Empirical Macroeconomics

Non “Recoverable” shocks

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Kalman Filter

Derivation

Consider a simple State Space Model (SSM)

$$S_{t+1} = AS_t + B\eta_{t+1}$$
$$X_t = CS_t$$

where S_t is the state vector (unobservable variables like the factors in our previous paper), η_t are the shocks to the states and X_t are observables variables.

Kalman Filter

Forecasts Based on Linear projection

Define the linear least square forecats of the state vector on the basis of the data observed,

$$\hat{S}_{t+1|t} = E\left(S_{t+1}|\mathcal{X}_t\right)$$

is the linear projection of S_{t+1} on $\mathcal{X}_t = (X_t', X_{t-1}', ...X_1')$. The Kalman calculates recursively $\hat{S}_{1|0}, \hat{S}_{2|1,...}\hat{S}_{T|T-1}$ in succession. Associated with each forecast there is a MSE

$$\Sigma_{t+1,t} = E\left(\left(S_{t+1} - \hat{S}_{t+1|t}\right)\left(S_{t+1} - \hat{S}_{t+1|t}\right)'\right).$$

Key results of linear projections

Forecasts Based on Linear projection

Linear projection:

$$\hat{Y}_{t+1|t} = \alpha' \mathcal{Z}_t$$

where α is such that the forecast error

$$E\left(\left(Y_{t+1} - \alpha' \mathcal{Z}_t\right) \mathcal{Z}_t'\right) = 0'$$

is uncorrelated with $\mathcal{Z}_t$. (smallest mean squared error among linear forecasting rules). Notice

$$E\left(Y_{t+1} \mathcal{Z}_t'\right) = \alpha' E\left(\mathcal{Z}_t \mathcal{Z}_t'\right)$$

which implies

$$\alpha' = E\left(Y_{t+1} \mathcal{Z}_t'\right) \left[E\left(\mathcal{Z}_t \mathcal{Z}_t'\right)\right]^{-1}$$

Notice similarity with OLS formula. However it is constructed with Population moments while OLS with sample moments.

Key results of linear projections

Updating a linear projectopn

You have obtained α through the Linear projection:

$$\hat{Y}_{t+1|t} = \alpha' \mathcal{Z}_t = E\left(Y_{t+1}\mathcal{Z}_t'\right)\left[E\left(\mathcal{Z}_t\mathcal{Z}_t'\right)\right]^{-1}\mathcal{Z}_t$$

write it as $P\left(Y_{t+1}|\mathcal{Z}_t\right)$. Now you receive new information W_t that you can use to update the forecast. You can show that the new forecast is

$$\begin{aligned} P\left(Y_{t+1}|\mathcal{Z}_t, W_t\right) = {} & P\left(Y_{t+1}|\mathcal{Z}_t\right) \\ & + E\left\{\left[Y_{t+1} - P\left(Y_{t+1}|\mathcal{Z}_t\right)\right]\left[W_{t+1} - P\left(W_{t+1}|\mathcal{Z}_t\right)\right]\right\} \\ & \times \left[E\left\{\left[W_{t+1} - P\left(W_t|\mathcal{Z}_t\right)\right]\left[W_{t+1} - P\left(W_t|\mathcal{Z}_t\right)\right]'\right\}\right]^{-1} \\ & \times \left[W_{t+1} - P\left(W_t|\mathcal{Z}_t\right)\right] \end{aligned}$$

Which intuitively add to the projection on the initial information a term which is the linear projection the initial projection error on the projection error of obtained by projecting the new information on the initial information.

Kalman Filter Derivation

Start the Recursion

Start with the unconditional mean of the States:

$$\hat{S}_{1|0} = E\left(S_1\right)$$

and the associated MSE

$$\Sigma_{1,0} = E\left(\left(S_1 - \hat{S}_{1|0}\right)\left(S_1 - \hat{S}_{1|0}\right)'\right)$$

The initial values can be motivated by the property of the model or be guesses.

Kalman Filter Derivation

Forecasting X_t

with conditional mean of the States:

$$\hat{X}_{t|t-1} = E\left(X_t | \mathcal{X}_{t-1}\right) = C\hat{S}_{t|t-1}$$

Kalman Filter Derivation

Forecasting X_t

The error of the forecast:

$$X_t - \hat{X}_{t|t-1} = CS_t - C\hat{S}_{t|t-1}$$

The variance of the the forecast error:

$$\begin{aligned} E\left[\left(X_t - \hat{X}_{t|t-1}\right)\left(X_t - \hat{X}_{t|t-1}\right)'\right] &= E\left[C\left(S_t - \hat{S}_{t|t-1}\right)\left(C\left(S_t - \hat{S}_{t|t-1}\right)\right)'\right] \\ &= C\Sigma_{t|t-1}C' \end{aligned}$$

Kalman Filter Derivation

Updating S_t

We can now update our inference of S_t using X_t, $\hat{S}_{t|t} = E\left(S_t | \mathcal{X}_{t-1}, X_t\right)$ using the the formula to update a linear projection

$$\hat{S}_{t|t} = \hat{S}_{t|t-1} +$$
$$E\left[\left(S_t - \hat{S}_{t|t-1}\right)\left(X_t - \hat{X}_{t|t-1}\right)'\right] E\left[\left(X_t - \hat{X}_{t|t-1}\right)\left(X_t - \hat{X}_{t|t-1}\right)'\right]^{-1} \times$$
$$\left(X_t - \hat{X}_{t|t-1}\right).$$

Kalman Filter Derivation

Updating S_t

Notice the terms of the big matrix of the projection:
$E\left[\left(S_t - \hat{S}_{t|t-1}\right)\left(X_t - \hat{X}_{t|t-1}\right)'\right] =$
$E\left[\left(S_t - \hat{S}_{t|t-1}\right)\left(C\left(S_t - \hat{S}_{t|t-1}\right)\right)'\right] = \Sigma_{t,t-1}C'$ and the second is the variance of the the forecast error

$$E\left[\left(X_t - \hat{X}_{t|t-1}\right)\left(X_t - \hat{X}_{t|t-1}\right)'\right]^{-1} = \left[C\Sigma_{t,t-1}C'\right]^{-1}$$

Therefore

$$\hat{S}_{t|t} = \hat{S}_{t|t-1} + \Sigma_{t,t-1}C'\left[C\Sigma_{t,t-1}C'\right]^{-1}\left(X_t - C\hat{S}_{t|t-1}\right)$$

Kalman Filter Derivation

Updating Σ_t

Now you can update : $\Sigma_{t|t} = E\left[\left(S_t - \hat{S}_{t|t}\right)\left(S_t - \hat{S}_{t|t}\right)'\right]$ which using the previous results gives

$$\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1} C' \left[C \Sigma_{t,t-1} C'\right]^{-1} C \Sigma_{t|t-1}$$

Kalman Filter Derivation

Forecasting S_t

We can now forecast S_{t+1} using our forecast of S_t based on information at time t:

$$\hat{S}_{t+1|t} = A\hat{S}_{t|t}$$

now

$$\hat{S}_{t+1|t} = A\hat{S}_{t|t-1} + A\Sigma_{t,t-1}C' \left[C\Sigma_{t,t-1}C'\right]^{-1} \left(X_t - C\hat{S}_{t|t-1}\right)$$

$$\hat{S}_{t+1|t} = A\hat{S}_{t|t-1} + K_t \left(X_t - C\hat{S}_{t|t-1}\right)$$

$$\hat{S}_{t+1|t} = (A - K_t C)\, \hat{S}_{t|t-1} + K_t X_t$$

where $K_t = A\Sigma_{t,t-1}C' \left[C\Sigma_{t,t-1}C'\right]^{-1}$ is known as the gain matrix

Kalman Filter Derivation

MSE of the Forecast S_t

Now that we have $\hat{S}_{t+1|t}$ we van compute $\Sigma_{t+1|t}$ (Big matrix)

$$\Sigma_{t+1|t} = E\left[\left(S_{t+1} - \hat{S}_{t+1|t}\right)\left(S_{t+1} - \hat{S}_{t+1|t}\right)'\right]$$
$$\Sigma_{t+1|t} = E\left[\left(AS_t + B\eta_{t+1} - A\hat{S}_{t|t}\right)\left(AS_t + B\eta_{t+1} - A\hat{S}_{t|t}\right)'\right]$$
$$\Sigma_{t+1|t} = A\Sigma_{t|t}A' + B\Sigma_\eta B'$$
$$\Sigma_{t+1|t} = \left(A - K_t C'\right)\Sigma_{t|t-1}\left(A' - CK_t'\right) + B\Sigma_\eta B'$$

and we have the values to start the next iteration. Iterating we obtain $\left\{\hat{S}_{t+1|t}\right\}_{t=0}^{T}$ which are called the filtered values of the states.

Kalman Filter Derivation

Kalman with Noise

You can extend the State Space Model (SSM) to noise in the obervations

$$S_{t+1} = AS_t + B\eta_{t+1}$$
$$X_t = CS_t + Dv_t$$

The formulas now must take into account the noise in the forecast error of X_t. Let $R = D\Omega D'$ where $\Omega$is the covariance of v_t

$$E\left[\left(X_t - \hat{X}_{t|t-1}\right)\left(X_t - \hat{X}_{t|t-1}\right)'\right] = C\Sigma_{t|t-1}C' + R$$

Paper: News and Noise shocks In Macroeconomics

DGP of the productivity process and signal

Let productivity a_t be the sum of a transitory and a permanent component

$$a_t = x_t + z_t$$

$$\Delta x_t = \rho_x \Delta x_{t-1} + \eta_{x,t}$$

$$z_t = \rho_z z_{t-1} + \eta_{z,t}$$

the agents observe only a_t and a noisy signal on x_t

$$s_t = x_t + v_t$$

Properties the observed process

Special case

Consider the variance of $\Delta a_t = \Delta x_t + \Delta z_t$

$$Var\left[\Delta a_t\right] = \frac{1}{1-\rho_x^2}\sigma_\epsilon^2 + \frac{2}{1+\rho_z}\sigma_\eta^2$$

$$Cov\left[\Delta a_t, \Delta a_{t-j}\right] = \rho_x^j \frac{1}{1-\rho_x^2}\sigma_\epsilon^2 - \rho_z^{j-1}\frac{1-\rho_z}{1+\rho_z}\sigma_\eta^2$$

by setting $\rho_x^j \frac{1}{1-\rho_x^2}\sigma_\epsilon^2 = \rho_z^{j-1}\frac{1-\rho_z}{1+\rho_z}\sigma_\eta^2$ namely $\rho_x = \rho_z = \rho$ and $\rho\sigma_\epsilon^2 = (1-\rho)^2\sigma_\eta^2$ you obtain that the productivity is a random walk.

Economic model

Limit NKM with frequency of price adjustment goes to zero

The model is

$$c_t = lim_{j \to \infty} E_t \left[a_{t+j} \right]$$

Output is determined by consumption

$$y_t = c_t$$

and employment

$$n_t = y_t - a_t$$

Closing the model

Equilibrium from the long run expecaction

Using the process of $\Delta x_t = \rho \Delta x_{t-1} + \eta_{x,t}$

$$E_t\left[x_{t+1} - x_t\right] = \rho E_t\left[x_t - x_{t-1}\right]$$

$$E_t\left[x_{t+2} - x_{t+1}\right] = \rho E_t\left[x_{t+1} - x_t\right]$$

$$E_t\left[x_{t+2}\right] = \left(1 + \rho + \rho^2\right) E_t\left[x_t\right] - (1+\rho)\rho E_t\left[x_{t-1}\right]$$

$$E_t\left[x_{t+3}\right] = \left(1 + \rho + \rho^2 + \rho^3\right) E_t\left[x_t\right] - \rho\left(1 + \rho + \rho^2\right) E_t\left[x_{t-1}\right]$$

$$lim_{j\to\infty} E_t\left[x_{t+j}\right] = \frac{1}{1-\rho}\left(E_t\left[x_t\right] - \rho E_t\left[x_{t-1}\right]\right)$$

Closing the model

Equilibrium

$$c_t = \frac{1}{1-\rho}\left(E_t\left[x_t\right] - \rho E_t\left[x_{t-1}\right]\right)$$

and shows that the agent need to form an expectation on $E_t\left[x_t\right]$ and $E_t\left[x_{t-1}\right]$. Remember that x_t is not observed and that both expectations are endogenous variables.

Find the expectations

Express the model in State Space form

The agent's have $S_t = (x_t, x_{t-1}, z_t)'$ and $X_t = (a_t, s_t)$:

$$S_t = AS_{t-1} + B\eta_t$$
$$X_t = CS_t + Dv_t$$

Find the expectations

Express the model in State Space form

The consumer filtering problem (the consumer solves a signal extraction problem to for expectations):

$$S_t = \begin{bmatrix} x_t \\ x_{t-1} \\ z_t \end{bmatrix} = \begin{bmatrix} 1+\rho & -\rho & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix} S_{t-1} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_t \\ \eta_t \end{bmatrix}$$

$$X_t = \begin{bmatrix} a_t \\ s_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} S_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_t$$

Find the expectations

Initialization for non stationary processes

The intial state is

$$\hat{S}_{1|0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \delta + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \eta_0$$

where $\eta_0 = z_1 \sim N(0, Q_0)$ and $Q_0 = \frac{\sigma_\eta^2}{1-\rho^2}$ therefore

$$\Sigma_{1|0} = \kappa \Sigma_\infty + \Sigma_*$$

where $\Sigma_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\Sigma_* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q_0 \end{bmatrix}$

Find the expectations

Solve the signal extration problem

The above Kalman gives you values for $\hat{S}_{t|t-1}$ *and* $\hat{S}_{t|t}$. Here we are intereste in the latter $\hat{S}_{t|t} = \begin{bmatrix} \hat{x}_{t|t} \\ \hat{x}_{t-1|t-1} \\ \hat{z}_{t|t} \end{bmatrix}$.

Simulate the model

Consumer solution

The consumer problem has the following solution:

$$S_t^e = \begin{bmatrix} x_t \\ x_{t-1} \\ z_t \\ \hat{x}_{t|t} \\ \hat{x}_{t-1|t-1} \\ \hat{z}_{t|t} \end{bmatrix} = \begin{bmatrix} A & 0 \\ KCA & (I - KC)\,A \end{bmatrix} S_{t-1}^e + \begin{bmatrix} B & 0 \\ KCB & KD \end{bmatrix} \begin{bmatrix} \epsilon_t \\ \eta_t \\ v_t \end{bmatrix}$$

$$X_t^e = \begin{bmatrix} a_t \\ c_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1-\rho} & -\frac{\rho}{1-\rho} & 0 \end{bmatrix} S_t^e$$

Solution

IRF

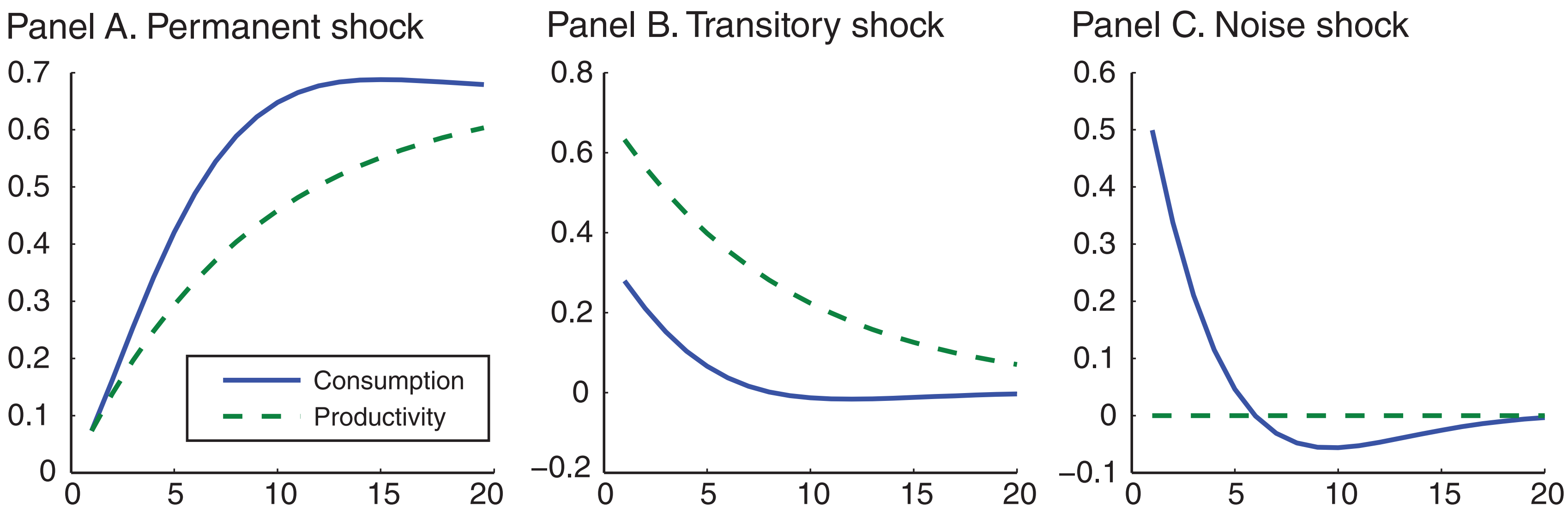

FIGURE 1. IMPULSE RESPONSES

Figure: Shocks in the benchmark model

Reduced Form VAR

Revovering the shocks

The VAR represntation of the model is
$c_t = lim_{j \to \infty} E_t [a_{t+j}]$,$c_{t+1} = lim_{j \to \infty} E_{t+1} [a_{t+1+j}]$ implies
$c_t = E_t [c_{t+1}]$

$$c_t = c_{t-1} + u_t^c$$
$$a_t = \rho a_{t-1} + (1-\rho) c_{t-1} + u_t^a$$

The univariate representation of productivity is a random walk, by assumption. But in the multivariate representation, past consumption helps to predict productivity.

Reduced Form VAR

Revovering the shocks

When $\sigma_v = 0$ the VAR represntation of the model is

$$c_t = c_{t-1} + \frac{1}{1-\rho}\epsilon_t$$
$$a_t = \rho a_{t-1} + (1-\rho)c_{t-1} + \epsilon_t + \eta_t$$

When $\sigma_v \to \infty$ the VAR represntation of the model is

$$c_t = c_{t-1} + u_t$$
$$a_t = a_{t-1} + u_t$$

SVAR

Revovering the shocks with BQ decomposition

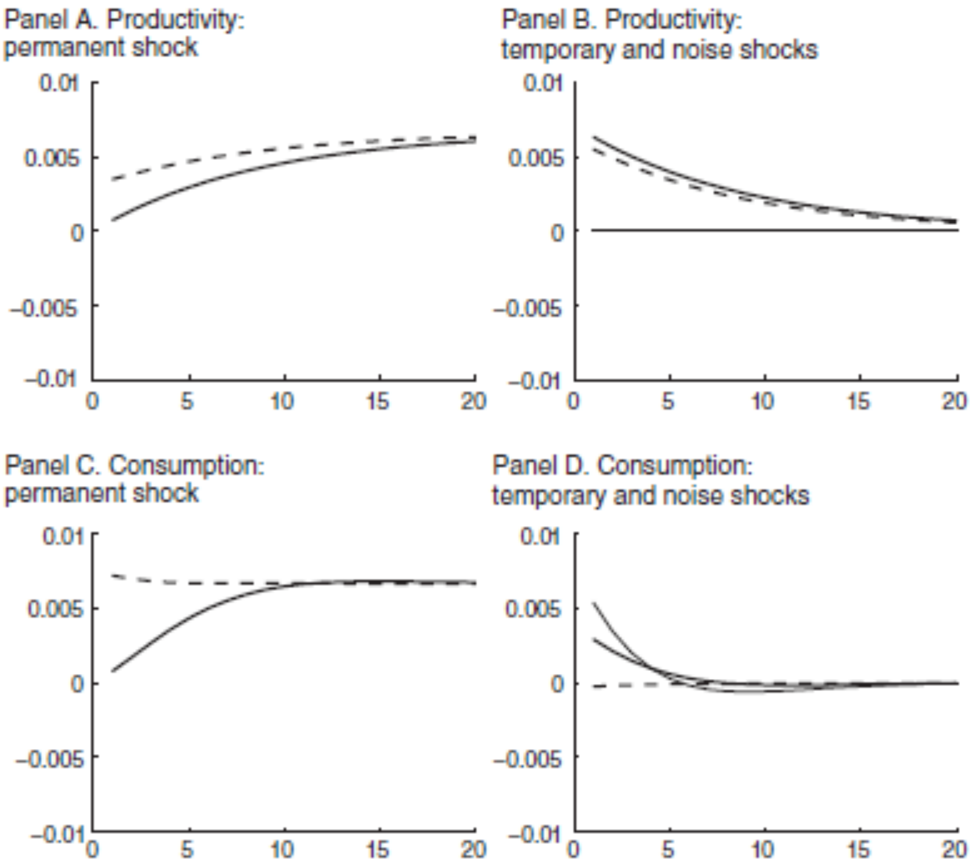

Figure 2. Model and SVAR-Identified Impulse Responses

Figure: SVAR and true responses

Reduced Form VAR

Revovering the shocks

Can you recover the structural shocks? without noise yes. otherwise no.The reason is that agents' decisions are functions of their expectations, so even if the econometrician observes the three variables (c, a, s), the first variable is a function of the other two, which implies that there are only two independent innovations driving the system. But you can estimate the structural model to which we now turn.

Matching moments

Revovering the parameters

You can recover ρ from

$$a_t = \rho a_{t-1} + (1-\rho)c_{t-1} + u_t^a$$
$$\Delta a_t = (1-\rho)\left(c_{t-1} - a_{t-1}\right) + u_t^a$$

notice you can exploit the corrlation of productivity growth and consumption at different horizons

$$a_{t+j} - a_t = \left(1-\rho^j\right)\left(c_{t-1} - a_{t-1}\right) + u_t^{a,j}$$

Matching moments

Revovering the parameters

Using the represntation of a_t we get

$$\sigma_\epsilon^2 = Var\left[\Delta a_t\right](1-\rho)^2$$
$$\sigma_\eta^2 = Var\left[\Delta a_t\right]\rho$$

finally to recover σ_v they exploit the correlation between u_t^a and Δc_t

Maximum Likelihood

Revovering the parameters

Solve the agents signal extraction and next build the econometrician's Kalman Filter including in the list of unobservable state variables the consumer's expectations.

Kalman for the econometrician

Econometrician

The econometrician has $S_t^e = \left(S_t, \hat{S}_{t|t}\right)'$ and $X_t^e = (a_t, c_t)$:

$$S_t^e = \begin{bmatrix} x_t \\ x_{t-1} \\ z_t \\ \hat{x}_{t|t} \\ \hat{x}_{t-1|t-1} \\ \hat{z}_{t|t} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ KCA & (I - KC)\,A & 0 \end{bmatrix} S_{t-1}^e + \begin{bmatrix} B & 0 \\ KCB & KD \end{bmatrix} \begin{bmatrix} \epsilon_t \\ \eta_t \\ v_t \end{bmatrix}$$

$$X_t^e = \begin{bmatrix} a_t \\ c_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1-\rho} & -\frac{\rho}{1-\rho} & 0 & 0 \end{bmatrix} S_t^e$$

The Likelihood
using the Kalman filter

If the inital state S_1 and η_t and v_t are multivariate Gaussian we have:

$$f(X_t|X_{t-1}, X_{t-2}, ...) \sim N\left(C\hat{S}_{t|t-1}, C\Sigma_{t|t-1}C' + D\Sigma_v D'\right) \text{ } for \text{ } t = 1, 2, ..., T$$

from which you can compute the sample loh likelihhod and maximize wrt to $A, B, C, D, \Sigma, \Sigma_v$

Smoothing

Forming an inference based on the sample

In some application we are interested in having an inference os S_t based on the full sample,

$$\hat{S}_{t|T} = E\left[S_t | X_T\right]$$

to do obtain it you need to compute the filtered quantities (above) $\hat{S}_{t|t}, \hat{S}_{t+1|t}, \Sigma_{t|t}, \Sigma_{t+1|t}$ and then

$$\hat{S}_{T-1|T} = \hat{S}_{T-1|T-1} + J_{T-1}\left(\hat{S}_{T|T} - \hat{S}_{T|T-1}\right)$$

where $J_{T-1} = \Sigma_{T-1|T-1} A' \Sigma_{T|T-1}^{-1}$ and proceed backwards. You can also compute the MSE assciated with the smoothed estimates.

Smoothing

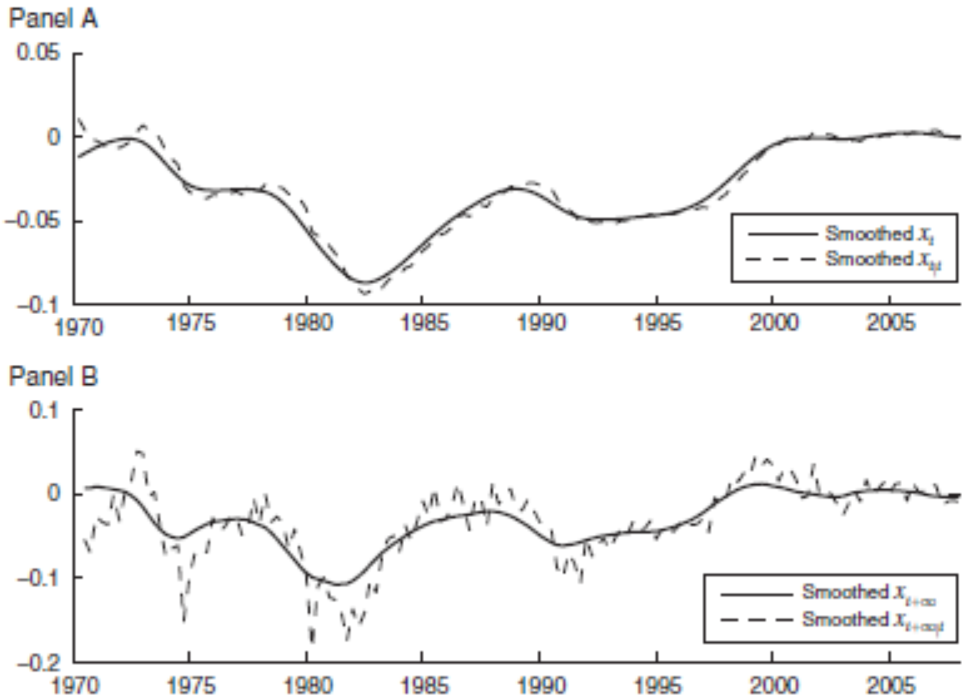

Figure 3. Smoothed Estimates of the Permanent Component of Productivity, of Long-Run Productivity, and of Consumers' Real Time Expectations

Figure: Smoothed states

Smoothing

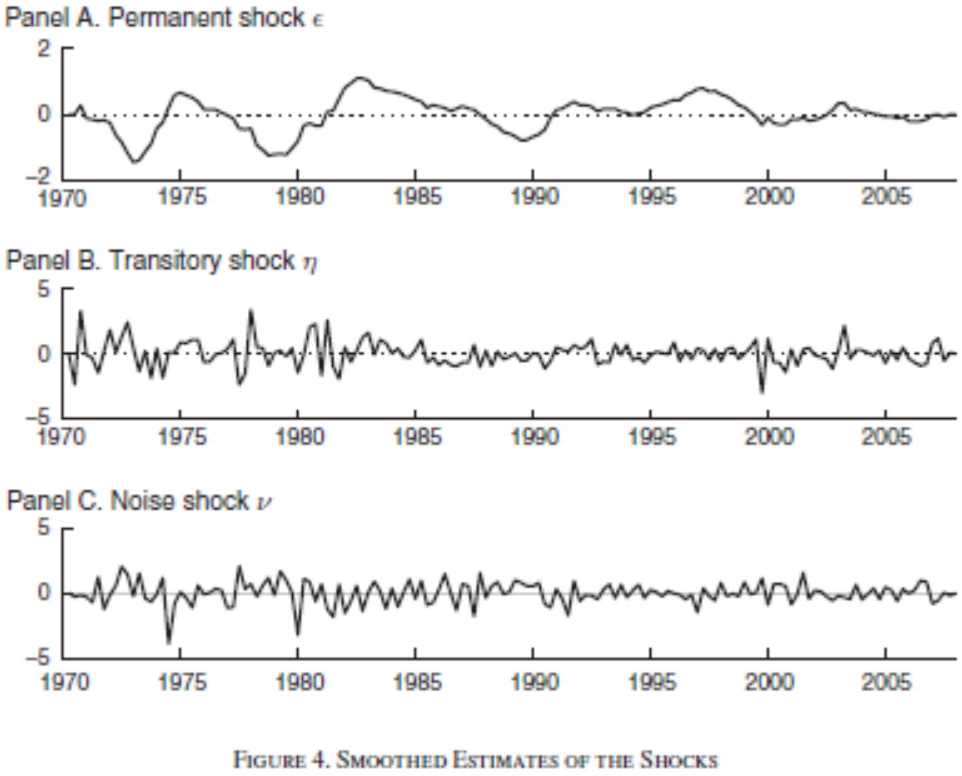

Figure: Smoothed shocks