

Draft_Application_Stochastic_Volatility

1 Overview

In the following sections, we will try to replicate some of the analyses which were previously conducted on synthetic data, generated by a basic stochastic volatility model; this time, the same model specification and filters will be used on real data. However, as we will see, real data require different strategies to evaluate the effectiveness of several Sequential Monte Carlo techniques. Intuitively, such differences are due to the fact that now the latent process driving the Hidden Markov Model is not observed, nor do we know exactly its law of motion. This part of the work is structured as follows: section *fill with the number of the section* provides a review of recent research in the stochastic volatility literature, including extensions of the basic specification and the use of particle filters; section *fill with the number of the section* describes the data which we employ and the transformations that were applied to them. In section *fill with the number of the section* we briefly compare the series of filtered states that are obtained from different particle filters; moreover, several PFs are compared in terms of computational cost in approximating the exact filtering distribution implied by our SV model. Section *fill with the number of the section* features a forecast comparison between two SV specifications and a model that assumes constant mean and variance for the financial returns, namely a forecasting exercise that would not be possible were it not for particle filtering techniques. In section *fill with the number of the section* we conclude by employing a non-parametric measure of realised volatility as a proxy for the latent volatility process, allowing a RMSE and MAE comparison between filters which is similar to that of section *fill with the number of the section* for synthetic data.

2 SV models + Literary Review

In section *fill with the number of the section* a simple specification of a Stochastic Volatility model was presented and used as the data generating process of a simulated dataset. Denoting by y_t the log returns at time t and by x_t the latent volatility stochastic process, such state-space model was presented as follows,

$$\begin{aligned} y_t | x_t &\sim N(0, e^{x_t}) \\ x_t | x_{t-1}, \alpha, \beta, \tau^2 &\sim N(\alpha + \beta x_{t-1}, \tau^2) \end{aligned}$$

This specification - which we will employ in this section on a real dataset - can be deemed as the standard version of the SV model. Throughout the recent

decades, several extensions have been proposed, mostly (but not exclusively) in the field of financial econometrics. On the one hand, for simplicity we will stick to the standard version, one that will suffice for the scope of this work on Sequential Monte Carlo methods. On the other hand, it is worth mentioning some of these extensions: indeed, one can rather safely assume that, through their higher degree of sophistication, such extensions might well improve on the performance of our standard SV model in the analyses that will follow.

Stochastic Volatility Models, whose early formulation is commonly attributed to Taylor (1982, 1986), allow to account for time-varying and autoregressive volatility in financial returns, posing themselves as a valid alternative to ARCH (Engle 1982) or GARCH (Bollerslev 1986) models in dealing with non-constant volatility. Kim, Shephard and Chib (1998) define the canonical model for regularly spaced data as

$$\begin{aligned} y_t &= \psi e^{\frac{x_t}{2}} \epsilon_t \\ x_{t+1} &= \mu + \beta(x_t - \mu) + \tau \eta_t \\ x_1 | \sigma, \beta &\sim N\left(\mu, \frac{\sigma^2}{1 - \beta^2}\right) \\ \epsilon_t &\sim N(0, 1) \\ \eta_t &\sim N(0, 1) \end{aligned}$$

where the log volatility x_t is assumed to follow a stationary process ($|\beta| < 1$), ψ is a constant scaling factor, β is the persistence in the volatility and τ is the volatility of the log-volatility.¹ Most importantly, ϵ_t and η_t , the Gaussian white noise processes that drive the canonical model, are assumed to be uncorrelated. This latter assumption justifies another definition of this specification, namely “discrete SV model without leverage”.²

In order to accommodate for leverage effects, namely capture the increase in volatility that follows a drop in the returns, the model can be extended as in Omori

¹For identifiability reasons, either $\psi = 1$ or $\mu = 0$. Kim, Shephard and Chib (1998) prefer $\psi = 1$.

²Note that in this review we do not focus on continuous time SV models. In fact, such models have attracted a considerable amount of research in financial econometrics and mathematical finance, especially after Hull and White (1987) considered stochastic volatility for option pricing. Arguably, the most influential model was then proposed for option pricing by Heston (1993), a SV model with leverage effects and square root diffusion driving volatility. Diffusion-based SV models enjoyed increasing popularity, see for example Barndorff-Nielsen and Shephard (2001), or Christoffersen et al. (2010), who investigate alternatives to the entrenched affine square root SV model. Eraker (2004) proposed a SV model with correlated jumps in prices and volatility, extending Heston’s model, while Comte and Renault (1998) extended Hull and White’s model as to feature long memory properties. More recently, Gatheral et al. (2018) further built on Comte and Renault’s fractional SV model to propose the popular “rough volatility models” (see also Friz et al., 2021).

et al. (2007),

$$\begin{aligned} y_t &= e^{\frac{x_t}{2}} \epsilon_t \\ x_{t+1} &= \mu + \beta(x_t - \mu) + \eta_t \\ \begin{pmatrix} \epsilon_t \\ \eta_t \end{pmatrix} \Big| \rho, \tau &\stackrel{i.i.d.}{\sim} N_2(\mathbf{0}, \begin{pmatrix} 1 & \rho\tau \\ \rho\tau & \tau^2 \end{pmatrix}), \end{aligned}$$

where $\rho < 0$ captures the negative correlation. Such specification, which can be referred to as “discrete SV model with leverage”, captures the asymmetric response of volatility to returns of different signs, so that similar specifications are sometimes also deemed as “asymmetric SV models” (e.g. Harvey and Shephard 1996, Mao et al. 2020).

Note that so far the assumptions on ϵ_t and η_t implied that the returns are conditionally normally distributed. The discrete time SV models can also be extended to allow for heavy-tailed or asymmetric conditional returns distributions: symmetric or skewed Student- t , Generalised Hyperbolic (GH) distribution, Generalised Error Distribution (GED) and scale mixtures of normals feature as popular choices (Kim and Stoffer 2008, Nakajima and Omori 2012, Mao et al. 2020). In fact, SV with heavy tailed return distributions were shown to better meet empirical regularities like the leptokurtic distribution of the returns and slowly decaying autocorrelation functions of the squared returns (Liesenfeld and Jung 2000).

Assuming that ϵ_t follows a Student- t distribution, and exploiting the fact that ϵ_t can then be written as $\lambda_t^{-1/2} \zeta_t$, where $\zeta_t \sim N(0, 1)$ and $v\lambda_t \sim \chi_v^2$ (Harvey et al. 1994, Chib et al. 2002), we have the following SV model with both fat tails and leverage effect (Jacquier et al., 2004),

$$\begin{aligned} y_t &= e^{\frac{x_t}{2}} \lambda_t^{-1/2} \zeta_t \\ x_{t+1} &= \alpha + \beta x_t + \eta_t \\ \begin{pmatrix} \zeta_t \\ \eta_t \end{pmatrix} \Big| \rho, \tau &\stackrel{i.i.d.}{\sim} N_2(\mathbf{0}, \begin{pmatrix} 1 & \rho\tau \\ \rho\tau & \tau^2 \end{pmatrix}) \\ v\lambda_t &\sim \chi_v^2 \end{aligned}$$

To capture other elements of the behaviour of financial data, several other extensions have been proposed. For example, SV models have been extended to include conditional heteroskedasticity in the mean returns (Koopman and Hol Uspensky 2002) to capture potential volatility feedback effects, or to feature autoregressive moving average innovations (Chan 2013, Zhang et al. 2020), allowing better goodness of fit and out-of-sample forecasts.^{3 4} The model

³Dimitrakopoulos and Kolossiatis (2020) note that “the moving average component, the leverage effect and the conditional heteroscedasticity in mean have been considered separately in the stochastic volatility literature” and provide two specifications, one featuring an MA component and leverage effects, the other an MA component and conditional heteroskedasticity in mean.

⁴Other extensions, though ones for which we do not report the specifications, feature

proposed by Koopman and Hol Uspensky is specified as

$$\begin{aligned} y_t &= \nu_t + \psi e^{\frac{x_t}{2}} \epsilon_t \\ \nu_t &= a + b y_{t-1} + d \psi^2 e^{x_t} \\ x_t &= \beta x_{t-1} + \tau \eta_t \\ \begin{pmatrix} \epsilon_t \\ \eta_t \end{pmatrix} &\overset{i.i.d.}{\sim} N_2\left(\mathbf{0}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \end{aligned}$$

while the state space representation of the ARMA(p,q)-SV framework, as in Zhang et al. (2020), reads

$$\begin{aligned} y_t &= \nu_t + \gamma_t \\ \gamma_t &= \phi_1 \gamma_{t-1} + \dots + \phi_p \gamma_{t-p} + u_t + \varphi_1 u_{t-1} + \dots + \varphi_q u_{t-q} \\ u_t | x_t &\sim N(0, e^{x_t}) \\ x_t &= x_{t-1} + \eta_t \\ \eta_t | \tau &\sim N(0, \tau^2) \end{aligned}$$

where the error terms u_t and η_t are independent across all leads and lags, while ν_t follows an unspecified time-varying process.

Interestingly enough, the path along which the SV models evolved coincides with that suggested by optimal portfolio findings. Johannes, Korteweg and Polson (2014) found that, in order to generate statistically significant portfolio improvements in a Bayesian learning problem, the model employed by the investor should incorporate both time-varying expected returns and stochastic volatility: indeed, either of these features alone did not lead to statistically significant gains with respect to employing models with time-constant expected returns and volatility.⁵

Finally, before moving to the application of Sequential Monte Carlo techniques to a SV model, we conclude by going through some references for the Bayesian analysis proposed for such models. Starting from the seminal work of Jacquier et al. (1994), the use of MCMC methods has become increasingly popular for parameter estimation and smoothing exercises in SV models (e.g. Kastner 2019, presenting the R package *stochvol* for Bayesian parameter estimation,

modelling the latent volatility process x_t as an ARFIMA process (Long Memory Stochastic Volatility model, Breidt et al. 1998) or as governed by a first-order Markov process (Markov Switching Stochastic Volatility model, So et al. 1998). Recently, Luo et al. (2018) incorporated neural networks in the stochastic volatility model (Neural Stochastic Volatility Model), while Xu and Chen (2021) employ deep learning models (Deep Stochastic Volatility Model).

⁵One could then argue that caution is needed when employing basic specifications of stochastic volatility models. For instance, Poon and Granger (2005) found that historical volatility and ARCH models both achieved better volatility forecasting performance than SV models. Similarly, Allen and McAleer (2020, see also Allen, 2020) found that, using realised volatility as benchmark, neither the canonical SV model or a GARCH(1,1) specification could forecast better than a simple form of historical volatility model.

and Chopin and Papaspiliopoulos 2020, who use MCMC to sample from the smoothing distribution of a SV model). As regards filtering exercises, the adoption of particle filters was rather rapid:⁶ indeed, latent volatilities in Kim et al. (1998) were already filtered by employing the particle filter suggested in Pitt and Shephard (1999), paving the way for subsequent applications (*inter alia*) in Chib et al. (2006), Omori et al. (2007), Kim and Stoffer (2008) and Nakajima and Omori (2012).

3 Data

As previously mentioned, we will analyze the behaviour of the described filtering tools associated to a simple stochastic volatility model. In particular, we are evaluating the performance of such model on real data. For the observable process, namely financial returns in the SV model, we consider the continuously compounded daily returns (also called logarithmic returns) of three indices, S&P500, DOW JONES and STOXX50, in a time interval from June 1st, 2017 to May 30th, 2021. From these, we estimate the daily volatility, as computed by the model.

The indices have been selected as representatives of the global economic trends in the US and EU markets. Specifically, the S&P500 is a market-capitalization-weighted stock-price index tracing the performance of the 500 largest companies listed on US stock exchanges (NYSE and Nasdaq Exchange). The DOW JONES, instead, is a price-weighted stock-market index and accounts for the 30 major companies listed on US stock exchanges, characterized for being “blue-chip”. Also the EURO STOXX 50 follows blue-chip stocks representing leading firms in regions of the Eurozone.

The three plots below represent the time series of the log returns calculated for the three indices during our period of interest. In particular, each series is a sequence of daily observations representing the logarithm of the ratio between the closing price of the index for a given day and the closing price of the day before.

At a first visual inspection, we can see a common behaviour in the volatility of all three indices. Especially, it is worth pointing out the similarity between the pattern of DJI and that of S&P500 (especially in terms of peaks), which both differ from the STOXX50E, presumably because the first two describe the US stock market, while the last index describes the European one. For example, a difference that can be seen by having a glance at the plots is that the US indices log returns displays differences in fluctuation magnitude across different

⁶Actually, SV models are now often used as straightforward applications of particle filters on non-linear state space models, see for example Andrieu et al. (2010), Douc et al. (2014) or Chopin and Papaspiliopoulos (2020).

periods that are more marked than for the EU index, for which log returns display, in general, wider fluctuations. Anyways, what stands out the most in all three plots is the very wide fluctuations present from February - March 2020 to around September 2020, which, very intuitively, are connected with the Covid-19 Pandemic Crisis.

As regards the series that we will employ in section *fill with the number of the section*, namely the series of realised volatilities which we will employ as a proxy for the true latent volatilities, the data were retrieved from the Oxford-Man Institute's Realized Library, which provides several different daily non-parametric measures of past volatility.⁷ We consider a proper rescaling of the *rv5* series, *i.e.* model-free daily volatility estimates based on 5 min intraday return intervals. Also in this case, the chosen interval spanned from June 1st, 2017 to May 30th, 2021. *Includere grafico e commento del grafico della realised volatility*

3.1 Rescaling the data

When specifying the parameters of a stochastic volatility model, one should clarify whether the series of observed financial returns is one of log returns or of percentage log returns. Indeed, let y_t denote the returns at time t , so that a basic SV specification would prescribe $y_t|x_t \sim N(0, e^{x_t})$: clearly, for each period t , e^{x_t} needs to be higher to replicate the observed *percentage* log return rather than the observed log return. This implies that the latent volatility x should fluctuate around higher values when using percentage log returns as y_t . This feature, when adopting the aforementioned canonical SV model as defined in Kim et al. (1998), would be represented by a higher μ . For example, as noted in Kastner (2019), daily log returns often have a variance of 0.0001 or less, which implies that μ should lie somewhere around $\ln(0.0001) \approx -9$; instead, when considering daily percentage log returns, one then has a variance of 1, so that $\mu = \ln(1) = 0$. Note that when $\mu = 0$, the canonical model boils down to the basic specification we employed on the simulated dataset, so that we prefer using percentage log returns for the 3 indices. In other terms, one might end up with the basic specification of the SV model when (roughly) calibrating the parameters of the canonical model on a wider set of financial returns.

Note that rescaling the log returns has analogous implications for the section in which we employ measures of realised volatility as a proxy for the latent volatility process. Let us denote by $rv5_t$ the daily realised variance at time t , which we obtain from the *rv5* series provided in the Oxford-Man database. Now, note that in a SV framework one should not directly compare $rv5_t$ to x_t , as $rv5_t$ can be rather seen as an estimate of the variance of the log returns at time t , *i.e.* of e^{x_t} in the SV model. Moreover, by construction of $rv5_t$,⁸ the comparison

⁷Such class of measures will be better presented in section *fill with the number of the section*, when reviewing the use of realised volatility estimators as proxies for latent volatilities.

⁸Nota sulla formula, che considera returns, non returns in percentuale. Spiegare qui perché possa avere senso paragonare questa varianza a quella dei log returns

should feature a SV model where the log returns did *not* undergo the percentage transformation. Thus, in order to achieve a common scale with the percentage log returns, we need to multiply $rv5$ by 100^2 . To ease the interpretation and representation of the comparison between differently approximated filtering distributions, we finally take the square root of the rescaled $rv5$, so that we compare it to $e^{x_t/2}$ (*In realtà detto così è poco preciso, perchè in realtà andiamo a confrontare con la trasformazione della media dello stato filtrato*).

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