Microlocal Analysis Seminar

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1 Reminder: definitions and notations

1.1 Symbols

We shall here list the definition of the space of symbols of order $m \in \mathbb{N}$ in Euclidean space $\mathbb{R}^n$ that one encounters in the literature. The main motivation is based on the property of linear differential operators of order $m \in \mathbb{N}$ with smooth coefficient that, after Fourier transform gives the polynomial of ξ with smooth coefficient

$$P(x, \xi) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\xi^\alpha.$$

Definition 1.1. The **space of symbols of order** m, denoted $S^m_\infty(\mathbb{R}^p; \mathbb{R}^n)$, is the space of smooth functions $a \in C^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ such that for all multi-index $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^n$

$$\left| D^\alpha_x D^\beta_\xi a(x, \xi) \right| \leqslant C_{\alpha,\beta} \langle\xi\rangle^{m-|\beta|}$$

uniformly on $\mathbb{R}^p \times \mathbb{R}^n$. We can also defined the space of symbol, $S^m_\infty(\Omega; \mathbb{R}^n)$ on a set with non-empty interior $\Omega \subset \mathbb{R}^p$, $\Omega \subset \overline{\mathrm{Int}(\Omega)}$ such that the bound above is satisfied uniformly in $(x, \xi) \in \mathrm{Int}(\Omega) \times \mathbb{R}^n$. The subscript ∞ refers the uniform boundedness condition in x. Together with the family of seminorms (indexed by $N \in \mathbb{N}$)

$$\|a\|_{N,m} = \sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \max_{|\alpha|+|\beta|\leqslant N} \frac{D^\alpha_x D^\beta_\xi a(x, \xi)}{\langle\xi\rangle^{m-|\beta|}} \tag{1}$$

gives a Frechet topology to $S^m_\infty(\Omega; \mathbb{R}^n)$.

Furthermore, we define the total symbol space as

$$S^\infty_\infty(\Omega; \mathbb{R}^n) = \bigcup_{m\in\mathbb{R}} S^m_\infty(\Omega; \mathbb{R}^n)$$

and the residual *residual* space of the filtration as

$$S^{-\infty}_\infty(\Omega; \mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} S^m_\infty(\Omega; \mathbb{R}^n).$$

Note: In defining pseudodifferential operators, we shall focus on the case where $p = 2n$, i.e. $a(x, y, \xi) \in S^m_\infty(\mathbb{R}^{2n}_{x,y}; \mathbb{R}^n)$.

1.2 Quantisation

Pseudodifferential operators are defined using symbols. The main gadget is the following oscillatory integral:

$$S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n) \ni a \mapsto I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi)\, \mathrm{d}\xi \tag{2}$$

with action on Schwartz functions $u \in S(\mathbb{R}^n)$ given by

$$I(a)(u) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi)u(y)\, \mathrm{d}y\, \mathrm{d}\xi. \tag{3}$$

The integral 3 above might be divergent unless $m < -n$, but it can be interpreted as a tempered distribution, i.e. a linear function on $S(\mathbb{R}^n)$, with action

$$S(\mathbb{R}^n) \ni v \mapsto I(a)(u)(v) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi)u(y)v(x)\, \mathrm{d}y\, \mathrm{d}\xi\, \mathrm{d}x \in \mathbb{C}. \tag{4}$$

The process of turning the symbol a into an operator $S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ is known as the quantisation procedure. The goal of this talk is the following:

Goal :
To establish that the procedure above is well-defined, so that for each $a \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$

$$\begin{aligned} I(a) : S(\mathbb{R}^n) &\to S'(\mathbb{R}^n) \\ u &\mapsto I(a)(u) \quad : S(\mathbb{R}^n) \to \mathbb{C} \\ &\qquad v \mapsto I(a)(uv) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi) u(y) v(x) \, \mathrm{d}y \, \mathrm{d}\xi \, \mathrm{d}x \end{aligned}$$

is a continuous linear map between Frechet spaces.

Remark. Given $a \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$, we sometimes write $A = Op(a) = I(a)$ for the operator $S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ defined by quantising the symbol a. Also, once the procedure above is proven to be well-defined, we will, with abuse of notation, identify the integral

$$I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi) \, \mathrm{d}\xi \in S'(\mathbb{R}^n \times \mathbb{R}^n)$$

to be the *Schwartz Kernel* of the operator $I(a) : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$.

2 Properties of Symbols

In this section, we shall establish the following summarising theorem:

Theorem 2.1 (Summary)**.** *Given $m \in \mathbb{R}$, $p, n \in \mathbb{N}$, then*

1. $S^m_\infty(\mathbb{R}^p; \mathbb{R}^n)$ *is a Frechet space, hence completely metrisable.*
2. $S^\infty_\infty(\mathbb{R}^p; \mathbb{R}^n)$ *is a graded commutative *-algebra over $\mathbb{C}$ with continuous inclusion*

$$\iota : S^m_\infty(\mathbb{R}^p; \mathbb{R}^n) \to S^{m'}_\infty(\mathbb{R}^p; \mathbb{R}^n)$$

for all $m \leqslant m'$.

3. $S^{-\infty}_\infty(\mathbb{R}^p; \mathbb{R}^n)$ *is dense in* $S^m_\infty(\mathbb{R}^p; \mathbb{R}^n)$ *in the topology of* $S^{m+\epsilon}_\infty(\mathbb{R}^p; \mathbb{R}^n)$ *for any* $\epsilon \in \mathbb{R}_{>0}$.

Exercise : Show that symbol spaces are Frechet spaces. That is, show that the family of seminorms in 1 separates points and that if a sequence is Cauchy in each seminorm, then there exist a unique symbol where the sequence converges in each seminorm.

2.1 $S^\infty_\infty(\mathbb{R}^p; \mathbb{R}^n)$ is a graded commutative *-algebra with continuous inclusion

We first prove continuous inclusion of lower order into higher order symbol space.

Proposition 2.2. *Let $p, n \in \mathbb{N}$ be given and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$. If $m, m' \in \mathbb{R}$ such that $m \leqslant m'$, then $S^m_\infty(\Omega; \mathbb{R}^n) \subset S^{m'}_\infty(\mathbb{R}^)$. Furthermore, the inclusion map*

$$\iota : S^m_\infty(\Omega; \mathbb{R}^n) \to S^{m'}_\infty(\Omega; \mathbb{R}^n)$$

is continuous.

Proof. Let the real numbers $m \leqslant m'$ be given. We note that for any $\xi \in \mathbb{R}^n$

$$\langle \xi \rangle^m \leqslant 1 \cdot \langle \xi \rangle^{m'}$$

and thus if $a \in S^m_\infty(\Omega; \mathbb{R}^n)$, we have that $\forall \alpha \in \mathbb{N}^p, \forall \beta \in \mathbb{N}^n$

$$\left| D^\alpha_x D^\beta_\xi a(x, \xi) \right| \leqslant C \langle \xi \rangle^{m - |\beta|} \leqslant C \langle \xi \rangle^{m' - |\beta|}$$

which show that $a \in S^{m'}_\infty(\Omega; \mathbb{R}^n)$ as well.

To show that ι is a continuous inclusion, it suffices to show that

$$\|\iota(a)\|_{N,m'} \leqslant C \|a\|_{N,m}$$

for any $a \in S^m_\infty(\Omega; \mathbb{R}^n)$ and $N \in \mathbb{N}$. Indeed, this bound holds since

$$\frac{D^\alpha_x D^\beta_\xi a(x, \xi)}{\langle \xi \rangle^{m' - |\beta|}} \leqslant \frac{D^\alpha_x D^\beta_\xi a(x, \xi)}{\langle \xi \rangle^{m - |\beta|}}.$$

□

Proposition 2.3. *Let $p, n \in \mathbb{N}$ be given. Let $\Omega \subset \mathbb{R}^p$ be such that $\Omega \subset \overline{\operatorname{Int}(\Omega)}$. Then, for any $m, m' \in \mathbb{R}$, we have*
$$S^m_\infty(\Omega;\mathbb{R}^n) \cdot S^{m'}_\infty(\Omega;\mathbb{R}^n) = S^{m+m'}_\infty(\Omega;\mathbb{R}^n)$$

Proof. Let $a \in S^m_\infty(\Omega;\mathbb{R}^n)$ and $b \in S^{m'}_\infty(\Omega;\mathbb{R}^n)$ be given. By (general) Leibinz formula, we have that for all multi-index α, β,

$$\begin{aligned}
\sup_{(x,\xi)\in\operatorname{Int}(\Omega)\times\mathbb{R}^n} \frac{\left| D^\alpha_x D^\beta_\xi a(x,\xi) b(x,\xi) \right|}{\langle\xi\rangle^{(m+m')-|\beta|}} &\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} \sup_{(x,\xi)\in\operatorname{Int}(\Omega)\times\mathbb{R}^n} \frac{\left| D^\mu_x D^\gamma_\xi a(x,\xi)\right| \left| D^{\alpha-\mu}_x D^{\beta-\gamma}_\xi b(x,\xi) \right|}{\langle\xi\rangle^{(m+m')-|\beta|}} \\
&\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \sup_{\xi\in\mathbb{R}^n} \frac{\langle\xi\rangle^{m-|\gamma|}\langle\xi\rangle^{m'-|\beta-\gamma|}}{\langle\xi\rangle^{(m+m')-|\beta|}} \\
&= \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{|\beta|-(|\beta-\gamma|+|\gamma|)} \\
&\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \\
&< \infty
\end{aligned}$$

where we have use the property of multi-index that $|\beta| = |\beta-\mu| + |\mu|$. We have thus shown that $S^m_\infty(\Omega;\mathbb{R}^n) \cdot S^{m'}_\infty(\Omega;\mathbb{R}^n) \subset S^{m+m'}_\infty(\Omega;\mathbb{R}^n)$

For the reverse inclusion, let $c \in S^{m+m'}_\infty(\Omega;\mathbb{R}^n)$ be given. Define

$$\begin{aligned}
a &: (x,\xi) \mapsto \langle\xi\rangle^m \\
b &: (x,\xi) \mapsto \frac{c(x,\xi)}{a(x,\xi))}
\end{aligned}$$

and observe that

- $a \in S^m_\infty(\Omega;\mathbb{R}^n)$. It is clear that a is smooth in both x and ξ. It is independent of x and thus any x derivative gives 0. We need only to check that for all $\beta \in \mathbb{N}^n$,
$$\left| D^\beta_\xi \langle\xi\rangle^m \right| \leqslant C \langle\xi\rangle^{m-|\beta|}$$
which can be proven by induction on n and β. We shall only prove the base case where $n = 1$ and $\beta = 1$. We have
$$|D_\xi \langle\xi\rangle^m| = \left|\partial_\xi (1+\xi^2)^{m/2}\right| = \left| m\xi \langle\xi\rangle^{m-2}\right| = \left| m\frac{\xi}{\langle\xi\rangle}\right| \langle\xi\rangle^{m-1} \leqslant |m| \langle\xi\rangle^{m-1}$$
where we have used the fact that $|\xi| \leqslant \langle\xi\rangle$ for all ξ.
- $b \in S^{m'}_\infty(\Omega;\mathbb{R}^n)$. We note first that $\langle\xi\rangle^m \neq 0$ for all $\xi \in \mathbb{R}^n$ and thus b is well-defined. Since division by $\langle\xi\rangle^m$ does not affect any of the x derivative, we only need to show that for any $\beta \in \mathbb{N}^n$, we have
$$\left| D^\beta_\xi b(x,\xi) \right| \leqslant C \langle\xi\rangle^{m+m'-|\beta|}$$

for some constant $C > 0$ uniformly in ξ. Indeed, observe that by the Leibinz formula

$$\begin{aligned}\left| D_\xi^\beta b(x,\xi) \right| &\leqslant \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \left| D_\xi^\mu c(x,\xi) \right| \left| D^{\beta-\mu} \langle \xi \rangle^{-m} \right| \\ &\leqslant C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m+m'-|\mu|} \langle \xi \rangle^{-m-|\beta-\mu|} \\ &\leqslant C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m'-(|\mu|+|\beta-\mu|)} \\ &= C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m'-|\beta|} \\ &= C 2^\beta \langle \xi \rangle^{m'-|\beta|}\end{aligned}$$

where we have use the definition of c and applied the result proven for a with $m \mapsto -m$. Thus, $b \in S_\infty^{m'}(\Omega; \mathbb{R}^n)$.

It is clear that $a \cdot b = c$ and we have therefore shown that $S_\infty^{m+m'}(\Omega; \mathbb{R}^n) \subset S_\infty^m(\Omega; \mathbb{R}^n) \cdot S_\infty^{m'}(\Omega; \mathbb{R}^n)$. ☐

The results above, together with the easily proven fact $a^*(x,\xi) := \overline{a(x,\xi)} \in S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n) \iff a \in S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n)$, gives the desired algebraic structure for $S_\infty^\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$

2.2 Density of residual space, $S_\infty^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$

Next, we have a rather technical density result : the residual space, $S_\infty^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$, is dense in $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$, but only with the topology of $S_\infty^{m+\epsilon}(\mathbb{R}^p; \mathbb{R}^n)$. The main strategy in this proof is to approximate any symbol with the very same symbol but cut off by a compactly supported function. As such, the main reason we **cannot** have density of $S_\infty^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$ in $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ is the same reason to the fact that Schwartz functions are not dense in the space of smooth bounded functions, in particular, $1 \in S_\infty^0(\Omega; \mathbb{R}^n)$ is not in the closure of $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$.

Lemma 2.4. *Given any $m \in \mathbb{R}$, $n, p \in \mathbb{N}$ and $a \in S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$, there exist a sequence in $S_\infty^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$ that is bounded in $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ and converges to a in the topology of $S_\infty^{m+\epsilon}(\mathbb{R}^p; \mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$.*

Proof. Let $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ and $\epsilon \in \mathbb{R}_{>0}$ be given. Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be a non-negative smooth cut-off function, i.e. $\chi \geqslant 0$ and satisfies

$$\chi(\xi) = \begin{cases} 1 & , |\xi| < 1 \\ 0 & , |\xi| > 2. \end{cases}$$

Then, for each $k \in \mathbb{N}$, we define

$$a_k(x,\xi) = \chi\left(\frac{\xi}{k}\right) a(x,\xi).$$

Now, given arbitrary $N, k \in \mathbb{N}$, observe that

$$|a_k| \leqslant C \langle \xi \rangle^{-N}$$

since a_k is compactly supported in ξ (as χ is compactly supported). Furthermore, by Leibinz formula and the symbol estimates on $a \in S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$, we have

$$\begin{aligned}\left| D_x^\alpha D_\xi^\beta a_k(x,\xi) \right| &\leqslant \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} k^{-|\mu|} \left(D_\xi^\mu \chi \right) \left(\frac{\xi}{k} \right) \left| D_x^\alpha D_\xi^{\beta-\mu} a(x,\xi) \right| \\ &\leqslant C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} k^{-|\mu|} \left(D_\xi^\mu \chi \right) \left(\frac{\xi}{k} \right) \langle \xi \rangle^{m-|\beta-\mu|}.\end{aligned}$$

Since χ and all its derivatives are compactly supported, each term above is bounded in ξ and thus a_k is bounded in $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ and that

$$\left| D^\alpha_x D^\beta_\xi a_k(x,\xi) \right| \leqslant C' \langle \xi \rangle^{-N}$$

which allow us to conclude that $a_k \in S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$.

It remains to show that $\lim_{k\to\infty} a_k = a$ in $S^{m+\epsilon}_\infty(\Omega;\mathbb{R}^n)$. In the first symbol norm, we observe that, using the symbol estimate for a

$$\begin{aligned}
\|a_k - a\|_{0,m+\epsilon} &= \sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{|\, a_k(x,\xi)\,|}{\langle \xi \rangle^{m+\epsilon}} \\
&= \sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{|\,(1-\chi(\xi/k))\,|\,|\,a(x,\xi)\,|}{\langle \xi \rangle^{m+\epsilon}} \\
&\leqslant \|a\|_{0,m} \sup_{\xi\in\mathbb{R}^n} \frac{|\,(1-\chi(\xi/k))\,|}{\langle \xi \rangle^{\epsilon}} \\
&\leqslant \|a\|_{0,m} \langle k \rangle^{-\epsilon} \\
&\to 0
\end{aligned}$$

as $k\to\infty$, since $|\,(1-\chi(\xi/k))\,|$ is 0 in the region $|\,\xi\,| \leqslant k$ and bounded by 1 otherwise. We remark upon the necessity of the extra decay given by $\langle \xi \rangle^{-\epsilon}$ factor. For other symbol norms we shall again use Leibinz formula:

$$\begin{aligned}
\sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{\left| D^\alpha_x D^\beta_\xi a_k(x,\xi) \right|}{\langle \xi \rangle^{m+\epsilon-|\,\beta\,|}} &\leqslant \sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{1}{\langle \xi \rangle^{m+\epsilon-|\,\beta\,|}} \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D^\mu(1-\chi)\right)\left(\frac{\xi}{k}\right) \left| D^\alpha_x D^{\beta-\mu}_\xi a(x,\xi) \right| \\
&\leqslant \sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{C}{\langle \xi \rangle^{m+\epsilon-|\,\beta\,|}} \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D^\mu(1-\chi)\right)\left(\frac{\xi}{k}\right) \langle \xi \rangle^{m-|\,\beta-\mu\,|} \\
&= C \sup_{\xi\in\mathbb{R}^n} \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D^\mu(1-\chi)\right)\left(\frac{\xi}{k}\right) \langle \xi \rangle^{-\epsilon-|\,\mu\,|} \\
&\leqslant C' k^{-\epsilon} \\
&\to 0
\end{aligned}$$

as $k\to\infty$ by the same argument as before. Thus, we have proven that $a_k \to a$ as $k\to\infty$ in $S^{m+\epsilon}_\infty(\mathbb{R}^p;\mathbb{R}^n)$.

$\square$

3 Quantisation

3.1 Continuity of $I(a) : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$

We first note that, if $m < -n$ (write $m = -n - \epsilon$ for some $\epsilon > 0$), the oscillatory integral 3, is absolutely convergent and defines a continuous linear operator

$$\begin{aligned} &I : S_\infty^{-n-\epsilon}(\mathbb{R}^{2n}; \mathbb{R}^n) \to S'(\mathbb{R}^{2n}) \\ &\qquad a \mapsto I(a) : S(\mathbb{R}^{2n}) \ni \varphi \mapsto I(a)(\varphi) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi)\varphi(x, y)\, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y. \end{aligned}$$

The map above is clearly linear. Continuity comes from the bound given by the following computation: $\forall M \in \mathbb{N}$, $\forall a \in S_\infty^{-n-\epsilon}(\mathbb{R}^{2n}; \mathbb{R}^n)$, $\forall \varphi \in S(\mathbb{R}^n)$

$$\begin{aligned} |\, I(a)(\varphi) \,| &\leqslant \frac{1}{(2\pi)^n} \int |\, a(x, y, \xi)\varphi(x, y) \,|\, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y \\ &\leqslant \frac{\|a\|_{0,-n-\epsilon}}{(2\pi)^n} \int \langle \xi \rangle^{-n-\epsilon} \langle (x, y) \rangle^{-M} \langle (x, y) \rangle^{M} \,|\, \varphi(x, y) \,|\, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y \\ &\leqslant \frac{\|a\|_{0,-n-\epsilon} \|\varphi\|_M}{(2\pi)^n} \int \langle \xi \rangle^{-n-\epsilon} \langle (x, y) \rangle^{-M}\, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y \end{aligned}$$

for any $M \in \mathbb{N}$, where

$$\|\varphi\|_M := \sum_{|\,\alpha\,| \leqslant M} \sup_{(x,y) \in \mathbb{R}^{2n}} \langle (x, y) \rangle^M \,\big|\, D^\alpha_{x,y}\varphi(x, y) \,\big| \tag{5}$$

is the Schwartz seminorm on $S(\mathbb{R}^{2n})$. If we choose $M \geqslant 2n + 1$, the x, y integrals are convergent and since $m = -n - \epsilon < -n$, the ξ integral converges as well, hence we have

$$|\, I(a)(\varphi) \,| \leqslant C \,\|a\|_{0,m} \,\|\varphi\|_M$$

which implies continuity.

The proposition below extend this result to general $m \in \mathbb{R}$.

Proposition 3.1. *The continuous linear map*

$$I : S_\infty^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n) \to S'(\mathbb{R}^{2n})$$

extends uniquely to a linear map

$$I : S_\infty^{m}(\mathbb{R}^{2n}; \mathbb{R}^n) \to S'(\mathbb{R}^{2n})$$

which is continuous as linear map from $S_\infty^{m'}(\mathbb{R}^{2n}; \mathbb{R}^n)$ to $S'(\mathbb{R}^{2n})$ for arbitrary $m \in \mathbb{R}$ and $m' > m$.

Proof. Let $m, m' \in \mathbb{R}$, $n \in \mathbb{N}$ with $m < m'$ be given. For any $a \in S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n)$, the density of $S_\infty^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ in $S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n)$ with the topology of $S_\infty^{m'}(\mathbb{R}^{2n}; \mathbb{R}^n)$ means that there exist a sequence $a_k \in S_\infty^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ so that $a_k \to a \in S_\infty^{m'}(\mathbb{R}^{2n}; \mathbb{R}^n)$. Together with the completeness of $S'(\mathbb{R}^{2n})$ (being a continuous linear map into $\mathbb{C}$ which is complete), we have unique continuous linear extension of $I : S_\infty^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n) \to S'(\mathbb{R}^{2n})$ to $S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n)$ given by

$$I(a) := \lim_{k \to \infty} I(a_k)$$

which is continuous in the $S^{m'}_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$ topology. Therefore, it is enough to show that for any $a \in S^{-\infty}_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$ and $\varphi \in S(\mathbb{R}^{2n})$, there exist $N, M \in \mathbb{N}$, such that

$$|\, I(a)(\varphi)\,| \leqslant C\, \|a\|_{N,m'}\, \|\varphi\|_M\,.$$

Let a, φ as above be given. Note that

$$e^{i(x-y)\xi} = \langle\xi\rangle^{-2q}\,(1+\xi\cdot D_x)^q\, e^{i(x-y)\xi} = \langle\xi\rangle^{-2q}\,(1-\xi\cdot D_y)^q\, e^{i(x-y)\xi}.$$

Thus, using integration by parts, for any $q \in \mathbb{N}$,

$$\begin{aligned}
I(a)(\varphi) &= \frac{1}{(2\pi)^n}\int e^{i(x-y)\xi} a(x,y,\xi)\varphi(x,y)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{(2\pi)^n}\int \langle\xi\rangle^{-4q}\,(1-\xi\cdot D_y)^q\,(1+\xi\cdot D_x)^q\, e^{i(x-y)\xi} a(x,y,\xi)\varphi(x,y)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{(2\pi)^n}\int \langle\xi\rangle^{-4q}\, e^{i(x-y)\xi}\,(1-\xi\cdot D_y)^q\,(1+\xi\cdot D_x)^q\,[a(x,y,\xi)\varphi(x,y)]\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{(2\pi)^n}\int \langle\xi\rangle^{-4q}\, e^{i(x-y)\xi}\left(\sum_{|\,\gamma\,|\leqslant 2q} a_\gamma(x,y,\xi) D^\gamma_{x,y}\varphi(x,y)\right)\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

where

$$a_\gamma(x,y,\xi) = \sum_{|\,\mu\,|,|\,\nu\,|\leqslant q} C_{\mu\nu}\xi^{\mu+\nu} D^\mu_x D^\nu_y a(x,y,\xi)$$

for some combinatorial constants $C_{\mu\nu}$. Now, using the symbol estimate for $a \in S^{-\infty}_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$, and that $|\,\mu\,| + |\,\nu\,| \leqslant 2q$

$$\begin{aligned}
|\, a_\gamma(x,y,\xi)\,| &\leqslant \sum_{|\,\mu\,|,|\,\nu\,|\leqslant q} C_{\mu\nu}\,|\,\xi\,|^{\mu+\nu}\,\left|\, D^\mu_x D^\nu_y a(x,y,\xi)\,\right|\\
&= \sum_{|\,\mu\,|,|\,\nu\,|\leqslant q} C_{\mu\nu}\,|\,\xi\,|^{\mu+\nu}\,\langle\xi\rangle^{m'}\,\frac{\left|\, D^\mu_x D^\nu_y a(x,y,\xi)\,\right|}{\langle\xi\rangle^{m'}}\\
&\leqslant \|a\|_{2q,m'}\,\langle\xi\rangle^{m'} \sum_{|\,\mu\,|,|\,\nu\,|\leqslant q} C_{\mu\nu}\,|\,\xi\,|^{\mu+\nu}\\
&\leqslant \|a\|_{2q,m'}\,\langle\xi\rangle^{m'} \sum_{|\,\mu\,|,|\,\nu\,|\leqslant q} C_{\mu\nu}\,\langle\xi\rangle^{\mu+\nu}\\
&\leqslant \|a\|_{2q,m'}\,\langle\xi\rangle^{m'}\,\langle\xi\rangle^{2q} \sum_{|\,\mu\,|,|\,\nu\,|\leqslant q} C_{\mu\nu}\\
&\leqslant C_q\,\|a\|_{2q,m'}\,\langle\xi\rangle^{m'+2q}
\end{aligned}$$

and since $|\,\gamma\,| \leqslant 2q$,

$$\begin{aligned}
\left|\, D^\gamma_{x,y}\varphi(x,y)\,\right| &= \langle(x,y)\rangle^{-2q-2n-1}\,\langle(x,y)\rangle^{2q+2n+1}\,\left|\, D^\gamma_{x,y}\varphi(x,y)\,\right|\\
&\leqslant \langle(x,y)\rangle^{-2q-2n-1} \sum_{|\,\alpha\,|\leqslant 2q+2n+1}\ \sup_{(x,y)\in\mathbb{R}^{2n}} \langle(x,y)\rangle^{2q+2n+1}\,\left|\, D^\alpha_{x,y}\varphi(x,y)\,\right|\\
&\leqslant \langle(x,y)\rangle^{-2q-2n-1}\,\|\varphi\|_{2q+2n+1}\,.
\end{aligned}$$

Bring together both bounds, we have

$$
\begin{aligned}
|\, I(a)(\varphi)\,| &\leqslant \frac{1}{(2\pi)^n}\int \langle\xi\rangle^{-4q}\left(\sum_{|\,\gamma\,|\leqslant 2q} \left|\, a_\gamma(x,y,\xi) D^\gamma_{x,y}\varphi(x,y)\,\right|\right) \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y \\
&\leqslant C' \, \|a\|_{2q,m'} \, \|\varphi\|_{2q+2n+1} \int \langle\xi\rangle^{-4q} \langle\xi\rangle^{m'+2q} \langle(x,y)\rangle^{-2q-2n-1} \, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y \\
&= C' \, \|a\|_{2q,m'} \, \|\varphi\|_{2q+2n+1} \int \langle\xi\rangle^{m'-2q} \langle(x,y)\rangle^{-2q-2n-1} \, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y
\end{aligned}
$$

Thus, as long as $m' - 2q < -n$, i.e. $q > \max\left(\frac{m'+n}{2}, 0\right)$, the integral above converges. Finally, set $N = 2q$, $M = 2q + 2n + 1$, we have

$$
|\, I(a)(\varphi)\,| \leqslant C \, \|a\|_{N,m'} \, \|\varphi\|_M
$$

as required. □

By the Schwartz Kernel theorem, each $a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ defines a continuous linear operator

$$
I(a) : S(\mathbb{R}^n) \to S'(\mathbb{R}^n).
$$

We can now define the space of m-order pseudo-differential operators as the space

$$
\Psi^m_\infty(\mathbb{R}^n) := \left\{ A = I(a) \,|\, a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \right\}
$$

with the total space $\Psi^\infty_\infty(\mathbb{R}^n) := \cup_{m\in\mathbb{R}} \Psi^m_\infty(\mathbb{R}^n)$ and the residual space $\Psi^{-\infty}_\infty(\mathbb{R}^n) := \cap_m \Psi^m_\infty(\mathbb{R}^n)$ defined similarly.

3.2 Composition theorem

In this section we shall prove that, just like symbol spaces, $\Psi^\infty_\infty(\mathbb{R}^n)$ forms a graded *-algebra. The difference being, this time, the algebra is *non-commutative*. That is, we shall show that following theorem holds.

Theorem 3.2 (Summary). *Given $n \in \mathbb{N}$, $\Psi^\infty_\infty(\mathbb{R}^n)$ is a graded *-algebra over $\mathbb{C}$ with continuous inclusion*

$$
\iota : \Psi^m_\infty(\mathbb{R}^n) \to \Psi^{m'}_\infty(\mathbb{R}^n)
$$

for any $m \leqslant m'$.

We shall prove this theorem by first accumulate several technical lemmas, of which the most important is the reduction lemma that allow us remove the dependence of either x or y in the symbol $a(x,y,\xi) \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$.

3.2.1 Asymptotic Summation

Suppose we are given a sequence of symbols with decreasing order, $a_j \in S^{m-j}_\infty(\mathbb{R}^p;\mathbb{R}^n)$, $j \in \mathbb{N}$, we know that $a_j(x,\xi)$ has ever higher rate of decay for large $|\,\xi\,|$ with increasing j. However, the series $\sum_{j\in\mathbb{N}} a_j(x,\xi)$ need not converge. However, we have the following notion of asymptotic convergence.

Definition 3.3 (Asymptotic summation)**.** A sequence of symbols with decreasing order, $a_j \in S^{m-j}_\infty(\mathbb{R}^p;\mathbb{R}^n)$, $j \in \mathbb{N}$ is **asymptotically summable** if there exist $a \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ such that for all $N \in \mathbb{N}$,

$$a - \sum_{j=0}^{N-1} a_j \in S^{m-N}_\infty(\mathbb{R}^p;\mathbb{R}^n).$$

We write

$$a \sim \sum_{j\in\mathbb{N}} a_j.$$

Lemma 3.4. *Every sequence of symbols with decreasing order is asymptotically summable. Furthermore, the sum is unique up to an additive term in* $S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$*.*

Sketch. Let $a_j \in S^{m-j}_\infty(\mathbb{R}^p;\mathbb{R}^n)$, $j \in \mathbb{N}$ be given. For uniqueness, suppose $a, a' \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ are both asymptotic sums of the sequence. We need to show that $a - a' \in S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$. Indeed, for any $N \in \mathbb{N}$,

$$a - a' = \left(a - \sum_{j=0}^{N-1} a_j\right) - \left(a' - \sum_{j=0}^{N-1} a_j\right) \in S^{m-N}_\infty(\mathbb{R}^p;\mathbb{R}^n)$$

since both terms on the right are elements of $S^{m-N}_\infty(\mathbb{R}^p;\mathbb{R}^n)$. Thus,

$$a - a' \in \cap_{n\in\mathbb{N}} S^{m-N}_\infty(\mathbb{R}^p;\mathbb{R}^n) = S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n).$$

For existence, we construct $aS^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ by Borel's method []. Let $\chi \in C^\infty_c(\mathbb{R}^p)$ be a bump function and define

$$a = \sum_{j\in\mathbb{N}} (1-\chi)(\epsilon_j\xi) a_j(x,\xi)$$

where $\mathbb{R}_{>0} \ni \epsilon_j \to 0$ is a strictly monotonic decreasing sequence that converges to 0. We note that the sequence is a finite sum for any input (x,ξ) and hence define a smooth function. It remains to show that, for some choice of ϵ_j with sufficiently rapid decay,

$$\sum_{j\geqslant N} (1-\chi)(\epsilon_j\xi) a_j(x,\xi)$$

converges in $S^{m-N}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ for any $N \in \mathbb{N}$.

Note: This is again an exercise in using symbol seminorms and Leibniz formula. □

3.2.2 Reduction

We will now show that $\Psi^m_\infty(\mathbb{R}^n)$ is exactly the range of $I : S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \to S'(\mathbb{R}^{2n})$ restricted to $S^m_\infty(\mathbb{R}^n;\mathbb{R}^n) \subset S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$.

Definition 3.5. Let

$$\pi_L : \mathbb{R}^{3n}_{x,y,\xi} \to \mathbb{R}^{2n}_{x,\xi}$$

be the projection map $(x,y,\xi) \mapsto (x,\xi)$. We define the **left quantisation map** as

$$q_L := I \circ \pi_L^* : S^m_\infty(\mathbb{R}^n;\mathbb{R}^n) \to \Psi^m_\infty(\mathbb{R}^n)$$

with elements $a_L \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ known as the **left reduced symbols**.

Lemma 3.6 (Reduction). *For any $a(x,y,\xi) \in S^m_\infty(\mathbb{R}^{2n}_{x,y};\mathbb{R}^n_\xi)$ there exist unique $a_L(x,\xi) \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that $I(a) = q_L(a_L) = I(a_L \circ \pi_L)$. Furthermore, with $\iota : \mathbb{R}^{2n} \ni (x,\xi) \mapsto (x,x,\xi) \in \mathbb{R}^{3n}$ being the diagonal inclusion map, we have*

$$a_L(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\iota^* D^\alpha_y D^\alpha_\xi a(x,y,\xi). \tag{6}$$

Sketch. Note that

$$D^\alpha_\xi e^{i(x-y)\xi} = (x-y)^\alpha e^{i(x-y)\xi} \implies I((x-y)^\alpha a) = I((-1)^{|\alpha|D^\alpha_\xi a})$$

where we have extended the identity that is true using integration by parts in $S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ to general $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ using the density result of symbol space. Now, if we Taylor expand a around the diagonal $x = y$, we get

$$a(x,y,\xi) = \sum_{|\alpha| \leqslant N-1} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha D^\alpha_y a(x,x,\xi) + r_N(x,y,\xi)$$

where

$$r_N(x,y,\xi) = \sum_{|\alpha|=N} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha \int_0^1 (1-t)^{N-1} D^\alpha_y a(x,(1-t)x+ty,\xi)\,\mathrm{d}t$$

for any $N \in \mathbb{N}$. Applying the identity above give us

$$\begin{aligned} I(a) &= \sum_{j=0}^{N-1} A_j + R_N \\ A_j &= I\left(\sum_{|\alpha|=j} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_y D^\alpha_\xi a(x,x,\xi)\right) \in \Psi^{m-j}_\infty(\mathbb{R}^n) \\ R_N &\in \Psi^{m-N}_\infty(\mathbb{R}^n) \end{aligned}$$

Asymptotic summation lemma give us

$$b(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_y D^\alpha_\xi a(x,x,\xi) \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$$

so that $I(a) - I(b) \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. It remains to show that $A \in \Psi^{-\infty}_\infty(\mathbb{R}^n) \iff A = I(c), c \in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n)$. □

3.2.3 Composition theorem

Theorem 3.7 (Composition). *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$ for some $m, m' \in \mathbb{R}$. Then,*

1. *$A^* \in \Psi^m_\infty(\mathbb{R}^n)$.*
2. *$A \circ B \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$.*

Sketch. Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$ for some $m, m' \in \mathbb{R}$ be given. Since $A : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ (**??**), we have the adjoint operator $A^* : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ defined by

$$A^*u(\varphi) = u(\overline{A\varphi}), \quad u \in S'(\mathbb{R}^n), \varphi \in S(\mathbb{R}^n).$$

We check that A^*u is indeed an element of $S'(\mathbb{R}^n)$ since it is the composition of the maps $u \in S'(\mathbb{R}^n)$ and $S(\mathbb{R}^n) \ni \varphi \mapsto \overline{A\varphi}$ which are both continuous. Let $a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ be such that $A = I(a)$. Observe that,

$$\begin{aligned}
\langle Au, \varphi \rangle_{L^2} &= \int Au(x)\overline{\varphi(x)}\,\mathrm{d}x \\
&= \int u(y)\overline{\int e^{i(x-y)\xi}\overline{a(x,y,\xi)}\varphi(x)\,\mathrm{d}x\,\mathrm{d}\xi}\,\mathrm{d}y \\
&= \int u(y)\overline{I(b)\varphi(y)}\,\mathrm{d}y \\
&= \langle u, A^*\varphi \rangle_{L^2}
\end{aligned}$$

where $b(x,y,\xi) = \overline{a(y,x,\xi)} \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$. Thus, $A^* \in \Psi^m_\infty(\mathbb{R}^n)$.

For composition, applying the reduction lemma twice to get $a_L \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ and $b_L \in S^{m'}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ so that

$$\begin{aligned}
A\varphi(x) &= \frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}a(x,\xi)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi \\
B^*\varphi(x) &= \frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}\overline{b(x,\xi)}\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi
\end{aligned}$$

which shows that

$$AB\varphi(x) = \frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}a(x,\xi)b(y,\xi)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi$$

and thus $AB = I(a(x,\xi)b(y,\xi))$. Since $a(x,\xi)b(y,\xi) \in S^{m+m'}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$, we have the result $AB \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$ as required.

$\square$

4 Appendix: Functional Analysis

Theorem 4.1 (Continuous Linear extension)**.** *Let $T \in \mathcal{L}(V,W)$ be a continuous linear map between normed vector spaces V and W with W completely metrisable. Then, there exist unique extension $\widetilde{T} \in \mathcal{L}(\widetilde{V},W)$ of T, i.e. $\widetilde{T}|_V = T$ where $\widetilde{V}$ is the completion of V.*

Theorem 4.2. *Let normed vector spaces V, W be given. If W is complete, then $\mathcal{L}(V,W)$ is complete.*

Theorem 4.3 (Schwartz Kernel Theorem [**?**, Chapter 4.6, p. 345])**.** *Let M, N be compact manifold and*

$$T : C^\infty(M) \to \mathcal{D}'(N)$$

be a continuous linear map ($C^\infty(M)$ being given Frechet space topology and $D'(N)$ the weak topology). Define a bilinear map*

$$\begin{aligned} B : C^\infty(M) \times C^\infty(N) &\to \mathbb{C} \\ (u,v) &\mapsto B(u,v) = \langle v, Tu \rangle \,. \end{aligned}$$

Then, there exist a distribution $k \in \mathcal{D}'(M \times N)$ such that for all $(u,v) \in C^\infty(M) \times C^\infty(N)$

$$B(u,v) = \langle u \otimes v, k \rangle \,.$$

We call such k the kernel of T.

Definition 4.4 (Frechet space)**.**