Aerospace Course Notes
Classical ThermoMechanics

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Chapter 1

Mathematical Elements

1.1 Calculus

Theorem 1 (**Gauss Theorem**).

$$\int\limits_{\mathcal{V}} \frac{\partial T_{jk\ldots}}{\partial x_i} dV = \int\limits_{\mathcal{A}} T_{jk\ldots} n_i dA \tag{1.1}$$

$$\int\limits_{\mathcal{V}} \boldsymbol{\nabla T} dV = \int\limits_{\mathcal{A}} \boldsymbol{T} \otimes \boldsymbol{n} dA \tag{1.2}$$

Theorem 2 (**Stokes Theorem**).

$$\int\limits_{\mathcal{A}} e_{ijk} \frac{\partial T_{kl\ldots}}{\partial x_j} n_i dA = \oint\limits_{\mathcal{C}} T_{kl\ldots} dx_k \tag{1.3}$$

$$\int\limits_{\mathcal{A}} \boldsymbol{\nabla} \times \boldsymbol{T} \cdot \boldsymbol{n} dA = \oint\limits_{\mathcal{C}} \boldsymbol{T} \cdot d\boldsymbol{x} \tag{1.4}$$

Theorem 3 (**Reynolds Transport Theorem**).

$$\frac{d}{dt} \int\limits_{\mathcal{V}(t)} T_{jk\ldots}(t, x_i) dV = \int\limits_{\mathcal{V}(t)} \frac{\partial T_{jk\ldots}}{\partial t} dV + \int\limits_{\mathcal{A}(t)} T_{jk\ldots} v_i n_i dA \tag{1.5}$$

$$\frac{d}{dt} \int\limits_{\mathcal{V}} \boldsymbol{T}(t, \boldsymbol{x}) dV = \int\limits_{\mathcal{V}} \frac{\partial \boldsymbol{T}}{\partial t} dV + \int\limits_{\mathcal{A}} \boldsymbol{T v} \cdot \boldsymbol{n} dA \tag{1.6}$$

1.2 Variational Calculus

Variational principles are of the most important concepts and tools in the development of modern physics and its application extends from solid and fluid mechanics, gravitation, electromagnetism, optics, geodesy, to quantum mechanics and string theory. The variational principles are set upon variational calculus, which reflect upon entities such as functors, functionals and it's variations, which are defined next, in a very basic way.

Definition 1 (**High-Order Function or Functor**). *Its an operator $F[f]$ which has for its domain a function f from a given set of functions i.e. $f \subset \mathcal{A} \subset \mathbb{R}^n$. The codomain of F is also a function. Thus a functor $F : \mathcal{A} \to \mathbb{R}^m$, is called a function of functions that "outputs" another function.*

Definition 2 (**Functional**). *Corresponds to the case where the functor F has a codomain witch is the real number set, i.e. $F : \mathcal{A} \to \mathbb{R}$, therefore a function of functions that "output" a real number*

A common class of functionals I are the ones that are obtain has integrals of functors F which makes them immediately comply with a real codomain requisite.

$$I[f(x)] = \int_{x_1}^{x_2} F[f(x)]dx \tag{1.7}$$

In calculus the concepts of **differential** d and **derivative** $\frac{d}{dx}$ are crucial for the determination of maximum and minimum of function, the analogy of these operations to variational calculus are called **variations** δ and **functional derivative** $\frac{\delta}{\delta f}$. Curiously enough the first variation has very similar rules to those of differential, so without proving them we show some of the most important rules in (1.2)

$$\delta F[f] = \frac{\partial F}{\partial f}\delta f \tag{1.8}$$

$$\delta F[f, g] = \frac{\partial F}{\partial f}\delta f + \frac{\partial F}{\partial g}\delta g \tag{1.9}$$

$$\delta(\alpha I) = \alpha \delta I \tag{1.10}$$

$$\delta(I + J) = \delta I + \delta J \tag{1.11}$$

$$\delta(I \times J) = J\delta I + I\delta J \tag{1.12}$$

$$\delta\left(\frac{I}{J}\right) = \frac{J\delta I - I\delta J}{(\delta J)^2} \tag{1.13}$$

Therefore the first variation of (1.7), will be the following.

$$\delta I = \delta \int_{x_1}^{x_2} F[f(x)]dx = \int_{x_1}^{x_2} \delta F[f(x)]dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial f}\delta f \; dx \tag{1.14}$$

Theorem 4 (**Fundamental Theorem of Calculus of Variations**).

$$\int_{x_1}^{x_2} f(x)h(x)dx = 0 \Leftrightarrow f(x) = 0 \quad \forall\, f, h \subset \mathcal{C}^k \;\wedge h(x_1) = h(x_2) = 0 \wedge x \in [x_1, x_2] \tag{1.15}$$

Variational calculus is not only important in the determination of physical laws, but is all applied in the numerical solutions of those sames laws, being the foundation of methods like the Ritz and Galerkin and the finite element method.

Chapter 2

Fundamental Conservation Laws

2.1 Conservation of Mass

The mass is an extensive property that can be given in terms of a intensive property, density ρ, by volume integration.

$$m = \int_{\mathcal{V}} \rho dV \quad \Leftrightarrow \quad \frac{dm}{dV} = \rho \tag{2.1}$$

In classical mechanics mass is axiomatically considered constant therefore its total temporal derivative is zero. Continuing from there using the Reynodls transport theorem (1.5), and (1.1):

$$\frac{dm}{dt} = 0 \tag{2.2}$$

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} dV + \int_{\mathcal{A}} \rho v_j n_j dA = 0 \tag{2.3}$$

$$\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} \right) dV = 0 \tag{2.4}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} = 0 \tag{2.5}$$

The previous set of equation is crucial in all fluid dynamics, it contains the the Lagrangian mass conservation equation, the Eulerian mass conservation equation and also the Integral mass conservation equation. Most if not all problem in fluid dynamics and thermodynamics can be solved by using the most appropriate form.

2.2 Conservation of Species

2.3 Conservation of Linear Momentum

Here we also introduce the linear momentum vector $\boldsymbol{p}$, has the volume integration of density ρ, and velocity vector $\boldsymbol{v}$.

$$p_i = \int_{\mathcal{V}} \rho v_i dV \quad \Leftrightarrow \quad \frac{dp_i}{dV} = \rho v_i \tag{2.6}$$

Before proceeding to momentum equation it is extremely useful to introduce Cauchy stress vector and stress tensor definitions

$$T_i = \sigma_{ij} n_j \tag{2.7}$$

$$\sigma_{ij} = \tau_{ij} - p\delta_{ij} \tag{2.8}$$

The conservation of linear momentum is no more than the statement of Newton law of motion, i.e. total temporal derivative of linear momentum equals the exterior forces acting in the system. The external forces F_i are expressed in an equivalent of the volumetric and surface forces that are expressed by Cauch Stress tensor $\boldsymbol{\sigma}$. This same tensor can be separated in it's bulk or pressure and shear components according to (2.8).

$$\frac{dp_i}{dt} = \Sigma F_i \tag{2.9}$$

$$\int\limits_{\mathcal{V}} \frac{\partial \rho v_i}{\partial t} dV + \int\limits_{\mathcal{A}} \rho v_i v_j n_j dA = \int\limits_{\mathcal{V}} \rho f_i dV + \int\limits_{\mathcal{A}} T_i dA \tag{2.10}$$

$$\int\limits_{\mathcal{V}} \frac{\partial \rho v_i}{\partial t} dV + \int\limits_{\mathcal{A}} \rho v_i v_j n_j dA = \int\limits_{\mathcal{V}} \rho f_i dV + \int\limits_{\mathcal{A}} \sigma_{ij} n_j dA \tag{2.11}$$

$$\int\limits_{\mathcal{V}} \left(\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} \right) dV = \int\limits_{\mathcal{V}} \left(\rho f_i + \frac{\partial \sigma_{ij}}{\partial x_j} \right) dV \tag{2.12}$$

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} = \rho f_i + \frac{\partial \sigma_{ij}}{\partial x_j} \tag{2.13}$$

2.4 Conservation of Angular Momentum

2.5 Conservation of Energy

Again defining energy E has the volume integration of several energy types (internal, kinematic and potential) per unit mass.

$$E = U + K + \ldots \quad \Leftrightarrow \quad \int\limits_{\mathcal{V}} \rho e dV = \int\limits_{\mathcal{V}} (\rho u + \rho \frac{v_i v_i}{2} + \ldots) dV \quad \Leftrightarrow \quad \frac{d(U + K + \ldots)}{dV} = \rho u + \rho \frac{v_i v_i}{2} + \ldots \tag{2.14}$$

The conservation of energy is the renowned 1$^{\text{st}}$ law of thermodynamics, to which we apply the Reynolds and Gauss theorems (1.5),(1.1), and also the Fourier Law for heat conduction $\dot{\vec{q}}_A = k\frac{\partial T}{\partial \vec{x}}$.

$$dE = \delta Q - \delta W \tag{2.15}$$

$$\frac{dE}{dt} = \dot{Q} - \dot{W} \tag{2.16}$$

$$\int_{\mathcal{V}} \frac{\partial \rho u + \rho \frac{v_i v_i}{2}}{\partial t} dV + \int_{\mathcal{A}} (\rho u + \rho \frac{v_i v_i}{2}) v_j n_j dA = \int_{\mathcal{V}} \dot{q} dV + \int_{\mathcal{A}} k \frac{\partial T}{d x_j} n_j dA - \int_{\mathcal{V}} \rho f_i v_i dV + \int_{\mathcal{A}} (\tau_{ij} - p\delta_{ij}) v_i n_j dA \tag{2.17}$$

$$\int_{\mathcal{V}} \left(\rho \frac{\partial \left(u + \rho \frac{v_i v_i}{2}\right)}{\partial t} + \frac{\partial \left(\rho e_u + \rho \frac{v_i v_i}{2}\right) v_j}{\partial x_j} \right) dV = \int_{\mathcal{V}} \left(\dot{q} + \frac{\partial}{\partial x_j} \left(k \frac{\partial T}{d x_j} \right) - \rho f_i v_i + \frac{\partial \left(\tau_{ij} v_i - p v_j \right)}{\partial x_j} \right) dV \tag{2.18}$$

$$\frac{\partial \rho \left(u + \frac{v_i v_i}{2}\right)}{\partial t} + \frac{\partial \left(u + \frac{v_i v_i}{2}\right) \rho v_j}{\partial x_j} = \dot{q} + \frac{\partial}{\partial x_j} \left(k \frac{\partial T}{d x_j} \right) - \rho f_i v_i - \frac{\partial \left(\tau_{ij} v_i - p v_j \right)}{\partial x_j} \tag{2.19}$$

$$\frac{\partial \rho \left(u + \frac{v_i v_i}{2}\right)}{\partial t} + \frac{\partial \left(e_u + \frac{v_i v_i}{2} + p\right) \rho v_j}{\partial x_j} = \dot{q} + \frac{\partial}{\partial x_j} \left(k \frac{\partial T}{d x_j} \right) - \rho f_i v_i - \frac{\partial \tau_{ij} v_i}{\partial x_j} \tag{2.20}$$

2.6 Entropy

Again defining entropy S has quantity that can be integrated in volume for its entropy per unit mass s.

$$S = \int_{\mathcal{V}} \rho s dV \quad \Leftrightarrow \quad \frac{dS}{dV} = \rho s \tag{2.21}$$

And according to the 2$^{\text{nd}}$ law of thermodynamics, in it's differencial forms, using the Reynodls transport theorem (1.5), and (1.1).

$$dS = \frac{\delta Q}{T} + \delta S_{\text{irev}} \tag{2.22}$$

$$\frac{dS}{dt} = \frac{\dot{Q}_{ext}}{T} + \dot{S}_{\text{irev}} \tag{2.23}$$

$$\int_{\mathcal{V}} \frac{\partial \rho s}{\partial t} dV + \int_{\mathcal{A}} \rho s v_j n_j dA = \frac{1}{T} \left(\int_{\mathcal{V}} \dot{q} dV + \int_{\mathcal{A}} k \frac{\partial T}{d x_j} n_j dA \right) + \int_{\mathcal{V}} \dot{s}_{\text{irev}} dV \tag{2.24}$$

$$\int_{\mathcal{V}} \frac{\partial \rho s}{\partial t} dV + \int_{\mathcal{V}} \frac{\partial \rho s v_j}{\partial x_j} dV = \frac{1}{T} \left(\int_{\mathcal{V}} \dot{q} dV + \int_{\mathcal{V}} \frac{\partial}{\partial x_j} \left(k \frac{\partial T}{d x_j} \right) dV \right) + \int_{\mathcal{V}} \dot{s}_{\text{irev}} dV \tag{2.25}$$

$$\frac{\partial \rho s}{\partial t} + \frac{\partial \rho s v_j}{\partial x_j} = \frac{1}{T} \left(\dot{q} + \frac{\partial}{\partial x_j} \left(k \frac{\partial T}{d x_j} \right) \right) + \dot{s}_{\text{irev}} \tag{2.26}$$

Chapter 3

Thermodynamics

3.1 Thermodynamic Potentials

We start by the definition of entalpy H, and entalpy per unit mass h, and stagation entalpy h_0:

$$H \triangleq U + pV \tag{3.1}$$

$$h = u + \frac{p}{\rho} \tag{3.2}$$

$$H_0 \triangleq H + K = H + m\frac{v_i v_i}{2} \tag{3.3}$$

$$h_0 = h + K = h + \frac{v_i v_i}{2} \tag{3.4}$$

Other potencial widley used are the Gibbs G and Helmholtz Ψ potentials and their corresponding per unit mass:

$$G \triangleq H - TS \tag{3.5}$$

$$g = h - Ts \tag{3.6}$$

$$\Psi \triangleq U - TS \tag{3.7}$$

$$\psi = u - Ts \tag{3.8}$$

In order to develop $p-\rho-T$ dependencies of the potential functions, we need the Tds differential equation. It can obtained by solving the entropy differential equation (2.22) without irreversabilities $s_{\text{irev}} = 0$ to the heat term and using it to replace the heat term in the energy differential equation (2.15). If we futher consider that only pressure forces are acting on the body i.e. no external forces and no viscous forces, and also that only internal energy is considered, i.e. negleting other forms of energy like kinetic or potencial gravitic.

$$dE = TdS - \delta W \tag{3.9}$$

$$dU = TdS - pdV \tag{3.10}$$

$$du = Tds + \frac{p}{\rho^2}d\rho \tag{3.11}$$

$$\tag{3.12}$$

Now writing differentials of the potencials, and using the previous equation to replace the du terms

$$du = Tds + \frac{p}{\rho^2}d\rho \tag{3.13}$$

$$dh = Tds + \frac{dp}{\rho} \tag{3.14}$$

$$dg = \frac{dp}{\rho} - sdT \tag{3.15}$$

$$d\psi = \frac{p}{\rho^2}d\rho - sdT \tag{3.16}$$

From the definitions of differential this implies that the potential functions have the following dependencies

$$u = u(s, \rho) \tag{3.17}$$

$$h = h(s, p) \tag{3.18}$$

$$g = g(p, T) \tag{3.19}$$

$$\psi = \psi(\rho, T) \tag{3.20}$$

The famous Maxwell thermodynamic relations follow easily from the previous definitions.

3.1.1 Other Thermodynamic Relations

It is possible to define any thermodynamic property from 2 of the 3 intensive properties $p{-}\rho{-}T$ Therefore will look at some of interresting relation that will apear if some properties are defined in terms of others.

Specific Heats

The specific heats can be calculated from internal energy as function of temperature and density $u(T, \rho)$, and the entalpy as function of temperature and pressure $h(T, p)$

$$du = \left(\frac{\partial u}{\partial T}\right)_\rho dT + \left(\frac{\partial u}{\partial \rho}\right)_T d\rho \tag{3.21}$$

$$c_v(T, \rho) \triangleq \left(\frac{\partial u}{\partial T}\right)_\rho \qquad \text{(specific heat at constant volume)} \tag{3.22}$$

$$dh = \left(\frac{\partial h}{\partial T}\right)_p dT + \left(\frac{\partial h}{\partial p}\right)_T dp \tag{3.23}$$

$$c_p(T, p) \triangleq \left(\frac{\partial h}{\partial T}\right)_p \qquad \text{(specific heat at constant pressure)} \tag{3.24}$$

The ratio of specific heats is of extreme importance

$$\gamma \triangleq \frac{c_p}{c_v} \tag{3.25}$$

However if the internal energy and entalpy are functions of temperature alone, in this situation the fuild is called thermaly perfect

$$c_v(T) = \frac{du}{dT} \tag{3.26}$$

$$c_p(T) = \frac{dh}{dT} \tag{3.27}$$

And if futher considered constat the substance is calorically perfect

$$c_v(T) = c_v \tag{3.28}$$
$$c_p(T) = c_p \tag{3.29}$$

Speed of Sound

If pressure is defined as function of density and entropy $p(\rho, s)$, and futher consider it isentropic

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_s d\rho + \left(\frac{\partial p}{\partial s}\right)_\rho ds \tag{3.30}$$
$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s \tag{3.31}$$
$$a^2 \triangleq \frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s \qquad \text{(speed of sound)} \tag{3.32}$$
$$\tag{3.33}$$

Volume expansivity, Isothermal and Isentropic Compressibility

If density is defined as a function of temperature and pressure $\rho(T, p)$

$$d\rho = \left(\frac{\partial \rho}{\partial T}\right)_p dT + \left(\frac{\partial \rho}{\partial p}\right)_T dp \tag{3.34}$$
$$\beta \triangleq -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \qquad \text{(volume expansivity)} \tag{3.35}$$
$$\kappa \triangleq \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T \qquad \text{(isothermal compressibility)} \tag{3.36}$$

3.1.2 Isentropic Relations

relations are obtained using calorific ideal gas properties using and also a process that is reversible and adiabatic, i.e. isentropic.

$$\frac{T_1}{T_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma} = \left(\frac{p_1}{p_2}\right)^{\frac{\gamma}{\gamma-1}} \tag{3.37}$$

Chapter 4

Fluid Mechanics

The motion of a fluid is completely described by the conservation laws for the three basic properties: mass, momentum and energy. Indeed,how complicated the detailed evolution of a system might be, not only are the basic properties mass, momentum and energy conserved during the whole process at all times but more than that, these three conditions completely determine the behavior of the system without any additional dynamical law. This is a very remarkable property, indeed. The only additional information concerns the specification of the nature of the fluid (e.g. incompressible fluid, perfect gas, condensable fluid, viscoelastic material, etc.). A fluid flow is considered as known if, at any instant of time, the velocity field and a minimum number of static properties are known at every point. The number of static properties to be known is dependent on the nature of the fluid. This number will be equal to one for an isothermal incompressible fluid (e.g. the pressure), two (e.g. pressure and density) for a perfect gas or any real compressible fluid in thermodynamic equilibrium.

We will consider that a separate analysis has provided the necessary knowledge enabling to define the nature of the fluid. This is obtained from the study of the behavior of the various types of continua and the corresponding information is summarized in the constitutive laws and in some other parameters such as viscosity and heat conduction coefficients. This study also provides the information on the nature and properties of the internal forces acting on the fluid since, by definition, a deformable continuum such as a fluid, requires the existence of internal forces connected to the nature of the constitutive law. Besides, separate studies are needed in order to distinguish the various external forces that influence the motion of the system in addition to the internal ones. These external forces could be, e.g. gravity, buoyancy, Coriolis and centrifugal forces in rotating systems, electromagnetic forces in electrical conducting fluids.

Let us now move to the derivation of the basic fluid dynamic equations, by applying the general expressions derived in the previous chapter, to the specific quantities mass, momentum and energy. The equation for mass conservation is also called the continuity equation, while the momentum conservation law is the expression of the generalized Newton law, defining the equation of motion of a fluid. The energy conservation law is also referred to as the expression of the first principle of Thermodynamics.

4.1 Forms of the Fundamental Equations

Vorticity $\vec{\omega}$

$$\Omega_i \triangleq e_{ijk} \frac{\partial v_k}{\partial x_j} \triangleq (\nabla \times \vec{v})_i \tag{4.1}$$

Circulation Γ

$$\Gamma \triangleq \oint v_i dx_i \triangleq \oint \vec{v} d\vec{x} \tag{4.2}$$

4.2 Navier-Stokes Flow

Mass:

Momentum:

Energy:

4.3 Thin Shear Layers Flow

4.3.1 Boundary Layer Flow

4.4 Euler Flow

The Euler flow can be taken directly from the fundamental equations of chapter 2, however they are derivable from the Navier-Stokes flow equation by assuming that there are no viscous/shear forces present i.e. $\tau_{ij} = 0$.

Mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} = 0 \tag{4.3}$$

Momentum:

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} = \rho f_i - \frac{\partial p}{\partial x_i} \tag{4.4}$$

Energy:

$$\frac{\partial \rho e_u + \rho \frac{v_i v_i}{2}}{\partial t} + \frac{\partial (\rho e_u + \rho \frac{v_i v_i}{2} v_j)}{\partial x_j} = \dot{q} + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{d x_j}\right) - \rho f_i v_i + \frac{\partial p v_j}{\partial x_j} \tag{4.5}$$

4.4.1 Steady Quasi-1D Adiabatic Euler Flow

4.4.2 Steady 1D Adiabatic Euler Flow

An case of extreme engineering importance is the Euler flow equations for a steady, uni-dimensional and adiabatic produces the following simplified equations. This situation can be used to explain properties of flows in supersonic conditions, and their shock waves. talk about what means 1D vs Quasi 1D

Mass:

$$\int_{\mathcal{A}} \rho v_j n_j dA = 0 \tag{4.6}$$

$$\rho_1 v_1 = \rho_2 v_2 \tag{4.7}$$

Momentum:

$$\int_{\mathcal{A}} \rho v_i v_j n_j dA = \int_{\mathcal{A}} -p n_i dA \tag{4.8}$$

$$\rho_1 v_1^2 + p_1 = \rho_2 v_2^2 + p_2 \tag{4.9}$$

Energy:

$$\int_{\mathcal{A}} (\rho u + \rho \frac{v_i v_i}{2}) v_j n_j dA = \int_{\mathcal{A}} -p v_j n_j dA \tag{4.10}$$

$$\rho_1 e_{I_1} + \rho_1 \frac{v_1^2}{2} v_1 + p_1 v_1 = \rho_2 e_{I_2} + \rho_2 \frac{v_2^2}{2} v_2 + p_2 v_2 \tag{4.11}$$

Normal Shock Waves

the mach numbers after M_2 and before M_1, are related between themselves by:

$$M_2^2 = \frac{1 + \frac{\gamma-1}{2} M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}} \tag{4.12}$$

While the pressure, density, velocity, temperature and entalpy ratios across the normal shock are given by:

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1} \left(M_1^2 - 1 \right) \tag{4.13}$$

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma+1) M_1^2}{2 + (\gamma-1) M_1^2} \tag{4.14}$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} = \frac{\rho_1}{\rho_2} = \frac{h_2}{h_1} \tag{4.15}$$

This resumes the fundamental equations for the calculation of flows properties across normal shock waves.

Oblique Shock Waves

Prandtl-Meyer Expansion Waves

The expansion waves are the opposite of shock waves, hence trough an expansion wave the Mach number increases and pressure, density and temperature decreases.

The expansion fan is itself a continuous expansion region, composed of an infinite number of Mach waves, bounded upstream and downstream. The all expansion is composed of infinitesimal Mach waves we can consider the process isentropic, all the way through the wave, and therefore the stagnations relations are applicable.

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \left(\sqrt{\frac{\gamma-1}{\gamma+1} (M^2 - 1)} \right) - \tan^{-1} \left(\sqrt{M^2 - 1} \right) \tag{4.16}$$

Interaction of Shock Waves

4.4.3 Fanno Flow

Adiabatic flow through a constant area duct where the effect of friction is considered is also know as Fanno flow.

4.4.4 Steady 1D Non-Adiabatic Euler Flow (Rayleigh Flow)

Non-adiabatic flow through a constant area duct where the effect of heat addition or rejection is also know as the Rayleigh. The only significant difference relative to the 1D adiabatic Inviscid Flow of 4.4.2 is the addition of heat to the energy equation.

Mass:

$$\int_{\mathcal{A}} \rho v_j n_j dA = 0 \tag{4.17}$$

$$\rho_1 v_1 = \rho_2 v_2 \tag{4.18}$$

Momentum:

$$\int_{\mathcal{A}} \rho v_i v_j n_j dA = \int_{\mathcal{A}} -p n_i dA \tag{4.19}$$

$$\rho_1 v_1^2 + p_1 = \rho_2 v_2^2 + p_2 \tag{4.20}$$

Energy:

$$\int_{\mathcal{A}} (\rho u + \rho \frac{v_i v_i}{2}) v_j n_j dA = \int_{\mathcal{V}} \dot{q} dV + \int_{\mathcal{A}} -p v_j n_j dA \tag{4.21}$$

$$\rho_1 e_{I_1} + \rho_1 \frac{v_1^2}{2} v_1 + p_1 v_1 = q + \rho_2 e_{I_2} + \rho_2 \frac{v_2^2}{2} v_2 + p_2 v_2 \tag{4.22}$$

Solving the energy equation the heat per unit mass.

$$\rho_1 e_{I_1} + \rho_1 \frac{v_1^2}{2} v_1 + p_1 v_1 = q + \rho_2 e_{I_2} + \rho_2 \frac{v_2^2}{2} v_2 + p_2 v_2 \tag{4.23}$$

4.5 Steady Incompressible Potential Flow

The potential flow is definitely very important since, when further linearized, provides one of the only analytical methods of determining aerodynamic properties in airfoils and wing for either subsonic and supersonic. We start by presenting flow equation, that are obtainable from the Euler flow by assuming no volumetric forces $\vec{f} = 0$, adiabatic i.e. no heat transfers and constant density.

Mass:

$$\frac{\partial v_j}{\partial x_j} = 0 \tag{4.24}$$

Momentum:

$$\rho \frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} \tag{4.25}$$

Scalar Potential:

The flow gains its name from the assumption that the flow velocity can be obtained from the gradient of scalar potential field ϕ.

$$v_i \triangleq \frac{\partial \phi}{\partial x_i} = (\nabla \phi)_i \tag{4.26}$$

Replacing the (4.26) into the the mass equation (4.24) we see that the scalar potential verifies as Laplace equation.

$$\begin{aligned} \frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_j}\right) &= 0 \\ \frac{\partial^2 \phi}{\partial x_j^2} &= 0 \\ \nabla^2 \phi &= 0 \end{aligned} \tag{4.27}$$

Replacing (4.26) into the definition of vorticity (4.1) we verify that a potential flow is irrotational.

$$\begin{aligned} \Omega_i &= e_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_k}\right) \\ \Omega_i &= e_{ijk}\frac{\partial^2 \phi}{\partial x_j \partial x_k} \\ \Omega_i &= 0 \end{aligned} \tag{4.28}$$

Vector Potential:

A vector potential $\vec{\Psi}$ can also be used to determine the flow velocity .

$$v_i \triangleq e_{ijk}\frac{\partial \Psi_k}{\partial x_j} = \left(\nabla \times \vec{\Psi}\right)_i \tag{4.29}$$

Replacing the (4.29) into the the mass equation (4.24) is immediately verified.

$$\begin{aligned}\frac{\partial}{\partial x_i}\left(e_{ijk}\frac{\partial \Psi_k}{\partial x_j}\right) &= 0\\ e_{ijk}\frac{\partial^2 \Psi_k}{\partial x_i \partial x_j} &= 0\\ 0 &= 0\end{aligned}$$

Replacing the (4.29) into the vorticity definition (4.1)

$$\begin{aligned}\Omega_i &= e_{ijk}\frac{\partial}{\partial x_j}\left(e_{klm}\frac{\partial \Psi_m}{\partial x_l}\right)\\ \Omega_i &= e_{ijk}e_{klm}\frac{\partial^2 \Psi_m}{\partial x_j \partial x_l}\\ \Omega_i &= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\frac{\partial^2 \Psi_m}{\partial x_j \partial x_l}\\ \Omega_i &= \frac{\partial^2 \Psi_j}{\partial x_j \partial x_i} - \frac{\partial^2 \Psi_i}{\partial x_j \partial x_j}\\ \Omega_i &= \frac{\partial^2 \Psi_j}{\partial x_i \partial x_j} - \frac{\partial^2 \Psi_i}{\partial x_j^2} = \left(\nabla\left(\nabla \cdot \vec{\Psi}\right) - \nabla^2\vec{\Psi}\right)_i \qquad (4.30)\end{aligned}$$

So the vector potential $\vec{\Psi}$ must be chosen so that if a irrotatonal flow is desired then the right hand side of (4.30) must be equate to zero.

In a more general 3 dimensional case, as shown by [] , the vector potential can be written as function of two scalar fields, know as stream functions ψ and χ.

$$\Psi_i = \psi\frac{\partial \chi}{\partial x_i} = \left(\psi\nabla\chi\right)_i \tag{4.31}$$

For **2D Cartesian** the vector potential is then given by

$$\psi = \psi(x, y) \tag{4.32}$$
$$\chi = z \tag{4.33}$$
$$\vec{\Psi} = \begin{bmatrix}0 & 0 & \psi(x, y)\end{bmatrix} \tag{4.34}$$

For **3D Axisymmetric** the vector potential is then given by

$$\psi = \psi(x, y) \tag{4.35}$$
$$\chi = z \tag{4.36}$$
$$\vec{\Psi} = \begin{bmatrix}0 & 0 & \psi(x, y)\end{bmatrix} \tag{4.37}$$

Complex Potential:

The potential flow can be easily manipulated in a more compact form, if we use a analytical complex function $W(z)$ as function of the potential $\phi(x,y)$ and of the stream function $\psi(x,y)$.

$$W(z) = \phi + i\psi \qquad z \in \mathbb{C} \tag{4.38}$$
$$z = x + iy \qquad x, y \in \mathbb{R} \tag{4.39}$$

The differentials of (4.39) and (4.38)

$$dz = dx + idy \tag{4.40}$$
$$dW = \frac{\partial W}{\partial z} dz \tag{4.41}$$
$$dW = \frac{\partial W}{\partial z}(dx + idy) \tag{4.42}$$
$$dW = \frac{\partial W}{\partial z} dx + i\frac{\partial W}{\partial z} dy \tag{4.43}$$
$$\tag{4.44}$$

However the Complex potential is also a function $W(z) = W(x,y)$ therefore it's differential can also be given by

$$dW = \frac{\partial W}{\partial x} dx + \frac{\partial W}{\partial y} dy \tag{4.45}$$

Therefore equating both differentials we reach the following equality.

$$\frac{\partial W}{\partial z} = \frac{\partial W}{\partial x} \tag{4.46}$$
$$\frac{\partial W}{\partial z} = \frac{1}{i}\frac{\partial W}{\partial y} \tag{4.47}$$

Summing both equations we reach what is called commonly as the Wirtinger complex differential operator over function W

$$\frac{\partial W}{\partial z} = \frac{1}{2}\left(\frac{\partial W}{\partial x} - i\frac{\partial W}{\partial y}\right) \tag{4.48}$$

However if we subtracted both equations we would obtain the following equality

$$\frac{\partial W}{\partial x} = -i\frac{\partial W}{\partial y} \tag{4.49}$$

Expanding the x and y differentials of W

$$\frac{\partial W}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \tag{4.50}$$
$$\frac{\partial W}{\partial y} = \frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y} \tag{4.51}$$

Replacing back

$$\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = -i\left(\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right) \tag{4.52}$$
$$\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y} \tag{4.53}$$

Thus we obtain what is know as the Cauchy-Rienman condition.

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{4.54}$$
$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{4.55}$$

4.5.1 Complex Singularity Models

In this section we present set of singular flows that can be compounded by linear superposition to form more complex flows.

Uniform Flow

Complex Potential:

$$W(z) = Uze^{-i\alpha} \tag{4.56}$$
$$W(z) = Ure^{i(\theta-\alpha)} \tag{4.57}$$
$$W(z) = Ur\cos(\theta-\alpha) + iUr\sin(\theta-\alpha) \tag{4.58}$$

Potential and Stream Functions:

$$\begin{cases} \phi = Ur\cos(\theta-\alpha) \\ \psi = Ur\sin(\theta-\alpha) \end{cases} \tag{4.59}$$

Velocity:

$$\frac{dW}{dz} = Ue^{-i\alpha} = U\cos\alpha - iU\sin\alpha \tag{4.60}$$

$$\begin{cases} U_x = U\cos\alpha \\ U_y = U\sin\alpha \end{cases} \tag{4.61}$$

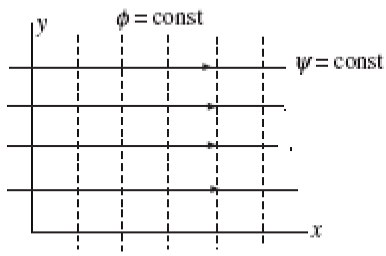

Figure 4.1: *Uniform flow with equipotential lines, and streamlines drawn.*

Corner Flows

Complex Potential:

$$W(z) = Az^n \tag{4.62}$$

Line of Sources

Complex Potential:

$$W(z) = \frac{Q}{2\pi}\ln(z - z_0) \tag{4.63}$$

$$W(z) = \frac{Q}{2\pi}\ln(r^* e^{i\theta^*}) \tag{4.64}$$

$$W(z) = \frac{Q}{2\pi}\left(\ln r^* + \ln(e^{i\theta^*})\right) \tag{4.65}$$

$$W(z) = \frac{Q}{2\pi}\left(\ln r^* + i\theta^*\right) \tag{4.66}$$

Potential and Stream Functions:

$$\begin{cases} \phi = \dfrac{Q}{2\pi}\ln r^* \\ \psi = \dfrac{Q}{2\pi} i\theta^* \end{cases} \tag{4.67}$$

Velocity:

$$\frac{dW}{dz} = \frac{Q}{2\pi}\frac{1}{z - z_0} = \frac{Q}{2\pi r^*}e^{-i\theta^*} \tag{4.68}$$

$$\begin{cases} U_x = \dfrac{Q}{2\pi r^*}\cos\theta^* \\ U_y = \dfrac{Q}{2\pi r^*}\sin\theta^* \end{cases} \Leftrightarrow \begin{cases} U_r = \dfrac{Q}{2\pi r^*} \\ U_\theta = 0 \end{cases} \tag{4.69}$$

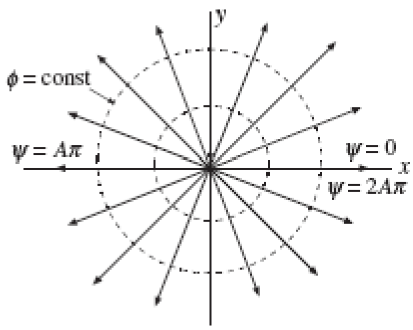

Figure 4.2: *Line of sources/sinks flow with equipotential lines, and streamlines drawn.*

Line of Vortexes

Complex Potential:

$$W(z) = -\frac{i\Gamma}{2\pi}\ln(z - z_0) \tag{4.70}$$

$$W(z) = -\frac{i\Gamma}{2\pi}\ln(r^* e^{i\theta^*}) \tag{4.71}$$

$$W(z) = -\frac{i\Gamma}{2\pi}\left(\ln r^* + \ln(e^{i\theta^*})\right) \tag{4.72}$$

$$W(z) = -\frac{i\Gamma}{2\pi}\left(\ln r^* + i\theta^*\right) \tag{4.73}$$

$$W(z) = \frac{\Gamma}{2\pi}\left(\theta^* - i\ln r^*\right) \tag{4.74}$$

Potential and Stream Functions:

$$\begin{cases} \phi = \dfrac{\Gamma}{2\pi}\theta^* \\ \psi = -\dfrac{\Gamma}{2\pi}\ln r^* \end{cases} \tag{4.75}$$

Velocity:

$$\frac{dW}{dz} = \frac{i\Gamma}{2\pi}\frac{1}{z - z_0} = \frac{i\Gamma}{2\pi r^*}e^{-i\theta^*} = \frac{\Gamma}{2\pi r^*}\left(\sin\theta^* + i\cos\theta^*\right) \tag{4.76}$$

$$\begin{cases} U_x = \dfrac{\Gamma}{2\pi r^*}\sin\theta^* \\ U_y = -\dfrac{\Gamma}{2\pi r^*}\cos\theta^* \end{cases} \Leftrightarrow \begin{cases} U_r = 0 \\ U_\theta = -\dfrac{\Gamma}{2\pi r^*} \end{cases} \tag{4.77}$$

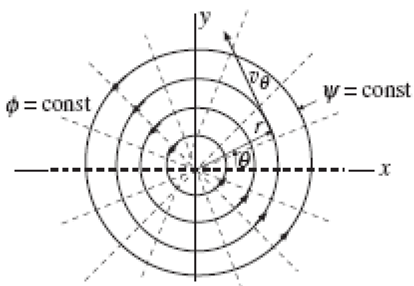

Figure 4.3: *Line of vortexes flow with equipotential lines, and streamlines drawn.*

Line of Dipoles

Complex Potential:

$$W(z) = -\frac{\mu}{z}e^{i\alpha} \tag{4.78}$$

$$W(z) = -\frac{\mu}{r}e^{i(\alpha-\theta)} \tag{4.79}$$

$$W(z) = -\frac{\mu}{r}\left(\cos(\alpha - \theta) + i\sin(\alpha - \theta)\right) \tag{4.80}$$

Potential and Stream Functions:

$$\begin{cases} \phi = -\dfrac{\mu}{r}\cos(\alpha - \theta) \\ \psi = -\dfrac{\mu}{r}\sin(\alpha - \theta) \end{cases} \tag{4.81}$$

Velocity:

$$\frac{dW}{dz} = \frac{\mu}{z^2}e^{i\alpha} = \frac{\mu}{r^2}e^{i(\alpha-2\theta)} = \frac{\mu}{r^2}\left(\cos(\alpha-2\theta) + i\sin(\alpha-2\theta)\right) \tag{4.82}$$

$$\begin{cases} U_x = \dfrac{\mu}{r^2}\cos(\alpha - 2\theta) \\ U_y = -\dfrac{\mu}{r^2}\sin(\alpha - 2\theta) \end{cases} \tag{4.83}$$

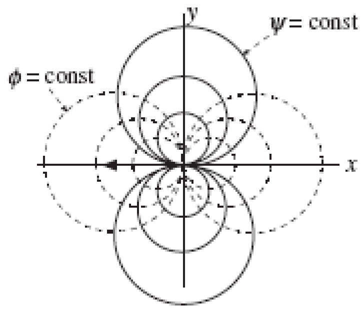

Figure 4.4: *Line of flow with equipotential lines, and streamlines drawn.*

Line of Npoles

Complex Potential:

$$W(z) = -\frac{k}{z^{n-1}} \tag{4.84}$$

Velocity:

$$\frac{dW}{dz} = \frac{k}{z^n} \tag{4.85}$$

Chapter 5

Airfoil Theory

5.1 Generalized Conformal Mapping

In order to obtain the flow around airfoils, a set of transformations can be applied to the complex potential field. In a general form these can be of the following type.

$$z = \zeta + \sum_{n=1} \frac{a_n}{\zeta^n} \qquad \zeta, z, a_n \in \mathbb{C} \tag{5.1}$$

The properties of these conformal transforms are quite interesting and provide powerful mechanisms to generate airfoil shapes.

5.2 Kármán-Treftz Airfoils

An airfoil family for which the Joukowski is actually a special case, is the Kármán-Trefetz transformations.

$$z = kb\frac{(\zeta + b)^k + (\zeta - b)^k}{(\zeta + b)^k - (\zeta - b)^k} \tag{5.2}$$

Kármán-Treftz Airfoils have some advantageous properties relative to the Joukowski airfoils, mainly the angle of the trailing edge. Since the trailing edge angle is different from the Joukowski 2π which is unfeasible to physically build.

5.3 Joukowski Airfoils

The Joukowski airfoil can be derived from the general transform (5.1) using $a_1 = b \vee a_n = 0$ with $n \geq 2$. Or from the Kármán-Treftz transform (5.2) with $k = 2$.

$$z = \zeta + \frac{b^2}{\zeta} \tag{5.3}$$

The Joukowski airfoil has very interesting properties since a circle in the ζ plane will transform into an airfoil in the z plane. The geometrical properties of this airfoil will be determined, has we shall seen, on the offset from the center of the ζ plane origin.

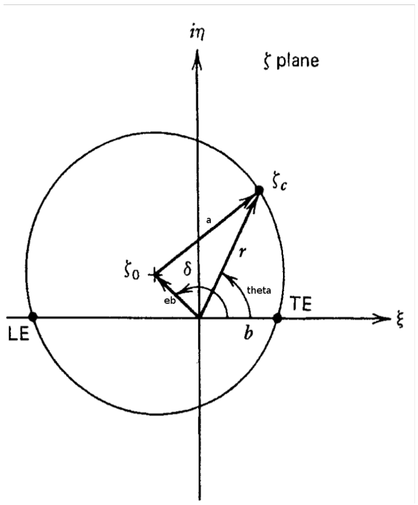

Figure 5.1: *Cylinder in the ζ plane, i.e. the transformation domain. The cylinder if offset from the origin by ζ_0*

Let's start by defining the center of that cylinder.

$$\zeta_0 = \epsilon b e^{i\delta} \tag{5.4}$$

The ϵ quantity is used to emphasize that the decentring needs to be "small" so that the airfoil shape can be linearized, since a closed an explicit equation for the full non linear airfoil is not known. The cylinder points can be given by

$$\zeta_c = re^{i\theta} \tag{5.5}$$

From the cosine law we can write a length using two inscribed triangles in 5.1 (note one of them isn't drawn).

$$a^2 = (\epsilon b)^2 + r^2 - 2\epsilon br\cos(\delta - \theta) \tag{5.6}$$

$$a^2 = (\epsilon b)^2 + b^2 - 2\epsilon b^2\cos(\delta) \tag{5.7}$$

Equating both equations

$$(\epsilon b)^2 + r^2 - 2\epsilon br\cos(\delta - \theta) = (\epsilon b)^2 + b^2 - 2\epsilon b^2\cos(\delta) \tag{5.8}$$

$$r^2 - 2\epsilon br\cos(\delta - \theta) = b^2 - 2\epsilon b^2\cos(\delta) \tag{5.9}$$

$$\left(\frac{r}{b}\right)^2 - 2\epsilon\frac{r}{b}\cos(\delta - \theta) = 1 - 2\epsilon\cos(\delta) \tag{5.10}$$

$$\left(\frac{r}{b}\right)^2 - \left(\frac{r}{b}\right)2\epsilon\cos(\delta - \theta) - 1 + 2\epsilon\cos(\delta) = 0 \tag{5.11}$$

Solving for r/b.

$$\frac{r}{b} = \frac{2\epsilon\cos(\delta - \theta) \pm \sqrt{4\epsilon^2\cos^2(\delta - \theta) - 4(2\epsilon\cos(\delta) - 1)}}{2} \tag{5.12}$$

$$\frac{r}{b} = \epsilon\cos(\delta - \theta) \pm \sqrt{\epsilon^2\cos^2(\delta - \theta) - 2\epsilon\cos(\delta) + 1} \tag{5.13}$$

$$\frac{r}{b} = 1 + \epsilon\left(\cos(\delta - \theta) - \cos\delta\right) + \mathcal{O}[\epsilon^2] \tag{5.14}$$

$$\frac{r}{b} = 1 + \epsilon\left(\sin\delta\sin\theta - \cos\delta\left(1 - \cos\theta\right)\right) + \mathcal{O}[\epsilon^2] \tag{5.15}$$

$$\frac{r}{b} = 1 + \epsilon B + \mathcal{O}[\epsilon^2] \tag{5.16}$$

$$r \approx b\left(1 + \epsilon B\right) \tag{5.17}$$

To get the shape of the airfoil let us pass the circle coordinates through the Joukowski transform.

$$z_a = \zeta_c + \frac{b^2}{\zeta_c} \tag{5.18}$$

$$z_a = re^{i\theta} + \frac{b^2}{re^{i\theta}} \tag{5.19}$$

$$z_a \approx b\,(1+\epsilon B)\,e^{i\theta} + \frac{b^2}{b\,(1+\epsilon B)}e^{-i\theta} \tag{5.20}$$

$$z_a \approx b\,(1+\epsilon B)\,e^{i\theta} + b\frac{1}{1+\epsilon B}e^{-i\theta} \tag{5.21}$$

Developing in power series the last equation $\dfrac{1}{1+\epsilon B} = 1 - \epsilon B + \mathcal{O}[\epsilon^2]$.

$$z_a \approx b\,(1+\epsilon B)\,e^{i\theta} + b\,(1-\epsilon B)\,e^{-i\theta} \tag{5.22}$$

$$z_a \approx b\,[(1+\epsilon B)\,(\cos\theta + i\sin\theta) + (1-\epsilon B)\,(\cos\theta - i\sin\theta)] \tag{5.23}$$

$$z_a \approx b\,(\cos\theta + i\sin\theta + \epsilon B\cos\theta + i\epsilon B\sin\theta + \cos\theta - i\sin\theta - \epsilon B\cos\theta + i\epsilon B\sin\theta) \tag{5.24}$$

$$z_a \approx 2b\,(\cos\theta + i\epsilon B\sin\theta) \tag{5.25}$$

$$z_a \approx 2b\left(\cos\theta + i\epsilon\left(\sin\delta\sin^2\theta - \cos\delta\sin\theta\,(1-\cos\theta)\right)\right) \tag{5.26}$$

The airfoil points in the transformed plane $z_a = (x_a, y_a)$ are given by:

$$\begin{cases} x_a \approx 2b\cos\theta \\ y_a \approx 2b\epsilon\left(\sin\delta\sin^2\theta - \cos\delta\sin\theta\,(1-\cos\theta)\right) \end{cases} \tag{5.27}$$

The airfoil cord can be calculated has:

$$c = z_a(\theta = 0) - z_a(\theta = \pi) \tag{5.28}$$

$$c \approx 2b - (-2b)$$

$$c \approx 4b \tag{5.29}$$

Rewriting (5.27) in terms of the airfoil cord (5.29).

$$\begin{cases} \bar{x}_a \triangleq \dfrac{x_a}{c} \approx \dfrac{\cos\theta}{2} \\ \bar{y}_a \triangleq \dfrac{y_a}{c} \approx \dfrac{\epsilon}{2}\left(\sin\delta\sin^2\theta - \cos\delta\sin\theta\,(1-\cos\theta)\right) \end{cases} \tag{5.30}$$

It's possible do define an implicit equation for the airfoil shape, by replacing all the θ in the y_a/c in equation (5.30), as variables of x_a/c instead.

$$\begin{cases} \cos\theta = 2\bar{x}_a \quad \vee \quad \sin\theta = \pm\sqrt{1-4\bar{x}_a^2} \\ \bar{y}_a = \dfrac{\epsilon}{2}\sin\delta\left(1-4\bar{x}_a^2\right) \pm \dfrac{\epsilon}{2}\cos\delta\sqrt{1-4\bar{x}_a^2}\,(1-2\bar{x}_a) \end{cases} \tag{5.31}$$

The linearized Joukowski airfoil given by (5.31) can be understood as a camber contribution plus a thickness contribution applies on the upper or lower camber line.

$$\bar{y}_a = \underbrace{\frac{\epsilon}{2}\sin\delta\left(1-4\bar{x}_a^2\right)}_{\text{Camber Line}} \pm \underbrace{\frac{\epsilon}{2}\cos\delta\,(1-2\bar{x}_a)\sqrt{1-4\bar{x}_a^2}}_{\text{Thickness Distribution}} \tag{5.32}$$

$\hookrightarrow$ +Upper, −Lower airfoil

Calculating the maximum of the camber line of (5.32).

$$\bar{y}_c = \frac{\epsilon}{2} \sin\delta \left(1 - 4\bar{x}_a^2\right) \tag{5.33}$$

$$\frac{h}{c} \triangleq (\bar{y}_c)_{max} \Leftrightarrow \frac{d\bar{y}_c}{d\bar{x}_a} = 0 \tag{5.34}$$

$$\frac{d\bar{y}_c}{d\bar{x}_a} = \frac{d}{d\bar{x}_a}\left(\frac{\epsilon}{2} \sin\delta \left(1 - 4\bar{x}_a^2\right)\right) = 0 \tag{5.35}$$

$$\frac{\epsilon}{2} \sin\delta \left(0 - 8\bar{x}_a\right) = 0 \tag{5.36}$$

$$\bar{x}_a = 0 \tag{5.37}$$

$$\bar{h} \triangleq \frac{h}{c} = \frac{\epsilon}{2} \sin\delta \tag{5.38}$$

Chapter 6

Wing Theory

Lets start by defining a few geometric quantities useful in wing theory.

$$\bar{\bar{c}} = \frac{1}{S} \int_{-b/2}^{b/2} c^2 dy \tag{6.1}$$

$$\bar{c} = \frac{S}{b} \tag{6.2}$$

$$AR = \frac{b}{\bar{c}} \tag{6.3}$$

6.1 Lifting Line Theory

Chapter 7

Computational Fluid Dynamics

7.1 Mathematical and Physical Nature of Flow Equations

In the computational fluid dynamics, the mechanism to numericaly solve a specific flow equation cannot be separated from the mathematical and physical nature of the base equation. Concept like convection and difusion play a crucial role in choosing appropriate numerical schemes. These concept are also intrinsincly linked to the mathematical structure of the equation, for exmaple, diffusive fluxes appear through second orderderivative terms in space, as a consequence of the generalized Fick law, equation which expresses the essence of the molecular diffusion phenomenon as a tendency to smooth out gradients. The convective fluxes, on the other hand, appear as first order derivative terms in space and express the transport properties of a flow system.

For simplicity we start by studing them in the 1D Convection-Difusion equation then try to apply this concepts to more complex partial differential equations and lastly systems of these equations.

7.1.1 1D Convection-Difusion Equation

The one dimensional convection-difusion equation is as follows:

$$\frac{\partial f}{\partial t} + a(f)\frac{\partial f}{\partial x} = \alpha\frac{\partial^2 f}{\partial x^2} \tag{7.1}$$

The convection term is represent a transport of quanty f in space and is therfore represented by first dirivative in space $\frac{\partial f}{\partial x}$ the convection velocity is the nonlinear term $a(f)$

7.1.2 Classification of Partial Differential Equations

The classification of PDE is inextricably connected to the method of the characteristics, therefore will present them both.

7.2 Properties of Numerical Solution Methods

In other to analyse the quality of numerical method, one has to inspect, in a summarized way, the following aspects:

- **Consistency**: The discretization of the PDE, introduces truncation errors, in space and time. For a scheme to be consistent it should not have truncation error when $\Delta x \to 0$ and $\Delta t \to 0$. A consistent scheme is one that actually tries to solves the differential equation it was set to and not a different one.
- **Stability**: A scheme is said to be stable when errors aren't amplified over the iterations, i.e. the are bounded. Stability is difficult to test but there are a couple of method used, for example the Von Neumann stability criterion.
- **Convergence**: A numerical method is said to be convergent if the solution of the discretized equations tends to the exact solution of the differential equation as the grid spacing tends to zero. The *Lax Equivalence Theorem* implies that for a linear well posed initial value problem, if consistency is verified then stability is the necessary and sufficient condition for convergence. For nonlinear problems stability and convergence are difficult to demonstrate, and sometimes the only alternative is to numerically test for a wide range of boundary conditions and grid layouts and spacings.
- **Conservation**: Since the equations to be solved are conservation laws, the numerical scheme should also - on both a local and a global basis - respect these laws. This means that, a t steady state and in the absence of sources, the amount of a conserved quantity leaving a closed volume is equal to the amount entering that volume. If the strong conservation form of equations and a finite volume method are used, this is guaranteed for each individual control volume and for the solution domain as a whole. Other discretization methods can be made conservative if care is taken in the choice of approximations. The treatment of sources or sink terms should be consistent so that the total source or sink in the domain is equal to the net flux of the conserved quantity through the boundaries. This is an important property of the solution method, since it imposes a constraint on the solution error. If the conservation of mass, momentum and energy are insured, the error can only improperly distribute these quantities over the solution domain. Non-conservative schemes can produce artificial sources and sinks, changing the balance both locally and globally. However, non-conservative schemes can be consistent and stable and therefore lead to correct solutions in the limit of very fine grids. The errors due to non- conservation are in most cases appreciable only on relatively coarse grids. The problem is that it is difficult to know on which grid are these errors small enough. Conservative schemes are therefore preferred.
- **Boundedness**: Numerical solutions should lie within proper bounds. Physically non-negative quantities (like density, kinetic energy of turbulence) must always be positive; other quantities, such as concentration, must lie between 0% and 100%. In the absence of sources, some equations (e.g. the heat equation for the tem- perature when no heat sources are present) require that the minimum and maximum values of the variable be found on the boundaries of the domain. These conditions should be inherited by the numerical approximation. Boundedness is difficult to guarantee. We shall show later on that only some first order schemes guarantee this property. All higher-order schemes can produce unbounded solutions; fortunately, this usually happens only on grids that are too coarse, so a solution with undershoots and overshoots is usually an indication that the errors in the solution are large and the grid needs some refinement (at least locally). The problem is that schemes prone to producing unbounded solutions may have stability and convergence problems. These methods should be avoided, if possible.
- **Realizability**: Models of phenomena which are too complex to treat directly (for example, turbulence, combustion, or multiphase flow) should be designed to guarantee physically realistic solutions. This is not a numerical issue per se but models that are not realizable may result in unphysical

solutions or cause numerical methods to diverge. We shall not deal with these issues in this book, but if one wants to implement a model in a CFD code, one has t o be careful about this property.

- **Accuracy**: Numerical solutions of fluid flow and heat transfer problems are only ap- proximate solutions. In addition to the errors that might be introduced in the course of the development of the solution algorithm, in programming or setting up the boundary conditions, numerical solutions always include three kinds of systematic errors:

  Modeling errors: which are defined as the difference between the actual flow and the exact solution of the mathematical model;

  Discretization errors: defined as the difference between the exact solution of the conservation equations and the exact solution of the algebraic system of equations obtained by discretizing these equations, and

  Iteration errors: defined as the difference between the iterative and exact solutions of the algebraic equations systems.

  Iteration errors are often called convergence errors (which was the case in the earlier editions of this book). However, the term convergence is used not only in conjunction with error reduction in iterative solution methods, but is also (quite appropriately) often associated with the convergence of numerical solutions towards a grid-independent solution, in which case it is closely linked t o discretization error. To avoid confusion, we shall adhere t o the above definition of errors and, when discussing issues of convergence, always indicate which type of convergence we are talking about. It is important to be aware of the existence of these errors, and even more to try t o distinguish one from another. Various errors may cancel each other, so that sometimes a solution obtained on a coarse grid may agree better with the experiment than a solution on a finer grid - which, by definition, should be more accurate. Modeling errors depend on the assumptions made in deriving the trans- port equations for the variables. They may be considered negligible when laminar flows are investigated, since the Navier-Stokes equations represent a sufficiently accurate model of the flow. However, for turbulent flows, two- phase flows, combustion etc., the modeling errors may be very large - the exact solution of the model equations may be qualitatively wrong. Modeling errors are also introduced by simplifying the geometry of the solution do- main, by simplifying boundary conditions etc. These errors are not known a priori; they can only be evaluated by comparing solutions in which the dis- cretization and convergence errors are negligible with accurate experimental data or with data obtained by more accurate models (e g data from direct simulation of turbulence, etc.). It is essential to control and estimate the convergence and discretization errors before the models of physical phenomena (like turbulence models) can be judged. We mentioned above that discretization approximations introduce errors which decrease as the grid is refined, and that the order of the approximation is a measure of accuracy. However, on a given grid, methods of the same order may produce solution errors which differ by as much as an order of magnitude. This is because the order only tells us the rate at which the error decreases as the mesh spacing is reduced - it gives no information about the error on a single grid. We shall show how discretization errors can be estimated in the next chapter. Errors due to iterative solution and round-off are easier to control; we shall see how this can be done in Chap. 5 , where iterative solution methods are introduced. There are many solution schemes and the developer of a CFD code may have a difficult time deciding which one to adopt. The ultimate goal is to obtain desired accuracy with least effort, or the maximum accuracy with the available resources. Each time we describe a particular scheme we shall point out its advantages or disadvantages with respect to these criteria.

Chapter 8

Aerospace Vehicle Dynamics

8.1 Aerospace Reference Frames

8.1.1 Earth Coordinate Systems

The aircraft system definition usually starts with the definitions of earth axes system since it will define "how inertial" will be the aircraft system be.

NED

ENU

8.1.2 Body Coordinate System

These axes are characterized by having ϕ , θ and ψ angles, The transformation from Body Axes to Earth Axis is given by a rotation of ϕ around the x-axis (roll angle), followed by a rotation of θ around the y-axis (pitch angle) and finally a rotation of ψ around the z-axis (yaw angle).

$$
\begin{aligned}
T_{BE} &= R_z(\psi)R_y(\theta)R_x(\phi) \\
T_{BE} &= \begin{bmatrix} cos(\psi) & -sin(\psi) & 0 \\ sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi) & cos(\phi) \end{bmatrix} \\
T_{BE} &= \begin{bmatrix} cos(\theta)cos(\psi) & sin(\theta)sin(\phi)cos(\psi) - cos(\phi)sin(\psi) & sin(\theta)cos(\phi)cos(\psi) + sin(\phi)sin(\psi) \\ cos(\theta)sin(\psi) & sin(\theta)sin(\phi)sin(\psi) + cos(\phi)cos(\psi) & sin(\theta)cos(\phi)sin(\psi) - sin(\phi)cos(\psi) \\ -sin(\theta) & cos(\theta)sin(\phi) & cos(\theta)cos(\phi) \end{bmatrix}
\end{aligned}
$$

The inverse transformation from Earth Axes to Body Axes will be given by:

$$T_{EB} = {T_{BE}}^{-1} = {T_{BE}}^{T}$$
$$T_{EB} = (R_z(\psi)R_y(\theta)R_x(\phi))^T$$
$$T_{EB} = R_x(\phi)^T R_y(\theta)^T R_z(\psi)^T$$
$$T_{EB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\phi) & sin(\phi) \\ 0 & -sin(\phi) & cos(\phi) \end{bmatrix} \begin{bmatrix} cos(\theta) & 0 & -sin(\theta) \\ 0 & 1 & 0 \\ sin(\theta) & 0 & cos(\theta) \end{bmatrix} \begin{bmatrix} cos(\psi) & sin(\psi) & 0 \\ -sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$T_{EB} = \begin{bmatrix} cos(\theta)cos(\psi) & cos(\theta)sin(\psi) & -sin(\theta) \\ sin(\theta)sin(\phi)cos(\psi) - cos(\phi)sin(\psi) & sin(\theta)sin(\phi)sin(\psi) + cos(\phi)cos(\psi) & cos(\theta)sin(\phi) \\ sin(\theta)cos(\phi)cos(\psi) + sin(\phi)sin(\psi) & sin(\theta)cos(\phi)sin(\psi) - sin(\phi)cos(\psi) & cos(\theta)cos(\phi) \end{bmatrix}$$

8.1.3 Wind Coordinate System

These axes are characterized by having α and β angles between them and the body axes. The transformation from Wind Axes to Body Axes (T_{WB}) is given by a rotation of β around the z-axis of the wind frame, followed by a rotation of $-\alpha$ around y-axis.

$$T_{WB} = R_y(-\alpha)R_z(\beta) \tag{8.1}$$
$$T_{WB} = \begin{bmatrix} cos(\alpha) & 0 & -sin(\alpha) \\ 0 & 1 & 0 \\ sin(\alpha) & 0 & cos(\alpha) \end{bmatrix} \begin{bmatrix} cos(\beta) & -sin(\beta) & 0 \\ sin(\beta) & cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.2}$$
$$T_{WB} = \begin{bmatrix} cos(\alpha)cos(\beta) & -cos(\alpha)sin(\beta) & -sin(\alpha) \\ sin(\beta) & cos(\beta) & 0 \\ sin(\alpha)cos(\beta) & -sin(\alpha)sin(\beta) & cos(\alpha) \end{bmatrix} \tag{8.3}$$

The inverse transformation from Body Axes to Wind Axes is given by:

$$T_{BW} = {T_{WB}}^{-1} = {T_{WB}}^{T} \tag{8.4}$$
$$T_{BW} = (R_y(-\alpha)R_z(\beta))^T \tag{8.5}$$
$$T_{BW} = R_z(\beta)^T R_y(-\alpha)^T \tag{8.6}$$
$$T_{BW} = \begin{bmatrix} cos(\beta) & sin(\beta) & 0 \\ -sin(\beta) & cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\alpha) & 0 & sin(\alpha) \\ 0 & 1 & 0 \\ -sin(\alpha) & 0 & cos(\alpha) \end{bmatrix} \tag{8.7}$$
$$T_{BW} = \begin{bmatrix} cos(\alpha)cos(\beta) & sin(\beta) & sin(\alpha)cos(\beta) \\ -cos(\alpha)sin(\beta) & cos(\beta) & -sin(\alpha)sin(\beta) \\ -sin(\alpha) & 0 & cos(\alpha) \end{bmatrix} \tag{8.8}$$

Transformation Earth-Wind

The transformation from Earth Axes to Wind Axes is the composition of Earth to Body then Body to Wind, i.e.:

$$T_{EW} = T_{BW}T_{EB} \tag{8.9}$$
$$T_{EW} = (R_z(\beta)^T R_y(-\alpha)^T)(R_x(\phi)^T R_y(\theta)^T R_z(\psi)^T) \tag{8.10}$$

The transformation from Wind to Earth Axes would be:

$$T_{WE} = T_{BE}T_{WB} \tag{8.11}$$

$$T_{W\to NED} = (R_z(\psi)R_y(\theta)R_x(\phi))(R_y(-\alpha)R_z(\beta)) \tag{8.12}$$

$$\tag{8.13}$$

8.2 Aerospace Vehicles Forces

8.2.1 Inertial Forces

The inertial forces occur on all aircraft components (since all have mass), but from Newtonian mechanics the choice of reference can also "introduce" inertial forces. The inertial forces that act on the airframe ma_i are represented on the center of gravity (c.g.).

$$\vec{F}_{inertial}|_{NED} = m \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \tag{8.14}$$

8.2.2 Aerodynamic Forces

The aerodynamic force on a powered airplane is commonly represented by three vectors: thrust, lift and drag **??**. Has a simplification we only include the wings+fuselage lift (L) and drag (D) engine thrust (T) and when applicable tail lift (L_t) These aerodynamic forces are usually represented in the aerodynamic center (a.c.) since in this point the aerodynamic pitching moment M_0 is constant and does not change with angle of attack (α).

$$\vec{F}_{aero}|_W = \begin{bmatrix} T - D \\ 0 \\ -(L + L_t) \end{bmatrix} \tag{8.15}$$

8.2.3 Gravitic Forces

The gravity force or weight is easily expressed in the NED reference frame and is considered has acting in center of gravity (c.g.).

$$\vec{F}_g|_{NED} = \vec{W}|_{NED} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} \tag{8.16}$$

Chapter 9

Aerospace Structures

9.1 Airworthiness

9.1.1 Regulations

some very condensed regulation can go here. yada yada yada

9.2 Airframe Loads

9.2.1 Load Factors

The definition of load factor is usually given to students as the ratio of lift over weight i.e. $n = \frac{L}{W}$, although this is can be true, it is only so in very simplified cases, since in some flight condition or axis there might not even exist lift force component. The more general definition for load factor for a given axis, is ratio of resultant forces over weight. An equivalent statement for load factor is the factor, for which the weight is multiplied so that a dynamic problem could be reduced to a static problem. As we will see these last definitions cover all possible flight cases from steady fight to landings, and also automatically define load factors for each reference frame axis. Some of the important properties of the load factor, according to these definitions are they depend on the all the forces that are present in the aircraft in a certain instant and also that the load factor sign is dependent on the choice of reference frame, since it is tied to sign of acceleration in that axis.

$$\Sigma F_i = ma_i \tag{9.1}$$
$$\tag{9.2}$$

A since the mass can be written as weight.

$$W = mg \tag{9.3}$$
$$\Sigma F_i = W\frac{a_i}{g} \tag{9.4}$$

Leading to two definitions of load factor.

$$n_i \triangleq \frac{\Sigma F_i}{W} \tag{9.5}$$
$$n_i \triangleq \frac{a_i}{g} \tag{9.6}$$

Although useful this definition it includes a subtlety, in the sum of all forces i.e. resultant force, the weight force can be included or not in that resultant, therefore a new load factor can be defined by excluding that weight factor from the resultant force and adding it to the inertial force.

$$n_i = \frac{\Sigma F_i \mp k_i W}{W} \tag{9.7}$$
$$n_i \pm k_i = \frac{\Sigma F_i}{W} \tag{9.8}$$
$$\bar{n}_i \triangleq n_i \pm k_i \tag{9.9}$$

Therefore the new definition of load factor differs from the previous by weight factor $\pm k_i$ depending again on the reference frame orientation.

Despite the fact that this chapter deals with aircraft loads we went into no details about the resultant of the forces that act on the airframe (ΣF_i). This section focused only in representing the usual equation formatting the is done in terms for loads factors.

9.2.2 Flight Maneouvers

Before we start babbling about all the force and momentum equation and the corresponding load factors for each flight conditions it's far more important to show how to consistently deduce them. So we adopt a serious treatment of aircraft axes systems since they are the base for all aircraft dynamic equations, and an usually poor explained topic.

Steady Climb/Descent

Level Flight

Steady Pull-Out

Correctly Banked Turn

In this scenario the only acceleration component is inwards to turn radius R, and is exclusively generated by lift component.

Landing

Gust Loads

9.2.3 V-n Diagrams

9.3 Bending, Shear and Torsion of Thin Walled Beams

9.4 Structural Idealization

9.5 Aeroelasticity

Aeroelasticity is essentially the coupling of aerodynamics loads and the structural elastic behavior of the airplane's fuselage, and specially, tail and wings. What happens in the wing for example is the lift generated bends the wing upward changing the local incidence angle thus adding more lift, thus more bending, thus more lift, till a equilibrium is reached, if the wing stiffness if enough or till the material load limit and therefore destruction of the wing.

Aeroelasticity can subdivided into steady and dynamic phenomena. In steady aeroeslaticity, we assume that a equilibrium of aerodynamic loading distribution and the deformed structured has been reached thus assuming that that configuration exists by providing enough stiffness. In dynamic aeroelasticity were interest in analyzing the transient behavior of the structure, thus inertial forces must be included in the models.

9.5.1 Airfoil Torsional Divergence

Coupling of torsional load and the divergence is one of the most important in wing aeroelasticity because a twist of the wing greatly impacts local incidence angle of the airfoil and thus the lift and looping back the the structure twist. Simple 2-D wing tunnel models are presented next:

Wall Mounted Model

Sting Mounted Model

Strut Mounted Model

9.5.2 Finite Straight Wing Torsional Divergence

Making an equilibrium of moments in spanwise wing element of length Δz one obtains the following equation

$$\left(T + \frac{dT}{dz}\Delta z\right) - T + ec\Delta L + \Delta M_{ac} = 0 \tag{9.10}$$

$$\Delta L = \frac{1}{2}\rho U^2 \Delta S C_L = \frac{1}{2}\rho U^2 c \Delta z \left(C_{L_0} + \frac{\partial C_L}{\partial \alpha}(\alpha + \theta)\right) \tag{9.11}$$

$$\Delta M_{ac} = \frac{1}{2}\rho U^2 \Delta S c C m_{ac} = \frac{1}{2}\rho U^2 c^2 \Delta z C m_{ac} \tag{9.12}$$

$$T = GJ\frac{d\theta}{dz} \tag{9.13}$$

Combining the above equations one obtains

$$GJ\frac{d^2\theta}{dz^2} + \frac{1}{2}\rho U^2 e c^2 \left(C_{L_0} + \frac{\partial C_L}{\partial \alpha}(\alpha + \theta)\right) + \frac{1}{2}\rho U^2 c^2 C m_{ac} = 0 \tag{9.14}$$

Equation (9.14)is actually an second order differential equation in $\theta(z)$, and it admits a solution of following type.

$$\theta = Asin(\lambda z) + Bcos(\lambda z) - \left(\frac{C_{m_0}}{eC_{l_\alpha}} + \alpha\right) \tag{9.15}$$

9.5.3 Finite Swept Wing Torsional Divergence

The swept back wing decreases the the