{R}^n)$ such that

$$A \circ B - 1 \in \Psi_\infty^{-\infty}(\mathbb{R}^n).$$

The approximate inverse B is known as the (global) elliptic parametrix of A.

A canonical example of elliptic differential operator is the Helmholtz operator

$$1 + \Delta = 1 - \sum_{j=1}^{n} \partial_{x_j}^2 \in \Psi_\infty^2(\mathbb{R}^n)$$

which has explicitly invertible left reduced symbol $|\,\xi\,|^2 + 1$ and principal symbol $|\,\xi\,|^2$. Using the calculus of symbols, we know that $(1+\Delta)^{-1}$ is simply the pseudodifferential operator that acts as

$$(1+\Delta)^{-1}u(x) = I\left(\frac{1}{1+|\,\xi\,|^2}\right)u(x) = \frac{1}{(2\pi)^n}\int e^{i(x-y)\cdot\xi}\frac{1}{1+|\,\xi\,|^2}u(y)\,\mathrm{d}y\,\mathrm{d}\xi$$

We will find out later that ellipticity is in fact a property of the *principal symbol* only. Thus, the Laplacian Δ which has the same principal symbol $|\,\xi\,|^2$ is also elliptic as expected from traditional theory on differential operators.
motivation for elliptic symbols

Definition 4.5. Given $p, n \in \mathbb{N}$ and $m \in \mathbb{R}$, an order m symbol $a \in S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ is (globally) **elliptic** if there exist $\epsilon \in \mathbb{R}_{>0}$ such that

$$\inf_{|\,\xi\,| \geqslant 1/\epsilon} |\,a(x,\xi)\,| \geqslant \epsilon\,\langle\xi\rangle^m\,.$$

The importance of elliptic symbol is that they are invertible modulo $S_\infty^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$ as shown in the next lemma.

Lemma 4.3. *Let $p, n \in \mathbb{N}$, $m \in \mathbb{R}$ be given and let $a \in S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ be an elliptic symbol of order m. Then there exist a symbol $b \in S_\infty^{-m}(\mathbb{R}^p; \mathbb{R}^n)$ such that*

$$a \cdot b - 1 \in S_\infty^{-\infty}(\mathbb{R}^p; \mathbb{R}^n).$$

Proof. We shall follow the general strategy of inverting the symbol outside of a compact set. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a smooth cut off function, i.e $0 \leqslant \phi \leqslant 1$ and $\phi(\xi) = 1$

for $|\,\xi\,| < 1$ and $\phi(\xi) = 0$ for $|\,\xi\,| > 2$.

Let $a \in S^m_\infty(\Omega; \mathbb{R}^n)$ be an elliptic symbol, that is, for any fixed $\epsilon \in \mathbb{R}_{>0}$, we have

$$|\,a(x,\xi)\,| \geqslant \epsilon\,\langle\xi\rangle^m$$

for any $|\,\xi\,| \geqslant 1/\epsilon$. Thus, we can define

$$b(x,\xi) = \begin{cases} \frac{1-\phi(\epsilon\xi/2)}{a(x,\xi)} & |\,\xi\,| \geqslant 1/\epsilon \\ 0 & |\,\xi\,| < 1/\epsilon. \end{cases}$$

We check:

b is well-defined and smooth.
We note that $|\,a(x,\xi)\,| > 0$ whenever $|\,\xi\,| \geqslant 1/\epsilon$ and therefore b is well defined in that region. For smoothness, we note first that b is smooth in the regions $|\,\xi\,| > 1/\epsilon$ and $|\,\xi\,| < 1/\epsilon$. Set $\delta = 1/(2\epsilon)$. In the region where $1/\epsilon - \delta < |\,\xi\,| < 1/\epsilon + \delta$, we have $|\,\epsilon\xi/2\,| < 1/\epsilon$ and therefore $b(x,\xi) \equiv 0$ in this region and is thus smooth. Since the we have covered $\Omega \times \mathbb{R}^n$ by the three chart domain above, b is smooth by the (smooth) gluing lemma.

b is a symbol of order $-m$.
We can prove by induction that in the region $|\,\xi\,| \geqslant 1/\epsilon$

$$D^\alpha_x D^\beta_\xi b = a^{-1-|\,\alpha\,|-|\,\beta\,|} G_{\alpha\beta}$$

for all multi-index α, β, where $G_{\alpha\beta}$ is a symbol of order $(|\,\alpha\,| + |\,\beta\,|)m - |\,\beta\,|$. Therefore, using the ellipticity estimate for a, we get

$$\begin{aligned} \|b\|_{k,-m} &= \sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{\left|\, D^\alpha_x D^\beta_\xi b(x,\xi)\,\right|}{\langle\xi\rangle^{-m-k}} \\ &= \sup_{|\,\xi\,|\geqslant 1/\epsilon} \left|\, a^{-1-|\,\alpha\,|-|\,\beta\,|} G_{\alpha\beta}\,\right| \langle\xi\rangle^{m+k} \\ &\leqslant \frac{\|G_{\alpha\beta}\|_{0,(|\,\alpha\,|+|\,\beta\,|)m-|\,\beta\,|}}{\epsilon} \sup_{|\,\xi\,|\geqslant 1/\epsilon^{1+|\,\alpha\,|+|\,\beta\,|}} \langle\xi\rangle^{-m(1+|\,\alpha\,|+|\,\beta\,|)} \langle\xi\rangle^{m+k} \\ &< \infty \end{aligned}$$

as required.

b is an inverse of a modulo $S^{-\infty}_\infty(\Omega; \mathbb{R}^n)$.
The main observation is that the set where b fails to be the multiplicative inverse of a is a compact set (in ξ) and thus $a \cdot b - 1$ is in fact a compactly supported smooth function of ξ which is a subset of $S^{-\infty}_\infty(\Omega; \mathbb{R}^n)$.

Explicitly, for any $N \in \mathbb{N}$

$$\sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{\left|\, D^\alpha_x D^\beta_\xi (a\cdot b - 1)\,\right|}{\langle\xi\rangle^{-N}} \leqslant \sup_{|\,\xi\,|\leqslant 1/\epsilon} \langle\xi\rangle^N \left|\, D^\alpha_x D^\beta_\xi (\phi(\xi\epsilon/2))\,\right| < \infty.$$

□

The main theorem regarding globally elliptic pseudodifferential operator is the following.

Theorem 4.4. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ be a pseudodifferential operator. Then, the following are equivalent*

1. *A is an elliptic pseudodifferential operator.*
2. *$\sigma_L(A) \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$ is an elliptic symbol.*
3. *$\exists b \in S^{-m}_\infty(\mathbb{R}^n; \mathbb{R}^n)$, s.t. $\sigma_L(A) \cdot b - 1 \in S^{-\infty}_\infty(\mathbb{R}^n; \mathbb{R}^n)$.*
4. *the principal symbol of A is invertible in the quotient symbol space, i.e.*

$$\exists [b] \in S^{m-[1]}_\infty(\mathbb{R}^n; \mathbb{R}^n), \quad s.t. \quad \sigma_m(A) \cdot [b] = [1] \in S^{0-[1]}_\infty(\mathbb{R}^n; \mathbb{R}^n)$$

where $S^{m-[1]}_\infty(\mathbb{R}^n; \mathbb{R}^n)$ denotes the quotient space $S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)/S^{m-1}_\infty(\mathbb{R}^n; \mathbb{R}^n)$.

Proof. Note that we already have (2) $\iff$ (3) from the previous lemma. We remark that (1) $\implies$ (4) is simply the application of the assuming property of principal symbol under composition on the elliptic parametrix of A. For the rest of the proof, see reference**??**.

□

An important characteristic of elliptic pseudodifferential operators is that they are completely regularising, meaning solutions to $Au = f$ have to be smooth if we know that f is smooth.

Proposition 4.5. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ be elliptic. For any $u \in \mathcal{S}'(\mathbb{R}^n)$,*

$$Au \in \mathcal{S}(\mathbb{R}^n) \implies u \in \mathcal{S}(\mathbb{R}^n).$$

Proof. Let $B \in \Psi^{-m}_\infty(\mathbb{R}^n)$ be the elliptic parametrix to A so that $E := BA - 1 \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. We have

$$u = 1 \cdot u = (BA + E)u = BAu + Eu.$$

Since $Eu \in \mathcal{S}(\mathbb{R}^n)$ by **??**, and we have assumed $Au \in \mathcal{S}(\mathbb{R}^n)$, we conclude that $u \in \mathcal{S}(\mathbb{R}^n)$.

□

Proposition 4.6. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ be elliptic and $u \in H^N(\mathbb{R}^n)$ for some $N \in \mathbb{R}$. Then, for any $s \in \mathbb{R}$*

$$Au \in H^s(\mathbb{R}^n) \implies u \in H^{s+m}(\mathbb{R}^n)$$

and u satisfies the estimates: $\exists C > 0$

$$\|u\|_{H^{s+m}} \leqslant C\left(\|Au\|_{H^s} + \|u\|_{H^N}\right).$$

Proof. Again, let $B \in \Psi_\infty^{-m}(\mathbb{R}^n)$ be the elliptic parametrix so that $E := BA - 1 \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$. We know that $B : H^s \to H^{s+m}$ and $E : H^N \to H^{s+m}$ are bounded linear map. Using $u = BAu + Eu$, we have

$$\|u\|_{H^{s+m}} \leqslant \|BAu\|_{H^{s+m}} + \|u\|_{H^{s+m}} \leqslant C\left(\|Au\|_{H^s} + \|u\|_{H^N}\right)$$

for some $C > 0$. □

4.3 Microlocalisation

The principal symbol σ_m captures the leading order behaviour of a pseudodifferential operator when the fibre variable ξ is large. In this chapter, however, we will develop further concepts that describes behaviour of a pseudodifferential operator in different *direction* in the phase space, $T^*\mathbb{R}^n$. Of central importance are

Characteristic set $\Sigma^m(A) \subset T^*_{x,\xi}\mathbb{R}^n$ of an operator $A \in \Psi^m_\infty(\mathbb{R}^n)$ describes points in phase space where A is not locally elliptic. This will lead to the notion of microlocal ellipticity.

Operator wavefront set $\mathrm{WF}'(A) \subset T^*_{x,\xi}\mathbb{R}^n$ describes the directions ξ in the fibre of x where A is not "trivial".

4.3.1 Elliptic set of pseudodifferential operator

Instead of global ellipticity, we will now define a notion of *ellipticity at a point* in phase space which allow up to define various microlocal contructions that focus on localised (conically in phase space) behaviour.

Definition 4.6. A pseudodifferential operator, $A \in \Psi^m_\infty(\mathbb{R}^n), m \in \mathbb{R}$ is **elliptic at a point** $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ if there exist $\epsilon > 0$ such that its left-reduced symbol satisfies the lower bound

$$|\,\sigma_L(A)(x,\xi)\,| \geqslant \epsilon\,\langle\xi\rangle^m$$

in the region

$$\overline{U}_\epsilon = \left\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \,:\, |\,x - x_0\,| \leqslant \epsilon, \left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \epsilon, |\,\xi\,| \geqslant 1/\epsilon\right\}$$

where $\widehat{\xi} = \xi/|\,\xi\,|$ denotes the unit vector in the direction of ξ for any non-zero $\xi \in \mathbb{R}^n$. We denote the set of all elliptic points of A as

$$\mathit{Ell}^m(A) = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \,:\, A \text{ is elliptic of order } m \text{ at } (x,\xi)\}$$

and its complement in $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ as

$$\begin{aligned}\Sigma^m(A) &= Ell^m(A)^c \setminus \{(x,0) \,:\, x \in \mathbb{R}^n\} \\ &= \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \,:\, A \text{ is } \textbf{not} \text{ elliptic of order } m \text{ at } (x,\xi)\}\end{aligned}$$

Lemma 4.7. *Let $A \in \Psi^m_\infty(\mathbb{R}^n), m \in \mathbb{R}$.*

1. *If $\sigma_m(A)(x,\xi)$ is homogeneous of degree m in ξ, then*

$$Ell^m(A) = \{(x_0,\xi_0) \,:\, \xi_0 \neq 0, \sigma_m(A)(x_0,\xi_0) \neq 0\}.$$

2. *$Ell^m(A)$ is open in $\mathbb{R}^n \times \mathbb{R}^n$.*
3. *$Ell^m(A)$ is conic in $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$, in the sense that*

$$(x_0,\xi_0) \in Ell^m(A) \implies (x_0,t\xi_0) \in Ell^m(A), \forall t \in \mathbb{R}_{>0}.$$

4. *$\Sigma^m(A)$ is closed conic.*
5. *if $B \in \Psi^{m'}(\mathbb{R}^n)$, then*

$$\Sigma^{m+m'}(A \circ B) = \Sigma^m(A) \cup \Sigma^{m'}(B).$$

Proof. Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $m \in \mathbb{R}$ be given.

1. Suppose the principal symbol $\sigma_m(A)(x,\xi)$ is homogeneous of order m in ξ. We need to show that

$$(x_0,\xi_0) \in Ell^m(A) \iff \xi_0 \neq 0,\ \sigma_m(A)(x_0,\xi_0) \neq 0.$$

If $\xi_0 = 0$, $(x_0,\xi_0) \notin Ell^m_\infty$ by definition of ellipticity. If $\sigma_m(x_0,\xi_0) = 0$, by homogeneity, we have

$$\sigma_m(x_0,t\xi_0) = t^m\sigma_m(x_0,\xi_0) = t^m \cdot 0 = 0$$

for all $t \in \mathbb{R}_{>0}$. By definition of principal symbol, we can write the left symbol of A as

$$\sigma_L(A) = \sigma_m(A) + a$$

where $a \in S^{m-1}_\infty(\mathbb{R}^n;\mathbb{R}^n)$. Now, observe that for any $\epsilon > 0$, the set

$$\overline{U}_\epsilon = \left\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \,:\, |\,x - x_0\,| \leqslant \epsilon, \left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \epsilon, |\,\xi\,| \geqslant 1/\epsilon\right\}$$

contains the(open) half-line starting at $\widehat{\xi_0}/\epsilon$, i.e. the set $\{(x_0, t\xi_0/(|\,\xi_0\,|\,\epsilon) \,:\, t > 0\}$.

However, by the symbol estimate of a,

$$\begin{aligned}
\left| \sigma_L(A)\left(x_0, \frac{t\xi_0}{|\,\xi_0\,|\,\epsilon}\right) \right| &\leqslant \left(\frac{t}{\epsilon\,|\,\xi_0\,|}\right)^m |\,\sigma_m(x_0,\xi_0)\,| + \left| a\left(x_0, \frac{t\xi_0}{|\,\xi_0\,|\,\epsilon}\right) \right| \\
&= 0 + \left| a\left(x_0, \frac{t\xi_0}{|\,\xi_0\,|\,\epsilon}\right) \right| \\
&\leqslant C \left\langle \frac{t\xi_0}{|\,\xi_0\,|\,\epsilon} \right\rangle^{m-1} \\
&= C \left\langle t/\epsilon \right\rangle^{m-1}
\end{aligned}$$

and therefore

$$\begin{aligned}
\inf_{(x,\xi)\in\overline{U}_\epsilon} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} &\leqslant \inf_{t>0} \frac{\left| \sigma_L(A)\left(x_0, \frac{t\xi_0}{|\,\xi_0\,|\epsilon}\right) \right|}{\langle t/\epsilon\rangle^m} \\
&\leqslant \inf_{t>0} \frac{C\,\langle t/\epsilon\rangle^{m-1}}{\langle t/\epsilon\rangle^m} \\
&= C \inf_{t>0} \langle t/\epsilon\rangle^{-1} \\
&= 0
\end{aligned}$$

which means that $(x_0, \xi_0) \notin \mathit{Ell}^m(A)$.

Conversely, if $\sigma_m(A)(x_0,\xi_0) \neq 0$, by continuity and homogeneity, $\sigma_m(A)$, is non-zero in a (closed) conic neighbourhood, i.e. there exist $\epsilon > 0$ such that $\sigma_m(A) \neq 0$ in

$$\overline{U}_\epsilon = \left\{ (x,\xi)\ :\ |\,x - x_0\,| \leqslant \epsilon, \left| \widehat{\xi} - \widehat{\xi_0} \right| \leqslant \epsilon, |\,\xi\,| \geqslant 1/\epsilon \right\}.$$

Again, writing the left symbol as a sum of the principal symbol an a lower order term, we observe that in $\overline{U}_\epsilon$,

$$\begin{aligned}
\frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} &\geqslant \frac{|\,|\,\sigma_m(A)(x,\xi)\,| - |\,a(x,\xi)\,|\,|}{\langle\xi\rangle^m} \\
&= \left| \frac{|\,\xi\,|^m}{\langle\xi\rangle^m} \left| \sigma_m(A)(x,\widehat{\xi}) \right| - \frac{|\,a(x,\xi)\,|}{\langle\xi\rangle^m} \right|
\end{aligned}$$

By the symbol estimate of a, the second term is tending to 0 which the first term is bounded below by $C = \inf_{(x,\xi)\in\overline{U}_\epsilon} |\,\sigma_m(A)(x,\xi)\,| > 0$. Therefore, choosing a smaller ϵ if necessary, we have $|\,a(x,\xi)\,| / \langle\xi\rangle^m < C$ and thus

$$\inf_{(x,\xi)\in\overline{U}_\epsilon} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} \geqslant C' \geqslant \epsilon.$$

and therefore $(x_0, \xi_0) \in \mathit{Ell}^m(A)$.

2. We note first that if the principal symbol is homogeneous of degree m, the previous result applies and continuity of the principal symbol guarantee that the elliptic set is open (if $\sigma_m(A)$ is non-zero at a point, it is non-zero in an open neighbourhood of the point).

   For the general case, suppose $(x_0, \xi_0) \in Ell^m(A)$. We therefore have for some $\epsilon > 0$,

$$|\,\sigma_L(A)(x,\xi)\,| \geqslant \epsilon\, \langle \xi \rangle^m$$

   in the region

$$\overline{U}_\epsilon(x_0,\xi_0) = \left\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \,:\, |\,x - x_0\,| \leqslant \epsilon, \left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \epsilon, |\,\xi\,| \geqslant 1/\epsilon\right\}.$$

   It suffice to show that there is an open neighbourhood of (x_0, ξ_0) where A remains elliptic. We can take desired open neighbourhood to be

$$V = \left\{(x',\xi') \,:\, \xi' \neq 0,\, |\,x' - x_0\,| < \epsilon/2, \left|\,\widehat{\xi'} - \widehat{\xi_0}\,\right| < \epsilon/2\right\}.$$

   Then, we can check that for every $(x', \xi') \in V$, A satisfies the elliptic estimate in $\overline{U}_{\epsilon/2}(x',\xi')$. Indeed, if $(x,\xi) \in \overline{U}_{\epsilon/2}(x',\xi')$, then

$$\begin{aligned}
&|\,x - x_0\,| \leqslant |\,x - x'\,| + |\,x' - x_0\,| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\
&\left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \left|\,\widehat{\xi} - \widehat{\xi'}\,\right| + \left|\,\widehat{\xi'} - \widehat{\xi_0}\,\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\
&|\,\xi\,| \geqslant 2/\epsilon \geqslant 1/\epsilon
\end{aligned}$$

   which shows that $\overline{U}_{\epsilon/2}(x',\xi') \subset \overline{U}_\epsilon(x_0,\xi_0)$. Therefore,

$$\inf_{(x,\xi)\in\overline{U}_{\epsilon/2}(x',\xi')} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} \geqslant \inf_{(x,\xi)\in\overline{U}_{\epsilon}(x_0,\xi_0)} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} \geqslant \epsilon \geqslant \epsilon/2$$

   as required.

3. Again, this result is immediate if the principal symbol is homogeneous in ξ. In general, this result come from the observation that only $\widehat{\xi} = \xi/\,|\,\xi\,|$ appears in $\overline{U}_\epsilon$ in the definition of $Ell^m(A)$, i.e. only the *direction* in the dual variable is important.

   Explicitly, let $(x_0, \xi_0) \in Ell^m(A)$ and $t \in \mathbb{R}_{>0}$. Clearly $t\xi_0 \neq 0$. And note that

$$\overline{U}_\epsilon(x_0,\xi_0) = \overline{U}_\epsilon(x_0,t\xi_0)$$

   since $\widehat{\xi} = \widehat{t\xi}$.

4. $\Sigma^m(A) = Ell^m(A)^c$ where $Ell^m(A)$ is open and conic. Since complement of conic set is conic, and complement of open is closed, we conclude that $\Sigma^m(A)$ is closed conic.

5. If both principal symbols are homoegenous of degree m, m' respectively, we can applied the result above and by symbol calculus, we have

$$\begin{aligned} Ell^{m+m'}(A\circ B) &= \{(x,\xi)\,:\,\xi\neq 0, \sigma_{m+m'}(A\circ B) = \sigma_m(A)\sigma_{m'}(B)\neq 0\} \\ &= \{(x,\xi)\,:\,\xi\neq 0, \sigma_m(A)\neq 0\}\cap\{(x,\xi)\,:\,\xi\neq 0, \sigma_{m'}(B)\neq 0\} \\ &= Ell^m(A)\cap Ell^{m'}(B). \end{aligned}$$

Taking complement give the desired result.

In general,

□

Definition 4.7. The **wavefront set** of a compactly supported tempered distribution

$$u\in C_c^{-\infty}(\mathbb{R}^n) = \{u\in\mathcal{S}'(\mathbb{R}^n)\,:\,\operatorname{supp}(u)\Subset\mathbb{R}^n\}$$

is given by

$$\mathrm{WF}(u) = \bigcap\left\{\Sigma^0(A)\,:\,A\in\Psi^0_\infty(\mathbb{R}^n), Au\in C^\infty(\mathbb{R}^n)\right\}.$$

For general tempered distribution $u\in\mathcal{S}'(\mathbb{R}^n)$, its wavefront set is given by

$$\mathrm{WF}(u) = \bigcup_{\chi\in C_c^\infty(\mathbb{R}^n)}\mathrm{WF}(\chi u).$$

Proposition 4.8. *For compactly supported tempered distribution, $u\in C_c^{-\infty}(\mathbb{R}^n)$,*

$$\pi(\mathrm{WF}(u)) = \operatorname{singsupp}(u).$$

where $\pi(x,y) = x$ is the projection map.

Proof. To show $\pi(\mathrm{WF}(u))\subset\operatorname{singsupp}(u)$, we observe that, by definition of singular support,

$$x_0\notin\operatorname{singsupp}(u)\implies\exists\phi\in\mathcal{S}(\mathbb{R}^n),\ \phi(x_0)\neq 0,\ \phi u\in\mathcal{S}(\mathbb{R}^n).$$

But since multiplication by ϕ gives an operator in $\Psi^0_\infty(\mathbb{R}^n)$ which is elliptic at (x_0,ξ) for any $\xi\neq 0$ (ϕ is its own principal symbol which happens to be homogeneous and non-zero for any $(x_0,\xi), \xi\neq 0$). Therefore, $x_0\notin\pi(\mathrm{WF}(u))$.

Conversely, if $x_0\notin\pi(\mathrm{WF}(u))$, then for all $\xi\neq 0$, there exist $A_\xi\in\Psi^0_\infty(\mathbb{R}^n)$ such that A_ξ is elliptic at (x_0,ξ) and $A_\xi u\in C^\infty(\mathbb{R}^n)$. Since elliptic set $Ell^0(A_\xi)$ is open and conic, we know that there exist $\epsilon = \epsilon(\xi)$ such that A_ξ is elliptic in the open conic set

$$V_\xi = \left\{(x',\xi')\in\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})\,:\,|x'-x_0|<\epsilon, \left|\widehat{\xi'}-\widehat{\xi}\right|<\epsilon\right\}.$$

Compactness of the sphere (note that $\xi' \mapsto \widehat{\xi'}$ is an embedding of $\mathbb{R}^n \setminus \{0\}$ into S^n) allow us to cover $\{x_0\} \times (\mathbb{R}^n \setminus \{0\})$ with finite number of $V_{\xi_j}, j = 1, \ldots, N$ with corresponding operators A_{ξ_j}.
Now, consider the operator

$$A = \sum_{j=1}^{N} A^*_{\xi_j} A_{\xi_j}.$$

By the pseudolocality of pseudodifferential operators, we know that $A_{\xi_j} u \in C^\infty(\mathbb{R}^n) \implies A^*_{\xi_j} A_{\xi_j} u \in \mathbb{C}^\infty(\mathbb{R}^n)$. Therefore, $Au \in C^\infty(\mathbb{R}^n)$ and A is elliptic at $(x_0, \xi), \forall \xi \neq 0$ with non-negative symbol. We can pick a smooth cut-off χ, $\chi \equiv 1$ when resticted to an $\epsilon/2$-ball around x_0 forming an operator

$$A + (1 - \chi) \in \Psi^0_\infty(\mathbb{R}^n)$$

that is globally elliptic. We can construct a (global) elliptic parametrix E so that

$$1 - (E \circ A + E(1 - \chi)) \in \Psi^{-\infty}_\infty(\mathbb{R}^n).$$

Finally, we can choose any smooth cut-off ϕ with support subordinate to that of χ, i.e. $\operatorname{supp}(\phi) \subset \operatorname{supp}(\chi)$ and note that

$$\phi \circ E \circ (1 - \chi) \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$$

making it a smoothing operator []. Thus, we conclude

$$\phi u = \phi E \circ Au + \phi \circ E \circ (1 - \chi) u \in C^\infty(\mathbb{R}^n)$$

as required.

□

Definition 4.8. Let $a \in S^m_\infty(\mathbb{R}^p; \mathbb{R}^n)$ for some $m \in \mathbb{R}$, $p, n \in \mathbb{N}$ be a symbol. We say a is of order $-\infty$ at a point $(x_0, \xi_0) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\}$ (write $a = O(\langle \xi \rangle^{-\infty})$) if there exist $\epsilon \in \mathbb{R}_{>0}$ such that for all $M \in \mathbb{R}$, there is a constant $C_M > 0$ such that

$$|\, a(x, \xi) \,| \leqslant C_M \, \langle \xi \rangle^{-M}$$

in the neigbourhood of (x_0, ξ_0) given by

$$\overline{U}_{(x_0, \xi_0)} = \left\{ (x, \xi) \in \mathbb{R}^p \times \mathbb{R}^n \,:\, |\, x - x_0 \,| \leqslant \epsilon, \left| \,\widehat{\xi} - \widehat{\xi_0}\, \right| \leqslant \epsilon \right\}.$$

We define the cone support of the symbol a to be all the points in phase space that where it fails to be $O(\langle \xi \rangle^{-\infty})$.

$$\operatorname{conesupp}(a) = \left\{ (x, \xi) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\} \,:\, a = O(\langle \xi \rangle^{-\infty}) \text{ at } (x, \xi) \right\}^c.$$

Lemma 4.9. *Let $a \in S^{\infty}_{\infty}(\mathbb{R}^p; \mathbb{R}^n)$, then*

1. conesupp(a) *is a closed conic set in* $\mathbb{R}^p \times \mathbb{R}^n$.
2. *If $a = O(\langle \xi \rangle^{-\infty})$ at $(x_0, \xi_0) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\}$, then so is $D^\alpha_x D^\beta_\xi a(x, \xi)$ for any multi-index α, β*

With the lemma above, we can in fact define the cone support of a symbol to be the smallest closed conic subset of the phase space (with $\xi \neq 0$) such that, in the complement, a and all its derivatives are of order $-\infty$.

Definition 4.9. Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $m \in \mathbb{R}$ be pseudodifferential operator. We define the **essential support**, $\mathrm{WF}'(A)$, of A to be the cone support of its left symbol, i.e.

$$\mathrm{WF}'(A) = \mathrm{conesupp}(\sigma_L(A)) \subset \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$$

Using symbol calculus we can prove the following result.

Lemma 4.10. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$ be pseudifferential operators. Then*

1. $\mathrm{WF}'(A) = \mathrm{conesupp}(\sigma_R(A))$.
2. $\mathrm{WF}'(A \circ B) \subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B)$.
3. $\mathrm{WF}'(A + B) = \mathrm{WF}'(A) \cup \mathrm{WF}'(B)$.

With the concept of essentail support we can define the notion of *microlocal elliptic parametrix* which can be thought of as local inverse at an elliptic point of ΨDO's.

Proposition 4.11. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ and $z \notin \Sigma^m(A)$. Then there exist a (two-sided) microlocal parametrix $B \in \Psi^{-m}(\mathbb{R}^n)$ such that*

$$z \notin \mathrm{WF}'(1 - AB) \text{ and } z \notin \mathrm{WF}'(1 - BA).$$

Proof. Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ is elliptic at $(x_0, \xi_0) \in \mathit{Ell}^m(A)$. For each $\epsilon \in \mathbb{R}_{>0}$ we define

$$\gamma_\epsilon(x, \xi) = \chi\left(\frac{x - x_0}{\epsilon}\right)(1 - \chi(\epsilon\xi))\chi\left(\frac{\widehat{\xi} - \widehat{\xi_0}}{\epsilon}\right)$$

where $\chi \in C^\infty(\mathbb{R}^n)$ is a smooth cut-off function that is identically 1/2-ball but supported only in the unit ball. Notice that $\gamma_\epsilon \in S^0_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$ with support given by

$$\mathrm{supp}(\gamma_\epsilon) \subset \left\{(x, \xi) \,:\, |\, x - x_0 \,| \leqslant \epsilon,\, |\, \xi \,| \geqslant \frac{1}{2\epsilon},\, \left|\, \widehat{\xi} - \widehat{\xi_0} \,\right| \leqslant \epsilon\right\}.$$

Restricted to the conic set

$$\overline{U}_\epsilon = \left\{ (x,\xi) \,:\, |\, x - x_0 \,| \leqslant \frac{\epsilon}{2},\, \left| \widehat{\xi} - \widehat{\xi_0} \right| \leqslant \frac{\epsilon}{2},\, |\, \xi \,| \geqslant \frac{1}{\epsilon} \right\} \subset \operatorname{supp}(\gamma_\epsilon)$$

it is identically 1 and therefore γ_ϵ is elliptic at (x_0,ξ_0). Let $L_\epsilon = \mathrm{Op}_L(\gamma_\epsilon)$ be the corresponding pseudodifferential operator. Observe that, by construction

$$(x_0,\xi_0) \notin \mathrm{WF}'(1 - L_\epsilon)$$

since $1-\gamma_\epsilon$ is supported away from an ϵ-neighbourhood of $x = x_0$ and the wavefront set of L_ϵ is contained in an ϵ-neighbourhood of (x_0,ξ_0), i.e.

$$\mathrm{WF}'(L_\epsilon) \subset N_\epsilon(x_0,\xi_0) := \left\{ (x,\xi) \,:\, |\, x - x_0 \,| \leqslant \epsilon,\, \left| \widehat{\xi} - \widehat{\xi_0} \right| \leqslant \epsilon \right\}$$

since γ_ϵ is bounded below in some conic neighbourhood of every point in $N_\epsilon(x_0,\xi_0)$.

Now, let $G_s = \mathrm{Op}_L(\langle\xi\rangle^s)$ for each $s \in \mathbb{R}$. Note that G_s is globally elliptic with positive principal symbol. Consider the following operator,

$$J = (1 - L_\epsilon) \circ G_{2m} + A^*A \in \Psi^{2m}_\infty(\mathbb{R}^n)$$

with principal symbol

$$\sigma_{2m}(J) = (1 - \gamma_\epsilon)\, \langle\xi\rangle^{2m} + |\, \sigma_m(A) \,|^2 \,.$$

Since $Ell^m(A)$ is open conic, we can choose ϵ is small enough so that $Ell^m(A) \subset \operatorname{supp}(\gamma_\epsilon)$. Then, by positivity of all terms involved, we have

$$\frac{|\, \sigma_{2m}(J) \,|}{\langle\xi\rangle^{2m}} = (1 - \gamma_\epsilon) + \frac{|\, \sigma_m(A) \,|^2}{\langle\xi\rangle^{2m}}$$

where the first term is bounded below by 1 outside of $\operatorname{supp}(\gamma_\epsilon)$ while in $\operatorname{supp}(\gamma_\epsilon)$ the second term is bounded below by ϵ since A is elliptic (of order m) at every point in $\operatorname{supp}(\gamma_\epsilon)$. Therefore J is globally elliptic and thus have a global elliptic parametrix $H \in \Psi^{-2m}_\infty(\mathbb{R}^n)$. We shall claim that

$$B = H \circ A^* \in \Psi^m_\infty(\mathbb{R}^n)$$

is a (right) microlocal elliptic parametrix to A. Indeed,

$$\begin{aligned} B \circ A - 1 &= HA^*A - 1 \\ &= H\,(J - (1 - L_\epsilon)G_{2m}) - 1 \\ &= (HJ - 1) - H(1 - L_\epsilon)G_{2m}. \end{aligned}$$

Since H is a global parametrix to J, the first term above is a smoothing operator (i.e. an element of $\Psi^{-\infty}_\infty(\mathbb{R}^n)$ and therefore have empty wavefront set. Furthermore, by [?] the wavefront set of the second term is a subste of $\mathrm{WF}'(1 - L_\epsilon)$ which does not contain (x_0,ξ_0) by construction. $\square$

Proposition 4.12. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. If for some $Q' \in \Psi^0_\infty(\mathbb{R}^n)$, $Q'Au \in H^s(\mathbb{R}^n)$, then for any other $Q \in \Psi^0_\infty(\mathbb{R}^n)$ such that $\mathrm{WF}'(Q) \subset Ell^m(A) \cap Ell^0(Q')$ we have $Qu \in H^{s+m}(\mathbb{R}^n)$ and it satisfies the estimate: $\forall N \in \mathbb{R}$, $\exists C > 0$*

$$\|Qu\|_{H^{s+m}} \leqslant C \left(\|Q'Au\|_{H^s} + \|u\|_{H^N}\right).$$

Proof. At any point $\alpha \in \mathrm{WF}'(Q)$, $Q'A$ is microlocally elliptic, since $Ell^2(Q'A) = Ell^2(A) \cap Ell^0(Q')$. Therefore, we can construct a microlocal elliptic parametrix B for $Q'A$ near α so that if $E = BQ'A - 1$ denotes the residue, we have

$$\mathrm{WF}'(QE) = \mathrm{WF}'(E) \cap \mathrm{WF}'(Q) = \emptyset$$

This shows that $QE \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. Also, by definition of E,

$$Qu = Q(BQ'A - E)u = QBQ'Au - QEu$$

where we know that $QB \in \Psi^{-m}_\infty(\mathbb{R}^n)$. Using the Sobolev boundedness of the operators

$$QB : H^{s+m} \to H^s$$
$$E : H^{s+m} \to H^N$$

we get

$$\|Qu\|_{H^{s+m}} \leqslant \|QBQ'Au\|_{H^{s+m}} + \|QEu\|_{H^{s+m}} \leqslant C \left(\|Q'Au\|_{H^s} + \|u\|_{H^N}\right)$$

for some $C > 0$.

□

Proposition 4.13. *Pseudodifferential operators are microlocal in the following sense: Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ and $u \in C_c^{-\infty}(\mathbb{R}^n)$, then*

$$\mathrm{WF}(Au) \subset \mathrm{WF}(u). \tag{4.2}$$

In fact, we have

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}(u).$$

Proof. □

A partial converse to the above is given by the following proposition.

Proposition 4.14. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ and $u \in C_c^{-\infty}(\mathbb{R}^n)$, then*

$$\mathrm{WF}(u) \subset \mathrm{WF}(Au) \cup \Sigma^m(A).$$

Chapter 5

Fredholm problem of wave operator on torus

Having develop a substantial theory for elliptic pseudodifferential operators, we shall presently turn our attention to a simple non-elliptic problem. Specifically, we will consider the Fredholm problem of the wave operator on the torus. We shall first give the definitions of the manifolds and Hilbert spaces that will feature in this problem.

Definition 5.1. Let $\mathbb{S}^1 = [0,1]/(0 \sim 1)$ denote the circle and for any $k \in \mathbb{N}$ let

$$\mathbb{T}^k := \underbrace{\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{k}$$

denote the k-dimensional torus. We shall study the totally periodic wave operator, on $M :== \mathbb{T}^1_t \times \mathbb{T}^n_x$ given by

$$\Box := \partial_t^2 - \sum_{j=1}^{n-1} \partial_{x_j}^2 \tag{5.1}$$

where $(t, x_1, \ldots, x_n)$ are the local coordinates on M.

Note first that $\Box$ is a second order differential (and thus pseudodifferential) operator with *homogeneous* principal symbol given by

$$\sigma_2(\Box)(t,x,\tau,\xi) = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 - \tau^2 = |\,\xi\,|^2 - \tau^2.$$

Thus, the characteristic set is given by the zero set of the principal symbol (excluding the zero section), i.e.

$$\Sigma^2(\Box) = \{\sigma_2(\Box) = 0\} \setminus \{(t,x,\tau,\xi) \,:\, \tau = \xi = 0\}$$

which lies precisely on the 'light cone' emanated from every point $(t,x) \in M$, i.e.

$$\Sigma^2(\Box) := \{(t,x,\tau,\xi) \,:\, 0 \neq |\,\xi\,| = |\,\tau\,|\}$$

So, clearly the operator is not elliptic everywhere. However, using the results we have accumulated thus far and a further result described below, we shall show that we can perturb the wave operator by a "complex absorbing potential" **??**, iQ so that $\Box - iQ$ is a Fredholm operator as a map

$$\Box - iQ : \mathcal{X}^s \to H^{s-1}(M)$$

where $\mathcal{X}^s$ is a non-trivial subspace of $H^s(M)$. This will be the goal of this chapter.

Strategy of the proof

The crucial step in the proof is to show that the following *Fredholm estimates* holds: $\forall s, N \in \mathbb{R}$, $\forall u \in \mathcal{X}^s$, there exist positive real number $C > 0$, such that

$$\begin{aligned} \|u\|_{H^s} &\leqslant C \left(\|(\Box - iQ)u\|_{H^{s-1}} + \|u\|_{H^N} \right) \\ \|u\|_{H^s} &\leqslant C \left(\|(\Box + iQ^*)u\|_{H^{s-1}} + \|u\|_{H^N} \right). \end{aligned}$$

First, we note that in the elliptic set $Ell^2(\Box)$, microlocal elliptic regularity estimate **??** provide us with sharper estimates. However, to obtain the estimates above near elements in the characteristic set $\Sigma^2(\Box)$, we will need to "propagate" results from elliptic regions along the Hamiltonian flow of $p = \sigma_2(\Box)$. This is where we will employ propagation of singularity theorem described in the next section. Indeed, this method will prompt us to construct Q to introduce a larger region in T^*M where $\Box - iQ$ is elliptic, so that every point in the characteristic set of $\Box - iQ$ is a point along the Hamiltonian flow that begins in $Ell^2(\Box - iQ)$.

Significance

One characterisation of Fredholm operators is that they are operators that are invertible up to a compact operator. Moreover, our result will allow us conclude the regularity of the solutions to differential equation of the form

$$(\Box - iQ)u = f, \quad u, f \in H^{-\infty}(M)$$

in terms of the regularity of f. One immediate consequence of the estimates given above is that if f is smooth, we know that $\|(\Box - iQ)u\|_{H^{s-1}}$ (i.e. right hand side of estimate) is bounded for all $s \in \mathbb{R}$ and therefore $\|u\|_{H^s}$ (left hand side) is bounded for all $s \in \mathbb{R}$. In short, we have the result

$$f \in C^\infty(M) \iff u \in C^\infty(M).$$

In particular, solutions to the homogeneous equation $(\Box - iQ)u = 0$ is guarantee to be smooth.

Furthermore, the method we shall employ relies only on the *principal symbol* of $\Box$ which is second order. This allows us to extend the result to any first order pertubation, i.e. any choice of $A \in \Psi^1_\infty(\mathbb{R}^n)$ give rise to a continuous (in fact, analytic) operator valued map

$$f : z \mapsto (\Box - iQ + zA)$$

where $f(z) : \mathcal{X}^s \to H^{s-1}(M)$ is Fredhom for any $z \in \mathbb{C}$. Recall that the Fredholm index is defined by

$$\mathrm{Ind}(T) := \dim \ker T - \dim \mathrm{coker} T, \quad T \text{ Fredholm}$$

and it can be proven that Ind is a continuous as a map from the space of Fredholm maps between Hilbert spaces to $\mathbb{Z}$ **??**. Therefore, the composition $\mathrm{Ind} \circ f : \mathbb{C} \to \mathbb{Z}$ is continuous and thus maps connected sets to conneted sets. Since the only connected sets in $\mathbb{Z}$ are the singletons, we conclude that the family

$$\{\Box - iQ + zA\}_{z\in\mathbb{C}}$$

is in fact a family of Fredholm operators with constant index for any choice of $A \in \Psi^1_\infty(\mathbb{R}^n)$.
something about constant index

5.1 Propagation of singularities

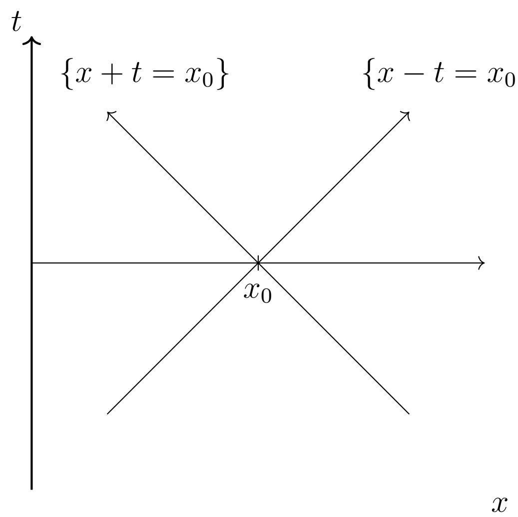
To motivate this result, we will first look at the example of the wave operator in $2D$ Euclidean space, i.e. the operator

$$\partial_t^2 - \partial_x^2.$$

Using the factorisation $(\partial_t^2 - \partial_x^2) = (\partial_t + \partial_x)(\partial_t - \partial_x)$, we can easily obtain the solution to the homogeneous differential equation

$$(\partial_t^2 - \partial_x^2)u = 0 \iff u(t,x) = g(x-t) + h(x+t) \tag{5.2}$$

where g, h are continuous functions defining the right and left moving waves respectively. Notice that if $g(x)$ is not differentiable at $x_0 \in \mathbb{R}$, then $u(t,x)$ is not differentiable on the light cone $\{x - t = x_0\}$ emanated from the initial point $(t,x) = (0, x_0)$. Similarly, any initial singularity $x_1 \in \mathbb{R}$ of h, will be propagated to points on the light cone $\{x + t = x_1\}$.



Thus, we can conjecture that for any solution u to the homogeneous equation 5.2, either $u(t,x)$ is differentiable at (t_0, x_0), or the same singular behaviour (i.e. non-differentiability) will be present on all of the light cone

$$\{(t_0 + s, x_0 + s) \,:\, s \in \mathbb{R}\} \cup \{(t_0 - s, x_0 + s) \,:\, s \in \mathbb{R}\}\,.$$

We shall see that this phenomenon generalise to arbitrary manifolds, with "light cone" replaced by the null-bicharacteristics of the principal symbol of the operator under investigation. More precisely, we have the following result due to Hormander ??:

Theorem 5.1. *Let $P \in \Psi^m_{cl}(\mathbb{R}^n)$ be a classical pseudifferential operator whose principal symbol $\sigma_m(P)$ has real homogeneous representative p. Then,* $\mathrm{WF}(u) \setminus \mathrm{WF}(Pu)$ *is a union of maximally extended null-bicharacteristics of p, i.e. it is the union of the sets of the form*

$$\{\exp(tH_p(x_0, \xi_0)) \,:\, t \in (t_0, t_1) \subset \mathbb{R}, \quad p(x_0, \xi_0) = 0\}$$

where H_p is the Hamilton vector field of p.

The argument involved to establish the propagation theorem can be adapted so that we can accomodate operators with not necessarily real homogeneous principal. The theorem can also be stated in quantitative terms as below:
example on euclidean wave operator in one space dimension insert discussion + proof sketch of prop of sing theorem

Theorem 5.2 (Propagation of singularities)**.** *Suppose we have*

1. *$P \in \Psi^k_{cl}(\mathbb{R}^n)$ a properly supported operator,*
2. *$\sigma_k(P) = p - iq$ for real polyhomogeneous symbols $p, q \in S^k_{ph}(\mathbb{R}^{2n}; \mathbb{R}^n)$,*
3. *$A, B, B' \in \Psi^0_{cl}(\mathbb{R}^n)$ compactly supported and $q \geqslant 0$ on $\mathrm{WF}(B')$,*
4. *for all $(x, \xi) \in \mathrm{WF}(A)$, there exists $\sigma \geqslant 0$ such that for all $t \in [-\sigma, 0]$*

$$\exp(-t \langle \xi \rangle^{1-k} H_p)(x, \xi) \in Ell(B)$$

then for all $s, N \in \mathbb{R}$ and $u \in C^\infty(\mathbb{R}^n)$, there exist $C > 0$ such that

$$\|Au\|_{H^s} \leqslant C\left(\|Bu\|_{H^s} + \|B'Pu\|_{H^{s-k+1}} + \|u\|_{H^{-N}}\right).$$

5.2 Fredholm problem of totally periodic wave operator

We will now define the required operator Q and the subspace $\mathcal{X}^s \subset H^s(M)$ that make $\Box - iQ : \mathcal{X}^s \to H^{s-1}$ Fredholm. As mentioned we will proceed via the "complex absorption" method **??**. We define

$$Q := \chi(t)\partial_t^2$$

where $\chi : \mathbb{S}^1 \to [0,1]$ is a smooth cut-off satisfying

$$\chi(t) = \begin{cases} 1 & t \geqslant 1-\delta \text{ or } t \leqslant \delta \\ 0 & \delta+\delta' < t < 1-\delta-\delta' \end{cases}$$

where $\delta, \delta' \in (0, 1/8)$ (ensuring that $\delta + \delta' < 1/4$).

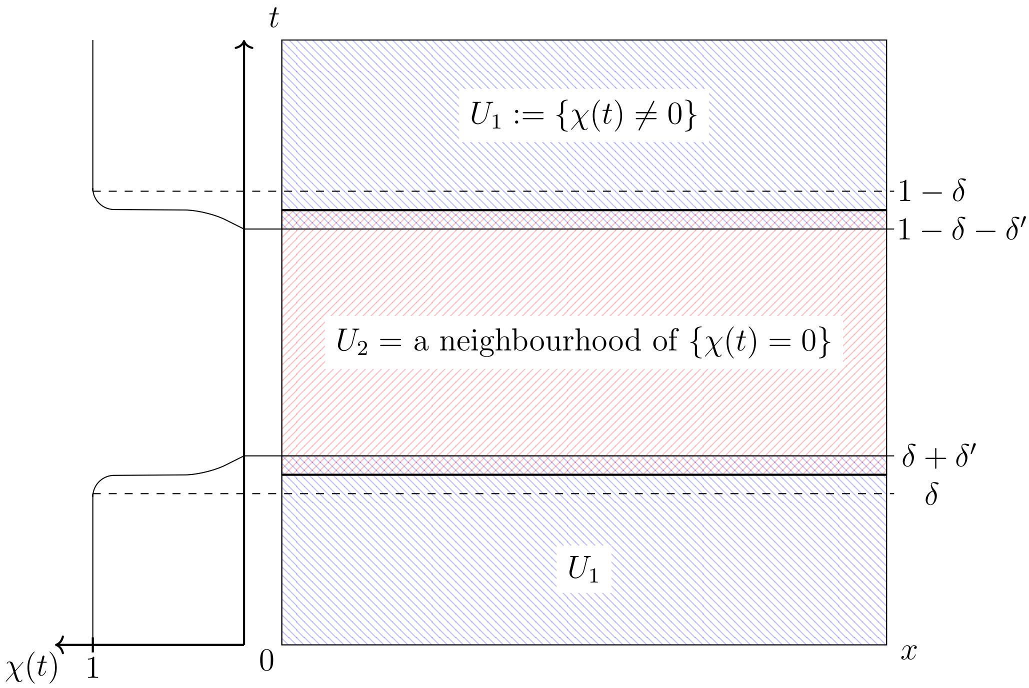


Figure 5.1: The square above represents the fundamental domain of the $n+1$-torus with the periodic time dimension , $t \in [0,1]/\sim$ runing vertically and the space dimensions $x \in ([0,1]/\sim)^n$ collapsed into one-dimensional running horizotally. The graph on the left is the graph of the smooth cut-off function that is supported in U_1, the blue region. The complement of the blue region, U_1^c is the projection (from the cotangent bundle T^*M) of the characteristic set of $\Box - iQ$ onto the manifold M. The red region U_2 is an open neighbourhood of U_1^c.

We note that $\Box - iQ$ is again a second order differential operator, but its principal symbol changes to

$$\sigma_2(\Box - iQ)(t, x, \tau, \xi) = |\,\xi\,|^2 - \tau^2 + i\chi(t)\tau^2 \tag{5.3}$$

which, like $\sigma_2(\Box)$, remains homogeneous to second degree in the dual variables, (τ, ξ). The characteristic set of $\Box - iQ$ is given by the zero set of (5.3) minus the 0 section. In U_2, where $\chi(t) = 0$:

$$\sigma_2(\Box - iQ) = 0 \iff |\,\xi\,| = |\,\tau\,|\,.$$

On the other hand, outside of U_2, $\chi(t) \neq 0$. Thus, the imaginary part of $\sigma_2(\Box - iQ)$ is zero only if $\tau = 0$ which implies that $\xi = 0$ if the real part were to be zero as well. Thus, $\sigma_2(\Box - iQ) = 0$ in U_2^c only if $\xi = \tau = 0$ which is the zero section excluded from the characteristic set. In short, the characteristic set is given by the close set

$$\begin{aligned}\Sigma^2(\Box - iQ) &= \{\sigma_2(\Box - iQ) = 0\} \setminus 0 \\ &= \{(t, x, \tau, \xi) \, : \, \chi(t) \neq 0 \text{ and } 0 \neq |\,\xi\,| = |\,\tau\,|\} \subset T^*M \setminus 0.\end{aligned}$$

We see now that the pertubation Q introduced a region, $T^*U_1 \subset T^*M$, called the absorption region where $\Box - iQ$ is (microlocally) elliptic.
Mention the sympletic structure on the torus and the Hamiltonian flow $\exp(sH_p)(t, x, \tau, \xi) = (t + s\tau, x + s\xi, \tau, \xi)$. As expected, this is tangential to the characteristic set.
The estimate provided by the propagation of singularity theorem suggest that we give the following definition for $\mathcal{X}^s$.

Definition 5.2. Let M be the manifold and $\Box - iQ$ be the operator defined above **??**. For each $s \in \mathbb{R}$, we define a subspace of $H^s(M)$ given by

$$\mathcal{X}^s = \left\{u \in H^s(M) \, : \, (\Box - iQ)u \in H^{s-1}(M)\right\}.$$

Note that $\mathcal{X}^s$ is a Hilbert space (see lemma **??**) with inner product and associated norm given by

$$\begin{aligned}\langle u, v\rangle_{\mathcal{X}^s} &:= \langle u, v\rangle_{H^s} + \langle(\Box - iQ)u, (\Box - iQ)v\rangle_{H^{s-1}} \\ \|u\|^2_{\mathcal{X}^s} &= \langle u, u\rangle_{\mathcal{X}^s} = \|u\|^2_{H^s} + \|(\Box - iQ)u\|^2_{H^{s-1}}\end{aligned}$$

for all $u, v \in \mathcal{X}^s$.

Lemma 5.3. *Let $A_j \in \Psi^{m_j}_\infty(M)$ and $s_j \in \mathbb{R}$ for $j \in \{1, \ldots, N\}$. Define the linear subspace of $H^s(M)$ by*

$$\mathcal{X} := \{u \in H^s \, : \, A_j u \in H^{s_j}, j = 1, \ldots, N\},$$

then with the norm

$$\|u\|^2_{\mathcal{X}} := \|u\|^2_{H^s} + \sum_{j=1}^N \|A_j u\|^2_{H^{s_j}}$$

$\mathcal{X}$ is complete.

Proof. Let $\{u_k\}_{k=1}^\infty$ be a Cauchy sequence in $\mathcal{X}$, i.e. for any $\epsilon > 0$

$$\|u_k\|^2_{H^s} + \sum_{j=1}^N \|A_j u_k\|^2_{H^{s_j}} < \epsilon$$

which in turn implies that each $A_j u_k \in H^{s_j}$ are Cauchy. By completeness of H^s and $H^{s_j}, j = 1, \ldots, N$, we get v and v_j's such that

$$
\begin{aligned}
u_k &\to v \in H^s \\
A_j u_k &\to v_j \in H^{s_j}.
\end{aligned}
$$

Then, by continuity of $A_j : H^s \to H^{s-m_j}$, we have

$$
v_j = \lim_{k\to\infty} A_j u_k = A_j \lim_{k\to\infty} u_k = A_j v.
$$

Since $v_j \in H^{s_j}$, the equality above gives $A_j v \in H^{s_j}$ for each $j = 1, \ldots, N$. Therefore, $v \in \mathcal{X}$. Furthurmore, for any $\epsilon > 0$, the convergence in H^s and H^{s_j} above give the bounds that $\|u_k - v\|_{H^s}^2 < \epsilon/(N+1)$ and $\|A_j u_k - v_j\|_{H^{s_j}}^2 < \epsilon/(N+1)$ for large enough $k \in \mathbb{N}$. Therefore,

$$
\begin{aligned}
\|u_k - v\|_{\mathcal{X}} &= \|u_k - v\|_{H^s}^2 + \sum_{j=1}^{N} \|A_j u_k - A_j v\|_{H^{s_j}}^2 \\
&= \|u_k - v\|_{H^s}^2 + \sum_{j=1}^{N} \|A_j u_k - v_j\|_{H^{s_j}}^2 \\
&< (N+1) \cdot \frac{\epsilon}{N+1} \\
&= \epsilon.
\end{aligned}
$$

Hence, every $\mathcal{X}$-Cauchy sequence converges in $\mathcal{X}$ as required.

□

With the definitions, in place, we are ready to prove:

Theorem 5.4. *For each $s \in \mathbb{R}$, the operator*

$$
(\Box - iQ) : \mathcal{X}^s \to H^{s-1}(\mathbb{T}^n)
$$

is Fredholm.

Proof. As mentioned, we need to show that $\forall s, N \in \mathbb{R}$, $\forall u \in \mathcal{X}^s$, there exist positive real number $C > 0$, such that

$$
\|u\|_{H^s} \leqslant C\left(\|(\Box - iQ)u\|_{H^{s-1}} + \|u\|_{H^N}\right) \tag{5.4}
$$

$$
\|u\|_{H^s} \leqslant C\left(\|(\Box + iQ^*)u\|_{H^{s-1}} + \|u\|_{H^N}\right). \tag{5.5}
$$

Since the estimates above will be proven for all $s \in \mathbb{R}$, adding the term involving H^{s-1}-norm on both sides and recalling that the dual of Sobolev spaces are given by $(H^{s-1})^* = H^{1-s}$the estimates above translate to : $\exists C > 0$,

$$
\|u\|_{\mathcal{X}^s} = \|u\|_{H^s} + \|(\Box - iQ)u\|_{H^{s-1}} \leqslant C\left(\|(\Box - iQ)u\|_{H^{s-1}} + \|u\|_{H^N}\right)
$$

Also, since the torus M is a compact manifold, $H^r(M) \hookrightarrow H^{r'}(M)$ is a compact inclusion for any $r' < r$ [] . Thus, by **??**, $\Box - iQ$ has finite dimensional kernel and closed range.

Furthermore, by the isomorphism between Hilbert space and its continuous dual, the dual map of $\Box - iQ$

$$\begin{array}{ccc} (\Box - iQ)' : (H^{s-1}(M))' & \longrightarrow & (\mathcal{X}^s)' \\ \downarrow \cong & & \downarrow \cong \\ (\Box - iQ)^* : H^{1-s}(M) & \longrightarrow & (\mathcal{X}^s)^* \end{array}$$

is given by a map $(\Box - iQ)^* = \Box + iQ^*$ that maps from $H^{1-s}(M)$. Applying the second estimate to $1-s$, we know that $\Box + iQ^*$ has finite dimensinal kernel. Using **??**, we have

$$\operatorname{coker}(\Box - iQ) \cong \ker(\Box + iQ^*).$$

allowing us to conclude that $\operatorname{coker}(\Box - iQ)$ is finite dimensional as well. Jointly, these allow us to conclude that $\Box - iQ : \mathcal{X}^s \to H^{s-1}(M)$ is Fredholm.

It remains to prove the aforementioned Fredholm estimates. Let $s, N \in \mathbb{R}$, $u \in \mathcal{X}^s$ be given. First, we will establish a sharper estimate in the absorption region where $\Box - iQ$ is elliptic. Let $\chi_1 \in C_c^\infty(T^*M)$ be a smooth compactly supported cut-off with $\operatorname{supp}\chi_1 \subset U_1 \subset Ell^2(\Box - iQ)$. Explicitly, we can choose

$$\chi_1(t, x, \tau, \xi) = \chi(2t), \quad \operatorname{supp}(\chi_1) \subset \{\chi(2t) \neq 0\} \subset \{\chi(t) \neq 0\} = U_1$$

We note that multiplication by χ_1 act as a 0^{th} order pseudodifferential operator. Its wavefront set are points where it is non-zero, i.e.

$$\mathrm{WF}'(\chi_1) = \{(t, x, \tau, \xi) \,:\, \chi(2t) \neq 0\} \subset Ell^2(\Box - iQ)$$

and thus microlocal elliptic regularity estimate applies to give remember to include Vasy's microlocal estimate **??** : $\exists C > 0$ such that

$$\|\chi_1 u\|_{H^s} \leqslant C\left(\|(\Box - iQ)u\|_{H^{s-2}} + \|u\|_{H^N}\right)$$

Outside the absorption region, however, we will rely on propagation of singularity estimate. The crucial geometric observation is that every light ray eminating from a point in U_1^c can be traced, in both forward and backward direction, to a source in U_1. In other words, every element of $\Sigma^2(\Box - iQ)$ outside of the absorption region lies along the Hamiltonian flow of the principal symbol of $\Box$ that begins in the absorption region. In symbols,

$$\begin{aligned} & (t, x, \tau, \xi) \in \Sigma^2(\Box - iQ) \\ \iff & \chi(t) = 0 \text{ and } |\,\tau\,| = |\,\xi\,| \\ \iff & \exists (t', x', \tau', \xi') = (t', x', \tau, \xi) \in Ell^0(\chi_1), \\ & \exp(\sigma H_p)(t', x', \tau', \xi') = (t' + \sigma t, x + \sigma\xi, \tau, \xi) = (t, x, \tau, \xi) \end{aligned}$$

for some $\sigma \in \mathbb{R}$. We have use the fact that $Ell^0(\chi_1) = \{\chi_1 \neq 0\} \subset U_1$. In fact, we can always choose $t' = 0$, since the light ray eminating from the line $t' = 0$ will cover every

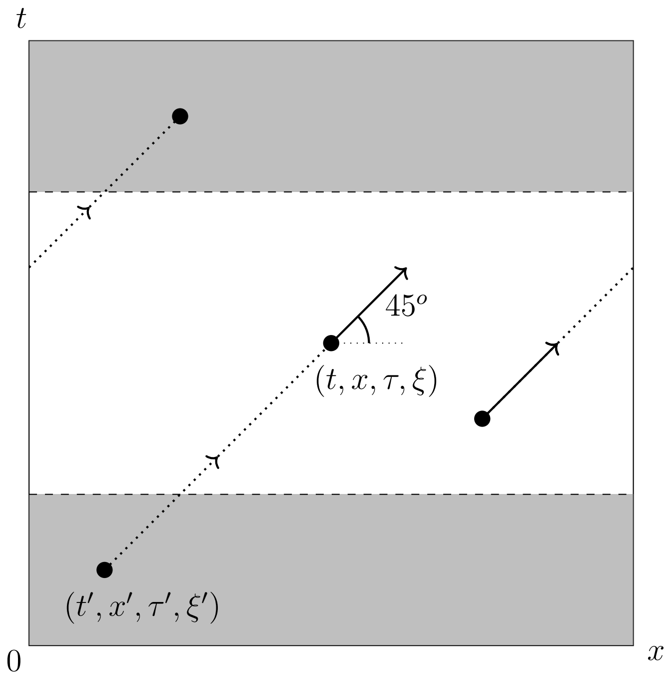


Figure 5.2: The hamiltonian flow associated with the principal symbol $p := \sigma_2(\Box)$ of the wave operator is given by the map $\exp(\sigma H_p)(t, x, \tau, \xi) = (t + \sigma\tau, x + \sigma\xi, \tau, \xi)$ where $\sigma \in \mathbb{R}$ is the flow parameter. As depicted above, every point in the characteristic set of $\Box - iQ$, namely points in U_1^c (white region) pointing in the 45^o direction, i.e. the dual variables satisfies $|\,\xi\,| = |\,\tau\,|$, are points that flows from and will flow into the absorption region (dark region).

As such, by **??**, we obtain the estimate : $\exists C > 0$

$$\|u\|_{H^s} \leqslant C \left(\|\chi_1 u\|_{H^s} + \|(\Box - iQ)u\|_{H^{s-1}} + \|u\|_{H^N} \right).$$

Substituting **??**, we have: $\exists C' > 0$

$$\begin{aligned} \|u\|_{H^s} &\leqslant C' \left(C \,\|(\Box - iQ)u\|_{H^{s-2}} + C\,\|u\|_{H^N} + \|(\Box - iQ)u\|_{H^{s-1}} + \|u\|_{H^N} \right) \\ &\leqslant C'' \left(\|(\Box - iQ)u\|_{H^{s-1}} + \|u\|_{H^N} \right) \end{aligned}$$

since $\|v\|_{H^{s-2}} \leqslant \|v\|_{H^{s-1}}$ for any $v \in H^{s-2}$.

For the second estimate, observe that

$$Q^* = (\chi(t)^2 \partial_t^2)^* = \chi(t)\partial_t^2 + \chi'(t)\partial_t + \chi''(t)$$

which has the same principal symbol $\chi(t)\tau^2$ as that of Q. Therefore, using the same argument, replacing $-iQ$ with $+iQ^*$, reversing the direction of the hamiltonian flow $\exp(-\sigma H_p)$, we get a similar estimate for $\Box + iQ^*$, namely, : $\exists C > 0$

$$\|u\|_{H^s} \leqslant C \left(\|(\Box + iQ^*)u\|_{H^{s-1}} + \|u\|_{H^N} \right).$$

We have thus obtain both Fredholm estimates for $\Box - iQ$.

$\Box$

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