1/4-BPS Wilson loops in $\mathcal{N}=4$ SYM at strong coupling

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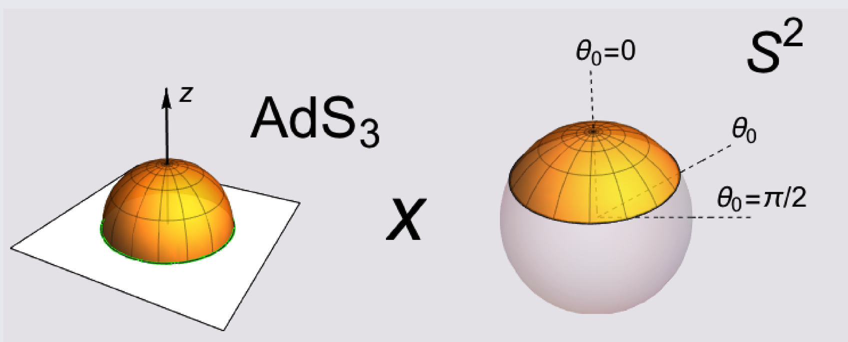
Motivations

- The precise match between semiclassical quantization [1] and the exact prediction obtained via supersymmetric localization [2] does not go beyond leading order for the **1/2-BPS circular Wilson loop**.
- Smooth BPS Wilson loops are finite observables, but the dual string calculation is intrinsically plagued by divergencies, calling for a **regularization scheme consistent with supersymmetry**.

Goal

We study the **strong-coupling** behaviour of a **one-parameter family of** 1/4-**BPS latitude Wilson loop operators** in planar $\mathcal{N}=4$ SYM theory, computing the one-loop correction to the classical string solution in $AdS_5\times S^5$ in **string sigma-model perturbation theory**. The minimal-area surface ends on the Wilson loop (green path) and is composed of a dome-like surface in Poincaré $AdS_3\subset AdS_5$ and of a cup-like embedding wrapping the north pole of $S^2\subset S^5$ (orange surfaces) [3].



- We compute the path integral weighted by the type IIB Green-Schwarz superstring action in $AdS_5\times S^5$

$$Z(\lambda,\theta_0)\equiv\int[\mathcal{D}\delta x][\mathcal{D}\Psi]\,e^{-S(x_{\rm cl}+\delta x,\Psi)}$$

where $x_{\rm cl}$ collects the coordinates of the classical worldsheet and $\delta x,\Psi$ the bosonic and fermionic semiclassical fluctuations around it. This equals the vev of the Wilson loop in the 4D conformal boundary

$$\boldsymbol{Z(\lambda,\theta_0)=\langle\mathcal{W}(\lambda,\theta_0)\rangle}\,.$$

- In gauge theory these Wilson loop operators couple to three scalars of $\mathcal{N}=4$ SYM. The family interpolates between two notable cases: the **1/2-BPS circular Wilson loop** (the S^2-part shrinks to the north pole for $\theta_0=0$) [4, 1] and the **1/4-BPS Zarembo Wilson loop** (it extends over the maximal circle for $\theta_0=\frac{\pi}{2}$) [5].
- The localization answer for the circular Wilson loop is a Gaussian matrix model [2], expressible as a modified Bessel function of the 't Hooft coupling λ

$$\langle\mathcal{W}(\lambda,0)\rangle=\frac{2}{\sqrt{\lambda}}I_1\left(\sqrt{\lambda}\right)\overset{\lambda\gg1}{\approx}\sqrt{\frac{2}{\pi}}\lambda^{-3/4}e^{\sqrt{\lambda}}.$$

The same all-loop formula accommodates the generic latitude upon the substitution $\lambda\to\lambda\cos^2\theta_0$, so the **perturbative approach to the string sigma-model is expected to reproduce**

$$\frac{\langle\mathcal{W}(\lambda,\theta_0)\rangle}{\langle\mathcal{W}(\lambda,0)\rangle}\approx\cos^{-3/2}\theta_0\,e^{\sqrt{\lambda}(\cos\theta_0-1)}\qquad\lambda\gg1.$$

The **normalization** has the advantage of washing out any θ_0-independent factors [1], in principle computable from diffeomorphism-ghost zero modes, and provides a natural reference solution within the same class of Wilson loops.

The worldsheet geometry

- Endowing the $AdS_5\times S^5$ space with a Lorentzian metric in global coordinates (conveniently adapted to facilitate the semiclassical analysis)

$$ds^2_{\rm 10D}\equiv G_{MN}dx^Mdx^N=-\cosh^2\rho\,dt^2+d\rho^2+\sinh^2\rho\,\frac{dy_m^2}{(1+\frac{y^2}{4})^2}+\frac{dz_n^2}{(1+\frac{z^2}{4})^2},\quad y^2\equiv\sum_{m=1}^{3}y_m^2,\quad z^2\equiv\sum_{n=1}^{5}z_n^2,$$

the classical configuration $x_{\rm cl}$ is the **spacelike surface** [3]

$$\begin{aligned}&t=0,\qquad \rho=\rho(\sigma),\qquad y_1=2\sin\tau,\qquad y_2=2\cos\tau,\qquad y_3=0\,,\\ &z_1=z_2=0,\qquad z_3=2\cos\theta(\sigma),\qquad z_4=2\sin\theta(\sigma)\sin\tau,\qquad z_5=2\sin\theta(\sigma)\cos\tau\,,\end{aligned}$$

and implements the correct boundary geometry and minimizes the area functional for the choice

$$\begin{aligned}&\sinh\rho(\sigma)=\frac{1}{\sinh\sigma},\qquad \cosh\rho(\sigma)=\frac{1}{\tanh\sigma},\\ &\sin\theta(\sigma)=\frac{1}{\cosh(\sigma+\sigma_0)},\qquad \cos\theta(\sigma)=\tanh(\sigma+\sigma_0)\,,\\ &\cos\theta_0\equiv\tanh\sigma_0,\qquad \tau\in[0,2\pi),\qquad \sigma\in[0,\infty).\end{aligned}$$

- The induced worldsheet metric

$$ds^2_{\rm 2D}\equiv h_{\tau\tau}d\tau^2+h_{\sigma\sigma}d\sigma^2=\Omega^2(\sigma)\left(d\tau^2+d\sigma^2\right)$$

shows a conformal factor $\Omega^2(\sigma)\equiv\sinh^2\rho(\sigma)+\sin^2\theta(\sigma)$ depending on the latitude polar angle θ_0.

The superstring action

The string dynamics is governed by the **type IIB Green-Schwarz action**, whose bosonic part is the usual Nambu-Goto action

$$S_B=\frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma\sqrt{-h}\,,$$

while in the fermionic sector the quadratic truncation in fermions suffices for the one-loop analysis

$$S_F=i\frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma\left(\sqrt{-h}h^{ij}\delta^{IJ}-\epsilon^{ij}s^{IJ}\right)\bar{\Psi}^I\rho_iD_j^{JK}\Psi^K.$$

Here above Ψ^I $(I=1,2)$ are two 10D Majorana-Weyl spinors with the same chirality, $s^{IJ}=(\sigma_3)^{IJ}$, $\epsilon^{ij}=(i\sigma_2)^{ij}$, $\rho_i=e^A_M\partial_ix^M\Gamma_A$ are 10D Dirac matrices projected onto the worldsheet and D_j^{JK} is the 2D pullback $\partial_jx^M D_M^{JK}$ of the 10D covariant derivative, sum of an ordinary spinor covariant derivative and the coupling to the Ramond-Ramond flux background

$$D_j^{JK}=\delta^{JK}\left(\partial_j+\frac{1}{4}\partial_jx^M\omega_M^{AB}\Gamma_{AB}\right)+\frac{1}{2}\epsilon^{JK}\Gamma_{01234}\rho_j.$$

The action, written here for a 2D metric with Lorentzian signature, needs to be analytically rotated to the Euclidean case of interest.

Classical action

The leading contribution to the Wilson loop is given by the classical *regularized* area

$$S_B^{(0)}(\lambda,\theta_0)=\frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi}d\tau\int_\epsilon^\infty d\sigma\Omega^2(\sigma)=\sqrt{\lambda}\left(\frac{1}{\epsilon}-\cos\theta_0\right).$$

The $1/\epsilon$ pole is reminiscent of the boundary singularity in the *AdS* metric and cured by a large radial cutoff at $\rho\sim1/\epsilon$. Since we are interested in the normalized latitude partition function, divergences cancel out and the leading behaviour reads

$$\frac{Z_{\rm cl}(\lambda,\theta_0)}{Z_{\rm cl}(\lambda,0)}=\frac{\exp\left[-S_B^{(0)}(\lambda,\theta_0)\right]}{\exp\left[-S_B^{(0)}(\lambda,0)\right]}=\exp\left[\sqrt{\lambda}(\cos\theta_0-1)\right].$$

One-loop fluctuation determinants

The perturbative approach to the string sigma-model prescribes to split each field, say $X(\tau,\sigma)$, into its classical value on the string solution $X_{\rm cl}(\tau,\sigma)$ and its semiclassical fluctuation $\delta X(\tau,\sigma)$. We readily carry out the fluctuation analysis around the minimal-area surface using the general approach outlined in [6].

- Only eight bosonic fluctuations are true degrees of freedom. In **static gauge** the two longitudinal ones are removed by demanding the fluctuation vector δx^M $(M=1,\dots10)$ to be orthogonal to the classical surface $x^M_{\rm cl}$

$$G_{MN}\partial_\tau x^M_{\rm cl}\delta x^N=G_{MN}\partial_\sigma x^M_{\rm cl}\delta x^N=0$$

while we single out the physical bosons ζ_i $(i=1,\dots8)$ by projecting $\delta x^M_{\rm cl}$ onto eight normal directions.

- A natural choice for the **κ-symmetry gauge-fixing** is to set $\Psi^1=\Psi^2\equiv\Psi$.
- We expand the action to quadratic order in the fluctuations (8 transverse bosonic ζ_i + 16 real fermionic dof in Ψ, accommodated into 8 2D spinors Ψ^{p_1,p_2,p_3} labeled by $p_1,p_2,p_3=\pm1$).

$$S_B^{(2)}(\theta_0)=\int d\tau d\sigma\Omega^2(\sigma)\left[\zeta_1\mathcal{O}_1\zeta_1+\zeta_5\mathcal{O}_1\zeta_5+\zeta_6\mathcal{O}_1\zeta_6+\zeta_2\mathcal{O}_2\zeta_2+\zeta_7\mathcal{O}_2\zeta_7+\zeta_8\mathcal{O}_2\zeta_8+\begin{pmatrix}\zeta_3\\ \zeta_4\end{pmatrix}^T\mathcal{O}_3\begin{pmatrix}\zeta_3\\ \zeta_4\end{pmatrix}\right]$$

$$S_F^{(2)}(\theta_0)=2i\int d\tau d\sigma\Omega^2(\sigma)\sum_{p_1,p_2,p_3=\pm1}\bar{\Psi}^{(p_1,p_2,p_3)}\mathcal{O}_F^{(p_1,p_2,p_3)}(\theta_0)\,\Psi^{(p_1,p_2,p_3)}$$

The scalar/matrix-valued operators read in a convenient Γ-matrix representation

$$\mathcal{O}_1=-\partial_\sigma^2-\partial_\tau^2+\frac{2}{\sinh^2\sigma}\qquad\mathcal{O}_2(\theta_0)=-\partial_\sigma^2-\partial_\tau^2-\frac{2}{\cosh^2(\sigma+\sigma_0)}$$

$$\mathcal{O}_3(\theta_0)=\begin{pmatrix}-\partial_\sigma^2-\partial_\tau^2-2+3\tanh^2(2\sigma+\sigma_0) & 2\tanh(2\sigma+\sigma_0)\,\partial_\tau\\ -2\tanh(2\sigma+\sigma_0)\,\partial_\tau & -\partial_\sigma^2-\partial_\tau^2-2+3\tanh^2(2\sigma+\sigma_0)\end{pmatrix}$$

$$\mathcal{O}_F^{(p_1,p_2,p_3)}(\theta_0)=\begin{pmatrix}\frac{1}{\Omega^2(\sigma)}\left(p_1\sinh^2\rho(\sigma)-p_1p_2\sin^2\theta(\sigma)\right) & \frac{i}{\Omega(\sigma)}\left(\partial_\sigma+i\partial_\tau-a_1(\sigma)+p_2a_2(\sigma)\right)\\ \frac{i}{\Omega(\sigma)}\left(\partial_\sigma-i\partial_\tau-a_1(\sigma)-p_2a_2(\sigma)\right) & \frac{1}{\Omega^2(\sigma)}\left(-p_1\sinh^2\rho(\sigma)-p_1p_2\sin^2\theta(\sigma)\right)\end{pmatrix}.$$

- The τ-translational invariance allows to Fourier transform $\partial_\tau\to i\omega$ and to apply the **Gel'fand-Yaglom method** (and derived technology [7]) for the evaluation of the worldsheet determinants on $\sigma\in[\epsilon,R]$ with R large. At the endpoints we impose **Dirichlet boundary conditions** on all the bosonic and (the squares of) fermionic operators.

$$\frac{Z_B(\theta_0)}{Z_B(0)}\equiv\frac{\cancel{\mathrm{Det}^{3/2}\mathcal{O}_1}\,\mathrm{Det}^{3/2}\mathcal{O}_2(\theta_0)\,\mathrm{Det}^{1/2}\mathcal{O}_3(\theta_0)}{\cancel{\mathrm{Det}^{3/2}\mathcal{O}_1}\,\mathrm{Det}^{3/2}\mathcal{O}_2(0)\,\mathrm{Det}^{1/2}\mathcal{O}_3(0)}=\frac{\tanh^{3/2}\sigma_0}{(1+e^{-2\sigma_0})^{1/2}}\prod_{\omega\geq1}\left(\frac{\omega+\tanh\sigma_0}{\omega+1}\right)^4$$

$$\frac{Z_F(\theta_0)}{Z_F(0)}\equiv\prod_{p_1,p_2,p_3=\pm1}\frac{\mathrm{Det}^{1/4}\left[\mathcal{O}_F^{(p_1,p_2,p_3)}(\theta_0)\right]^2}{\mathrm{Det}^{1/4}\left[\mathcal{O}_F^{(p_1,p_2,p_3)}(0)\right]^2}=\frac{1}{1+e^{-2\sigma_0}}\prod_{\omega\geq1}\left(\frac{\omega+\tanh\sigma_0}{\omega+1}\right)^4$$

$$\frac{Z_{\text{1-loop}}(\theta_0)}{Z_{\text{1-loop}}(0)}=\left(\frac{Z_B(\theta_0)}{Z_B(0)}\right)^{-1}\frac{Z_F(\theta_0)}{Z_F(0)}=\cos^{-\frac{3}{2}}\theta_0\,(1+e^{-2\sigma_0})^{-\frac{1}{2}}\qquad\text{with}\qquad\cos\theta_0\equiv\tanh\sigma_0$$

Conclusions

- Fermionic frequencies are already integers, not half-integers as in [1], so there is **no need of supersymmetry-preserving regularization**.
- The cancellation of bosonic against fermionic $\omega\neq0$ frequencies is an expected **vestige of supersymmetry**.
- The one-loop correction arising from $\omega=0$ frequencies **matches the prediction** $\cos^{-\frac{3}{2}}\theta_0$ **up to a θ_0-dependent factor**. The existing discrepancy between sigma-model perturbation theory and exact predictions via supersymmetric localization for the normalized latitude Wilson loop **remains an interesting problem** which needs to be clarified.
- In the Polyakov description in conformal gauge, the additional **longitudinal-modes and diffeo-ghosts determinants** (supplemented by reasonable boundary conditions) cannot account for the cancellation of the extra factor.
- Same remark for the pre-exponential $\lambda^{-3/4}$ in $\langle\mathcal{W}(\lambda,\theta_0)\rangle$: even though not included above, it is purely topological (see also related discussion in [8]).
- The mismatch might be an artifact of the sigma-model regularization (the unphysical boundary $\sigma=R$).

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