Chapter 3: Ellipticity and Microlocalisation

1 Microlocalisation

Roughly, the support of a distribution in $\mathbb{R}^n$ consist of points $x \in \mathbb{R}^n$ where the distribution is non-zero after any smooth cut-offs near x.

Definition 1.1. The **support of a tempered distribution** $u \in S'(\mathbb{R}^n)$ is given by the set

$$\text{supp}(u) = \{x \in \mathbb{R}^n \mid \exists \phi \in S(\mathbb{R}^n), \phi(x) \neq 0, \phi u = 0\}^c .$$

And the singular support of a distribution is the set of points where the distribution fails to behave like an element of $S(\mathbb{R}^n)$.

Definition 1.2. The **singular support of a tempered distribution** $u \in S'(\mathbb{R}^n)$ is given by the set

$$\text{singsupp}(u) = \{x \in \mathbb{R}^n \mid \exists \phi \in S(R^n), \phi(x) \neq 0, \phi(u) \in S(\mathbb{R}^n)\}^c$$

Note that if the singular support of a tempered distribution is empty, it is then an element of $C^\infty(\mathbb{R}^n)$ The support of an operator is given by the support of its Schwartz kernel.

Definition 1.3. The **support of a continuous linear operator** $A : S(R^n) \to S'(\mathbb{R}^n)$ is given by

$$\text{supp}(A) = \text{supp}(K_A) \subset \mathbb{R}^n \times \mathbb{R}^n$$

where $K_A \in S'(\mathbb{R}^n \times \mathbb{R}^n)$ is the Schwartz kernel of A.

We note from the above that supports or singular supports are complement of open sets, therefore they are closed. We have the following result relating the support of a smooth function after the action of a continuous linear operator.

Proposition 1.4 (Calculus of support)**.** *Let $A : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ be a continuous linear operator and $\phi \in C_c^\infty(\mathbb{R}^n)$, then*

$$\text{supp}(A\phi) \subset \text{supp}(A) \circ \text{supp}(\phi) := \{x \in \mathbb{R}^n \mid \exists y \in \text{supp}(\phi), (x, y) \in \text{supp}(A)\} .$$

Proof. We shall show the contrapositive statement:

$$x \notin \operatorname{supp}(A) \circ \operatorname{supp}(\phi) \implies x \notin \operatorname{supp}(A\phi).$$

Suppose $x \notin \operatorname{supp}(A) \circ \operatorname{supp}(\phi)$. Observe that

$$\operatorname{supp}(A) \circ \operatorname{supp}(\phi) = \pi_x(\pi_y^{-1}(\operatorname{supp}(\phi)) \cap \operatorname{supp}(A))$$

where $\pi_{x,y} : \mathbb{R}^2 \to \mathbb{R}$ are the projection map to the respective coordinates. Since $\operatorname{supp}(A)$ is closed and $\operatorname{supp}(\phi)$ is compact, we have that $\operatorname{supp}(A) \circ \operatorname{supp}(\phi)$ is closed and thus x belongs to an open set. We can therefore choose a smooth cutt-off function $\chi \in C_c^\infty(\mathbb{R}^n)$ supported at x ($\chi(x) \neq 0$) but away from $\operatorname{supp}(A) \circ \operatorname{supp}(\phi)$. Thus,

$$\operatorname{supp}(A) \cap (\operatorname{supp}(\chi) \times \operatorname{supp}(\phi)) = \emptyset$$

and hence $\chi(x)K_A(x,y)\phi(y) = 0 \implies \chi A\phi = 0$, as required. □

1.1 Pseudolocality

We shall show two results on the singular support of pseudodifferential operators. The first result is that the singular support of any ΦDO is contained within the diagonal, i.e. they are smooth away from $x = y$. The second result is teh pseudolocality result that says that action ΨDO's do not increase singular support of distributions.

Proposition 1.5. *Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ for some $m \in \mathbb{R}$, then*

$$\operatorname{singsupp}(A) \subset \left\{(x,y) \in \mathbb{R}^{2n} \,|\, x = y\right\}.$$

Proof. We shall prove this theorem for elements of $\Psi_\infty^{-\infty}(\mathbb{R}^n)$ and then extend by continuity to all orders. Let $A \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ with symbol $a \in S_\infty^{-\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$. Its singular support is given by the singular support of the kernel. Since all derivatives of a are $O(\langle\xi\rangle^{-\infty})$, the oscillatory integral below representing the kernel is absolutely convergent and, using integration by parts, we get

$$\begin{aligned} I(a) &= \frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}a(x,y,\xi)\,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n}\int \frac{1}{(x-y)^\alpha}(-D_\xi^\alpha)\left(e^{i(x-y)\xi}\right)a(x,y,\xi)\,\mathrm{d}\xi \\ &= \frac{1}{(x-y)^\alpha}\frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}(-D_\xi^\alpha)a(x,y,\xi)\,\mathrm{d}\xi \\ &= \frac{1}{(x-y)^\alpha}I((-D_\xi^\alpha)a) \end{aligned}$$

which is true for all multi-index α of any order. Since all x,y-derivatives of a are uniformly bounded by $\langle\xi\rangle^{-N}$ for any $N \in \mathbb{N}$, we can differentiate under the integral sign to get the equation

$$\begin{aligned} D_x^\beta D_y^\gamma (x-y)^\alpha I(a) &= \frac{1}{(2\pi)^n}\int D_x^\beta D_y^\gamma e^{i(x-y)\xi}(-D_\xi^\alpha)a(x,y,\xi)\,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n}\int (-1)^{|\beta|}\xi^{\beta+\gamma}e^{i(x-y)\xi}(-D_\xi^\alpha)a(x,y,\xi)\,\mathrm{d}\xi \end{aligned}$$

where the last integral gives a smooth function, thus showing that $(x-y)^\alpha I(a)$ is smooth for all α, and hence $I(a)$ is smooth away from $x=y$.

Now, for a general $A \in \Psi_\infty^m(\mathbb{R}^n)$, $m \in \mathbb{R}$, we shall use the density of $S_\infty^{-\infty}(\mathbb{R}^{2n};\mathbb{R}^n) \subset S_\infty^m(\mathbb{R}^{2n};\mathbb{R}^n)$ and that I extends by continuity to a map $I : S_\infty^m(\mathbb{R}^{2n};\mathbb{R}^n) \to S'(\mathbb{R}^{2n})$ in the topology $S_\infty^{m+\epsilon}(\mathbb{R}^{2n};\mathbb{R}^n)$ for any $\epsilon > 0$ **??**.

□

Proposition 1.6. *Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ for some $m \in \mathbb{R}$ and $u \in C^{-\infty}(\mathbb{R}^n)$, then*

$$\operatorname{singsupp}(Au) \subset \operatorname{singsupp}(u).$$

We call operators that satisfies the above property pseudolocal

Proof. Again we shall prove the contrapositive statement that

$$x \notin \operatorname{singsupp}(u) \implies x \notin \operatorname{singsupp}(Au)$$

Let $u \in S'(\mathbb{R}^n)$ be compactly supported and $x_0 \notin \operatorname{singsupp}(u)$. We can choose $\chi \in S(\mathbb{R}^n)$, (normalised) so that $\chi \equiv 1$ in a neighbourhood of x_0 and that $\chi u \in S(\mathbb{R}^n)$. Observe that

$$Au = A(\chi u + (1-\chi)u) = A(\chi u) + A(1-\chi)u.$$

Since $\chi x u \in S(\mathbb{R}^n) \implies A\chi u \in S(\mathbb{R}^n)$ [**?**], we have that

$$\operatorname{singsupp}(Au) = \operatorname{singsupp}(A(1-\chi)u).$$

Furthermore, we know that $x_0 \notin \operatorname{supp}((1-\chi)u)$. Now, we shall further cut-off near x_0 by choosing a $\phi \in S(\mathbb{R}^n)$ compactly supported away from $\operatorname{supp}(1-\chi)$ and $\phi \equiv 1$ near x_0, i.e.

$$\operatorname{supp}(1-\chi) \cap \operatorname{supp}\phi = \emptyset.$$

We now have an operator $\phi A(1-\chi)$ with kernel

$$\phi(x)K_A(x,y)(1-\phi(y))$$

that vanishes (in particular smooth) on the diagonal. Since we have shown that the singular support of a pseudodifferential operator have to be contained in the diagonal, we conclude that $\phi A(1-\chi)$ is a smoothing operator, and thus $\phi A(1-\chi)u \in C^\infty(\mathbb{R}^n)$ as required. . □

1.2 Elliptic, Characteristic, Wavefront sets

We will now define *ellipticity at a point* in phase space which allow up to define various microlocal contructions that focus on localised (conically in phase space) behaviour ΨDO's and distributions.

Definition 1.7. A pseudodifferential operator, $A \in \Psi^m_\infty(\mathbb{R}^n), m \in \mathbb{R}$ is **elliptic at a point** $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ if there exist $\epsilon > 0$ such that its left-reduced symbol satisfies the lower bound

$$|\,\sigma_L(A)(x,\xi)\,| \geqslant \epsilon\,\langle\xi\rangle^m$$

in the region

$$\overline{U}_\epsilon = \left\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \;\middle|\; |\,x - x_0\,| \leqslant \epsilon, \left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \epsilon, |\,\xi\,| \geqslant 1/\epsilon\right\}$$

where $\widehat{\xi} = \xi/|\,\xi\,|$ for any non-zero $\xi \in \mathbb{R}^n$. We denote the set of all elliptic points of A as

$$Ell^m(A) = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid A \text{ is elliptic of order } m \text{ at } (x,\xi)\}$$

and its complement in $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ as

$$\begin{aligned}\Sigma^m(A) &= Ell^m(A)^c \setminus \{(x,0) \,|\, x \in \mathbb{R}^n\} \\ &= \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid A \text{ is } \textbf{not} \text{ elliptic of order } m \text{ at } (x,\xi)\}\end{aligned}$$

Lemma 1.8. *Let $A \in \Psi^m_\infty(\mathbb{R}^n), m \in \mathbb{R}$.*

1. *If $\sigma_m(A)(x,\xi)$ is homogeneous of degree m in ξ, then*

$$Ell^m(A) = \{(x_0,\xi_0) \,|\, \xi_0 \neq 0, \sigma_m(A)(x_0,\xi_0) \neq 0\}\,.$$

2. *$Ell^m(A)$ is open in $\mathbb{R}^n \times \mathbb{R}^n$.*

3. *$Ell^m(A)$ is conic in $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$, in the sense that*

$$(x_0,\xi_0) \in Ell^m(A) \implies (x_0, t\xi_0) \in Ell^m(A), \forall t \in \mathbb{R}_{>0}.$$

4. *$\Sigma^m(A)$ is closed conic.*

5. *if $B \in \Psi^{m'}(\mathbb{R}^n)$, then*

$$\Sigma^{m+m'}(A \circ B) = \Sigma^m(A) \cup \Sigma^{m'}(B).$$

Proof. Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $m \in \mathbb{R}$ be given.

1. Suppose the principal symbol $\sigma_m(A)(x,\xi)$ is homogeneous of order m in ξ. We need to show that

$$(x_0,\xi_0) \in Ell^m(A) \iff \xi_0 \neq 0,\ \sigma_m(A)(x_0,\xi_0) \neq 0.$$

If $\xi_0 = 0$, $(x_0,\xi_0) \notin Ell^m_\infty$ by definition of ellipticity. If $\sigma_m(x_0,\xi_0) = 0$, by homogeneity, we have

$$\sigma_m(x_0, t\xi_0) = t^m \sigma_m(x_0,\xi_0) = t^m \cdot 0 = 0$$

for all $t \in \mathbb{R}_{>0}$. By definition of principal symbol, we can write the left symbol of A as

$$\sigma_L(A) = \sigma_m(A) + a$$

where $a \in S^{m-1}_\infty(\mathbb{R}^n;\mathbb{R}^n)$. Now, observe that for any $\epsilon > 0$, the set

$$\overline{U}_\epsilon = \left\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid |\, x - x_0 \,| \leqslant \epsilon, \left| \widehat{\xi} - \widehat{\xi_0} \right| \leqslant \epsilon, |\, \xi \,| \geqslant 1/\epsilon \right\}$$

contains the(open) half-line starting at $\widehat{\xi_0}/\epsilon$, i.e. the set $\{(x_0, t\xi_0/(|\,\xi_0\,|\,\epsilon) \mid t > 0\}$. However, by the symbol estimate of a,

$$\begin{aligned}
\left| \sigma_L(A) \left(x_0, \frac{t\xi_0}{|\,\xi_0\,|\,\epsilon} \right) \right| &\leqslant \left(\frac{t}{\epsilon\,|\,\xi_0\,|} \right)^m |\,\sigma_m(x_0,\xi_0)\,| + \left| a \left(x_0, \frac{t\xi_0}{|\,\xi_0\,|\,\epsilon} \right) \right| \\
&= 0 + \left| a \left(x_0, \frac{t\xi_0}{|\,\xi_0\,|\,\epsilon} \right) \right| \\
&\leqslant C \left\langle \frac{t\xi_0}{|\,\xi_0\,|\,\epsilon} \right\rangle^{m-1} \\
&= C \left\langle t/\epsilon \right\rangle^{m-1}
\end{aligned}$$

and therefore

$$\begin{aligned}
\inf_{(x,\xi)\in\overline{U}_\epsilon} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle \xi \rangle^m} &\leqslant \inf_{t>0} \frac{\left| \sigma_L(A) \left(x_0, \frac{t\xi_0}{|\,\xi_0\,|\epsilon} \right) \right|}{\langle t/\epsilon \rangle^m} \\
&\leqslant \inf_{t>0} \frac{C \langle t/\epsilon \rangle^{m-1}}{\langle t/\epsilon \rangle^m} \\
&= C \inf_{t>0} \langle t/\epsilon \rangle^{-1} \\
&= 0
\end{aligned}$$

which means that $(x_0,\xi_0) \notin Ell^m(A)$.

Conversely, if $\sigma_m(A)(x_0,\xi_0) \neq 0$, by continuity and homogeneity, $\sigma_m(A)$, is non-zero in a (closed) conic neighbourhood, i.e. there exist $\epsilon > 0$ such that $\sigma_m(A) \neq 0$ in

$$\overline{U}_\epsilon = \left\{(x,\xi) \mid |\, x - x_0 \,| \leqslant \epsilon, \left| \widehat{\xi} - \widehat{\xi_0} \right| \leqslant \epsilon, |\, \xi \,| \geqslant 1/\epsilon \right\}.$$

Again, writing the left symbol as a sum of the principal symbol an a lower order term, we observe that in $\overline{U}_\epsilon$,

$$\begin{aligned}\frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} &\geqslant \frac{|\,|\,\sigma_m(A)(x,\xi)\,| - |\,a(x,\xi)\,|\,|}{\langle\xi\rangle^m} \\ &= \left|\,\frac{|\,\xi\,|^m}{\langle\xi\rangle^m}\left|\,\sigma_m(A)(x,\widehat{\xi})\,\right| - \frac{|\,a(x,\xi)\,|}{\langle\xi\rangle^m}\,\right|\end{aligned}$$

By the symbol estimate of a, the second term is tending to 0 which the first term is bounded below by $C = \inf_{(x,\xi)\in\overline{U}_\epsilon} |\,\sigma_m(A)(x,\xi)\,| > 0$. Therefore, choosing a smaller ϵ if necessary, we have $|\,a(x,\xi)\,| \,/\, \langle\xi\rangle^m < C$ and thus

$$\inf_{(x,\xi)\in\overline{U}_\epsilon} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} \geqslant C' \geqslant \epsilon.$$

and therefore $(x_0,\xi_0) \in \mathit{Ell}^m(A)$.

2. We note first that if the principal symbol is homogeneous of degree m, the previous result applies and continuity of the principal symbol guarantee that the elliptic set is open (if $\sigma_m(A)$ is non-zero at a point, it is non-zero in an open neighbourhood of the point).

   For the general case, suppose $(x_0,\xi_0) \in \mathit{Ell}^m(A)$. We therefore have for some $\epsilon > 0$,

   $$|\,\sigma_L(A)(x,\xi)\,| \geqslant \epsilon\,\langle\xi\rangle^m$$

   in the region

   $$\overline{U}_\epsilon(x_0,\xi_0) = \left\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \;\middle|\; |\,x - x_0\,| \leqslant \epsilon, \left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \epsilon, |\,\xi\,| \geqslant 1/\epsilon\right\}.$$

   It suffice to show that there is an open neighbourhood of (x_0,ξ_0) where A remains elliptic. We can take desired open neighbourhood to be

   $$V = \left\{(x',\xi')\,|\,\xi' \neq 0,\; |\,x' - x_0\,| < \epsilon/2,\; \left|\,\widehat{\xi'} - \widehat{\xi_0}\,\right| < \epsilon/2\right\}.$$

   Then, we can check that for every $(x',\xi') \in V$, A satisfies the elliptic estimate in $\overline{U}_{\epsilon/2}(x',\xi')$. Indeed, if $(x,\xi) \in \overline{U}_{\epsilon/2}(x',\xi')$, then

   $$\begin{aligned}&|\,x - x_0\,| \leqslant |\,x - x'\,| + |\,x' - x_0\,| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\ &\left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \left|\,\widehat{\xi} - \widehat{\xi'}\,\right| + \left|\,\widehat{\xi'} - \widehat{\xi_0}\,\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\ &|\,\xi\,| \geqslant 2/\epsilon \geqslant 1/\epsilon\end{aligned}$$

   which shows that $\overline{U}_{\epsilon/2}(x',\xi') \subset \overline{U}_\epsilon(x_0,\xi_0)$. Therefore,

   $$\inf_{(x,\xi)\in\overline{U}_{\epsilon/2}(x',\xi')} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} \geqslant \inf_{(x,\xi)\in\overline{U}_\epsilon(x_0,\xi_0)} \frac{|\,\sigma_L(A)(x,\xi)\,|}{\langle\xi\rangle^m} \geqslant \epsilon \geqslant \epsilon/2$$

   as required.

3. Again, this result is immediate if the principal symbol is homogeneous in ξ. In general, this result come from the observation that only $\widehat{\xi} = \xi / \,|\, \xi \,|$ appears in $\overline{U}_\epsilon$ in the definition of $Ell^m(A)$, i.e. only the *direction* in the dual variable is important.

   Explicitly, let $(x_0, \xi_0) \in Ell^m(A)$ and $t \in \mathbb{R}_{>0}$. Clearly $t\xi_0 \neq 0$. And note that
$$\overline{U}_\epsilon(x_0, \xi_0) = \overline{U}_\epsilon(x_0, t\xi_0)$$
   since $\widehat{\xi} = \widehat{t\xi}$.

4. $\Sigma^m(A) = Ell^m(A)^c$ where $Ell^m(A)$ is open and conic. Since complement of conic set is conic, and complement of open is closed, we conclude that $\Sigma^m(A)$ is closed conic.

5. If both principal symbols are homoegenous of degree m, m' respectively, we can applied the result above and by symbol calculus, we have
$$\begin{aligned} Ell^{m+m'}(A \circ B) &= \{(x, \xi) \,|\, \xi \neq 0, \sigma_{m+m'}(A \circ B) = \sigma_m(A)\sigma_{m'}(B) \neq 0\} \\ &= \{(x, \xi) \,|\, \xi \neq 0, \sigma_m(A) \neq 0\} \cap \{(x, \xi) \,|\, \xi \neq 0, \sigma_{m'}(B) \neq 0\} \\ &= Ell^m(A) \cap Ell^{m'}(B). \end{aligned}$$
   Taking complement give the desired result.

   In general,

□

Definition 1.9. The **wavefront set** of a compactly supported tempered distribution
$$u \in C_c^{-\infty}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) \,|\, \mathrm{supp}(u) \Subset \mathbb{R}^n\}$$
is given by
$$\mathrm{WF}(u) = \bigcap \left\{\Sigma^0(A) \,|\, A \in \Psi^0_\infty(\mathbb{R}^n), Au \in C^\infty(\mathbb{R}^n)\right\}.$$
For general tempered distribution $u \in S'(\mathbb{R}^n)$, its wavefront set is given by
$$\mathrm{WF}(u) = \bigcup_{\chi \in C_c^\infty(\mathbb{R}^n)} \mathrm{WF}(\chi u).$$

Proposition 1.10. *For compactly supported tempered distribution, $u \in C_c^{-\infty}(\mathbb{R}^n)$,*
$$\pi(\mathrm{WF}(u)) = \mathrm{singsupp}(u).$$

where $\pi(x, y) = x$ is the projection map.

Proof. To show $\pi(\mathrm{WF}(u)) \subset \mathrm{singsupp}(u)$, we observe that, by definition of singular support,

$$x_0 \notin \mathrm{singsupp}(u) \implies \exists \phi \in S(\mathbb{R}^n),\, \phi(x_0) \neq 0,\, \phi u \in S(R^n).$$

But since multiplication by ϕ gives an operator in $\Psi^0_\infty(\mathbb{R}^n)$ which is elliptic at (x_0, ξ) for any $\xi \neq 0$ (ϕ is its own principal symbol which happens to be homogeneous and non-zero for any $(x_0, \xi), \xi \neq 0$). Therefore, $x_0 \notin \pi(\mathrm{WF}(u))$.

Conversely, if $x_0 \notin \pi(\mathrm{WF}(u))$, then for all $\xi \neq 0$, there exist $A_\xi \in \Psi^0_\infty(\mathbb{R}^n)$ such that A_ξ is elliptic at (x_0, ξ) and $A_\xi u \in C^\infty(\mathbb{R}^n)$. Since elliptic set $Ell^0(A_\xi)$ is open and conic, we know that there exist $\epsilon = \epsilon(\xi)$ such that A_ξ is elliptic in the open conic set

$$V_\xi = \left\{ (x', \xi') \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \mid |\, x' - x_0 \,| < \epsilon, \left| \widehat{\xi'} - \widehat{\xi} \right| < \epsilon \right\}.$$

Compactness of the sphere (note that $\xi' \mapsto \widehat{\xi'}$ is an embedding of $\mathbb{R}^n \setminus \{0\}$ into S^n) allow us to cover $\{x_0\} \times (\mathbb{R}^n \setminus \{0\})$ with finite number of $V_{\xi_j}, j = 1, \ldots, N$ with corresponding operators A_{ξ_j}.
Now, consider the operator

$$A = \sum_{j=1}^{N} A^*_{\xi_j} A_{\xi_j}.$$

By the pseudolocality of pseudodifferential operators, we know that $A_{\xi_j} u \in C^\infty(\mathbb{R}^n) \implies A^*_{\xi_j} A_{\xi_j} u \in \mathbb{C}^\infty(\mathbb{R}^n)$. Therefore, $Au \in C^\infty(\mathbb{R}^n)$ and A is elliptic at $(x_0, \xi), \forall \xi \neq 0$ with non-negative symbol. We can pick a smooth cut-off χ, $\chi \equiv 1$ when resticted to an $\epsilon/2$-ball around x_0 forming an operator

$$A + (1 - \chi) \in \Psi^0_\infty(\mathbb{R}^n)$$

that is globally elliptic. We can construct a (global) elliptic parametrix E so that

$$1 - (E \circ A + E(1 - \chi)) \in \Psi^{-\infty}_\infty(\mathbb{R}^n).$$

Finally, we can choose any smooth cut-off ϕ with support subordinate to that of χ, i.e. $\mathrm{supp}(\phi) \subset \mathrm{supp}(\chi)$ and note that

$$\phi \circ E \circ (1 - \chi) \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$$

making it a smoothing operator []. Thus, we conclude

$$\phi u = \phi E \circ Au + \phi \circ E \circ (1 - \chi) u \in C^\infty(\mathbb{R}^n)$$

as required.

□

Definition 1.11. Let $a \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ for some $m \in \mathbb{R}$, $p, n \in \mathbb{N}$ be a symbol. We say a is of order $-\infty$ at a point $(x_0,\xi_0) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\}$ (write $a = O(\langle\xi\rangle^{-\infty})$) if there exist $\epsilon \in \mathbb{R}_{>0}$ such that for all $M \in \mathbb{R}$, there is a constant $C_M > 0$ such that

$$|\, a(x,\xi)\,| \leqslant C_M \,\langle\xi\rangle^{-M}$$

in the neigbourhood of (x_0,ξ_0) given by

$$\overline{U}_{(x_0,\xi_0)} = \left\{(x,\xi) \in \mathbb{R}^p \times \mathbb{R}^n \mid |\, x - x_0\,| \leqslant \epsilon, \left|\,\widehat{\xi} - \widehat{\xi_0}\,\right| \leqslant \epsilon\right\}.$$

We define the cone support of the symbol a to be all the points in phase space that where it fails to be $O(\langle\xi\rangle^{-\infty})$.

$$\operatorname{conesupp}(a) = \left\{(x,\xi) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\} \mid a = O(\langle\xi\rangle^{-\infty}) \text{ at } (x,\xi)\right\}^c.$$

Lemma 1.12. *Let* $a \in S^\infty_\infty(\mathbb{R}^p;\mathbb{R}^n)$*, then*

1. $\operatorname{conesupp}(a)$ *is a closed conic set in* $\mathbb{R}^p \times \mathbb{R}^n$.
2. *If* $a = O(\langle\xi\rangle^{-\infty})$ *at* $(x_0,\xi_0) \in \mathbb{R}^p \times \mathbb{R}^n \setminus \{0\}$*, then so is* $D^\alpha_x D^\beta_\xi a(x,\xi)$ *for any multi-index* α, β

With the lemma above, we can in fact define the cone support of a symbol to be the smallest closed conic subset of the phase space (with $\xi \neq 0$) such that, in the complement, a and all its derivatives are of order $-\infty$.

Definition 1.13. Let $A \in \Psi^m_\infty(\mathbb{R}^n)$, $m \in \mathbb{R}$ be pseudodifferential operator. We define the **essential support**, $\mathrm{WF}'(A)$, of A to be the cone support of its left symbol, i.e.

$$\mathrm{WF}'(A) = \operatorname{conesupp}(\sigma_L(A)) \subset \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$$

Using symbol calculus we can prove the following result.

Lemma 1.14. *Let* $A \in \Psi^m_\infty(\mathbb{R}^n)$*,* $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$ *be pseudifferential operators. Then*

1. $\mathrm{WF}'(A) = \operatorname{conesupp}(\sigma_R(A))$.
2. $\mathrm{WF}'(A \circ B) \subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B)$.
3. $\mathrm{WF}'(A + B) = \mathrm{WF}'(A) \cup \mathrm{WF}'(B)$.

With the concept of essentail support we can define the notion of *microlocal elliptic parametrix* which can be thought of as local inverse at an elliptic point of ΨDO's.

Proposition 1.15. *Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ and $z \notin \Sigma^m(A)$. Then there exist a (two-sided) microlocal parametrix $B \in \Psi^{-m}(\mathbb{R}^n)$ such that*

$$z \notin \mathrm{WF}'(1 - AB) \text{ and } z \notin \mathrm{WF}'(1 - BA).$$

Proof. Let $A \in \Psi^m_\infty(\mathbb{R}^n)$ is elliptic at $(x_0, \xi_0) \in \mathit{Ell}^m(A)$. For each $\epsilon \in \mathbb{R}_{>0}$ we define

$$\gamma_\epsilon(x, \xi) = \chi\left(\frac{x - x_0}{\epsilon}\right)(1 - \chi(\epsilon\xi))\chi\left(\frac{\widehat{\xi} - \widehat{\xi_0}}{\epsilon}\right)$$

where $\chi \in C^\infty(\mathbb{R}^n)$ is a smooth cut-off function that is identically 1/2-ball but supported only in the unit ball. Notice that $\gamma_\epsilon \in S^0_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$ with support given by

$$\mathrm{supp}(\gamma_\epsilon) \subset \left\{(x, \xi) \mid |\, x - x_0 \,| \leqslant \epsilon,\, |\, \xi \,| \geqslant \frac{1}{2\epsilon},\, \left|\, \widehat{\xi} - \widehat{\xi_0} \,\right| \leqslant \epsilon \right\}.$$

Restricted to the conic set

$$\overline{U}_\epsilon = \left\{(x, \xi) \mid |\, x - x_0 \,| \leqslant \frac{\epsilon}{2},\, \left|\, \widehat{\xi} - \widehat{\xi_0} \,\right| \leqslant \frac{\epsilon}{2},\, |\, \xi \,| \geqslant \frac{1}{\epsilon}\right\} \subset \mathrm{supp}(\gamma_\epsilon)$$

it is identically 1 and therefore γ_ϵ is elliptic at (x_0, ξ_0). Let $L_\epsilon = \mathrm{Op}_L(\gamma_\epsilon)$ be the corresponding pseudodifferential operator. Observe that, by construction

$$(x_0, \xi_0) \notin \mathrm{WF}'(1 - L_\epsilon)$$

since $1 - \gamma_\epsilon$ is supported away from an ϵ-neighbourhood of $x = x_0$ and the wavefront set of L_ϵ is contained in an ϵ-neighbourhood of (x_0, ξ_0), i.e.

$$\mathrm{WF}'(L_\epsilon) \subset N_\epsilon(x_0, \xi_0) := \left\{(x, \xi) \mid |\, x - x_0 \,| \leqslant \epsilon,\, \left|\, \widehat{\xi} - \widehat{\xi_0} \,\right| \leqslant \epsilon\right\}$$

since γ_ϵ is bounded below in some conic neighbourhood of every point in $N_\epsilon(x_0, \xi_0)$.

Now, let $G_s = \mathrm{Op}_L(\langle\xi\rangle^s)$ for each $s \in \mathbb{R}$. Note that G_s is globally elliptic with positive principal symbol. Consider the following operator,

$$J = (1 - L_\epsilon) \circ G_{2m} + A^* A \in \Psi^{2m}_\infty(\mathbb{R}^n)$$

with principal symbol

$$\sigma_{2m}(J) = (1 - \gamma_\epsilon)\,\langle\xi\rangle^{2m} + |\, \sigma_m(A) \,|^2.$$

Since $\mathit{Ell}^m(A)$ is open conic, we can choose ϵ is small enough so that $\mathit{Ell}^m(A) \subset \mathrm{supp}(\gamma_\epsilon)$. Then, by positivity of all terms involved, we have

$$\frac{|\, \sigma_{2m}(J) \,|}{\langle\xi\rangle^{2m}} = (1 - \gamma_\epsilon) + \frac{|\, \sigma_m(A) \,|^2}{\langle\xi\rangle^{2m}}$$

where the first term is bounded below by 1 outside of $\mathrm{supp}(\gamma_\epsilon)$ while in $\mathrm{supp}(\gamma_\epsilon)$ the second term is bounded below by ϵ since A is elliptic (of order m) at every point in $\mathrm{supp}(\gamma_\epsilon)$. Therefore J is globally elliptic and thus have a global elliptic parametrix $H \in \Psi_\infty^{-2m}(\mathbb{R}^n)$. We shall claim that

$$B = H \circ A^* \in \Psi_\infty^m(\mathbb{R}^n)$$

is a (right) microlocal elliptic parametrix to A. Indeed,

$$\begin{aligned} B \circ A - 1 &= HA^*A - 1 \\ &= H\left(J - (1 - L_\epsilon)G_{2m}\right) - 1 \\ &= (HJ - 1) - H(1 - L_\epsilon)G_{2m}. \end{aligned}$$

Since H is a global parametrix to J, the first term above is a smoothing operator (i.e. an element of $\Psi_\infty^{-\infty}(\mathbb{R}^n)$ and therefore have empty wavefront set. Furthermore, by [**?**] the wavefront set of the second term is a subste of $\mathrm{WF}'(1 - L_\epsilon)$ which does not contain (x_0, ξ_0) by construction. □

Proposition 1.16. *Pseudodifferential operators are microlocal in the following sense: Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ and $u \in C_c^{-\infty}(\mathbb{R}^n)$, then*

$$\mathrm{WF}(Au) \subset \mathrm{WF}(u). \tag{1}$$

In fact, we have

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}(u).$$

Proof. □

A partial converse to the above is given by the following proposition.

Proposition 1.17. *Let $A \in \Psi_\infty^m(\mathbb{R}^n)$ and $u \in C_c^{-\infty}(\mathbb{R}^n)$, then*

$$\mathrm{WF}(u) \subset \mathrm{WF}(Au) \cup \Sigma^m(A).$$