

We shall follow the presentation given in [?].

1 Motivation for Pseudodifferential operators

- Solving PDEs via Fourier transform. For example, in Euclidean space, \mathbb{R}^n , constant coefficient linear PDE

$$P(D)u = \sum_{|\alpha| \leq n} c_\alpha D^\alpha u = f, \quad c_\alpha \in \mathbb{R}$$

where $P \in \mathbb{R}[x]$, can be solved by applying Fourier transform which gives a solution of the form

$$u(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} f(y) \frac{1}{P(\xi)} dy d\xi$$

due to the observation that

$$\mathcal{F}P(D)u = P(\xi)\mathcal{F}u.$$

Moreover, for linear differential operators with smooth coefficients

$$P(x, D) : u \mapsto \sum_{|\alpha| \leq n} a_\alpha D^\alpha u, \quad a_\alpha \in C^\infty(\mathbb{R}^n)$$

we have

$$P(x, D)u = \frac{1}{(2\pi)^n} \int P(x, \xi) e^{i(x-y)\cdot\xi} u(y) dy d\xi.$$

We would like to generalise the above so that $P(x, \xi)$ are smooth functions satisfying certain uniform bounds, called *symbols*, instead of just polynomials in ξ . This gives us a class of operators, called pseudodifferential operators, that acts as

$$A_a u(x) = \frac{1}{(2\pi)^n} \int a(x, \xi) e^{i(x-y)\cdot\xi} u(y) dy d\xi$$

for each symbol a .

- There isn't enough differential operators with smooth coefficient in the sense that elliptic differential operators are not, in general, invertible in this class. For example, the operator

$$u \mapsto (\Delta + 1)u$$

has inverse that acts as (using construction via Fourier transform shown above)

$$(\Delta + 1)^{-1}f = \frac{1}{(2\pi)^n} \int \frac{1}{1 + |\xi|^2} e^{i(x-y)\cdot\xi} f(y) dy d\xi$$

which is a pseudodifferential operator with symbol $a(x, \xi) = (1 + |\xi|^2)^{-1}$.

- Motivation from quantum mechanics. The notion of “quantisation” in quantum mechanics can be formalised as the map that sends a symbol a (a smooth function that represent deterministic observable in classical mechanics) to its corresponding pseudodifferential operator (i.e. the corresponding quantum observable)

$$A_a : \psi \mapsto \frac{1}{(2\pi)^n} \int \frac{1}{1 + |\xi|^2} e^{i(x-y)\xi} \psi(y) dy d\xi$$

that acts on the wavefunction ψ .

- Used in the formulation and proof of Atiyah-Singer Index theorem.

We shall define, on Euclidean space, the space of symbols, $S^m(\mathbb{R}_{x,y}^{2n}; \mathbb{R}_\xi^n)$ and the corresponding space of pseudodifferential operators, $\Psi^m(\mathbb{R}^n)$ which acts on distributions via the Schwartz kernel given by the oscillatory integral

$$I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) d\xi.$$

We note that we have introduced an extra variable y which will help in explicating the properties of pseudodifferential operators. However, the extra variable does not change the essence of the theory.

2 Symbols

We shall here list the definition of the space of symbols of order $m \in \mathbb{N}$ in Euclidean space \mathbb{R}^n that one encounter in the literature. The main motivation is again based on the property of linear differential operators of order $m \in \mathbb{N}$ with smooth coefficient that, after Fourier transform gives the polynomial of ξ with smooth coefficient

$$P(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha.$$

It has the property that

$$\left| D_x^\alpha D_\xi^\beta P(x, \xi) \right| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\beta|}$$

i.e. $P(x, \xi)$ is smooth and decreases in order as $\xi \rightarrow \infty$ with successive ξ -derivative.

Definition 2.1. The space $S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ of order m is the space of smooth functions $a \in C^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ such that for all multi-index $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\beta|}$$

uniformly on $\mathbb{R}^p \times \mathbb{R}^n$. We can also defined the space of symbol, $S_\infty^m(\Omega; \mathbb{R}^n)$ on a set with non-empty interior $\Omega \subset \mathbb{R}^p$, $\Omega \subset \overline{\text{Int}(\Omega)}$ such that the bound above is satisfied uniformly in $(x, \xi) \in \text{Int}(\Omega) \times \mathbb{R}^n$. The subscript ∞ refers the uniform boundedness condition. Together with the family of seminorm (indexed

by $N \in \mathbb{N}$)

$$\|a\|_{N,m} = \sup_{(x,\xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \max_{|\alpha|+|\beta| \leq N} \frac{D_x^\alpha D_\xi^\beta a(x, \xi)}{\langle \xi \rangle^{m-|\beta|}}$$

gives a Frechet topology to $S_\infty^m(\Omega; \mathbb{R}^n)$.

Note: In defining pseudodifferential operators, we shall focus on the case where $p = 2n$.

Definition 2.2. A **symbol** of type $S_{\delta,\delta'}^{m,l_1,l_2}$ where $m, l_1, l_2 \in \mathbb{R}$ and $\delta, \delta' \in [0, 1/2)$ is an element of $C^\infty(\mathbb{R}_x^n; \mathbb{R}_y^n; \mathbb{R}_\xi^n)$ satisfying

$$\frac{|D_x^\alpha D_y^\beta D_\xi^\gamma a(x, y, \xi)|}{\langle \xi \rangle^{m-|\gamma|} \langle x \rangle^{l_1-|\alpha|} \langle y \rangle^{l_2-|\beta|} \langle \xi \rangle^{|\delta|(\alpha,\beta,\gamma)|} \langle x, y \rangle^{\delta'|\alpha,\beta,\gamma|}} \leq C_{\alpha,\beta,\gamma}$$

uniformly in \mathbb{R}^{3n} . Taking the supremum over \mathbb{R}^{3n} , we get a family of seminorms, indexed by $N \in \mathbb{N}$ defined by

$$\|a\|_{S_{\delta,\delta'}^{m,l_1,l_2},N} := \sum_{|(\alpha,\beta,\gamma)| \leq N} \inf C_{\alpha,\beta,\gamma}$$

which gives $S_{\delta,\delta'}^{m,l_1,l_2}$ a Frechet topology.

Definition 2.3. A (Kohn-Nirenberg) **symbol** of order $m \in \mathbb{R}$ on $T^*\mathbb{R}^n \cong \mathbb{R}_{x,\xi}^{2n}$ is a smooth function $a = a(x, \xi)$ satisfying

$$\forall \alpha, \beta \in \mathbb{N}^n, \exists C \in \mathbb{R}_{\geq 0} : |D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C \langle \xi \rangle^{m-|\beta|}$$

uniformly in x . The **space of symbol of order m** on $T^*\mathbb{R}^n$

Definition 2.4. Let $n \in \mathbb{N}$ be given. An **order function** $g \in C^\infty(\mathbb{R}^n; \mathbb{R}_{\geq 0})$ is a *non-negative* function satisfying

$$\forall \alpha \in \mathbb{N}^n \exists C \in \mathbb{R}_{\geq 0} : \partial^\alpha g \leq Cg$$

uniformly on \mathbb{R}^n , i.e. $\partial^\alpha g = O(g)$ uniformly on \mathbb{R}^n .

Given an order function g , a **symbol** of order g is a smooth function $a = a(x, \xi) \in C^\infty(T^*\mathbb{R}^n)$ satisfying

$$\forall \alpha, \beta \in \mathbb{N}^n : |D_x^\alpha D_\xi^\beta a(x, \xi)| \leq Cg(\xi)$$

uniformly in x .

2.1 Properties of Symbols

Proposition 2.5. *Let $p, n \in \mathbb{N}$ be given and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\text{Int}(\Omega)}$. If $m, m' \in \mathbb{R}$ such that $m \leq m'$, then $S_\infty^m(\Omega; \mathbb{R}^n) \subset S_\infty^{m'}(\mathbb{R}^n)$. Furthermore, the inclusion map*

$$\iota : S_\infty^m(\Omega; \mathbb{R}^n) \rightarrow S_\infty^{m'}(\Omega; \mathbb{R}^n)$$

is continuous.

Proof. Let the real numbers $m \leq m'$ be given. We note that for any $\xi \in \mathbb{R}^n$

$$\langle \xi \rangle^m \leq 1 \cdot \langle \xi \rangle^{m'}$$

and thus if $a \in S_\infty^m(\Omega; \mathbb{R}^n)$, we have that $\forall \alpha \in \mathbb{N}^p, \forall \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x, \xi) \right| \leq C \langle \xi \rangle^{m-|\beta|} \leq C \langle \xi \rangle^{m'-|\beta|}$$

which show that $a \in S_\infty^{m'}(\Omega; \mathbb{R}^n)$ as well.

To show that ι is a continuous inclusion, it suffices to show that

$$\|\iota(a)\|_{N, m'} \leq C \|a\|_{N, m}$$

for any $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ and $N \in \mathbb{N}$. Indeed, this bound holds since

$$\frac{D_x^\alpha D_\xi^\beta a(x, \xi)}{\langle \xi \rangle^{m'-|\beta|}} \leq \frac{D_x^\alpha D_\xi^\beta a(x, \xi)}{\langle \xi \rangle^{m-|\beta|}}.$$

□

This inclusion property allow us to consider $S_\infty^m(\Omega; \mathbb{R}^n)$ as the filtration of the space

$$S_\infty^\infty(\Omega; \mathbb{R}^n) = \bigcup_{m \in \mathbb{R}} S_\infty^m(\Omega; \mathbb{R}^n)$$

and we shall denote the *residual* space of the filtration as

$$S_\infty^{-\infty}(\Omega; \mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} S_\infty^m(\Omega; \mathbb{R}^n).$$

We have a rather technical result of the density of the residual space in $S_\infty^m(\Omega; \mathbb{R}^n)$.

Lemma 2.6. *Given any $m \in \mathbb{R}$ and $a \in S_\infty^m(\Omega; \mathbb{R}^n)$, there exist a sequence in $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$ such that bounded in $S_\infty^m(\Omega; \mathbb{R}^n)$ and converges to a in the topology of $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$. In other words, for any $m \in \mathbb{R}$ and $\epsilon > 0$, $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$ is dense in $S_\infty^m(\Omega; \mathbb{R}^n)$ with the topology of $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$.*

Proof. The main strategy in this proof is to approximate any symbol with the very same symbol but cut off by a compactly supported function. As such, the main reason we can't have density of $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$ in $S_\infty^m(\Omega; \mathbb{R}^n)$ is the same reason to the fact that Schwartz functions are not dense in the space of smooth bounded functions, in particular, $1 \in S_\infty^0(\Omega; \mathbb{R}^n)$ is not in the closure of $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$.

Now, let $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ and $\epsilon \in \mathbb{R}_{>0}$ be given. Take any smooth cut off functions supported in the unit ball, i.e. take $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \phi \leq 1$ and $\phi(\xi) = 1$ if $|\xi| < 1$ and $\phi(\xi) = 0$ if $|\xi| > 2$. We define for each $k \in \mathbb{N}$

$$a_k(x, \xi) = \phi\left(\frac{\xi}{k}\right) a(x, \xi)$$

and we check the following

1. $a_k \in S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$ for all $k \in \mathbb{N}$;
2. a_k are bounded in $S_\infty^m(\Omega; \mathbb{R}^n)$ for all $k \in \mathbb{N}$;
3. $a_k \rightarrow a$ as $k \rightarrow \infty$ in $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$.

Given arbitrary $N, k \in \mathbb{N}$, observe that

$$|a_k| \leq C \langle \xi \rangle^{-N}$$

since a_k is compactly supported in ξ (as ϕ is compactly supported) and by Leibniz formula and symbol estimates on $a \in S_\infty^m(\Omega; \mathbb{R}^n)$

$$\left| D_x^\alpha D_\xi^\beta a_k(x, \xi) \right| \leq \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu \phi) \left(\frac{\xi}{k} \right) \left| D_x^\alpha D_\xi^{\beta-\mu} a(x, \xi) \right| \leq C \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu \phi) \left(\frac{\xi}{k} \right) \langle \xi \rangle^{m-|\mu|}$$

Since ϕ and all its derivatives are compactly supported, each term above is bounded in ξ and thus a_k is bounded in $S_\infty^m(\Omega; \mathbb{R}^n)$ and that

$$\left| D_x^\alpha D_\xi^\beta a_k(x, \xi) \right| \leq C' \langle \xi \rangle^{-N}$$

which allow us to conclude that $a_k \in S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$.

It remains to show that $\lim_{k \rightarrow \infty} a_k = a$ in $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$. In the first symbol norm, we observe that, using the symbol estimate for a

$$\begin{aligned} \|a_k - a\|_{0, m+\epsilon} &= \sup_{(x, \xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{|a_k(x, \xi)|}{\langle \xi \rangle^{m+\epsilon}} \\ &= \sup_{(x, \xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{|(1 - \phi(\xi/k))| |a(x, \xi)|}{\langle \xi \rangle^{m+\epsilon}} \\ &\leq \|a\|_{0, m} \sup_{\xi \in \mathbb{R}^n} \frac{|(1 - \phi(\xi/k))|}{\langle \xi \rangle^\epsilon} \\ &\leq \|a\|_{0, m} \langle k \rangle^{-\epsilon} \\ &\rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$, since $|(1 - \phi(\xi/k))|$ is 0 in the region $|\xi| \leq k$ and bounded by 1 otherwise. We remark upon the necessity of the extra decay given by $\langle \xi \rangle^{-\epsilon}$ factor. For other symbol norm we shall again use Leibinz formula to obtain

$$\begin{aligned}
\sup_{(x,\xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{|D_x^\alpha D_\xi^\beta a_k(x, \xi)|}{\langle \xi \rangle^{m+\epsilon-|\beta|}} &\leq \sup_{(x,\xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{1}{\langle \xi \rangle^{m+\epsilon-|\beta|}} \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu(1 - \phi)) \left(\frac{\xi}{k} \right) |D_x^\alpha D_\xi^{\beta-\mu} a| \\
&\leq \sup_{(x,\xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{C}{\langle \xi \rangle^{m+\epsilon-|\beta|}} \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu(1 - \phi)) \left(\frac{\xi}{k} \right) \langle \xi \rangle^{m-|\beta-\mu|} \\
&= C \sup_{\xi \in \mathbb{R}^n} \sum_{\mu \leq \beta} \binom{\beta}{\mu} k^{-|\mu|} (D^\mu(1 - \phi)) \left(\frac{\xi}{k} \right) \langle \xi \rangle^{-\epsilon-|\mu|} \\
&\leq C' k^{-\epsilon} \\
&\rightarrow 0
\end{aligned}$$

as $k \rightarrow \infty$ by the same argument as before. Thus, we have proven that $a_k \rightarrow a$ as $k \rightarrow \infty$ in $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$. □

Proposition 2.7. *Let $p, n \in \mathbb{N}$ be given. Let $\Omega \subset \mathbb{R}^p$ be such that $\Omega \subset \overline{\text{Int}(\Omega)}$. Then, for any $m, m' \in \mathbb{R}$, we have*

$$S_\infty^m(\Omega; \mathbb{R}^n) \cdot S_\infty^{m'}(\Omega; \mathbb{R}^n) = S_\infty^{m+m'}(\Omega; \mathbb{R}^n)$$

Proof. Let $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ and $b \in S_\infty^{m'}(\Omega; \mathbb{R}^n)$ be given. By (general) Leibinz formula, we have that for all multi-index α, β ,

$$\begin{aligned}
\sup_{(x,\xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{|D_x^\alpha D_\xi^\beta a(x, \xi) b(x, \xi)|}{\langle \xi \rangle^{(m+m')-|\beta|}} &\leq \sum_{\mu \leq \alpha, \gamma \leq \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} \sup_{(x,\xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{|D_x^\mu D_\xi^\gamma a(x, \xi)|}{\langle \xi \rangle^{(m+m')-|\beta|}} |D_x^{\alpha-\mu} D_\xi^{\beta-\gamma} b(x, \xi)| \\
&\leq \sum_{\mu \leq \alpha, \gamma \leq \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} C \sup_{\xi \in \mathbb{R}^n} \frac{\langle \xi \rangle^{m-|\gamma|} \langle \xi \rangle^{m'-|\beta-\gamma|}}{\langle \xi \rangle^{(m+m')-|\beta|}} \\
&= \sum_{\mu \leq \alpha, \gamma \leq \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} C \sup_{\xi \in \mathbb{R}^n} \langle \xi \rangle^{|\beta|-(|\beta-\gamma|+|\gamma|)} \\
&\leq \sum_{\mu \leq \alpha, \gamma \leq \beta} \binom{\alpha}{\mu} \binom{\beta}{\gamma} C \\
&< \infty
\end{aligned}$$

where we have use the property of multi-index that $|\beta| = |\beta - \mu| + |\mu|$. We have thus shown that $S_\infty^m(\Omega; \mathbb{R}^n) \cdot S_\infty^{m'}(\Omega; \mathbb{R}^n) \subset S_\infty^{m+m'}(\Omega; \mathbb{R}^n)$

For the reverse inclusion, let $c \in S_\infty^{m+m'}(\Omega; \mathbb{R}^n)$ be given. Define

$$\begin{aligned}
a &: (x, \xi) \mapsto \langle \xi \rangle^m \\
b &: (x, \xi) \mapsto \frac{c(x, \xi)}{a(x, \xi)}
\end{aligned}$$

and observe that

- $a \in S_\infty^m(\Omega; \mathbb{R}^n)$. It is clear that a is smooth in both x and ξ . It is independent of x and thus any x derivative gives 0. We need only to check that for all $\beta \in \mathbb{N}^n$,

$$\left| D_\xi^\beta \langle \xi \rangle^m \right| \leq C \langle \xi \rangle^{m-|\beta|}$$

which can be proven by induction on n and β . We shall only prove the base case where $n = 1$ and $\beta = 1$. We have

$$\left| D_\xi \langle \xi \rangle^m \right| = \left| \partial_\xi (1 + \xi^2)^{m/2} \right| = \left| m\xi \langle \xi \rangle^{m-2} \right| = \left| m \frac{\xi}{\langle \xi \rangle} \right| \langle \xi \rangle^{m-1} \leq |m| \langle \xi \rangle^{m-1}$$

where we have used the fact that $|\xi| \leq \langle \xi \rangle$ for all ξ .

- $b \in S_\infty^{m'}(\Omega; \mathbb{R}^n)$. We note first that $\langle \xi \rangle^m \neq 0$ for all $\xi \in \mathbb{R}^n$ and thus b is well-defined. Since division by $\langle \xi \rangle^m$ does not affect any of the x derivative, we only need to show that for any $\beta \in \mathbb{N}^n$, we have

$$\left| D_\xi^\beta b(x, \xi) \right| \leq C \langle \xi \rangle^{m+m'-|\beta|}$$

for some constant $C > 0$ uniformly in ξ . Indeed, observe that by the Leibinz formula

$$\begin{aligned} \left| D_\xi^\beta b(x, \xi) \right| &\leq \sum_{\mu \leq \beta} \binom{\beta}{\mu} \left| D_\xi^\mu c(x, \xi) \right| \left| D^{\beta-\mu} \langle \xi \rangle^{-m} \right| \\ &\leq C \sum_{\mu \leq \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m+m'-|\mu|} \langle \xi \rangle^{-m-|\beta-\mu|} \\ &\leq C \sum_{\mu \leq \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m'-(|\mu|+|\beta-\mu|)} \\ &= C \sum_{\mu \leq \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m'-|\beta|} \\ &= C 2^\beta \langle \xi \rangle^{m'-|\beta|} \end{aligned}$$

where we have use the definition of c and applied the result proven for a with $m \mapsto -m$. Thus, $b \in S_\infty^{m'}(\Omega; \mathbb{R}^n)$.

It is clear that $a \cdot b = c$ and we have therefore shown that $S_\infty^{m+m'}(\Omega; \mathbb{R}^n) \subset S_\infty^m(\Omega; \mathbb{R}^n) \cdot S_\infty^{m'}(\Omega; \mathbb{R}^n)$. □

A summarising theorem:

Theorem 2.8. *Given $p, n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\text{Int}(\Omega)}$. Let*

$$S_\infty^\infty(\Omega; \mathbb{R}^n) = \bigcup_{m \in \mathbb{R}} S_\infty^m(\Omega; \mathbb{R}^n).$$

Then $S_\infty^\infty(\Omega; \mathbb{R}^n)$ is a graded algebra over \mathbb{R} with continuous inclusion $S_\infty^m(\Omega; \mathbb{R}^n) \rightarrow S_\infty^{m'}(\Omega; \mathbb{R}^n)$ for all $m \leq m'$.

2.2 Ellipticity of symbols

Definition 2.9. Given $p, n \in \mathbb{N}$, $m \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\text{Int}(\Omega)}$, an order m symbol $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ is (globally) **elliptic** if there exist $\epsilon \in \mathbb{R}_{>0}$ such that

$$\inf_{|\xi| \geq 1/\epsilon} |a(x, \xi)| \geq \epsilon \langle \xi \rangle^m.$$

The importance of elliptic symbol is that they are invertible modulo $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$.

Lemma 2.10. Given $p, n \in \mathbb{N}$, $m \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\text{Int}(\Omega)}$. Let $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ be an elliptic symbol of order m . Then there exist a symbol $b \in S_\infty^{-m}(\Omega; \mathbb{R}^n)$ such that

$$a \cdot b - 1 \in S_\infty^{-\infty}(\Omega; \mathbb{R}^n).$$

Proof. We shall follow the general strategy of inverting the symbol outside of a compact set. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a smooth cut off function, i.e $0 \leq \phi \leq 1$ and $\phi(\xi) = 1$ for $|\xi| < 1$ and $\phi(\xi) = 0$ for $|\xi| > 2$.

Let $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ be an elliptic symbol, that is, for any fixed $\epsilon \in \mathbb{R}_{>0}$, we have

$$|a(x, \xi)| \geq \epsilon \langle \xi \rangle^m$$

for any $|\xi| \geq 1/\epsilon$. Thus, we can define

$$b(x, \xi) = \begin{cases} \frac{1-\phi(\epsilon\xi/2)}{a(x, \xi)} & |\xi| \geq 1/\epsilon \\ 0 & |\xi| < 1/\epsilon. \end{cases}$$

We check:

b is well-defined and smooth.

We note that $|a(x, \xi)| > 0$ whenever $|\xi| \geq 1/\epsilon$ and therefore b is well defined in that region. For smoothness, we note first that b is smooth in the regions $|\xi| > 1/\epsilon$ and $|\xi| < 1/\epsilon$. Set $\delta = 1/(2\epsilon)$. In the region where $1/\epsilon - \delta < |\xi| < 1/\epsilon + \delta$, we have $|\epsilon\xi/2| < 1/\epsilon$ and therefore $b(x, \xi) \equiv 0$ in this region and is thus smooth. Since we have covered $\Omega \times \mathbb{R}^n$ by the three chart domain above, b is smooth by the (smooth) gluing lemma.

b is a symbol of order $-m$.

We can prove by induction that in the region $|\xi| \geq 1/\epsilon$

$$D_x^\alpha D_\xi^\beta b = a^{-1-|\alpha|-|\beta|} G_{\alpha\beta}$$

for all multi-index α, β , where $G_{\alpha\beta}$ is a symbol of order $(|\alpha| + |\beta|)m - |\beta|$. Therefore, using the ellipticity estimate for a , we get

$$\begin{aligned} \|b\|_{k,-m} &= \sup_{(x,\xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{|D_x^\alpha D_\xi^\beta b(x, \xi)|}{\langle \xi \rangle^{-m-k}} \\ &= \sup_{|\xi| \geq 1/\epsilon} |a^{-1-|\alpha|-|\beta|} G_{\alpha\beta}| \langle \xi \rangle^{m+k} \\ &\leq \frac{\|G_{\alpha\beta}\|_{0, (|\alpha|+|\beta|)m-|\beta|}}{\epsilon} \sup_{|\xi| \geq 1/\epsilon^{1+|\alpha|+|\beta|}} \langle \xi \rangle^{-m(1+|\alpha|+|\beta|)} \langle \xi \rangle^{m+k} \\ &< \infty \end{aligned}$$

as required.

b is an inverse of a modulo $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$.

The main observation is that the set where b fails to be the multiplicative inverse of a is a compact set (in ξ) and thus $a \cdot b - 1$ is in fact a compactly supported smooth function of ξ which is a subset of $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$.

Explicitly, for any $N \in \mathbb{N}$

$$\sup_{(x,\xi) \in \text{Int}(\Omega) \times \mathbb{R}^n} \frac{|D_x^\alpha D_\xi^\beta (a \cdot b - 1)|}{\langle \xi \rangle^{-N}} \leq \sup_{|\xi| \leq 1/\epsilon} \langle \xi \rangle^N |D_x^\alpha D_\xi^\beta (\phi(\xi\epsilon/2))| < \infty.$$

□

3 Pseudodifferential Operators (Ψ DO's)

As mentioned in section ??, we wanted to generalise the action of differential operators

$$P(x, D)u = \frac{1}{(2\pi)^n} \int P(x, \xi) e^{i(x-y)\xi} u(y) dy d\xi$$

where P is an m^{th} order polynomial in ξ with C^∞ coefficient, to the actions of symbols $a \in S_\infty^m(\mathbb{R}^n; \mathbb{R}^n)$

$$A_a u = \frac{1}{(2\pi)^n} \int a(x, \xi) e^{i(x-y)\xi} u(y) dy d\xi$$

or $a \in S_\infty^m(\mathbb{R}_{(x,y)}^{2n}; \mathbb{R}^n)$ with action

$$A_a u = \frac{1}{(2\pi)^n} \int a(x, y, \xi) e^{i(x-y)\xi} u(y) dy d\xi.$$

One of the result we will prove is that action of $a(x, y, \xi)$ as in the later case can always be reduced to the action of some other $a(x, \xi)$ as in the former case.

Here we shall introduce a slightly more general symbol space, $\langle x - y \rangle^w S_\infty^m(\Omega; \mathbb{R}^n)$, to allow for polynomial growth perpendicular to the diagonal.

Definition 3.1. Given $m, w \in \mathbb{R}$, a w -weighted symbol space of order m , $\langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n)$ is given by

$$a \in \langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n) \iff a(x, y, \xi) = \langle x - y \rangle^w \tilde{a}(x, y, \xi), \tilde{a} \in S_\infty^m(\Omega; \mathbb{R}^n)$$

or equivalently, $a \in \langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n)$ if and only if for all multi-index α, β, γ ,

$$|D_x^\alpha D_y^\beta D_\xi^\gamma a(x, y, \xi)| \leq C \langle x - y \rangle^w \langle \xi \rangle^{m-|\gamma|}.$$

We shall show that the elements $a \in \langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n)$ acts on $S(\mathbb{R}^n)$ via the Schwartz kernel

$$I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) d\xi.$$

Proposition 3.2. Let $n \in \mathbb{N}$ and $m, w \in \mathbb{R}$ with $m < -n$, then the map

$$\begin{aligned} I : \langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n) &\rightarrow (1 + |x|^2 + |y|^2) C_\infty^0(\mathbb{R}^{2n}) \\ a \mapsto I(a) &= \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) d\xi \end{aligned}$$

extends by continuity to

$$I : \langle x - y \rangle^w S_\infty^m(\mathbb{R}_{x,y}^{2n}, \mathbb{R}^n) \rightarrow S'(\mathbb{R}^{2n})$$

in the topology of $S_\infty^{m+\epsilon}(\Omega; \mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$.

Proof. □

4 Microlocalisation

Roughly, the support of a distribution in \mathbb{R}^n consist of points $x \in \mathbb{R}^n$ where the distribution is non-zero after any smooth cut-offs near x .

Definition 4.1. The **support of a tempered distribution** $u \in S'(\mathbb{R}^n)$ is given by the set

$$\text{supp}(u) = \{x \in \mathbb{R}^n \mid \exists \phi \in S(\mathbb{R}^n), \phi(x) \neq 0, \phi u = 0\}^c.$$

And the singular support of a distribution is the set of points where the distribution fails to behave like an element of $S(\mathbb{R}^n)$.

Definition 4.2. The **singular support of a tempered distribution** $u \in$

$S'(\mathbb{R}^n)$ is given by the set

$$\text{singsupp}(u) = \{x \in \mathbb{R}^n \mid \exists \phi \in S(\mathbb{R}^n), \phi(x) \neq 0, \phi(u) \in S(\mathbb{R}^n)\}^c$$

Note that if the singular support of a tempered distribution is empty, it is then an element of $C^\infty(\mathbb{R}^n)$. The support of an operator is given by the support of its Schwartz kernel.

Definition 4.3. The **support of a continuous linear operator** $A : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ is given by

$$\text{supp}(A) = \text{supp}(K_A) \subset \mathbb{R}^n \times \mathbb{R}^n$$

where $K_A \in S'(\mathbb{R}^n \times \mathbb{R}^n)$ is the Schwartz kernel of A .

We note from the above that supports or singular supports are complement of open sets, therefore they are closed. We have the following result relating the support of a smooth function after the action of a continuous linear operator.

Proposition 4.4 (Calculus of support). *Let $A : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ be a continuous linear operator and $\phi \in C_c^\infty(\mathbb{R}^n)$, then*

$$\text{supp}(A\phi) \subset \text{supp}(A) \circ \text{supp}(\phi) := \{x \in \mathbb{R}^n \mid \exists y \in \text{supp}(\phi), (x, y) \in \text{supp}(A)\}.$$

Proof. We shall show the contrapositive statement:

$$x \notin \text{supp}(A) \circ \text{supp}(\phi) \implies x \notin \text{supp}(A\phi).$$

Suppose $x \notin \text{supp}(A) \circ \text{supp}(\phi)$. Observe that

$$\text{supp}(A) \circ \text{supp}(\phi) = \pi_x(\pi_y^{-1}(\text{supp}(\phi)) \cap \text{supp}(A))$$

where $\pi_{x,y} : \mathbb{R}^2 \rightarrow \mathbb{R}$ are the projection map to the respective coordinates. Since $\text{supp}(A)$ is closed and $\text{supp}(\phi)$ is compact, we have that $\text{supp}(A) \circ \text{supp}(\phi)$ is closed and thus x belongs to an open set. We can therefore choose a smooth cutt-off function $\chi \in C_c^\infty(\mathbb{R}^n)$ supported at x ($\chi(x) \neq 0$) but away from $\text{supp}(A) \circ \text{supp}(\phi)$. Thus,

$$\text{supp}(A) \cap (\text{supp}(\chi) \times \text{supp}(\phi)) = \emptyset$$

and hence $\chi(x)K_A(x, y)\phi(y) = 0 \implies \chi A\phi = 0$, as required. \square

4.1 Pseudolocality

We shall show two results on the singular support of pseudodifferential operators. The first result is that the singular support of any Φ DO is contained within the diagonal, i.e. they are smooth away from $x = y$. The second result is the pseudolocality result that says that action Ψ DO's do not increase singular support of distributions.

Proposition 4.5. *Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ for some $m \in \mathbb{R}$, then*

$$\text{singsupp}(A) \subset \{(x, y) \in \mathbb{R}^{2n} \mid x = y\}.$$

Proof. We shall prove this theorem for elements of $\Psi_\infty^{-\infty}(\mathbb{R}^n)$ and then extend by continuity to all orders. Let $A \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ with symbol $a \in S_\infty^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$. Its singular support is given by the singular support of the kernel. Since all derivatives of a are $O(\langle \xi \rangle^{-\infty})$, the oscillatory integral below representing the kernel is absolutely convergent and, using integration by parts, we get

$$\begin{aligned} I(a) &= \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) d\xi \\ &= \frac{1}{(2\pi)^n} \int \frac{1}{(x-y)^\alpha} (-D_\xi^\alpha) (e^{i(x-y)\xi}) a(x, y, \xi) d\xi \\ &= \frac{1}{(x-y)^\alpha} \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} (-D_\xi^\alpha) a(x, y, \xi) d\xi \\ &= \frac{1}{(x-y)^\alpha} I((-D_\xi^\alpha) a) \end{aligned}$$

which is true for all multi-index α of any order. Since all x, y -derivatives of a are uniformly bounded by $\langle \xi \rangle^{-N}$ for any $N \in \mathbb{N}$, we can differentiate under the integral sign to get the equation

$$\begin{aligned} D_x^\beta D_y^\gamma (x-y)^\alpha I(a) &= \frac{1}{(2\pi)^n} \int D_x^\beta D_y^\gamma e^{i(x-y)\xi} (-D_\xi^\alpha) a(x, y, \xi) d\xi \\ &= \frac{1}{(2\pi)^n} \int (-1)^{|\beta|} \xi^{\beta+\gamma} e^{i(x-y)\xi} (-D_\xi^\alpha) a(x, y, \xi) d\xi \end{aligned}$$

where the last integral gives a smooth function, thus showing that $(x-y)^\alpha I(a)$ is smooth for all α , and hence $I(a)$ is smooth away from $x = y$.

Now, for a general $A \in \Psi_\infty^m(\mathbb{R}^n)$, $m \in \mathbb{R}$, we shall use the density of $S_\infty^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n) \subset S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n)$ and that I extends by continuity to a map $I : S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n) \rightarrow S'(\mathbb{R}^{2n})$ in the topology $S_\infty^{m+\epsilon}(\mathbb{R}^{2n}; \mathbb{R}^n)$ for any $\epsilon > 0$??.

□

Proposition 4.6. *Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ for some $m \in \mathbb{R}$ and $u \in C^{-\infty}(\mathbb{R}^n)$, then*

$$\text{singsupp}(Au) \subset \text{singsupp}(u).$$

We call operators that satisfies the above property pseudolocal

Proof. Again we shall prove the contrapositive statement that

$$x \notin \text{singsupp}(u) \implies x \notin \text{singsupp}(Au)$$

Let $u \in S'(\mathbb{R}^n)$ be compactly supported and $x_0 \notin \text{singsupp}(u)$. We can choose $\chi \in S(\mathbb{R}^n)$, (normalised) so that $\chi \equiv 1$ in a neighbourhood of x_0 and that $\chi u \in S(\mathbb{R}^n)$. Observe that

$$Au = A(\chi u + (1 - \chi)u) = A(\chi u) + A(1 - \chi)u.$$

Since $\chi u \in S(\mathbb{R}^n) \implies A\chi u \in S(\mathbb{R}^n)$ [?], we have that

$$\text{singsupp}(Au) = \text{singsupp}(A(1 - \chi)u).$$

Furthermore, we know that $x_0 \notin \text{supp}((1 - \chi)u)$. Now, we shall further cut-off near x_0 by choosing a $\phi \in S(\mathbb{R}^n)$ compactly supported away from $\text{supp}(1 - \chi)$ and $\phi \equiv 1$ near x_0 , i.e.

$$\text{supp}(1 - \chi) \cap \text{supp}\phi = \emptyset.$$

We now have an operator $\phi A(1 - \chi)$ with kernel

$$\phi(x)K_A(x, y)(1 - \phi(y))$$

that vanishes (in particular smooth) on the diagonal. Since we have shown that the singular support of a pseudodifferential operator have to be contained in the diagonal, we conclude that $\phi A(1 - \chi)$ is a smoothing operator, and thus $\phi A(1 - \chi)u \in C^\infty(\mathbb{R}^n)$ as required. \square

4.2 Elliptic, Characteristic, Wavefront sets

We will now define *ellipticity at a point* in phase space which allow us to define various microlocal constructions that focus on localised (conically in phase space) behaviour Ψ DO's and distributions.

Definition 4.7. A pseudodifferential operator, $A \in \Psi_\infty^m(\mathbb{R}^n)$, $m \in \mathbb{R}$ is **elliptic at a point** $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ if there exist $\epsilon > 0$ such that its left-reduced symbol satisfies the lower bound

$$|\sigma_L(A)(x, \xi)| \geq \epsilon \langle \xi \rangle^m$$

in the region

$$\overline{U}_\epsilon = \left\{ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid |x - x_0| \leq \epsilon, \left| \widehat{\xi} - \widehat{\xi}_0 \right| \leq \epsilon, |\xi| \geq 1/\epsilon \right\}$$

where $\widehat{\xi} = \xi/|\xi|$ for any non-zero $\xi \in \mathbb{R}^n$. We denote the set of all elliptic points of A as

$$Ell^m(A) = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid A \text{ is elliptic of order } m \text{ at } (x, \xi)\}$$

and its complement in $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ as

$$\begin{aligned}\Sigma^m(A) &= Ell^m(A)^c \setminus \{(x, 0) \mid x \in \mathbb{R}^n\} \\ &= \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid A \text{ is \textbf{not} elliptic of order } m \text{ at } (x, \xi)\}\end{aligned}$$

Lemma 4.8. *Let $A \in \Psi_\infty^m(\mathbb{R}^n)$, $m \in \mathbb{R}$.*

1. *If $\sigma_m(A)(x, \xi)$ is homogeneous of degree m in ξ , then*

$$Ell^m(A) = \{(x_0, \xi_0) \mid \xi_0 \neq 0, \sigma_m(A)(x_0, \xi_0) \neq 0\}.$$

2. *$Ell^m(A)$ is open in $\mathbb{R}^n \times \mathbb{R}^n$.*

3. *$Ell^m(A)$ is conic in $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$, in the sense that*

$$(x_0, \xi_0) \in Ell^m(A) \implies (x_0, t\xi_0) \in Ell^m(A), \forall t \in \mathbb{R}_{>0}.$$

4. *$\Sigma^m(A)$ is closed conic.*

5. *if $B \in \Psi^{m'}(\mathbb{R}^n)$, then*

$$\Sigma^{m+m'}(A \circ B) = \Sigma^m(A) \cup \Sigma^{m'}(B).$$

Proof. Let $A \in \Psi_\infty^m(\mathbb{R}^n)$, $m \in \mathbb{R}$ be given.

1. Suppose the principal symbol $\sigma_m(A)(x, \xi)$ is homogeneous of order m in ξ . We need to show that

$$(x_0, \xi_0) \in Ell^m(A) \iff \xi_0 \neq 0, \sigma_m(A)(x_0, \xi_0) \neq 0.$$

If $\xi_0 = 0$, $(x_0, \xi_0) \notin Ell_\infty^m$ by definition of ellipticity. If $\sigma_m(x_0, \xi_0) = 0$, by homogeneity, we have

$$\sigma_m(x_0, t\xi_0) = t^m \sigma_m(x_0, \xi_0) = t^m \cdot 0 = 0$$

for all $t \in \mathbb{R}_{>0}$. By definition of principal symbol, we can write the left symbol of A as

$$\sigma_L(A) = \sigma_m(A) + a$$

where $a \in S_\infty^{m-1}(\mathbb{R}^n; \mathbb{R}^n)$. Now, observe that for any $\epsilon > 0$, the set

$$\overline{U}_\epsilon = \left\{ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid |x - x_0| \leq \epsilon, \left| \widehat{\xi} - \widehat{\xi}_0 \right| \leq \epsilon, |\xi| \geq 1/\epsilon \right\}$$

contains the (open) half-line starting at $\widehat{\xi}_0/\epsilon$, i.e. the set $\{(x_0, t\xi_0/(|\xi_0|\epsilon)) \mid t > 0\}$.

However, by the symbol estimate of a ,

$$\begin{aligned}
\left| \sigma_L(A) \left(x_0, \frac{t\xi_0}{|\xi_0|\epsilon} \right) \right| &\leq \left(\frac{t}{\epsilon|\xi_0|} \right)^m |\sigma_m(x_0, \xi_0)| + \left| a \left(x_0, \frac{t\xi_0}{|\xi_0|\epsilon} \right) \right| \\
&= 0 + \left| a \left(x_0, \frac{t\xi_0}{|\xi_0|\epsilon} \right) \right| \\
&\leq C \left\langle \frac{t\xi_0}{|\xi_0|\epsilon} \right\rangle^{m-1} \\
&= C \langle t/\epsilon \rangle^{m-1}
\end{aligned}$$

and therefore

$$\begin{aligned}
\inf_{(x,\xi) \in \bar{U}_\epsilon} \frac{|\sigma_L(A)(x, \xi)|}{\langle \xi \rangle^m} &\leq \inf_{t>0} \frac{\left| \sigma_L(A) \left(x_0, \frac{t\xi_0}{|\xi_0|\epsilon} \right) \right|}{\langle t/\epsilon \rangle^m} \\
&\leq \inf_{t>0} \frac{C \langle t/\epsilon \rangle^{m-1}}{\langle t/\epsilon \rangle^m} \\
&= C \inf_{t>0} \langle t/\epsilon \rangle^{-1} \\
&= 0
\end{aligned}$$

which means that $(x_0, \xi_0) \notin Ell^m(A)$.

Conversely, if $\sigma_m(A)(x_0, \xi_0) \neq 0$, by continuity and homogeneity, $\sigma_m(A)$, is non-zero in a (closed) conic neighbourhood, i.e. there exist $\epsilon > 0$ such that $\sigma_m(A) \neq 0$ in

$$\bar{U}_\epsilon = \left\{ (x, \xi) \mid |x - x_0| \leq \epsilon, \left| \hat{\xi} - \hat{\xi}_0 \right| \leq \epsilon, |\xi| \geq 1/\epsilon \right\}.$$

Again, writing the left symbol as a sum of the principal symbol and a lower order term, we observe that in \bar{U}_ϵ ,

$$\begin{aligned}
\frac{|\sigma_L(A)(x, \xi)|}{\langle \xi \rangle^m} &\geq \frac{||\sigma_m(A)(x, \xi)| - |a(x, \xi)||}{\langle \xi \rangle^m} \\
&= \left| \frac{|\xi|^m}{\langle \xi \rangle^m} \left| \sigma_m(A)(x, \hat{\xi}) \right| - \frac{|a(x, \xi)|}{\langle \xi \rangle^m} \right|
\end{aligned}$$

By the symbol estimate of a , the second term is tending to 0 which the first term is bounded below by $C = \inf_{(x,\xi) \in \bar{U}_\epsilon} |\sigma_m(A)(x, \xi)| > 0$. Therefore, choosing a smaller ϵ if necessary, we have $|a(x, \xi)| / \langle \xi \rangle^m < C$ and thus

$$\inf_{(x,\xi) \in \bar{U}_\epsilon} \frac{|\sigma_L(A)(x, \xi)|}{\langle \xi \rangle^m} \geq C' \geq \epsilon.$$

and therefore $(x_0, \xi_0) \in Ell^m(A)$.

2. We note first that if the principal symbol is homogeneous of degree m , the previous result applies and continuity of the principal symbol guarantee that the elliptic set is open (if $\sigma_m(A)$ is non-zero at a point, it is non-zero in an open neighbourhood of the point).

For the general case, suppose $(x_0, \xi_0) \in Ell^m(A)$. We therefore have for some $\epsilon > 0$,

$$|\sigma_L(A)(x, \xi)| \geq \epsilon \langle \xi \rangle^m$$

in the region

$$\overline{U}_\epsilon(x_0, \xi_0) = \left\{ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid |x - x_0| \leq \epsilon, \left| \widehat{\xi} - \widehat{\xi}_0 \right| \leq \epsilon, |\xi| \geq 1/\epsilon \right\}.$$

It suffice to show that there is an open neighbourhood of (x_0, ξ_0) where A remains elliptic. We can take desired open neighbourhood to be

$$V = \left\{ (x', \xi') \mid \xi' \neq 0, |x' - x_0| < \epsilon/2, \left| \widehat{\xi}' - \widehat{\xi}_0 \right| < \epsilon/2 \right\}.$$

Then, we can check that for every $(x', \xi') \in V$, A satisfies the elliptic estimate in $\overline{U}_{\epsilon/2}(x', \xi')$. Indeed, if $(x, \xi) \in \overline{U}_{\epsilon/2}(x', \xi')$, then

$$\begin{aligned} |x - x_0| &\leq |x - x'| + |x' - x_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\ \left| \widehat{\xi} - \widehat{\xi}_0 \right| &\leq \left| \widehat{\xi} - \widehat{\xi}' \right| + \left| \widehat{\xi}' - \widehat{\xi}_0 \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\ |\xi| &\geq 2/\epsilon \geq 1/\epsilon \end{aligned}$$

which shows that $\overline{U}_{\epsilon/2}(x', \xi') \subset \overline{U}_\epsilon(x_0, \xi_0)$. Therefore,

$$\inf_{(x, \xi) \in \overline{U}_{\epsilon/2}(x', \xi')} \frac{|\sigma_L(A)(x, \xi)|}{\langle \xi \rangle^m} \geq \inf_{(x, \xi) \in \overline{U}_\epsilon(x_0, \xi_0)} \frac{|\sigma_L(A)(x, \xi)|}{\langle \xi \rangle^m} \geq \epsilon \geq \epsilon/2$$

as required.

3. Again, this result is immediate if the principal symbol is homogeneous in ξ . In general, this result come from the observation that only $\widehat{\xi} = \xi/|\xi|$ appears in \overline{U}_ϵ in the definition of $Ell^m(A)$, i.e. only the *direction* in the dual variable is important.

Explicitly, let $(x_0, \xi_0) \in Ell^m(A)$ and $t \in \mathbb{R}_{>0}$. Clearly $t\xi_0 \neq 0$. And note that

$$\overline{U}_\epsilon(x_0, \xi_0) = \overline{U}_\epsilon(x_0, t\xi_0)$$

since $\widehat{\xi} = \widehat{t\xi}$.

4. $\Sigma^m(A) = Ell^m(A)^c$ where $Ell^m(A)$ is open and conic. Since complement of conic set is conic, and complement of open is closed, we conclude that $\Sigma^m(A)$ is closed conic.

5. If both principal symbols are homoeogenous of degree m, m' respectively, we can applied the result above and by symbol calculus, we have

$$\begin{aligned} Ell^{m+m'}(A \circ B) &= \{(x, \xi) \mid \xi \neq 0, \sigma_{m+m'}(A \circ B) = \sigma_m(A)\sigma_{m'}(B) \neq 0\} \\ &= \{(x, \xi) \mid \xi \neq 0, \sigma_m(A) \neq 0\} \cap \{(x, \xi) \mid \xi \neq 0, \sigma_{m'}(B) \neq 0\} \\ &= Ell^m(A) \cap Ell^{m'}(B). \end{aligned}$$

Taking complement give the desired result.

In general,

□

Definition 4.9. The **wavefront set** of a compactly supported tempered distribution

$$u \in C_c^{-\infty}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) \mid \text{supp}(u) \Subset \mathbb{R}^n\}$$

is given by

$$\text{WF}(u) = \bigcap \{ \Sigma^0(A) \mid A \in \Psi_{\infty}^0(\mathbb{R}^n), Au \in C^{\infty}(\mathbb{R}^n) \}.$$

For general tempered distribution $u \in S'(\mathbb{R}^n)$, its wavefront set is given by

$$\text{WF}(u) = \bigcup_{\chi \in C_c^{\infty}(\mathbb{R}^n)} \text{WF}(\chi u).$$

Proposition 4.10. For compactly supported tempered distribution, $u \in C_c^{-\infty}(\mathbb{R}^n)$,

$$\pi(\text{WF}(u)) = \text{singsupp}(u).$$

where $\pi(x, y) = x$ is the projection map.

Proof. To show $\pi(\text{WF}(u)) \subset \text{singsupp}(u)$, we observe that, by definition of singular support,

$$x_0 \notin \text{singsupp}(u) \implies \exists \phi \in S(\mathbb{R}^n), \phi(x_0) \neq 0, \phi u \in S(\mathbb{R}^n).$$

But since multiplication by ϕ gives an operator in $\Psi_{\infty}^0(\mathbb{R}^n)$ which is elliptic at (x_0, ξ) for any $\xi \neq 0$ (ϕ is its own principal symbol which happens to be homogeneous and non-zero for any $(x_0, \xi), \xi \neq 0$). Therefore, $x_0 \notin \pi(\text{WF}(u))$.

Conversely, if $x_0 \notin \pi(\text{WF}(u))$, then for all $\xi \neq 0$, there exist $A_{\xi} \in \Psi_{\infty}^0(\mathbb{R}^n)$ such that A_{ξ} is elliptic at (x_0, ξ) and $A_{\xi}u \in C^{\infty}(\mathbb{R}^n)$. Since elliptic set $Ell^0(A_{\xi})$ is open and conic, we know that there exist $\epsilon = \epsilon(\xi)$ such that A_{ξ} is elliptic in the open conic set

$$V_{\xi} = \left\{ (x', \xi') \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \mid |x' - x_0| < \epsilon, \left| \widehat{\xi'} - \widehat{\xi} \right| < \epsilon \right\}.$$

Compactness of the sphere (note that $\xi' \mapsto \widehat{\xi'}$ is an embedding of $\mathbb{R}^n \setminus \{0\}$ into S^n) allow us to cover $\{x_0\} \times (\mathbb{R}^n \setminus \{0\})$ with finite number of $V_{\xi_j}, j = 1, \dots, N$ with corresponding operators A_{ξ_j} .

Now, consider the operator

$$A = \sum_{j=1}^N A_{\xi_j}^* A_{\xi_j}.$$

By the pseudolocality of pseudodifferential operators, we know that $A_{\xi_j} u \in C^\infty(\mathbb{R}^n) \implies A_{\xi_j}^* A_{\xi_j} u \in C^\infty(\mathbb{R}^n)$. Therefore, $Au \in C^\infty(\mathbb{R}^n)$ and A is elliptic at $(x_0, \xi), \forall \xi \neq 0$ with non-negative symbol. We can pick a smooth cut-off χ , $\chi \equiv 1$ when restricted to an $\epsilon/2$ -ball around x_0 forming an operator

$$A + (1 - \chi) \in \Psi_\infty^0(\mathbb{R}^n)$$

that is globally elliptic. We can construct a (global) elliptic parametrix E so that

$$1 - (E \circ A + E(1 - \chi)) \in \Psi_\infty^{-\infty}(\mathbb{R}^n).$$

Finally, we can choose any smooth cut-off ϕ with support subordinate to that of χ , i.e. $\text{supp}(\phi) \subset \text{supp}(\chi)$ and note that

$$\phi \circ E \circ (1 - \chi) \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$$

making it a smoothing operator \square . Thus, we conclude

$$\phi u = \phi E \circ Au + \phi \circ E \circ (1 - \chi)u \in C^\infty(\mathbb{R}^n)$$

as required. \square

Definition 4.11. Let $a \in S_\infty^m(\mathbb{R}^p; \mathbb{R}^n)$ for some $m \in \mathbb{R}$, $p, n \in \mathbb{N}$ be a symbol. We say a is of order $-\infty$ at a point $(x_0, \xi_0) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\}$ (write $a = O(\langle \xi \rangle^{-\infty})$) if there exist $\epsilon \in \mathbb{R}_{>0}$ such that for all $M \in \mathbb{R}$, there is a constant $C_M > 0$ such that

$$|a(x, \xi)| \leq C_M \langle \xi \rangle^{-M}$$

in the neighbourhood of (x_0, ξ_0) given by

$$\overline{U}_{(x_0, \xi_0)} = \left\{ (x, \xi) \in \mathbb{R}^p \times \mathbb{R}^n \mid |x - x_0| \leq \epsilon, \left| \widehat{\xi} - \widehat{\xi_0} \right| \leq \epsilon \right\}.$$

We define the cone support of the symbol a to be all the points in phase space that where it fails to be $O(\langle \xi \rangle^{-\infty})$.

$$\text{conesupp}(a) = \left\{ (x, \xi) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\} \mid a = O(\langle \xi \rangle^{-\infty}) \text{ at } (x, \xi) \right\}^c.$$

Lemma 4.12. *Let $a \in S_\infty^\infty(\mathbb{R}^p; \mathbb{R}^n)$, then*

1. *conesupp(a) is a closed conic set in $\mathbb{R}^p \times \mathbb{R}^n$.*
2. *If $a = O(\langle \xi \rangle^{-\infty})$ at $(x_0, \xi_0) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\}$, then so is $D_x^\alpha D_\xi^\beta a(x, \xi)$ for any multi-index α, β*

With the lemma above, we can in fact define the cone support of a symbol to be the smallest closed conic subset of the phase space (with $\xi \neq 0$) such that, in the complement, a and all its derivatives are of order $-\infty$.

Definition 4.13. Let $A \in \Psi_\infty^m(\mathbb{R}^n)$, $m \in \mathbb{R}$ be pseudodifferential operator. We define the **essential support**, $\text{WF}'(A)$, of A to be the cone support of its left symbol, i.e.

$$\text{WF}'(A) = \text{conesupp}(\sigma_L(A)) \subset \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$$

Using symbol calculus we can prove the following result.

Lemma 4.14. *Let $A \in \Psi_\infty^m(\mathbb{R}^n)$, $B \in \Psi_\infty^{m'}(\mathbb{R}^n)$ be pseudifferential operators. Then*

1. $\text{WF}'(A) = \text{conesupp}(\sigma_R(A))$.
2. $\text{WF}'(A \circ B) \subset \text{WF}'(A) \cap \text{WF}'(B)$.
3. $\text{WF}'(A + B) = \text{WF}'(A) \cup \text{WF}'(B)$.

With the concept of essential support we can define the notion of *microlocal elliptic parametrix* which can be thought of as local inverse at an elliptic point of ΨDO 's.

Proposition 4.15. *Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ and $z \notin \Sigma^m(A)$. Then there exist a (two-sided) microlocal parametrix $B \in \Psi^{-m}(\mathbb{R}^n)$ such that*

$$z \notin \text{WF}'(1 - AB) \text{ and } z \notin \text{WF}'(1 - BA).$$

Proof. Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ is elliptic at $(x_0, \xi_0) \in \text{Ell}^m(A)$. For each $\epsilon \in \mathbb{R}_{>0}$ we define

$$\gamma_\epsilon(x, \xi) = \chi\left(\frac{x - x_0}{\epsilon}\right) (1 - \chi(\epsilon\xi)) \chi\left(\frac{\widehat{\xi} - \widehat{\xi}_0}{\epsilon}\right)$$

where $\chi \in C^\infty(\mathbb{R}^n)$ is a smooth cut-off function that is identically 1/2-ball but supported only in the unit ball. Notice that $\gamma_\epsilon \in S_\infty^0(\mathbb{R}^{2n}; \mathbb{R}^n)$ with support given by

$$\text{supp}(\gamma_\epsilon) \subset \left\{ (x, \xi) \mid |x - x_0| \leq \epsilon, |\xi| \geq \frac{1}{2\epsilon}, \left| \widehat{\xi} - \widehat{\xi}_0 \right| \leq \epsilon \right\}.$$

Restricted to the conic set

$$\overline{U}_\epsilon = \left\{ (x, \xi) \mid |x - x_0| \leq \frac{\epsilon}{2}, \left| \widehat{\xi} - \widehat{\xi}_0 \right| \leq \frac{\epsilon}{2}, |\xi| \geq \frac{1}{\epsilon} \right\} \subset \text{supp}(\gamma_\epsilon)$$

it is identically 1 and therefore γ_ϵ is elliptic at (x_0, ξ_0) . Let $L_\epsilon = \text{Op}_L(\gamma_\epsilon)$ be the corresponding pseudodifferential operator. Observe that, by construction

$$(x_0, \xi_0) \notin \text{WF}'(1 - L_\epsilon)$$

since $1 - \gamma_\epsilon$ is supported away from an ϵ -neighbourhood of $x = x_0$ and the wavefront set of L_ϵ is contained in an ϵ -neighbourhood of (x_0, ξ_0) , i.e.

$$\text{WF}'(L_\epsilon) \subset N_\epsilon(x_0, \xi_0) := \left\{ (x, \xi) \mid |x - x_0| \leq \epsilon, \left| \widehat{\xi} - \widehat{\xi}_0 \right| \leq \epsilon \right\}$$

since γ_ϵ is bounded below in some conic neighbourhood of every point in $N_\epsilon(x_0, \xi_0)$.

Now, let $G_s = \text{Op}_L(\langle \xi \rangle^s)$ for each $s \in \mathbb{R}$. Note that G_s is globally elliptic with positive principal symbol. Consider the following operator,

$$J = (1 - L_\epsilon) \circ G_{2m} + A^* A \in \Psi_\infty^{2m}(\mathbb{R}^n)$$

with principal symbol

$$\sigma_{2m}(J) = (1 - \gamma_\epsilon) \langle \xi \rangle^{2m} + |\sigma_m(A)|^2.$$

Since $\text{Ell}^m(A)$ is open conic, we can choose ϵ is small enough so that $\text{Ell}^m(A) \subset \text{supp}(\gamma_\epsilon)$. Then, by positivity of all terms involved, we have

$$\frac{|\sigma_{2m}(J)|}{\langle \xi \rangle^{2m}} = (1 - \gamma_\epsilon) + \frac{|\sigma_m(A)|^2}{\langle \xi \rangle^{2m}}$$

where the first term is bounded below by 1 outside of $\text{supp}(\gamma_\epsilon)$ while in $\text{supp}(\gamma_\epsilon)$ the second term is bounded below by ϵ since A is elliptic (of order m) at every point in $\text{supp}(\gamma_\epsilon)$. Therefore J is globally elliptic and thus have a global elliptic parametrix $H \in \Psi_\infty^{-2m}(\mathbb{R}^n)$. We shall claim that

$$B = H \circ A^* \in \Psi_\infty^m(\mathbb{R}^n)$$

is a (right) microlocal elliptic parametrix to A . Indeed,

$$\begin{aligned} B \circ A - 1 &= H A^* A - 1 \\ &= H (J - (1 - L_\epsilon) G_{2m}) - 1 \\ &= (H J - 1) - H(1 - L_\epsilon) G_{2m}. \end{aligned}$$

Since H is a global parametrix to J , the first term above is a smoothing operator (i.e. an element of $\Psi_\infty^{-\infty}(\mathbb{R}^n)$) and therefore have empty wavefront set. Furthermore, by [?] the wavefront set of the second term is a subste of $\text{WF}'(1 - L_\epsilon)$ which does not contain (x_0, ξ_0) by construction. \square

Proposition 4.16. *Pseudodifferential operators are microlocal in the following sense: Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ and $u \in C_c^{-\infty}(\mathbb{R}^n)$, then*

$$\text{WF}(Au) \subset \text{WF}(u). \quad (1)$$

In fact, we have

$$\text{WF}(Au) \subset \text{WF}'(A) \cap \text{WF}(u).$$

Proof. □

A partial converse to the above is given by the following proposition.

Proposition 4.17. *Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ and $u \in C_c^{-\infty}(\mathbb{R}^n)$, then*

$$\text{WF}(u) \subset \text{WF}(Au) \cup \Sigma^m(A).$$

5 Appendix

5.1 Stationary phase lemma

In the study of pseudodifferential operators, we often encounter integral of highly oscillatory functions of the form

$$I(h) = \int_{\mathbb{R}} a(x) e^{i\varphi(x)/h} dx$$

where $a \in C_c^\infty(\mathbb{R})$, $\varphi \in C^\infty(\mathbb{R})$ and we are interested in the asymptotic behaviour as $h \rightarrow 0$. We note that if φ is linear (or constant), i.e. $\varphi(x) = \alpha x + \beta$, $\alpha, \beta \in \mathbb{R}$, then,

$$|I(h)| = \left| \int_{\mathbb{R}} a(x) e^{i(\alpha x + \beta)/h} dx \right| = |e^{i\beta/h}| \left| \int_{\mathbb{R}} a(x) e^{i\alpha x/h} dx \right| = \left| \int_{\mathbb{R}} a(x) e^{i\alpha x/h} dx \right| \rightarrow 0$$

as $h \rightarrow 0$ by Riemann-Lebesgue lemma. That is to say, as the length scale of the oscillation tends to zero, the values of the integrand achieve perfect cancellation. In general, if $\varphi'(x) \neq 0$, we expect $e^{i\varphi(x)/h}$ to oscillate at length scale of order h and thus as $h \rightarrow 0$,

References

[Taylor, 2011] Taylor, M. (2011). *Partial Differential Equations I*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2 edition.