Microlocal Analysis Seminar : Symbols and ΨDOs

September 11, 2018

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1 Reminder: definitions and notations

1.1 Symbols

We shall here list the definition of the space of symbols of order $m \in \mathbb{N}$ in Euclidean space $\mathbb{R}^n$ that one encounter in the literature. The main motivation is again based on the property of linear differential operators of order $m \in \mathbb{N}$ with smooth coefficient that, after Fourier transform gives the polynomial of ξ with smooth coefficient

$$P(x,\xi) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\xi^\alpha.$$

Definition 1.1. The **space of symbol of order** m, denoted $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$, is the space of smooth functions $a \in C^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ such that for all multi-index $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}$$

uniformly on $\mathbb{R}^p \times \mathbb{R}^n$. We can also defined the space of symbol, $S_\infty^m(\Omega; \mathbb{R}^n)$ on a set with non-empty interior $\Omega \subset \mathbb{R}^p$, $\Omega \subset \overline{\text{Int}(\Omega)}$ such that the bound above is satisfied uniformly in $(x,\xi) \in \text{Int}(\Omega) \times \mathbb{R}^n$. The subscript ∞ refers the uniform boundedness condition in x. Together with the family of seminorms (indexed by $N \in \mathbb{N}$)

$$\|a\|_{N,m} = \sup_{(x,\xi)\in\text{Int}(\Omega)\times\mathbb{R}^n} \max_{|\alpha|+|\beta|\leqslant N} \frac{D_x^\alpha D_\xi^\beta a(x,\xi)}{\langle\xi\rangle^{m-|\beta|}} \tag{1}$$

gives a Frechet topology to $S_\infty^m(\Omega; \mathbb{R}^n)$.

Furthermore, we define the total symbol space as

$$S_\infty^\infty(\Omega; \mathbb{R}^n) = \bigcup_{m\in\mathbb{R}} S_\infty^m(\Omega; \mathbb{R}^n)$$

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and the residual *residual* space of the filtration as

$$S_\infty^{-\infty}(\Omega; \mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} S_\infty^m(\Omega; \mathbb{R}^n).$$

Note: In defining pseudodifferential operators, we shall focus on the case where $p = 2n$, i.e. $a(x, y, \xi) \in S_\infty^m(\mathbb{R}_{x,y}^{2n}; \mathbb{R}^n)$.

1.2 Quantisation

Pseudodifferential operators are defined using symbols. The main gadget is the following oscillatory integral:

$$S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n) \ni a \mapsto I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) \, d\xi \tag{2}$$

with action on Schwartz functions $u \in S(\mathbb{R}^n)$ given by

$$I(a)(u) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) u(y) \, dy \, d\xi. \tag{3}$$

The integral 3 above might be divergent unless $m < -n$, but it can be interpreted as a tempered distribution, i.e. a linear function on $S(\mathbb{R}^n)$, with action

$$S(\mathbb{R}^n) \ni v \mapsto I(a)(u)(v) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) u(y) v(x) \, dy \, d\xi \, dx \in \mathbb{C}. \tag{4}$$

The process of turning the symbol a into an operator $S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ is known as the quantisation procedure. The goal of this talk is the following:

Goal :
To establish that the procedure above is well-defined, so that for each $a \in S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n)$

$$I(a) : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$$
$$u \mapsto I(a)(u) \quad : S(\mathbb{R}^n) \to \mathbb{C}$$
$$v \mapsto I(a)(uv) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) u(y) v(x) \, dy \, d\xi \, dx$$

is a continuous linear map between Frechet spaces.

Remark. Given $a \in S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n)$, we sometimes write $A = Op(a) = I(a)$ for the operator $S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ defined by quantising the symbol a. Also, once the procedure above is proven to be well-defined, we will, with abuse of notation, identify the integral

$$I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) \, d\xi \in S'(\mathbb{R}^n \times \mathbb{R}^n)$$

to be the *Schwartz Kernel* of the operator $I(a) : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$.

2 Properties of Symbols

In this section, we shall establish the following summarising theorem:

Theorem 2.1 (Summary). *Given $m \in \mathbb{R}$, $p, n \in \mathbb{N}$, then*

1. *$S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ is a Frechet space, hence completely metrisable.*

2. *$S_\infty^\infty(\mathbb{R}^p; \mathbb{R}^n)$ is a graded commutative *-algebra over $\mathbb{C}$ with continuous inclusion*

$$\iota : S_\infty^m(\mathbb{R}^p; \mathbb{R}^n) \to S_\infty^{m'}(\mathbb{R}^p; \mathbb{R}^n)$$

   for all $m \leqslant m'$.

3. *$S_\infty^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$ is dense in $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ in the topology of $S_\infty^{m+\epsilon}(\mathbb{R}^p; \mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$.*

Exercise : Show that symbol spaces are Frechet spaces. That is, show that the family of seminorms in 1 separates points and that if a sequence is Cauchy in each seminorm, then there exist a unique symbol where the sequence converges in each seminorm.

Now, we prove continuous inclusion of lower order into higher order symbol space.

Proposition 2.2. *Let $p, n \in \mathbb{N}$ be given and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$. If $m, m' \in \mathbb{R}$ such that $m \leqslant m'$, then $S_\infty^m(\Omega; \mathbb{R}^n) \subset S_\infty^{m'}(\mathbb{R}^)$. Furthermore, the inclusion map*

$$\iota : S_\infty^m(\Omega; \mathbb{R}^n) \to S_\infty^{m'}(\Omega; \mathbb{R}^n)$$

is continuous.

Proof. Let the real numbers $m \leqslant m'$ be given. We note that for any $\xi \in \mathbb{R}^n$

$$\langle \xi \rangle^m \leqslant 1 \cdot \langle \xi \rangle^{m'}$$

and thus if $a \in S_\infty^m(\Omega; \mathbb{R}^n)$, we have that $\forall \alpha \in \mathbb{N}^p, \forall \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x, \xi) \right| \leqslant C \langle \xi \rangle^{m - |\beta|} \leqslant C \langle \xi \rangle^{m' - |\beta|}$$

which show that $a \in S_\infty^{m'}(\Omega; \mathbb{R}^n)$ as well.

To show that ι is a continuous inclusion, it suffices to show that

$$\|\iota(a)\|_{N, m'} \leqslant C \|a\|_{N, m}$$

for any $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ and $N \in \mathbb{N}$. Indeed, this bound holds since

$$\frac{D_x^\alpha D_\xi^\beta a(x, \xi)}{\langle \xi \rangle^{m' - |\beta|}} \leqslant \frac{D_x^\alpha D_\xi^\beta a(x, \xi)}{\langle \xi \rangle^{m - |\beta|}}.$$

$\square$

Proposition 2.3. *Let $p, n \in \mathbb{N}$ be given. Let $\Omega \subset \mathbb{R}^p$ be such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$. Then, for any $m, m' \in \mathbb{R}$, we have*

$$S_\infty^m(\Omega; \mathbb{R}^n) \cdot S_\infty^{m'}(\Omega; \mathbb{R}^n) = S_\infty^{m+m'}(\Omega; \mathbb{R}^n)$$

Proof. Let $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ and $b \in S_\infty^{m'}(\Omega; \mathbb{R}^n)$ be given. By (general) Leibinz formula, we have that for all multi-index α, β,

$$\sup_{(x,\xi) \in \mathrm{Int}(\Omega) \times \mathbb{R}^n} \frac{\left| D_x^\alpha D_\xi^\beta a(x,\xi) b(x,\xi) \right|}{\langle \xi \rangle^{(m+m')-|\beta|}} \leqslant \sum_{\mu \leqslant \alpha, \gamma \leqslant \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} \sup_{(x,\xi) \in \mathrm{Int}(\Omega) \times \mathbb{R}^n} \frac{\left| D_x^\mu D_\xi^\gamma a(x,\xi) \right| \left| D_x^{\alpha-\mu} D_\xi^{\beta-\gamma} b(x,\xi) \right|}{\langle \xi \rangle^{(m+m')-|\beta|}}$$

$$\leqslant \sum_{\mu \leqslant \alpha, \gamma \leqslant \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} C \sup_{\xi \in \mathbb{R}^n} \frac{\langle \xi \rangle^{m-|\gamma|} \langle \xi \rangle^{m'-|\beta-\gamma|}}{\langle \xi \rangle^{(m+m')-|\beta|}}$$

$$= \sum_{\mu \leqslant \alpha, \gamma \leqslant \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} C \sup_{\xi \in \mathbb{R}^n} \langle \xi \rangle^{|\beta|-(|\beta-\gamma|+|\gamma|)}$$

$$\leqslant \sum_{\mu \leqslant \alpha, \gamma \leqslant \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} C$$

$$< \infty$$

where we have use the property of multi-index that $|\beta| = |\beta - \mu| + |\mu|$. We have thus shown that $S_\infty^m(\Omega; \mathbb{R}^n) \cdot S_\infty^{m'}(\Omega; \mathbb{R}^n) \subset S_\infty^{m+m'}(\Omega; \mathbb{R}^n)$

For the reverse inclusion, let $c \in S_\infty^{m+m'}(\Omega; \mathbb{R}^n)$ be given. Define

$$a : (x, \xi) \mapsto \langle \xi \rangle^m$$

$$b : (x, \xi) \mapsto \frac{c(x, \xi)}{a(x, \xi))}$$

and observe that

- $a \in S_\infty^m(\Omega; \mathbb{R}^n)$. It is clear that a is smooth in both x and ξ. It is independent of x and thus any x derivative gives 0. We need only to check that for all $\beta \in \mathbb{N}^n$,

$$\left| D_\xi^\beta \langle \xi \rangle^m \right| \leqslant C \langle \xi \rangle^{m-|\beta|}$$

  which can be proven by induction on n and β. We shall only prove the base case where $n = 1$ and $\beta = 1$. We have

$$|D_\xi \langle \xi \rangle^m| = \left| \partial_\xi (1 + \xi^2)^{m/2} \right| = \left| m \xi \langle \xi \rangle^{m-2} \right| = \left| m \frac{\xi}{\langle \xi \rangle} \right| \langle \xi \rangle^{m-1} \leqslant |m| \langle \xi \rangle^{m-1}$$

  where we have used the fact that $|\xi| \leqslant \langle \xi \rangle$ for all ξ.

- $b \in S_\infty^{m'}(\Omega; \mathbb{R}^n)$. We note first that $\langle \xi \rangle^m \neq 0$ for all $\xi \in \mathbb{R}^n$ and thus b is well-defined. Since division by $\langle \xi \rangle^m$ does not affect any of the x derivative, we only need to show that for any $\beta \in \mathbb{N}^n$, we have

$$\left| D_\xi^\beta b(x, \xi) \right| \leqslant C \langle \xi \rangle^{m+m'-|\beta|}$$

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for some constant $C > 0$ uniformly in ξ. Indeed, observe that by the Leibinz formula

$$\left| D_\xi^\beta b(x,\xi) \right| \leqslant \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \left| D_\xi^\mu c(x,\xi) \right| \left| D^{\beta-\mu} \langle\xi\rangle^{-m} \right|$$

$$\leqslant C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \langle\xi\rangle^{m+m'-|\mu|} \langle\xi\rangle^{-m-|\beta-\mu|}$$

$$\leqslant C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \langle\xi\rangle^{m'-(|\mu|+|\beta-\mu|)}$$

$$= C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \langle\xi\rangle^{m'-|\beta|}$$

$$= C 2^\beta \langle\xi\rangle^{m'-|\beta|}$$

where we have use the definition of c and applied the result proven for a with $m \mapsto -m$. Thus, $b \in S_\infty^{m'}(\Omega;\mathbb{R}^n)$.

It is clear that $a \cdot b = c$ and we have therefore shown that $S_\infty^{m+m'}(\Omega;\mathbb{R}^n) \subset S_\infty^m(\Omega;\mathbb{R}^n) \cdot S_\infty^{m'}(\Omega;\mathbb{R}^n)$. $\square$

We have a rather technical density result : the residual space is dense in $S_\infty^m(\mathbb{R}^p;\mathbb{R}^n)$, but only with the topology of $S_\infty^{m+\epsilon}(\mathbb{R}^p;\mathbb{R}^n)$. The main strategy in this proof is to approximate any symbol with the very same symbol but cut off by a compactly supported function. As such, the main reason we **cannot** have density of $S_\infty^{-\infty}(\mathbb{R}^p;\mathbb{R}^n)$ in $S_\infty^m(\mathbb{R}^p;\mathbb{R}^n)$ is the same reason to the fact that Schwartz functions are not dense in the space of smooth bounded functions, in particular, $1 \in S_\infty^0(\Omega;\mathbb{R}^n)$ is not in the closure of $S_\infty^{-\infty}(\Omega;\mathbb{R}^n)$.

Lemma 2.4. *Given any $m \in \mathbb{R}$, $n,p \in \mathbb{N}$ and $a \in S_\infty^m(\mathbb{R}^p;\mathbb{R}^n)$, there exist a sequence in $S_\infty^{-\infty}(\mathbb{R}^p;\mathbb{R}^n)$ that is bounded in $S_\infty^m(\mathbb{R}^p;\mathbb{R}^n)$ and converges to a in the topology of $S_\infty^{m+\epsilon}(\mathbb{R}^p;\mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$.*

Proof. Let $a \in S_\infty^m(\Omega;\mathbb{R}^n)$ and $\epsilon \in \mathbb{R}_{>0}$ be given. Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be a non-negative smooth cut-off function, i.e. $\chi \geqslant 0$ and satisfies

$$\chi(\xi) = \begin{cases} 1 & ,|\xi| < 1 \\ 0 & ,|\xi| > 2. \end{cases}$$

Then, for each $k \in \mathbb{N}$, we define

$$a_k(x,\xi) = \chi\left(\frac{\xi}{k}\right) a(x,\xi).$$

Now, given arbitrary $N,k \in \mathbb{N}$, observe that

$$|a_k| \leqslant C \langle\xi\rangle^{-N}$$

since a_k is compactly supported in ξ (as χ is compactly supported). Furthermore, by Leibinz formula and the symbol estimates on $a \in S_\infty^m(\mathbb{R}^p;\mathbb{R}^n)$, we have

$$\left| D_x^\alpha D_\xi^\beta a_k(x,\xi) \right| \leqslant \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} k^{-|\mu|} \left(D_\xi^\mu \chi \right) \left(\frac{\xi}{k}\right) \left| D_x^\alpha D_\xi^{\beta-\mu} a(x,\xi) \right|$$

$$\leqslant C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} k^{-|\mu|} \left(D_\xi^\mu \chi \right) \left(\frac{\xi}{k}\right) \langle\xi\rangle^{m-|\beta-\mu|}.$$

Since χ and all its derivatives are compactly supported, each term above is bounded in ξ and thus a_k is bounded in $S_\infty^m(\mathbb{R}^p;\mathbb{R}^n)$ and that

$$\left| D_x^\alpha D_\xi^\beta a_k(x,\xi) \right| \leqslant C' \langle\xi\rangle^{-N}$$

which allow us to conclude that $a_k \in S_\infty^{-\infty}(\mathbb{R}^p; \mathbb{R}^n)$.

It remains to show that $\lim_{k \to \infty} a_k = a$ in $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$. In the first symbol norm, we observe that, using the symbol estimate for a

$$
\begin{aligned}
\|a_k - a\|_{0,m+\epsilon} &= \sup_{(x,\xi) \in \mathbb{R}^p \times \mathbb{R}^n} \frac{|a_k(x,\xi)|}{\langle \xi \rangle^{m+\epsilon}} \\
&= \sup_{(x,\xi) \in \mathbb{R}^p \times \mathbb{R}^n} \frac{|(1 - \chi(\xi/k))| \, |a(x,\xi)|}{\langle \xi \rangle^{m+\epsilon}} \\
&\leqslant \|a\|_{0,m} \sup_{\xi \in \mathbb{R}^n} \frac{|(1 - \chi(\xi/k))|}{\langle \xi \rangle^\epsilon} \\
&\leqslant \|a\|_{0,m} \langle k \rangle^{-\epsilon} \\
&\to 0
\end{aligned}
$$

as $k \to \infty$, since $|(1 - \chi(\xi/k))|$ is 0 in the region $|\xi| \leqslant k$ and bounded by 1 otherwise. We remark upon the necessity of the extra decay given by $\langle \xi \rangle^{-\epsilon}$ factor. For other symbol norms we shall again use Leibinz formula:

$$
\begin{aligned}
\sup_{(x,\xi) \in \mathbb{R}^p \times \mathbb{R}^n} \frac{\left| D_x^\alpha D_\xi^\beta a_k(x,\xi) \right|}{\langle \xi \rangle^{m+\epsilon-|\beta|}} &\leqslant \sup_{(x,\xi) \in \mathbb{R}^p \times \mathbb{R}^n} \frac{1}{\langle \xi \rangle^{m+\epsilon-|\beta|}} \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu(1-\chi)) \left(\frac{\xi}{k}\right) \left| D_x^\alpha D_\xi^{\beta-\mu} a(x,\xi) \right| \\
&\leqslant \sup_{(x,\xi) \in \mathbb{R}^p \times \mathbb{R}^n} \frac{C}{\langle \xi \rangle^{m+\epsilon-|\beta|}} \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu(1-\chi)) \left(\frac{\xi}{k}\right) \langle \xi \rangle^{m-|\beta-\mu|} \\
&= C \sup_{\xi \in \mathbb{R}^n} \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu(1-\chi)) \left(\frac{\xi}{k}\right) \langle \xi \rangle^{-\epsilon-|\mu|} \\
&\leqslant C' k^{-\epsilon} \\
&\to 0
\end{aligned}
$$

as $k \to \infty$ by the same argument as before. Thus, we have proven that $a_k \to a$ as $k \to \infty$ in $S_\infty^{m+\epsilon}(\mathbb{R}^p; \mathbb{R}^n)$.
$\square$

We shall show that the elements $a \in \langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n)$ acts on $S(\mathbb{R}^n)$ via the Schwartz kernel

$$
I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi) \, d\xi.
$$

Proposition 2.5. *Let $n \in \mathbb{N}$ and $m, w \in \mathbb{R}$ with $m < -n$, then the map*

$$
I : \langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n) \to (1 + |x|^2 + |y|^2) C_\infty^0(\mathbb{R}^{2n})
$$

$$
a \mapsto I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi) \, d\xi
$$

extends by continuity to

$$
I : \langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n) \to S'(\mathbb{R}^{2n})
$$

in the topology of $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$.

Proof.
$\square$

3 Appendix

3.1 Functional Analysis background