Euler-Maclaurin formula (Mathematica notation, $p \geq 1$)

$$\sum_{i=m+1}^{n} g(i)$$
$$= \int_m^n g(\omega)\, d\omega + \frac{g(n) - g(m)}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \left[\frac{d^{2k-1}}{d\omega^{2k-1}} g(n) - \frac{d^{2k-1}}{d\omega^{2k-1}} g(m) \right] - \int_m^n \frac{d^{2p}}{d\omega^{2p}} g(\omega)\, \frac{B_{2p}(\omega - \lfloor \omega \rfloor)}{(2p)!} d\omega$$
$$\overset{e.g.\ p=1}{=} \int_m^n g(\omega)\, d\omega + \frac{g(n) - g(m)}{2} + \frac{\frac{d}{d\omega} g(n) - \frac{d}{d\omega} g(m)}{12} - \frac{1}{2} \int_m^n \frac{d^2}{d\omega^2} g(\omega)\ B_2(\omega - \lfloor \omega \rfloor)\, d\omega$$
$$\overset{e.g.\ p=\infty}{=} \int_m^n g(\omega)\, d\omega + \frac{g(n) - g(m)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left[\frac{d^{2k-1}}{d\omega^{2k-1}} g(n) - \frac{d^{2k-1}}{d\omega^{2k-1}} g(m) \right]$$

We pose and calculate

$$f(0) + 2f(1) + 2f(2) = summand1$$
$$F(\omega) \equiv f^2(\omega) = summand2 \qquad \omega \geq 3$$
$$F(\omega) = \sum_{i=1}^{order} \frac{F_i}{\omega^i} + O\left(\frac{1}{\omega^{order+1}}\right) \qquad order \geq 1,\ \omega \to \infty$$
$$F_1 = \chi_v - \chi_v^{cir}$$
$$\frac{d^{2k-1}}{d\omega^{2k-1}} \left(\frac{1}{\omega^i}\right) = -\frac{(i+2k-2)!}{(i-1)!} \frac{1}{\omega^{i+2k-1}} \qquad k \geq 1,\ i \geq 1$$
$$\sum_{\omega=3}^{\infty} \frac{1}{\omega^i} = -1 - \frac{1}{2^i} + \zeta(i) \qquad i \geq 2$$
$$\sum_{\omega=\omega_0}^{\infty} \log(\omega + \omega_1) \overset{\zeta-reg.}{=} \log \frac{\sqrt{2\pi}}{\Gamma(\omega_0 + \omega_1)}$$

The sum proportional to μ is neglibile, so we concentrate on the main sum.

Cutoff function (not used)

$$G(\omega) = \begin{cases} 0 & \omega \to 2 \\ \frac{1}{2} & \omega \to O\left(\frac{1}{\epsilon_0}\right) \\ 1 + O\left(e^{-\#\omega}\right) & \omega \to \infty \end{cases}$$
$$e.g. \qquad G(\omega) = 1 - e^{\epsilon_0(2-\omega)}$$
$$F(\omega)\, G(\omega) = \begin{cases} 0 & \omega \to 2 \\ \sum_{i=1}^{order} \frac{F_i}{\omega^i} + O\left(\frac{1}{\omega^{order+1}}\right) + O\left(e^{-\#\omega}\right) & \omega \to \infty \end{cases}$$

1)

$$
\begin{aligned}
&\log\frac{Z\left(\sigma_0\right)}{Z\left(\infty\right)}\\
=&\sum_{\omega=-\Lambda}^{\Lambda} f\left(\omega\right)-\left(\chi_v-\chi_v^{cir}\right)\log\Lambda\\
=&f\left(0\right)+2f\left(1\right)+2f\left(2\right)+\sum_{\omega=3}^{\Lambda}F\left(\omega\right)-\left(\chi_v-\chi_v^{cir}\right)\log\Lambda\\
=&f\left(0\right)+2f\left(1\right)+2f\left(2\right)+\int_2^{\Lambda}F\left(\omega\right)d\omega\\
&-\frac{F\left(2\right)}{2}-\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}F^{(2k-1)}\left(2\right)-\left(\chi_v-\chi_v^{cir}\right)\log\Lambda\\
=&f\left(0\right)+2f\left(1\right)+2f\left(2\right)+\int_2^{\infty}\left[F\left(\omega\right)-\sum_{i=1}^{order}\frac{F_i}{\omega^i}\right]d\omega-\left(\chi_v-\chi_v^{cir}\right)\log 2-\sum_{i=2}^{order}F_i\frac{2^{1-i}}{1-i}\\
&-\frac{F\left(2\right)}{2}-\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}F^{(2k-1)}\left(2\right)\\
=&f\left(0\right)+2f\left(1\right)+2f\left(2\right)+\int_2^{\bar\omega}F\left(\omega\right)d\omega+\int_{\bar\omega}^{\infty}\left[F\left(\omega\right)-\sum_{i=1}^{order}\frac{F_i}{\omega^i}\right]d\omega-\left(\chi_v-\chi_v^{cir}\right)\log\bar\omega-\sum_{i=2}^{order}F_i\frac{\bar\omega^{1-i}}{1-i}\\
&-\frac{F\left(2\right)}{2}-\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}F^{(2k-1)}\left(2\right)\\
=&f\left(0\right)+2f\left(1\right)+2f\left(2\right)+\int_2^{\bar\omega}F\left(\omega\right)d\omega+\int_{\bar\omega}^{\infty}\left[F\left(\omega\right)-\sum_{i=1}^{order}\frac{F_i}{\omega^i}\right]d\omega-\left(\chi_v-\chi_v^{cir}\right)\log\bar\omega-\sum_{i=2}^{order}F_i\frac{\bar\omega^{1-i}}{1-i}\\
&-\frac{F\left(2\right)}{2}-\sum_{k=1}^{p}\frac{B_{2k}}{(2k)!}\frac{d^{2k-1}}{d\omega^{2k-1}}F\left(2\right)-\int_2^{\infty}\frac{d^{2p}}{d\omega^{2p}}F\left(\omega\right)\frac{B_{2p}\left(\omega-\lfloor\omega\rfloor\right)}{(2p)!}d\omega
\end{aligned}
$$

2)

$$
\begin{aligned}
&\log\frac{Z\left(\sigma_0\right)}{Z\left(\infty\right)}\\
=&\sum_{\omega=-\Lambda}^{\Lambda} f\left(\omega\right)-\left(\chi_v-\chi_v^{cir}\right)\log\Lambda\\
=&f\left(0\right)+2f\left(1\right)+2f\left(2\right)+\sum_{\omega=3}^{\Lambda}F\left(\omega\right)-\left(\chi_v-\chi_v^{cir}\right)\log\Lambda\\
=&f\left(0\right)+2f\left(1\right)+2f\left(2\right)+\int_2^{\Lambda}F\left(\omega\right)d\omega\\
&-\frac{F\left(2\right)}{2}-\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}F^{(2k-1)}\left(2\right)-\left(\chi_v-\chi_v^{cir}\right)\log\Lambda\\
=&f\left(0\right)+2f\left(1\right)+2f\left(2\right)+\int_2^{\infty}\left[F\left(\omega\right)-\frac{\chi_v-\chi_v^{cir}}{\omega}-\sum_{i=2}^{order}G\left(\omega\right)\frac{F_i}{\omega^i}\right]d\omega-\left(\chi_v-\chi_v^{cir}\right)\log 2+\int_2^{\infty}\sum_{i=2}^{order}G\left(\omega\right)\frac{F_i}{\omega^i}d\omega\\
&-\frac{F\left(2\right)}{2}-\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}F^{(2k-1)}\left(2\right)\\
=&f\left(0\right)+2f\left(1\right)+2f\left(2\right)+\int_2^{\infty}\left[F\left(\omega\right)-\frac{\chi_v-\chi_v^{cir}}{\omega}-\sum_{i=2}^{order}G\left(\omega\right)\frac{F_i}{\omega^i}\right]d\omega-\left(\chi_v-\chi_v^{cir}\right)\log 2+\int_2^{\infty}\sum_{i=2}^{order}G\left(\omega\right)\frac{F_i}{\omega^i}d\omega\\
&-\frac{F\left(2\right)}{2}-\sum_{k=1}^{p}\frac{B_{2k}}{(2k)!}\frac{d^{2k-1}}{d\omega^{2k-1}}F\left(2\right)-\int_2^{\infty}\frac{d^{2p}}{d\omega^{2p}}F\left(\omega\right)\frac{B_{2p}\left(\omega-\lfloor\omega\rfloor\right)}{(2p)!}d\omega
\end{aligned}
$$

2reg) We prove that zeta-reg. = cut-off reg. EVEN on the latitude!

$$
\begin{aligned}
&\log\frac{Z(\sigma_0)}{Z(\infty)}\\
=&\sum_{\omega=-\Lambda}^{\Lambda} f(\omega)\\
=&f(0)+2f(1)+2f(2)+\sum_{\omega=3}^{\infty}\left[F(\omega)-\log\left(1+\frac{\chi_v-\chi_v^{cir}}{\omega}\right)\right]+\sum_{\omega=3}^{\infty}\log\left(1+\frac{\chi_v-\chi_v^{cir}}{\omega}\right)\\
\overset{\zeta-reg.}{=}&f(0)+2f(1)+2f(2)+\sum_{\omega=3}^{\infty}\left[F(\omega)-\log\left(1+\frac{\chi_v-\chi_v^{cir}}{\omega}\right)\right]+\log\frac{\Gamma(3)}{\Gamma\left(3+\chi_v-\chi_v^{cir}\right)}\\
=&f(0)+2f(1)+2f(2)+\sum_{\omega=3}^{\infty}\left[F(\omega)+\sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{\chi_v-\chi_v^{cir}}{\omega}\right)^k\right]+\log\frac{2}{\Gamma\left(3+\chi_v-\chi_v^{cir}\right)}\\
=&f(0)+2f(1)+2f(2)+\sum_{\omega=3}^{\infty}\left[F(\omega)-\frac{\chi_v-\chi_v^{cir}}{\omega}\right]+\sum_{\omega=3}^{\infty}\sum_{k=2}^{\infty}\frac{1}{k}\left(-\frac{\chi_v-\chi_v^{cir}}{\omega}\right)^k+\log\frac{2}{\Gamma\left(3+\chi_v-\chi_v^{cir}\right)}\\
=&f(0)+2f(1)+2f(2)+\sum_{\omega=3}^{\Lambda}F(\omega)+\left(\chi_v-\chi_v^{cir}\right)\left(\frac{3}{2}-\gamma-\log\Lambda\right)+\sum_{k=2}^{\infty}\frac{-1-2^{-k}+\zeta(k)}{k}(-)^k\left(\chi_v-\chi_v^{cir}\right)^k+\log\frac{2}{\Gamma\left(3+\chi_v-\chi_v^{ci}\right.}\\
=&f(0)+2f(1)+2f(2)+\sum_{\omega=3}^{\Lambda}F(\omega)-\left(\chi_v-\chi_v^{cir}\right)\log\Lambda+\log\frac{2\left(1+\chi_v-\chi_v^{cir}\right)\left(1+\frac{\chi_v-\chi_v^{cir}}{2}\right)\Gamma\left(1+\chi_v-\chi_v^{cir}\right)}{\Gamma\left(3+\chi_v-\chi_v^{cir}\right)}\\
=&f(0)+2f(1)+2f(2)+\sum_{\omega=3}^{\Lambda}F(\omega)-\left(\chi_v-\chi_v^{cir}\right)\log\Lambda\\
=&\sum_{\omega=-\Lambda}^{\Lambda} f(\omega)-\left(\chi_v-\chi_v^{cir}\right)\log\Lambda
\end{aligned}
$$

2der)

$$
\begin{aligned}
&\frac{d}{d\sigma_0}\log\frac{Z(\sigma_0)}{Z(\infty)}\\
=&\frac{df(0)}{d\sigma_0}+2\frac{df(1)}{d\sigma_0}+2\frac{df(2)}{d\sigma_0}+\int_2^{\infty}\left[\frac{d}{d\sigma_0}F(\omega)-\frac{\frac{d}{d\sigma_0}\chi_v}{\omega}-\sum_{i=2}^{order}G(\omega)\frac{\frac{d}{d\sigma_0}F_i}{\omega^i}\right]d\omega-\frac{d\chi_v}{d\sigma_0}\log 2+\int_2^{\infty}\sum_{i=2}^{order}G(\omega)\frac{\frac{d}{d\sigma_0}F_i}{\omega^i}d\omega\\
&-\frac{\frac{d}{d\sigma_0}F(2)}{2}-\sum_{k=1}^{p}\frac{B_{2k}}{(2k)!}\frac{d^{2k-1}}{d\omega^{2k-1}}\frac{d}{d\sigma_0}F(2)-\int_2^{\infty}\frac{d^{2p}}{d\omega^{2p}}\frac{d}{d\sigma_0}F(\omega)\frac{B_{2p}(\omega-\lfloor\omega\rfloor)}{(2p)!}d\omega
\end{aligned}
$$