# Microlocal Analysis
## with Applications to Non-Elliptic Fredholm Problems

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19 October 2018

# Introduction

A linear partial differential operator of order $k \in \mathbb{N}$ in $\mathbb{R}^n$ :

$$P = P(x, D_x) = \sum_{|\alpha| \leqslant k} c_\alpha(x) D_x^\alpha, \quad c_\alpha \in C_\infty^\infty(\mathbb{R}^n)$$

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where

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n \qquad \text{multi-index}$$

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \qquad \text{order of multi-index}$$

$$D_x^\alpha = (-i\partial_{x_1})^{\alpha_1} (-i\partial_{x_2})^{\alpha_2} \ldots (-i\partial_{x_n})^{\alpha_n}$$

$$D_{x_i} = -i\partial_{x_i} \implies \mathcal{F} D_x^\alpha = \xi^\alpha \mathcal{F}$$

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Examples:

$$\Delta = D_{x_1}^2 + \cdots + D_{x_n}^2 \qquad \text{Laplace operator}$$

$$\Box u = D_{x_1}^2 + \cdots + D_{x_n}^2 - D_t^2 \qquad \text{Wave operator}$$

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Weak solution and forcing:

$$\begin{aligned} u : \mathcal{S}(\mathbb{R}^n) &\to \mathbb{C} \\ \varphi &\mapsto u(\varphi) \end{aligned}$$

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \iff \varphi \in C^\infty \text{ and } \sup_x \left| x^\beta D_x^\alpha \varphi(x) \right| < \infty$$

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**Fredholm** theory tackles all three simultaneously!

# Overview


1. Introduction
2. Fredholm Operators and Regularity
3. “Elliptic operators are Fredholm”
4. A Non-elliptic Fredholm problem

# Fredholm Operators

## Definition (Fredholm operators)

A continuous linear operator $T : \mathcal{X} \to \mathcal{Y}$ between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ is Fredholm, if

- $T$ has closed range, i.e. $T(\mathcal{X})$ is closed in $\mathcal{Y}$,
- $\ker(T) \subset \mathcal{X}$ is finite dimensional,
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Suppose $Tx = y$ for a given $y \in \mathcal{Y}$.

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$T$ Fredholm

$\rightsquigarrow$ existence and uniqueness reduce to finite dimensional linear algebra.

# Fredholm Estimate

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**Theorem (Fredholm Estimate)**

*Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be Banach spaces. If*

- $T : \mathcal{X} \to \mathcal{Y}$ *is continuous,*
- $\mathcal{X}$ *is compactly contained in* $\mathcal{Z}$*, i.e.* $\iota : \mathcal{X} \hookrightarrow \mathcal{Z}$ *is compact,*
- *for all* $x \in \mathcal{X}$*, there exist* $C > 0$ *such that the following estimate hold*

$$\|x\|_{\mathcal{X}} \leqslant C \left( \|Tx\|_{\mathcal{Y}} + \|x\|_{\mathcal{Z}} \right) \tag{1}$$

*then* $T$ *is semi-Fredholm*

- *the image,* $T(\mathcal{X})$ *is closed, and*
- $T$ *has finite dimensional kernel.*

# Constructing a *Fredholm problem*

What's a Fredholm differential operator? ... what's $\mathcal{X}$ and $\mathcal{Y}$?

**Fredholm Problem**

Given a differential operator $P$, can we construct solution spaces $\mathcal{X}$ and $\mathcal{Y}$, so that

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What's the link to *regularity*? Sobolev Space!

# Sobolev Space

## Definition

The Sobolev space of order $k \in \mathbb{N}$ on $\mathbb{R}^n$, $H^k(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ is defined by

$$\begin{aligned} u \in H^k(\mathbb{R}^n) &\iff D^\alpha u \in L^2(\mathbb{R}^n) \text{ whenever } |\alpha| \leqslant k \\ &\iff \langle \xi \rangle^k \mathcal{F}u(\xi) \in L^2(\mathbb{R}^n). \end{aligned}$$

$$\langle \xi \rangle := \left(1 + |\xi|^2\right)^{1/2} = \left(1 + |\xi_1|^2 + \cdots + |\xi_n|^2\right)^{1/2}$$

Hilbert space structure that keeps track of (global) regularity data of $u$.

$$\|u\|_{H^k} = \underbrace{\|u\|_{L^2}}_{\text{global decay}} + \underbrace{\sum_{|\alpha| \leqslant k} \|D^\alpha u\|_{L^2}}_{k \text{ times differentiable}}$$

# Sobolev Space on Closed Manifold

Let $M$ be a smooth closed $n$-manifold (compact without boundary), $s \in \mathbb{R}$, $u \in (C^\infty(M))'$, then

$$u \in H^s(M) \iff (\chi \cdot u) \circ \Phi^{-1} \in H^s(U)$$

for any chart $\Phi : \widetilde{U} \to U \subset \mathbb{R}^n$ and smooth bump function $\chi \in C^\infty(M)$ compactly supported in the chart domain $\widetilde{U}$.

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Henceforth, $M$ is either $\mathbb{R}^n$ or a closed $n$-manifold.

# General Strategy

Existence, uniqueness, regularity $\rightsquigarrow$

For what $s, s' \in \mathbb{R}$ can we prove

$$\|u\|_{H^s} \leqslant C\left(\|Pu\|_{H^{s'}} + \|u\|_{H^N}\right).$$

so that $P : H^s(M) \to H^{s'}(M)$ is (semi-) Fredholm?

For **elliptic operators**:

$$\text{any } s + m \text{ and } s' = s.$$

For **non-elliptic operators**:

$$\text{any } s + m \text{ and } s' = s + 1.$$

Only certain subsets of Sobolev spaces allowed.

# "Elliptic operators are Fredholm"

How do we get such an estimate?

**Theorem (Elliptic regularity)**

*Let $P$ be an order $m \in \mathbb{R}$ elliptic differential operator on an $n$-manifold, $M$. Suppose we know a priori that $u \in H^N(M)$ for some $N \in \mathbb{R}$. Then, for any $s \in \mathbb{R}$*

$$(f =)Pu \in H^s(M) \implies u \in H^{s+m}(M)$$

*and $u$ satisfies the estimates: $\exists C > 0$*

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$$(\Delta + 1)\, u = \mathcal{F}^{-1} \mathcal{F} (\Delta + 1) u = \mathcal{F}^{-1} (1 + |\, \xi \,|^2) \mathcal{F} u$$

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We call $\left(1 + |\,\xi\,|^2\right)$ is the **symbol** for $(\Delta + 1)$.
We expect an inverse ...

$$(\Delta + 1)^{-1}\,(\Delta + 1)\,u(x) = \mathcal{F}^{-1}(1 + |\,\xi\,|^2)^{-1}(1 + |\,\xi\,|^2)\mathcal{F}u = u$$

# Pseudodifferential operator

Question: What is $(\Delta + 1)^{-1}$? Answer: **pseudo**differential operator.

$$P(x, D_x)u = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} p(x,\xi) u(y) \, \mathrm{d}y \, \mathrm{d}\xi$$

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## Definition

A smooth function $p(x,\xi)\in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ is a symbol of order $m\in\mathbb{R}$, i.e. $p\in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$, if

$$\left| D_x^\alpha D_\xi^\beta p(x,\xi)\right| \leqslant C_{\alpha,\beta,\gamma}\left\langle\xi\right\rangle^{m-|\beta|}, \quad C_{\alpha,\beta}>0$$

for any multi-index $\alpha,\beta\in\mathbb{N}^n$.
A pseudodifferential operator, $P\in\Psi^m_\infty(\mathbb{R}^n)$ of order $m$ with (left reduced) symbol $p\in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ has action on $u\in\mathcal{S}'(\mathbb{R}^n)$ given by the integral above.

# Pseudodifferential operators

**Lemma**

*If $P \in \Psi^m_\infty(M)$ for some $m \in \mathbb{R}$,*

1. $P : H^s(M) \to H^{s-m}(M)$ *is continuous for any $s \in \mathbb{R}$.*
2. *If $P$ is* **elliptic** *of order $m$, i.e. its symbol $p$ satisfies*

$$|\, p(x,\xi)\,| \geqslant \epsilon\,\langle\xi\rangle^m \qquad \text{in } |\,\xi\,| > \epsilon \text{ for some } \epsilon > 0$$

*then there exist parametrix $Q \in \Psi^{-m}_\infty(M)$ such that*

$$QP - 1 : H^s(M) \to H^{s'}(M)$$

*is continuous for any $s, s' \in \mathbb{R}$.*

# Proof of Elliptic Regularity

$P$ elliptic with parametrix $Q$. Given $u \in H^N(M)$. Given any $u \in H^N(M)$. Write $u = QPu - (QP - 1)\,u$.

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$$\|u\|_{H^{s+m}} \leqslant \underbrace{\|QPu\|_{H^{s+m}}}_{\leqslant C\|Pu\|_{H^s}} + \underbrace{\|(QP-1)\,u\|_{H^{s+m}}}_{\leqslant C\|u\|_{H^N}}$$

using continuity $Q : H^{s+m} \to H^s$ and $(QP - 1) : H^{s+m} \to H^N$.

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We get

$$\|u\|_{H^{s+m}} \leqslant C\,\|Pu\|_{H^s} + C\,\|u\|_{H^N}\,.$$

# Non-elliptic Fredholm problem

Recent work by [**?**, **?**] show that we can construct Fredholm problem for **non-elliptic** operators too!

## Theorem (Main theorem)

*There exist a perturbation $Q$ of the wave operator $\square$ on $\mathbb{T}^{1+n}$ and a subspace $\mathcal{X}^{s+2} \subset H^{s+2}(\mathbb{T}^{1+n})$ for each $s \in \mathbb{R}$, such that the operator:*

$$(\square - iQ) : \mathcal{X}^{s+2} \to H^{s+1}(\mathbb{T}^n)$$

*is Fredholm.*

$$\mathbb{T}^{1+n} := \mathbb{S}^1_t \times \underbrace{\mathbb{S}^1_{x_1} \times \mathbb{S}^1_{x_2} \times \cdots \times \mathbb{S}^1_{x_n}}_{n}$$

$$\square := \partial_t^2 - \sum_{j=1}^{n-1} \partial_{x_j}^2, \quad p(t, x, \tau, \xi) = |\,\xi\,|^2 - \tau^2$$

# Microlocal Viewpoint

Global ellipticity

$\iff \; |\, p(t,x,\tau,\xi) \,| \geqslant \epsilon \, \langle (\tau,\xi) \rangle^2$ whenever $|\, (\tau,\xi) \,| > 1/\epsilon$.

# Microlocal Viewpoint

Microlocal ellipticity at a point $(t_0, x_0, \tau_0, \xi_0) \in T^*\mathbb{T}^{1+n} \setminus 0$
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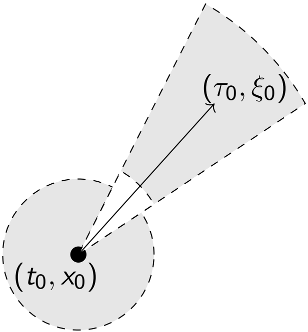

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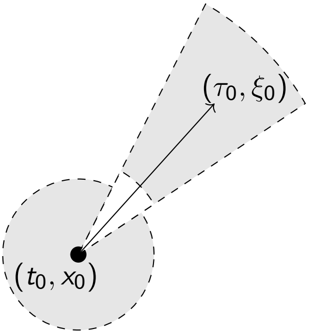


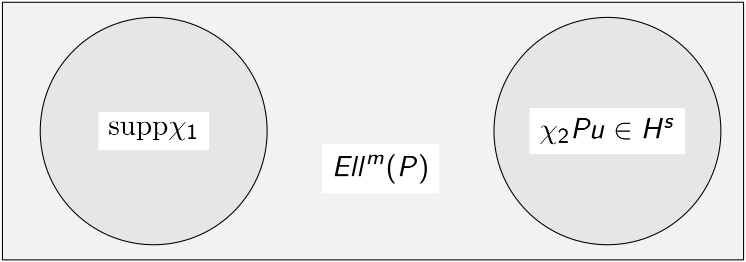

$Ell^2 = \{$points in phase space where $p$ is elliptic$\} \setminus 0$
$\Sigma^2 = Ell^m(\Box)^c \setminus 0.$
Note: For differential operators: Elliptic $\iff$ principal symbol is non-zero (outside of zero section).

# Two Major Ingredients

## Theorem (Microlocal elliptic regularity)

*Let* $P \in \Psi^m_\infty(\mathbb{R}^n)$ *and* $u \in \mathcal{S}'(\mathbb{R}^n)$. *If for some* $\chi_2 \in \Psi^0_\infty(\mathbb{R}^n)$, $\chi_2 P u \in H^s(\mathbb{R}^n)$, *then for any other* $\chi_1 \in \Psi^0_\infty(\mathbb{R}^n)$ *such that* $\mathrm{WF}'(\chi_1) \subset Ell^m(P) \cap Ell^0(\chi_2)$ *we have* $\chi_1 u \in H^{s+m}(\mathbb{R}^n)$ *and it satisfies the estimate:* $\forall N \in \mathbb{R}$, $\exists C > 0$

$$\|\chi_1 u\|_{H^{s+m}} \leqslant C \left( \|\chi_2 P u\|_{H^s} + \|u\|_{H^N} \right).$$

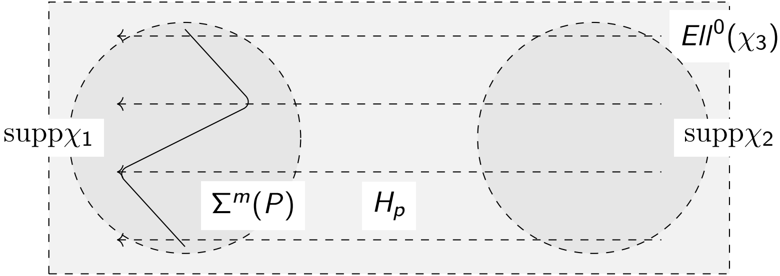


# Two Major Ingredients

## Theorem (Propagation of singularities)

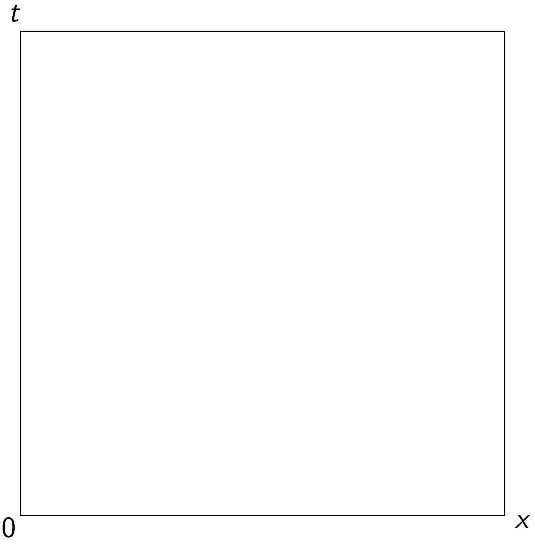
*Let $P \in \Psi^m_\infty(\mathbb{R}^n)$ is a properly supported pseudodifferential operator with polyhomogeneous principal $\sigma_m(P) = p - iq$ with real $p, q$. If we have $\chi_1, \chi_2, \chi_3 \in \Psi^0_\infty(\mathbb{R}^n)$ and $q \geqslant 0$ on $\mathrm{WF}'(\chi_3)$ and every $(x, \xi) \in WF'(P)$ is in the integral curve of $H_p$ originating from $Ell^0(\chi_2)$, then for all $s, N \in \mathbb{R}$ and $u \in C^\infty(\mathbb{R}^n)$, there exist $C > 0$ such that*

$$\|\chi_1 u\|_{H^{s+m}} \leqslant C\left(\|\chi_2 u\|_{H^s} + \|\chi_3 Pu\|_{H^{s+1}} + \|u\|_{H^N}\right).$$



# Constructions

Main idea : Create enough elliptic region! $\Box - iQ = \Box - i\chi(t)\partial_t^2$.



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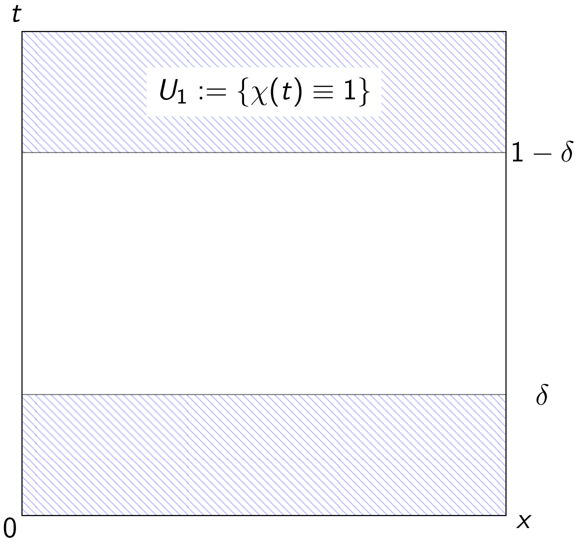

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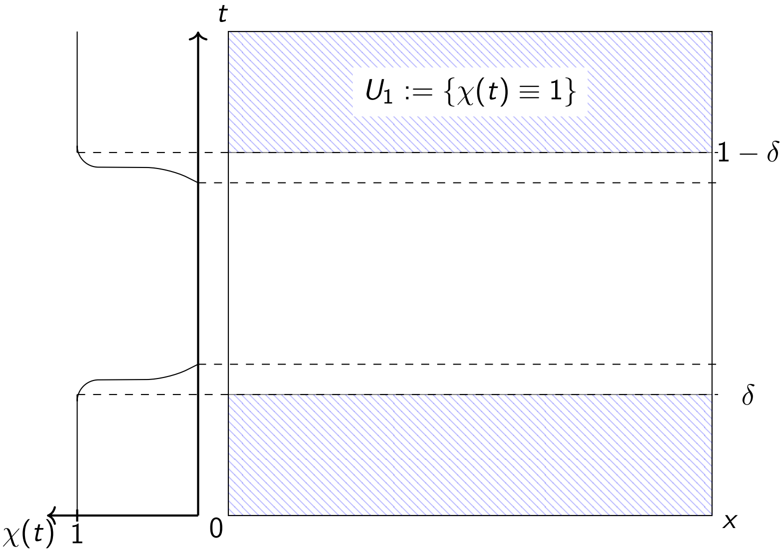

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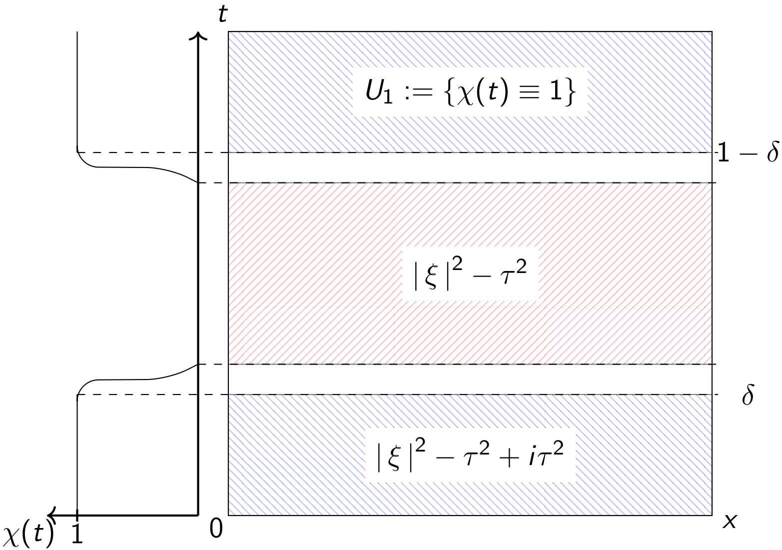
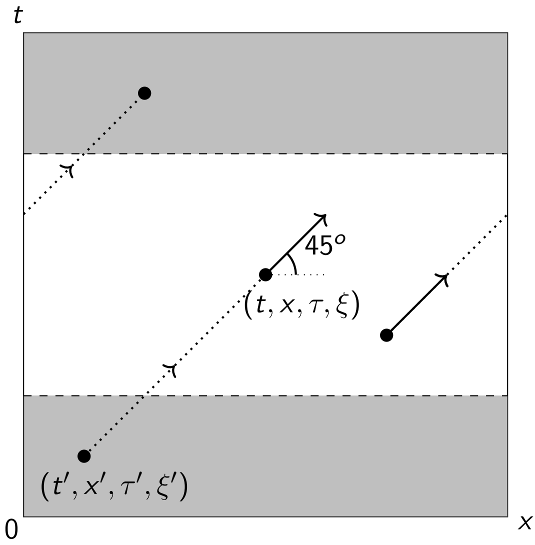


# Hamilton flow of $\sigma_2(\Box)$

Hamiltonian flow: $\exp(sH_p)(t, x, \tau, \xi) = (t + s\tau, x + s\xi, \tau, \xi)$



# Constructions

- (principal) symbol of the form $p - iq$, $p = \sigma_2(\Box)$, $q \geqslant 0$. $(\checkmark)$
- Elliptic region that propagates to hit every point in $\Sigma^2(\Box - iQ)$. $(\checkmark)$

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Propagation of singularity $\implies$

$$\left\|\chi_1 u\right\|_{H^{s+m}} \leqslant C \left\|\chi_2 u\right\|_{H^s} + C \left\|\chi_3 P u\right\|_{H^{s+1}} + \left\|u\right\|_{H^N}$$

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$$\|u\|_{H^{s+2}} \leqslant C''\,(\|(\Box - iQ)u\|_{H^{s+1}} + \|u\|_{H^N})$$

Almost there!

$$\|u\|_{H^{s+2}} \leqslant C'' \left( \|(\Box - iQ)u\|_{H^{s+1}} + \|u\|_{H^N} \right)$$

Which suggest the Hilbert space domain that we want is

$$\mathcal{X}^s = \left\{ u \in H^{s+2} \,:\, (\Box - iQ)u \in H^{s+} \right\}.$$

And

$$\Box - iQ : \mathcal{X}^s \to H^{s+1}$$

is (semi-) Fredholm for any $s \in \mathbb{R}$.

# The End

Thank you!

Questions are welcomed!