1 Polyhomogeneity of Riemann Map for polygonal region

Theorem 1.1 (Riemann Mapping Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a simply connected region which is not the whole plane and $z_0 \in \Omega$. There exists a unique one-to-one analytic function $f : \Omega \to D$, with $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$ being the open unit disk, such that $f(z_0) = 0$ and $f'(z_0) > 0$.*

It can also be shown that if the boundary, $\partial\Omega$ of the region is a Jordan Curve, the Riemann Map can be extended to an analytic one-to-one function on $\overline{\Omega}$ onto the closed unit disk, i.e. $f : \overline{\Omega} \to \overline{D}$. When extended, map $f : \Omega \to D$, simply by virtue of being a topological map (i.e. homeomorphism), will map boundary to boundary.

1.1 Riemann Map for polygonal region

In this section we shall exhibit an explicit formula for the (inverse of) Riemann map for a polygonal region $\Omega \subset \mathbb{C}$. An n-gon can be specified by an ordered sequence of n distinct complex numbers $(z_k)_{1 \leqslant k \leqslant n}$. We shall let $(\alpha_k\,\pi)_{1 \leqslant k \leqslant n}$ denote the interior angles at z_k, and $(\beta_k\,\pi)$ the corresponding exterior angles. Since the (extended) Riemann Map will map boundary to boundary, the points z_k will be mapped to $w_k \in S^1 \subset \overline{D}$. With these notations in place, we shall give the following formula for the conformal of Ω to D.

Theorem 1.2 (Schwarz-Christoffel Formula)**.** *The function $z = F(w)$ which map D, the open unit disk, conformally onto an n-gon defined by $(z_k)_{1 \leqslant k \leqslant n}$ with exterior angles $(\beta_k\,\pi)_{1 \leqslant k \leqslant n}$ is given by*

$$F(w) = C \int_0^w \prod_{k=1}^{n} (\eta - w_k)^{-\beta_k}\, d\eta + C' \tag{1}$$

for some $C, C' \in \mathbb{C}$, with $z_k = \lim_{w \to w_k} F(w)$.

1.2 Polyhomegeity

In order to understand the behaviour of the conformal map as we approach a corner of the polygon, we shall seek asymptotic expansion of the map F in terms of r, the distance from a particular $w \in \{w_1, w_2, \ldots, w_n\}$. Rename the points w_k if necessary, we may assume $w = w_1$ and $\beta = \beta_1 \in (-1, 1)$. Let $I(\omega)$ denote the integral in the expression of F. Observe that for $\alpha \in D$

$$I(\alpha) = \int_0^\alpha (\eta - w)^{-\beta} \prod_{k=2}^{n} (\eta - w_k)^{-\beta_k}\, d\eta.$$

Let $\epsilon > 0$ be the minimum distance between w and w_k, $k \in \{2, 3, \ldots, n\}$, we know that the product

$$p(\eta) = p(\eta; w_2, \ldots, w_n, \beta_2, \ldots, \beta_n) = \prod_{k=2}^{n} (\eta - w_k)^{-\beta_k}$$

is holomorphic in the domain $B_\epsilon(w) = \{z \,|\, |z - w| < \epsilon\}$, and thus have an absolutely and uniformly convergent Taylor expansion around $\eta = w$ given by

$$p(\eta) = \sum_{m=0}^{\infty} a_m (\eta - w)^m. \tag{2}$$

In other words, the radius of convergence of (**??**) is precisely ϵ since it is the distance to the nearest branch point.

Now, fix $a \in D \cup B_\epsilon(w) \neq \emptyset$ and observe that for any $\alpha(r,\theta) = w + re^{i\theta}$, $r \in (0,\epsilon)$, we have

$$
\begin{aligned}
I(\alpha) &= I(a) + \int_a^\alpha (\eta - w)^{-\beta} \sum_{m=0}^{\infty} a_m (\eta - w)^m \, d\eta \\
&= I(a) + \int_a^\alpha \sum_{m=0}^{\infty} a_m (\eta - w)^{m-\beta} \, d\eta \\
&= I(a) + \sum_{m=0}^{\infty} a_m \int_a^\alpha (\eta - w)^{m-\beta} \, d\eta \\
&= I(a) + \sum_{m=0}^{\infty} \frac{a_m}{m-\beta+1} \left[(\alpha - w)^{m-\beta+1} - (a - w)^{m-\beta+1} \right]
\end{aligned}
$$

where the swapping of integral and infinite sum is justified by Lebesgue dominated convergence theorem [1]

[1] For an elementary argument, observe that since $\beta \in (-1,1)$, for $m \geqslant 1$, the integrand has a holomorphic branch