Microlocal Analysis

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Abstract

Acknowledgements

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Introduction

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Functional analytic background

In addition to fixing notations, this chapter serves to introduces concepts and theorems that are integral to the theory of microlocal analysis and it's application to Fredholm problems for pseudodifferential operators.

2.1 Fourier transform and tempered distribution

Definition 2.1.1 (Schwartz space)**.** The space of Schwartz (test) functions of rapidly decaying functions on $\mathbb{R}^n$, denoted $S(\mathbb{R}^n)$, is the space of smooth functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ such that for any multi-index $\alpha, \beta \in \mathbb{N}^n$,

$$\sup_{x \in \mathbb{R}^n} \left| \, x^\beta D_x^\alpha \varphi(x) \, \right| < \infty. \tag{2.1}$$

We can define a countable family of seminorm on $S(\mathbb{R}^n)$ by

$$\|\varphi\|_k := \max_{|\,\alpha\,|+|\,\beta\,| \leqslant k} \sup_{x \in \mathbb{R}^n} \left| \, x^\beta D_x^\alpha \varphi(x) \, \right| \tag{2.2}$$

for $k \in \mathbb{N}$, $\varphi \in S(\mathbb{R}^n)$. This makes $S(\mathbb{R}^n)$ into a Frechet space with metric

$$d(\varphi, \psi) = \sum_{k \in \mathbb{N}} 2^{-k} \frac{\|\varphi - \psi\|_k}{1 + \|\varphi - \psi\|_k}$$

for any $\varphi, \psi \in \S(\mathbb{R}^n)$, which defines a complete metric topology on $S(\mathbb{R}^n)$.

Remark.

1. We note that the space $S(\mathbb{R}^n)$ is non-empty since it contains all the compactly supported smooth functions $\chi : \mathbb{R}^n \to \mathbb{C}$. In fact,

   $$C_c^\infty(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) \,|\, \exists C \in \mathbb{R}_{>0}, \, |\, x \,| > R \implies u(x) = 0\} \subset S(\mathbb{R}^n)$$

   is a dense inclusion.

2. $S(\mathbb{R}^n)$ with pointwise multiplication and addition is a commutative algebra over $\mathbb{C}$ without identity since $1 \notin S(\mathbb{R}^n)$. It is also closed under several useful

elementary operations including coordinate multiplication and partial differentiation

$$x_j : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$$
$$D_{x_j} : S(\mathbb{R}^n) \to S(\mathbb{R}^n).$$

Definition 2.1.2 (Tempered distribution)**.**

Lemma 2.1.3. *The injection $\iota : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ given by (**??**) has dense image in the weak topology.*

Lemma 2.1.4 (Extension by duality)**.**

Theorem 2.1.5 (Schwartz represenatation theorem)**.**

Theorem 2.1.6 (Schwartz Kernel Theorem [?, Chapter 4.6, p. 345])**.** *Let M, N be compact manifold and*

$$T : C^\infty(M) \to \mathcal{D}'(N)$$

be a continuous linear map ($C^\infty(M)$ being given Frechet space topology and $D'(N)$ the weak topology). Define a bilinear map*

$$B : C^\infty(M) \times C^\infty(N) \to \mathbb{C}$$
$$(u, v) \mapsto B(u, v) = \langle v, Tu \rangle .$$

Then, there exist a distribution $k \in \mathcal{D}'(M \times N)$ such that for all $(u, v) \in C^\infty(M) \times C^\infty(N)$

$$B(u, v) = \langle u \otimes v, k \rangle .$$

We call such k the kernel of T.

2.2 Sobolev Spaces [?, Chapter 4]

Definition 2.2.1 (Sobolev Spaces)**.** Let $p \in \mathbb{R}$ and $n, k \in \mathbb{N}$ be given. We define the k^{th}-order L^p-based Sobolev space on $\mathbb{R}^n$ as the Banach space

$$W^{p,k}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) \mid D^\alpha u \in L^p(\mathbb{R}^n),\ |\,\alpha\,| \leqslant k\}$$

with the norm

$$\|u\|_{W^{p,k}} = \|u\|_{L^p} + \left\|D^k u\right\|_{L^p}.$$

For $p = 2$, we have denote $H^k := W^{2,k}$ and note that result from Fourier analysis gives

$$H^k(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n) \mid \langle\xi\rangle^k \hat{u} \in L^2(\mathbb{R}^n)\right\}$$

allowing us to extend the definition to each real order $s \in \mathbb{R}$,

$$H^s(\mathbb{R}^n) = \left\{u \in S'(\mathbb{R}^n) \mid \langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^n)\right\} = \Lambda^{-s} L^2(\mathbb{R}^n)$$

where $S'(\mathbb{R}^n)$ is the space of tempered distribution on $\mathbb{R}^n$ and $\Lambda^s : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ being the operator $\Lambda^s u = \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u})$. This forms a Hilbert space with inner product given by

$$\langle u, v\rangle_{H^s} = \langle\Lambda^s u, \Lambda^s v\rangle_{L^2}.$$

Remark. It is straightforward to show that the derivative operator D_{x_j} is a continuous linear operator $D_{x_j} : H^s(\mathbb{R}^n) \to H^{s-1}(\mathbb{R}^n)$ and thus by induction, for any multi-index $\alpha \in \mathbb{N}^n$, $D^\alpha : H^s(\mathbb{R}^n) \to H^{s-|\,\alpha\,|}(\mathbb{R}^n)$.

Proposition 2.2.2 (Sobolev Embedding theorem)**.** *Let $s \in \mathbb{R}$, $n \in \mathbb{N}$ be given. If $s > n/2$, then every $u \in H^s(\mathbb{R}^n)$ is bounded and continuous.*

2.3 Compact and Fredholm operators

In this section we shall restrict our attention to just maps between Banach spaces (unless specified otherwise). As per definition (**??**), a compact operator between Banach spaces is one where the image of all bounded sets are precompact. We shall denote the set of all compact (continuous) operators between V and W as $\mathcal{K}(V, W) \subset \mathcal{L}(V, W)$. Some elementary result pertaining to compact operator are given below.

Lemma 2.3.1. *Let V, W be Banach spaces.*

1. *$\mathcal{K}(V, W)$ is a closed linear subspace in $\mathcal{L}(V, W)$ in the (operator-)norm topology, i.e. $\mathcal{K}$ is closed and closed under linear combination.*

2. *If* $T \in \mathcal{L}(V, W)$ *and* $T(V)$ *is finite dimensional, then* T *is compact.*

3. *If* $T \in \mathcal{K}(V, W)$ *then* $T' \in \mathcal{K}(W', V')$.

Theorem 2.3.2. *Let* V, W, Y *be Banach spaces,* $T \in \mathcal{L}(V, W)$ *and* $K \in \mathcal{K}(V, Y)$. *If for all* $u \in V$, *the estimate*

$$\|u\|_V \leqslant C\left(\|Tu\|_W + \|Ku\|_Y\right)$$

holds for some positive real constant $C \in \mathbb{R}_{>0}$, *then the image,* $T(V)$ *is closed.*

Proof. Let $\{Tu_n \in T(V) \mid n \in \mathbb{N}, u_n \in V\}$ be a convergent sequence in $T(V)$ with limit $w \in W$. We need to show that there exist $v \in V$ such that $Tv = w$. Let $L = \ker T$. There are two cases

Case 1 ($\forall n \in \mathbb{N}$, $d(u_n, L) \leqslant a < \infty$).
By definition of distance of a point to a set, for each n there exist $x_n \in L$ such that $\|u_n - x_n\| \leqslant 2a$. We can therefore define, for each n, $v_n = u_n - x_n$. Note that $\|v_n\| \leqslant 2a$ and $\lim_{n\to\infty} Tv_n = \lim_{n\to\infty} Tu_n + Tx_n = \lim_{n\to\infty} Tu_n + 0 = w$. Since the sequence v_n is bounded and K is compact, there exist a subsequence $\left\{v_{n_j}\right\}_{j\in\mathbb{N}}$ such that $Kv_{n_j} \to y_0 \in Y$. Then, applying the estimate on $v_{n_j} - v_{n_{j+k}}$, we get, as $j \to \infty$

$$\begin{aligned}\left\|v_{n_j} - v_{n_{j+k}}\right\|_V &\leqslant C\left(\left\|Tv_{n_j} - Tv_{n_{j+k}}\right\|_W + \left\|Kv_{n_j} - Kv_{n_{j+k}}\right\|_Y\right)\\ &\to \left(\|w - w\|_W + \|y_0 - y_0\|_Y\right)\\ &= 0\end{aligned}$$

which shows that $\left\{v_{n_j}\right\}_j$ is a Cauchy and therefore has a limit $v \in V$. Using continuity we get $w = \lim_{n\to\infty} Tv_n = \lim_{j\to\infty} Tu_{n_j} = T\lim_{j\to\infty} u_{n_j} = Tv$ as required.

Case 2 ($d(u_n, L) \to \infty$ as $n \to \infty$).
We can assume without loss of generality that $d(u_n, L) \geqslant 1, \forall n$. For each n, there exist $x_n \in L$ such that $1 \leqslant d(u_n, L) \leqslant \|v_n\| \leqslant d(u_n, L) + 1$ where $v_n := u_n - x_n$. Define $w_n = v_n / \|v_n\|$. Since w_n is a bounded sequence (bounded by 1), there is a subsequence Kw_{n_j} that converges, with limit $y_0 \in Y$. Furthermore, $T(w_n) = T(u_n - x_n)/\|u_n - x_n\| \to 0$ as $n \to \infty$ since $\|u_n - x_n\| \geqslant d(u_n, L) \to \infty$. Therefore, the estimate applied on $w_{n_j} - w_{n_{j+k}}$ gives

$$\begin{aligned}\left\|w_{n_j} - w_{n_{j+k}}\right\|_V &\leqslant C\left(\left\|Tw_{n_j} - Tw_{n_{j+k}}\right\|_W + \left\|Kw_{n_j} - Kw_{n_{j+k}}\right\|_Y\right)\\ &\to \left(\|0 - 0\|_W + \|y_0 - y_0\|_Y\right)\\ &= 0\end{aligned}$$

as $j \to \infty$, showing that $\left\{w_{n_j}\right\}_j$ is a Cauchy sequence and therefore have a limit

$w \in V$. But, $Tw = \lim_{j\to\infty} Tw_{n_j} = 0 \implies w \in L$, yet

$$\begin{aligned} d(w_n, L) &= \inf_{x\in L} \left\| \frac{v_n}{\|v_n\|} - x \right\| \\ &= \|v_n\| \inf_{x\in L} \|v_n - x\| \\ &= \|v_n\| \inf_{x\in L} \|u_n - x\| \\ &= \|v_n\| \, d(u_n, L) \\ &\geqslant 1 \end{aligned}$$

implying that, in the limit as $n \to \infty$, $d(w, L) \geqslant 1$ which is a contraction.

□

Lemma 2.3.3 (Riez's inequality)**.** *Let X be a normed linear space. Given a non-dense subspace (or closed proper subspace) $Y \subset X$ and any $r \in (0,1)$, then there exist $x \in X$ with $\|x\| = 1$ such that*

$$\inf_{y\in Y} \|x - y\| \geqslant r.$$

Proof.
Let $x_0 \in \overline{Y}^c$ and $R = \inf_{y\in Y} \|y - x_0\| > 0$. Given any $\epsilon > 0$ we can pick (by definition of inf) a $y_0 \in Y$ such that

$$\|y_0 - x_0\| < R + \epsilon.$$

Take $\epsilon < R\frac{1-r}{r}$ and define $x \in X$ to be

$$x = \frac{y_0 - x_0}{\|y_0 - x_0\|}.$$

Observe that $\|x\| = 1$ and

$$\begin{aligned} \inf_{y\in Y} \|x - y\| &= \frac{1}{\|x_0 - y_0\|} \inf_{y\in Y} \|y_0 - x_0 - y\,\|x_0 - y_0\|\| \\ &= \frac{1}{\|x_0 - y_0\|} \inf_{y\in Y} \|x_0 - y\| && \text{since } \alpha y - y_0 \in Y \text{ for any scalar } \alpha \\ &\geqslant \frac{R}{R + \epsilon} \\ &\geqslant \frac{R}{R + R\frac{1-r}{r}} \\ &= r \end{aligned}$$

as required. □

Riez's lemma gives us a clear distinction between finite and infinite dimensional Banach spaces.

Corollary. *The closed unit ball in a Banach Space X is compact iff X is finite dimensional.*

Proof. Let X be a Banach space and $\overline{B}$ be closed unit ball.
Case 3 ($\Longleftarrow$). If X is finite dimensional, it is isometrically isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$, where, by Heine-Borel theorem, the closed unit ball is compact.
Case 4 ($\Longrightarrow$). We will prove the contrapositive. Suppose, X is infinite dimensional. Let $x_0 \in \partial\overline{B}$ be an element in the unit sphere. For each $n \in \mathbb{N}$, we will use Riez Lemma to obtain a unit vector x_n such that

$$\inf_{y \in \operatorname{span}\{x_0, \ldots, x_{n-1}\}} \|x_n - y\| \geqslant \frac{1}{2}.$$

It is clear that $\{x_n \,|\, n \in \mathbb{N}\}$ is a sequence of element in $\overline{B}$ that has no convergent subsequence. Therefore $\overline{B}$ is not compact.

$\square$

Chapter 3

Symbols and Pseudodifferential Operators

3.1 Symbols

We shall first give the definition of the space of symbols of order $m \in \mathbb{N}$ in Euclidean space $\mathbb{R}^n$. The main motivation is again based on the property of linear differential operators of order $m \in \mathbb{N}$ with smooth coefficient that, after Fourier transform gives the polynomial of ξ with smooth coefficient

$$P(x,\xi) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\xi^\alpha.$$

It has the property that

$$\left| D_x^\alpha D_\xi^\beta P(x,\xi) \right| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}$$

i.e. $P(x,\xi)$ is smooth and decreases in order as $\xi \to \infty$ with successive ξ-derivative.

Definition 3.1.1. The space $S_\infty^m(\mathbb{R}^p;\mathbb{R}^n)$ of order m is the space of smooth functions $a \in C^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ such that for all multi-index $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^n$

$$\left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}$$

uniformly on $\mathbb{R}^p \times \mathbb{R}^n$. We can also defined the space of symbol, $S_\infty^m(\Omega;\mathbb{R}^n)$ on a set with non-empty interior $\Omega \subset \mathbb{R}^p$, $\Omega \subset \overline{\mathrm{Int}(\Omega)}$ such that the bound above is satisfied uniformly in $(x,\xi) \in \mathrm{Int}(\Omega) \times \mathbb{R}^n$. The subscript ∞ refers the uniform boundedness condition. Together with the family of seminorm (indexed by $N \in \mathbb{N}$)

$$\|a\|_{N,m} = \sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \max_{|\alpha|+|\beta|\leqslant N} \frac{D_x^\alpha D_\xi^\beta a(x,\xi)}{\langle \xi \rangle^{m-|\beta|}}$$

gives a Frechet topology to $S_\infty^m(\Omega;\mathbb{R}^n)$.

Note: In defining pseudodifferential operators, we shall focus on the case where $p = 2n$.

3.2 Properties of Symbols

In this section, we shall establish the following summarising theorem:

Theorem 3.2.1 (Summary). *Given $m \in \mathbb{R}$, $p, n \in \mathbb{N}$, then*

1. $S^m_\infty(\mathbb{R}^p; \mathbb{R}^n)$ *is a Frechet space, hence completely metrisable.*
2. $S^\infty_\infty(\mathbb{R}^p; \mathbb{R}^n)$ *is a graded commutative *-algebra over $\mathbb{C}$ with continuous inclusion*
$$\iota : S^m_\infty(\mathbb{R}^p; \mathbb{R}^n) \to S^{m'}_\infty(\mathbb{R}^p; \mathbb{R}^n)$$
for all $m \leqslant m'$.
3. $S^{-\infty}_\infty(\mathbb{R}^p; \mathbb{R}^n)$ *is dense in $S^m_\infty(\mathbb{R}^p; \mathbb{R}^n)$ in the topology of $S^{m+\epsilon}_\infty(\mathbb{R}^p; \mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$.*

We first prove continuous inclusion of lower order into higher order symbol space.

Proposition 3.2.2. *Let $p, n \in \mathbb{N}$ be given and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$. If $m, m' \in \mathbb{R}$ such that $m \leqslant m'$, then $S^m_\infty(\Omega; \mathbb{R}^n) \subset S^{m'}_\infty(\mathbb{R}^)$. Furthermore, the inclusion map*

$$\iota : S^m_\infty(\Omega; \mathbb{R}^n) \to S^{m'}_\infty(\Omega; \mathbb{R}^n)$$

is continuous.

Proof. Let the real numbers $m \leqslant m'$ be given. We note that for any $\xi \in \mathbb{R}^n$

$$\langle \xi \rangle^m \leqslant 1 \cdot \langle \xi \rangle^{m'}$$

and thus if $a \in S^m_\infty(\Omega; \mathbb{R}^n)$, we have that $\forall \alpha \in \mathbb{N}^p, \forall \beta \in \mathbb{N}^n$

$$\left| D^\alpha_x D^\beta_\xi a(x, \xi) \right| \leqslant C \langle \xi \rangle^{m - |\beta|} \leqslant C \langle \xi \rangle^{m' - |\beta|}$$

which show that $a \in S^{m'}_\infty(\Omega; \mathbb{R}^n)$ as well.

To show that ι is a continuous inclusion, it suffices to show that

$$\|\iota(a)\|_{N,m'} \leqslant C \|a\|_{N,m}$$

for any $a \in S^m_\infty(\Omega; \mathbb{R}^n)$ and $N \in \mathbb{N}$. Indeed, this bound holds since

$$\frac{D^\alpha_x D^\beta_\xi a(x, \xi)}{\langle \xi \rangle^{m' - |\beta|}} \leqslant \frac{D^\alpha_x D^\beta_\xi a(x, \xi)}{\langle \xi \rangle^{m - |\beta|}}.$$

∎

Proposition 3.2.3. *Let $p, n \in \mathbb{N}$ be given. Let $\Omega \subset \mathbb{R}^p$ be such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$. Then, for any $m, m' \in \mathbb{R}$, we have*

$$S^m_\infty(\Omega;\mathbb{R}^n) \cdot S^{m'}_\infty(\Omega;\mathbb{R}^n) = S^{m+m'}_\infty(\Omega;\mathbb{R}^n)$$

Proof. Let $a \in S^m_\infty(\Omega;\mathbb{R}^n)$ and $b \in S^{m'}_\infty(\Omega;\mathbb{R}^n)$ be given. By (general) Leibinz formula, we have that for all multi-index α, β,

$$\begin{aligned}
\sup_{(x,\xi)\in\mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{\left| D^\alpha_x D^\beta_\xi a(x,\xi) b(x,\xi) \right|}{\langle \xi \rangle^{(m+m')-|\beta|}} &\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} \sup_{(x,\xi)\in\mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{\left| D^\mu_x D^\gamma_\xi a(x,\xi) \right| \Big| D^{\alpha-\mu}_x D^{\beta-\gamma}_\xi b(x,}{\langle \xi \rangle^{(m+m')-|\beta|}} \\
&\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \sup_{\xi\in\mathbb{R}^n} \frac{\langle \xi \rangle^{m-|\gamma|} \langle \xi \rangle^{m'-|\beta-\gamma|}}{\langle \xi \rangle^{(m+m')-|\beta|}} \\
&= \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \sup_{\xi\in\mathbb{R}^n} \langle \xi \rangle^{|\beta|-(|\beta-\gamma|+|\gamma|)} \\
&\leqslant \sum_{\mu\leqslant\alpha,\gamma\leqslant\beta} \binom{\alpha}{\mu}\binom{\beta}{\gamma} C \\
&< \infty
\end{aligned}$$

where we have use the property of multi-index that $|\beta| = |\beta - \mu| + |\mu|$. We have thus shown that $S^m_\infty(\Omega;\mathbb{R}^n) \cdot S^{m'}_\infty(\Omega;\mathbb{R}^n) \subset S^{m+m'}_\infty(\Omega;\mathbb{R}^n)$

For the reverse inclusion, let $c \in S^{m+m'}_\infty(\Omega;\mathbb{R}^n)$ be given. Define

$$\begin{aligned}
a &: (x,\xi) \mapsto \langle \xi \rangle^m \\
b &: (x,\xi) \mapsto \frac{c(x,\xi)}{a(x,\xi))}
\end{aligned}$$

and observe that

- $a \in S^m_\infty(\Omega;\mathbb{R}^n)$. It is clear that a is smooth in both x and ξ. It is independent of x and thus any x derivative gives 0. We need only to check that for all $\beta \in \mathbb{N}^n$,

  $$\left| D^\beta_\xi \langle \xi \rangle^m \right| \leqslant C \langle \xi \rangle^{m-|\beta|}$$

  which can be proven by induction on n and β. We shall only prove the base case where $n = 1$ and $\beta = 1$. We have

  $$|D_\xi \langle \xi \rangle^m| = \left| \partial_\xi (1+\xi^2)^{m/2} \right| = \left| m\xi \langle \xi \rangle^{m-2} \right| = \left| m \frac{\xi}{\langle \xi \rangle} \right| \langle \xi \rangle^{m-1} \leqslant |m| \langle \xi \rangle^{m-1}$$

  where we have used the fact that $|\xi| \leqslant \langle \xi \rangle$ for all ξ.

- $b \in S^{m'}_{\infty}(\Omega;\mathbb{R}^n)$. We note first that $\langle\xi\rangle^m \neq 0$ for all $\xi \in \mathbb{R}^n$ and thus b is well-defined. Since division by $\langle\xi\rangle^m$ does not affect any of the x derivative, we only need to show that for any $\beta \in \mathbb{N}^n$, we have

$$\left| D^{\beta}_{\xi} b(x,\xi) \right| \leqslant C \langle\xi\rangle^{m+m'-|\beta|}$$

for some constant $C > 0$ uniformly in ξ. Indeed, observe that by the Leibinz formula

$$\begin{aligned}
\left| D^{\beta}_{\xi} b(x,\xi) \right| &\leqslant \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} \left| D^{\mu}_{\xi} c(x,\xi) \right| \left| D^{\beta-\mu} \langle\xi\rangle^{-m} \right| \\
&\leqslant C \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} \langle\xi\rangle^{m+m'-|\mu|} \langle\xi\rangle^{-m-|\beta-\mu|} \\
&\leqslant C \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} \langle\xi\rangle^{m'-(|\mu|+|\beta-\mu|)} \\
&= C \sum_{\mu\leqslant\beta} \binom{\beta}{\mu} \langle\xi\rangle^{m'-|\beta|} \\
&= C 2^{\beta} \langle\xi\rangle^{m'-|\beta|}
\end{aligned}$$

where we have use the definition of c and applied the result proven for a with $m \mapsto -m$. Thus, $b \in S^{m'}_{\infty}(\Omega;\mathbb{R}^n)$.

It is clear that $a \cdot b = c$ and we have therefore shown that $S^{m+m'}_{\infty}(\Omega;\mathbb{R}^n) \subset S^{m}_{\infty}(\Omega;\mathbb{R}^n) \cdot S^{m'}_{\infty}(\Omega;\mathbb{R}^n)$. □

The results above, together with the easily proven fact $a^*(x,\xi) := \overline{a(x,\xi)} \in S^m_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n) \iff a \in S^m_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$, gives the desired algebraic structure for $S^{\infty}_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$
Next, we have a rather technical density result : the residual space, $S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$, is dense in $S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$, but only with the topology of $S^{m+\epsilon}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$. The main strategy in this proof is to approximate any symbol with the very same symbol but cut off by a compactly supported function. As such, the main reason we **cannot** have density of $S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ in $S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ is the same reason to the fact that Schwartz functions are not dense in the space of smooth bounded functions, in particular, $1 \in S^0_{\infty}(\Omega;\mathbb{R}^n)$ is not in the closure of $S^{-\infty}_{\infty}(\Omega;\mathbb{R}^n)$.

Lemma 3.2.4. *Given any $m \in \mathbb{R}$, $n, p \in \mathbb{N}$ and $a \in S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$, there exist a sequence in $S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ that is bounded in $S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ and converges to a in the topology of $S^{m+\epsilon}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ for any $\epsilon \in \mathbb{R}_{>0}$.*

Proof. Let $a \in S^m_{\infty}(\Omega;\mathbb{R}^n)$ and $\epsilon \in \mathbb{R}_{>0}$ be given. Let $\chi \in C^{\infty}_c(\mathbb{R}^n)$ be a non-negative smooth cut-off function, i.e. $\chi \geqslant 0$ and satisfies

$$\chi(\xi) = \begin{cases} 1 & , |\xi| < 1 \\ 0 & , |\xi| > 2. \end{cases}$$

Then, for each $k \in \mathbb{N}$, we define

$$a_k(x,\xi) = \chi\left(\frac{\xi}{k}\right) a(x,\xi).$$

Now, given arbitrary $N, k \in \mathbb{N}$, observe that

$$|\, a_k \,| \leqslant C \langle \xi \rangle^{-N}$$

since a_k is compactly supported in ξ (as χ is compactly supported). Furthermore, by Leibinz formula and the symbol estimates on $a \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$, we have

$$\begin{aligned}
\left| D_x^\alpha D_\xi^\beta a_k(x,\xi) \right| &\leqslant \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D_\xi^\mu \chi\right)\left(\frac{\xi}{k}\right) \left| D_x^\alpha D_\xi^{\beta-\mu} a(x,\xi) \right| \\
&\leqslant C \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D_\xi^\mu \chi\right)\left(\frac{\xi}{k}\right) \langle \xi \rangle^{m-|\,\beta-\mu\,|}.
\end{aligned}$$

Since χ and all its derivatives are compactly supported, each term above is bounded in ξ and thus a_k is bounded in $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ and that

$$\left| D_x^\alpha D_\xi^\beta a_k(x,\xi) \right| \leqslant C' \langle \xi \rangle^{-N}$$

which allow us to conclude that $a_k \in S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$.

It remains to show that $\lim_{k\to\infty} a_k = a$ in $S^{m+\epsilon}_\infty(\Omega;\mathbb{R}^n)$. In the first symbol norm, we observe that, using the symbol estimate for a

$$\begin{aligned}
\|a_k - a\|_{0,m+\epsilon} &= \sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{|\, a_k(x,\xi)\,|}{\langle \xi \rangle^{m+\epsilon}} \\
&= \sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{|\,(1-\chi(\xi/k))\,|\,|\, a(x,\xi)\,|}{\langle \xi \rangle^{m+\epsilon}} \\
&\leqslant \|a\|_{0,m} \sup_{\xi\in\mathbb{R}^n} \frac{|\,(1-\chi(\xi/k))\,|}{\langle \xi \rangle^{\epsilon}} \\
&\leqslant \|a\|_{0,m} \langle k \rangle^{-\epsilon} \\
&\to 0
\end{aligned}$$

as $k \to \infty$, since $|\,(1-\chi(\xi/k))\,|$ is 0 in the region $|\,\xi\,| \leqslant k$ and bounded by 1 otherwise. We remark upon the necessity of the extra decay given by $\langle \xi \rangle^{-\epsilon}$ factor. For other symbol norms we shall again use Leibinz formula:

$$\begin{aligned}
\sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{\left| D_x^\alpha D_\xi^\beta a_k(x,\xi) \right|}{\langle \xi \rangle^{m+\epsilon-|\,\beta\,|}} &\leqslant \sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{1}{\langle \xi \rangle^{m+\epsilon-|\,\beta\,|}} \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D^\mu(1-\chi)\right)\left(\frac{\xi}{k}\right) \left| D_x^\alpha D_\xi^{\beta-\mu} a(x,\xi) \right| \\
&\leqslant \sup_{(x,\xi)\in\mathbb{R}^p\times\mathbb{R}^n} \frac{C}{\langle \xi \rangle^{m+\epsilon-|\,\beta\,|}} \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D^\mu(1-\chi)\right)\left(\frac{\xi}{k}\right) \langle \xi \rangle^{m-|\,\beta-\mu\,|} \\
&= C \sup_{\xi\in\mathbb{R}^n} \sum_{\mu \leqslant \beta} \binom{\beta}{\mu} k^{-|\,\mu\,|} \left(D^\mu(1-\chi)\right)\left(\frac{\xi}{k}\right) \langle \xi \rangle^{-\epsilon-|\,\mu\,|} \\
&\leqslant C' k^{-\epsilon} \\
&\to 0
\end{aligned}$$

as $k \to \infty$ by the same argument as before. Thus, we have proven that $a_k \to a$ as $k \to \infty$ in $S^{m+\epsilon}_{\infty}(\mathbb{R}^p; \mathbb{R}^n)$.

$\square$

Definition 3.2.5. Given $p, n \in \mathbb{N}$, $m \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$, an order m symbol $a \in S^m_{\infty}(\Omega; \mathbb{R}^n)$ is (globally) **elliptic** if there exist $\epsilon \in \mathbb{R}_{>0}$ such that

$$\inf_{|\,\xi\,| \geqslant 1/\epsilon} |\, a(x,\xi)\,| \geqslant \epsilon \,\langle \xi \rangle^m \,.$$

The importance of elliptic symbol is that they are invertible modulo $S^{-\infty}_{\infty}(\Omega; \mathbb{R}^n)$.

Lemma 3.2.6. *Given $p, n \in \mathbb{N}$, $m \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^p$ such that $\Omega \subset \overline{\mathrm{Int}(\Omega)}$. Let $a \in S^m_{\infty}(\Omega; \mathbb{R}^n)$ be an elliptic symbol of order m. Then there exist a symbol $b \in S^{-m}_{\infty}(\Omega; \mathbb{R}^n)$ such that*

$$a \cdot b - 1 \in S^{-\infty}_{\infty}(\Omega; \mathbb{R}^n).$$

Proof. We shall follow the general strategy of inverting the symbol outside of a compact set. Let $\phi \in C^{\infty}_c(\mathbb{R}^n)$ be a smooth cut off function, i.e $0 \leqslant \phi \leqslant 1$ and $\phi(\xi) = 1$ for $|\,\xi\,| < 1$ and $\phi(\xi) = 0$ for $|\,\xi\,| > 2$.

Let $a \in S^m_{\infty}(\Omega; \mathbb{R}^n)$ be an elliptic symbol, that is, for any fixed $\epsilon \in \mathbb{R}_{>0}$, we have

$$|\, a(x,\xi)\,| \geqslant \epsilon \,\langle \xi \rangle^m$$

for any $|\,\xi\,| \geqslant 1/\epsilon$. Thus, we can define

$$b(x,\xi) = \begin{cases} \frac{1-\phi(\epsilon\xi/2)}{a(x,\xi)} & |\,\xi\,| \geqslant 1/\epsilon \\ 0 & |\,\xi\,| < 1/\epsilon. \end{cases}$$

We check:

b is well-defined and smooth.
We note that $|\,a(x,\xi)\,| > 0$ whenever $|\,\xi\,| \geqslant 1/\epsilon$ and therefore b is well defined in that region. For smoothness, we note first that b is smooth in the regions $|\,\xi\,| > 1/\epsilon$ and $|\,\xi\,| < 1/\epsilon$. Set $\delta = 1/(2\epsilon)$. In the region where $1/\epsilon - \delta < |\,\xi\,| < 1/\epsilon + \delta$, we have $|\,\epsilon\xi/2\,| < 1/\epsilon$ and therefore $b(x,\xi) \equiv 0$ in this region and is thus smooth. Since the we have covered $\Omega \times \mathbb{R}^n$ by the three chart domain above, b is smooth by the (smooth) gluing lemma.

b is a symbol of order $-m$.
We can prove by induction that in the region $|\,\xi\,| \geqslant 1/\epsilon$

$$D^{\alpha}_x D^{\beta}_{\xi} b = a^{-1-|\,\alpha\,|-|\,\beta\,|} G_{\alpha\beta}$$

for all multi-index α, β, where $G_{\alpha\beta}$ is a symbol of order $(|\alpha| + |\beta|)m - |\beta|$. Therefore, using the ellipticity estimate for a, we get

$$\begin{aligned}
\|b\|_{k,-m} &= \sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{\left| D_x^\alpha D_\xi^\beta b(x,\xi) \right|}{\langle \xi \rangle^{-m-k}} \\
&= \sup_{|\xi|\geqslant 1/\epsilon} \left| a^{-1-|\alpha|-|\beta|} G_{\alpha\beta} \right| \langle \xi \rangle^{m+k} \\
&\leqslant \frac{\|G_{\alpha\beta}\|_{0,(|\alpha|+|\beta|)m-|\beta|}}{\epsilon} \sup_{|\xi|\geqslant 1/\epsilon^{1+|\alpha|+|\beta|}} \langle \xi \rangle^{-m(1+|\alpha|+|\beta|)} \langle \xi \rangle^{m+k} \\
&< \infty
\end{aligned}$$

as required.

b is an inverse of a modulo $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$.
The main observation is that the set where b fails to be the multiplicative inverse of a is a compact set (in ξ) and thus $a \cdot b - 1$ is in fact a compactly supported smooth function of ξ which is a subset of $S_\infty^{-\infty}(\Omega; \mathbb{R}^n)$.

Explicitly, for any $N \in \mathbb{N}$

$$\sup_{(x,\xi)\in \mathrm{Int}(\Omega)\times\mathbb{R}^n} \frac{\left| D_x^\alpha D_\xi^\beta (a \cdot b - 1) \right|}{\langle \xi \rangle^{-N}} \leqslant \sup_{|\xi|\leqslant 1/\epsilon} \langle \xi \rangle^N \left| D_x^\alpha D_\xi^\beta (\phi(\xi\epsilon/2)) \right| < \infty.$$

□

3.3 Quantisation

Pseudodifferential operators are defined using symbols. The main gadget is the following oscillatory integral:

$$S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n) \ni a \mapsto I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi)\, \mathrm{d}\xi \tag{3.1}$$

with action on Schwartz functions $u \in S(\mathbb{R}^n)$ given by

$$I(a)(u) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi) u(y)\, \mathrm{d}y\, \mathrm{d}\xi. \tag{3.2}$$

The integral 3.2 above might be divergent unless $m < -n$, but it can be interpreted as a tempered distribution, i.e. a linear function on $S(\mathbb{R}^n)$, with action

$$S(\mathbb{R}^n) \ni v \mapsto I(a)(u)(v) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi) u(y) v(x)\, \mathrm{d}y\, \mathrm{d}\xi\, \mathrm{d}x \in \mathbb{C}. \tag{3.3}$$

The process of turning the symbol a into an operator $S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ is known as the quantisation procedure. Our first goal is the following:

Goal :
To establish that the procedure above is well-defined, so that for each $a \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$

$$\begin{aligned} I(a) : S(\mathbb{R}^n) &\to S'(\mathbb{R}^n) \\ u &\mapsto I(a)(u) \quad : S(\mathbb{R}^n) \to \mathbb{C} \\ &\qquad v \mapsto I(a)(uv) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi) u(y) v(x)\, \mathrm{d}y\, \mathrm{d}\xi\, \mathrm{d}x \end{aligned}$$

is a continuous linear map between Frechet spaces.

Remark. Given $a \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$, we sometimes write $A = Op(a) = I(a)$ for the operator $S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ defined by quantising the symbol a. Also, once the procedure above is proven to be well-defined, we will, with abuse of notation, identify the integral

$$I(a) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi)\, \mathrm{d}\xi \in S'(\mathbb{R}^n \times \mathbb{R}^n)$$

to be the *Schwartz Kernel* of the operator $I(a) : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$.
We first note that, if $m < -n$ (write $m = -n - \epsilon$ for some $\epsilon > 0$), the oscillatory integral 3.2, is absolutely convergent and defines a continuous linear operator

$$\begin{aligned} I : S^{-n-\epsilon}_\infty(\mathbb{R}^{2n}; \mathbb{R}^n) &\to S'(\mathbb{R}^{2n}) \\ a &\mapsto I(a) : S(\mathbb{R}^{2n}) \ni \varphi \mapsto I(a)(\varphi) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi) \varphi(x,y)\, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y. \end{aligned}$$

The map above is clearly linear. Continuity comes from the bound given by the following computation: $\forall M \in \mathbb{N}$, $\forall a \in S^{-n-\epsilon}_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$, $\forall \varphi \in S(\mathbb{R}^n)$

$$\begin{aligned} |\, I(a)(\varphi)\,| &\leqslant \frac{1}{(2\pi)^n} \int |\, a(x,y,\xi)\varphi(x,y)\,|\, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y \\ &\leqslant \frac{\|a\|_{0,-n-\epsilon}}{(2\pi)^n} \int \langle \xi \rangle^{-n-\epsilon} \langle (x,y) \rangle^{-M} \langle (x,y) \rangle^{M} |\, \varphi(x,y)\,|\, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y \\ &\leqslant \frac{\|a\|_{0,-n-\epsilon} \|\varphi\|_M}{(2\pi)^n} \int \langle \xi \rangle^{-n-\epsilon} \langle (x,y) \rangle^{-M}\, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y \end{aligned}$$

for any $M \in \mathbb{N}$, where

$$\|\varphi\|_M := \sum_{|\,\alpha\,| \leqslant M} \sup_{(x,y) \in \mathbb{R}^{2n}} \langle (x,y) \rangle^M \left|\, D^\alpha_{x,y} \varphi(x,y)\,\right| \tag{3.4}$$

is the Schwartz seminorm on $S(\mathbb{R}^{2n})$. If we choose $M \geqslant 2n+1$, the x, y integrals are convergent and since $m = -n - \epsilon < -n$, the ξ integral converges as well, hence we have

$$|\, I(a)(\varphi)\,| \leqslant C \|a\|_{0,m} \|\varphi\|_M$$

which implies continuity.

The proposition below extend this result to general $m \in \mathbb{R}$.

Proposition 3.3.1. *The continuous linear map*

$$I : S_{\infty}^{-\infty}(\mathbb{R}^{2n};\mathbb{R}^n) \to S'(\mathbb{R}^{2n})$$

extends uniquely to a linear map

$$I : S_{\infty}^{m}(\mathbb{R}^{2n};\mathbb{R}^n) \to S'(\mathbb{R}^{2n})$$

which is continuous as linear map from $S_{\infty}^{m'}(\mathbb{R}^{2n};\mathbb{R}^n)$ to $S'(\mathbb{R}^{2n})$ for arbitrary $m \in \mathbb{R}$ and $m' > m$.

Proof. Let $m, m' \in \mathbb{R}$, $n \in \mathbb{N}$ with $m < m'$ be given. For any $a \in S_{\infty}^{m}(\mathbb{R}^{2n};\mathbb{R}^n)$, the density of $S_{\infty}^{-\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$ in $S_{\infty}^{m}(\mathbb{R}^{2n};\mathbb{R}^n)$ with the topology of $S_{\infty}^{m'}(\mathbb{R}^{2n};\mathbb{R}^n)$ means that there exist a sequence $a_k \in S_{\infty}^{-\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$ so that $a_k \to a \in S_{\infty}^{m'}(\mathbb{R}^{2n};\mathbb{R}^n)$. Together with the completeness of $S'(\mathbb{R}^{2n})$ (being a continuous linear map into $\mathbb{C}$ which is complete), we have unique continuous linear extension of $I : S_{\infty}^{-\infty}(\mathbb{R}^{2n};\mathbb{R}^n) \to S'(\mathbb{R}^{2n})$ to $S_{\infty}^{m}(\mathbb{R}^{2n};\mathbb{R}^n)$ given by

$$I(a) := \lim_{k\to\infty} I(a_k)$$

which is continuous in the $S_{\infty}^{m'}(\mathbb{R}^{2n};\mathbb{R}^n)$ topology. Therefore, it is enough to show that for any $a \in S_{\infty}^{-\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$ and $\varphi \in S(\mathbb{R}^{2n})$, there exist $N, M \in \mathbb{N}$, such that

$$|\, I(a)(\varphi)\,| \leqslant C \left\| a \right\|_{N,m'} \left\| \varphi \right\|_{M}.$$

Let a, φ as above be given. Note that

$$e^{i(x-y)\xi} = \langle \xi \rangle^{-2q} \left(1 + \xi \cdot D_x\right)^q e^{i(x-y)\xi} = \langle \xi \rangle^{-2q} \left(1 - \xi \cdot D_y\right)^q e^{i(x-y)\xi}.$$

Thus, using integration by parts, for any $q \in \mathbb{N}$,

$$\begin{aligned}
I(a)(\varphi) &= \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,y,\xi)\varphi(x,y)\, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y \\
&= \frac{1}{(2\pi)^n} \int \langle \xi \rangle^{-4q} \left(1 - \xi \cdot D_y\right)^q \left(1 + \xi \cdot D_x\right)^q e^{i(x-y)\xi} a(x,y,\xi)\varphi(x,y)\, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y \\
&= \frac{1}{(2\pi)^n} \int \langle \xi \rangle^{-4q} e^{i(x-y)\xi} \left(1 - \xi \cdot D_y\right)^q \left(1 + \xi \cdot D_x\right)^q \left[a(x,y,\xi)\varphi(x,y)\right] \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y \\
&= \frac{1}{(2\pi)^n} \int \langle \xi \rangle^{-4q} e^{i(x-y)\xi} \left(\sum_{|\,\gamma\,|\leqslant 2q} a_\gamma(x,y,\xi) D_{x,y}^{\gamma}\varphi(x,y) \right) \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y
\end{aligned}$$

where

$$a_\gamma(x,y,\xi) = \sum_{|\,\mu\,|,|\,\nu\,|\leqslant q} C_{\mu\nu} \xi^{\mu+\nu} D_x^{\mu} D_y^{\nu} a(x,y,\xi)$$

for some combinatorial constants $C_{\mu\nu}$. Now, using the symbol estimate for $a \in S^{-\infty}_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$, and that $|\,\mu\,| + |\,\nu\,| \leqslant 2q$

$$
\begin{aligned}
|\,a_\gamma(x,y,\xi)\,| &\leqslant \sum_{|\,\mu\,|,|\,\nu\,|\leqslant q} C_{\mu\nu}\,|\,\xi\,|^{\mu+\nu}\,\left|\,D_x^\mu D_y^\nu a(x,y,\xi)\,\right| \\
&= \sum_{|\,\mu\,|,|\,\nu\,|\leqslant q} C_{\mu\nu}\,|\,\xi\,|^{\mu+\nu}\,\langle\xi\rangle^{m'}\,\frac{\left|\,D_x^\mu D_y^\nu a(x,y,\xi)\,\right|}{\langle\xi\rangle^{m'}} \\
&\leqslant \|a\|_{2q,m'}\,\langle\xi\rangle^{m'} \sum_{|\,\mu\,|,|\,\nu\,|\leqslant q} C_{\mu\nu}\,|\,\xi\,|^{\mu+\nu} \\
&\leqslant \|a\|_{2q,m'}\,\langle\xi\rangle^{m'} \sum_{|\,\mu\,|,|\,\nu\,|\leqslant q} C_{\mu\nu}\,\langle\xi\rangle^{\mu+\nu} \\
&\leqslant \|a\|_{2q,m'}\,\langle\xi\rangle^{m'}\,\langle\xi\rangle^{2q} \sum_{|\,\mu\,|,|\,\nu\,|\leqslant q} C_{\mu\nu} \\
&\leqslant C_q\,\|a\|_{2q,m'}\,\langle\xi\rangle^{m'+2q}
\end{aligned}
$$

and since $|\,\gamma\,| \leqslant 2q$,

$$
\begin{aligned}
\left|\,D^\gamma_{x,y}\varphi(x,y)\,\right| &= \langle(x,y)\rangle^{-2q-2n-1}\,\langle(x,y)\rangle^{2q+2n+1}\,\left|\,D^\gamma_{x,y}\varphi(x,y)\,\right| \\
&\leqslant \langle(x,y)\rangle^{-2q-2n-1} \sum_{|\,\alpha\,|\leqslant 2q+2n+1}\ \sup_{(x,y)\in\mathbb{R}^{2n}} \langle(x,y)\rangle^{2q+2n+1}\,\left|\,D^\alpha_{x,y}\varphi(x,y)\,\right| \\
&\leqslant \langle(x,y)\rangle^{-2q-2n-1}\,\|\varphi\|_{2q+2n+1}\,.
\end{aligned}
$$

Bring together both bounds, we have

$$
\begin{aligned}
|\,I(a)(\varphi)\,| &\leqslant \frac{1}{(2\pi)^n}\int \langle\xi\rangle^{-4q}\left(\sum_{|\,\gamma\,|\leqslant 2q}\left|\,a_\gamma(x,y,\xi)D^\gamma_{x,y}\varphi(x,y)\,\right|\right)\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant C'\,\|a\|_{2q,m'}\,\|\varphi\|_{2q+2n+1}\int \langle\xi\rangle^{-4q}\,\langle\xi\rangle^{m'+2q}\,\langle(x,y)\rangle^{-2q-2n-1}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y \\
&= C'\,\|a\|_{2q,m'}\,\|\varphi\|_{2q+2n+1}\int \langle\xi\rangle^{m'-2q}\,\langle(x,y)\rangle^{-2q-2n-1}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

Thus, as long as $m' - 2q < -n$, i.e. $q > \max\left(\frac{m'+n}{2}, 0\right)$, the integral above converges. Finally, set $N = 2q$, $M = 2q + 2n + 1$, we have

$$|\,I(a)(\varphi)\,| \leqslant C\,\|a\|_{N,m'}\,\|\varphi\|_M$$

as required. □

By the Schwartz Kernel theorem, each $a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ defines a continuous linear operator

$$I(a) : S(\mathbb{R}^n) \to S'(\mathbb{R}^n).$$

We can now define the space of m-order pseudo-differential operators as the space

$$\Psi^m_\infty(\mathbb{R}^n) := \left\{A = I(a) \mid a \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)\right\}$$

with the total space $\Psi^\infty_\infty(\mathbb{R}^n) := \cup_{m\in\mathbb{R}}\Psi^m_\infty(\mathbb{R}^n)$ and the residual space $\Psi^{-\infty}_\infty(\mathbb{R}^n) := \cap_m \Psi^m_\infty(\mathbb{R}^n)$ defined similarly.

3.3.1 Composition theorem

In this section we shall prove that, just like symbol spaces, $\Psi^\infty_\infty(\mathbb{R}^n)$ forms a graded *-algebra. The difference being, this time, the algebra is *non-commutative*. That is, we shall show that following theorem holds.

Theorem 3.3.2 (Summary)**.** *Given $n \in \mathbb{N}$, $\Psi^\infty_\infty(\mathbb{R}^n)$ is a graded *-algebra over $\mathbb{C}$ with continuous inclusion*

$$\iota : \Psi^m_\infty(\mathbb{R}^n) \to \Psi^{m'}_\infty(\mathbb{R}^n)$$

for any $m \leqslant m'$.

We shall prove this theorem by first accumulate several technical lemmas, of which the most important is the reduction lemma that allow us remove the dependence of either x or y in the symbol $a(x, y, \xi) \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$.

Asymptotic Summation

Suppose we are given a sequence of symbols with decreasing order, $a_j \in S^{m-j}_\infty(\mathbb{R}^p; \mathbb{R}^n)$, $j \in \mathbb{N}$, we know that $a_j(x, \xi)$ has ever higher rate of decay for large $|\,\xi\,|$ with increasing j. However, the series $\sum_{j\in\mathbb{N}} a_j(x, \xi)$ need not converge. However, we have the following notion of asymptotic convergence.

Definition 3.3.3 (Asymptotic summation)**.** A sequence of symbols with decreasing order, $a_j \in S^{m-j}_\infty(\mathbb{R}^p; \mathbb{R}^n)$, $j \in \mathbb{N}$ is **asymptotically summable** if there exist $a \in S^m_\infty(\mathbb{R}^p; \mathbb{R}^n)$ such that for all $N \in \mathbb{N}$,

$$a - \sum_{j=0}^{N-1} a_j \in S^{m-N}_\infty(\mathbb{R}^p; \mathbb{R}^n).$$

We write

$$a \sim \sum_{j\in\mathbb{N}} a_j.$$

Lemma 3.3.4. *Every sequence of symbols with decreasing order is asymptot-*

ically summable. Furthermore, the sum is unique up to an additive term in $S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$.

Sketch. Let $a_j \in S^{m-j}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$, $j \in \mathbb{N}$ be given. For uniqueness, suppose $a, a' \in S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ are both asymptotic sums of the sequence. We need to show that $a-a' \in S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$. Indeed, for any $N \in \mathbb{N}$,

$$a - a' = \left(a - \sum_{j=0}^{N-1} a_j\right) - \left(a' - \sum_{j=0}^{N-1} a_j\right) \in S^{m-N}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$$

since both terms on the right are elements of $S^{m-N}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$. Thus,

$$a - a' \in \cap_{n\in\mathbb{N}} S^{m-N}_{\infty}(\mathbb{R}^p;\mathbb{R}^n) = S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n).$$

For existence, we construct $aS^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ by Borel's method []. Let $\chi \in C^{\infty}_c(\mathbb{R}^p)$ be a bump function and define

$$a = \sum_{j\in\mathbb{N}} (1-\chi)\,(\epsilon_j\xi) a_j(x,\xi)$$

where $\mathbb{R}_{>0} \ni \epsilon_j \to 0$ is a strictly monotonic decreasing sequence that converges to 0. We note that the sequence is a finite sum for any input (x,ξ) and hence define a smooth function. It remains to show that, for some choice of ϵ_j with sufficiently rapid decay,

$$\sum_{j\geqslant N} (1-\chi)\,(\epsilon_j\xi) a_j(x,\xi)$$

converges in $S^{m-N}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ for any $N \in \mathbb{N}$.

Note: This is again an exercise in using symbol seminorms and Leibniz formula. □

Reduction

We will now show that $\Psi^m_{\infty}(\mathbb{R}^n)$ is exactly the range of $I : S^m_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n) \to S'(\mathbb{R}^{2n})$ restricted to $S^m_{\infty}(\mathbb{R}^n;\mathbb{R}^n) \subset S^m_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$.

Definition 3.3.5. Let

$$\pi_L : \mathbb{R}^{3n}_{x,y,\xi} \to \mathbb{R}^{2n}_{x,\xi}$$

be the projection map $(x,y,\xi) \mapsto (x,\xi)$. We define the **left quantisation map** as

$$q_L := I \circ \pi_L^* : S^m_{\infty}(\mathbb{R}^n;\mathbb{R}^n) \to \Psi^m_{\infty}(\mathbb{R}^n)$$

with elements $a_L \in S^m_{\infty}(\mathbb{R}^n;\mathbb{R}^n)$ known as the **left reduced symbols**.

Lemma 3.3.6 (Reduction)**.** *For any $a(x,y,\xi) \in S^m_\infty(\mathbb{R}^{2n}_{x,y};\mathbb{R}^n_\xi)$ there exist unique $a_L(x,\xi) \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that $I(a) = q_L(a_L) = I(a_L \circ \pi_L)$. Furthermore, with $\iota : \mathbb{R}^{2n} \ni (x,\xi) \mapsto (x,x,\xi) \in \mathbb{R}^{3n}$ being the diagonal inclusion map, we have*

$$a_L(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\iota^* D_y^\alpha D_\xi^\alpha a(x,y,\xi). \tag{3.5}$$

Sketch. Note that

$$D_\xi^\alpha e^{i(x-y)\xi} = (x-y)^\alpha e^{i(x-y)\xi} \implies I((x-y)^\alpha a) = I((-1)^{|\alpha|D_\xi^\alpha a})$$

where we have extended the identity that is true using integration by parts in $S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ to general $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ using the density result of symbol space. Now, if we Taylor expand a around the diagonal $x=y$, we get

$$a(x,y,\xi) = \sum_{|\alpha|\leqslant N-1} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha D_y^\alpha a(x,x,\xi) + r_N(x,y,\xi)$$

where

$$r_N(x,y,\xi) = \sum_{|\alpha|=N} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha \int_0^1 (1-t)^{N-1} D_y^\alpha a(x,(1-t)x+ty,\xi)\,\mathrm{d}t$$

for any $N \in \mathbb{N}$. Applying the identity above give us

$$\begin{aligned} I(a) &= \sum_{j=0}^{N-1} A_j + R_N \\ A_j &= I\left(\sum_{|\alpha|=j} \frac{i^{|\alpha|}}{\alpha!} D_y^\alpha D_\xi^\alpha a(x,x,\xi)\right) \in \Psi^{m-j}_\infty(\mathbb{R}^n) \\ R_N &\in \Psi^{m-N}_\infty(\mathbb{R}^n) \end{aligned}$$

Asymptotic summation lemma give us

$$b(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_y^\alpha D_\xi^\alpha a(x,x,\xi) \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$$

so that $I(a) - I(b) \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. It remains to show that $A \in \Psi^{-\infty}_\infty(\mathbb{R}^n) \iff A = I(c), c \in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n)$. □

Composition theorem

Theorem 3.3.7 (Composition). *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$ for some $m, m' \in \mathbb{R}$. Then,*

1. $A^* \in \Psi^m_\infty(\mathbb{R}^n)$.
2. $A \circ B \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$.

Sketch. Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$ for some $m, m' \in \mathbb{R}$ be given. Since $A : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ (**??**), we have the adjoint operator $A^* : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ defined by

$$A^*u(\varphi) = u(\overline{A\varphi}), \quad u \in S'(\mathbb{R}^n), \varphi \in S(\mathbb{R}^n).$$

We check that A^*u is indeed an element of $S'(\mathbb{R}^n)$ since it is the composition of the maps $u \in S'(\mathbb{R}^n)$ and $S(\mathbb{R}^n) \ni \varphi \mapsto \overline{A\varphi}$ which are both continuous. Let $a \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$ be such that $A = I(a)$. Observe that,

$$\begin{aligned}
\langle Au, \varphi \rangle_{L^2} &= \int Au(x)\overline{\varphi(x)}\,\mathrm{d}x \\
&= \int u(y) \overline{\int e^{i(x-y)\xi}\overline{a(x,y,\xi)}\varphi(x)\,\mathrm{d}x\,\mathrm{d}\xi}\,\mathrm{d}y \\
&= \int u(y)\overline{I(b)\varphi(y)}\,\mathrm{d}y \\
&= \langle u, A^*\varphi \rangle_{L^2}
\end{aligned}$$

where $b(x,y,\xi) = \overline{a(y,x,\xi)} \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$. Thus, $A^* \in \Psi^m_\infty(\mathbb{R}^n)$.

For composition, applying the reduction lemma twice to get $a_L \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$ and $b_L \in S^{m'}_\infty(\mathbb{R}^n; \mathbb{R}^n)$ so that

$$\begin{aligned}
A\varphi(x) &= \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,\xi)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi \\
B^*\varphi(x) &= \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} \overline{b(x,\xi)}\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi
\end{aligned}$$

which shows that

$$AB\varphi(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,\xi) b(y,\xi)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi$$

and thus $AB = I(a(x,\xi)b(y,\xi))$. Since $a(x,\xi)b(y,\xi) \in S^{m+m'}_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$, we have the result $AB \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$ as required.

$\square$

Chapter 4

Ellipticity and Microlocalisation

4.1 Microlocalisation

Roughly, the support of a distribution in $\mathbb{R}^n$ consist of points $x \in \mathbb{R}^n$ where the distribution is non-zero after any smooth cut-offs near x.

Definition 4.1.1. The **support of a tempered distribution** $u \in S'(\mathbb{R}^n)$ is given by the set

$$\operatorname{supp}(u) = \{x \in \mathbb{R}^n \mid \exists \phi \in S(\mathbb{R}^n), \phi(x) \neq 0, \phi u = 0\}^c .$$

And the singular support of a distribution is the set of points where the distribution fails to behave like an element of $S(\mathbb{R}^n)$.

Definition 4.1.2. The **singular support of a tempered distribution** $u \in S'(\mathbb{R}^n)$ is given by the set

$$\operatorname{singsupp}(u) = \{x \in \mathbb{R}^n \mid \exists \phi \in S(R^n), \phi(x) \neq 0, \phi(u) \in S(\mathbb{R}^n)\}^c$$

Note that if the singular support of a tempered distribution is empty, it is then an element of $C^\infty(\mathbb{R}^n)$ The support of an operator is given by the support of its Schwartz kernel.

Definition 4.1.3. The **support of a continuous linear operator** $A : S(R^n) \to S'(\mathbb{R}^n)$ is given by

$$\operatorname{supp}(A) = \operatorname{supp}(K_A) \subset \mathbb{R}^n \times \mathbb{R}^n$$

where $K_A \in S'(\mathbb{R}^n \times \mathbb{R}^n)$ is the Schwartz kernel of A.

We note from the above that supports or singular supports are complement of open sets, therefore they are closed. We have the following result relating the support of a smooth function after the action of a continuous linear operator.

Proposition 4.1.4 (Calculus of support)**.** *Let $A : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ be a continuous linear operator and $\phi \in C_c^\infty(\mathbb{R}^n)$, then*

$$\operatorname{supp}(A\phi) \subset \operatorname{supp}(A) \circ \operatorname{supp}(\phi) := \{x \in \mathbb{R}^n \,|\, \exists y \in \operatorname{supp}(\phi), (x,y) \in \operatorname{supp}(A)\}\,.$$

Proof. We shall show the contrapositive statement:

$$x \notin \operatorname{supp}(A) \circ \operatorname{supp}(\phi) \implies x \notin \operatorname{supp}(A\phi).$$

Suppose $x \notin \operatorname{supp}(A) \circ \operatorname{supp}(\phi)$. Observe that

$$\operatorname{supp}(A) \circ \operatorname{supp}(\phi) = \pi_x(\pi_y^{-1}(\operatorname{supp}(\phi)) \cap \operatorname{supp}(A))$$

where $\pi_{x,y} : \mathbb{R}^2 \to \mathbb{R}$ are the projection map to the respective coordinates. Since $\operatorname{supp}(A)$ is closed and $\operatorname{supp}(\phi)$ is compact, we have that $\operatorname{supp}(A) \circ \operatorname{supp}(\phi)$ is closed and thus x belongs to an open set. We can therefore choose a smooth cutt-off function $\chi \in C_c^\infty(\mathbb{R}^n)$ supported at x ($\chi(x) \neq 0$) but away from $\operatorname{supp}(A) \circ \operatorname{supp}(\phi)$. Thus,

$$\operatorname{supp}(A) \cap (\operatorname{supp}(\chi) \times \operatorname{supp}(\phi)) = \emptyset$$

and hence $\chi(x)K_A(x,y)\phi(y) = 0 \implies \chi A\phi = 0$, as required. □

4.1.1 Pseudolocality

We shall show two results on the singular support of pseudodifferential operators. The first result is that the singular support of any ΦDO is contained within the diagonal, i.e. they are smooth away from $x = y$. The second result is teh pseudolocality result that says that action ΨDO's do not increase singular support of distributions.

Proposition 4.1.5. *Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ for some $m \in \mathbb{R}$, then*

$$\operatorname{singsupp}(A) \subset \left\{(x,y) \in \mathbb{R}^{2n} \,|\, x = y\right\}.$$

Proof. We shall prove this theorem for elements of $\Psi_\infty^{-\infty}(\mathbb{R}^n)$ and then extend by continuity to all orders. Let $A \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ with symbol $a \in S_\infty^{-\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$. Its singular support is given by the singular support of the kernel. Since all derivatives of a are $O(\langle\xi\rangle^{-\infty})$, the oscillatory integral below representing the kernel is absolutely convergent and, using integration by parts, we get

$$\begin{aligned}
I(a) &= \frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}a(x,y,\xi)\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n}\int \frac{1}{(x-y)^\alpha}(-D_\xi^\alpha)\left(e^{i(x-y)\xi}\right)a(x,y,\xi)\,\mathrm{d}\xi \\
&= \frac{1}{(x-y)^\alpha}\frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}(-D_\xi^\alpha)a(x,y,\xi)\,\mathrm{d}\xi \\
&= \frac{1}{(x-y)^\alpha}I((-D_\xi^\alpha)a)
\end{aligned}$$

which is true for all multi-index α of any order. Since all x, y-derivatives of a are uniformly bounded by $\langle\xi\rangle^{-N}$ for any $N \in \mathbb{N}$, we can differentiate under the integral sign to get the equation

$$\begin{aligned} D_x^\beta D_y^\gamma (x-y)^\alpha I(a) &= \frac{1}{(2\pi)^n} \int D_x^\beta D_y^\gamma e^{i(x-y)\xi} (-D_\xi^\alpha) a(x,y,\xi)\, \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n} \int (-1)^{|\beta|} \xi^{\beta+\gamma} e^{i(x-y)\xi} (-D_\xi^\alpha) a(x,y,\xi)\, \mathrm{d}\xi \end{aligned}$$

where the last integral gives a smooth function, thus showing that $(x-y)^\alpha I(a)$ is smooth for all α, and hence $I(a)$ is smooth away from $x = y$.

Now, for a general $A \in \Psi_\infty^m(\mathbb{R}^n)$, $m \in \mathbb{R}$, we shall use the density of $S_\infty^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n) \subset S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n)$ and that I extends by continuity to a map $I : S_\infty^m(\mathbb{R}^{2n}; \mathbb{R}^n) \to S'(\mathbb{R}^{2n})$ in the topology $S_\infty^{m+\epsilon}(\mathbb{R}^{2n}; \mathbb{R}^n)$ for any $\epsilon > 0$ **??**.

$\square$

Proposition 4.1.6. *Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ for some $m \in \mathbb{R}$ and $u \in C^{-\infty}(\mathbb{R}^n)$, then*

$$\mathrm{singsupp}(Au) \subset \mathrm{singsupp}(u).$$

We call operators that satisfies the above property pseudolocal

Proof. Again we shall prove the contrapositive statement that

$$x \notin \mathrm{singsupp}(u) \implies x \notin \mathrm{singsupp}(Au)$$

Let $u \in S'(\mathbb{R}^n)$ be compactly supported and $x_0 \notin \mathrm{singsupp}(u)$. We can choose $\chi \in S(\mathbb{R}^n)$, (normalised) so that $\chi \equiv 1$ in a neighbourhood of x_0 and that $\chi u \in S(\mathbb{R}^n)$. Observe that

$$Au = A(\chi u + (1-\chi)u) = A(\chi u) + A(1-\chi)u.$$

Since $\chi x u \in S(\mathbb{R}^n) \implies A\chi u \in S(\mathbb{R}^n)$ [**?**], we have that

$$\mathrm{singsupp}(Au) = \mathrm{singsupp}(A(1-\chi)u).$$

Furthermore, we know that $x_0 \notin \mathrm{supp}((1-\chi)u)$. Now, we shall further cut-off near x_0 by choosing a $\phi \in S(\mathbb{R}^n)$ compactly supported away from $\mathrm{supp}(1-\chi)$ and $\phi \equiv 1$ near x_0, i.e.

$$\mathrm{supp}(1-\chi) \cap \mathrm{supp}\phi = \emptyset.$$

We now have an operator $\phi A(1-\chi)$ with kernel

$$\phi(x) K_A(x,y)(1-\phi(y))$$

that vanishes (in particular smooth) on the diagonal. Since we have shown that the singular support of a pseudodifferential operator have to be contained in the diagonal, we conclude that $\phi A(1-\chi)$ is a smoothing operator, and thus $\phi A(1-\chi)u \in C^\infty(\mathbb{R}^n)$ as required. . $\square$

4.1.2 Elliptic, Characteristic, Wavefront sets

We will now define *ellipticity at a point* in phase space which allow up to define various microlocal contructions that focus on localised (conically in phase space) behaviour ΨDO's and distributions.

Definition 4.1.7. A pseudodifferential operator, $A \in \Psi^m_\infty(\mathbb{R}^n), m \in \mathbb{R}$ is **elliptic at a point** $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ if there exist $\epsilon > 0$ such that its left-reduced symbol satisfies the lower bound

$$|\,\sigma_L(A)(x,\xi)\,| \geqslant \epsilon\,\langle\xi\rangle^m$$

in the region

$$\overline{U}_\epsilon = \left\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid |\,x - x_0\,| \leqslant \epsilon, \left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \epsilon, |\,\xi\,| \geqslant 1/\epsilon\right\}$$

where $\widehat{\xi} = \xi/|\,\xi\,|$ for any non-zero $\xi \in \mathbb{R}^n$. We denote the set of all elliptic points of A as

$$Ell^m(A) = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid A \text{ is elliptic of order } m \text{ at } (x,\xi)\}$$

and its complement in $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ as

$$\begin{aligned}\Sigma^m(A) &= Ell^m(A)^c \setminus \{(x,0) \mid x \in \mathbb{R}^n\} \\ &= \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid A \text{ is } \textbf{not} \text{ elliptic of order } m \text{ at } (x,\xi)\}\end{aligned}$$

Lemma 4.1.8. *Let $A \in \Psi^m_\infty(\mathbb{R}^n), m \in \mathbb{R}$.*

1. *If $\sigma_m(A)(x,\xi)$ is homogeneous of degree m in ξ, then*

$$Ell^m(A) = \{(x_0,\xi_0) \mid \xi_0 \neq 0, \sigma_m(A)(x_0,\xi_0) \neq 0\}\,.$$

2. *$Ell^m(A)$ is open in $\mathbb{R}^n \times \mathbb{R}^n$.*
3. *$Ell^m(A)$ is conic in $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$, in the sense that*

$$(x_0,\xi_0) \in Ell^m(A) \implies (x_0, t\xi_0) \in Ell^m(A), \forall t \in \mathbb{R}_{>0}.$$

4. *$\Sigma^m(A)$ is closed conic.*
5. *if $B \in \Psi^{m'}(\mathbb{R}^n)$, then*

$$\Sigma^{m+m'}(A \circ B) = \Sigma^m(A) \cup \Sigma^{m'}(B).$$

Proof. Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $m \in \mathbb{R}$ be given.

1. Suppose the principal symbol $\sigma_m(A)(x,\xi)$ is homogeneous of order m in ξ. We need to show that

$$(x_0,\xi_0) \in Ell^m(A) \iff \xi_0 \neq 0,\ \sigma_m(A)(x_0,\xi_0) \neq 0.$$

If $\xi_0 = 0$, $(x_0,\xi_0) \notin Ell^m_\infty$ by definition of ellipticity. If $\sigma_m(x_0,\xi_0) = 0$, by homogeneity, we have

$$\sigma_m(x_0,t\xi_0) = t^m\sigma_m(x_0,\xi_0) = t^m \cdot 0 = 0$$

for all $t \in \mathbb{R}_{>0}$. By definition of principal symbol, we can write the left symbol of A as

$$\sigma_L(A) = \sigma_m(A) + a$$

where $a \in S^{m-1}_\infty(\mathbb{R}^n;\mathbb{R}^n)$. Now, observe that for any $\epsilon > 0$, the set

$$\overline{U}_\epsilon = \left\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid |\, x - x_0\,| \leqslant \epsilon, \left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \epsilon, |\,\xi\,| \geqslant 1/\epsilon\right\}$$

contains the(open) half-line starting at $\widehat{\xi_0}/\epsilon$, i.e. the set $\{(x_0, t\xi_0/(|\,\xi_0\,|\,\epsilon) \mid t > 0\}$. However, by the symbol estimate of a,

$$\begin{aligned}
\left|\,\sigma_L(A)\left(x_0,\frac{t\xi_0}{|\,\xi_0\,|\,\epsilon}\right)\right| &\leqslant \left(\frac{t}{\epsilon\,|\,\xi_0\,|}\right)^m |\,\sigma_m(x_0,\xi_0)\,| + \left|\,a\left(x_0,\frac{t\xi_0}{|\,\xi_0\,|\,\epsilon}\right)\right| \\
&= 0 + \left|\,a\left(x_0,\frac{t\xi_0}{|\,\xi_0\,|\,\epsilon}\right)\right| \\
&\leqslant C\left\langle\frac{t\xi_0}{|\,\xi_0\,|\,\epsilon}\right\rangle^{m-1} \\
&= C\,\langle t/\epsilon\rangle^{m-1}
\end{aligned}$$

and therefore

$$\begin{aligned}
\inf_{(x,\xi)\in\overline{U}_\epsilon} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} &\leqslant \inf_{t>0} \frac{\left|\,\sigma_L(A)\left(x_0,\frac{t\xi_0}{|\,\xi_0\,|\,\epsilon}\right)\right|}{\langle t/\epsilon\rangle^m} \\
&\leqslant \inf_{t>0} \frac{C\,\langle t/\epsilon\rangle^{m-1}}{\langle t/\epsilon\rangle^m} \\
&= C \inf_{t>0} \langle t/\epsilon\rangle^{-1} \\
&= 0
\end{aligned}$$

which means that $(x_0,\xi_0) \notin Ell^m(A)$.

Conversely, if $\sigma_m(A)(x_0,\xi_0) \neq 0$, by continuity and homogeneity, $\sigma_m(A)$, is non-zero in a (closed) conic neighbourhood, i.e. there exist $\epsilon > 0$ such that $\sigma_m(A) \neq 0$ in

$$\overline{U}_\epsilon = \left\{(x,\xi) \mid |\, x - x_0\,| \leqslant \epsilon, \left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \epsilon, |\,\xi\,| \geqslant 1/\epsilon\right\}.$$

Again, writing the left symbol as a sum of the principal symbol an a lower order term, we observe that in $\overline{U}_\epsilon$,

$$\begin{aligned}\frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} &\geqslant \frac{|\,|\,\sigma_m(A)(x,\xi)\,| - |\,a(x,\xi)\,|\,|}{\langle\xi\rangle^m} \\ &= \left| \frac{|\,\xi\,|^m}{\langle\xi\rangle^m} \left|\sigma_m(A)(x,\widehat{\xi})\right| - \frac{|\,a(x,\xi)\,|}{\langle\xi\rangle^m} \right|\end{aligned}$$

By the symbol estimate of a, the second term is tending to 0 which the first term is bounded below by $C = \inf_{(x,\xi)\in\overline{U}_\epsilon} |\,\sigma_m(A)(x,\xi)\,| > 0$. Therefore, choosing a smaller ϵ if necessary, we have $|\,a(x,\xi)\,| \,/\, \langle\xi\rangle^m < C$ and thus

$$\inf_{(x,\xi)\in\overline{U}_\epsilon} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} \geqslant C' \geqslant \epsilon.$$

and therefore $(x_0,\xi_0) \in Ell^m(A)$.

2. We note first that if the principal symbol is homogeneous of degree m, the previous result applies and continuity of the principal symbol guarantee that the elliptic set is open (if $\sigma_m(A)$ is non-zero at a point, it is non-zero in an open neighbourhood of the point).

   For the general case, suppose $(x_0,\xi_0) \in Ell^m(A)$. We therefore have for some $\epsilon > 0$,

$$|\,\sigma_L(A)(x,\xi)\,| \geqslant \epsilon\,\langle\xi\rangle^m$$

   in the region

$$\overline{U}_\epsilon(x_0,\xi_0) = \left\{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\} \;\middle|\; |\,x-x_0\,| \leqslant \epsilon, \left|\widehat{\xi}-\widehat{\xi_0}\right| \leqslant \epsilon, |\,\xi\,| \geqslant 1/\epsilon\right\}.$$

   It suffice to show that there is an open neighbourhood of (x_0,ξ_0) where A remains elliptic. We can take desired open neighbourhood to be

$$V = \left\{(x',\xi')\,|\,\xi'\neq 0,\ |\,x'-x_0\,| < \epsilon/2,\ \left|\widehat{\xi'}-\widehat{\xi_0}\right| < \epsilon/2\right\}.$$

   Then, we can check that for every $(x',\xi')\in V$, A satisfies the elliptic estimate in $\overline{U}_{\epsilon/2}(x',\xi')$. Indeed, if $(x,\xi)\in\overline{U}_{\epsilon/2}(x',\xi')$, then

$$\begin{aligned}&|\,x-x_0\,| \leqslant |\,x-x'\,| + |\,x'-x_0\,| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon \\ &\left|\widehat{\xi}-\widehat{\xi_0}\right| \leqslant \left|\widehat{\xi}-\widehat{\xi'}\right| + \left|\widehat{\xi'}-\widehat{\xi_0}\right| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon \\ &|\,\xi\,| \geqslant 2/\epsilon \geqslant 1/\epsilon\end{aligned}$$

   which shows that $\overline{U}_{\epsilon/2}(x',\xi') \subset \overline{U}_\epsilon(x_0,\xi_0)$. Therefore,

$$\inf_{(x,\xi)\in\overline{U}_{\epsilon/2}(x',\xi')} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} \geqslant \inf_{(x,\xi)\in\overline{U}_\epsilon(x_0,\xi_0)} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} \geqslant \epsilon \geqslant \epsilon/2$$

   as required.

3. Again, this result is immediate if the principal symbol is homogeneous in ξ. In general, this result come from the observation that only $\widehat{\xi} = \xi / \left| \xi \right|$ appears in $\overline{U}_\epsilon$ in the definition of $Ell^m(A)$, i.e. only the *direction* in the dual variable is important.

   Explicitly, let $(x_0, \xi_0) \in Ell^m(A)$ and $t \in \mathbb{R}_{>0}$. Clearly $t\xi_0 \neq 0$. And note that

$$\overline{U}_\epsilon(x_0, \xi_0) = \overline{U}_\epsilon(x_0, t\xi_0)$$

   since $\widehat{\xi} = \widehat{t\xi}$.

4. $\Sigma^m(A) = Ell^m(A)^c$ where $Ell^m(A)$ is open and conic. Since complement of conic set is conic, and complement of open is closed, we conclude that $\Sigma^m(A)$ is closed conic.

5. If both principal symbols are homoegenous of degree m, m' respectively, we can applied the result above and by symbol calculus, we have

$$\begin{aligned} Ell^{m+m'}(A \circ B) &= \{(x, \xi) \mid \xi \neq 0, \sigma_{m+m'}(A \circ B) = \sigma_m(A)\sigma_{m'}(B) \neq 0\} \\ &= \{(x, \xi) \mid \xi \neq 0, \sigma_m(A) \neq 0\} \cap \{(x, \xi) \mid \xi \neq 0, \sigma_{m'}(B) \neq 0\} \\ &= Ell^m(A) \cap Ell^{m'}(B). \end{aligned}$$

   Taking complement give the desired result.

   In general,

□

Definition 4.1.9. The **wavefront set** of a compactly supported tempered distribution

$$u \in C_c^{-\infty}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) \mid \operatorname{supp}(u) \Subset \mathbb{R}^n\}$$

is given by

$$\mathrm{WF}(u) = \bigcap \left\{ \Sigma^0(A) \mid A \in \Psi^0_\infty(\mathbb{R}^n), Au \in C^\infty(\mathbb{R}^n) \right\}.$$

For general tempered distribution $u \in S'(\mathbb{R}^n)$, its wavefront set is given by

$$\mathrm{WF}(u) = \bigcup_{\chi \in C_c^\infty(\mathbb{R}^n)} \mathrm{WF}(\chi u).$$

Proposition 4.1.10. *For compactly supported tempered distribution, $u \in C_c^{-\infty}(\mathbb{R}^n)$,*

$$\pi(\mathrm{WF}(u)) = \operatorname{singsupp}(u).$$

where $\pi(x, y) = x$ is the projection map.

Proof. To show $\pi(\mathrm{WF}(u)) \subset \mathrm{singsupp}(u)$, we observe that, by definition of singular support,

$$x_0 \notin \mathrm{singsupp}(u) \implies \exists \phi \in S(\mathbb{R}^n),\ \phi(x_0) \neq 0,\ \phi u \in S(R^n).$$

But since multiplication by ϕ gives an operator in $\Psi^0_\infty(\mathbb{R}^n)$ which is elliptic at (x_0, ξ) for any $\xi \neq 0$ (ϕ is its own principal symbol which happens to be homogeneous and non-zero for any $(x_0, \xi), \xi \neq 0$). Therefore, $x_0 \notin \pi(\mathrm{WF}(u))$.

Conversely, if $x_0 \notin \pi(\mathrm{WF}(u))$, then for all $\xi \neq 0$, there exist $A_\xi \in \Psi^0_\infty(\mathbb{R}^n)$ such that A_ξ is elliptic at (x_0, ξ) and $A_\xi u \in C^\infty(\mathbb{R}^n)$. Since elliptic set $Ell^0(A_\xi)$ is open and conic, we know that there exist $\epsilon = \epsilon(\xi)$ such that A_ξ is elliptic in the open conic set

$$V_\xi = \left\{(x', \xi') \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \mid |\, x' - x_0\,| < \epsilon, \left|\widehat{\xi'} - \widehat{\xi}\right| < \epsilon\right\}.$$

Compactness of the sphere (note that $\xi' \mapsto \widehat{\xi'}$ is an embedding of $\mathbb{R}^n \setminus \{0\}$ into S^n) allow us to cover $\{x_0\} \times (\mathbb{R}^n \setminus \{0\})$ with finite number of $V_{\xi_j}, j = 1, \ldots, N$ with corresponding operators A_{ξ_j}.
Now, consider the operator

$$A = \sum_{j=1}^{N} A^*_{\xi_j} A_{\xi_j}.$$

By the pseudolocality of pseudodifferential operators, we know that $A_{\xi_j} u \in C^\infty(\mathbb{R}^n) \implies A^*_{\xi_j} A_{\xi_j} u \in \mathbb{C}^\infty(\mathbb{R}^n)$. Therefore, $Au \in C^\infty(\mathbb{R}^n)$ and A is elliptic at $(x_0, \xi), \forall \xi \neq 0$ with non-negative symbol. We can pick a smooth cut-off χ, $\chi \equiv 1$ when resticted to an $\epsilon/2$-ball around x_0 forming an operator

$$A + (1 - \chi) \in \Psi^0_\infty(\mathbb{R}^n)$$

that is globally elliptic. We can construct a (global) elliptic parametrix E so that

$$1 - (E \circ A + E(1 - \chi)) \in \Psi^{-\infty}_\infty(\mathbb{R}^n).$$

Finally, we can choose any smooth cut-off ϕ with support subordinate to that of χ, i.e. $\mathrm{supp}(\phi) \subset \mathrm{supp}(\chi)$ and note that

$$\phi \circ E \circ (1 - \chi) \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$$

making it a smoothing operator []. Thus, we conclude

$$\phi u = \phi E \circ Au + \phi \circ E \circ (1 - \chi) u \in C^\infty(\mathbb{R}^n)$$

as required.

□

Definition 4.1.11. Let $a \in S^m_\infty(\mathbb{R}^p; \mathbb{R}^n)$ for some $m \in \mathbb{R}$, $p, n \in \mathbb{N}$ be a symbol. We say a is of order $-\infty$ at a point $(x_0, \xi_0) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\}$ (write $a = O(\langle\xi\rangle^{-\infty})$) if there exist $\epsilon \in \mathbb{R}_{>0}$ such that for all $M \in \mathbb{R}$, there is a constant $C_M > 0$ such that

$$|\, a(x,\xi)\,| \leqslant C_M \, \langle\xi\rangle^{-M}$$

in the neigbourhood of (x_0, ξ_0) given by

$$\overline{U}_{(x_0,\xi_0)} = \left\{(x,\xi) \in \mathbb{R}^p \times \mathbb{R}^n \,\middle|\, |\, x - x_0\,| \leqslant \epsilon, \left|\, \widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \epsilon\right\}.$$

We define the cone support of the symbol a to be all the points in phase space that where it fails to be $O(\langle\xi\rangle^{-\infty})$.

$$\operatorname{conesupp}(a) = \left\{(x,\xi) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\} \mid a = O(\langle\xi\rangle^{-\infty}) \text{ at } (x,\xi)\right\}^c.$$

Lemma 4.1.12. *Let $a \in S^\infty_\infty(\mathbb{R}^p; \mathbb{R}^n)$, then*

1. conesupp(a) *is a closed conic set in* $\mathbb{R}^p \times \mathbb{R}^n$.
2. *If $a = O(\langle\xi\rangle^{-\infty})$ at $(x_0,\xi_0) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\}$, then so is $D^\alpha_x D^\beta_\xi a(x,\xi)$ for any multi-index α, β*

With the lemma above, we can in fact define the cone support of a symbol to be the smallest closed conic subset of the phase space (with $\xi \neq 0$) such that, in the complement, a and all its derivatives are of order $-\infty$.

Definition 4.1.13. Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $m \in \mathbb{R}$ be pseudodifferential operator. We define the **essential support**, $\mathrm{WF}'(A)$, of A to be the cone support of its left symbol, i.e.

$$\mathrm{WF}'(A) = \operatorname{conesupp}(\sigma_L(A)) \subset \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$$

Using symbol calculus we can prove the following result.

Lemma 4.1.14. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$ be pseudifferential operators. Then*

1. $\mathrm{WF}'(A) = \operatorname{conesupp}(\sigma_R(A))$.
2. $\mathrm{WF}'(A \circ B) \subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B)$.
3. $\mathrm{WF}'(A + B) = \mathrm{WF}'(A) \cup \mathrm{WF}'(B)$.

With the concept of essentail support we can define the notion of *microlocal elliptic parametrix* which can be thought of as local inverse at an elliptic point of ΨDO's.

Proposition 4.1.15. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ and $z \notin \Sigma^m(A)$. Then there exist a (two-sided) microlocal parametrix $B \in \Psi^{-m}(\mathbb{R}^n)$ such that*

$$z \notin \mathrm{WF}'(1 - AB) \text{ and } z \notin \mathrm{WF}'(1 - BA).$$

Proof. Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ is elliptic at $(x_0, \xi_0) \in \mathit{Ell}^m(A)$. For each $\epsilon \in \mathbb{R}_{>0}$ we define

$$\gamma_\epsilon(x,\xi) = \chi\left(\frac{x - x_0}{\epsilon}\right)(1 - \chi(\epsilon\xi))\chi\left(\frac{\widehat{\xi} - \widehat{\xi_0}}{\epsilon}\right)$$

where $\chi \in C^\infty(\mathbb{R}^n)$ is a smooth cut-off function that is identically 1/2-ball but supported only in the unit ball. Notice that $\gamma_\epsilon \in S^0_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$ with support given by

$$\mathrm{supp}(\gamma_\epsilon) \subset \left\{(x,\xi) \mid |\, x - x_0 \,| \leqslant \epsilon,\ |\, \xi \,| \geqslant \frac{1}{2\epsilon},\ \left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \epsilon\right\}.$$

Restricted to the conic set

$$\overline{U}_\epsilon = \left\{(x,\xi) \mid |\, x - x_0 \,| \leqslant \frac{\epsilon}{2},\ \left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \frac{\epsilon}{2},\ |\, \xi \,| \geqslant \frac{1}{\epsilon}\right\} \subset \mathrm{supp}(\gamma_\epsilon)$$

it is identically 1 and therefore γ_ϵ is elliptic at (x_0, ξ_0). Let $L_\epsilon = \mathrm{Op}_L(\gamma_\epsilon)$ be the corresponding pseudodifferential operator. Observe that, by construction

$$(x_0, \xi_0) \notin \mathrm{WF}'(1 - L_\epsilon)$$

since $1 - \gamma_\epsilon$ is supported away from an ϵ-neighbourhood of $x = x_0$ and the wavefront set of L_ϵ is contained in an ϵ-neighbourhood of (x_0, ξ_0), i.e.

$$\mathrm{WF}'(L_\epsilon) \subset N_\epsilon(x_0,\xi_0) := \left\{(x,\xi) \mid |\, x - x_0 \,| \leqslant \epsilon,\ \left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \epsilon\right\}$$

since γ_ϵ is bounded below in some conic neighbourhood of every point in $N_\epsilon(x_0, \xi_0)$.

Now, let $G_s = \mathrm{Op}_L(\langle\xi\rangle^s)$ for each $s \in \mathbb{R}$. Note that G_s is globally elliptic with positive principal symbol. Consider the following operator,

$$J = (1 - L_\epsilon) \circ G_{2m} + A^*A \in \Psi^{2m}_\infty(\mathbb{R}^n)$$

with principal symbol

$$\sigma_{2m}(J) = (1 - \gamma_\epsilon)\,\langle\xi\rangle^{2m} + |\,\sigma_m(A)\,|^2\,.$$

Since $\mathit{Ell}^m(A)$ is open conic, we can choose ϵ is small enough so that $\mathit{Ell}^m(A) \subset \mathrm{supp}(\gamma_\epsilon)$. Then, by positivity of all terms involved, we have

$$\frac{|\,\sigma_{2m}(J)\,|}{\langle\xi\rangle^{2m}} = (1 - \gamma_\epsilon) + \frac{|\,\sigma_m(A)\,|^2}{\langle\xi\rangle^{2m}}$$

where the first term is bounded below by 1 outside of $\mathrm{supp}(\gamma_\epsilon)$ while in $\mathrm{supp}(\gamma_\epsilon)$ the second term is bounded below by ϵ since A is elliptic (of order m) at every point in $\mathrm{supp}(\gamma_\epsilon)$. Therefore J is globally elliptic and thus have a global elliptic parametrix $H \in \Psi_\infty^{-2m}(\mathbb{R}^n)$. We shall claim that

$$B = H \circ A^* \in \Psi_\infty^m(\mathbb{R}^n)$$

is a (right) microlocal elliptic parametrix to A. Indeed,

$$\begin{aligned} B \circ A - 1 &= HA^*A - 1 \\ &= H\left(J - (1 - L_\epsilon)G_{2m}\right) - 1 \\ &= (HJ - 1) - H(1 - L_\epsilon)G_{2m}. \end{aligned}$$

Since H is a global parametrix to J, the first term above is a smoothing operator (i.e. an element of $\Psi_\infty^{-\infty}(\mathbb{R}^n)$ and therefore have empty wavefront set. Furthermore, by [?] the wavefront set of the second term is a subste of $\mathrm{WF}'(1 - L_\epsilon)$ which does not contain (x_0, ξ_0) by construction. □

Proposition 4.1.16. *Pseudodifferential operators are microlocal in the following sense: Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ and $u \in C_c^{-\infty}(\mathbb{R}^n)$, then*

$$\mathrm{WF}(Au) \subset \mathrm{WF}(u). \tag{4.1}$$

In fact, we have

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}(u).$$

Proof. □

A partial converse to the above is given by the following proposition.

Proposition 4.1.17. *Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ and $u \in C_c^{-\infty}(\mathbb{R}^n)$, then*

$$\mathrm{WF}(u) \subset \mathrm{WF}(Au) \cup \Sigma^m(A).$$

Chapter 5

Fredholm problem of wave operator on torus

Let $\mathbb{T} = [0,1]/0 \sim 1$ denote the torus and $\mathbb{T}^n$ the n-dimensional torus[1]. We shall study the d'Alembertian, i.e. the totally periodic wave operator, on $\mathbb{T}^n = \mathbb{T}_t \times \mathbb{T}_x^{n-1}$

$$\Box := \partial_t^2 - \sum_{j=1}^{n-1} \partial_{x_j}^2. \tag{5.1}$$

We first note that the symbol of the operator,

$$\sigma(\Box) = \tau^2 - (\xi_1^2 + \xi_2^2 + \cdots + \xi_{n-1}^2) =: \tau^2 - |\xi|^2$$

is 0 precisely on the light cone $L = \{|\tau| = |\xi|\}$. The operator is therefore not elliptic everywhere in $\mathbb{T}^n$. We shall proceed by using the "complex absorption" method, i.e. we will perturb the operator by some operator $-iQ$ so that $\Box - iQ$ is elliptic on a "large" enough subset of $\mathbb{T}^n$. Specifically, we can take

$$Q = \chi(t)\partial_t^2 \tag{5.2}$$

where $\chi : \mathbb{T}^n \to \mathbb{R}_{\geqslant 0}$ is a smooth cut-off function supported away from $(-\delta+1/2, \delta+1/2)$ for some $\delta \in (0, 1/2)$. Our goal will to prove the following theorem,

Theorem 5.0.1. *Let $s \in \mathbb{R}$ be given and define*

$$\chi^s = \left\{ u \in H^s(\mathbb{T}^n) \,|\, (\Box - iQ)u \in H^{s-1}(\mathbb{T}^n) \right\}.$$

Then, the operator,

$$(\Box - iQ) : \chi^s \to H^{s-1}(\mathbb{T}^n)$$

is a Fredholm operator.

[1] we shall variously use, without comment, the identifications $\mathbb{T} \cong S^1 \cong \mathbb{R}/\mathbb{Z}$ and $\mathbb{T}^n \cong S^1 \times S^1 \times \cdots \times S^1 \cong \mathbb{R}^n/\mathbb{Z}^n$

Lemma 5.0.2 (Riez's inequality). *Let X be a normed linear space. Given a non-dense subspace (or closed proper subspace) $Y \subset X$ and any $r \in (0,1)$, then there exist $x \in X$ with $\|x\| = 1$ such that*

$$\inf_{y \in Y} \|x - y\| \geqslant r.$$

Theorem 5.0.3. *Let X, Y be Hilbert spaces and $T : X \to Y \in \mathcal{L}(X,Y)$ be a continuous (therefore bounded) linear operator. Suppose T satisfies*

$$\begin{aligned} \forall u \in X, \quad \|u\|_X &\leqslant C\left(\|Tu\|_Y + \|u\|_Z\right) \\ \forall v \in Y, \quad \|v\|_Y &\leqslant C'\left(\|T^* v\|_X + \|v\|_{Z^*}\right) \end{aligned}$$

where $Z \Subset X$ and $Z^* \Subset Y$ are compact subsets, then T is Fredholm, i.e. $T(X)$ is closed in Y and both $\mathrm{ker}T, \mathrm{coker}T$ *are finite dimensional.*

proof sketch.

□

Appendix A

Appendix