Wilson loops in supersymmetric gauge theories

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June 12, 2020

Plan

Quoted references

- Carroll, Notes on GR, arXiv+book
- Blau, Notes on GR, personal webpage
- Bertlmann, Anomalies in QFT, book
- Peskin Schröder, Intro To QFT, book
- Makeenko, Introduction to Wilson Loops at large N, arXiv
- Mariño, Les Houches lectures, arXiv
- Cremonesi, Introduction to localisation, POS Sissa
- Pestun et al., Review of localization, arXiv
- Zarembo, Localization and AdS/CFT correspondence, arXiv
- Jackson, Classical electrodynamics, book
- other papers, arXiv

Extra

- Trancanelli, School on Supersymmetric Localization, ICTP website

  WL in fund. repr. in gauge theory
  no high-rank representation and holography

- Münkler, thesis + review, arXiv
- Preti, thesis, online

master student

1 WL in gauge theories

1.1 Definition

Notion of parallel transport of vector along curve in diff. geom.

$x^\mu = x^\mu(\tau)$, $\tau \in [\tau_0, \tau_1]$

No sense to ask if 2 vectors defined at 2pt are parallel,
but given metric+curve, one can drag one vector along the curve using ∇_μ and compare

parallel transport equation

$$\dot{x}^\nu \left(\nabla_\nu V\right)^\mu \equiv \dot{x}^\nu \left(\partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda\right) = 0 \quad \rightarrow \quad \begin{cases} \frac{d}{d\tau} V^\mu(\tau) = -\left(\Gamma^\mu_{\nu\lambda}\dot{x}^\nu V^\lambda\right)(\tau) \\ V^\mu(\tau_0) = \bar{V}^\mu \end{cases}$$

Linear 1st-order diff. eq.; there exists a unique solution

One thing GR textbooks don't say is formal solution [Carroll ch.3]

$$V^\mu(\tau) \equiv \underbrace{U_{P\ \nu}^{\ \mu}(\tau, \tau_0)}_{\text{parallel propagator}} \bar{V}^\nu$$

$$\begin{cases} \frac{d}{d\tau} U_{P\ \nu}^{\ \mu}(\tau,\tau_0) = -\Gamma^\mu_{\sigma\lambda}(\tau)\,\dot{x}^\sigma(\tau)\, U_{P\ \nu}^{\ \lambda}(\tau,\tau_0) \\ U_{P\ \nu}^{\ \mu}(\tau_0,\tau_0) = \delta^\mu_\nu \end{cases}$$

solve iteratively by plugging lhs into rhs repeatedly

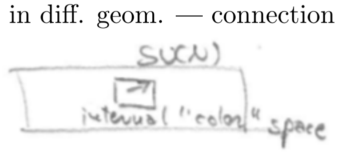
$$U_{P\ \nu}^{\ \mu}(\tau,\tau_0) = \mathcal{P}\exp\left(-\int_{\tau_0}^{\tau} \Gamma^\mu_{\sigma\nu}\dot{x}^\sigma d\tau\right)$$

μ, ν are outside

P = path-ordering operator $= \mathcal{P}\left(\Gamma^\mu_{\sigma_1\rho}(\tau_1)\,\Gamma^\rho_{\sigma_2\nu}(\tau_2)\right) = \begin{cases} \Gamma^\mu_{\sigma_1\rho}(\tau_1)\,\Gamma^\rho_{\sigma_2\nu}(\tau_2) & \tau_1 > \tau_2 \\ \Gamma^\mu_{\sigma_2\rho}(\tau_2)\,\Gamma^\rho_{\sigma_1\nu}(\tau_1) & \tau_2 > \tau_1 \end{cases}$
matrix product, ordered in such a way that the largest τ is on the left, and each subsequent value of τ is less than the previous one.

similar to time evolution operator in QM (Dyson's formula - e^{iHt} - $\mathcal{T}$ appears when Hamiltonian at different times does not commute)
another occurrence is in heat kernel propagator of spinor fields

Empty example?
cf. connection in Riemannian geometry in diff. geom. — connection in gauge theories in flat space



[Carroll ch.3] [Blau 10.13] [Bertlmann]

$$A_\mu = A^a_\mu T^a_R \qquad T^a_R \in \mathfrak{g} = \mathrm{Lie}(G) \qquad G = SU(N) \qquad R = \text{fundamental repr.}$$

we can parallel transport along paths

$$\dot{x}^{\nu}(\tau)\left(D_{\nu}V(\tau)\right)^{i} \equiv \dot{x}^{\nu}\left(\partial_{\nu}V^{i} - iA^{i}_{\nu j}V^{j}\right) = 0 \quad \rightarrow \quad \begin{cases} \frac{d}{d\tau}V^{i}(\tau) = \left(iA^{i}_{\nu j}\dot{x}^{\nu}V^{j}\right)(\tau) \\ V^{i}(\tau_0) = \bar{V}^{i} \end{cases}$$

$$V^{i}(\tau) = U_{P\,j}^{\,i}(\tau,\tau_0)\,\bar{V}^{j}$$

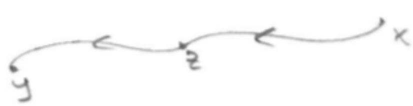
$$\begin{cases} \frac{d}{d\tau}U_{P\,j}^{\,i}(\tau,\tau_0) = iA^{i}_{\mu k}(\tau)\,\dot{x}^{\mu}(\tau)\,U_{P\,j}^{\,k}(\tau,\tau_0) \\ U_{P\,j}^{\,i}(\tau_0,\tau_0) = \delta^{i}_{j} \end{cases}$$

$$\begin{aligned} U_{P\,j}^{\,i}(\tau,\tau_0) &= \mathcal{P}\exp\left(i\int_{\tau_0}^{\tau} A^{i}_{\mu j}(\tau)\,\dot{x}^{\mu}(\tau)\,d\tau\right) \\ &= \text{ parallel propagator, gauge compensator, Wilson line} \end{aligned}$$

properties

1) U_P meets criteria of comparing vectors (= quarks = Dirac spinors in the fund of G):
$U \in G$
$\psi(x) \rightarrow \psi'(x) = U(x)\,\psi(x)$
$A_{\mu}(x) \rightarrow A'_{\mu}(x) = U(x)\,A_{\mu}(x)\,U^{\dagger}(x) - i\partial_{\mu}U(x)\,U^{\dagger}(x)$
$U_P(y,x) \rightarrow U'_P(y,x) = U(y)\,U_P(y,x)\,U^{\dagger}(x)$ because U' obeys par-tr eq with A' in place of A + uniqueness
$U_P(y,x)\,\psi(x) \rightarrow U(y)\,U_P(y,x)\,\psi(x)$

2) concatenation property $U_P(y,x) = U_P(y,z)\,U_P(z,x)$
from uniqueness



to convert U into gauge-invariant object
Wilson loop operator = trace of matrix obtained by parallel transporting a quark around a closed curve

$$\mathcal{W}_R[\gamma] \equiv \mathrm{tr}_R\,U(x,x) = \mathrm{tr}_R\mathcal{P}\exp\left(i\int_{\gamma} A_{\mu}dx^{\mu}\right)$$

[Peskin]

- non-local, gauge-invariant operator
- depends on closed γ; γ-independent iff $F_{\mu\nu} = 0$ (non-abelian Stokes theorem relates par.tr. around infinitesimal loop to flux of F through loop; complete analogy with differential geometry)
- depends on R
(so far $R = \square$, obtained by considering the g.tr. of quark)
(construct by considering other combinations of eigenvalues)
(e.g. quarks in R = 2-index anti-symmetric repr. of $SU(3)$ [Armoni, Shifman,Veneziano 0307097])

pert.th. vev leads to divergences [Polyakov 80], which require renormalization:
- linear divergence, interpret as a mass renormalization of the external particle
- extra due to cusps and self-intersections.

physical interpretation in Maxwell theory
$G = U(1) \qquad \rightarrow \qquad$ no tr$\mathcal{P}$

$$\langle \mathcal{W}_R [\gamma] \rangle = \frac{1}{Z} \int \mathcal{D}A \, e^{iS_{\text{Maxwell}} + i \int_\gamma A_\mu dx^\mu}$$

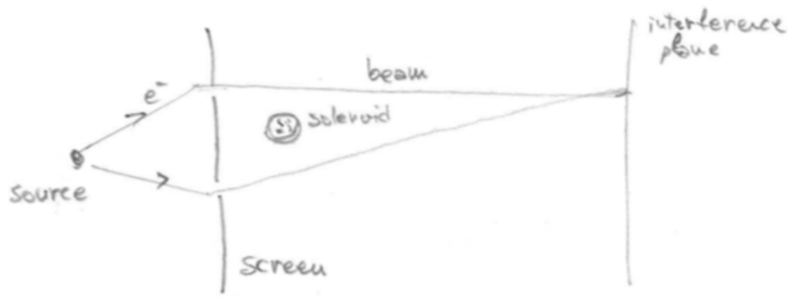
WL = phase acquired by charge moving along path

$$A_\mu(x) \, j^\mu(x) = A_\mu(x) \, \dot{x}^\mu(\tau) \, \delta^{(d)}(x^\mu - x^\mu(\tau)) = \text{minimal-coupling interaction}$$

no $-m \int_\gamma ds$ because quark is non-dynamical

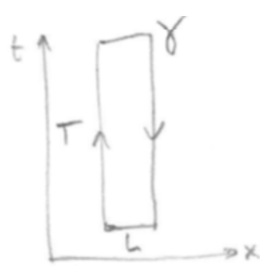
interesting contours [Makeenko]

Aharonov-Bohm effect
phase becomes detectable observable in not-simply-connected spaces



math and experiment [Orasch - thesis 2017]
experiment vs idealization [Earman 2017]

1.2 Interparticle potential



Euclidean space
$S_{\text{Maxwell}} = \int d^4x \left(\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2e^2} (\partial^\mu A_\mu)^2 \right)$ = pure QED in Feynman gauge with no dynamical fermions
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \langle A_\mu(x) \, A_\nu(y) \rangle = \Delta_{xy} \delta_{\mu\nu} = \frac{e^2}{4\pi^2 (x-y)^2} \delta_{\mu\nu} \qquad \mathcal{W}[\gamma] = e^{i \int_\gamma A_\mu dx^\mu}$
absence of interactions allow to compute vev to all orders in perturbation theory.

γ = rectangle, which includes
$x^\mu(\tau_1) = (\tau_1, 0, 0, 0) \qquad \tau_1, (0, T)$

$x^\mu(\tau_2) = (-\tau_2, L, 0, 0) \qquad \tau_1, \tau_2 \in (-T, 0) \qquad T \gg L$

$$\langle \mathcal{W}[\text{rect}] \rangle = \frac{1}{Z} \int \mathcal{D}A \, e^{-S_{\text{Maxwell}} + i \int_\gamma A_\mu dx^\mu}$$

$$\text{rewrite as source} \qquad = \frac{1}{Z} \int \mathcal{D}A \, e^{\int d^4x \left(-\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2e^2} \partial^\mu A_\mu \partial^\nu A_\nu - A_\mu J^\mu \right)}$$

$$J^\mu(x) = -i \dot{x}^\mu(\tau) \, \delta^{(4)}(x^\mu - x^\mu(\tau))$$

$$= \frac{1}{Z} \int \mathcal{D}A \, e^{\int d^4x \left(\frac{1}{2e^2} A^\mu \Box A_\mu - A_\mu J^\mu \right)}$$

$$\text{[Peskin]} \qquad = \frac{Z[J]}{Z[0]}$$

$$\sim e^{-V(L)T}$$

in num/den: QM derivation of Z is vacuum-to-vacuum transition amplitude,
described by a superposition of energy eigenstates
in ratio, $V(L)$ = ground-state-energy variation of 2 Hamiltonians

compare to

$$\langle \mathcal{W}[\text{rect}] \rangle = \frac{1}{Z} \int \mathcal{D}A \, e^{\int d^4x \left(\frac{1}{2e^2} A^\mu \Box A_\mu - A_\mu J^\mu \right)}$$

$$\text{EXERCISE Gauss. int. [Peskin ch.9]} \qquad = e^{\int d^4x d^4y \, \frac{-1}{2} J^\mu(x) \Delta_{xy} J^\mu(y)}$$

$$= e^{\int_0^T d\tau_1 \int_0^{-T} d\tau_2 \, \underbrace{2}_{\tau_1 \leftrightarrow \tau_2} \times \frac{e^2 \dot{x}(\tau_1) \cdot \dot{x}(\tau_2)}{8\pi^2 (x(\tau_1) - x(\tau_2))^2} + \text{"self-energies"}}$$

$$= e^{\int_0^T d\tau_1 \int_0^{-T} d\tau_2 \, \frac{-e^2}{4\pi^2 \left[(\tau_1 + \tau_2)^2 + L^2 \right]}}$$

$$= \exp\left(\frac{e^2}{4\pi} \frac{T}{L} \right)$$

by comparison

$$V(L) = -\frac{e^2}{4\pi L} = \text{Coulomb potential}$$

$\propto 1/L$ from scale invariance

In general, rectangular WL diagnoses phase transitions between confining and non-confining phases of gauge theory when loop size is scaled up to infinity

perimeter law	$\langle \mathcal{W}[\text{rect}] \rangle \sim e^{-kP} \to V(L) = k$	saturates constant = Coulomb phase
area law	$\langle \mathcal{W}[\text{rect}] \rangle \sim e^{-\sigma A} \to V(L) = \sigma L$	grows linearly = confining phase

e.g. order parameter of hadronic matter, between confined and deconfined (= quark-gluon plasma) phase

2 Supersymmetric WL

focus on one SUSY theory instead of many
counterpart in Chern-Simons-matter (ABJM) theory
crash course on action, field content and symmetries

2.1 N=4 super Yang-Mills theory in 4d

- Euclidean signature $(++++)$, $G = SU(N)$

$$S = \int d^4x \left(\frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2g^2} \partial_\mu \phi^a_i \partial_\mu \phi^a_i + \text{fermions, interactions, g.f. term} \right)$$
$$a = 1, ...d(G) = N^2 - 1$$

Field content = one $\mathcal{N} = 4$ vector multiplet

1 gluon	A_μ	$\mu = 1, \ldots 4$	
6 real scalars	ϕ_I	$I = 1, \ldots 6$	
4 complex Weyl fermions	ψ^A_α	$\alpha = 1, \ldots 4$	$A = 1, \ldots 4$

fields in adjoint of G = matrix-valued fields $N \times N$

- from dimensional reduction of N=1 SYM in 10d

$$S_{(10)} = \frac{1}{g^2_{(10)}} \int d^{10}x \left(\frac{1}{4} F^a_{MN} F^a_{MN} - \frac{1}{2} \Psi^a \Gamma^M D_M \Psi^a \right) \qquad M, N = 1, ...10$$

dimensional reduction = demand fields to depend on 4 coords + compactifying other 6.

field content reflects 10d origin
$A_M \to (A_\mu, \phi_I)$
MW spinor in 10d $\Psi \to \psi^A_\alpha$
MW means $\Psi^T C_{10} = \Psi^\dagger \Gamma^0$ and $\Gamma^{11}\Psi = +\Psi$
$\Gamma^M \to (\Gamma^\mu, \Gamma^I)$ $\quad \{\Gamma^M, \Gamma^N\} = 2\delta^{MN}$ $\quad 32 \times 32$

- symmetries of action:
gauge group $U(N)$
global symmetry group $PSU(2,2|4) \supset SO(2,4)_{\text{spacetime}} \times SO(6)_{\text{R-symmetry}}$
(go back and comment on fields' indices)
+ 16 Q's + 16 S's
(Q's inherited from 10d, S's from additional conformal invariance in presence of SUSY)

these fermionic transformations on fields are

$$\delta_\epsilon A_\mu = \epsilon \Gamma_\mu \Psi$$
$$\delta_\epsilon \phi_I = \epsilon \Gamma_I \Psi$$
$$\delta_\epsilon \Psi = \ldots$$

fermionic tr. parametrized by $\epsilon(x) = \epsilon_Q + x^\mu \Gamma_\mu \epsilon_S$
ϵ_Q, ϵ_S are constant MW 10d spinors with 16 d.o.f. each
so total 32 linearly indep. tr.

Most symmetric, non-gravitational, interacting theory in 4d
family of theories parametrized by g (symmetries protects it from renormalization) and gauge group G, i.e. N
't Hooft coupling constant $\lambda = g^2 N$
AdS/CFT, see Wiseman's lectures

2.2 Maldacena-Wilson loop

problem: there is no quark!
introduce massive quark via spontaneous symmetry breaking/Higgs mechanism
break $U(N+1) \to U(N) \times U(1)$; $U(1)$ decouples when physics for energy scales much lower than mass

infinitely massive quark "W-bosons" transform in fundamental of $U(N)$
reason name:
- W-bosons in SM = gauge field made massive by a Higgs mechanism
- "W-bosons" in SYM = vector field made massive by a Higgs mechanism

ansatz for SUSY WL should include ϕ:
- ϕ and A belong to same multiplet
- dimensional reduction
$\mathcal{W}_R\left[\gamma_{(10)}\right] = \mathrm{tr}_R \mathcal{P} \exp\left(i \int_{\gamma_{(10)}} A_M dx^M\right)$
$A_M dx^M \to A_\mu dx^\mu + \phi_I dx^I$
$x^M(\tau) \to \left(x^\mu(\tau), \theta^I(\tau)\right)$
- $\delta_\epsilon \phi_I$ may cancel $\delta_\epsilon A_\mu$

$$\mathcal{W}_R[\gamma,\theta] \equiv \frac{1}{d(R)} \mathrm{tr}_R \mathcal{P} \exp\left[i \int_\gamma \underbrace{\left(A_\mu \dot{x}^\mu - i|\dot{x}|\theta^I \phi_I\right)}_{\text{generalized connection } \mathcal{A} \equiv \mathcal{A}_\mu \dot{x}^\mu} d\tau\right]$$

[Maldacena 98]

- depends on γ, θ, R
- τ common parameter spanning spacetime and internal loop

- $|\dot{x}|$ for parametrization invariance
- normalization ensures $\langle \mathcal{W}_R[\gamma] \rangle = \frac{1}{d(R)} \mathrm{tr}_R(\mathbb{I}) = 1 + O\left(g^2\right)$ in free theory
- not a phase in Euclidean space (relative i plays a role in supersymmetry and finiteness)

constrain ansatz (γ, θ, R) imposing invariance under SUSY and superconformal tr.
caveat: say "SUSY-invariant" in place of "invariant under SUSY and superconformal tr"
we know variation of fields, so variation of composite operators

- WL are locally 1/2-BPS for any γ, R and $\theta^2 = \theta^I \theta^I = 1$

$$0 = \delta_\epsilon \mathcal{W}_R[\gamma,\theta] \propto \mathrm{tr}_R \mathcal{P}\left[\left(\int_\gamma \delta_\epsilon \mathcal{A}_\mu dx^\mu\right) \exp\left(i \int_\gamma \mathcal{A}_\mu dx^\mu\right)\right]$$

$$0 = \delta_\epsilon \mathcal{A}_\mu = \epsilon\left(i\Gamma_\mu \dot{x}^\mu + |\dot{x}| \Gamma_I \theta^I\right)\Psi$$

ABJM is different: ansatz → merely bosonic, not employ fully gauge group, not symmetry of dual string
make ansatz fermions, replace by weaker requirement and find higher supersymmetry and holo description

$$\text{swap } \epsilon, \Psi \text{ and remove } \Psi \qquad \left(i\Gamma_\mu \dot{x}^\mu(\tau) + |\dot{x}(\tau)| \Gamma_I \theta^I(\tau)\right)\epsilon(\tau) = 0$$

local (= 1 eq for each τ and in general matrices not commute) BPS (= WL preserves supercharges) equation

still have to explain 1/2-BPS and $\theta^2 = 1$: homogeneous system of 16 eqs must have non-zero solution!
matrix is degenerate when $\theta^2 = 1 \to$ there exists solution
to see this,

$$\left(i\Gamma_\mu \dot{x}^\mu + |\dot{x}| \Gamma_I \theta^I\right)^2 = 0$$

use it to define orthogonal projectors

$$P^{\pm} = \frac{1}{\sqrt{2}}\left(1 \pm i\frac{\dot{x}^\mu}{|\dot{x}|}\Gamma_\mu \Gamma_I \theta^I\right)$$

resume

$$i\Gamma_\nu \dot{x}^\nu P^- \epsilon = 0 \qquad \rightarrow \qquad P^- \epsilon = 0 \qquad \rightarrow \qquad \epsilon = P^+ \underbrace{\bar{\epsilon}}_{\text{arbitrary constant spinor}}$$

1/2-BPS = half of 16+16 components in ϵ are independent = half of supercharges that annihilate vacuum also annihilate WL

- WL preserve SUSY locally. Can they preserve rigid SUSY, namely $\forall \tau \; \epsilon(\tau) = \epsilon$?
  - local SUSY is not an honest symmetry of action

  WL that preserve SUSY globally
  - not entirely protected from quantum corrections
  - but invariance impose massive diagrammatical cancellations and leaves behind a relatively simple result
  - often computed by SUSY localization techniques

  Some pairs (γ, θ); R plays no role

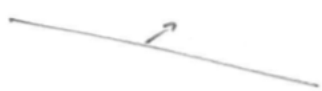
2.3 Examples

Complete classification of superconformally invariant WL [Dymarsky, Pestun 2009]

simplest option θ= constant along curve

- Straight line, 1/2-BPS

$$x^\mu = (\tau, 0, 0, 0) \qquad \theta^I = (0, 0, 1, 0, 0, 0)$$



  gauge invariance restored if gauge tr. go to zero at infinity
  EXERCISE: bosonic and fermionic symmetries

- Circular (DGRT) loop, 1/2-BPS

$$x^\mu = (\cos\tau, \sin\tau, 0, 0) \qquad \theta^I = (0, 0, 1, 0, 0, 0)$$

  EXERCISE: bosonic and fermionic symmetries [Bianchi, Green, Kovacs 02, sec.2] [Drukker, Giombi, Ricci, Trancanelli 07]

If one allows θ to vary along the curve, it is possible to preserve some SUSY globally for special curves.
fall under 2 classes, both 1/16-BPS

ingenious way to eliminate local dependence

- Zarembo loops in $\mathbb{R}^4$ [Zarembo 2002]
  topological twist that identifies Euclidean Lorentz group $SO(4)$ and $SO(4) \subset SO(6)_R$

$$\mathcal{A} = A_\mu \dot{x}^\mu + i\,|\dot{x}| \underbrace{M^I_{\ \mu}}_{\text{assigns tg vector to scalar combination}} \frac{\dot{x}^\mu}{|\dot{x}|}\phi_I\,, \qquad M^I_{\ \mu} M^I_{\ \nu} = \delta_{\mu\nu}$$

correlation dimensionality loop and degree of BPSness

	$\gamma \in$ ...	...-BPS	
	$\mathbb{R}^4$		
superspace formalism and topological arguments	$\mathbb{R}^3$	1/8-BPS	
proves non-renormalization theorems	$\mathbb{R}^2$	1/4-BPS	e.g. circular (Zarembo) loop
$\langle \mathcal{W}\,[\text{Zarembo}] \rangle = 1$	$\mathbb{R}$	1/2-BPS	familiar straight line

SUSY not so strong to trivialize vev
a richer variety that is no longer protected from quantum corrections

- DGRT loops on S^3, 1/16-BPS [Drukker, Giombi, Ricci, Trancanelli 07]
  different topological twist

$$x^\mu x^\mu = 1$$

$$\mathcal{A} = A_\mu dx^\mu - \frac{i}{2} \underbrace{\sigma^R_i}_{SU(2)\text{ right-invariant 1-forms on } S^3} M^i_{\ I}\Phi^I$$

$$M^i_{\ I} M^j_{\ I} = \delta^{ij}$$

loops on S^2, 1/8-BPS

$$x^\mu = \left(x^i, 0\right) \qquad x^i x^i = 1 \qquad |\dot{x}|\,\theta^I = \left(\epsilon^{ijk} x^j \dot{x}^j, 0, 0, 0\right) = \text{suggestively rewritten in a geometric fashion}$$

EXERCISE calculate BPS for latitude or wedge/two-longitude

DGRT loops on S^2 show peculiar localization properties that allow for the exact evaluation of vev [Pestun 07, 09]
instrumental in deriving the all-loop expression for a non-BPS quantity called Bremsstrahlung function, see lecture part 4.2

so far operators, now
Compute vev:
- strong coupling (dual description in AdS/CFT in terms of fundamental strings, branes, bubbling geometries)
- finite coupling (integrability, supersymmetric localization)
- weak coupling (ordinary perturbation theory)

2.4 Perturbation theory

color algebra for $G = U(N)$ [Peskin appendix]

$$\begin{aligned}\left[T^a, T^b\right] &= i f^{abc} T^c \\ \mathrm{tr}\left(T^a T^b\right) &= C(R)\,\delta^{ab} \\ \mathrm{tr}\mathbb{I} &= \delta^{ii} = d(R) \\ N^2 &= \delta^{aa} = d(G)\end{aligned}$$

tons of daughter relations for higher loop orders

weak-coupling expansion at 1 loop

$$\begin{aligned}\mathcal{W}_R[\gamma,\theta] &= \frac{1}{d(R)}\mathrm{tr}_R \mathcal{P}\exp\left(i\int \mathcal{A}(\tau)\,d\tau\right) \\ \text{def. exp} &= \frac{1}{d(R)}\mathrm{tr}_R\mathcal{P}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(i\int\mathcal{A}(\tau)\,d\tau\right)^n\right] \\ \text{at fixed n, use an identity} &= \frac{1}{d(R)}\mathrm{tr}_R\sum_{n=0}^{\infty}\left(i^n\int_{\tau_1>\tau_2>\ldots>\tau_n} d\tau_1 d\tau_2 \ldots d\tau_n\ \mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_n\right) \qquad \mathcal{A}_1 \equiv \mathcal{A}(\tau_1),\ldots \\ \text{terms for leading correction} &= \frac{1}{d(R)}\mathrm{tr}\mathbb{I} + \frac{i}{d(R)}\int_{\tau_1} d\tau_1\ \mathrm{tr}\mathcal{A}_1 - \frac{1}{d(R)}\int_{\tau_1>\tau_2} d\tau_1 d\tau_2\ \mathrm{tr}_R(\mathcal{A}_1\mathcal{A}_2) + \ldots\end{aligned}$$

$$\begin{aligned}&\langle\mathcal{W}_R[\gamma,\theta]\rangle \\ \text{color identities} &= 1 + 0 - \frac{C(R)}{d(R)}\int_{\tau_1>\tau_2} d\tau_1 d\tau_2\ \langle\mathcal{A}_1^a\mathcal{A}_2^a\rangle + O\left(g^4\right) \\ \text{def. }\mathcal{A} &= 1 - \frac{C(R)}{d(R)}\int_{\tau_1>\tau_2} d\tau_1 d\tau_2\ \left[\left\langle A^a_{\mu_1}A^a_{\mu_2}\right\rangle \dot{x}_1^{\mu_1}\dot{x}_2^{\mu_2} - |\dot{x}_1|\,|\dot{x}_2|\,\theta_1^{I_1}\theta_2^{I_2}\left\langle\phi^a_{I_1}\phi^a_{I_2}\right\rangle\right] + O\left(g^4\right)\end{aligned}$$

free Wick contractions since aim at g^2

$$\begin{aligned}\left\langle A^a_\mu(x)\,A^b_\nu(y)\right\rangle &= \frac{g^2}{4\pi^2(x-y)^2}\delta_{\mu\nu}\delta^{ab} \\ \left\langle\phi^a_I(x)\,\phi^b_J(y)\right\rangle &= \frac{g^2}{4\pi^2(x-y)^2}\delta^{IJ}\delta^{ab}\end{aligned}$$

$$\begin{aligned}&= 1 - \frac{1}{2}\frac{C(R)}{d(R)}\int d\tau_1 d\tau_2\ \frac{g^2}{4\pi^2(x_1-x_2)^2}N^2\left(\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1|\,|\dot{x}_2|\,\theta_1\cdot\theta_2\right) + O\left(g^4\right) \\ &= 1 + \frac{g^2N^2}{8\pi^2}\frac{C(R)}{d(R)}\int d\tau_1 d\tau_2\underbrace{\frac{-\dot{x}_1\cdot\dot{x}_2 + |\dot{x}_1|\,|\dot{x}_2|\,\theta_1\cdot\theta_2}{(x_1-x_2)^2}}_{\text{1-loop propagator}} + O\left(g^4\right)\end{aligned}$$

from 2 loops onwards:

- find max number of $\mathcal{A}$ and compute correlators to certain loop precision
- T-ordering does not override P-ordering: former affects the coefficients A^a_μ, the latter refers to T^a.

- WL's vev is finite when γ is smooth and non-self-interesecting and θ is smooth
  under these assumptions, 1-loop propagator can diverge only in coinciding limit $\tau_2 = \tau_1 + \epsilon$
  loop everywhere spacelike $\rightarrow$ arc-length parametrization $\dot{x}^2 = 1$

  $$\text{expand num/den to } \epsilon^2 \qquad \frac{-\dot{x}_1 \cdot \dot{x}_2 + \theta_1 \cdot \theta_2}{(x_1 - x_2)^2} = O\left(\epsilon^0\right)$$

  gluon and scalar pole cancel! better UV property than ordinary WL.

  all-loop argument [Drukker, Gross, Ooguri 99, sec. 2.2]
  when not smooth, still interesting, see lecture part 4.2

EXERCISE LINE
1-loop propagator is constant on circle!

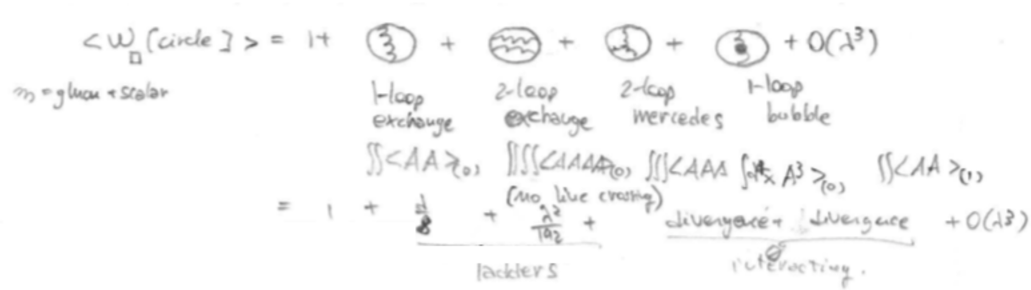
$$C(\Box) = \frac{1}{2} \qquad d(\Box) = N$$

$$\langle \mathcal{W}_\Box (\text{circle}) \rangle = 1 + \frac{\lambda}{8} + O\left(\lambda^2\right)$$

EXERCISE LATITUDE (conjecture: relate latitude's vev to circle's vev)

3 Circular WL

3.1 Exact vev from perturbation theory



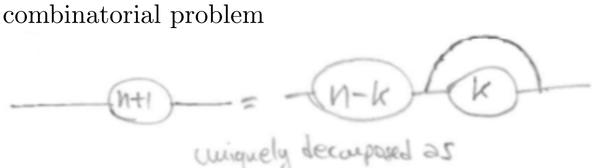
Conjecture at large N [Erickson Semenoff Zarembo 2000]
- only ladders, which have constant integrands, contribute to vev,
- interaction diagrams cancel at all loops.

$$\langle \mathcal{W}_\Box [\text{circle}] \rangle = \underbrace{\sum_{n=0}^{\infty}}_{\text{loop-counting variable}} \underbrace{\left(\frac{g^2}{4\pi^2} \times \frac{1}{2}\right)^n}_{\text{propagator}} \times \underbrace{\frac{(2\pi)^{2n}}{(2n)!}}_{\int_{2\pi > \tau_1 > \tau_2 > \ldots > \tau_{2n} > 0} d\tau_1 d\tau_2 \ldots d\tau_{2n}} \times \underbrace{\left(\frac{N}{2}\right)^n}_{\text{color factor - repeated app. of } T^a T^a = \frac{N}{2}\mathbb{I}}$$

$$\times \underbrace{1}_{\frac{1}{N}\text{tr}\mathbb{I}} \times \underbrace{A_n}_{\#\text{ planar graphs with n propagators}}$$

$$= \sum_{n=0}^{n} \frac{A_n}{(2n)!} \left(\frac{\lambda}{4}\right)^n$$

combinatorial problem



$$\begin{cases} A_{n+1} = \sum_{k=0}^{n} A_{n-k} A_k & \text{from planarity} \\ A_0 = 1 & \text{normalization WL} \end{cases}$$

generating function

$$f(z) = \sum_{k=0}^{\infty} A_k z^k$$

then

$$\begin{aligned} f^2(z) &= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} A_k z^k A_p z^p \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_k z^k A_{n-k} z^{n-k} \\ &= \sum_{n=0}^{\infty} A_{n+1} z^n \\ &= \frac{f(z) - 1}{z} \end{aligned}$$

$$f(0) = 1$$

so

$$f(z) = \frac{1 - \sqrt{1-4z}}{2z} = \sum_{n=0}^{\infty} \underbrace{\frac{(2n)!}{(n+1)!n!}}_{A_n} z^n$$

lo and behold

$$\begin{aligned} \forall \lambda,\ N \gg 1 \qquad \langle \mathcal{W}_{\Box}[\text{circle}] \rangle &= \sum_{n=0}^{n} \frac{1}{(n+1)!n!} \left(\frac{\lambda}{4}\right)^n = \frac{2}{\sqrt{\lambda}} I_1\left(\sqrt{\lambda}\right) \\ &= \begin{cases} \text{sqrt fictitious to some extent} \qquad 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \dots & \lambda \ll 1 \\ \sqrt{\frac{2}{\pi}} \lambda^{-3/4} e^{\sqrt{\lambda}} (1 + \dots) & \lambda \gg 1 \end{cases} \end{aligned}$$

modified Bessel function - exact in λ and large N.

check from holography

[Drukker Gross 00] extend to include non-planar corrections

$$\forall \lambda,\ N \qquad \langle \mathcal{W}_{\Box}[\text{circle}] \rangle = \frac{1}{N} L_{N-1}^{1}\left(-\frac{\lambda}{4N}\right) e^{\frac{\lambda}{8N}}$$

generalized Laguerre polynomial
(anomaly arising from the singular mapping of trivial line to circle + conjecture interaction diagrams cancel)

constant propagator is that of an effective 0d field theory
they argued it can be computed by Gaussian matrix model

analytic proof of this conjecture is inspiration to Pestun for developing localization principle
nowadays to explore non-perturbative aspects of SUSY gauge theories

3.2 Localize the circular WL

Localization technique that allows to reduce an ∞-dim integral to a lower (typically finite)-dim integral

physicists' version of equivariant localization

SUSY localization principle in QFT
[Cremonesi 13] [Pestun et al. 16]

QFT in Euclidean, compact manifold ($\to$ solve IR div), collection fields ϕ and $Z = \int \mathcal{D}\phi\, e^{-S[\phi]}$

goal is evaluate Z exactly, we need 2 things

fermionic operator δ:
- off-shell symmetry of action $\delta S[\phi] = 0$ (i.e. true without e.o.m.)
- not anomalous symmetry $\delta(\mathcal{D}\phi) = 0$ (cf. chiral anomalies in the style of Fujikawa)
- $\delta^2 =$ squares to linear combination of bosonic symmetries of S

fermionic functional/potential $V[\phi]$:
- $\delta V[\phi]$ is closed $\delta^2 V[\phi] = 0$,
- $\delta V[\phi]$ has positive-definite bosonic part $(\delta V[\phi])_{\text{bos}} \geq 0$

deform action by δ-exact term $\to$ 1-parameter family of theories

$$\begin{aligned} Z(t) &= \int \mathcal{D}\phi\, e^{-S[\phi]-t\,\delta V[\phi]} \\ \frac{d\,Z(t)}{dt} &= -\int \mathcal{D}\phi\, \delta V[\phi]\; e^{-S[\phi]-t\,\delta V[\phi]} \\ &= -\int \mathcal{D}\phi\, \delta\left(V[\phi]\; e^{-S(\phi)-t\,\delta V[\phi]}\right) \end{aligned}$$

analogue of Stokes' th. for path-int
of δ-exact terms in superQFT $= 0$

Z does not change under the deformation for any "coupling" of δ-exact term!
$Z(0) =$ undeformed Z
$Z(\infty) \to$ hope for drastic simplification to occur

$$Z = \underbrace{Z(+\infty)}_{\text{saddle-point approx on non-suppressed config. s.t. } t\,\delta V[\phi_0]=\text{finite}}$$

- find localization locus $= \phi_0$ s.t. $(\delta V[\phi_0])_{\text{bos}}=0$ = critical points of localizing potential
- decompose field $\phi = \phi_0 + \frac{\tilde{\phi}}{\sqrt{t}}$ $\quad \int \mathcal{D}\phi = \int \mathcal{D}\phi_0 \mathcal{D}\tilde{\phi}$
- $S[\phi] + t\,\delta V[\phi] = S[\phi_0] + 0 + \tilde{\phi}\cdot(\text{operator}[\phi_0])\cdot\tilde{\phi} + O\left(t^{-1/2}\right)$
- Gaussian integrate the quadratic $\tilde{\phi}$ (give rise to determinants of Laplace+Dirac operators)

localization formula

$$Z = \sum_{\phi_0\in\text{locus}} \int \mathcal{D}\phi_0 \quad \underbrace{\frac{1}{\sqrt{\text{SDet}\,(\text{operator}\,[\phi_0])}}}_{\text{1-loop det counts flucuations normal to locus, index theorems}} \quad \underbrace{e^{-S[\phi_0]}}_{\text{classical action, just spectator}}$$

favorable cases: locus = constant field config, localized integral if often finite-dim. integral of 0d QFT = matrix model

repeat for vev of BPS operator $\mathcal{O}[\phi]$ ($\delta(\mathcal{O}[\phi]) = 0$ under same charge),
$\langle \mathcal{O}[\phi] \rangle$ = insertion into localized path integral

Pestun's pioneering work [Pestun 07]
first nontrivial example of explicit prescription for superQFT on curved space, from the construction of theory to the evaluation of SUSY observables.

- $\mathcal{N} = 4$ SYM in $\mathbb{R}^4$

$$\langle \mathcal{W}_R[\text{circle}] \rangle = \frac{1}{d(R)} \mathrm{tr}_R \mathcal{P} \exp\left[\int_0^{2\pi} \left(iA_1 \sin\tau - iA_2 \cos\tau + \phi^1\right) d\tau\right]$$
$$\delta_\epsilon = \epsilon \cdot Q \qquad \delta_\epsilon S = \delta_\epsilon\left(\mathcal{W}_R[\text{circle}]\right) = 0$$

- Conformal map to S^4

has 2 consequences

- mod. action: kinetic term for ϕ must be deformed as $\partial_\mu \phi^I \partial_\mu \phi^I + \frac{R}{6} \phi^I \phi^I$ because of the curvature $R = \frac{12}{r^2}$
indeed this is now invariant under Weyl rescaling of metric and conformal invariance is preserved

$$S = \frac{1}{g^2} \int d^4x \sqrt{g}\, \mathrm{tr}\left(\frac{1}{2} F_{MN} F_{MN} - \frac{1}{2} \Psi \Gamma^M D_M \Psi + \underbrace{\frac{1}{r^2} \phi_I \phi_I}_{\text{mass = geom. coupling curv-scalar}}\right) \qquad M, N = 1, \ldots 10$$

- mod. on-shell SUSY tr. with r^{-1}- terms.
- close δ_ϵ off-shell with aux scalar fields K_i with free quadratic action $-K_i K_i$
- localization principle
$V = \int d^4x \sqrt{g}\, (Q\Psi)^\dagger\, \Psi =$ use SUSY tr.
$(QV)_{\text{bos}} = \int d^4x \sqrt{g}\, (Q\Psi)^\dagger\, Q\Psi \geq 0$
- find critical points of $(QV)_{\text{bos}}$
strategy is $(QV)_{\text{bos}} = \sum \underbrace{|\ldots|^2}_{\text{semi-def. terms}} \quad \rightarrow \quad$ locus = $\{\phi^1$ = constant zero-mode M, $A_\mu = 0$, $\phi^2 = \ldots \phi^6 = 0$, $K_i = \ldots\}$
- (gauge fixing, index theorems for determinants)
-

$$\langle \mathcal{W}_R[\text{circle}] \rangle_{\mathbb{R}^4} = \langle \mathcal{W}_R[\text{circle}] \rangle_{S^4} = \underbrace{\left\langle \frac{1}{d(R)} \mathrm{tr}_R e^{2\pi M} \right\rangle_{\text{Gaussian m.m.}}}_{\text{m.m. is simplest example of QGT}}$$
$$\equiv \frac{1}{Z} \int_{\text{basic field is N}\times\text{N hermitian matrix}} \underbrace{dM} \underbrace{\exp\left(-\frac{8\pi^2 N}{\lambda} \mathrm{tr}_R M^2\right)}_{\text{from vol}(S^4)} \underbrace{1}_{\text{1-loop det}\to\text{Gaussian m.m.}} \underbrace{\frac{1}{d(R)} \mathrm{tr}_R e^{2\pi M}}_{\text{classical insertion}}$$

3.3 Exact vev from matrix model

solve exactly $R = \Box$ and finite λ

2 methods [Mariño, ch.2] [lectures in ICTP]:
- orthogonal polynomials (finite N) [Drukker, Gross 00, app.A] (Hermite polynomials are orthogonal under Gaussian weight)
- saddle-point approximation ($N \gg 1$) (advantage: not to find orthogonal pol.)

start with partition function itself $$Z = \int dM \exp\left(-\frac{8\pi^2 N}{\lambda}\mathrm{tr}M^2\right)$$

---

1) Diagonalize M: use ~~gauge~~ symmetry

this is like a Faddeev-Popov method details on pages 12 - 13 of [Mariño]

N^2 dof $M \to UMU^\dagger = D$

N dof $= \mathrm{diag}(m_1, \dots m_N)$

$$Z = \int \left(\prod_i dm_i\right) \underbrace{\left(\prod_{i<j}(m_i - m_j)\right)^2}_{\text{FP determinant} = (\text{Vandermonde determinant})^2} e^{-\frac{8\pi^2 N}{\lambda}\sum_i m_i^2}$$

$$= \int \left(\prod_i dm_i\right) e^{-N^2 S_{eff}(m_i)}$$

$$S_{eff}(m_i) = \frac{8\pi^2}{\lambda N}\sum_i m_i^2 - \frac{2}{N^2}\sum_{i<j}\log|m_i - m_j|$$

m_i = coords of classical system of N particles on line

common harmonic potential (attractive)

repulsive pair-wise interaction

$\sum_i \sim N$, $\sum_{i<j} \sim N^2$ $\Rightarrow$ $S_{eff} \sim \mathcal{O}(1)$ in N

---

2) N^2 is like $1/\hbar$

when $N \to \infty$ the saddle point of S_{eff} dominates extremize action to find equilibrium configuration

$$\frac{\delta S_{eff}}{\delta m_i} = 0 = \boxed{\frac{16\pi^2}{\lambda N} m_i - \frac{2}{N^2}\Sigma_{j\neq i}\frac{1}{m_i - m_j} \quad (i=1,\dots,N)}$$

Eigenvalue distribution density: $\rho(m) = \frac{1}{N}\Sigma_{i=1}^{N}\delta(m-m_i)$

$N\to\infty$: continuous distribution

we expect

$$\frac{1}{N}\Sigma_{i=1}^{N} f(m_i) \to \int_I f(m)\rho(m)\, dm$$

I = interval

$\Sigma_{i=1}^{N} 1 = 1 \Rightarrow$ $\rho(m) = 0$ for $m\notin I$ — one-cut solution why?

$$\int_I \rho(m)\, dm = 1$$

Saddle point equation:

$$\boxed{\frac{8\pi^2}{\lambda} m = \fint \frac{\rho(m')\, dm'}{m-m'}}$$ integral equation

Principal value $(i\neq j)$

way to solve it

3) Introduce an auxiliary function: "resolvent" genus-0

$$\omega(m) = \int \frac{\rho(m')\, dm'}{m-m'}$$

Use Sokhotski–Plemelj theorem in complex analysis

version over real line

$$\fint \frac{f(x)}{x} dx = \int \frac{f(x)}{x\pm i\epsilon} \pm i\pi f(0) \quad (\epsilon\to 0^+)$$

$$= \frac{1}{2}\left(\int \frac{f(x)}{x+i\epsilon}dx + \int \frac{f(x)}{x-i\epsilon}dx\right)$$

Then $$\boxed{\frac{8\pi^2}{\lambda} m = -\frac{1}{2}\left(\omega(m+i\epsilon) + \omega(m-i\epsilon)\right)}$$

Moreover 3 properties

*) ω is analytic in $\mathbb{C}$ plane except along I

*) $m\to\infty$: $\omega(m)\sim\frac{1}{m}$ due to normalization of ρ

*) discontinuity equation across I (from residue's theorem)

$$\rho(m) = -\frac{1}{2\pi i}\left(\omega(m+i\epsilon) - \omega(m-i\epsilon)\right) \text{ for } m\in I$$

$$\omega(m+i\epsilon) - \omega(m-i\epsilon) = \int\frac{\rho(m')dm'}{m+i\epsilon-m'} - \int\frac{\rho(m')dm'}{m-i\epsilon-m'} = \int_{\mathbb{R}-i\epsilon}\frac{\rho(m')dm'}{m-m'} - \int_{\mathbb{R}+i\epsilon}\frac{\rho(m')dm'}{m-m'}$$

$= -2\pi i\, \mathrm{Res}(\cdot\cdot, m'=m) = -2\pi i\, \rho(m)$

To find ω:

$$\Sigma_i \frac{1}{m_i - m}\left(\frac{8\pi^2}{\lambda N} m_i = \frac{1}{N^2}\Sigma_{j\neq i}\frac{1}{m_i-m_j}\right)$$

for $N\to\infty$ this gives:

$$(\omega(m))^2 - \frac{16\pi^2}{\lambda} m\,\omega(m) + \frac{16\pi^2}{\lambda} = 0$$

which is a quadratic equation for $\omega(m)$:

principal branch of $\sqrt{\ }$

$$\omega(m) = \frac{8\pi^2}{\lambda}\left(m - \sqrt{m^2 - \frac{\lambda}{4\pi^2}}\right)$$

to guarantee correct behavior $\sim\frac{1}{m}$ as $m\to\infty$

Check:

$$\boxed{\frac{1}{N}\frac{8\pi^2}{\lambda}\Sigma_i \frac{m_i}{m_i - m} = \frac{1}{N^2}\Sigma_i\frac{1}{m_i-m}\Sigma_{j\neq i}\frac{1}{m_i-m_j}}$$

LHS: $$\frac{8\pi^2}{\lambda}\int\frac{\rho(m')\,m'\,dm'}{m'-m} = \frac{8\pi^2}{\lambda}\int\frac{\rho(m')(m'-m+m)\,dm'}{m'-m}$$

$$= \frac{8\pi^2}{\lambda}\left(1 - m\,\omega(m)\right)$$

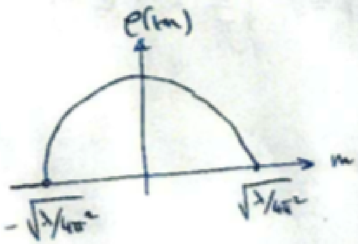
RHS use Sokhotski-Plemelj to remove the principal value in

$$\int \frac{\rho(m')\,dm'}{m'-m} ⨍ \frac{\rho(m'')\,dm''}{m'-m''} = -\frac{1}{2}\left(\omega(m)\right)^2$$

4) Discontinuity equation

$$\Rightarrow \rho(m) = -\frac{1}{2\pi i}\,\frac{8\pi^2}{\lambda}\left(-2\sqrt{m^2-\frac{\lambda}{4\pi^2}}\right)$$

$$= \frac{8\pi}{\lambda}\sqrt{\frac{\lambda}{4\pi^2}-m^2}$$ Wigner semi-circle distribution



Check: $\int_{-\sqrt{\lambda/4\pi^2}}^{+\sqrt{\lambda/4\pi^2}} \rho(m)\,dm = 1$

5) Back to the Wilson loop:

Key observation: $\mathrm{tr}\, e^{2\pi M} \to \sum_i e^{2\pi m_i}$ does not change the saddle point equation because it goes like $O(N^0)$

$\Rightarrow$ one can still use the Wigner distribution as background

$O(N^2)$ boxes $\to$ bubbling geometries change background

$$\langle W_\square(\text{circle})\rangle = \frac{1}{N}\langle \mathrm{tr}\, e^{2\pi M}\rangle = \int_{-\sqrt{\lambda/4\pi^2}}^{+\sqrt{\lambda/4\pi^2}} \rho(m)\, e^{2\pi m}\, dm = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

Mathematica

4 Related observables

Smooth, globally-BPS WL are simplest operators to study:
- line is protected from quantum corrections
- localization computes circle in closed-form.

WL become physically interesting when cusps or local operators are inserted into path develop divergences!
[Zarembo 16]

4.1 Cusp anomalous dimension

$\theta_1 \cdot \theta_2 = \cos\theta$

this divergence is a local effect, so studied by zooming onto the vicinity of the cusp

cutoff regularization
ϵ = UV cutoff = cut off the cusp
L = IR cutoff = scale at which it starts deviating from line

$$\begin{aligned}
\langle \mathcal{W}_\square\,[\text{cusp}]\rangle &= 1 + \frac{\lambda}{8\pi^2}\int_{-L}^{L}\int_{-L}^{L} \frac{-\dot{x}_1\cdot\dot{x}_2 + |\dot{x}_1|\,|\dot{x}_2|\,\theta_1\cdot\theta_2}{(x_1-x_2)^2} d\tau_1 d\tau_2 + \mathcal{O}\left(\lambda^2\right) \\
&= 1 + 0 + 0 + \frac{\lambda}{8\pi^2}\int_{\epsilon}^{L}\int_{-L}^{-\epsilon} \frac{\cos\theta - \cos\phi}{\tau_1^2+\tau_2^2-2\tau_1\tau_2\cos\phi} d\tau_1 d\tau_2 + \mathcal{O}\left(\lambda^2\right) \\
&= 1 + \frac{\lambda}{8\pi^2}\frac{\cos\theta-\cos\phi}{\sin\phi}\left(\phi\log\frac{L}{\epsilon} + \text{finite}\right) + \mathcal{O}\left(\lambda^2\right)
\end{aligned}$$

logarithmic divergent behavior [Polyakov 80] defines cusp an. dim.

$$\langle \mathcal{W}_\square\,[\text{cusp}]\rangle \sim \exp\left(-\underbrace{\Gamma_{\text{cusp}}(\lambda,\phi,\theta)}_{\text{cusp anomalous dimension}}\log\frac{L}{\epsilon}\right)$$

in general gauge theory, many cusps
cusped WL is multiplicatively renormalizable [Brandt, Neri, Sato 81]

$$\underbrace{\mathcal{W}_{\text{ren}}\,[\text{cusp}]}_{\text{ren. WL}} = \underbrace{Z_{\text{cusp}}\left(\{\phi_1\ldots\phi_n\}\right)}_{\substack{\text{counterterms}\\ \text{local and factorized}}}\underbrace{\mathcal{W}\,[\text{cusp}]}_{\text{bare WL}}$$

$$Z_{\text{cusp}}\left(\{\phi_1\ldots\phi_n\}\right) = Z_{\text{cusp}}(\phi_1)\ldots Z_{\text{cusp}}(\phi_n)$$

if path open, there is log-div associated to the endpoints, need second renormalization Z_{open} ($Z_{\text{open}} = 1$ in SYM)

reason name
usual renormalization passes through the intro of a mass scale μ
RG equation for single-cusp WL [Korchemsky, Radyushkin 87]

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\left(g_{\text{ren}}\right)\frac{\partial}{\partial g_{\text{ren}}} + \Gamma_{\text{cusp}}\left(g_{\text{ren}},\mu,\phi\right)\right]\mathcal{W}_{\text{ren}}\,[\text{cusp}] = 0$$

$$\beta\left(g_{\text{ren}}\right)=\mu\frac{\partial}{\partial\mu}g_{\text{ren}}\left(\mu\right)$$
$$\Gamma_{\text{cusp}}\left(g_{\text{ren}},\mu,\phi,\theta\right)=\mu\frac{\partial}{\partial\mu}Z_{\text{cusp}}\left(g_{\text{ren}},\mu,\phi\right)$$

similar renormalization of 2-pt functions of local operators
Callan-Symanzik equation = evolution of n-pt function under variation of the energy scale at which the theory is defined
Γ_{cusp} plays role of an. dim. of the (non-local) quantum operator

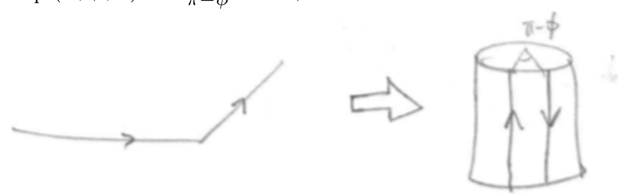
implies exponentiation

Relation $\Gamma_{\text{cusp}}\left(\lambda,\phi,\theta\right)$ to SYM physics
most apply to any CFT
[3 papers by Correa Maldacena Sever 12] [Zarembo 16]

- characterizes IR divergences of scattering amplitude of massive colored particles (W-bosons)
  (φ = boost angle between 2 external particles
  massive particles can be obtained by non-zero Higgs vevs
  θ = angle between Higgs vevs of 2 external particles
  get one for each consecutive pair of lines in the color-ordered diagram)

- $\theta=0$ $\qquad \phi=i\varphi\to i\infty$ (changing the Euclidean cusp into Minkowskian cusp)
  $\Gamma_{\text{cusp}}\left(\lambda,i\varphi,0\right)\sim\frac{\varphi}{2}f\left(\lambda\right)$
  "universal" or "angle-independent" cusp an. dim.
  [Beisert, Eden, Staudacher 07]

- $\Gamma_{\text{cusp}}\left(\lambda,\phi,\theta\right)=\frac{v(\lambda,\theta)}{\pi-\phi}$ $\qquad \phi\sim\pi$

  cusp in $\mathbb{R}^4$
  $\langle\mathcal{W}\rangle\sim e^{-\Gamma_{\text{cusp}}(\lambda,\phi,\theta)\log\frac{L}{\epsilon}}$
  map to antiparallel lines in cylinder $\mathbb{R}\times S^3$
  it measures potential here
  $\langle\mathcal{W}\rangle\sim e^{-V(\lambda,\phi,\theta)T}=e^{-\frac{v(\lambda,\theta)}{\pi-\phi}T}$
  for small distances, it measures potential in flat space $V\left(\lambda,\theta\right)=\frac{v(\lambda,\theta)}{L}$
  [Gromov, Levkovich-Maslyuk 16]

- cusped WL with $\theta=\pm\phi$ is the conformal projection of 1/4-BPS wedge on S^2
  so it BPS configuration
  divergence controlled by $\Gamma_{\text{cusp}}\left(\lambda,\phi,\pm\phi\right)=0$

  near-BPS limit
  $\Gamma_{\text{cusp}}\left(\lambda,\phi,\theta\right)\sim\left(\theta^2-\phi^2\right)B\left(\lambda\right)$ $\qquad \phi\sim\theta\sim0$ $\qquad$ only 1 overall coefficient

  TBA system of integral equations for $\Gamma_{\text{cusp}}\left(\lambda,\phi,\theta\right)$, solvable for B [Drukker 12][Correa Maldacena Sever 12]
  localization computes B too [Correa Henn Maldacena Sever 12]
  $B\left(\lambda,N\right)=\frac{\lambda}{2\pi^2}\frac{\partial}{\partial\lambda}\log\left\langle\mathcal{W}_{\Box}\left[\text{circle}\right]\right\rangle$

  check integrability coupling constant
  open problem for ABJM

baptized Bremsstrahlung function

reason that prompted Correa et al. to call it so
relation to the amount of power radiated by a moving quark

4.2 Bremsstrahlung radiation

German for "braking radiation"

$\Delta E = 2\pi B \int |\ddot{x}|^2 \, d\tau \qquad |\dot{x}| \ll 1 \qquad =$ Larmor formula
relativistic generalization = Liénard formula
[Jackson]

depends on theory's parameter
in QED $B = \frac{2\alpha}{3\pi}$

in SYM B measures:
(1) 2-pt function of displacement operator on Wilson line
(2) energy radiated by moving quark
(3) wavy Wilson line
(4) near-BPS limit of cusped Wilson line
(5) 1-pt function of stress-energy tensor in presence of Wilson line

most things valid any line defect operator in any CFT

- (1) excite CFT with extended probe $\to$ break symmetry $\to$ break current $\to$ conservation law is modified by d.o.f. to compensate for non-conservation

  WL is a defect that breaks 3 translations
  $\partial_\mu T_{\mu i} = \delta^{(3)}\left(x^i\right)\mathbb{D}_i\left(x^0\right)$ WL along x^0 $\quad i = 1, 2, 3 \quad \to \quad$ protected dim, 2pt coefficient is physical
  $\to \quad \langle \mathbb{D}_i(\tau)\, \mathbb{D}_j(0)\rangle_{\mathcal{W}} = \frac{12B}{\tau^4}\delta_{ij}$ with $B > 0$ for reflection positivity

  primary definition of B is as an important characterization of defect; will see how B appears in other cases

  plug into correlator of fields and Wilson lines (= vacuum)
  then $\mathbb{D}$ implements modifications of vev

  plug into correlator of nothing
  derive $\mathbb{D}_i = iF_{0i} + D_i\phi^3$ at weak coupling

- (2)

  absorption probability of energy quantum
  along line with small oscillatory displacement

  time-dependent perturbation theory in QM
  probability $= \left\| \epsilon \int e^{-i\omega\tau}\mathbb{D}(\tau)\,|0\rangle\, d\tau \right\|^2 \qquad \to \qquad \Delta E = 2\pi B \int |\ddot{x}|^2 \, d\tau$

- (3)

at 1 loop

$$\langle \mathcal{W}_\square [\text{wavy}] \rangle \overset{\text{1 loop}}{=} 1 + \underbrace{\frac{\lambda}{32} \int d\tau_1 d\tau_2 \frac{\left(\dot{\epsilon}_1 - \dot{\epsilon}_2\right)^2}{\left(\tau_1 - \tau_2\right)^2}}_{\text{1-loop quadratic order in waviness}} + O\left(\epsilon^4\right) \qquad \text{up to } \lambda,\, \epsilon^2$$

at all loops

$$\langle \mathcal{W}_\square [\text{wavy}] \rangle = 1 + \underbrace{\frac{B}{2}}_{\text{overall coefficient}} \qquad \underbrace{\int d\tau_1 d\tau_2 \frac{\left(\dot{\epsilon}_1 - \dot{\epsilon}_2\right)^2}{\left(\tau_1 - \tau_2\right)^2}}_{\substack{\text{ansatz for integral kernel fixed by} \\ \text{conf. inv. in 1d, which is preserved by line WL}}} \qquad + O\left(\epsilon^4\right)$$

[Semenoff Young 2004]
proof that coefficient is B uses-the fact that $\mathbb{D}$ implements modifications of vev
$\frac{\langle \mathcal{W}_\square [\text{wavy}] \rangle}{\langle \mathcal{W}_\square [\text{line}] \rangle} = 1 + \int_{\tau_1 > \tau_2} d\tau_1 d\tau_2 \, \langle \mathbb{D}_i \left(\tau_1\right) \mathbb{D}_j \left(\tau_2\right) \rangle_{\mathcal{W}} \, \epsilon_1^i \epsilon_2^i$

- (4) cusp with small deflection angle ϕ and $\theta = 0$

particular case of the wavy line $\dot{\epsilon}^\mu \left(\tau\right) = \theta\left(\tau\right) \phi n^\mu \qquad n \cdot n = 1 \qquad n \perp$ first segment

lift to $\theta \sim \phi$ using supersymmetry
e.g. $\Gamma_{\text{cusp}} \left(\phi, \theta, \lambda\right) \sim \left(\theta^2 - \phi^2\right) B\left(\lambda\right) \qquad \phi \sim \theta \sim 0$

- (5) $\frac{\langle T_{00}(r) \mathcal{W}[\text{line}] \rangle}{\langle \mathcal{W}[\text{line}] \rangle} = \frac{B}{3r^4}$ fixed by conf.inv. to a single coefficient
[Lewkowycz, Maldacena 13]

proof: see WL as a constantly accelerated probe links 2 concepts
emitted energy $\rightarrow$ Bremsstrahlung radiation
energy flux at large distances $\rightarrow$ stress tensor