1 Polyhomogeneous functions

Since we are interested in "singular" behaviours, the space of smooth functions is too restrictive. The space of polyhomogeneous functions $\mathcal{A}^E(M)$ of a mwc form a broad class of "nice" function suitable for the study of differential operators and functions on the space. Roughly, in analogy with the cases with smooth or analytic functions with expansion in powers of x, polyhomogeneous functions are functions that has expansions in terms of the form $x^z log^n x$ near a corner point in a mwc.

1.1 In model spaces $\mathbb{R}^n_k$

We shall first study the notion of polyhomogeneity in a simple model space $\mathbb{R}^2_2 = \mathbb{R}^2_+$. It is straight forward to generalise to $\mathbb{R}^n_k$ and from there generalising (locally using charts) to mwcs.

Definition 1.1 (Polyhomogeneous function on $\mathbb{R}^2_+$)**.** Set $M = \mathbb{R}^2_+$ to be the manifold with corners with and $H = \partial M$ be the boundary hypersurfaces and M^o the interior.

1. An **index set** is a discrete (in the product topology) set $E \subset \mathbb{C} \times \mathbb{N}$ such that for every $N \in \mathbb{R}$, the set $\{(z, p) \in E \,|\, \Re z < N\}$ is finite.
2. A function $f : M^0 \to \mathbb{R}$ is said to have asymptotic expansion in x as $x \to 0$ with index set E, i.e.

$$f(x, y) \sim \sum_{(z,p)\in E} a_{z,p}(y) x^z \log^p x$$

   if for all $N \in \mathbb{R}$, $\alpha, \beta \in \mathbb{N}$, there exist, uniformly for every compact subset of $\mathbb{R}_+$, a constant C that depends only on N, α, β, such that

$$\left| (x\partial_x)^\alpha \partial_y^\beta \left(f(x, y) - \sum_{\substack{(z,p)\in E \\ \Re z \leqslant N}} a_{z,p}(y) x^z \log^p x \right) \right| \leqslant C_{\alpha,\beta,N} x^N$$

3. Given an index sets E, F, a function $f : M^0 \to \mathbb{R}$ is **polyhomogeneous** with respect to E, F (denote $f \in \mathcal{A}^{E,F}(M)$) if $f \in C^\infty(M^0)$, if
   (a) f is smooth on the interior, $f \in C^\infty(M^0)$,
   (b) $\forall y > 0, f$ has asymptotic expansion in as $x \to 0$ with respect to E of the form

$$f \sim \sum_{(z,p)\in E} a_{z,p}(y) x^z \log^p x,$$

   (c) $\forall x > 0, f$ has asymptotic expansion as $y \to 0$ with respect to F of the form

$$f \sim \sum_{(w,k)\in E} b_{w,k}(y) y^w \log^k y,$$

   and
   (d) $\forall (z, p) \in E, a_{z,p} \in \mathcal{A}^F(\mathbb{R}_+)$ and $\forall (w, k) \in F, b_{w,k} \in \mathcal{A}^E(\mathbb{R}_+)$,

   where polyhomogeneity in $\mathbb{R}_+$, i.e. in defining $\mathcal{A}^E(\mathbb{R}_+)$, we have analogous definition with the coefficients in the asymptotic expansion replaces with constants in $\mathbb{R}$.

An important point to note in the above definition is that all the coefficients in the expansions lies in the *same singularity class*. This means that the euclidean norm function $r(x, y) = \sqrt{x^2 + y^2}$ is *not* polyhomogeneous because, when we take asymptotic expansion in x as $x \to 0$ [1]

$$r(x, y) = y\sqrt{1 + (x/y)^2} = \sum_{n=0}^{\infty} c_n y^{1-2n} x^{2n}$$

we find that the coefficient functions in y become more and more singular as $n \to \infty$. Since condition 1 in the definition above requires that the index set has bounded negative real part for z, r cannot be polyhomogeneous.

[1] Taylor theorem guarantee uniqueness of expansion

2 Blow up and resolution

Blow up can be informally describe as a coordinate independent way of introducing polar coordinate near a corner point in a mwc. The reason for such a construction is that we want to understand the singular behaviour of operators and maps near the corner points on mwc's by "looking through", i.e. resolve, them as less singular (e.g. smooth) objects in the blow up space. This way, we can appeal to and therefore focus on the well-studied theory of "nice" or smooth functions.

Definition 2.1 (Blow up)**.** Let M be a mwc and $S \subset M$ be a p-submanifold. The blow up, $[M, S]$ of M along S is locally given by the following construction. In coordinate (of M), the pair $(M, S) \cong (\mathbb{R}^n, \mathbb{R}^k \times \{0\})$. Thus, the blow up is locally modelled by the blow up of the model spaces

$$[\mathbb{R}^n, \mathbb{R}^k \times \{0\}^{n-k}] = \mathbb{R}^k \times [\mathbb{R}^{n-k}, 0]$$

where $[\mathbb{R}^j, 0] = [0, \infty)_r \times S^{j-1}$