Microlocal Analysis
with Applications to Non-Elliptic Fredholm Problems

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Overview

Introduction

Fredholm Operators

Theorem (Rank-nullity)

If $T : V \to W$ be a linear operator between finite dimensional vector space V and W, then

$$\mathrm{Ind}(T) := \dim \ker T - \dim \operatorname{coker} T = \dim W - \dim V.$$

Definition (Fredholm operators)

A continuous linear operator $T : X \to Y$ between Banach spaces X and Y is Fredholm, if

- T has closed range,
- $\ker(T)$ is finite dimensional,
- $\operatorname{coker}(T)$ is finite dimensional.

Fredholm operators

$$Tx = y \quad x \in X,\ y \in Y$$

- $\dim \mathrm{coker}(T) < \infty$ means that solutions exist if and only if

  $$\omega_1(y) = \omega_2(y) = \ldots \omega_n(y) = 0$$

  where $\{\omega_i\}_{i=1}^n$ is any basis of $\mathrm{coker}(T)$. with isomorphism to the perpendicular space
- $\dim \ker(T) < \infty$ means that solution is unique up to addition of element in a finite dimensional space.
- T is invertible modulo compact operators (limits of finite rank operators).

Fredholm Index

Definition

The index of a Fredholm operator T is defined by

$$\mathrm{Ind}(T) := \dim \ker(T) - \dim \operatorname{coker}(T).$$

- $\mathrm{Ind} : \mathrm{Fred}(X, Y) \to \mathbb{Z}$ is a continuous map.
- Unlike kernel and cokernel themselves, Ind is well-behaved. example on the circle.
- something about Atiyah-Singer index which only works for elliptic diff on compact manifolds

Fredholm differential operators

Theorem

Let V, W, Y be Banach spaces, $T \in \mathcal{L}(V,W)$ and $K \in \mathcal{K}(V,Y)$. If for all $u \in V$, the estimate

$$\|u\|_V \leqslant C\left(\|Tu\|_W + \|Ku\|_Y\right)$$

holds for some positive real constant $C \in \mathbb{R}_{>0}$, then the image, $T(V)$ is closed, and T has finite dimensional kernel.

Suppose we can show that the differential operator P is Fredholm as a map

$$P : H^s \to H^{s-m}$$

$$Pu = f \quad f \in H^s(M), u \in \mathcal{S}'$$

Significance of fredholm operator:
Finite dimension of restrictions and non-injectivity.
Connection to regularity to solutions of PDE. (definition of Sobolev spaces).
The "Fredholm Estimates".

“Ellitptic Operators are Fredholm”

Example (Laplacian)

$\Delta : H^s \to H^{s-2}$

$$\|u\|_{H^{s+2}} \leqslant C\left(\|\Delta u\|_{H^s} + \|u\|_{L^2}\right).$$

Proposition

Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ be elliptic and $u \in H^N(\mathbb{R}^n)$ for some $N \in \mathbb{R}$. Then, for any $s \in \mathbb{R}$

$$Au \in H^s(\mathbb{R}^n) \implies u \in H^{s+m}(\mathbb{R}^n)$$

and u satisfies the estimates: $\exists C > 0$

$$\|u\|_{H^{s+m}} \leqslant C\left(\|Au\|_{H^s} + \|u\|_{H^N}\right).$$

Elliptic regularity estimate. (compact and non-compact manifold case).

Propagation of singularities

Theorem (Propagation of singularities)

Suppose we have

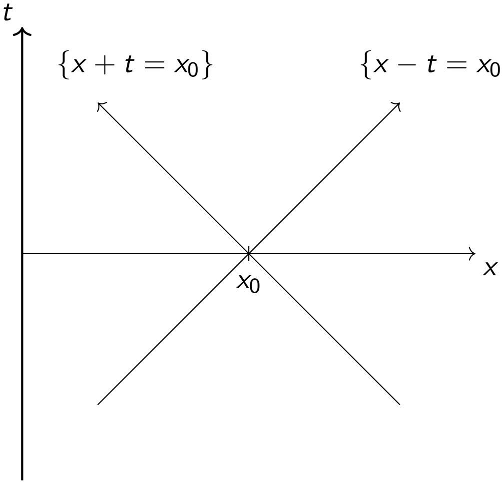
- $P \in \Psi^k_{cl}(\mathbb{R}^n)$ *a properly supported operator,*
- $\sigma_k(P) = p - iq$ *for real polyhomogeneous symbols* $p, q \in S^k_{ph}(\mathbb{R}^{2n}; \mathbb{R}^n)$,
- $A, B, B' \in \Psi^0_{cl}(\mathbb{R}^n)$ *compactly supported and* $q \geqslant 0$ *on* $\mathrm{WF}(B')$,
- *for all* $(x, \xi) \in \mathrm{WF}(A)$, *there exists* $\sigma \geqslant 0$ *such that for all* $t \in [-\sigma, 0]$

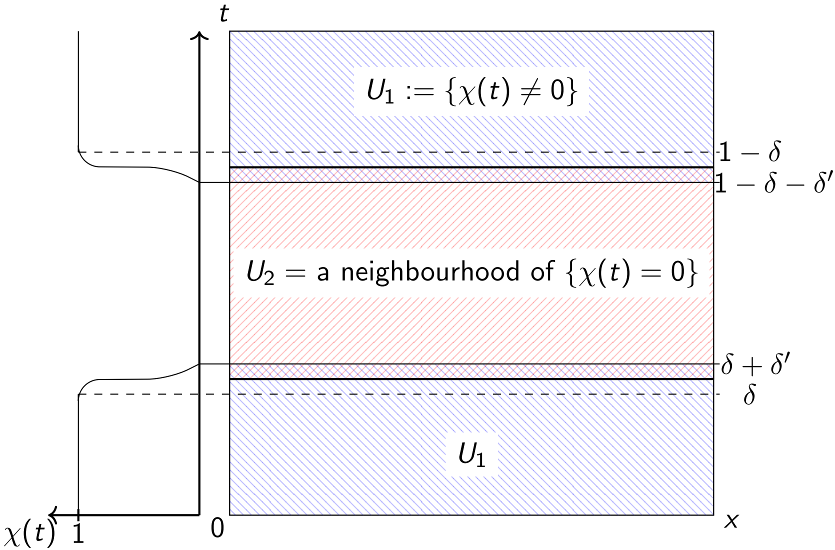
$$\exp(-t \langle \xi \rangle^{1-k} H_p)(x, \xi) \in Ell(B)$$

then for all $s, N \in \mathbb{R}$ *and* $u \in C^\infty(\mathbb{R}^n)$, *there exist* $C > 0$ *such that*

$$\|Au\|_{H^s} \leqslant C \left(\|Bu\|_{H^s} + \left\|B'Pu\right\|_{H^{s-k+1}} + \|u\|_{H^{-N}} \right).$$

Characteristic set of wave operator: the “light cone”




t
$U_1 := \{\chi(t) \neq 0\}$
$1-\delta$
$1-\delta-\delta'$
$U_2 =$ a neighbourhood of $\{\chi(t) = 0\}$
$\delta+\delta'$
δ
U_1
$\chi(t)$
1
0
x

Sandbox Frame

Fourier analysis relates the (global) regularity of functions to their fourier transform. e.g. in the 'superposition of wave' picture, only waves with high frequency can approximate jump discontinuity, linking continuity with the decay of fourier coefficients. Microlocal analysis also keeps track of the direction of decay.

Theorem (Atiyah-Singer index theorem)

Given

- X *a smooth comapct manifold,*
- E, F *smooth vector bundles over* X*,*
- $P : \Gamma(E) \to \Gamma(F)$ *be an elliptic differential operators between the space of sections of* E *and* F*.*

Then, P *is Fredholm and its Fredholm index is related to its topological index.*