

# 1 Polyhomogeneity of Riemann Map for polygonal region

**Theorem 1.1** (Riemann Mapping Theorem). *Let  $\Omega \subset \mathbb{C}$  be a simply connected region which is not the whole plane and  $z_0 \in \Omega$ . There exists a unique one-to-one analytic function  $f : \Omega \rightarrow D$ , with  $D = \{z \in \mathbb{C} \mid |z| < 1\}$  being the open unit disk, such that  $f(z_0) = 0$  and  $f'(z_0) > 0$ .*

It can also be shown that if the boundary,  $\partial\Omega$  of the region is a Jordan Curve, the Riemann Map can be extended to an analytic one-to-one function on  $\bar{\Omega}$  onto the closed unit disk, i.e.  $f : \bar{\Omega} \rightarrow \bar{D}$ . When extended, map  $f : \Omega \rightarrow D$ , simply by virtue of being a topological map (i.e. homeomorphism), will map boundary to boundary.

## 1.1 Riemann Map for polygonal region

In this section we shall exhibit an explicit formula for the (inverse of ) Riemann map for a polygonal region  $\Omega \subset \mathbb{C}$ . An  $n$ -gon can be specified by an ordered sequence of  $n$  distinct complex numbers  $(z_k)_{1 \leq k \leq n}$ . We shall let  $(\alpha_k \pi)_{1 \leq k \leq n}$  denote the interior angles at  $z_k$ , and  $(\beta_k \pi)$  the corresponding exterior angles. Since the (extended) Riemann Map will map boundary to boundary, the points  $z_k$  will be mapped to  $w_k \in S^1 \subset \bar{D}$ . With these notations in place, we shall give the following formula for the conformal of  $\Omega$  to  $D$ .

**Theorem 1.2** (Schwarz-Christoffel Formula). *The function  $z = F(w)$  which map  $D$ , the open unit disk, conformally onto an  $n$ -gon defined by  $(z_k)_{1 \leq k \leq n}$  with exterior angles  $(\beta_k \pi)_{1 \leq k \leq n}$  is given by*

$$F(w) = C \int_0^w \prod_{k=1}^n (\eta - w_k)^{-\beta_k} d\eta + C' \quad (1)$$

for some  $C, C' \in \mathbb{C}$ , with  $z_k = \lim_{w \rightarrow w_k} F(w)$ .

## 1.2 Polyhomegeity

In order to understand the behaviour of the conformal map as we approach a corner of the polygon, we shall seek asymptotic expansion of the map  $F$  in terms of  $r$ , the distance from a particular  $w \in \{w_1, w_2, \dots, w_n\}$ . Rename the points  $w_k$  if necessary, we may assume  $w = w_1$  and  $\beta = \beta_1 \in (-1, 1)$ . Let  $I(w)$  denote the integral in the expression of  $F$ . Observe that for  $\alpha \in D$

$$I(\alpha) = \int_0^\alpha (\eta - w)^{-\beta} \prod_{k=2}^n (\eta - w_k)^{-\beta_k} d\eta.$$

Let  $\epsilon > 0$  be the minimum distance between  $w$  and  $w_k$ ,  $k \in \{2, 3, \dots, n\}$ , we know that the product

$$p(\eta) = p(\eta; w_2, \dots, w_n, \beta_2, \dots, \beta_n) = \prod_{k=2}^n (\eta - w_k)^{-\beta_k}$$

is holomorphic in the domain  $B_\epsilon(w) = \{z \mid |z - w| < \epsilon\}$ , and thus have an absolutely and uniformly convergent Taylor expansion around  $\eta = w$  given by

$$p(\eta) = \sum_{m=0}^{\infty} a_m (\eta - w)^m. \quad (2)$$

In other words, the radius of convergence of (??) is precisely  $\epsilon$  since it is the distance to the nearest branch point.

Now, fix  $a \in D \cup B_\epsilon(w) \neq \emptyset$  and observe that for any  $\alpha(r, \theta) = w + re^{i\theta}$ ,  $r \in (0, \epsilon)$ , we have

$$\begin{aligned}
I(\alpha) &= I(a) + \int_a^\alpha (\eta - w)^{-\beta} \sum_{m=0}^{\infty} a_m (\eta - w)^m d\eta \\
&= I(a) + \int_a^\alpha \sum_{m=0}^{\infty} a_m (\eta - w)^{m-\beta} d\eta \\
&= I(a) + \sum_{m=0}^{\infty} a_m \int_a^\alpha (\eta - w)^{m-\beta} d\eta \\
&= I(a) + \sum_{m=0}^{\infty} \frac{a_m}{m - \beta + 1} [(\alpha - w)^{m-\beta+1} - (a - w)^{m-\beta+1}]
\end{aligned}$$

where the swapping of integral and infinite sum is justified by Lebesgue dominated convergence theorem

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<sup>1</sup>For an elementary argument, observe that since  $\beta \in (-1, 1)$ , for  $m \geq 1$ , the integrand has a holomorphic branch