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THE IAU 2000A AND IAU 2006 PRECESSION-NUTATION THEORIES AND THEIR IMPLEMENTATION

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The IAU 2000A precession-nutation theory relates the International Celestial Reference Frame to the International Terrestrial Reference Frame and has been effective since January 2003. In 2006, the IAU moved to adopt a more dynamically consistent precession model to complement the IAU 2000A nutation theory. This update – described as IAU 2006 precession in the 2009 Astronomical Almanac – is effective January 2009. Now there are multiple numerical standards relating the ICRF and ITRF precise to within a few ìas. In this paper, the impact of alternative (yet acceptable) ITRF-to-ICRF transformations is discussed, and an operational alternative is also proposed that is computationally faster and easier to maintain, while preserving precision.

INTRODUCTION

The ordinary differential equations that describe artificial satellite motion due to Newtonian gravitation are most simply defined relative to an "inertial" frame, where the time derivatives of the coordinate axes are negligible. However, gravity-field modeling and spacecraft observations are usually made with respect to the surface of the rotating Earth. Spacecraft orbit determination therefore requires that we relate a *celestial reference frame* (where the equations of spacecraft motion are simply expressed) with a *terrestrial reference frame* (where gravity is modeled and spacecraft observations tend to be taken).

The conventional reference systems in common use, and the relationships between them, are approved and maintained by various international organizations, such as the International Astronomical Union (IAU), and the International Union of Geodesy and Geophysics (IUGG) with its associations. The best supported global terrestrial reference frame - the *International Terrestrial Reference Frame* (ITRF) - has its origin at the Earth's center of mass and its fundamental axes are implied by the adopted coordinates of defining fundamental stations on the surface of the Earth. Since the relative station coordinates are affected by plate tectonic motion on the order of centimeters per year, the ITRF is regularly re-estimated as a weighted, global combination of several analysis center solutions, constrained so there is no net rotation or frame shift with respect to previous realizations of the ITRF.[1] The *International Celestial Reference Frame* (ICRF) has been the official fundamental celestial reference frame since 1 January, 1998. Its axes were established

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close to the mean equatorial pole and equinox at epoch J2000, and are realized from VLBI observations of quasi-stellar radio sources.

The ICRF is related to the ITRF through an *Earth-orientation model* that predicts the slowly changing direction of Earth's spin axis due to external gravitational torques caused by the Sun, Moon, and planets. Models for Earth orientation are based on semi-analytic solutions to the second-order differential equations describing the angular momentum of the oblate Earth orbiting through space, which are ultimately expressed relative to the axis of figure of the terrestrial system (where the Earth's inertia momenta are most simply characterized).[2, 3] Observed corrections to the conventional Earth-orientation model, known as *Earth-orientation parameters* (EOPs), are published by the International Earth Rotation and Reference System Service (IERS), a joint service of the IAU and IUGG. The uncertainty of the orientation between the ICRF and ITRF is already maintained at a level that tends to be better than the uncertainty of the spacecraft observations and estimated spacecraft locations; nevertheless, even small improvements in Earth's relative orientation (either by precise changes to authorized models, or by correction with EOPs) are noticeable to spacecraft analysts.

Analysts concerned with the precise reproducibility of conventional results should be aware of an IAU-sanctioned update to the conventional Earth-orientation model that became effective January 1, 2009. The collection of procedures for modeling the relationship between the ITRF and ICRF has become potentially more confusing with this introduction, because differences between the numerous available methods tend to differ far below the level of the uncertainty of Earth-orientation theory itself, *e.g.*, to within a few milliarcseconds (mas). Here the reader is reminded that 1 mas equates to a tangential displacement of about 3 cm at a distance of one Earth radius, and 31 microarcseconds (μas) is approximately 1 mm of displacement per Earth radii. Such small differences tend to be quite inconsequential for almost all spacecraft analysis applications; nevertheless, analysts responsible for implementing and maintaining standards must give changing conventions due consideration.

The events leading to the introduction of these latest updates to Earth-orientation theory present a complicated narrative, and regrettably, revisions to IERS Conventions were unavailable when the 2009 update went into effect (when this work was prepared).[4] Also, with regard to the pre-2009 model, the authors have noted small discrepancies between the model documented by the IERS Conventions 2003 and source-codes available via the IERS,[*] the Standards of Fundamental Astronomy (SOFA),[†] and the US Naval Observatory's Vector Astrometry Subroutines (NOVAS).[‡] Thus it becomes prudent to document some of our experiences addressing the question "What is current standard for Earth orientation, and how precise is it?"

THE EQUINOX METHOD OF EARTH ORIENTATION

A three-dimensional rotation may be represented with as few as three rotations relative to any basis.[5] However, the complete change of basis between the celestial and terrestrial frames tends to be separated into several sequential rotations (*i.e.*, precession, nutation, *etc.*).[§] Intermediate frames are byproducts of these partial rotations.

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[*] ftp://tai.bipm.org/iers/conv2003/chapter5/

[†] http://www.iau-sofa.rl.ac.uk/

[‡] http://aa.usno.navy.mil/software/novas/novas_info.php

[§] In this paper, $\mathbf{R_1}(\cdot)$, $\mathbf{R_2}(\cdot)$, and $\mathbf{R_3}(\cdot)$ are rotations about the $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ axes respectively.

A precessing and nutating pole and equinox have long been the bases of traditional astrometric observations measuring Earth's rotation. Conceptually, the equinox is defined as the line of intersection between the plane of the equator (defined by a conventional "intermediate" pole for the Earth) and the plane of the ecliptic (defined by the mean angular momentum of the Earth's orbital motion).[6] Traditionally, precession and nutation have been modeled as separated effects, Earth-orientation models having the form:[7]

$$\begin{aligned} \mathbf{r}_{\mathrm{ICRF}} &= [\mathbf{B}]\ [\mathbf{P}(t_i,t_0)]\ [\mathbf{N}(t_i)]\ [\mathbf{S}_{\mathbf{EQX}}(t_i)]\ [\mathbf{W}_{\mathbf{EQX}}(t_i)]\ \mathbf{r}_{\mathrm{ITRF}} \\ \mathbf{r}_{\mathrm{ITRF}} &= [\mathbf{W}_{\mathbf{EQX}}(t_i)]^{\mathrm{T}}\ [\mathbf{S}_{\mathbf{EQX}}(t_i)]^{\mathrm{T}}\ [\mathbf{N}(t_i)]^{\mathrm{T}}\ [\mathbf{P}(t_i,t_0)]^{\mathrm{T}}\ [\mathbf{B}]^{\mathrm{T}}\ \mathbf{r}_{\mathrm{ICRF}} \end{aligned} \tag{1}$$

where t_i is Julian date in the Terrestrial time scale (TT), $\mathbf{r}_{\mathrm{ICRF}}$ is direction (location) with respect the ICRF, $[\mathbf{B}]$ is a frame-bias matrix representing an offset from the conventionally prescribed origin at epoch t_0, $[\mathbf{P}(t_i,t_0)]$ is the precession matrix from date t_0 to t_i, $[\mathbf{N}(t_i)]$ is the nutation matrix of date t_i, $[\mathbf{S}_{\mathbf{EQX}}(t_i)]$ is the sidereal rotation matrix of date t_i, $[\mathbf{W}_{\mathbf{EQX}}(t_i)]$ is the polar motion matrix of date t_i, and $\mathbf{r}_{\mathrm{ITRF}}$ is direction (location) with respect to the ITRF.

The classical Earth-orientation paradigm models the precessional drift $[\mathbf{P}(t_i,t_0)]$ of the conventional pole and equinox, to which quasi-periodic nutations $[\mathbf{N}(t_i)]$ are then applied. A simple sidereal rotation $[\mathbf{S}_{\mathbf{EQX}}(t_i)] = \mathbf{R}_3(-\theta_{\mathrm{GAST}})$ relative to the equinox about the pole almost fulfills the change of basis from a celestial to a terrestrial framework; an additional set of small-angle rotations known as polar motion, $[\mathbf{W}_{\mathbf{EQX}}(t_i)] = \mathbf{R}_3(-s')\ \mathbf{R}_2(x_{\mathrm{p}})\ \mathbf{R}_1(y_{\mathrm{p}})$, compensates for the fact that the Earth's conventional pole is not predictably aligned with the ITRF.[8] Tables of Earth-orientation parameters x_{p}, y_{p}, and UT1–UTC, are available through Celestrak[*] and the USNO IERS service center[†] (presently, the angle s' is conventionally approximated as a secular drift of about 47 μas per century). A discussion of the equinox method in particular, together with some numerical examples, is given by Vallado, Seago, and Seidelmann (2006).[9]

Until 2002, the equinox method was the only one in effect, where the equinox and pole were officially modeled by the IAU 1976 Theory of Precession and the IAU 1980 Nutation model.[10] Upon the adoption of the ICRF after 1997, the IERS began maintaining tabulated corrections ($\delta\Delta\varepsilon$ and $\delta\Delta\psi$) to the IAU 1980 Nutation model to relate the ITRF to the ICRF, because a sufficiently accurate IAU model had yet to be developed (Figure 1).[11] In 2000, the XXIV$^{\mathrm{th}}$ IAU General Assembly passed additional resolutions that would significantly impact operational methods for modeling Earth orientation relative to the ICRF.[‡]

CELESTIAL INTERMEDIATE ORIGIN (CIO) METHOD OF EARTH ORIENTATION

The classical transformation uses a temporal equinox that has an intermediate dependence on the ecliptic of date. However, modern Earth orientation relies on VLBI observations that are rather insensitive to the ecliptic plane.[12] To alleviate this dependence, IAU Resolution B1.8 (2000) recommended the use of a "non-rotating" *Celestial Intermediate Origin* (CIO) and *Celestial Intermediate Pole* (CIP) that could replace the traditional equinox of date and Celestial Ephemeris Pole, respectively. At the same time, the IAU also resolved that the IERS continue to provide users with data and algorithms for the traditional transformations, thus advocating two parallel methods achieving practically the same outcome.

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[*] http://www.celestrak.com/SpaceData/
[†] ftp://maia.usno.navy.mil/ser7/finals.all
[‡] http://syrte.obspm.fr/IAU_resolutions/Resol-UAI.htm

The adopted direction of the CIO was set close to the mean equinox at epoch J2000.0, but as a consequence of precession-nutation, the CIO slowly moves according to the kinematical property of a *non-rotating origin.*[13] In this context, "non-rotating" means that the CIO is constrained to have no celestial motion within the plane of the instantaneous equator and only travels perpendicularly to the instantaneous equator. This is in contrast to the mean equinox, which has celestial motion along the instantaneous equator due to precession and therefore "rotates" about the pole at the average rate of equatorial precession.

The CIO-based method is often represented with precession and nutation in combination, *i.e.*:

$$\begin{aligned}\mathbf{r}_{\text{ICRF}} &= [\mathbf{BPN}_{\mathbf{CIO}}(t_i)]\ [\mathbf{S}_{\mathbf{CIO}}(t_i)]\ [\mathbf{W}_{\mathbf{CIO}}(t_i)]\ \mathbf{r}_{\text{ITRF}} \\ \mathbf{r}_{\text{ITRF}} &= [\mathbf{W}_{\mathbf{CIO}}(t_i)]^{\mathrm{T}}\ [\mathbf{S}_{\mathbf{CIO}}(t_i)]^{\mathrm{T}}\ [\mathbf{BPN}_{\mathbf{CIO}}(t_i)]^{\mathrm{T}}\ \mathbf{r}_{\text{ICRF}}\end{aligned} \tag{2}$$

where t_i is Julian date in the Terrestrial time scale (TT), $\mathbf{r}_{\text{ICRF}}$ is direction (location) with respect the ICRF, and $[\mathbf{BPN}_{\mathbf{CIO}}(t_i)]$ describes both the large-scale secular motion and the quasi-periodic variability of the CIO and CIP. Here $[\mathbf{S}_{\mathbf{CIO}}(t_i)]$ equals $\mathbf{R}_3(-\theta_{\text{ERA}})$ and implies sidereal rotation about the CIP to the CIO from the terrestrial origin, where θ_{ERA} is the *Earth Rotation Angle* in radians:

$$\theta_{\text{ERA}} = 2\pi\,(0.779057273264 + 1.00273781191135448 \cdot T_u)\,. \tag{3}$$

Here T_u is Universal time (UT1) measured in days since epoch JD 2451545.0. $[\mathbf{W}_{\mathbf{CIO}}(t_i)]$ is the polar motion matrix of date t_i, and $\mathbf{r}_{\text{ITRF}}$ is direction (location) with respect to the ITRF.

The CIO method is convenient since the combined $[\mathbf{BPN}_{\mathbf{CIO}}(t_i)]$ matrix may be represented as the orthogonal transformation[12]

$$[\mathbf{BPN}_{\mathbf{CIO}}(t_i)] = \begin{bmatrix} 1-a\left(X(t_i)\right)^2 & -aX(t_i)Y(t_i) & X(t_i) \\ -aX(t_i)Y(t_i) & 1-a\left(Y(t_i)\right)^2 & Y(t_i) \\ -X(t_i) & -Y(t_i) & 1-a\left(\left(X(t_i)\right)^2+\left(Y(t_i)\right)^2\right) \end{bmatrix} \mathbf{R}_3(s(t_i))$$

$$a = \frac{1}{1+Z(t_i)}\,; \quad Z(t_i) = \sqrt{1-\left(X(t_i)\right)^2+\left(Y(t_i)\right)^2} \tag{4}$$

where $X(t_i)$ and $Y(t_i)$ are the direction-cosine components (or "coordinates") of the CIP unit vector with respect to the ICRF at epoch t_i. Conventionally, numerical expressions for $X(t_i)$ and $Y(t_i)$ are multiplied by the factor 1296000"/2π in order to express their unitless values in terms of arcseconds, as if the cosines were angles in radians (IERS Conventions 2003, p. 35).[4]

Realization of the CIO Locator *s*

The sidereal rotation $\mathbf{R}_3(s(t_i))$ accounts for the slight angular difference between the ICRF right ascension and the intermediate right ascension of the intersection of the ICRF and intermediate equators.[14] The CIO locator $s(t_i)$ is defined according to the integral expression

$$s(t_i) = \int_{t_0}^{t_i} \frac{X(t)\dot{Y}(t) - \dot{X}(t)Y(t)}{1+Z(t)}\,dt - C_0 \tag{5}$$

where the value of C_0 = 94 μas has been fitted taking into account the continuity constraint in UT1 as its definition changed on January 1, 2003. For operational purposes, Eq. (5) may be reformulated as

$$s(t_i)+\frac{X(t_i)Y(t_i)}{2}=\int_{t_0}^{t_i}\dot{X}(t)Y(t)dt-C_0-\int_{t_0}^{t_i}\left(\frac{1-Z(t)}{1+Z(t)}\right)\frac{X(t)\dot{Y}(t)-Y(t)\dot{X}(t)}{2}dt \qquad (6)$$

and solved using semi-analytic techniques.[15] Maintaining terms having effect larger than 0.1 μas, the result is a series representation having the combined form of a fourth-order polynomial with additional Fourier and Poisson terms as a function of the fundamental (Delaunay) arguments, *i.e.*,

$$\begin{aligned} s(t_i)+\frac{X(t_i)Y(t_i)}{2}&=C_0+\sum_n s_n t^n \\ &+\sum_k\left[C_{s,0}\sin\alpha_k+C_{c,0}\cos\alpha_k\right]+\sum_{k,j}\left[C_{s,j}\sin\alpha_k+C_{c,j}\cos\alpha_k\right]t^j \end{aligned} \qquad (7)$$

The quantity $s(t_i)+X(t_i)\,Y(t_i)\,/\,2$ is a slowly-changing quantity whose magnitude remains less than 6 mas throughout 1975 – 2050. The CIO locator $s(t_i)$ is realized from Eq. (7) by subtracting $X(t_i)\,Y(t_i)\,/\,2$.

If the equinox method is employed, it also becomes possible to realize the CIO via the so-called *equation of the origins* (EO), the angle between the true equinox of date and the CIO along the intermediate equator, which can be realized using analytical approximations or numerical integrations. Use of the equinox and EO avoids explicit use of the CIO locator *s*, which is the approach adopted by NOVAS.*

THE IAU 2000A PRECESSION-NUTATION MODEL

IAU Resolution B1.6 (2000) was adopted at the same time as IAU Resolution B1.8 (2000). It stated that the IAU 1976 Precession Model and IAU 1980 Theory of Nutation should be replaced by the (so-called) MHB2000 precession-nutation model beginning January 1, 2003. It also stated that MHB2000 precession-nutation would be officially known as “IAU 2000A”.†

Resolution B1.6 (2000) cited a manuscript submitted to the *Journal of Geophysical Research* by Mathews, Herring and Buffett, as the basis of the MHB2000 transfer functions.[16] While that paper provides theoretical considerations and outlines the formalisms for the MHB2000 solution, it does not include sufficient information to apply the MHB2000 theory. Rather, Resolution B1.6 (2000) stated that the IAU 2000A theory is to be “as published in the IERS Conventions.”

Precession Model (Traditional Form)

The MHB2000 model employs the customary form of precession involving three Euler angles, *i.e.*,

$$[\mathbf{P}(t_i,t_0)]=\mathbf{R}_3(\zeta_A)\,\mathbf{R}_2(-\theta_A)\,\mathbf{R}_3(z_A)\,. \qquad (8)$$

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* http://aa.usno.navy.mil/software/novas/new_novas_f/NOVAS_F3.0g.f: subroutine EQXRA.

† A truncated version of the MHB2000 nutation theory – IAU 2000B – was also recommended, for those needing a less-accurate model (~1 mas).

It has the benefit of involving the minimum number of angles for evaluating precession alone, and its usage is most familiar. An illustration of the angles z_A, θ_A, and ζ_A on the celestial sphere can be found in a variety of references, such as Seidelmann (1992).[17]

Canonical 4-Rotation (Capitaine *et al.*) Form of Precession

There is a slight bias between the fundamental axes of the ICRF (defined by the adopted coordinates of quasi-stellar radio sources) and the directions of the CIP and CIO predicted by the MHB2000 precession-nutation theory at epoch J2000. Capitaine and Wallace (2006) suggest that the traditional Euler-angle form for precession $[\mathbf{P}(t_i,t_0)]$ is "no longer useful" due to the present need to apply an additional small rotation **[B]** to account for this bias, thus causing the traditional angles z_A and ζ_A undergo rapid changes around the J2000 epoch which are not as concisely approximated by a traditional time polynomial.[14] An alternative representation of the MHB2000 precession was therefore proposed by Capitaine *et al.* (2003) and published in the IERS Conventions 2003.[18] This form involves the so-called ecliptic-precession angles, which attempts to cleanly separate (luni-solar) precession of the equator from (planetary) precession of the ecliptic:

$$[\mathbf{P}(t_i,t_0)] = \mathbf{R_1}(-\varepsilon_0)\,\mathbf{R_3}(\psi_A)\,\mathbf{R_1}(\omega_A)\,\mathbf{R_3}(-\chi_A)\,. \tag{9}$$

An illustration of the angles χ_A, ω_A, ψ_A, and ε_0 on the celestial sphere is found in Figure 4 of Lieske *et al*. (1979).[19] The differences between the traditional form and the "canonical 4-rotation" form remain less than 1 μas after four centuries (IERS Conventions 2003, p. 45).[4] Updated expressions for these angles, consistent with the P03 precession model (to be discussed in the sequel), are given in the *Report of the IAU Division I Working Group on Precession and the Ecliptic* and in *USNO Circular № 179*, p. 43.[10, 20]

Nutation Model (Traditional)

The traditional expression for nutation $[\mathbf{N}(t)]$ is

$$[\mathbf{N}(t_i)] = \mathbf{R_1}(\varepsilon_A)\,\mathbf{R_3}(\Delta\psi)\,\mathbf{R_1}(\,\varepsilon_A+\Delta\varepsilon\,) \tag{10}$$

where ε_A is the obliquity of the ecliptic at date t_i, $\Delta\psi$ is nutation in longitude, and $\Delta\varepsilon$ is the change in obliquity due to nutation. The two nutation angles are represented by a rather lengthy geometric series having the form:

$$\Delta\psi = \sum_{i=1}^{1365}\left\{\left(S_i + \dot{S}_i\,T\right)\sin(\Phi_i) + \left(C_i' + \dot{C}'_i\,T\right)\cos(\Phi_i)\right\}$$

$$\Delta\varepsilon = \sum_{i=1}^{1365}\left\{\left(C_i + \dot{C}_i\,T\right)\cos(\Phi_i) + \left(S_i' + \dot{S}'_i\,T\right)\sin(\Phi_i)\right\} \tag{11}$$

$$\Phi_i = \sum_{j=1}^{14} M_{i,j}\,\varphi(T)$$

where T is number of Julian centuries (TT) since epoch J2000, $\varphi(T)$ are the fourteen fundamental arguments (five of which are luni-solar Delaunay arguments l, l', F, D, and Ω, and eight of which are planetary mean longitudes). Tables of M, S (in-phase terms), and C (out-of-phase terms) have been maintained as electronic supplements to the IERS Conventions 2003.[*] Tables are also published as an appendix to *USNO Circular № 179*.[10] Of the 1365 series terms, 678 are luni-solar contributions and 687 are planetary contributions.

Known Nutation Model Variations (IAU 2000A)

Prior to the publication of the IERS Conventions 2003, early adopters of the IAU 2000A theory relied on a *c*. 1999 FORTRAN source-code realization named MHB_2000.f.[†] This code introduced slight variations from the theory that was eventually published by the IERS Conventions 2003. These variations, which have been adopted into other published source-code realizations distributed by SOFA, IERS, and NOVAS, are now described.

Omission of Luni-Solar Out-of-Phase Rate Terms. The secular rates $\dot{C}'_i$ and $\dot{S}'_i$ in Eq. (11)[‡] are included in the online supplement of the IAU 2000A nutation series from the IERS Conventions. However, only seven of the 1365 out-of-phase rate terms have non-zero values, all which are luni-solar terms. Of these seven non-zero rate terms, the largest is only about 3 μas/century, such that their total effect does not exceed more than 1 μas/century within a few decades of the J2000 epoch.[§] Understanding that the overall accuracy of the MHB2000 theory was probably no better than 20 μas over time intervals longer than this, these out-of-phase rate terms were dropped from MHB_2000.f.[**]

Planetary Nutation. In the evaluation of planetary nutations, the Delaunay arguments l, l', F, D, and Ω are changed to linear expressions (*cf.*, Souchay *et al.*) in MHB_2000.f, but these arguments are *not* changed when evaluating luni-solar nutations. This is acknowledged in IERS Conventions 2003, which mentions (p. 48) that "simplified expressions are used for the planetary nutation" which appear to cause a maximum difference in the nutation amplitudes at "less than 0.1 μas." However, for the time period of interest in this paper (1975 – 2050), $|T| < 1$ in Eq. (11) such that the maximum effect on the CIP coordinates was observed to reach merely 0.003 μas by the year 2046.

Neptune's Mean Longitude. Planetary mean longitudes make up eight of the fundamental arguments $\varphi(T)$ in Eq. (11). The authors found that the mean longitude and rate of Neptune adopted by MHB_2000.f and subsequent source-code realizations, *i.e.*,

$$\lambda_{\text{Neptune}} = 5.321159000 + 3.8127774000\,T \tag{12}$$

differs notably from other basic sources when converted to units of radians, *i.e.*,[21]

$$\lambda_{\text{Neptune}} = 5.311886287 + 3.8133035638\,T\,. \tag{13}$$

The numerical effect of substituting Eq. (12) with Eq. (13) causes a sinusoidal difference on the CIP coordinates (primarily Y) of amplitude 0.035 μas and with a period of about 80 years.

The combined effects of these source-code changes (omission of non-zero out-of-phase rates terms, truncated Delaunay arguments for planetary nutations, and changing Neptune's mean longitude) is illustrated by Figure 2. The figure is dominated by the effect of the omitted luni-solar

[*] ftp://tai.bipm.org/iers/conv2003/chapter5/tab5.3b.txt, ftp://tai.bipm.org/iers/conv2003/chapter5/tab5.3c.txt
[†] http://www-gpsg.mit.edu/~tah/mhb2000/MHB_2000.f
[‡] These terms correspond to A‴ and B‴ of Eq. 5-29 in Table 5.3a of the IERS Conventions 2003.
[§] http://hpiers.obspm.fr/eop-pc/models/nutations/nut_MHB2000-UAI1980.txt
[**] Herring, T. (2008), Personal correspondence of May 8th. Professor of Geophysics, Department of Earth, Atmospheric and Planetary Sciences, Massachusetts Institute of Technology, Cambridge, MA.

out-of-phase rate terms, although the noticeable offset in ΔY at epoch J2000 of approximately +0.03 μas is attributable to the change in Neptune's mean longitude. This figure establishes that the precision of MHB_2000.f and its descendant realizations is no better than about 1 μas through 2050, and is mainly due to the omission of the out-of-phase rate terms.

Procedural Options for IAU 2000A Precession-Nutation

Figure 3 provides an overview of Earth-orientation modeling options after 2002, when the MHB2000 precession-nutation model came into official use and the CIO method was alternatively adopted. The *Traditional* path of Figure 3 is the classical equinox-based path analogous to Figure 1 employing the conventional precession angles used in Eq. (8). The resulting bias-precession matrix $[\mathbf{BP}]^T$ is multiplied with the nutation matrix $[\mathbf{N}]^T$ of Eq. (10) using MHB 2000-nutation angles to form the $[\mathbf{BPN}]^T$ matrix. The *4-rotation* path is identical to the *Traditional* path except one uses the "canonical 4-rotation" precession model to realize **[P]**, instead of the traditional 3-rotation model.

Direction cosines X and Y for the CIO method are realized from the (3,1) and (3,2) elements of the equinox-based **[BPN]** matrix. This is designated as the *CIO Full Theory* path in Figure 3, which branches from the $[\mathbf{BPN}]^T$ matrix (actually, two branched paths are indicated since **[P]** may be realized using two different forms of precession). A separate means of realizing X and Y is to use a time-series representation of these matrix elements, as provided via the IERS and SOFA.[14, 22] This is designated as the *Series* path in Figure 3. Terms smaller than 0.1 μas/century are reportedly omitted from these published series, such that the maximum difference between the equinox and CIP series representation exceeds 2 μas from 1975 – 2050 (Figure 4). Still another means of realizing X and Y is to interpolate them from **[BPN]**-matrix elements tabulated by time. This method is designated as the *Interpolation* path in Figure 3, which is discussed in the sequel.

2006 PRECESSION-NUTATION UPDATES

The original MHB2000 model used an updated IAU-76 precession model that applied empirically observed adjustments to the precession constant and to rate of change of obliquity. However, this corrected precession model did not constitute a new, dynamically-consistent precession theory, so IAU Resolution B1.6 (2000) encouraged the development of new expressions for precession consistent with the new IAU 2000A nutation model. Consequently, the XXVI[th] IAU General Assembly passed Resolution B1 (2006) adopting the (so-called) P03 precession model to replace the MHB2000 precession, effective 1 January 2009.*

By 2050, the differences between the MHB2000 and P03 precession models reach about 0.6 mas in CIP coordinates (Figure 5), and updates to Greenwich apparent sidereal time θ_{GAST} affect a change of about 1.8 mas (Figure 6). Updates to the 2006 series representation of s (as published by SOFA) are quite small (Figure 7). Comparing Figure 2 with Figure 7, the change in s due to the P03 precession update is presently below the precision of the nutation model.

Fukushima-Williams (F-W) Form of Precession

Resolution B1 (2006) acknowledged that the choice of precession parameters should be left to the user, because one cannot expect any consensus on what precession parameters may be "best" for particular studies.[18, 20] Consequently, the Fukushima-Williams (F-W) form of precession has been additionally proposed for use with the P03 theory. The F-W form is originally due to Wil-

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* http://www.iau.org/static/resolutions/IAU2006_Resol1.pdf

liams, but was later modified by Fukushima to avoid a singularity caused by bias offsets near the epoch of origin.[23, 24] It takes the form:

$$[\mathbf{BP}(t_i,t_0)] = \mathbf{R_3}(-\gamma)\ \mathbf{R_1}(-\varphi)\ \mathbf{R_3}(\psi)\ \mathbf{R_1}(\varepsilon_A)\ ^{*} \tag{14}$$

which is analogous to "Method 2" of Capitaine and Wallace (2006).[22] A physical interpretation of the angles ε_A, ψ, φ, and γ, is given by Fukushima (2003).[24]

The F-W form has an operational advantage since $\Delta\psi$ and $\Delta\varepsilon$ serve as corrections to the angles ε_A and ψ of Eq. (14). The result is that precession and nutation can be realized *in combination* using only four rotations, *i.e.*,

$$[\mathbf{BPN}(t_i,t_0)] = \mathbf{R_3}(-\gamma)\ \mathbf{R_1}(-\varphi)\ \mathbf{R_3}(\psi+\Delta\psi)\ \mathbf{R_1}(\varepsilon_A+\Delta\varepsilon)\ . \tag{15}$$

While each of the three forms mentioned (Eq. (8), Eq. (9), and Eq. (14)) are presented in the 2009 Astronomical Almanac (p. 52) for the P03 precession theory, the F-W form is implemented by SOFA and it is also the form included in numerical examples from the Almanac.[25] It is also explicitly recommended and used by Wallace and Capitaine (2006).[22]

Known Nutation Model Variations (2006)

The originators of the P03 precession theory recommend a scale correction to the nutation theory to make it more dynamically compatible with new P03 precession model.[26] The recommended adjustment is:

$$\begin{aligned} \Delta\psi_{2006} &= (1.0000004697 + f)\ \Delta\psi_{2000\mathrm{A}} \\ \Delta\varepsilon_{2006} &= (1 + f)\ \Delta\varepsilon_{2000\mathrm{A}} \end{aligned} \tag{16}$$

where $f \equiv -2.7774\times 10^{-6}\ T$, T being the time interval since epoch J2000 measured in Julian centuries TT.

Using $\Delta\psi_{2006}$ and $\Delta\varepsilon_{2006}$ results in what is effectively another nutation theory for use with P03 precession, which adjusts the CIP location up to 10 μas by the year 2034 (Figure 8). The SOFA initiative implements these adjustments, citing them as "the IAU 2006 adjustments to the IAU 2000A nutation model,"[†] and "P03-adjusted IAU 2000A nutation."[22] These adjustments were reportedly used to generate the tables of the 2009 Astronomical Almanac (although their effect is far below the precision maintained by its published tables).

According to originators of the MHB2000 nutation theory, its original accuracy was limited to "tens of μas" thirty years beyond epoch J2000.[‡] Also, the CIP location can only be accurately predicted to about 2mas/century.[14] If one assumes that the effect of the P03 scale update is near the level of nutation-model precision, and below the level of nutation-model accuracy, then the necessity of the scale update might be questioned. It might also be questioned for another reason; namely, IAU Resolutions B1.6 (2000) and B1 (2006) seemingly limit the scope of changes to the

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[*] Equation (14) can represent either **[P]** or **[BP]**, depending on the expressions adopted for ψ, φ, and γ. A "*j*" or "J2000" subscript is sometimes added to these variables for expressions representing precession alone, while an over-bar, prime, or "GCRF" subscript may imply that bias is included in the angular expressions for precession. Variable subscripts are omitted here for aesthetic reasons.

[†] http://www.iau-sofa.rl.ac.uk/2008_0301/sofa/manual.pdf, p. 15

[‡] http://hpiers.obspm.fr/eop-pc/models/nutations/nut_MHB2000-UAI1980.txt

precession model only. At this writing, the NOVAS 3.0 does not rescale the MHB2000 model, and recommendations via updated IERS Conventions are unavailable.

Procedural Options for IAU 2006 Precession-Nutation

Capitaine and Wallace (2006) have reportedly identified six ways of computing $[\mathbf{BPN}(t_i)]$, three ways of locating the CIP, and eight ways of locating the CIO.[14, 22] In all cases the "precision goals are a few microarcseconds over a time span of a few hundred years (*i.e.*, about three orders of magnitude better than the expected accuracy of the prediction), meeting the requirements of high-accuracy applications."[14] Some of these procedures are nominally illustrated in Figure 9, which primarily differs from Figure 3 in that another path has been added for the F-W form of precession. This addition also results in one more option through the *CIO Full Theory* path. (Although the P03-based scale adjustment to MHB2000 nutation effectively creates another nutation theory, the doubling of paths in Figure 9 has been omitted for clarity.)

TABULATION–INTERPOLATION METHOD (CIO-BASED)

Anticipating the initial release of the IAU 2000A theory, Seago and Vallado (2000) proposed tabulating and interpolating the direction-cosines of the bias-precession-nutation matrix.[27] Using the IERS 1996 nutation theory as a proxy, their timing trials suggested that a tabulation-interpolation method should be orders of magnitude faster than the complete IAU 2000A theory. Because GPS observations now permit the determination of sub-daily periodic motions of the CIP, and because periodic motions under two days (as experienced in a space-fixed system) are conventionally excluded from precession-nutation, no spectral content should be lost when precession-nutation data are tabulated at a daily frequency, per the Nyquist sampling theorem.[*]

Table 1. Loss of Precision in CIP Series and Lagrange Interpolation of Daily Nodes (1975 – 2050)[†]

	X			*Y*		
CIP Series Representations	Mean (μas)	Std. Dev. (μas)	Max. Dev. (μas)	Mean (μas)	Std. Dev. (μas)	Max. Dev. (μas)
2000A (SOFA)	-0.11	0.31	1.5	-0.31	0.65	2.5
2006 (SOFA)	-0.07	0.32	1.4	0.13	0.65	2.2
Interpolation						
5th-order	0.00	6.3	27	0.0	6.8	30
7th-order	0.00	0.88	4.4	0.0	0.94	4.8
9th-order	0.00	0.19	1.0	0.0	0.20	1.1
11th-order	0.00	0.06	0.30	0.0	0.06	0.33
13th-order	0.00	0.02	0.11	0.0	0.02	0.12

[*] http://www.iau.org/static/resolutions/IAU2006_Resol1.pdf
[†] The times of evaluation were chosen to always provide worst-case results for interpolation.

The tabulation-interpolation method is illustrated as the *Interpolation* paths in Figure 3 and Figure 9. When using the CIO method with Eq. (4), tabulation and interpolation of the CIP coordinates X and Y, and the CIO-locator s, is most efficient (and since s is a very small angle, $\mathbf{R_3}(s)$ could be reasonably approximated without sines and cosines, and the square root of Z could be replaced by a 4^{th}-order series approximation of a accurate to 0.1 μas).[14]

To evaluate the loss of precision due to interpolation (*i.e.*, interpolation error), the authors tabulated the values of X, Y, and s at 0^h TT daily (in units of arcseconds). The table was interpolated using a variable-order Lagrange interpolator evaluated at 12^h TT daily. The interpolated values were then compared to the fully precise results at 12^h, and the resulting differences from 1975 to 2050 are listed in Table 1 and illustrated in Figure 10. For comparison, the CIP series representations due to SOFA were also evaluated at these same times and included in Table 1; however, it is important to stress that the *interpolator results are worst cases by design*, since evaluation at 12^h is farthest from the 0^h interpolation nodes and provides maximum interpolation errors. Results from the series representations are representative only, and are not necessarily worst cases.

Table 2. Relative Timing of Different Methods.

Method	*Relative timing*	
	4 wks @ 5 sec	*75 yrs @ 1 hr*
IAU 2000A/2006 (9^{th} Order Lagrange interpolation)	*1.0*	*1.0*
IAU 1976/80/82 (JPL Chebychev interpolation)	1.7	2.5
IAU 1976/80/82 (half-hour constant)*	0.15	12
IAU 1976/80/82	12	12
IAU 2000A/2006	400	380
IAU 2000A/2006 (CIP-series representation)	580	560

Timing results using a typical desktop computer are noted in Table 2. The algorithms were evaluated by incrementing the independent variable every five (5) seconds for four weeks, and also by incrementing every hour for 75 years. The total evaluation times are reported relative to the timing of 9^{th}-order interpolation. Tests were run more than once to affirm that the reported timings are uncertain to only a few percent (because of other processes affecting computing hardware). Series representations of CIP coordinates X and Y tend to be slower because there are more trigonometric terms involved.

To reduce the computational burden of the IAU 2000A nutation theory, approximation models continue to be developed to varying degrees of accuracies.[28] However, the tabulation-interpolation method seems most profitable as it does not require additional theory development and precision losses can be controlled simply by adjusting the order of interpolation. Since precision losses can be made lower than the precision of the MHB2000 nutation model, interpolation

* The "half-hour constant" method refers to a default approximating option employed by prior versions of AGI's Satellite Tool Kit. Here the precession-nutation matrix was held constant for up to thirty minutes to reduce the number of evaluations on earlier, slower hardware. Time steps that are less than 30-minutes show a speed benefit under this mode, but are less accurate.

likely obviates the need for other approximations that are less precise, such as CIP series representations, or truncated nutation series (*e.g.*, IAU 2000B, NU2000K via NOVAS, *etc*.). Most importantly, the tabulation-interpolation method is able to vastly improve evaluation speed (at the minor expense of increased memory utilization). This conclusion may already be obvious, as the JPL developmental ephemerides (DE) series have optionally provided interpolated IAU 1980 nutation for many years, and the suitability of tabulation and interpolation has been evaluated recently for on-board satellite operations.[29]

AGI SOFTWARE IMPLEMENTATION

The Analytical Graphics, Inc. (AGI) software suite employs intensive mathematical calculations and graphical animations that tend to be CPU-demanding, such that performance improvements in precession-nutation computations are especially warranted. Beginning with software products released in 2009, AGI will introduce the IAU 2000A and IAU 2006 theories based on the CIO method to support realization of the ICRF. AGI's implementation of the 2000A model employs traditional precession per Eq. (8), while the 2006 implementation uses the F-W form per Eq. (15) and the P03 scale adjustments of Eq. (16). Software options include the use of the tabulation-interpolation method (very fast), the full IAU 2000A and IAU 2006 theories (slow), and CIP series representations (slower). Ninth-order interpolation of X, Y, and s is the default behavior, which maintains a maximum precision loss of 1 μas.

There is a rising opinion that source code is to be preferred over paper documentation, because source code is the only practical method of stating an algorithm unambiguously and without room for interpretation.[30] When faced with the decision to develop software based on public-domain documentation versus published source-codes (that appear to differ in precision at perhaps tens of μas), it seems prudent to favor source code behaviors, although not for the aforementioned reason. Rather, those who are obliged to validate system software will of necessity compare against the numerical results of published source codes instead of documentation. Having a design goal to emulate the behavior of SOFA routines, testing has shown that AGI implementations of precession-nutation match to at least 10 significant digits over the period of interest in this paper.

SOME OTHER POSSIBLE SOURCES OF PRECISION DIFFERENCES

Though the independent variable in precession-nutation theory is strictly TDB, TDB usually must be approximately related to TT. Throughout this paper, TDB ≈ TT has been assumed, because this is the most convenient and simplest approximation, and a more precise relationship would only be arbitrarily and insignificantly "better". A more accurate relationship for TDB is used by NOVAS; which causes differences at a few hundredths of a microarcsecond (IERS Conventions 2003, p. 45, 48).

There is another variant of MHB_2000.f (MHB_2000FT.f) which is a "fine-tuned" version of the nutation theory having the amplitudes of the five longest-period terms estimated (6798, 3399, 1615, 1305, 1095 days).* The differences between this version and the original MHB2000 model appear to be less than 50 μas, and it does not appear to be an officially sanctioned model.

With regard to the total transformation between the ICRF and ITRF, precision differences can also result from the handling of Earth-orientation parameters. These include variations in the methods for interpolating free-core nutation corrections, as well as UT1-UTC and polar motion.

* http://www-gpsg.mit.edu/~tah/mhb2000/MHB_2000ft.f

Of course the omission of free-core nutation corrections and sub-diurnal corrections to EOPs is commonplace; usually their effect is measureable in milliarcseconds and therefore above the precision differences primarily discussed in this paper.

CONCLUSIONS

Orbital analysts who seek precise compliance with the ICRF should be aware that the IAU precession model officially changed to the IAU 2006 (P03) precession model effective January 1, 2009. This update results in model changes on the order of a milliarcsecond within the next fifty years. Citing numerical results, published commentary regarding the precision goals of realized models, and differences of result from using alternative models (*e.g.*, series-representation methods), the authors deduce that realizations of IAU precession-nutation theory are precision limited to the microarcsecond level over the next half-century, while being uncertain to the milliarcsecond level.

Three different forms of precession now exist (traditional, “canonical 4-rotation”, and Fukushima-Williams), the numerical differences of which are inconsequential but no less confusing in their multiplicity. Surprising consistency has been maintained in source-code implementations of the MHB2000 nutation theory because they all descend from a common FORTRAN ancestor; however, this code omitted terms effective at the 1-μas level within the next fifty years. With the recent adoption of the P03 precession model, a scale adjustment to the nutation model also has been proposed, affecting results at the 10-μas level within the next fifty years (although it is not obvious that nutation changes are sanctioned by IAU resolutions). Meanwhile, the CIO locator s changed by about 0.1 μas within the next fifty years.

We conclude that efforts by system developers and software testers to precisely match any particular realization of precession-nutation (*e.g.*, SOFA, NOVAS, *etc.*) below the microarcsecond level are likely to be unprofitable, because digits maintained below this level are likely to be arbitrary. Taking this fact to full advantage, the authors propose to tabulate and interpolate the CIP coordinates X and Y, and CIO locator s, and show that this method can vastly improve computing speed without degrading model precision. Due to its practicality and simplicity, this approach has been adopted as the default method for modeling Earth orientation in upcoming releases of AGI’s software suite.

Ultimately, the question must be raised as to whether spacecraft applications need to update to the latest IAU 2006 standards. The answer depends on the requirements of the application. This paper assays some numerical differences that may be helpful in answering that question. However, differences caused by the update do not tend to exceed the effects of sub-diurnal EOP and unpredictable free-core nutation corrections (via dX and dY Earth-orientation parameters), and these effects are miniscule enough that they tend to be applied only to a minority of spacecraft applications.

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APPENDIX: FIGURES

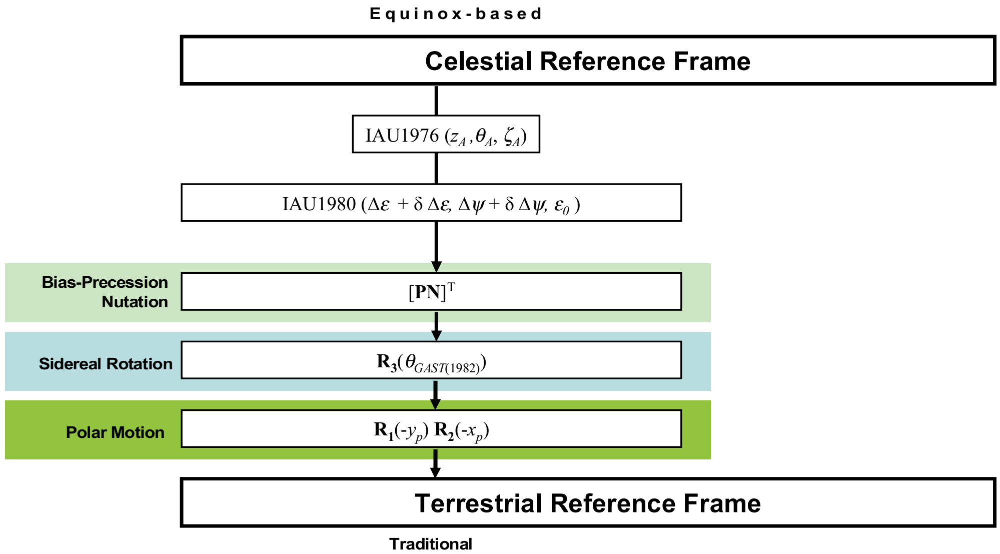

Figure 1. The Pre-IAU 2000A Procedure Relating the ICRF and the ITRF (1998).

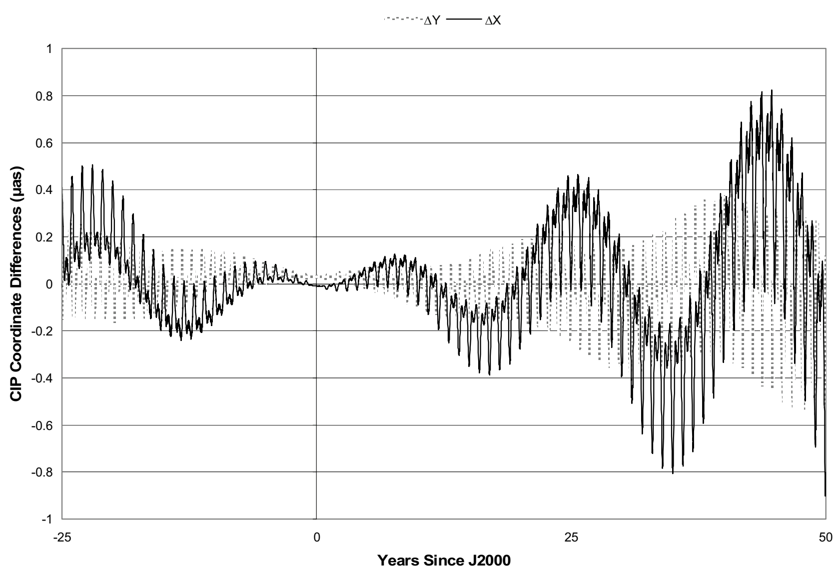

Figure 2. Known Losses of Precision in CIP Coordinates Inherited via MHB_2000.f.

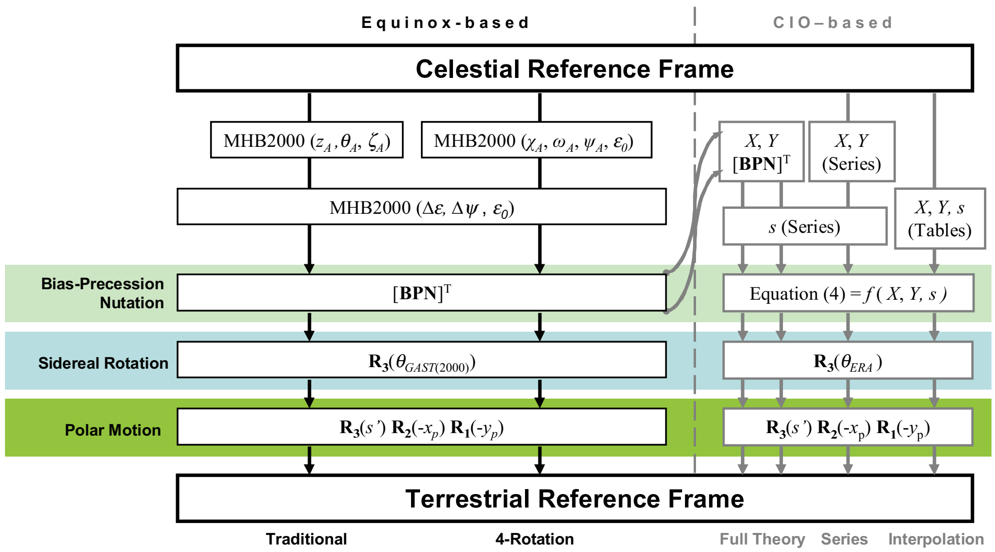

Figure 3. Some Procedures Relating the ICRF and the ITRF (2003).

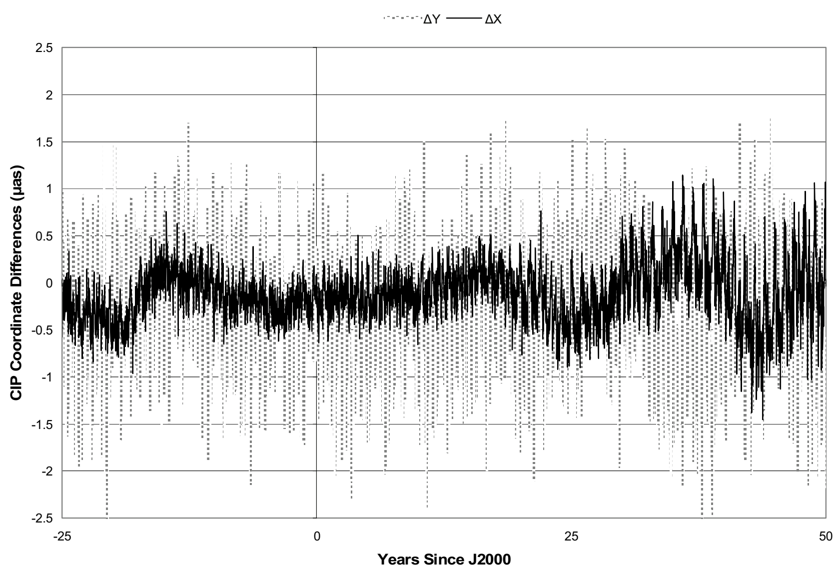

Figure 4. Precision Loss from Series Representation of CIP Coordinates (IAU 2000A).

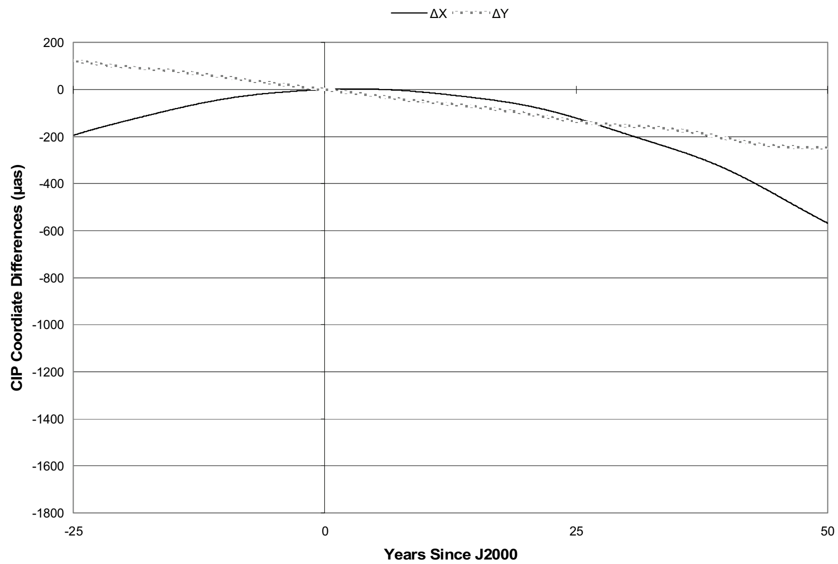

Figure 5. Change in CIP Coordinates Due to the Adoption of IAU 2006 (P03) Precession.

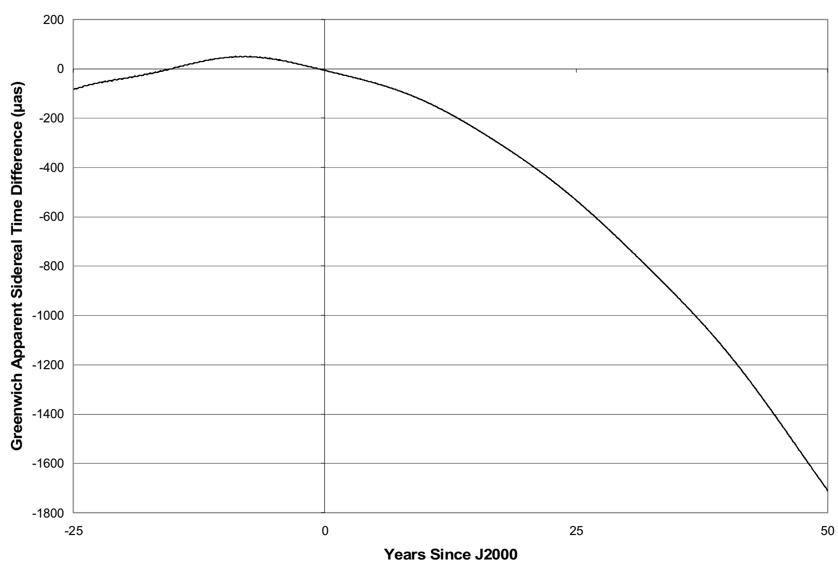

Figure 6. Change in Greenwich Apparent Sidereal Time Due to the Adoption of IAU 2006 (P03) Precession.

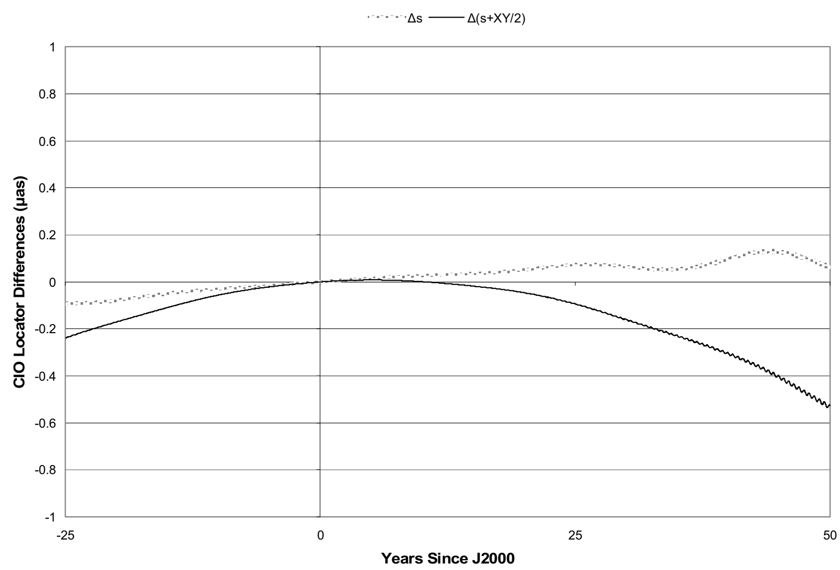

Figure 7. Change in s and (s + XY/2) Due to the Adoption of IAU 2006 (P03) Precession.

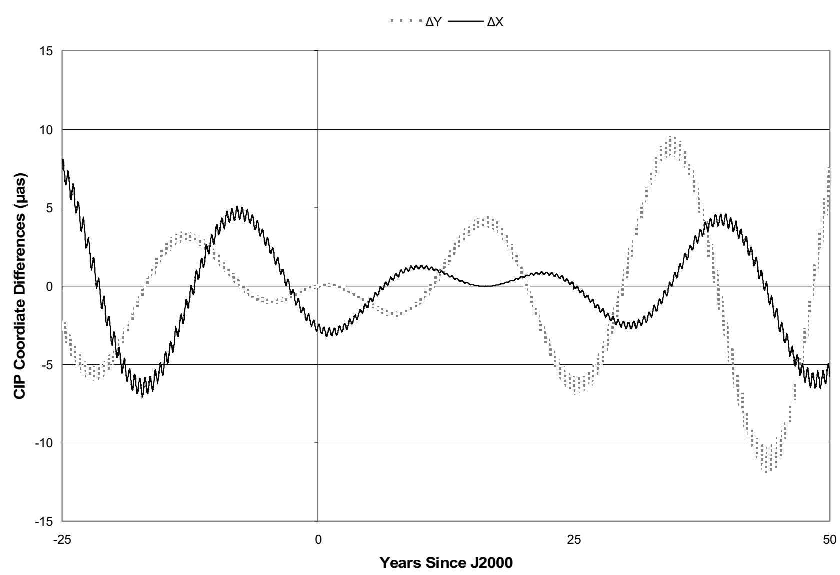

Figure 8. Differences Due to P03-Derived Scale Adjustments to IAU 2000A (MHB2000) Nutation.

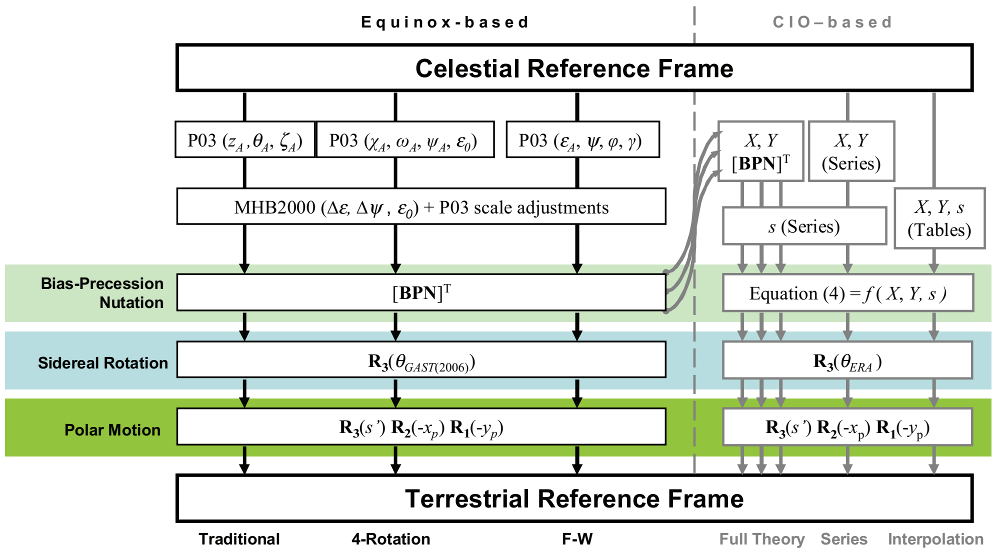

Figure 9. Some Procedures Relating the ICRF to the ITRF since 2009.

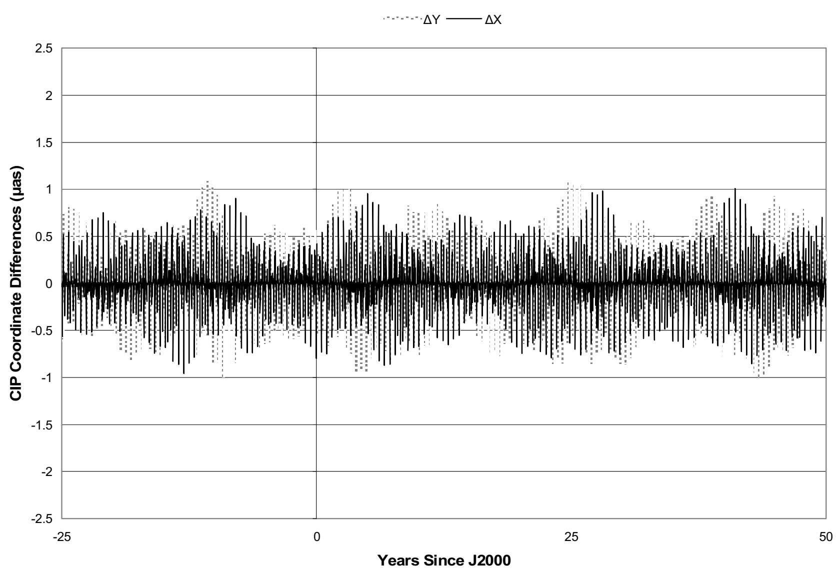

Figure 10. Precision Loss Due to 9th Order Interpolation of CIP Coordinates (IAU 2006).

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