Let $\mathbb{T} = [0,1]/0 \sim 1$ denote the torus and $\mathbb{T}^n$ the n-dimensional torus[1]. We shall study the d'Alembertian, i.e. the totally periodic wave operator, on $\mathbb{T}^n = \mathbb{T}_t \times \mathbb{T}_x^{n-1}$

$$\Box := \partial_t^2 - \sum_{j=1}^{n-1} \partial_{x_j}^2. \tag{1}$$

We first note that the symbol of the operator,

$$\sigma(\Box) = \tau^2 - (\xi_1^2 + \xi_2^2 + \cdots + \xi_{n-1}^2) =: \tau^2 - |\xi|^2$$

is 0 precisely on the light cone $L = \{|\tau| = |\xi|\}$. The operator is therefore not elliptic everywhere in $\mathbb{T}^n$. We shall proceed by using the "complex absorption" method, i.e. we will perturb the operator by some operator $-iQ$ so that $\Box - iQ$ is elliptic on a "large" enough subset of $\mathbb{T}^n$. Specifically, we can take

$$Q = \chi(t)\partial_t^2 \tag{2}$$

where $\chi : \mathbb{T}^n \to \mathbb{R}_{\geqslant 0}$ is a smooth cut-off function supported away from $(-\delta + 1/2, \delta + 1/2)$ for some $\delta \in (0, 1/2)$. Our goal will to prove the following theorem,

Theorem 0.1. *Let $s \in \mathbb{R}$ be given and define*

$$\chi^s = \left\{ u \in H^s(\mathbb{T}^n) \,|\, (\Box - iQ)u \in H^{s-1}(\mathbb{T}^n) \right\}.$$

Then, the operator,

$$(\Box - iQ) : \chi^s \to H^{s-1}(\mathbb{T}^n)$$

is a Fredholm operator.

Lemma 0.2 (Riez's inequality)**.** *Let X be a normed linear space. Given a non-dense subspace (or closed proper subspace) $Y \subset X$ and any $r \in (0,1)$, then there exist $x \in X$ with $\|x\| = 1$ such that*

$$\inf_{y \in Y} \|x - y\| \geqslant r.$$

Corollary. *The closed unit ball in a Banach Space X is compact iff X is finite dimensional.*

Theorem 0.3. *Let X, Y be Hilbert spaces and $T : X \to Y \in \mathcal{L}(X,Y)$ be a continuous (therefore bounded) linear operator. Suppose T satisfies*

$$\begin{aligned} \forall u \in X, \quad & \|u\|_X \leqslant C\left(\|Tu\|_Y + \|u\|_Z\right) \\ \forall v \in Y, \quad & \|v\|_Y \leqslant C'\left(\|T^*v\|_X + \|v\|_{Z^*}\right) \end{aligned}$$

where $Z \Subset X$ and $Z^* \Subset Y$ are compact subsets, then T is Fredholm, i.e. $T(X)$ is closed in Y and both $\ker T, \operatorname{coker} T$ are finite dimensional.

proof sketch.

$\Box$

[1]we shall variously use, without comment, the identifications $\mathbb{T} \cong S^1 \cong \mathbb{R}/\mathbb{Z}$ and $\mathbb{T}^n \cong S^1 \times S^1 \times \cdots \times S^1 \cong \mathbb{R}^n/\mathbb{Z}^n$