We shall follow the presentation given in [**?**].

1 Motivation for Pseudodifferential operators

- Solving PDEs via Fourier transform. For example, in Euclidean space, $\mathbb{R}^n$, constant coefficient linear PDE
$$P(D)u = \sum_{|\,\alpha\,| \leqslant n} c_\alpha D^\alpha u = f, \quad c_\alpha \in \mathbb{R}$$
where $P \in \mathbb{R}[x]$, can solved by applying Fourier transform which gives a solution of the form
$$u(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} f(y) \frac{1}{P(\xi)} \, \mathrm{d}y \, \mathrm{d}\xi$$
due to the observation that
$$\mathcal{F} P(D) u = P(\xi) \mathcal{F} u.$$
Moreover, for linear differential operators with smooth coefficients
$$P(x, D) : u \mapsto \sum_{|\,\alpha\,| \leqslant n} a_\alpha D^\alpha u, \quad a_\alpha \in C^\infty(\mathbb{R}^n)$$
we have
$$P(x, D)u = \frac{1}{(2\pi)^n} \int P(x, \xi) e^{i(x-y)\xi} u(y) \, \mathrm{d}y \, \mathrm{d}\xi.$$
We would like to generalise the above so that $P(x, \xi)$ are smooth functions satisfying certain uniform bounds, called *symbols*, instead of just polynomials in ξ. This gives us a class of operators, called pseudodifferential operators, that acts as
$$A_a u(x) = \frac{1}{(2\pi)^n} \int a(x, \xi) e^{i(x-y)\xi} u(y) \, \mathrm{d}y \, \mathrm{d}\xi$$
for each symbol a.
- There isn't enough differential operators with smooth coefficient in the sense that elliptic differential operators are not, in general, invertible in this class. For example, the operator
$$u \mapsto (\Delta + 1)u$$
has inverse that acts as (using construction via Fourier transform shown above)
$$(\Delta + 1)^{-1} f = \frac{1}{(2\pi)^n} \int \frac{1}{1 + |\,\xi\,|^2} e^{i(x-y)\xi} f(y) \, \mathrm{d}y \, \mathrm{d}\xi$$
which is a pseudodifferential operator with symbol $a(x, \xi) = (1 + |\,\xi\,|^2)^{-1}$.
- Motivation from quantum mechanics. The notion of "quantisation" in quantum mechanics can be formalised as the map that sends a symbol a (a smooth function that represent determistic observable in classical mechanics) to its corresponding pseudodifferential operator (i.e. the corresponding quantum observable)
$$A_a : \psi \mapsto \frac{1}{(2\pi)^n} \int \frac{1}{1 + |\,\xi\,|^2} e^{i(x-y)\xi} \psi(y) \, \mathrm{d}y \, \mathrm{d}\xi$$
that acts on the wavefunction ψ.
- Used in the formulation and proof of Atiyah-Singer Index theorem.

We shall define, on Euclidean space, the space of symbols, $S^m(\mathbb{R}^{2n}_{x,y}; \mathbb{R}^n_\xi)$ and the corresponding space of pseudodifferential operators, $\Psi^m(\mathbb{R}^n)$ which acts on distributions via the Schwartz kernel given by the oscilliatory integral
$$I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi) \, \mathrm{d}\xi.$$
We note that we have introduced an extra variable y which will help in explicating the properties of pseudodifferential operators. However, the extra variable does not change the essence of the theory.

2 Symbols

We shall here list the definition of the space of symbols of order $m \in \mathbb{N}$ in Euclidean space $\mathbb{R}^n$ that one encounter in the literature. The main motivation is again based on the property of linear differential operators of order $m \in \mathbb{N}$ with smooth coefficient that, after Fourier transform gives the polynomial of ξ with smooth coefficient

$$P(x,\xi) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\xi^\alpha.$$

It has the property that

$$\left| D_x^\alpha D_\xi^\beta P(x,\xi) \right| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}$$

i.e. $P(x,\xi)$ is smooth and decreases in order as $\xi \to \infty$ with successive ξ-derivative.

Definition 2.1. The space $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ of order m is the space of smooth functions $a \in C^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ such that for all multi-index $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}$$

uniformly on $\mathbb{R}^p \times \mathbb{R}^n$. We can also defined the space of symbol, $S^m_\infty(\Omega;\mathbb{R}^n)$ on a set with non-empty interior $\Omega \subset \mathbb{R}^p$, $\Omega \subset \overline{\mathrm{Int}(\Omega)}$ such that the bound above is satisfied uniformly in $(x,\xi) \in \mathrm{Int}(\Omega) \times \mathbb{R}^n$. The subscript ∞ refers the uniform boundedness condition. Together with the family of seminorm (indexed by $N \in \mathbb{N}$)

$$\|a\|_{N,m} = \sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \max_{|\alpha|+|\beta| \leqslant N} \frac{D_x^\alpha D_\xi^\beta a(x,\xi)}{\langle \xi \rangle^{m-|\beta|}}$$

gives a Frechet topology to $S^m_\infty(\Omega;\mathbb{R}^n)$.

Note: In defining pseudodifferential operators, we shall focus on the case where $p = 2n$.

Definition 2.2. A **symbol** of type $S^{m,l_1,l_2}_{\delta,\delta'}$ where $m, l_1, l_2 \in \mathbb{R}$ and $\delta, \delta' \in [0,1/2)$ is an element of $C^\infty(\mathbb{R}^n_x;\mathbb{R}^n_y;\mathbb{R}^n_\xi)$ satisfying

$$\frac{\left| D_x^\alpha D_y^\beta D_\xi^\gamma a(x,y,\xi) \right|}{\langle \xi \rangle^{m-|\gamma|} \langle x \rangle^{l_1-|\alpha|} \langle y \rangle^{l_2-|\beta|} \langle \xi \rangle^{\delta|(\alpha,\beta,\gamma)|} \langle x,y \rangle^{\delta'|(\alpha,\beta,\gamma)|}} \leqslant C_{\alpha,\beta,\gamma}$$

uniformly in $\mathbb{R}^{3n}$. Taking the supremum over $\mathbb{R}^{3n}$, we get a family of seminorms, indexed by $N \in \mathbb{N}$ defined by

$$\|a\|_{S^{m,l_1,l_2}_{\delta,\delta'},N} := \sum_{|(\alpha,\beta,\gamma)| \leqslant N} \inf C_{\alpha,\beta,\gamma}$$

which gives $S^{m,l_1,l_2}_{\delta,\delta'}$ a Frechet topology.

Definition 2.3. A (Kohn-Nirenberg) **symbol** of order $m \in \mathbb{R}$ on $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}_{x,\xi}$ is a smooth function $a = a(x,\xi)$ satisfying

$$\forall \alpha,\beta \in \mathbb{N}^n, \exists C \in \mathbb{R}_{\geqslant 0} \ : \ \left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \leqslant C \langle \xi \rangle^{m-|\beta|}$$

uniformly in x. The **space of symbol of order** m on $T^*\mathbb{R}^n$

Definition 2.4. Let $n \in \mathbb{N}$ be given. An **order function** $g \in C^\infty(\mathbb{R}^n; \mathbb{R}_{\geqslant 0})$ is a *non-negative* function satisfying

$$\forall \alpha \in N^n \exists C \in \mathbb{R}_{\geqslant 0} \ : \ \partial^\alpha g \leqslant Cg$$

uniformly on $\mathbb{R}^n$, i.e. $\partial^\alpha g = O(g)$ uniformly on $\mathbb{R}^n$.

Given an order function g, a **symbol** of order g is a smooth function $a = a(x, \xi) \in C^\infty(T^*\mathbb{R}^n)$ satisfying

$$\forall \alpha, \beta \in \mathbb{N}^n \ : \ \left| D_x^\alpha D_\xi^\beta a(x, \xi) \right| \leqslant Cg(\xi)$$

uniformly in x.

2.1 Properties of Symbols

Proposition 2.5. *Let $p, n \in \mathbb{N}$ be given and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$. If $m, m' \in \mathbb{R}$ such that $m \leqslant m'$, then $S_\infty^m(\Omega; \mathbb{R}^n) \subset S_\infty^{m'}(\mathbb{R}^)$. Furthermore, the inclusion map*

$$\iota : S_\infty^m(\Omega; \mathbb{R}^n) \to S_\infty^{m'}(\Omega; \mathbb{R}^n)$$

is continuous.

Proof. Let the real numbers $m \leqslant m'$ be given. We note that for any $\xi \in \mathbb{R}^n$

$$\langle \xi \rangle^m \leqslant 1 \cdot \langle \xi \rangle^{m'}$$

and thus if $a \in S_\infty^m(\Omega; \mathbb{R}^n)$, we have that $\forall \alpha \in \mathbb{N}^p, \forall \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x, \xi) \right| \leqslant C \langle \xi \rangle^{m - |\beta|} \leqslant C \langle \xi \rangle^{m' - |\beta|}$$

which show that $a \in S_\infty^{m'}(\Omega; \mathbb{R}^n)$ as well.

To show that ι is a continuous inclusion, it suffices to show that

$$\|\iota(a)\|_{N,m'} \leqslant C \|a\|_{N,m}$$

for any $a \in S_\infty^m(\Omega; \mathbb{R}^n)$ and $N \in \mathbb{N}$. Indeed, his bound holds since

$$\frac{D_x^\alpha D_\xi^\beta a(x, \xi)}{\langle \xi \rangle^{m' - |\beta|}} \leqslant \frac{D_x^\alpha D_\xi^\beta a(x, \xi)}{\langle \xi \rangle^{m - |\beta|}}.$$

□

This inclusion property allow us to consider $S_\infty^m(\Omega; \mathbb{R}^n)$ as the filtration of the space

$$S_\infty^\infty(\Omega; \mathbb{R}^n) = \bigcup_{m \in \mathbb{R}} S_\infty^m(\Omega; \mathbb{R}^n)$$

and we shall denote the *residual* space of the filtration as

$$S_\infty^{-\infty}(\Omega; \mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} S_\infty^m(\Omega; \mathbb{R}^n).$$

We have a rather technical result of the density of the residual space in $S_\infty^m(\Omega; \mathbb{R}^n)$.

Lemma 2.6. *Given any $m \in \mathbb{R}$ and $a \in S^m_\infty(\Omega;\mathbb{R}^n)$, there exist a sequence in $S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$ such that bounded in $S^m_\infty(\Omega;\mathbb{R}^n)$ and converges to a in the topology of $S^{m+\epsilon}_\infty(\Omega;\mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$. In other words, for any $m \in \mathbb{R}$ and $\epsilon > 0$, $S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$ is dense in $S^m_\infty(\Omega;\mathbb{R}^n)$ with the topology of $S^{m+\epsilon}_\infty(\Omega;\mathbb{R}^n)$.*

Proof. The main strategy in this proof is to approximate any symbol with the very same symbol but cut off by a compactly supported function. As such, the main reason we can't have density of $S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$ in $S^m_\infty(\Omega;\mathbb{R}^n)$ is the same reason to the fact that Schwartz functions are not dense in the space of smooth bounded functions, in particular, $1 \in S^0_\infty(\Omega;\mathbb{R}^n)$ is not in the closure of $S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$.

Now, let $a \in S^m_\infty(\Omega;\mathbb{R}^n)$ and $\epsilon \in \mathbb{R}_{>0}$ be given. Take anysmooth cut off functions supported in the unit ball, i.e. take $\phi \in C^\infty_c(\mathbb{R}^n)$ such that $0 \leqslant \phi \leqslant 1$ and $\phi(\xi) = 1$ if $|\,\xi\,| < 1$ and $\phi(\xi) = 0$ if $|\,\xi)\,| > 2$. We define for each $k \in \mathbb{N}$

$$a_k(x,\xi) = \phi\left(\frac{\xi}{k}\right) a(x,\xi)$$

and we check the following

1. $a_k \in S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$ for all $k \in \mathbb{N}$;
2. a_k are bounded in $S^m_\infty(\Omega;\mathbb{R}^n)$ for all $k \in \mathbb{N}$;
3. $a_k \to a$ as $k \to \infty$ in $S^{m+\epsilon}_\infty(\Omega;\mathbb{R}^n)$.

Given arbitrary $N, k \in \mathbb{N}$, observe that

$$|\,a_k\,| \leqslant C\,\langle\xi\rangle^{-N}$$

since a_k is compactly supported in ξ (as ϕ is compactly supported) and by Leibinz formula and symbol estimates on $a \in S^m_\infty(\Omega;\mathbb{R}^n)$

$$\left| D^\alpha_x D^\beta_\xi a_k(x,\xi) \right| \leqslant \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D^\mu\phi\right)\left(\frac{\xi}{k}\right) \left| D^\alpha_x D^{\beta-\mu}_\xi a(x,\xi) \right| \leqslant C \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D^\mu\phi\right)\left(\frac{\xi}{k}\right) \langle\xi\rangle^{m-|\,\beta-\mu\,|}.$$

Since ϕ and all its derivatives are compactly supported, each term above is bounded in ξ and thus a_k is bounded in $S^m_\infty(\Omega;\mathbb{R}^n)$ and that

$$\left| D^\alpha_x D^\beta_\xi a_k(x,\xi) \right| \leqslant C'\,\langle\xi\rangle^{-N}$$

which allow us to conclude that $a_k \in S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$.

It remains to show that $\lim_{k\to\infty} a_k = a$ in $S^{m+\epsilon}_\infty(\Omega;\mathbb{R}^n)$. In the first symbol norm, we observe that, using the symbol estimate for a

$$\begin{aligned}
\|a_k - a\|_{0,m+\epsilon} &= \sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{|\,a_k(x,\xi)\,|}{\langle\xi\rangle^{m+\epsilon}} \\
&= \sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{|\,(1-\phi(\xi/k))\,|\,|\,a(x,\xi)\,|}{\langle\xi\rangle^{m+\epsilon}} \\
&\leqslant \|a\|_{0,m} \sup_{\xi\in\mathbb{R}^n} \frac{|\,(1-\phi(\xi/k))\,|}{\langle\xi\rangle^{\epsilon}} \\
&\leqslant \|a\|_{0,m}\,\langle k\rangle^{-\epsilon} \\
&\to 0
\end{aligned}$$

as $k \to \infty$, since $|\,(1-\phi(\xi/k))\,|$ is 0 in the region $|\,\xi\,| \leqslant k$ and bounded by 1 otherwise. We remark upon the necessity of the extra decay given by $\langle\xi\rangle^{-\epsilon}$ factor. For other symbol norm we shall again use Leibinz formula to obtain

$$\begin{aligned}
\sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{\left| D_x^\alpha D_\xi^\beta a_k(x,\xi)\right|}{\langle\xi\rangle^{m+\epsilon-|\beta|}} &\leqslant \sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{1}{\langle\xi\rangle^{m+\epsilon-|\beta|}} \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} k^{-|\mu|} \left(D^\mu(1-\phi)\right)\left(\frac{\xi}{k}\right) \left| D_x^\alpha D_\xi^{\beta-\mu} a(x,\xi)\right| \\
&\leqslant \sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{C}{\langle\xi\rangle^{m+\epsilon-|\beta|}} \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} k^{-|\mu|} \left(D^\mu(1-\phi)\right)\left(\frac{\xi}{k}\right) \langle\xi\rangle^{m-|\beta-\mu|} \\
&= C \sup_{\xi\in\mathbb{R}^n} \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} k^{-|\mu|} \left(D^\mu(1-\phi)\right)\left(\frac{\xi}{k}\right) \langle\xi\rangle^{-\epsilon-|\mu|} \\
&\leqslant C' k^{-\epsilon} \\
&\to 0
\end{aligned}$$

as $k\to\infty$ by the same argument as before. Thus, we have proven that $a_k \to a$ as $k\to\infty$ in $S_\infty^{m+\epsilon}(\Omega;\mathbb{R}^n)$. □

Proposition 2.7. *Let $p, n \in \mathbb{N}$ be given. Let $\Omega\subset\mathbb{R}^p$ be such that $\Omega\subset\overline{\mathrm{Int}(\Omega)}$. Then, for any $m, m'\in\mathbb{R}$, we have*

$$S_\infty^m(\Omega;\mathbb{R}^n)\cdot S_\infty^{m'}(\Omega;\mathbb{R}^n) = S_\infty^{m+m'}(\Omega;\mathbb{R}^n)$$

Proof. Let $a\in S_\infty^m(\Omega;\mathbb{R}^n)$ and $b\in S_\infty^{m'}(\Omega;\mathbb{R}^n)$ be given. By (general) Leibinz formula, we have that for all multi-index α,β,

$$\begin{aligned}
\sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{\left| D_x^\alpha D_\xi^\beta a(x,\xi) b(x,\xi)\right|}{\langle\xi\rangle^{(m+m')-|\beta|}} &\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} \sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{\left| D_x^\mu D_\xi^\gamma a(x,\xi)\right| \left| D_x^{\alpha-\mu} D_\xi^{\beta-\gamma} b(x,\xi)\right|}{\langle\xi\rangle^{(m+m')-|\beta|}} \\
&\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \sup_{\xi\in\mathbb{R}^n} \frac{\langle\xi\rangle^{m-|\gamma|}\langle\xi\rangle^{m'-|\beta-\gamma|}}{\langle\xi\rangle^{(m+m')-|\beta|}} \\
&= \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{|\beta|-(|\beta-\gamma|+|\gamma|)} \\
&\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \\
&< \infty
\end{aligned}$$

where we have use the property of multi-index that $|\beta| = |\beta-\mu| + |\mu|$. We have thus shown that $S_\infty^m(\Omega;\mathbb{R}^n)\cdot S_\infty^{m'}(\Omega;\mathbb{R}^n)\subset S_\infty^{m+m'}(\Omega;\mathbb{R}^n)$

For the reverse inclusion, let $c\in S_\infty^{m+m'}(\Omega;\mathbb{R}^n)$ be given. Define

$$\begin{aligned}
a &: (x,\xi)\mapsto \langle\xi\rangle^m \\
b &: (x,\xi)\mapsto \frac{c(x,\xi)}{a(x,\xi))}
\end{aligned}$$

and observe that

- $a\in S_\infty^m(\Omega;\mathbb{R}^n)$. It is clear that a is smooth in both x and ξ. It is independent of x and thus any x derivative gives 0. We need only to check that for all $\beta\in\mathbb{N}^n$,

  $$\left| D_\xi^\beta \langle\xi\rangle^m \right| \leqslant C\langle\xi\rangle^{m-|\beta|}$$

  which can be proven by induction on n and β. We shall only prove the base case where $n=1$ and $\beta=1$. We have

  $$|D_\xi\langle\xi\rangle^m| = \left|\partial_\xi(1+\xi^2)^{m/2}\right| = \left| m\xi\langle\xi\rangle^{m-2}\right| = \left| m\frac{\xi}{\langle\xi\rangle}\right| \langle\xi\rangle^{m-1} \leqslant |m|\langle\xi\rangle^{m-1}$$

  where we have used the fact that $|\xi|\leqslant\langle\xi\rangle$ for all ξ.

- $b \in S^{m'}_{\infty}(\Omega;\mathbb{R}^n)$. We note first that $\langle \xi \rangle^m \neq 0$ for all $\xi \in \mathbb{R}^n$ and thus b is well-defined. Since division by $\langle \xi \rangle^m$ does not affect any of the x derivative, we only need to show that for any $\beta \in \mathbb{N}^n$, we have

$$\left| D^{\beta}_{\xi} b(x,\xi) \right| \leqslant C \langle \xi \rangle^{m+m'-|\beta|}$$

for some constant $C > 0$ uniformly in ξ. Indeed, observe that by the Leibinz formula

$$\begin{aligned}
\left| D^{\beta}_{\xi} b(x,\xi) \right| &\leqslant \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \left| D^{\mu}_{\xi} c(x,\xi) \right| \left| D^{\beta-\mu} \langle \xi \rangle^{-m} \right| \\
&\leqslant C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m+m'-|\mu|} \langle \xi \rangle^{-m-|\beta-\mu|} \\
&\leqslant C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m'-(|\mu|+|\beta-\mu|)} \\
&= C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} \langle \xi \rangle^{m'-|\beta|} \\
&= C 2^{\beta} \langle \xi \rangle^{m'-|\beta|}
\end{aligned}$$

where we have use the definition of c and applied the result proven for a with $m \mapsto -m$. Thus, $b \in S^{m'}_{\infty}(\Omega;\mathbb{R}^n)$.

It is clear that $a \cdot b = c$ and we have therefore shown that $S^{m+m'}_{\infty}(\Omega;\mathbb{R}^n) \subset S^{m}_{\infty}(\Omega;\mathbb{R}^n) \cdot S^{m'}_{\infty}(\Omega;\mathbb{R}^n)$.

□

A sumarising theorem:

Theorem 2.8. *Given $p, n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$. Let*

$$S^{\infty}_{\infty}(\Omega;\mathbb{R}^n) = \bigcup_{m \in \mathbb{R}} S^{m}_{\infty}(\Omega;\mathbb{R}^n).$$

Then $S^{\infty}_{\infty}(\Omega;\mathbb{R}^n)$ is a graded algebra over $\mathbb{R}$ with continuous inclusion $S^{m}_{\infty}(\Omega;\mathbb{R}^n) \to S^{m'}_{\infty}(\Omega;\mathbb{R}^n)$ for all $m \leqslant m'$.

2.2 Ellipticity of symbols

Definition 2.9. Given $p, n \in \mathbb{N}$, $m \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$, an order m symbol $a \in S^{m}_{\infty}(\Omega;\mathbb{R}^n)$ is (globally) **elliptic** if there exist $\epsilon \in \mathbb{R}_{>0}$ such that

$$\inf_{|\xi| \geqslant 1/\epsilon} |a(x,\xi)| \geqslant \epsilon \langle \xi \rangle^m .$$

The importance of elliptic symbol is that they are invertible modulo $S^{-\infty}_{\infty}(\Omega;\mathbb{R}^n)$.

Lemma 2.10. *Given $p, n \in \mathbb{N}$, $m \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$. Let $a \in S^{m}_{\infty}(\Omega;\mathbb{R}^n)$ be an elliptic symbol of order m. Then there exist a symbol $b \in S^{-m}_{\infty}(\Omega;\mathbb{R}^n)$ such that*

$$a \cdot b - 1 \in S^{-\infty}_{\infty}(\Omega;\mathbb{R}^n).$$

Proof. We shall follow the general strategy of inverting the symbol outside of a compact set. Let $\phi \in C^{\infty}_c(\mathbb{R}^n)$ be a smooth cut off function, i.e $0 \leqslant \phi \leqslant 1$ and $\phi(\xi) = 1$ for $|\xi| < 1$ and $\phi(\xi) = 0$ for $|\xi| > 2$.

Let $a \in S^{m}_{\infty}(\Omega;\mathbb{R}^n)$ be an elliptic symbol, that is, for any fixed $\epsilon \in \mathbb{R}_{>0}$, we have

$$|a(x,\xi)| \geqslant \epsilon \langle \xi \rangle^m$$

for any $|\,\xi\,| \geqslant 1/\epsilon$. Thus, we can define

$$b(x,\xi) = \begin{cases} \frac{1-\phi(\epsilon\xi/2)}{a(x,\xi)} & |\,\xi\,| \geqslant 1/\epsilon \\ 0 & |\,\xi\,| < 1/\epsilon. \end{cases}$$

We check:

b is well-defined and smooth.

We note that $|\,a(x,\xi)\,| > 0$ whenever $|\,\xi\,| \geqslant 1/\epsilon$ and therefore b is well defined in that region. For smoothness, we note first that b is smooth in the regions $|\,\xi\,| > 1/\epsilon$ and $|\,\xi\,| < 1/\epsilon$. Set $\delta = 1/(2\epsilon)$. In the region where $1/\epsilon - \delta < |\,\xi\,| < 1/\epsilon + \delta$, we have $|\,\epsilon\xi/2\,| < 1/\epsilon$ and therefore $b(x,\xi) \equiv 0$ in this region and is thus smooth. Since the we have covered $\Omega \times \mathbb{R}^n$ by the three chart domain above, b is smooth by the (smooth) gluing lemma.

b is a symbol of order $-m$.

We can prove by induction that in the region $|\,\xi\,| \geqslant 1/\epsilon$

$$D_x^\alpha D_\xi^\beta b = a^{-1-|\,\alpha\,|-|\,\beta\,|} G_{\alpha\beta}$$

for all multi-index α, β, where $G_{\alpha\beta}$ is a symbol of order $(|\,\alpha\,| + |\,\beta\,|)m - |\,\beta\,|$. Therefore, using the ellipticity estimate for a, we get

$$\begin{aligned} \|b\|_{k,-m} &= \sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{\left| D_x^\alpha D_\xi^\beta b(x,\xi) \right|}{\langle\xi\rangle^{-m-k}} \\ &= \sup_{|\,\xi\,|\geqslant 1/\epsilon} \left| a^{-1-|\,\alpha\,|-|\,\beta\,|} G_{\alpha\beta} \right| \langle\xi\rangle^{m+k} \\ &\leqslant \frac{\|G_{\alpha\beta}\|_{0,(|\,\alpha\,|+|\,\beta\,|)m-|\,\beta\,|}}{\epsilon} \sup_{|\,\xi\,|\geqslant 1/\epsilon^{1+|\,\alpha\,|+|\,\beta\,|}} \langle\xi\rangle^{-m(1+|\,\alpha\,|+|\,\beta\,|)} \langle\xi\rangle^{m+k} \\ &< \infty \end{aligned}$$

as required.

b is an inverse of a modulo $S_\infty^{-\infty}(\Omega;\mathbb{R}^n)$.

The main observation is that the set where b fails to be the multiplicative inverse of a is a compact set (in ξ) and thus $a \cdot b - 1$ is in fact a compactly supported smooth function of ξ which is a subset of $S_\infty^{-\infty}(\Omega;\mathbb{R}^n)$.

Explicitly, for any $N \in \mathbb{N}$

$$\sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{\left| D_x^\alpha D_\xi^\beta (a\cdot b - 1) \right|}{\langle\xi\rangle^{-N}} \leqslant \sup_{|\,\xi\,|\leqslant 1/\epsilon} \langle\xi\rangle^N \left| D_x^\alpha D_\xi^\beta (\phi(\xi\epsilon/2)) \right| < \infty.$$

□

3 Pseudodifferential Operators (ΨDO's)

As mentioned in section **??**, we wanted to generalise the action of differential operators

$$P(x,D)u = \frac{1}{(2\pi)^n} \int P(x,\xi) e^{i(x-y)\xi} u(y)\,\mathrm{d}y\,\mathrm{d}\xi$$

where P is an m^{th} order polynomial in ξ with C^∞ coefficient, to the actions of symbols $a \in S_\infty^m(\mathbb{R}^n;\mathbb{R}^n)$

$$A_a u = \frac{1}{(2\pi)^n} \int a(x,\xi) e^{i(x-y)\xi} u(y)\,\mathrm{d}y\,\mathrm{d}\xi$$

or $a \in S_\infty^m(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n)$ with action

$$A_a u = \frac{1}{(2\pi)^n} \int a(x,y,\xi) e^{i(x-y)\xi} u(y)\,\mathrm{d}y\,\mathrm{d}\xi.$$

One of the result we will prove is that action of $a(x, y, \xi)$ as in the later case can always be reduced to the action of some other $a(x, \xi)$ as in the former case.

Here we shall introduce a slightly more general symbol space, $\langle x-y\rangle^w S^m_\infty(\Omega; \mathbb{R}^n)$, to allow for polynomial growth perpendicular to the diagonal.

Definition 3.1. Given $m, w \in \mathbb{R}$, a **w-weighted symbol space of order** m, $\langle x-y\rangle^w S^m_\infty(\mathbb{R}^{2n}_{x,y}, \mathbb{R}^n)$ is given by

$$a \in \langle x-y\rangle^w S^m_\infty(\mathbb{R}^{2n}_{x,y}, \mathbb{R}^n) \iff a(x,y,\xi) = \langle x-y\rangle^w \tilde{a}(x,y,\xi),\ \tilde{a} \in S^m_\infty(\Omega; \mathbb{R}^n)$$

or equivalently, $a \in \langle x-y\rangle^w S^m_\infty(\mathbb{R}^{2n}_{x,y}, \mathbb{R}^n)$ if and only if for all multi-index α, β, γ,

$$\left| D^\alpha_x D^\beta_y D^\gamma_\xi a(x,y,\xi) \right| \leqslant C \langle x-y\rangle^w \langle \xi\rangle^{m-|\gamma|}.$$

We shall show that the elements $a \in \langle x-y\rangle^w S^m_\infty(\mathbb{R}^{2n}_{x,y}, \mathbb{R}^n)$ acts on $S(\mathbb{R}^n)$ via the Schwartz kernel

$$I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi)\,\mathrm{d}\xi.$$

Proposition 3.2. *Let $n \in \mathbb{N}$ and $m, w \in \mathbb{R}$ with $m < -n$, then the map*

$$I : \langle x-y\rangle^w S^m_\infty(\mathbb{R}^{2n}_{x,y}, \mathbb{R}^n) \to (1 + |x|^2 + |y|^2) C^0_\infty(\mathbb{R}^{2n})$$
$$a \mapsto I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi)\,\mathrm{d}\xi$$

extends by continuity to

$$I : \langle x-y\rangle^w S^m_\infty(\mathbb{R}^{2n}_{x,y}, \mathbb{R}^n) \to S'(\mathbb{R}^{2n})$$

in the topology of $S^{m+\epsilon}_\infty(\Omega; \mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$.

Proof. □

4 Microlocalisation

Roughly, the support of a distribution in $\mathbb{R}^n$ consist of points $x \in \mathbb{R}^n$ where the distribution is non-zero after any smooth cut-offs near x.

Definition 4.1. The **support of a tempered distribution** $u \in S'(\mathbb{R}^n)$ is given by the set

$$\operatorname{supp}(u) = \{x \in \mathbb{R}^n \mid \exists \phi \in S(\mathbb{R}^n), \phi(x) \neq 0, \phi u = 0\}^c.$$

And the singular support of a distribution is the set of points where the distribution fails to behave like an element of $S(\mathbb{R}^n)$.

Definition 4.2. The **singular support of a tempered distribution** $u \in S'(\mathbb{R}^n)$ is given by the set

$$\operatorname{singsupp}(u) = \{x \in \mathbb{R}^n \mid \exists \phi \in S(R^n), \phi(x) \neq 0, \phi(u) \in S(\mathbb{R}^n)\}^c$$

Note that if the singular support of a tempered distribution is empty, it is then an element of $C^\infty(\mathbb{R}^n)$ The support of an operator is given by the support of its Schwartz kernel.

Definition 4.3. The **support of a continuous linear operator** $A : S(R^n) \to S'(\mathbb{R}^n)$ is given by

$$\operatorname{supp}(A) = \operatorname{supp}(K_A) \subset \mathbb{R}^n \times \mathbb{R}^n$$

where $K_A \in S'(\mathbb{R}^n \times \mathbb{R}^n)$ is the Schwartz kernel of A.

We note from the above that supports or singular supports are complement of open sets, therefore they are closed. We have the following result relating the support of a smooth function after the action of a continuous linear operator.

Proposition 4.4 (Calculus of support)**.** *Let $A : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ be a continuous linear operator and $\phi \in C_c^\infty(\mathbb{R}^n)$, then*

$$\operatorname{supp}(A\phi) \subset \operatorname{supp}(A) \circ \operatorname{supp}(\phi) := \{x \in \mathbb{R}^n \mid \exists y \in \operatorname{supp}(\phi), (x, y) \in \operatorname{supp}(A)\}.$$

Proof. We shall show the contrapositive statement:

$$x \notin \operatorname{supp}(A) \circ \operatorname{supp}(\phi) \implies x \notin \operatorname{supp}(A\phi).$$

Suppose $x \notin \operatorname{supp}(A) \circ \operatorname{supp}(\phi)$. Observe that

$$\operatorname{supp}(A) \circ \operatorname{supp}(\phi) = \pi_x(\pi_y^{-1}(\operatorname{supp}(\phi)) \cap \operatorname{supp}(A))$$

where $\pi_{x,y} : \mathbb{R}^2 \to \mathbb{R}$ are the projection map to the respective coordinates. Since $\operatorname{supp}(A)$ is closed and $\operatorname{supp}(\phi)$ is compact, we have that $\operatorname{supp}(A) \circ \operatorname{supp}(\phi)$ is closed and thus x belongs to an open set. We can therefore choose a smooth cutt-off function $\chi \in C_c^\infty(\mathbb{R}^n)$ supported at x ($\chi(x) \neq 0$) but away from $\operatorname{supp}(A) \circ \operatorname{supp}(\phi)$. Thus,

$$\operatorname{supp}(A) \cap (\operatorname{supp}(\chi) \times \operatorname{supp}(\phi)) = \emptyset$$

and hence $\chi(x)K_A(x, y)\phi(y) = 0 \implies \chi A\phi = 0$, as required. ◻

4.1 Pseudolocality

We shall show two results on the singular support of pseudodifferential operators. The first result is that the singular support of any ΦDO is contained within the diagonal, i.e. they are smooth away from $x = y$. The second result is teh pseudolocality result that says that action ΨDO's do not increase singular support of distributions.

Proposition 4.5. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ for some $m \in \mathbb{R}$, then*

$$\operatorname{singsupp}(A) \subset \left\{(x, y) \in \mathbb{R}^{2n} \mid x = y\right\}.$$

Proof. We shall prove this theorem for elements of $\Psi^{-\infty}_\infty(\mathbb{R}^n)$ and then extend by continuity to all orders. Let $A \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$ with symbol $a \in S^{-\infty}_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$. Its singular support is given by the singular support of the kernel. Since all derivatives of a are $O(\langle\xi\rangle^{-\infty})$, the oscillatory integral below representing the kernel is absolutely convergent and, using integration by parts, we get

$$\begin{aligned}
I(a) &= \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \xi)\, \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int \frac{1}{(x-y)^\alpha} (-D^\alpha_\xi) \left(e^{i(x-y)\xi}\right) a(x, y, \xi)\, \mathrm{d}\xi \\
&= \frac{1}{(x-y)^\alpha} \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} (-D^\alpha_\xi) a(x, y, \xi)\, \mathrm{d}\xi \\
&= \frac{1}{(x-y)^\alpha} I((-D^\alpha_\xi)a)
\end{aligned}$$

which is true for all multi-index α of any order. Since all x, y-derivatives of a are uniformly bounded by $\langle\xi\rangle^{-N}$ for any $N \in \mathbb{N}$, we can differentiate under the integral sign to get the equation

$$\begin{aligned} D_x^\beta D_y^\gamma (x-y)^\alpha I(a) &= \frac{1}{(2\pi)^n} \int D_x^\beta D_y^\gamma e^{i(x-y)\xi} (-D_\xi^\alpha) a(x,y,\xi)\, \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n} \int (-1)^{|\,\beta\,|} \xi^{\beta+\gamma} e^{i(x-y)\xi} (-D_\xi^\alpha) a(x,y,\xi)\, \mathrm{d}\xi \end{aligned}$$

where the last integral gives a smooth function, thus showing that $(x-y)^\alpha I(a)$ is smooth for all α, and hence $I(a)$ is smooth away from $x = y$.

Now, for a general $A \in \Psi^m_\infty(\mathbb{R}^n)$, $m \in \mathbb{R}$, we shall use the density of $S^{-\infty}_\infty(\mathbb{R}^{2n}; \mathbb{R}^n) \subset S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$ and that I extends by continuity to a map $I : S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n) \to S'(\mathbb{R}^{2n})$ in the topology $S^{m+\epsilon}_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$ for any $\epsilon > 0$ **??**.

□

Proposition 4.6. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ for some $m \in \mathbb{R}$ and $u \in C^{-\infty}(\mathbb{R}^n)$, then*

$$\operatorname{singsupp}(Au) \subset \operatorname{singsupp}(u).$$

We call operators that satisfies the above property pseudolocal

Proof. Again we shall prove the contrapositive statement that

$$x \notin \operatorname{singsupp}(u) \implies x \notin \operatorname{singsupp}(Au)$$

Let $u \in S'(\mathbb{R}^n)$ be compactly supported and $x_0 \notin \operatorname{singsupp}(u)$. We can choose $\chi \in S(\mathbb{R}^n)$, (normalised) so that $\chi \equiv 1$ in a neighbourhood of x_0 and that $\chi u \in S(\mathbb{R}^n)$. Observe that

$$Au = A(\chi u + (1-\chi)u) = A(\chi u) + A(1-\chi)u.$$

Since $\chi x u \in S(\mathbb{R}^n) \implies A\chi u \in S(\mathbb{R}^n)$ [**?**], we have that

$$\operatorname{singsupp}(Au) = \operatorname{singsupp}(A(1-\chi)u).$$

Furthermore, we know that $x_0 \notin \operatorname{supp}((1-\chi)u)$. Now, we shall further cut-off near x_0 by choosing a $\phi \in S(\mathbb{R}^n)$ compactly supported away from $\operatorname{supp}(1-\chi)$ and $\phi \equiv 1$ near x_0, i.e.

$$\operatorname{supp}(1-\chi) \cap \operatorname{supp}\phi = \emptyset.$$

We now have an operator $\phi A(1-\chi)$ with kernel

$$\phi(x) K_A(x,y)(1-\phi(y))$$

that vanishes (in particular smooth) on the diagonal. Since we have shown that the singular support of a pseudodifferential operator have to be contained in the diagonal, we conclude that $\phi A(1-\chi)$ is a smoothing operator, and thus $\phi A(1-\chi)u \in C^\infty(\mathbb{R}^n)$ as required. . □

4.2 Elliptic, Characteristic, Wavefront sets

We will now define *ellipticity at a point* in phase space which allow up to define various microlocal contructions that focus on localised (conically in phase space) behaviour ΨDO's and distributions.

Definition 4.7. A pseudodifferential operator, $A \in \Psi^m_\infty(\mathbb{R}^n), m \in \mathbb{R}$ is **elliptic at a point** $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ if there exist $\epsilon > 0$ such that its left-reduced symbol satisfies the lower bound

$$|\,\sigma_L(A)(x,\xi)\,| \geqslant \epsilon \,\langle\xi\rangle^m$$

in the region

$$\overline{U}_\epsilon = \left\{ (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \;\middle|\; |\,x - x_0\,| \leqslant \epsilon, \left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \epsilon, |\,\xi\,| \geqslant 1/\epsilon \right\}$$

where $\widehat{\xi} = \xi / |\,\xi\,|$ for any non-zero $\xi \in \mathbb{R}^n$. We denote the set of all elliptic points of A as

$$Ell^m(A) = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid A \text{ is elliptic of order } m \text{ at } (x,\xi)\}$$

and its complement in $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ as

$$\begin{aligned}\Sigma^m(A) &= Ell^m(A)^c \setminus \{(x,0) \mid x \in \mathbb{R}^n\} \\ &= \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid A \text{ is } \textbf{not} \text{ elliptic of order } m \text{ at } (x,\xi)\}\end{aligned}$$

Lemma 4.8. *Let $A \in \Psi^m_\infty(\mathbb{R}^n), m \in \mathbb{R}$.*

1. *If $\sigma_m(A)(x,\xi)$ is homogeneous of degree m in ξ, then*
$$Ell^m(A) = \{(x_0,\xi_0) \mid \xi_0 \neq 0, \sigma_m(A)(x_0,\xi_0) \neq 0\}\,.$$
2. *$Ell^m(A)$ is open in $\mathbb{R}^n \times \mathbb{R}^n$.*
3. *$Ell^m(A)$ is conic in $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$, in the sense that*
$$(x_0,\xi_0) \in Ell^m(A) \implies (x_0, t\xi_0) \in Ell^m(A), \forall t \in \mathbb{R}_{>0}.$$
4. *$\Sigma^m(A)$ is closed conic.*
5. *if $B \in \Psi^{m'}(\mathbb{R}^n)$, then*
$$\Sigma^{m+m'}(A \circ B) = \Sigma^m(A) \cup \Sigma^{m'}(B).$$

Proof. Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $m \in \mathbb{R}$ be given.

1. Suppose the principal symbol $\sigma_m(A)(x,\xi)$ is homogeneous of order m in ξ. We need to show that
$$(x_0,\xi_0) \in Ell^m(A) \iff \xi_0 \neq 0,\ \sigma_m(A)(x_0,\xi_0) \neq 0.$$
If $\xi_0 = 0$, $(x_0,\xi_0) \notin Ell^m_\infty$ by definition of ellipticity. If $\sigma_m(x_0,\xi_0) = 0$, by homogeneity, we have
$$\sigma_m(x_0, t\xi_0) = t^m \sigma_m(x_0,\xi_0) = t^m \cdot 0 = 0$$
for all $t \in \mathbb{R}_{>0}$. By definition of principal symbol, we can write the left symbol of A as
$$\sigma_L(A) = \sigma_m(A) + a$$
where $a \in S^{m-1}_\infty(\mathbb{R}^n;\mathbb{R}^n)$. Now, observe that for any $\epsilon > 0$, the set
$$\overline{U}_\epsilon = \left\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid |\,x - x_0\,| \leqslant \epsilon, \left|\widehat{\xi} - \widehat{\xi_0}\right| \leqslant \epsilon, |\,\xi\,| \geqslant 1/\epsilon\right\}$$
contains the(open) half-line starting at $\widehat{\xi_0}/\epsilon$, i.e. the set $\{(x_0, t\xi_0/(|\,\xi_0\,|\,\epsilon) \mid t > 0\}$. However, by the symbol estimate of a,
$$\begin{aligned}\left|\sigma_L(A)\left(x_0, \frac{t\xi_0}{|\,\xi_0\,|\,\epsilon}\right)\right| &\leqslant \left(\frac{t}{\epsilon\,|\,\xi_0\,|}\right)^m |\,\sigma_m(x_0,\xi_0)\,| + \left|a\left(x_0, \frac{t\xi_0}{|\,\xi_0\,|\,\epsilon}\right)\right| \\ &= 0 + \left|a\left(x_0, \frac{t\xi_0}{|\,\xi_0\,|\,\epsilon}\right)\right| \\ &\leqslant C\left\langle \frac{t\xi_0}{|\,\xi_0\,|\,\epsilon}\right\rangle^{m-1} \\ &= C\,\langle t/\epsilon\rangle^{m-1}\end{aligned}$$

and therefore

$$\begin{aligned}\inf_{(x,\xi)\in\overline{U}_\epsilon} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} &\leqslant \inf_{t>0} \frac{\left|\,\sigma_L(A)\left(x_0, \frac{t\xi_0}{|\,\xi_0\,|\epsilon}\right)\,\right|}{\langle t/\epsilon\rangle^m}\\ &\leqslant \inf_{t>0} \frac{C\,\langle t/\epsilon\rangle^{m-1}}{\langle t/\epsilon\rangle^m}\\ &= C\inf_{t>0}\langle t/\epsilon\rangle^{-1}\\ &= 0\end{aligned}$$

which means that $(x_0,\xi_0)\notin \mathit{Ell}^m(A)$.

Conversely, if $\sigma_m(A)(x_0,\xi_0)\neq 0$, by continuity and homogeneity, $\sigma_m(A)$, is non-zero in a (closed) conic neighbourhood, i.e. there exist $\epsilon>0$ such that $\sigma_m(A)\neq 0$ in

$$\overline{U}_\epsilon = \left\{(x,\xi)\,|\,|\,x-x_0\,|\leqslant\epsilon, \left|\,\widehat{\xi}-\widehat{\xi_0}\,\right|\leqslant\epsilon, |\,\xi\,|\geqslant 1/\epsilon\right\}.$$

Again, writing the left symbol as a sum of the principal symbol an a lower order term, we observe that in $\overline{U}_\epsilon$,

$$\begin{aligned}\frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} &\geqslant \frac{|\,|\,\sigma_m(A)(x,\xi)\,|-|\,a(x,\xi)\,|\,|}{\langle\xi\rangle^m}\\ &= \left|\,\frac{|\,\xi\,|^m}{\langle\xi\rangle^m}\left|\,\sigma_m(A)(x,\widehat{\xi})\,\right| - \frac{|\,a(x,\xi)\,|}{\langle\xi\rangle^m}\,\right|\end{aligned}$$

By the symbol estimate of a, the second term is tending to 0 which the first term is bounded below by $C=\inf_{(x,\xi)\in\overline{U}_\epsilon}|\,\sigma_m(A)(x,\xi)\,|>0$. Therefore, choosing a smaller ϵ if necessary, we have $|\,a(x,\xi)\,|\,/\,\langle\xi\rangle^m<C$ and thus

$$\inf_{(x,\xi)\in\overline{U}_\epsilon}\frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m}\geqslant C'\geqslant\epsilon.$$

and therefore $(x_0,\xi_0)\in \mathit{Ell}^m(A)$.

2. We note first that if the principal symbol is homogeneous of degree m, the previous result applies and continuity of the principal symbol guarantee that the elliptic set is open (if $\sigma_m(A)$ is non-zero at a point, it is non-zero in an open neighbourhood of the point).

   For the general case, suppose $(x_0,\xi_0)\in \mathit{Ell}^m(A)$. We therefore have for some $\epsilon>0$,

$$|\,\sigma_L(A)(x,\xi)\,|\geqslant\epsilon\,\langle\xi\rangle^m$$

   in the region

$$\overline{U}_\epsilon(x_0,\xi_0) = \left\{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}\,\,|\,\,|\,x-x_0\,|\leqslant\epsilon, \left|\,\widehat{\xi}-\widehat{\xi_0}\,\right|\leqslant\epsilon, |\,\xi\,|\geqslant 1/\epsilon\right\}.$$

   It suffice to show that there is an open neighbourhood of (x_0,ξ_0) where A remains elliptic. We can take desired open neighbourhood to be

$$V=\left\{(x',\xi')\,|\,\xi'\neq 0,\,|\,x'-x_0\,|<\epsilon/2, \left|\,\widehat{\xi'}-\widehat{\xi_0}\,\right|<\epsilon/2\right\}.$$

   Then, we can check that for every $(x',\xi')\in V$, A satisfies the elliptic estimate in $\overline{U}_{\epsilon/2}(x',\xi')$. Indeed, if $(x,\xi)\in\overline{U}_{\epsilon/2}(x',\xi')$, then

$$\begin{aligned}&|\,x-x_0\,|\leqslant|\,x-x'\,|+|\,x'-x_0\,|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon\\ &\left|\,\widehat{\xi}-\widehat{\xi_0}\,\right|\leqslant\left|\,\widehat{\xi}-\widehat{\xi'}\,\right|+\left|\,\widehat{\xi'}-\widehat{\xi_0}\,\right|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon\\ &|\,\xi\,|\geqslant 2/\epsilon\geqslant 1/\epsilon\end{aligned}$$

which shows that $\overline{U}_{\epsilon/2}(x', \xi') \subset \overline{U}_{\epsilon}(x_0, \xi_0)$. Therefore,

$$\inf_{(x,\xi)\in \overline{U}_{\epsilon/2}(x',\xi')} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle \xi \rangle^m} \geqslant \inf_{(x,\xi)\in \overline{U}_{\epsilon}(x_0,\xi_0)} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle \xi \rangle^m} \geqslant \epsilon \geqslant \epsilon/2$$

as required.

3. Again, this result is immediate if the principal symbol is homogeneous in ξ. In general, this result come from the observation that only $\widehat{\xi} = \xi/\,|\,\xi\,|$ appears in $\overline{U}_\epsilon$ in the definition of $Ell^m(A)$, i.e. only the *direction* in the dual variable is important.

   Explicitly, let $(x_0, \xi_0) \in Ell^m(A)$ and $t \in \mathbb{R}_{>0}$. Clearly $t\xi_0 \neq 0$. And note that

   $$\overline{U}_\epsilon(x_0, \xi_0) = \overline{U}_\epsilon(x_0, t\xi_0)$$

   since $\widehat{\xi} = \widehat{t\xi}$.

4. $\Sigma^m(A) = Ell^m(A)^c$ where $Ell^m(A)$ is open and conic. Since complement of conic set is conic, and complement of open is closed, we conclude that $\Sigma^m(A)$ is closed conic.

5. If both principal symbols are homoegenous of degree m, m' respectively, we can applied the result above and by symbol calculus, we have

   $$\begin{aligned} Ell^{m+m'}(A \circ B) &= \{(x,\xi)\,|\,\xi \neq 0, \sigma_{m+m'}(A \circ B) = \sigma_m(A)\sigma_{m'}(B) \neq 0\} \\ &= \{(x,\xi)\,|\,\xi \neq 0, \sigma_m(A) \neq 0\} \cap \{(x,\xi)\,|\,\xi \neq 0, \sigma_{m'}(B) \neq 0\} \\ &= Ell^m(A) \cap Ell^{m'}(B). \end{aligned}$$

   Taking complement give the desired result.

   In general,

$\square$

Definition 4.9. The **wavefront set** of a compactly supported tempered distribution

$$u \in C_c^{-\infty}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n)\,|\,\mathrm{supp}(u) \Subset \mathbb{R}^n\}$$

is given by

$$\mathrm{WF}(u) = \bigcap \left\{\Sigma^0(A)\,|\,A \in \Psi^0_\infty(\mathbb{R}^n), Au \in C^\infty(\mathbb{R}^n)\right\}.$$

For general tempered distribution $u \in S'(\mathbb{R}^n)$, its wavefront set is given by

$$\mathrm{WF}(u) = \bigcup_{\chi \in C_c^\infty(\mathbb{R}^n)} \mathrm{WF}(\chi u).$$

Proposition 4.10. *For compactly supported tempered distribution, $u \in C_c^{-\infty}(\mathbb{R}^n)$,*

$$\pi(\mathrm{WF}(u)) = \mathrm{singsupp}(u).$$

where $\pi(x, y) = x$ is the projection map.

Proof. To show $\pi(\mathrm{WF}(u)) \subset \mathrm{singsupp}(u)$, we observe that, by definition of singular support,

$$x_0 \notin \mathrm{singsupp}(u) \implies \exists \phi \in S(\mathbb{R}^n),\ \phi(x_0) \neq 0,\ \phi u \in S(R^n).$$

But since multiplication by ϕ gives an operator in $\Psi^0_\infty(\mathbb{R}^n)$ which is elliptic at (x_0, ξ) for any $\xi \neq 0$ (ϕ is its own principal symbol which happens to be homogeneous and non-zero for any $(x_0, \xi), \xi \neq 0$).

Therefore, $x_0 \notin \pi(\mathrm{WF}(u))$.

Conversely, if $x_0 \notin \pi(\mathrm{WF}(u))$, then for all $\xi \neq 0$, there exist $A_\xi \in \Psi^0_\infty(\mathbb{R}^n)$ such that A_ξ is elliptic at (x_0, ξ) and $A_\xi u \in C^\infty(\mathbb{R}^n)$. Since elliptic set $Ell^0(A_\xi)$ is open and conic, we know that there exist $\epsilon = \epsilon(\xi)$ such that A_ξ is elliptic in the open conic set

$$V_\xi = \left\{(x', \xi') \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \mid |\, x' - x_0 \,| < \epsilon, \left|\widehat{\xi'} - \widehat{\xi}\right| < \epsilon\right\}.$$

Compactness of the sphere (note that $\xi' \mapsto \widehat{\xi'}$ is an embedding of $\mathbb{R}^n \setminus \{0\}$ into S^n) allow us to cover $\{x_0\} \times (\mathbb{R}^n \setminus \{0\})$ with finite number of $V_{\xi_j}, j = 1, \ldots, N$ with corresponding operators A_{ξ_j}.
Now, consider the operator

$$A = \sum_{j=1}^{N} A^*_{\xi_j} A_{\xi_j}.$$

By the pseudolocality of pseudodifferential operators, we know that $A_{\xi_j} u \in C^\infty(\mathbb{R}^n) \implies A^*_{\xi_j} A_{\xi_j} u \in \mathbb{C}^\infty(\mathbb{R}^n)$. Therefore, $Au \in C^\infty(\mathbb{R}^n)$ and A is elliptic at $(x_0, \xi), \forall \xi \neq 0$ with non-negative symbol. We can pick a smooth cut-off χ, $\chi \equiv 1$ when resticted to an $\epsilon/2$-ball around x_0 forming an operator

$$A + (1 - \chi) \in \Psi^0_\infty(\mathbb{R}^n)$$

that is globally elliptic. We can construct a (global) elliptic parametrix E so that

$$1 - (E \circ A + E(1 - \chi)) \in \Psi^{-\infty}_\infty(\mathbb{R}^n).$$

Finally, we can choose any smooth cut-off ϕ with support subordinate to that of χ, i.e. $\mathrm{supp}(\phi) \subset \mathrm{supp}(\chi)$ and note that

$$\phi \circ E \circ (1 - \chi) \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$$

making it a smoothing operator []. Thus, we conclude

$$\phi u = \phi E \circ Au + \phi \circ E \circ (1 - \chi) u \in C^\infty(\mathbb{R}^n)$$

as required.

□

Definition 4.11. Let $a \in S^m_\infty(\mathbb{R}^p; \mathbb{R}^n)$ for some $m \in \mathbb{R}$, $p, n \in \mathbb{N}$ be a symbol. We say a is of order $-\infty$ at a point $(x_0, \xi_0) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\}$ (write $a = O(\langle\xi\rangle^{-\infty})$) if there exist $\epsilon \in \mathbb{R}_{>0}$ such that for all $M \in \mathbb{R}$, there is a constant $C_M > 0$ such that

$$|\, a(x, \xi) \,| \leqslant C_M \langle\xi\rangle^{-M}$$

in the neigbourhood of (x_0, ξ_0) given by

$$\overline{U}_{(x_0, \xi_0)} = \left\{(x, \xi) \in \mathbb{R}^p \times \mathbb{R}^n \mid |\, x - x_0 \,| \leqslant \epsilon, \left|\widehat{\xi} - \widehat{\xi_0}\right| \leqslant \epsilon\right\}.$$

We define the cone support of the symbol a to be all the points in phase space that where it fails to be $O(\langle\xi\rangle^{-\infty})$.

$$\mathrm{conesupp}(a) = \left\{(x, \xi) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\} \mid a = O(\langle\xi\rangle^{-\infty}) \text{ at } (x, \xi)\right\}^c.$$

Lemma 4.12. *Let $a \in S^\infty_\infty(\mathbb{R}^p; \mathbb{R}^n)$, then*

1. $\mathrm{conesupp}(a)$ *is a closed conic set in* $\mathbb{R}^p \times \mathbb{R}^n$.
2. *If $a = O(\langle\xi\rangle^{-\infty})$ at $(x_0, \xi_0) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\}$, then so is $D^\alpha_x D^\beta_\xi a(x, \xi)$ for any multi-index α, β*

With the lemma above, we can in fact define the cone support of a symbol to be the smallest closed conic subset of the phase space (with $\xi \neq 0$) such that, in the complement, a and all its derivatives are of order $-\infty$.

Definition 4.13. Let $A \in \Psi_{\infty}^{m}(\mathbb{R}^n)$, $m \in \mathbb{R}$ be pseudodifferential operator. We define the **essential support**, $\mathrm{WF}'(A)$, of A to be the cone support of its left symbol, i.e.

$$\mathrm{WF}'(A) = \mathrm{conesupp}(\sigma_L(A)) \subset \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$$

Using symbol calculus we can prove the following result.

Lemma 4.14. *Let $A \in \Psi_{\infty}^{m}(\mathbb{R}^n)$, $B \in \Psi_{\infty}^{m'}(\mathbb{R}^n)$ be pseudifferential operators. Then*

1. $\mathrm{WF}'(A) = \mathrm{conesupp}(\sigma_R(A))$.
2. $\mathrm{WF}'(A \circ B) \subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B)$.
3. $\mathrm{WF}'(A + B) = \mathrm{WF}'(A) \cup \mathrm{WF}'(B)$.

With the concept of essentail support we can define the notion of *microlocal elliptic parametrix* which can be thought of as local inverse at an elliptic point of ΨDO's.

Proposition 4.15. *Let $A \in \Psi_{\infty}^{m}(\mathbb{R}^n)$ and $z \notin \Sigma^m(A)$. Then there exist a (two-sided) microlocal parametrix $B \in \Psi^{-m}(\mathbb{R}^n)$ such that*

$$z \notin \mathrm{WF}'(1 - AB) \text{ and } z \notin \mathrm{WF}'(1 - BA).$$

Proof. Let $A \in \Psi_{\infty}^{m}(\mathbb{R}^n)$ is elliptic at $(x_0, \xi_0) \in Ell^m(A)$. For each $\epsilon \in \mathbb{R}_{>0}$ we define

$$\gamma_\epsilon(x, \xi) = \chi\left(\frac{x - x_0}{\epsilon}\right)(1 - \chi(\epsilon\xi))\chi\left(\frac{\widehat{\xi} - \widehat{\xi_0}}{\epsilon}\right)$$

where $\chi \in C^\infty(\mathbb{R}^n)$ is a smooth cut-off function that is identically 1/2-ball but supported only in the unit ball. Notice that $\gamma_\epsilon \in S_{\infty}^{0}(\mathbb{R}^{2n}; \mathbb{R}^n)$ with support given by

$$\mathrm{supp}(\gamma_\epsilon) \subset \left\{(x, \xi) \mid |\, x - x_0 \,| \leqslant \epsilon,\, |\, \xi \,| \geqslant \frac{1}{2\epsilon},\, \left|\, \widehat{\xi} - \widehat{\xi_0} \,\right| \leqslant \epsilon \right\}.$$

Restricted to the conic set

$$\overline{U}_\epsilon = \left\{(x, \xi) \mid |\, x - x_0 \,| \leqslant \frac{\epsilon}{2},\, \left|\, \widehat{\xi} - \widehat{\xi_0} \,\right| \leqslant \frac{\epsilon}{2},\, |\, \xi \,| \geqslant \frac{1}{\epsilon} \right\} \subset \mathrm{supp}(\gamma_\epsilon)$$

it is identically 1 and therefore γ_ϵ is elliptic at (x_0, ξ_0). Let $L_\epsilon = \mathrm{Op}_L(\gamma_\epsilon)$ be the corresponding pseudodifferential operator. Observe that, by construction

$$(x_0, \xi_0) \notin \mathrm{WF}'(1 - L_\epsilon)$$

since $1 - \gamma_\epsilon$ is supported away from an ϵ-neighbourhood of $x = x_0$ and the wavefront set of L_ϵ is contained in an ϵ-neighbourhood of (x_0, ξ_0), i.e.

$$\mathrm{WF}'(L_\epsilon) \subset N_\epsilon(x_0, \xi_0) := \left\{(x, \xi) \mid |\, x - x_0 \,| \leqslant \epsilon,\, \left|\, \widehat{\xi} - \widehat{\xi_0} \,\right| \leqslant \epsilon \right\}$$

since γ_ϵ is bounded below in some conic neighbourhood of every point in $N_\epsilon(x_0, \xi_0)$.

Now, let $G_s = \mathrm{Op}_L(\langle\xi\rangle^s)$ for each $s \in \mathbb{R}$. Note that G_s is globally elliptic with positive principal symbol. Consider the following operator,

$$J = (1 - L_\epsilon) \circ G_{2m} + A^* A \in \Psi_{\infty}^{2m}(\mathbb{R}^n)$$

with principal symbol

$$\sigma_{2m}(J) = (1-\gamma_\epsilon)\langle\xi\rangle^{2m} + |\,\sigma_m(A)\,|^2.$$

Since $Ell^m(A)$ is open conic, we can choose ϵ is small enough so that $Ell^m(A) \subset \operatorname{supp}(\gamma_\epsilon)$. Then, by positivity of all terms involved, we have

$$\frac{|\,\sigma_{2m}(J)\,|}{\langle\xi\rangle^{2m}} = (1-\gamma_\epsilon) + \frac{|\,\sigma_m(A)\,|^2}{\langle\xi\rangle^{2m}}$$

where the first term is bounded below by 1 outside of $\operatorname{supp}(\gamma_\epsilon)$ while in $\operatorname{supp}(\gamma_\epsilon)$ the second term is bounded below by ϵ since A is elliptic (of order m) at every point in $\operatorname{supp}(\gamma_\epsilon)$. Therefore J is globally elliptic and thus have a global elliptic parametrix $H \in \Psi_\infty^{-2m}(\mathbb{R}^n)$. We shall claim that

$$B = H \circ A^* \in \Psi_\infty^m(\mathbb{R}^n)$$

is a (right) microlocal elliptic parametrix to A. Indeed,

$$\begin{aligned} B \circ A - 1 &= HA^*A - 1 \\ &= H\left(J - (1-L_\epsilon)G_{2m}\right) - 1 \\ &= (HJ - 1) - H(1-L_\epsilon)G_{2m}. \end{aligned}$$

Since H is a global parametrix to J, the first term above is a smoothing operator (i.e. an element of $\Psi_\infty^{-\infty}(\mathbb{R}^n)$ and therefore have empty wavefront set. Furthermore, by [?] the wavefront set of the second term is a subste of $\mathrm{WF}'(1-L_\epsilon)$ which does not contain (x_0,ξ_0) by construction. □

Proposition 4.16. *Pseudodifferential operators are microlocal in the following sense: Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ and $u \in C_c^{-\infty}(\mathbb{R}^n)$, then*

$$\mathrm{WF}(Au) \subset \mathrm{WF}(u). \tag{1}$$

In fact, we have

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}(u).$$

Proof. □

A partial converse to the above is given by the following proposition.

Proposition 4.17. *Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ and $u \in C_c^{-\infty}(\mathbb{R}^n)$, then*

$$\mathrm{WF}(u) \subset \mathrm{WF}(Au) \cup \Sigma^m(A).$$

5 Appendix

5.1 Stationary phase lemma

In the study of pseudodifferential operators, we often encounter integral of highly oscillatory functions of the form

$$I(h) = \int_{\mathbb{R}} a(x)e^{i\varphi(x)/h}dx$$

where $a \in C_c^\infty(\mathbb{R})$, $\varphi \in C^\infty(\mathbb{R})$ and we are interested in the asymptotic behaviour as $h \to 0$. We note that if φ is linear (or constant), i.e. $\varphi(x) = \alpha x + \beta$, $\alpha, \beta \in \mathbb{R}$, then,

$$|\,I(h)\,| = \left|\int_{\mathbb{R}} a(x)e^{i(\alpha x+\beta)/h}dx\right| = \left|e^{i\beta/h}\right|\left|\int_{\mathbb{R}} a(x)e^{i\alpha x/h}dx\right| = \left|\int_{\mathbb{R}} a(x)e^{i\alpha x/h}dx\right| \to 0$$

as $h \to 0$ by Riemann-Lebesgue lemma. That is to say, as the length scale of the oscillation tends to zero, the values of the integrand achieve perfect cancellation. In general, if $\varphi'(x) \neq 0$, we expect $e^{i\varphi(x)/h}$ to oscillate at length scale of order h and thus as $h \to 0$,

Theorem 5.1 (Schwartz Kernel Theorem [Taylor, 2011, Chapter 4.6, p. 345])**.** *Let* M, N *be compact manifold and*

$$T : C^\infty(M) \to \mathcal{D}'(N)$$

be a continuous linear map ($C^\infty(M)$ *being given Frechet space topology and* $D'(N)$ *the weak* topology). Define a bilinear map*

$$\begin{aligned} B : C^\infty(M) \times C^\infty(N) &\to \mathbb{C} \\ (u, v) &\mapsto B(u, v) = \langle v, Tu \rangle \, . \end{aligned}$$

Then, there exist a distribution $k \in \mathcal{D}'(M \times N)$ *such that for all* $(u, v) \in C^\infty(M) \times C^\infty(N)$

$$B(u, v) = \langle u \otimes v, k \rangle \, .$$

We call such k *the kernel of* T*.*

References

[Taylor, 2011] Taylor, M. (2011). *Partial Differential Equations I*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2 edition.