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Notes on correlators of determinant and trace operators

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1 Introduction and main results

We take a basis of complex scalars $\sum_{J=1}^{6} y_I^J \phi_J$ with $I = 1, \ldots 6$, where each one is a linear combination of the ϕ_J in the SYM action (C.1). The polarization vectors y_I are six-dimensional orthogonal null vectors:

$$y_I^J \equiv \sum_{K=1}^{6} y_I^K \left(y_J^K\right)^* = y_I^I \, \delta_I^J \,. \tag{1.1}$$

For example, $y_Z^{\bar{Z}} = y_Z \cdot y_{\bar{Z}} = 1$ and $y_{\bar{Z}}^{Z} = 0$. We define $X = (\phi_1 + i\phi_2)/\sqrt{2}$, $Y = (\phi_3 + i\phi_4)/\sqrt{2}$, $Z = (\phi_5 + i\phi_6)/\sqrt{2}$.

Giant gravitons that carry the maximum angular momentum in S^5 are dual to local operators of determinant type. We denote two such BPS operators at spacetime positions x_1 and x_2 by

$$\begin{aligned} \mathcal{G}\left(x_1\right) &\equiv \det\left(Z\left(x_1\right)\right) = \frac{1}{N!}\,\varepsilon_{i_1\ldots i_N}\varepsilon_{j_1\ldots j_N} Z_{i_1 j_1}\left(x_1\right)\ldots Z_{i_N j_N}\left(x_1\right) \\ \bar{\mathcal{G}}\left(x_2\right) &\equiv \det\left(\bar{Z}\left(x_2\right)\right) = \frac{1}{N!}\,\varepsilon_{k_1\ldots k_N}\varepsilon_{l_1\ldots l_N} \bar{Z}_{k_1 l_1}\left(x_2\right)\ldots \bar{Z}_{k_N l_N}\left(x_2\right)\,. \end{aligned} \tag{1.2}$$

In this paper we study the structure constants of their three-point functions with a single-trace operator made of complex scalars in the $SO(6)$ sector

$$\mathcal{O}_3^{I_1\ldots I_L}\left(x_3\right) \equiv \operatorname{tr}\left(y^{I_1}\phi_{I_1}\left(x_3\right)\ldots y^{I_L}\phi_{I_L}\left(x_3\right)\right)\,. \tag{1.3}$$

The polarization vectors y's are six-dimensional null vectors parametrizing the orientation of the external BPS operators which are inserted at four-dimensional positions x_3. We assume $I_i = I_{i+kL}$ for any $k \in \mathbb{N}$ instead of recasting many formulas in this paper into a more awkward way.

We also consider correlation functions involving two giant gravitons and an operator in the $SL(2)$ sector. The operators of twist L and spin S are linear combinations

$$\mathcal{O}_{(L,S)}\left(x\right) \equiv \sum_{1\le n_1\le n_2\le\cdots\le n_S\le L} \psi(n_1,\ldots n_S)\,\mathcal{O}_{n_1,\ldots n_S}\left(x\right) \tag{1.4}$$

of the basic operators

$$\mathcal{O}_{n_1,\ldots n_S}\left(x\right) \equiv \left(\prod_{i=1}^{L}\frac{1}{m_i!}\right)\operatorname{tr}\Big(\underbrace{\overbrace{Z\ldots Z}^{n_1-1}\,D_+ Z\ldots Z}_{n_2-1}\,D_+ Z\ldots\Big)(x)\,. \tag{1.5}$$

made of L scalar fields of one type and S covariant derivatives D_+ in a lightlike direction. The conventional normalization includes the m_i which count the derivatives acting on the scalar at position n_i with $i = 1,\ldots L$. For generic L and S, there are as many primaries (1.4) as the wave-functions ψ. The latter can be found by requiring that the state (1.4) diagonalizes the quantum dilatation operator of the theory. A wave-function is in one-to-one correspondence with a set of Bethe roots $\{u_1,\ldots u_S\}$ that solve the Bethe ansatz equations [1–4]

$$e^{ip(u_i)L}\prod_{j\neq i}^{S}\mathcal{S}\left(u_i,u_j\right) = 1\,, \qquad \prod_{i=1}^{S} e^{ip(u_i)} = 1\,. \tag{1.6}$$

Here, the momentum p and the S-matrix $\mathcal{S}$

$$e^{ip(u_i)} = \frac{x_i^+}{x_i^-}\,, \qquad \mathcal{S}\left(u_i,u_j\right) = \frac{u_i - u_j + i}{u_i - u_j - i}\left[\frac{1-\frac{1}{x_i^- x_j^+}}{1-\frac{1}{x_i^+ x_j^-}}\right]^2 \sigma^2(u_i,u_j) \tag{1.7}$$

are conveniently expressed in terms of the Zhukovsky variables $x(u)$

$$x(u) = \frac{u+\sqrt{u^2-4g^2}}{2g}\,, \qquad x_i^\pm = x(u_i \pm i/2)\,, \qquad g^2 = \frac{\lambda}{16\pi^2}\,. \tag{1.8}$$

The dressing phase σ^2 can be ignored in this paper as it deviates from 1 at three loops. The dimension of (1.4) is $L+S+\gamma$, where Bethe ansatz determines the anomalous part as

$$\gamma = 2ig \sum_{i=i}^{S} \left(\frac{1}{x_i^+} - \frac{1}{x_i^-} \right) . \tag{1.9}$$

2 Tree level

2.1 Partially-contracted giant graviton

Computing correlators of determinant operators in perturbation theory is technically hard. To see this, we write (1.2) as a multitrace operator by turning antisymmetric symbols into Kronecker deltas (B.11) and applying (B.7):

$$\mathcal{G}\bar{\mathcal{G}} = \sum_{k_1,k_2,\ldots k_N=0}^{N} \frac{(-)^N (N!)^2}{k_1!\ldots k_N!} \left(-\mathrm{tr}\,(Z\bar{Z})\right)^{k_1} \ldots \left(\frac{-1}{N}\mathrm{tr}\,(Z\bar{Z})^N\right)^{k_N} . \tag{2.1}$$

Each term contains $2N$ scalars split among a variable number of traces, i.e. $\sum_{n=1}^{N} nk_n = N$. When N is large, there are terms with a large number of traces with few scalars each (e.g. for $k_1 = N$) as well as terms with few long traces ($k_N = 1$). The problem arises when we compute planar Wick contractions between (2.1) and other operators in a correlation function. While the dominance of planar graphs is true for trace operators of small dimension, this is not the case when their number or their lengths become of order N because combinatorial factors in non-planar graphs overwhelm the suppression in powers of $1/N$.

An ingenious way to evade the impasse is to perform Wick contractions among determinants first. This takes out a number of scalars of order N. One can apply (B.7) on what results and arrange the uncontracted scalars into a finite number of traces of small length. Such expression is amenable to the usual diagrammatical expansion dominated by planar graphs once it is inserted into a correlator with non-determinant operators.

The first step involves only the determinants and it can be worked out irrespective of the correlator where they are inserted into. In this paper we are interested in correlators of $\mathcal{G}$ and $\bar{\mathcal{G}}$ (1.2), so it is natural to define the partially-contracted giant graviton (PCGG) of length $2l$ as the operator that results from the Wick contractions of all possible $N-l$ pairs of Z and $\bar{Z}$:

$$\mathrm{G}_{2l}\,(x_1,x_2) \equiv \left\langle \mathcal{G}\,(x_1)\,\bar{\mathcal{G}}\,(x_2)\right\rangle\Big|_{\substack{\text{Wick contractions}\\ \text{of } (Z\bar{Z})^{N-l} \text{ only}}} \qquad l = 0,1,\ldots N\,. \tag{2.2}$$

The PCGG contains l scalars Z and l scalars $\bar{Z}$. It is neither local (Z and $\bar{Z}$ sit in different points, see (1.2)) nor a single trace when $l \geq 2$ (see below). The fully-contracted case $l = 0$ is nothing but the two-point function of giant gravitons in appendix G. Contractions are made in an interacting theory and the PCGG is expandable at small coupling by means of Feynman diagrams

$$\mathrm{G}_{2l}\,(x_1,x_2) = \mathrm{G}_{2l}^{(0)}\,(x_1,x_2) + \mathrm{G}_{2l}^{(1)}\,(x_1,x_2) + O(\lambda^{N-l+2}) \tag{2.3}$$

with the leading and subleading terms of order λ^{N-l} and λ^{N-l+1} respectively. We derive the former in what follows and the latter in section 3.1, both at finite l and N. We also suppress the spacetime dependence.

At leading order the contractions in (2.2) are free:

$$\begin{aligned}
\mathrm{G}_{2l}^{(0)} &= \frac{1}{(N!)^2}\binom{N}{l}^2 (N-l)!\left(\frac{1}{2}\right)^{N-l}\left(\frac{\lambda}{N}I_{12}\right)^{N-l}((N-l)!)^2 \\
&\times \delta^{i_1\ldots i_l}_{l_1\ldots l_l}\delta^{j_1\ldots j_l}_{k_1\ldots k_l} Z_{i_1j_1}\ldots Z_{i_lj_l}\bar{Z}_{k_1l_1}\ldots\bar{Z}_{k_ll_l} \\
&= \frac{(N-l)!}{(l\,!)^2}\left(\frac{\lambda}{2N}I_{12}\right)^{N-l}\delta^{i_1\ldots i_l}_{l_1\ldots l_l}\delta^{j_1\ldots j_l}_{k_1\ldots k_l} Z_{i_1j_1}\ldots Z_{i_lj_l}\bar{Z}_{k_1l_1}\ldots\bar{Z}_{k_ll_l}\,.
\end{aligned} \tag{2.4}$$

In the first line, $(N!)^2$ in the denominator comes from (1.2), the binomials count the choices of $N-l$ scalars in $\mathcal{G}$ and of $N-l$ scalars in $\bar{\mathcal{G}}$, the factorial counts the ways to pair them up for contractions, the power of $1/2$ is from the factor in (A.2), the next factor is a power of the scalar propagator (C.2) and the last factorials arise in passing from the partial contraction of ε-tensors to Kronecker deltas (B.10). The identity (B.7) provides an expansion in non-local multitrace operators

$$\begin{aligned}
\mathrm{G}_{2l}^{(0)} &= (-)^l\,(N-l)!\left(\frac{\lambda}{2N}I_{12}\right)^{N-l} \\
&\times \sum_{k_1,k_2,\ldots k_l=0}^{l}\frac{1}{k_1!\ldots k_l!}\left(-\mathrm{tr}\,(Z\bar{Z})\right)^{k_1}\ldots\left(-\frac{1}{l}\mathrm{tr}\,(Z\bar{Z})^l\right)^{k_l} \\
&= \mathrm{G}_{2l,\,\mathrm{s.t.}}^{(0)} + \text{multitrace terms}\,,
\end{aligned} \tag{2.5}$$

with total length $2\sum_{n=1}^{l} nk_n = 2l$. The single trace reads

$$\mathrm{G}_{2l,\,\mathrm{s.t.}}^{(0)} \equiv -(-)^l\frac{1}{l}\,(N-l)!\left(\frac{\lambda}{2N}I_{12}\right)^{N-l}\mathrm{tr}\left(Z\bar{Z}\right)^{L/2}\,. \tag{2.6}$$

2.2 Structure constants in $SO(6)$ sector

We calculate the structure constant of two giant gravitons and an operator in the $SO(6)$ sector at weak coupling

$$\left\langle \mathcal{G}\,(x_1)\,\bar{\mathcal{G}}\,(x_2)\,\mathcal{O}_3^{I_1\ldots I_{2l}}\,(x_3)\right\rangle\,. \tag{2.7}$$

The leading order is (E.3). If we work with operators with unit two-point function, we have to multiply by $(\mathcal{N}_{12})^{-1}\left(\mathcal{N}^{I_1,\ldots I_{2l}}_{I_1,\ldots I_{2l}}\right)^{-1/2}$ in (G.1) and (G.3) and read off the tree-level structure constant as in (H.1)

$$C^{(0)}_{\mathcal{G}\bar{\mathcal{G}}}{}^{\mathcal{O}_3^{I_1\cdots I_{2l}}} = \frac{C_{I_1\ldots I_{2l}}}{2l}C^{(0)}_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}_3^{I_1\cdots I_{2l}}} = \frac{(-)^{l+1}}{2l}\left(C_{I_1\ldots I_{2l}}\right)^{1/2}\left[\delta^Z_{I_1}\delta^{\bar{Z}}_{I_2}\ldots\delta^Z_{I_{2l-1}}\delta^{\bar{Z}}_{I_{2l}} + (Z\leftrightarrow\bar{Z})\right] \tag{2.8}$$

with the R-symmetry conventions (1.1).

2.3 Structure constants in $SL(2)$ sector

We determine the structure constant of two giant gravitons and a single-trace operator in the $SL(2)$ sector. The presence of covariant derivatives complicates the calculation considerably when compared to section 3.2. Indeed, once the typical elements $D_+^m Z$ in (1.4)-(1.5) are expanded in partial derivatives acting on products of scalars and gluons, Wick contractions cannot be handled explicitly as long as the m's are unspecified.

An alternative way of obtaining structure constants relies on the power of operator product expansion (OPE): one decomposes an appropriate four-point function in conformal partial waves and reads off the structure constants of the infinitely-many operators flowing in such decomposition. In this approach, we first compute in perturbation theory the (connected) correlator of giant gravitons (1.2) and BPS-protected single traces $\mathcal{O}_3^{I\ldots I}(x_3) = \operatorname{tr}\left(\phi^I(x_3)\right)^{L'}$ and $\mathcal{O}_4^{J\ldots J}(x_4) = \operatorname{tr}\left(\phi^J(x_4)\right)^{L'}$

$$\left\langle \mathcal{G}(x_1)\,\bar{\mathcal{G}}(x_2)\,\mathcal{O}_3^{I\ldots I}(x_3)\,\mathcal{O}_4^{J\ldots J}(x_4)\right\rangle . \tag{2.9}$$

While this can be done for generic scalars, a careful choice of the polarizations allows to control the bridge length between the single traces, which in turn sets the units L of R-charge "exchanged" between giant gravitons and single traces. In a certain OPE limit, the correlator measures the product of two different structure constants: one ($C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}_{(L,S)}}$) refers to the three-point function of BPS single traces and a twist-L operator in the $SL(2)$ sector, the other ($C_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}_{(L,S)}}$) to the three-point function of giant gravitons and the same $SL(2)$ operator. We see shortly that our polarization choice implies that derivative operators are akin to those in (1.4)-(1.5) but with the complex scalar $(1+i)\left(Z + i\bar{Z} + iX - \bar{X}\right)$ in the place of Z. Finally, we extract the structure-constant product in a wide range of parameters L', L and S and match it against the prediction from integrability.

The focus of this section is the tree-level order. The operators in (2.9) do not fix the bridge lengths and the typical Feynman diagram (F.2) depends on the length ℓ of the PCGG built out of $\mathcal{G}$ and $\bar{\mathcal{G}}$. The geometric sum for $\ell = 2, 4, \ldots 2L'$ yields for canonically-normalized operators

$$\begin{aligned}
\mathcal{N}\left\langle \mathcal{G}\bar{\mathcal{G}}\mathcal{O}_3\mathcal{O}_4\right\rangle_{(0)} &= \frac{L'}{N x_{12}^{2N} x_{34}^{2L'}}\left(\frac{y_{IJ}^2}{y_I^I y_J^J}\right)^{L'/2} \frac{z\bar{z}}{z^2\bar{z}^2(1-\alpha)(1-\bar{\alpha}) - (1-z)(1-\bar{z})\alpha^2\bar{\alpha}^2} \\
&\times \Bigg[-\alpha\bar{\alpha}(1-\alpha)(1-\bar{\alpha}) + 2z\bar{z}(1-\alpha)(1-\bar{\alpha}) - (1-z)(1-\bar{z})\alpha\bar{\alpha} + \left(\frac{z\bar{z}}{\alpha\bar{\alpha}}\right)^{L'} \\
&\times \left(\frac{(1-\alpha)(1-\bar{\alpha})}{(1-z)(1-\bar{z})}\right)^{(L'+1)/2}\left[z\bar{z}(1-\alpha)(1-\bar{\alpha}) - 2(1-z)(1-\bar{z})\alpha\bar{\alpha} + z\bar{z}(1-z)(1-\bar{z})\right]\Bigg] .
\end{aligned} \tag{2.10}$$

The overall $\mathcal{N}$ includes the normalization factors (G.1) and (G.3)

$$\mathcal{N}^{-1} = (\mathcal{N}_{12})(\mathcal{N}^{I_1,\ldots I_{L'}}_{I_1,\ldots I_{L'}}\,\mathcal{N}^{J_1,\ldots J_{L'}}_{J_1,\ldots J_{L'}})^{1/2} = \frac{L'\,N!}{(8\pi^2)^{N+L'}\,N^N}\lambda^{N+L'}\left(y_I^I y_J^J\right)^{L'/2} , \tag{2.11}$$

while the $SO(4)$ and $SO(6)$ cross-ratios are

$$\begin{aligned} z\bar{z} &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad & (1-z)(1-\bar{z}) &= \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,, \\ \alpha\bar{\alpha} &= \frac{y_{IJ}}{y_{ZI}\, y_{\bar{Z}J}}\,, \qquad & (1-\alpha)(1-\bar{\alpha}) &= \frac{y_{ZJ}\, y_{\bar{Z}I}}{y_{ZI}\, y_{\bar{Z}J}}\,. \end{aligned} \tag{2.12}$$

The correlator combines diagrams with a variable number of propagators stretching between giant gravitons on one side and single traces on the other. A priori, it is impossible to disentangle their contributions from the final result (2.10). However, there exists a choice of the scalars in $\mathcal{O}_3$ and $\mathcal{O}_4$ that forbids certain Wick contractions and has the beneficial effect of neatly separating these contributions in the sum (2.10). The basic idea consists in assuming that polarizations depend linearly on some parameters β_3 and β_4, for example[1]

$$\begin{aligned} \phi_I(\beta_3) &= (1+i)\left(Z + i\bar{Z} + iX - \bar{X}\right) + 2i\beta_3 Y\,, \\ \phi_J(\beta_4) &= (1+i)\left(Z + i\bar{Z} + iX - \bar{X}\right) + 2i\beta_4 \bar{Y}\,. \end{aligned} \tag{2.13}$$

Then, according to the identity ($L \equiv 2l = 2, 4, \dots 2L' - 2$)

$$\begin{aligned} &\mathcal{N}' \Big\langle \mathcal{G}\bar{\mathcal{G}} \mathrm{tr}\left[\left((1+i)\left(Z + i\bar{Z} + iX - \bar{X}\right)\right)^l (2iY)^{L'-l}\right] \\ &\qquad \times \mathrm{tr}\left[\left((1+i)\left(Z + i\bar{Z} + iX - \bar{X}\right)\right)^l \left(2i\bar{Y}\right)^{L'-l}\right]\Big\rangle \\ &= \frac{\mathcal{N}'}{L'^2\left[(L'-l)!\right]^2}\left(\frac{\partial}{\partial\beta_3}\frac{\partial}{\partial\beta_4}\right)^{L'-l} \left\langle \mathcal{G}\bar{\mathcal{G}}\mathcal{O}_3\mathcal{O}_4\right\rangle\Big|_{\beta_3=\beta_4=0}\,, \end{aligned} \tag{2.14}$$

multiple derivatives of (2.9) deliver a four-point function with $(L'-l)$ propagators exchanged[2] between Y and $\bar{Y}$ in the single traces, hence forcing $2l$ propagators[3] to stretch from the giant gravitons' Z and $\bar{Z}$ to the single traces' $Z + i\bar{Z} + iX - \bar{X}$. In these Wick contractions the scalars X and $\bar{X}$ in (2.13) play a spectator role, but they ensure that the polarization of $Z + i\bar{Z} + iX - \bar{X}$ is a null vector. Also notice the rhs's dependence on the derivation parameters is implicit in $\mathcal{O}_3 = \mathrm{tr}\left(\phi_I(\beta_3)\right)^{L'}$ and $\mathcal{O}_4 = \mathrm{tr}\left(\phi_J(\beta_4)\right)^{L'}$ and the correlator is properly normalized by (G.1) and (G.3)

$$\begin{aligned} \left(\mathcal{N}'\right)^{-1} =& \frac{N!}{(8\pi^2)^{N+L'} N^N}\lambda^{N+L'}\left(y_I^I(0,0)\, y_J^J(0,0)\right)^{l/2} \\ &\times \left[\left(\frac{\partial}{\partial\beta_3} y_I(0) \cdot \frac{\partial}{\partial\beta_3} y_I^*(0)\right)\left(\frac{\partial}{\partial\beta_4} y_J(0) \cdot \frac{\partial}{\partial\beta_4} y_J^*(0)\right)\right]^{(L'-l)/2}\,. \end{aligned} \tag{2.15}$$

[1] We pose $\phi_I(\beta_3) = y_I(\beta_3)\cdot\phi$, $\phi_I(\beta_4) = y_J(\beta_4)\cdot\phi$ and place constraints on the six-vectors y_I and y_J. First, we have the null-norm conditions $y_I(0)\cdot y_I(0) = y_I'(0)\cdot y_I'(0) = 0$. Second, the permitted Wick contractions are stated by $y_I(0)\cdot y_J(0) = y_I'(0)\cdot y_Z = 0$, $y_I'(0)\cdot y_J'(0)\,,\, y_I(0)\cdot y_Z \neq 0$, as well as those equations obtained by replacing $I \to J$ or $Z \to \bar{Z}$. Third, the two species of scalars contracting with the giant gravitons must be the same, i.e. $y_I(0) = y_J(0)$, in order for the lhs of (2.14) to generate derivative operators made of a unique scalar in a OPE limit. Lastly, we adjust the overall phases to avoid square-root ambiguities in the formulas below.

[2] The exclusion of $l = L'$ in (2.14) guarantees at least a small bridge, i.e. $L' - l \geq 1$. This becomes important when OPE decomposing the single traces below (2.15) and its reason is explained in footnote 2 [5].

[3] We forbid in (2.14) the configuration $l = 0$ with no such propagators. Furthermore, $L = 2l$ is even because the PCGG built in (2.10) to give (F.2) can only possess an even number of scalars.

We are ready to describe the new four-point function (2.14) in terms of the infinite OPE series that governs the flow between giant gravitons and single-trace operators. Repeating the arguments in section 2.3 of [5], the OPE of $\mathcal{O}_3$ and $\mathcal{O}_4$ in the limit $x_1 \to x_2$ contains different operators with $2l$ units of R-charge in the direction $(1+i)\left(Z + i\bar{Z} + iX - \bar{X}\right)$, but not necessarily belonging to the $SL(2)$ sector. The expectation value in the lhs of (2.14) becomes a double series in small z and $\bar{z}$

$$\frac{1}{x_{12}^{2N} x_{34}^{2L'}} \frac{1}{N} \left[\bar{z}^l f_{2L',2l}(z,\tau) + O\left(\bar{z}^{l+1}\right)\right], \qquad \tau = \frac{1}{2}\log(z\bar{z}), \tag{2.16}$$

where the leading $\bar{z}$-power is completely governed by the smallest-twist operators at fixed conformal dimension. As this property coincides with the definition of $SL(2)$ operators, all what we have to do is to measure $f_{2L',2l}$. Before proceeding, it is useful to recall that they are polynomials of degree n in τ at perturbative loop order n

$$f_{2L',2l}(z,\tau) = \sum_{n=0}^{\infty} \left(\frac{\lambda}{16\pi^2}\right)^n \sum_{m=0}^{n} \sum_{S=0}^{\infty} \mathcal{P}^{n,m}_{2L',2l,S} f^{(m)}_{2l,S}(z,\tau) \tag{2.17}$$

and here the functions of cross-ratios are in terms of the conformal block expression

$$f^{(m)}_{2l,S}(z,\tau) = z^{l+S} \frac{\partial^m}{\partial\delta^m} \left[e^{\tau\delta} {}_2F_1\left(\frac{2l+2S+\delta}{2}, \frac{2l+2S+\delta}{2}; 2l+2S+\delta; z\right)\right]\Bigg|_{\delta=0}, \tag{2.18}$$

hence the contribution of leading-twist operators is captured by the infinite set of numbers $\mathcal{P}$'s. If one were able to compute Feynman diagrams for (2.9) to perturbative order λ^n, then (2.13)-(2.18) would make possible to find $\mathcal{P}^{0,0}_{2l,S}$, $\mathcal{P}^{1,0}_{2l,S}$, ... $\mathcal{P}^{n,n}_{2l,S}$ with arbitrary $2l$ and S. In particular, $\mathcal{P}$'s labeled by large values of $2l$ and S become accessible as one measures higher powers of z and τ in the Taylor expansion of $f_{2L',2l}$ in (2.16). The coefficients that arise in the OPE are related to the product of structure constants explained at the beginning of this section through the generating function [5]

$$\frac{1}{N} \sum_{n=0}^{\infty} \left(\frac{\lambda}{16\pi^2}\right)^n \sum_{m=0}^{n} y^m \mathcal{P}^{n,m}_{2L',2l,S} = \sum_{\text{solutions}} C_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}_{(2l,S)}} C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}_{(2l,S)}} e^{\gamma y}. \tag{2.19}$$

For the subsector of twist $2l$ and spin S, there are several $SL(2)$ operators with equal classical dimension $2l+S$ and different anomalous dimensions γ (1.9). The sum in (2.19) means that we cannot determine the structure-constant product due to a single Bethe state that flows in the OPE, but only the sum of structure-constant product due to all Bethe states in that subsector. Each state is indexed by a set of Bethe roots (in formula, "solutions") that solve the Bethe equations (1.6).

We are ready to we compare our integrability prediction for the rhs of (2.19) with the data obtained by direct perturbative computation of the $\mathcal{P}$'s in the lhs. For the scope of this section, the tree-level result (2.10) gives access to $\mathcal{P}^{0,0}_{2L',2l,S}$, whose values for $L'=5$, $l=1,2,3,4$ and lowest S are in table 1. Many of these coefficients can be measured after the tree-level value is used in (2.14): extracting $\mathcal{P}^{0,0}_{2L',2l,S}$ for $L'=1,2,\ldots,$ $l=1,2,\ldots 2L'-2$ and $S=0,1,\ldots S_{\max}$ requires retaining z^l, ... $z^{l+S_{\max}}$ in the Taylor series, whereas τ never

l \ S	0	2	4	6	8	10	12	...
1	$-\frac{1}{2}$	$-\frac{1}{12}$	$-\frac{1}{140}$	$\frac{1}{1848}$	$-\frac{1}{25740}$	$-\frac{1}{369512}$	$-\frac{1}{5408312}$	...
2	$-\frac{1}{8}$	$-\frac{1}{80}$	$-\frac{1}{1008}$	$-\frac{1}{13728}$	$-\frac{1}{194480}$	$-\frac{1}{2821728}$	$-\frac{1}{41602400}$	...
3	$-\frac{1}{32}$	$-\frac{1}{112}$	$-\frac{1}{704}$	$\frac{1}{5720}$	$-\frac{5}{268736}$	$-\frac{3}{1664096}$	$-\frac{7}{42791040}$	...
4	$-\frac{1}{128}$	$-\frac{1}{576}$	$-\frac{9}{36608}$	$-\frac{1}{35360}$	$-\frac{5}{1736448}$	$-\frac{9}{33281920}$	$-\frac{7}{291985920}$	...

Table 1: Values of $\mathcal{P}^{0,0}_{2L',2l,S}$ for $L'=5$. They vanish for odd S.

appears. Finally, the leading term of (2.19) in small λ and y can be used to quantify the structure-constant product at tree level

$$\frac{1}{N}\mathcal{P}^{0,0}_{2L',2l,S} = \sum_{\text{solutions}} \left[C_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}(2l,S)} C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(2l,S)} \right]_{(0)} . \tag{2.20}$$

[**Replace $\ell \to L'$ in the conjecture formula to align it to our notation.**] We find perfect agreement for all cases with $L' = 1,3,5,7$, $l = 1,2,\dots 2L'-2$ and $S = 0,1,\dots 12$. The non-trivial check makes us confident to claim that the match would persist in the other infinitely-many cases. [**To make the check less dependent on the particular choice (2.13), we may state that the R-symmetry dependence that involves L' and $2l$, irrespective of the choice, is**

$$\left(\frac{y_{ZI}(0)\, y_{\bar{Z}I}(0)\, y_{ZJ}(0)\, y_{\bar{Z}J}(0)}{y_I^I(0,0)\, y_J^J(0,0)} \right)^{l/2} \left[\frac{\left(\frac{\partial}{\partial \beta_3}\frac{\partial}{\partial \beta_4} y_{IJ}(0,0) \right)^2}{\left(\frac{\partial}{\partial \beta_3} y_I(0) \cdot \frac{\partial}{\partial \beta_3} y_I^*(0) \right) \left(\frac{\partial}{\partial \beta_4} y_J(0) \cdot \frac{\partial}{\partial \beta_4} y_J^*(0) \right)} \right]^{(L'-l)/2} .$$

] The vanishing of $\mathcal{P}^{0,0}_{2L',2l,S}$ for odd S, e.g. table 1, can be understood from a field-theoretical point of view if we recall that powers of z in $f_{2L',2l}$ are the conformal spin (classical dimension $2l+S$ plus spin S) of the exchanged operators. As $2l$ is even for the present setup, the empirical observation that the Taylor series of $f_{2L',2l}$ has powers z^{4n} implies that the spin must be even too. The vanishing for odd S is also distinctive feature of the integrability formula.

For concreteness, let us verify the conjecture for primaries with twist $2l = 6$ and spin $S = 2$. While we check this at leading order in λ now, intermediate expressions display subleading corrections that are useful for pushing the test to one loop in section 3.3. There are three solutions of the Bethe equations (1.6). For our conjecture, they all valid solutions as they are made of parity-symmetric roots and read at 10-digit precision

$$\begin{aligned} \text{solution 1} &= \{u_1 = 1.038260698 + 0.01131658446\,\lambda + O(\lambda^2)\,,\ u_2 = -u_1\}\,,\\ \text{solution 2} &= \{u_1 = 0.3987366944 + 0.01411155000\,\lambda + O(\lambda^2)\,,\ u_2 = -u_1\}\,,\\ \text{solution 3} &= \{u_1 = 0.1141217372 + 0.006280230573\,\lambda + O(\lambda^2)\,,\ u_2 = -u_1\}\,. \end{aligned} \tag{2.21}$$

We plug each solution (2.21) into our conjecture. Here we should also specify, for example, $L' = 5$, and evaluate the product of structure constants

$$\begin{aligned}
\left[C_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}_{(6,2)}} C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}_{(6,2)}}\right]_{\text{solution 1}} &= -0.8005536388 + 0.010541577780\,\lambda + O(\lambda^2) \\
\left[C_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}_{(6,2)}} C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}_{(6,2)}}\right]_{\text{solution 2}} &= 0.07056560106 - 0.001746338934\,\lambda + O(\lambda^2) \\
\left[C_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}_{(6,2)}} C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}_{(6,2)}}\right]_{\text{solution 3}} &= -0.4128691051 + 0.04186535298\,\lambda + O(\lambda^2)\,.
\end{aligned} \tag{2.22}$$

Combining the λ-independent terms in (3.13) and (2.22), integrability predicts the value of the sum

$$\sum_{\text{solutions}} \left[C_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}(6,2)} C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(6,2)}\right]_{(0)} = -0.008928571429 \frac{1}{N} = -\frac{1}{112N}\,, \tag{2.23}$$

We can cast the result into a rational number after the numerical solution of the Bethe equations adds sufficient digits to the roots (2.21). This is instrumental to achieve a precise matching with perturbation theory, whose data in table 1 comprises exact numbers. It is then straightforward to match the integrability prediction (2.23) with the number from perturbation theory

$$\frac{1}{N}\mathcal{P}^{0,0}_{10,6,2} = -\frac{1}{112N}\,. \tag{2.24}$$

3 One loop

3.1 Partially-contracted giant graviton

Beyond tree level the new ingredient is the one-loop part of the PCGG (2.3)

$$\begin{aligned}
\mathrm{G}^{(1)}_{2l} = &-\frac{2l\left[4N^2 - 4\,(2l+1)\,N + 2l\,(2l+2)\right]}{8N\,(N-l)} \lambda I^{-1}_{12}\,(Y_{112} + Y_{122})\,\mathrm{G}^{(0)}_{2l} \\
&+ \frac{(N-l)!}{(l!)^2}\left(\frac{\lambda}{2N}\right)^{N-l+1} (I_{12})^{N-l}\, I^{-1}_{12}\,(Y_{112} + Y_{122}) \\
&\times M^{i_1\ldots i_l;j_1\ldots j_l}_{l_1\ldots l_l;k_1\ldots k_l}\, Z_{i_1j_1}\ldots Z_{i_lj_l}\bar{Z}_{k_1l_1}\ldots\bar{Z}_{k_ll_l}\,.
\end{aligned} \tag{3.1}$$

The derivation is in appendix D. The first addend can be written as a multiplicative correction to the tree-level PCGG because it is proportional to the same Kronecker deltas in (2.4), whereas the other displays the new tensor M (D.1) in R-symmetry indices.

We build a multitrace expansion along the lines of section 2.1. The expansion of the tree-level PCGG (2.5) is what is needed in the first line in (3.1). The formula (B.7) comes in handy again for expanding the part with M. We start with (B.7) for $n = 2l+2$

$$\begin{aligned}
&\delta^{i_1,\ldots i_{l+1}}_{l_1,\ldots l_{l+1}}\delta^{j_1,\ldots j_{l+1}}_{k_1,\ldots k_{l+1}} Z_{i_1k_1}\ldots Z_{i_{l+1}k_{l+1}}\bar{Z}_{j_1l_1}\ldots\bar{Z}_{j_{l+1}l_{l+1}} \\
&= -\sum_{k_1,k_2,\ldots k_{l+1}=0}^{l+1} \frac{(-)^l\,((l+1)!)^2}{k_1!\ldots k_{l+1}!}\left(-\mathrm{tr}\,(Z\bar{Z})\right)^{k_1}\ldots\left(-\frac{\mathrm{tr}\,(Z\bar{Z})^{l+1}}{l+1}\right)^{k_{l+1}}
\end{aligned} \tag{3.2}$$

with the sum constrained by $\sum_{n=1}^{l+1} nk_n = l+1$. Next, one defines a replacement rule that turns a collection of Z and $\bar{Z}$ into the sum over all ways of substituting a pair $Z_{ij}\bar{Z}_{kl}$ in it with $\delta_{ij}\delta_{kl}$; for example

$$\begin{aligned} Z_{i_1k_1}Z_{i_2k_2}\bar{Z}_{j_1l_1}\bar{Z}_{j_2l_2}\big|_{\text{rule}} &= \delta_{i_1k_1}\delta_{j_1l_1}Z_{i_2k_2}\bar{Z}_{j_2l_2} + \delta_{i_1k_1}\delta_{j_2l_2}Z_{i_2k_2}\bar{Z}_{j_1l_1} \\ &+ \delta_{i_2k_2}\delta_{j_1l_1}Z_{i_1k_1}\bar{Z}_{j_2l_2} + \delta_{i_2k_2}\delta_{j_2l_2}Z_{i_1k_1}\bar{Z}_{j_1l_1}\,. \end{aligned} \tag{3.3}$$

Applying this operation on (3.2) must deliver a new identity. On the rhs, terms linear in N are produced when the rule sends $\mathrm{tr}(Z\bar{Z})$ into $\mathrm{tr}(\mathbb{I}) = N$. On the lhs, the $(l+1)^2$ ways of picking one Z and one $\bar{Z}$ are equivalent to select the last Z and $\bar{Z}$ thanks to the antisymmetry property of the generalized Kronecker delta (see text below (B.1)), hence they all feature an index contraction like (D.1)

$$(l+1)^2\left(N\delta^{i_1\ldots i_l}_{l_1\ldots l_l}\delta^{j_1\ldots j_l}_{k_1\ldots k_l} + M^{i_1,\ldots i_l;j_1,\ldots j_l}_{l_1,\ldots l_l;k_1,\ldots k_l}\right)Z_{i_1k_1}\ldots Z_{i_lk_l}\bar{Z}_{j_1l_1}\ldots\bar{Z}_{j_ll_l}\,. \tag{3.4}$$

After identifying the terms proportional to N^0 and N^1, we can discard the latter in order to read off the multitrace expansion of the tensor M. In practice, this is realized by a modified rule that takes the rhs of (3.2) and outputs the sum over all ways of substituting one Z_{ij} and one $\bar{Z}_{kl}$ *that have* $k \neq j$ *and* $l \neq i$ with $\delta_{ij}\delta_{kl}$. This exception has the effect of ignoring length-2 traces $\mathrm{tr}(Z\bar{Z})$ that would produce N upon the action of the original rule stated above (3.3). In conclusion, the multitrace expansion of the last line in (3.1) is

$$\begin{aligned} &M^{i_1,\ldots i_l;j_1,\ldots j_l}_{l_1,\ldots l_l;k_1,\ldots k_l}Z_{i_1k_1}\ldots Z_{i_lk_l}\bar{Z}_{j_1l_1}\ldots\bar{Z}_{j_ll_l} = -\,(l+1)^{-2} \\ &\times \sum_{k_1,k_2,\ldots k_{l+1}=0}^{l+1}\frac{(-)^l\left((l+1)!\right)^2}{k_1!\ldots k_{l+1}!}\left(-\mathrm{tr}\left(Z\bar{Z}\right)\right)^{k_1}\ldots\left(-\frac{\mathrm{tr}\left(Z\bar{Z}\right)^{l+1}}{l+1}\right)^{k_{l+1}}\Bigg|_{\substack{\text{modified}\\ \text{rule}}} \end{aligned} \tag{3.5}$$

subject to the constraint $\sum_{n=1}^{l+1} nk_n = l+1$. Since the modified rule removes two scalars at most in a trace and each trace displays the alternating pattern $\mathrm{tr}(\ldots Z\bar{Z}\ldots)$, it is easy to realize that (3.5) contains the non-alternating sequences

$$\mathrm{tr}(\ldots ZZ\ldots), \qquad \mathrm{tr}(\ldots ZZ\ldots ZZ\ldots), \qquad \mathrm{tr}(\ldots ZZ\ldots\bar{Z}\bar{Z}\ldots) \tag{3.6}$$

and their complex conjugates. The multitrace expansion of the one-loop PCGG (3.1) inherits this feature as well. In contrast, only the alternating pattern appears at tree level (2.5).

The *operators* (2.4) and (3.1) can be taken to large N with a word of caution as multitrace expansion and Wick contractions do not generally commute. Indeed inserting theit limit of (2.4) and (3.1) in a correlator overloooks the fact that Wick contractions of the neglected multitraces may produce powers of N that overcome their suppressions in N^{-1}. The correct procedure is instead to take the limit of the expectation value that is obtained using the (finite-N) expressions (2.4) and (3.1).

3.2 Structure constants in $SO(6)$ sector

In addition to the correction to the wave-functions, one has to quantify the one-loop correction to the three-point function with a Bethe state. To this end, in appendix E we calculate the planar diagrams (E.4), (E.6)-(E.8) in a point-splitting scheme that regulates short-distance divergences. The sum takes a rather suggestive form

$$\left\langle \mathcal{G}\bar{\mathcal{G}}\, \mathcal{O}_3^{I_1\dots I_{2l}} \right\rangle_{(1)} = \frac{1}{2}(-)^l\,(N-l)!\left(\frac{\lambda}{2N}I_{12}\right)^{N-l}\left(\frac{1}{4}I_{13}I_{23}\right)^l \\ \times\left(\log\frac{x_{13}^2x_{23}^2}{\epsilon^2x_{12}^2}+2\right)\sum_{k=1}^{2l}H_{k,k+1}\left(y_{I_1}^Z y_{I_2}^{\bar{Z}}\dots y_{I_{2l-1}}^Z y_{I_{2l}}^{\bar{Z}}\right)+(Z\leftrightarrow\bar{Z}) \tag{3.7}$$

in terms of the one-loop Hamiltonian density (G.5) of the $SO(6)$ spin-chain acting on all pairs in parenthesis. This apparent simplicity is far from being obvious before arranging the Feynman diagrams together. We return to this point below.

The "bare" $\tilde{C}^{(1)}$ still depends on the normalization of the operators and the regularization scheme. To extract the physical structure constant $C^{(1)}$, one has to divide by the square root of the two-point functions of the operators. This is accomplished by a compact formula (H.2) that we quote from [6]; we defer the reader to appendix H for a summary. When it is applied to scalar trace operators, it turns out that $C^{(1)}$ is the sum of the one-loop Hamiltonian density in the $SO(6)$ sector at the so-called splitting points of the three-point functions [6], see also (H.13). Here we prove that an analogue characterization holds for (2.7) when a pair of determinants takes the place of single-traces.

The expected structures (H.1) are to be compared with what is obtained by doing loop computations. Conformal dimensions and mixing matrices are tabulated in (H.5). From the three-point function above, one extracts in the R-symmetry conventions (1.1)

$$\tilde{C}^{(1)}_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}_3^{I_1\dots I_{2l}}} = \frac{(-)^l}{(C_{I_1\dots I_{2l}})^{1/2}}\sum_{k=1}^{2l}H_{k,k+1}\left[\delta_{I_1}^Z\delta_{I_2}^{\bar{Z}}\dots\delta_{I_{2l-1}}^Z\delta_{I_{2l}}^{\bar{Z}}+(Z\leftrightarrow\bar{Z})\right]. \tag{3.8}$$

This data determines the renormalization-invariant structure constant (H.2)

$$C^{(1)}_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}_3^{I_1\dots I_{2l}}} = \tilde{C}^{(1)}_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}_3^{I_1\dots I_{2l}}} - \frac{1}{2}\sum_{J_1,\dots J_{2l}=1}^{6} g_{\mathcal{O}_3^{I_1\dots I_{2l}}\bar{\mathcal{O}}^3_{J_1\dots J_{2l}}}\, C^{(0)}_{\mathcal{G}\bar{\mathcal{G}}}{}^{\mathcal{O}_3^{J_1\dots J_{2l}}} \\ = \frac{(-)^l}{2\,(C_{I_1\dots I_{2l}})^{1/2}}\sum_{k=1}^{2l}H_{k,k+1}\left[\delta_{I_1}^Z\delta_{I_2}^{\bar{Z}}\dots\delta_{I_{2l-1}}^Z\delta_{I_{2l}}^{\bar{Z}}+(Z\leftrightarrow\bar{Z})\right]. \tag{3.9}$$

It is worth pausing to understand the origin of this nice result. First, there is just one metric subtraction because the giant gravitons' anomalous dimensions are zero. Second, the pairs of insertion points $k,k+1$ in (3.9) span the full length of $\mathcal{O}_3$ as if all neighboring scalars here were splitting points, in a jargon that is explained below (H.13). This interpretation strengthen the analogy with (H.13) as figure 1 exemplifies. Third, the diagrammatical calculation of $\tilde{C}^{(1)}$ (3.8) produces the sum over Hamiltonians (3.7) that appears also in $g_{\mathcal{O}_3\bar{\mathcal{O}}_3}$ (H.5), so the metric subtraction in (3.9) does not spoil such structure. The reason

Figure 1: Hamiltonian insertions at the splitting points of $\mathcal{O}_3$. Left panel: one of the two possible insertions (H.13) that give the structure constant of the BPS states (H.8) with $L_1 = L_2 = 4$ and the non-BPS state (1.3) with $2l = 6$. Right panel: one of the $2l$ possible insertions (3.9) that give the structure constant of giant gravitons (1.2) and the same non-BPS operator. The alternating scalars stand for the tree-level PCGG.

behind the simplicity of $\tilde{C}^{(1)}$ for (BPS) giant gravitons (2.7) can be understood in analogy to what happens for a three-point function [6] with two BPS single-traces (H.8). The similarity between the structure constants (3.9) and (H.13) may be surprizing because determinant operators show fairly unique properties in field-theory computations (see section 2.1). However, it is also true that they effectively appear in the expectation value (3.7) in the guise of a single trace $\operatorname{tr}\left(Z(x_1)\bar{Z}(x_2)\right)^l$. This fact is visible throughout appendix E thanks to the label "s.t."[4].

3.3 Structure constants in $SL(2)$ sector

This section is devoted to extend the analysis of section 2.3 to one loop. We keep the discussion schematic as the strategy is fully outlined there. We start by computing (2.9) in appendix F. For a value of $L' = 3, 5, 7, \ldots$, the sum of diagrams (F.4)-(F.11) for $\ell = 2, 4, \ldots 2L'$ evaluates to

$$\begin{aligned}
&\mathcal{N}\left\langle \mathcal{G}\bar{\mathcal{G}}\mathcal{O}_3\mathcal{O}_4\right\rangle_{(1)} \\
=&\frac{L'\lambda}{8\pi^2 N x_{12}^{2N} x_{34}^{2L'}}\left(\frac{y_{IJ}^2}{y_I^I y_J^J}\right)^{L'/2}\frac{z\bar{z}\Phi^{(1)}(z,\bar{z})}{z^2\bar{z}^2(1-\alpha)(1-\bar{\alpha})-(1-z)(1-\bar{z})\alpha^2\bar{\alpha}^2} \\
&\times\left[1-\left(\frac{z^2\bar{z}^2(1-\alpha)(1-\bar{\alpha})}{\alpha^2\bar{\alpha}^2(1-z)(1-\bar{z})}\right)^{(L'-1)/2}\right]\{\alpha\bar{\alpha}(1-\alpha)(1-\bar{\alpha}) \\
&+z\bar{z}(1-\alpha)(1-\bar{\alpha})[-1-\alpha\bar{\alpha}+(1-\alpha)(1-\bar{\alpha})] \\
&+(1-z)(1-\bar{z})\alpha\bar{\alpha}[-1+\alpha\bar{\alpha}-(1-\alpha)(1-\bar{\alpha})]+z^2\bar{z}^2(1-\alpha)(1-\bar{\alpha}) \\
&+z\bar{z}(1-z)(1-\bar{z})[1-\alpha\bar{\alpha}-(1-\alpha)(1-\bar{\alpha})]+(1-z)^2(1-\bar{z})^2\alpha\bar{\alpha}\Big\}
\end{aligned} \tag{3.10}$$

[4] We acknowledge that the argument is not completely explanatory. First, Z and $\bar{Z}$ being located in different points allow interaction vertices to contract three scalars inside the trace (diagram (e) in figure 3) and not only one or two (diagrams (c)-(d)) as it would happen if $\operatorname{tr}\left(Z\bar{Z}\right)^l$ were local. Second, $\operatorname{tr}\left(Z\bar{Z}\right)^l$ is weighted differently in the leading (2.6) and subleading part (3.1) of the PCGG, which in turn "unbalances" the relative coefficient between diagrams (b) and (c)-(e) when they get summed in (3.7).

$l \backslash S$	0	2	4	6	8	10	...
1	0	1	$\frac{205}{1764}$	$\frac{553}{54450}$	$\frac{14380057}{18036018000}$	$\frac{3313402433}{55983859495200}$	...
2	0	0	0	0	0	0	...
3	0	$\frac{1}{16}$	$\frac{521}{34848}$	$\frac{10909}{4867200}$	$\frac{4415079}{16358787200}$	$\frac{20005895513}{700957672732800}$	...
4	0	0	0	0	0	0	...

Table 2: Values of $\mathcal{P}^{1,0}_{2L',2l,S}$ for $L' = 5$. They vanish for odd S like the tree-level coefficients in table 1. The vanishing for $S = 0$ is expected from (3.11) because the one-loop structure constants $\left[C_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}(2l,S)}\right]_{(1)}$, $\left[C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(2l,S)}\right]_{(1)}$ are of BPS operators.

where $\mathcal{N}$ and the cross-ratios are (2.11)-(2.12) and $\Phi^{(1)}$ is (C.5). The cancellation of Y integrals (C.6), which are not functions of cross-ratios, occurs in pairs between diagrams with a certain ℓ and those labeled by $\ell + 2$. This guarantees that (3.10) behaves well under conformal transformations. It is worth appreciating that the case $\ell = 2L'$, which would remain unpaired due to the constraint placed on the values of $2L'$, does not contribute at all to the final result.

The arguments between (2.13)-(2.19) hold at quantum level and lead to measure $\mathcal{P}^{1,0}_{2L',2l,S}$ and $\mathcal{P}^{1,1}_{2L',2l,S}$ when we make use of the one-loop result above. We list some of them in tables 2-3. The truncation of the Taylor series of the one-loop part of $f_{2L',2l}$ to the powers z^l, ... $z^{l+S_{\max}}$ and τz^l, ... $\tau z^{l+S_{\max}}$ is the way to extract the coefficients with $L' = 1, 2, \ldots, l = 1, 2, \ldots L' - 1$ and $S = 0, 1, \ldots S_{\max}$.

$l \backslash S$	0	2	4	6	8	10	...
1	0	-1	$-\frac{5}{42}$	$-\frac{7}{660}$	$-\frac{761}{900900}$	$-\frac{671}{10581480}$	...
2	0	0	0	0	0	0	...
3	0	$-\frac{1}{16}$	$-\frac{1}{66}$	$-\frac{43}{18720}$	$-\frac{101}{361760}$	$-\frac{4457}{149768640}$	...
4	0	0	0	0	0	0	...

Table 3: Values of $\mathcal{P}^{1,1}_{2L',2l,S}$ for $L' = 5$. They vanish for odd S. The vanishing for $S = 0$ is expected from (3.12) because spinless primaries are BPS states with $\gamma_{(1)} = 0$.

We look at the one-loop part of λ of (2.19) and relate $\mathcal{P}^{1,0}_{2L',2l,S}$ to the one-loop structure-constant product

$$\frac{1}{16\pi^2 N}\mathcal{P}^{1,0}_{2L',2l,S} = \sum_{\text{solutions}} \Big[C_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}(2l,S)} C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(2l,S)}\Big]_{(1)} \tag{3.11}$$

and $\mathcal{P}^{1,1}_{2L',2l,S}$ to a mixture of the tree-level structure-constant product and the one-loop part of the quantum anomalous dimension (1.9)

$$\frac{1}{16\pi^2 N}\mathcal{P}^{1,1}_{2L',2l,S} = \sum_{\text{solutions}} \Big[C_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}(2l,S)} C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(2l,S)}\Big]_{(0)} \gamma_{(1)}\,. \tag{3.12}$$

While the latter is nothing more than a consistency check, the former offers the direct way to test the integrability formula against the $\mathcal{P}$'s from the perturbative input (3.10). Both expressions perfectly match in a number of cases with $L' = 1, 3, 5, 7$, $l = 1, 2, \ldots L' - 1$ and $S = 0, 1, \ldots 12$, hence suggesting that our conjecture is valid even beyond this parameter range. As an example, we resume the exercise at the end of section 2.3 with the aim of extending the match for the $SL(2)$ operators of twist $2l = 6$ and spin $S = 2$ to one loop. We recall that the solutions of the Bethe equations are (1.6). They correspond to primaries with anomalous dimensions (1.9)

$$\begin{gathered}\gamma_{\text{solution 1}} = 0.01907422946\,\lambda + O(\lambda^2)\,, \qquad \gamma_{\text{solution 2}} = 0.06193363403\,\lambda + O(\lambda^2)\,, \\ \gamma_{\text{solution 3}} = 0.09630420788\,\lambda + O(\lambda^2)\,. \end{gathered} \tag{3.13}$$

Combining (2.22) and (3.13) and isolating the powers λ^0 and λ^1 as indicated by the subscript 0 and 1 respectively, integrability makes a prediction for the value of the sums

$$\begin{aligned}\sum_{\text{solutions}} \Big[C_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}(6,2)} C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(6,2)}\Big]_{(1)} &= 0.0003957858736\frac{1}{N} = \frac{1}{256\pi^2 N}\,, \\ \sum_{\text{solutions}} \Big[C_{\mathcal{G}\bar{\mathcal{G}}\mathcal{O}(6,2)} C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(6,2)}\Big]_{(0)} \gamma_{(1)} &= -0.0003957858736\frac{1}{N} = -\frac{1}{256\pi^2 N}\,. \end{aligned} \tag{3.14}$$

Numerical precision can be adjusted in order to express (3.14) in terms of rational numbers as done for (2.23), but with the difference that one has to isolate beforehand the π^{-2} of the relation (1.8) between the "integrability coupling" g^2 and the "perturbation-theory coupling" λ. It is easy to compare the integrability predictions to the data from perturbation theory in tables 2-3: the numbers in (3.14) agree respectively with

$$\frac{1}{16\pi^2 N}\mathcal{P}^{1,0}_{10,6,2} = \frac{1}{256\pi^2 N}\,, \qquad -\frac{1}{16\pi^2 N}\mathcal{P}^{1,1}_{10,6,2} = -\frac{1}{256\pi^2 N}\,. \tag{3.15}$$

A Matrix identities

The generators T^a in the fundamental representation of the algebra $\mathfrak{u}(N)$ are $N \times N$ Hermitian matrices. Those labelled by $a = 1, \ldots N^2 - 1$ span an $\mathfrak{su}(N)$ subalgebra while $T^{N^2} = \mathbb{I}/\sqrt{2N}$. Here we collect the properties useful in the text.

$$\operatorname{tr}(T^a T^b) = \frac{\delta^{ab}}{2} \qquad [T^a, T^b] = i f^{abc} T^c \qquad f^{acd} f^{bcd} = \begin{cases} N\delta^{ab} & \text{if } a, b \neq N^2 \\ 0 & \text{otherwise} \end{cases} \tag{A.1}$$

$$T^a_{ij}T^a_{kl} = \frac{1}{2}\delta_{il}\delta_{kj} \qquad\qquad f^{abc}T^bT^c = \begin{cases} \frac{iN}{2}T^a & \text{if } a \neq N^2 \\ 0 & \text{otherwise} \end{cases} \tag{A.2}$$

$$\begin{aligned}
f^{aa_1a_2}f^{aa_3a_4}T^{a_1}_{i_1j_1}T^{a_2}_{i_2j_2}T^{a_3}_{i_3j_3}T^{a_4}_{i_4j_4} &= \frac{1}{8}\left(-\delta_{i_1j_2}\delta_{i_2j_3}\delta_{i_3j_4}\delta_{i_4j_1} + \delta_{i_1j_2}\delta_{i_2j_4}\delta_{i_3j_1}\delta_{i_4j_3}\right. && \text{(A.3)}\\
&\quad \left. +\delta_{i_1j_3}\delta_{i_2j_1}\delta_{i_3j_4}\delta_{i_4j_2} - \delta_{i_1j_4}\delta_{i_2j_1}\delta_{i_3j_2}\delta_{i_4j_3}\right) \\
f^{aa_1a_2}f^{aa_3a_4}T^{a_1}_{i_1j_1}\left(T^{a_2}T^{a_3}T^{a_4}\right)_{i_2j_2} &= -\frac{N^2}{8}\left(\delta_{i_1j_2}\delta_{i_2j_1} - \frac{1}{N}\delta_{i_1j_1}\delta_{i_2j_2}\right) && \text{(A.4)}\\
f^{aa_1a_2}f^{aa_3a_4}\left(T^{a_1}T^{a_2}\right)_{i_1j_1}\left(T^{a_3}T^{a_4}\right)_{i_2j_2} &= \frac{N^2}{8}\left(\delta_{i_1j_1}\delta_{i_2j_2} - \delta_{i_1j_2}\delta_{i_2j_1}\right) && \text{(A.5)}\\
f^{aa_1a_2}f^{aa_3a_4}\left(T^{a_1}T^{a_3}\right)_{i_1j_1}\left(T^{a_2}T^{a_4}\right)_{i_2j_2} &= \frac{N}{4}\left(\delta_{i_1j_1}\delta_{i_2j_2} - \frac{1}{N}\delta_{i_1j_2}\delta_{i_2j_1}\right) && \text{(A.6)}\\
f^{aa_1a_2}f^{aa_3a_4}\left(T^{a_1}T^{a_2}T^{a_3}\right)_{i_1j_1}T^{a_4}_{i_2j_2} &= -\frac{N^2}{8}\left(\delta_{i_1j_2}\delta_{i_2j_1} - \frac{1}{N}\delta_{i_1j_1}\delta_{i_2j_2}\right) && \text{(A.7)}\\
f^{aa_1a_2}f^{aa_3a_4}T^{a_1}T^{a_2}T^{a_3}T^{a_4} &= -f^{aa_1a_4}f^{aa_2a_3}T^{a_1}T^{a_2}T^{a_3}T^{a_4} = -N\frac{N^2-1}{8}\mathbb{1} && \text{(A.8)}\\
f^{aa_1a_3}f^{aa_2a_4}T^{a_1}T^{a_2}T^{a_3}T^{a_4} &= 0 && \text{(A.9)}\\
& && \text{(A.10)}
\end{aligned}$$

The identity (A.3) plays a role to prove (3.1), but it does not seem to be reported anywhere in literature. The derivation uses the defining relation of structure constants to create pairs of T's with the same color index

$$f^{aa_1a_2}f^{aa_3a_4}T^{a_1}_{i_1j_1}T^{a_2}_{i_2j_2}T^{a_3}_{i_3j_3}T^{a_4}_{i_4j_4} = \left[T^{a_2}, T^a\right]_{i_1j_1}T^{a_2}_{i_2j_2}\left[T^a, T^{a_4}\right]_{i_3j_3}T^{a_4}_{i_4j_4} \tag{A.11}$$

and then the repeated application of the first relation in (A.2).

B Combinatorial identities

We organize the identities [7] for the systematic treatment of local operators of determinant type in sections 2.1 and 3.1. Indices $i, j, \ldots$ are $\mathfrak{u}(N)$ gauge-group indices, thus they range from 1 to N. We consider $0 \leq n \leq p \leq N$ in what follows.

The fundamental object is the generalized Kronecker delta

$$\delta^{i_1,\cdots i_n}_{j_1,\cdots j_n} \equiv \Sigma_\sigma(-)^{|\sigma|}\delta^{i_1}_{j_{\sigma(1)}}\ldots\delta^{i_n}_{j_{\sigma(n)}}\,, \qquad\qquad \delta \equiv 1 \quad \text{for} \quad n = 0\,. \tag{B.1}$$

The sum spans over the permutations σ of the set $\{1, 2, \ldots n\}$ and $|\sigma|$ is the parity of the permutation. The definition implies completely antisymmetry in upper and lower indices. We also have for a single delta

$$\begin{aligned}
\delta^{i_1,\cdots i_n}_{j_2,\cdots j_n,j_1} &= (-)^{n+1}\delta^{i_1,\cdots i_n}_{j_1,\cdots j_n}\,, \qquad & \delta^{i_1,\cdots i_n,i_{n+1},\cdots i_p}_{j_1,\cdots j_n,i_{n+1},\cdots i_p} &= \frac{(N-n)!}{(N-p)!}\delta^{i_1,\cdots i_n}_{j_1,\cdots j_n}\,,\\
\delta^{i_1,\cdots i_n}_{j_1,\cdots j_n} &= (N-n)!\,\delta^{i_1,\cdots i_n}_{j_1,\cdots j_n}\,, & \delta^{i_1,\cdots i_n}_{i_1,\cdots i_n} &= n!\binom{N}{n}
\end{aligned} \tag{B.2}$$

and for contractions of deltas

$$\delta^{i_1,\dots i_n}_{j_1,\dots j_n}\delta^{j_1,\dots j_n}_{k_1,\dots k_n} = n!\,\delta^{i_1,\dots i_n}_{k_1,\dots k_n}\,, \qquad \delta^{i_1,\dots i_n}_{j_1,\dots j_n}\delta^{j_1,\dots j_n}_{i_1,\dots i_n} = (n!)^2\binom{N}{n}\,. \tag{B.3}$$

The generalized Kronecker delta is a multi-linear operator with n upper and n lower indices when it acts on a sequence of n square matrices $A^{(1)},\dots A^{(n)}$

$$\begin{aligned}&\delta^{i_1,\dots i_n}_{j_1,\dots j_n}A^{(1)}_{i_1j_1}\dots A^{(n)}_{i_nj_n}\\ =&(-)^{n+1}\mathrm{tr}\left[A^{(1)}\left(A^{(2)}\dots A^{(n)} + \text{permutations of set }\{2,\dots n\}\right)\right] + \text{multitrace terms}\,.\end{aligned} \tag{B.4}$$

The index contraction gives rise to traces when permutations are written in the cycle representation. Here the terms made of a single trace are reported. Note that the order of the matrices is irrelevant because the exchange of two is equivalent to the swap of two index pairs, hence two minus signs that compensate. The action on n copies of the same matrix is expressible in closed form

$$\begin{aligned}\delta^{i_1,\dots i_n}_{j_1,\dots j_n}A_{i_1j_1}\dots A_{i_nj_n} &= \sum_{k_1,k_2,\dots k_n=0}^{n}\frac{(-)^n n!}{k_1!\dots k_n!}\left(-\mathrm{tr}A\right)^{k_1}\dots\left(-\frac{1}{n}\mathrm{tr}A^n\right)^{k_n}\\ &= (-)^{n+1}\,(n-1)!\,\mathrm{tr}A^n + \text{multitrace terms}\end{aligned} \tag{B.5}$$

where the sum extends over a set of integers obeying $k_1 + 2k_2 + \dots + nk_n = n\,.$ The derivation of this formula combines the expression of the determinant in traces [8, 9] and (B.9)-(B.10) below.

We are also interested in the cross-contraction of two sequences of square matrices through a pair of deltas

$$\begin{aligned}&\delta^{i_1,\dots i_n}_{l_1,\dots l_n}\delta^{j_1,\dots j_n}_{k_1,\dots k_n}A^{(1)}_{i_1k_1}\dots A^{(n)}_{i_nk_n}B^{(1)}_{j_1l_1}\dots B^{(n)}_{j_nl_n}\\ =&\Sigma_\sigma\Sigma_{\sigma'}(-)^{|\sigma'|}\delta^{i_1}_{j_{\sigma'(1)}}\dots\delta^{i_n}_{j_{\sigma'(n)}}\left(A^{(1)}B^{(\sigma(1))}\right)_{i_1j_1}\dots\left(A^{(n)}B^{(\sigma(n))}\right)_{i_nj_n}\end{aligned} \tag{B.6}$$

and in the particular case of all equal A's and B's

$$\begin{aligned}&\delta^{i_1,\dots i_n}_{l_1,\dots l_n}\delta^{j_1,\dots j_n}_{k_1,\dots k_n}A_{i_1k_1}\dots A_{i_nk_n}B_{j_1l_1}\dots B_{j_nl_n} = n!\delta^{i_1,\dots i_n}_{j_1,\dots j_n}(AB)_{i_1j_1}\dots(AB)_{i_nj_n}\\ =&\sum_{k_1,k_2,\dots k_n=0}^{n}\frac{(-)^n\,(n!)^2}{k_1!\dots k_n!}\left(-\mathrm{tr}\,(AB)\right)^{k_1}\dots\left(-\frac{1}{n}\mathrm{tr}\,(AB)^n\right)^{k_n}\\ =&(-)^{n+1}n!\,(n-1)!\,\mathrm{tr}\,(AB)^n + (-)^n\,(n!)^2\sum_{k=1}^{\lfloor n/2\rfloor}\frac{S_{k,n}}{k\,(n-k)}\mathrm{tr}\,(AB)^k\,\mathrm{tr}\,(AB)^{n-k}\\ &+ \text{more-than-2-trace terms}\,.\end{aligned} \tag{B.7}$$

The product of two equal-length traces occurs when n is even. The symmetry factor

$$S_{k,n} = \begin{cases}1/2 & \text{if } k = n/2\\ 1 & \text{otherwise}\end{cases} \tag{B.8}$$

prevents overcounting this term in summing over k.

The tensor $\varepsilon_{i_1,\dots i_N}$ is totally antisymmetric with $\varepsilon_{1,\dots N}=1$ and expresses the determinant of a square matrix

$$\det A=\frac{1}{N!}\,\varepsilon_{i_1\dots i_N}\varepsilon_{j_1\dots j_N}A_{i_1j_1}\dots A_{i_Nj_N}\,. \tag{B.9}$$

A similar object with two sequences of indices is built out of partially-contracted tensors

$$\varepsilon\varepsilon^{i_1,\dots i_n}_{j_1,\dots j_n}\equiv\varepsilon_{k_1,\dots k_{N-n}i_1,\dots i_n}\varepsilon_{k_1\dots k_{N-n}j_1\dots j_n}=(N-n)!\,\delta^{i_1,\dots i_n}_{j_1,\dots j_n}\,. \tag{B.10}$$

It inherits antisymmetry in all indices and the properties of the generalized Kronecker delta. The contraction of two such tensors delivers

$$\varepsilon\varepsilon^{i_1,\dots i_n}_{j_1,\dots j_n}\varepsilon\varepsilon^{i_1,\dots i_n}_{j_1,\dots j_n}=((N-n)!)^2\,\delta^{i_1,\dots i_n}_{j_1,\dots j_n}\delta^{i_1,\dots i_n}_{j_1,\dots j_n}=N!\,(N-n)!\,n!\,. \tag{B.11}$$

C Feynman rules

The action of $\mathcal{N}=4$ SYM in $\mathbb{R}^4$ and with gauge group $U(N)$ [10][5] reads in Feynman gauge

$$\begin{aligned}S=\frac{1}{2g^2_{\text{YM}}}\int d^4x\Big\{&\frac{1}{2}\left(F^a_{\mu\nu}\right)^2+\left(\partial_\mu\phi^a_I+f^{abc}A^b_\mu\phi^c_I\right)^2+i\bar\psi^a\Gamma^\mu\left(\partial_\mu\psi^a+f^{abc}A^b_\mu\psi^c\right)\\&+if^{abc}\bar\psi^a\Gamma^i\phi^b_I\psi^c+\frac{1}{2}f^{abc}f^{ade}\phi^b_I\phi^c_J\phi^d_I\phi^e_J+\partial_\mu\bar c^a\left(\partial_\mu c^a+f^{abc}A^b_\mu c^c\right)+(\partial_\mu A^a_\mu)^2\Big\}\end{aligned} \tag{C.1}$$

where the field-strength is $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+f^{abc}A^b_\mu A^c_\nu$, the matrices $\Gamma^A=(\Gamma^\mu,\Gamma^i)$ with $\mu=1,\dots 4$ and $I=5,\dots 10$ are 16×16 Dirac matrices in Majorana-Weyl representation and obey $\text{tr}\left(\Gamma^A\Gamma^B\right)=16\,\delta^{AB}$.

The relevant propagators for us are those of gluons and scalars

$$\left\langle A^a_\mu(x)\,A^b_\nu(y)\right\rangle_{(0)}=\frac{\lambda}{N}\delta^{ab}\delta_{\mu\nu}I_{xy}\,,\qquad\left\langle\phi^a_I(x)\,\phi^b_J(y)\right\rangle_{(0)}=\frac{\lambda}{N}\delta^{ab}\delta_{IJ}I_{xy} \tag{C.2}$$

given $I_{12}=(2\pi x_{12})^{-2}$ and $x_{ij}\equiv|x_i-x_j|$. The subscripts (0) and (1) of expectation values refer to their leading and subleading part in λ respectively[6].

Let us spell the integrals [12] preparatory for the interaction vertices.

$$\begin{aligned}&Y_{123}\equiv\int d^4x\,I_{1x}I_{2x}I_{3x}\,,\quad X_{1234}\equiv\int d^4x\,I_{1x}I_{2x}I_{3x}I_{4x}\,,\quad G_{1,23}\equiv Y_{123}\left(\frac{1}{I_{13}}-\frac{1}{I_{12}}\right)\,,\\&F_{12,34}\equiv\frac{X_{1234}}{I_{13}I_{24}}-\frac{X_{1234}}{I_{14}I_{23}}+G_{1,34}-G_{2,34}+G_{3,12}-G_{4,12}\,.\end{aligned} \tag{C.3}$$

The first two are totally symmetric and the others obey

$$G_{1,23}=-G_{1,32}\,,\qquad F_{12,34}=-F_{21,34}=-F_{12,43}=F_{34,12}\,. \tag{C.4}$$

[5] The sign of the scalar potential is corrected in [11].

[6] We work with $\lambda=g^2_{\text{YM}}N$ even when the planar limit is not assumed.

The integrals are finite for distinct points. The one-loop conformal integral is [13]

$$X_{1234} = \frac{\pi^2 \Phi(z,\bar{z})}{(2\pi)^8 x_{13}^2 x_{24}^2}\,, \qquad \Phi(z,\bar{z}) \equiv \frac{2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log(z\bar{z})\log\frac{1-z}{1-\bar{z}}}{z-\bar{z}} \tag{C.5}$$

with the cross-ratios $z\bar{z} = x_{12}^2 x_{34}^2 x_{13}^{-2} x_{24}^{-2}$ and $(1-z)(1-\bar{z}) = x_{14}^2 x_{23}^2 x_{13}^{-2} x_{24}^{-2}$. Some properties are consultable in [14]. The three-point integral is a limit of the latter

$$Y_{123} = \lim_{x_4 \to \infty} (2\pi)^2 x_4^2 X_{1234} = \frac{\pi^2 \Phi(z',\bar{z}')}{(2\pi)^6 x_{13}^2} \tag{C.6}$$

with $z'\bar{z}' = x_{12}^2 x_{13}^2$ and $(1-z')(1-\bar{z}') = x_{23}^2 x_{13}^2$. The integrals (C.3) diverge when two or more points concide. Loop computations of correlators are usually regularized by separating the colliding points by a distance ϵ [15][7]:

$$\begin{aligned}
Y_{112} &= -\frac{I_{12}}{16\pi^2}\left(\log\frac{\epsilon^2}{x_{12}^2} - 2\right), & X_{1123} &= -\frac{I_{12}I_{13}}{16\pi^2}\left(\log\frac{\epsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2\right), \\
X_{1122} &= -\frac{I_{12}^2}{8\pi^2}\left(\log\frac{\epsilon^2}{x_{12}^2} - 1\right), & F_{12,12} &= -\frac{1}{8\pi^2}\left(\log\frac{\epsilon^2}{x_{12}^2} - 3\right), \\
F_{12,13} &= -\frac{1}{16\pi^2}\left(\log\frac{\epsilon^2}{x_{23}^2} - 2\right) + Y_{123}\left(\frac{1}{I_{12}} + \frac{1}{I_{13}} - \frac{2}{I_{23}}\right).
\end{aligned} \tag{C.7}$$

In the paper the recurrent combinations that vanish in point-splitting regularization are

$$X_{1122} + I_{12}^2 F_{12,12} - 2I_{12}(Y_{112} + Y_{122}) = 0 \tag{C.8}$$

$$\begin{aligned}
&I_{12}^{-1}(Y_{112} + Y_{122}) + I_{13}^{-1}(Y_{113} + Y_{133}) + I_{23}^{-1}(Y_{223} + Y_{233}) - I_{13}^{-1}I_{23}^{-1}(X_{1233} + I_{13}I_{23}F_{13,23}) \\
&+ I_{12}^{-1}I_{13}^{-1}(-X_{1123} - I_{12}I_{13}F_{12,13}) + I_{12}^{-1}I_{23}^{-1}(-X_{1223} - I_{12}I_{23}F_{21,23}) = 0\,.
\end{aligned} \tag{C.9}$$

To compute correlators at small coupling, one brings down interaction terms by expanding e^{-S} in the path-integral and Wick-contracts fields using free propagators and interaction terms in (C.1). Here we compute the building blocks that occur at one loop for scalar operators: the corrections to the two-point function and to the four-point connected function of scalars.

The self-energy of the scalar propagator (C.2) is computed in [12].

$$\left\langle \phi_{I_1}^{a_1}(x_1)\,\phi_{I_2}^{a_2}(x_2)\right\rangle_O = -\frac{\lambda^2}{N}\delta_{I_1 I_2}\left(\delta^{a_1 a_2} - \delta^{a_1 N^2}\delta^{a_2 N^2}\right)(Y_{112} + Y_{122}) \tag{C.10}$$

The last color summand is due to the $\mathfrak{u}(N)$ generator with non-zero trace and it is absent for gauge group $SU(N)$ [10]. The reference illustrates the diagrams that give (C.10).

The connected four-point function receives contribution from the vertex ϕ^4

$$\begin{aligned}
\left\langle \phi_{I_1}^{a_1}(x_1)\,\phi_{I_2}^{a_2}(x_2)\,\phi_{I_3}^{a_3}(x_3)\,\phi_{I_4}^{a_4}(x_4)\right\rangle_X = \frac{\lambda^3}{N^3}&\left[-\left(f^{aa_1a_3}f^{aa_2a_4} + f^{aa_1a_4}f^{aa_2a_3}\right)\delta_{I_1I_2}\delta_{I_3I_4}\right. \\
+\left(f^{aa_1a_4}f^{aa_2a_3} - f^{aa_1a_2}f^{aa_3a_4}\right)\delta_{I_1I_3}\delta_{I_2I_4} &+ \left.\left(f^{aa_1a_2}f^{aa_3a_4} + f^{aa_1a_3}f^{aa_2a_4}\right)\delta_{I_1I_4}\delta_{I_2I_3}\right] X_{1234}
\end{aligned} \tag{C.11}$$

[7] An error in the paper is corrected.

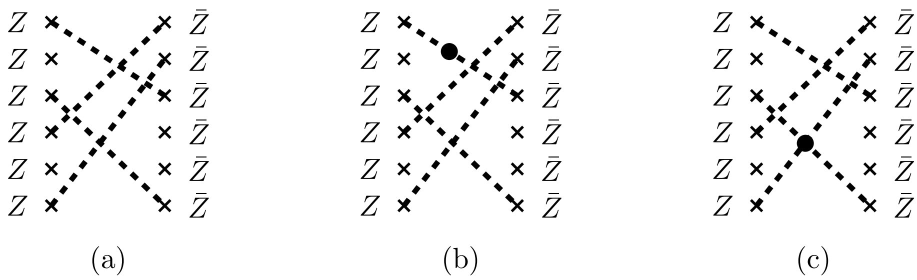

Figure 2: Typical diagrams that generate the one-loop PCGG (2.3). Crosses stand for scalar fields and dots for two-scalar (C.10) and four-scalar interactions (C.11)-(C.12). Here giant gravitons are made of $N = 6$ complex scalars and give rise to a PCGG with $2l = 4$ uncontracted fields. Diagram (a) is a Wick contraction at tree level (2.4), while diagrams (b)-(c) contain one-loop interactions (3.1). Topologies are planar despite the overlapping lines because giant gravitons are not trace operators.

and the gluon exchange made of two vertices $A\phi^2$

$$\left\langle \phi_{I_1}^{a_1}(x_1)\,\phi_{I_2}^{a_2}(x_2)\,\phi_{I_3}^{a_3}(x_3)\,\phi_{I_4}^{a_4}(x_4)\right\rangle_H = \frac{\lambda^3}{N^3}\left(f^{aa_1a_2}f^{aa_3a_4}\delta_{I_1I_2}\delta_{I_3I_4}I_{12}I_{34}F_{12,34}\right.$$
$$\left.+f^{aa_1a_3}f^{aa_2a_4}\delta_{I_1I_3}\delta_{I_2I_4}I_{13}I_{24}F_{13,24}+f^{aa_1a_4}f^{aa_2a_3}\delta_{I_1I_4}\delta_{I_2I_3}I_{14}I_{23}F_{14,23}\right)\,. \tag{C.12}$$

The expressions above serve also to define $\langle\ldots\rangle_X$ and $\langle\ldots\rangle_H$. We dub (C.10)-(C.12) the contributions of an effective O-, X- and H-vertex respectively. We also report color-traced correlators based on (A.8)-(A.9)

$$\langle \mathrm{tr}\left[\phi_{I_1}(x_1)\,\phi_{I_2}(x_2)\right]\rangle_O = -\frac{N^2-1}{2N}\lambda^2\left(Y_{112}+Y_{122}\right)\delta_{I_1I_2} \tag{C.13}$$

$$\langle \mathrm{tr}\left[\phi_{I_1}(x_1)\,\phi_{I_2}(x_2)\,\phi_{I_3}(x_3)\,\phi_{I_4}(x_4)\right]\rangle_{X+H} = \frac{N^2-1}{8N}\lambda^3\left[X_{1234}\left(-\delta_{I_1I_2}\delta_{I_3I_4}+2\delta_{I_1I_3}\delta_{I_2I_4}\right.\right.$$
$$\left.\left.-\delta_{I_1I_4}\delta_{I_2I_3}\right)-I_{12}I_{34}F_{12,34}\delta_{I_1I_2}\delta_{I_3I_4}+I_{14}I_{23}F_{14,23}\delta_{I_1I_4}\delta_{I_2I_3}\right] \tag{C.14}$$

with $\langle\ldots\rangle_{X+H} \equiv \langle\ldots\rangle_X + \langle\ldots\rangle_H$. The four-scalar trace is invariant under cyclic permutations as well as under the simultaneous swaps $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$.

D Derivation of (3.1)

The starting point is (B.3) with only two pairs of cross-contracted indices

$$\delta^{i_1\ldots i_l i}_{l_1\ldots l_l l}\,\delta^{j_1\ldots j_l i}_{k_1\ldots k_l l} \equiv N\delta^{i_1\ldots i_l}_{l_1\ldots l_l}\,\delta^{j_1\ldots j_l}_{k_1\ldots k_l}+M^{i_1\ldots i_l;j_1\ldots j_l}_{l_1\ldots l_l;k_1\ldots k_l}\,. \tag{D.1}$$

This defines the R-symmetry tensor M as the difference of the lhs and the deltas that appear in the tree-level PCGG (2.4). We set $M \equiv 0$ for $l = 0$ for consistency with (B.3).

There are two ways of dressing up the giant gravitons (2.2) with free propagators and interaction vertices in such a way that the Wick contractions leave l pairs of $Z\bar{Z}$ uncontracted and with one power of λ more than (2.4). The interaction vertices are basically worked out in (C.10)-(C.12).

One possibility is to lay down $N-l-1$ free propagators, which creates a PCGG of length $2l+2$, and insert one scalar self-energy (C.10), which takes 2 scalars out and leaves the remaining $2l$ uncontracted. This diagram is visualized in figure 2 (left panel) and results into

$$\begin{aligned}&\frac{(N-l-1)!}{(l+1)!^2}\left(\frac{\lambda}{2N}I_{12}\right)^{N-l-1}(l+1)^2\left[-\frac{\lambda^2}{2N}\left(Y_{112}+Y_{122}\right)\right] \\ &\times\left\{\left[(N-l)^2-1\right]\delta^{i_1\dots i_l}_{l_1\dots l_l}\delta^{j_1\dots j_l}_{k_1\dots k_l}-\frac{1}{N}M^{i_1\dots i_l;j_1\dots j_l}_{l_1\dots l_l;k_1\dots k_l}\right\}Z_{i_1j_1}\dots Z_{i_lj_l}\bar{Z}_{k_1l_1}\dots\bar{Z}_{k_ll_l}\,.\end{aligned}\tag{D.2}$$

The factorials and the power of the scalar propagator are those of a tree-level PCGG of length $2l+2$; one compares to (2.4) with $l\to l+1$. The factor $(l+1)^2$ is the number of pairs $Z\bar{Z}$ that can be picked from such tree-level structure. The square bracket is the self-energy of the chosen scalar pair; in particular $1/2$ is that of (A.2) and the rest comes from (C.10). The second line results from R-symmetry contraction when the self-energy is inserted in structure of the tree-level PCGG. Here is in detail:

$$\delta^{i_1\dots i_l i}_{l_1\dots l_l l}\delta^{j_1\dots j_l j}_{k_1\dots k_l k}\left(\delta_{il}\delta_{jk}-\frac{1}{N}\delta_{ij}\delta_{kl}\right)=\left[(N-l)^2-1\right]\delta^{i_1\dots i_l}_{l_1\dots l_l}\delta^{j_1\dots j_l}_{k_1\dots k_l}-\frac{1}{N}M^{i_1\dots i_l;j_1\dots j_l}_{l_1\dots l_l;k_1\dots k_l}\,.\tag{D.3}$$

The second identity in (B.2) and (D.1) take care of repeated indices.

The other contribution arises from a tree-level PCGG of length $2l+4$ made of $N-l-2$ free propagators. Then one chooses 4 scalars out of it and inserts the four-scalar vertices (C.11)-(C.12). These diagrams are depicted in figure 2 (right panel) and evaluate to

$$\begin{aligned}&\frac{(N-l-2)!}{(l+2)!^2}\left(\frac{\lambda}{2N}I_{12}\right)^{N-l-2}\frac{1}{4}\left(l+2\right)^2\left(l+1\right)^2\left[\frac{\lambda^3}{8N^3}\left(X_{1122}+I_{12}^2F_{12,12}\right)\right] \\ &\times\left\{4\left[(N-l)^2+N\right](N-l-1)^2\,\delta^{i_1\dots i_l}_{l_1\dots l_l}\delta^{j_1\dots j_l}_{k_1\dots k_l}+4\left(N-l-1\right)^2M^{i_1\dots i_l;j_1\dots j_l}_{l_1\dots l_l;k_1\dots k_l}\right\} \\ &\times Z_{i_1j_1}\dots Z_{i_lj_l}\bar{Z}_{k_1l_1}\dots\bar{Z}_{k_ll_l}\,.\end{aligned}\tag{D.4}$$

The two factorials and the propagator power are nothing but those of a tree-level PCGG of length $2l+4$, see again (2.4) with $l\to l+2$. The number $(l+2)^2\,(l+1)^2\,/4$ counts the way of choosing two Zs and two $\bar{Z}$s among the surviving Z^{2l+2} and $\bar{Z}^{2l+2}$. The second line stems from the R-symmetry contraction between the tree-level PCGG, the X- and H-vertices and uses (A.3)

$$\begin{aligned}&\delta^{i_1\dots i_l i i'}_{l_1\dots l_l l l'}\delta^{j_1\dots j_l j j'}_{k_1\dots k_l k k'}\left(-2\delta_{il}\delta_{i'l'}\delta_{jk'}\delta_{j'k}-2\delta_{il'}\delta_{i'l}\delta_{jk}\delta_{j'k'}+\delta_{ij'}\delta_{i'l}\delta_{jk'}\delta_{kl'}\right. \\ &\qquad\left.+\delta_{il}\delta_{i'j}\delta_{j'k'}\delta_{kl'}+\delta_{ij'}\delta_{i'l'}\delta_{jk}\delta_{k'l}+\delta_{il'}\delta_{i'j}\delta_{j'k}\delta_{k'l}\right) \\ &=4\left[(N-l)^2+N\right](N-l-1)^2\,\delta^{i_1\dots i_l}_{l_1\dots l_l}\delta^{j_1\dots j_l}_{k_1\dots k_l}+4\left(N-l-1\right)^2M^{i_1\dots i_l;j_1\dots j_l}_{l_1\dots l_l;k_1\dots k_l}\,.\end{aligned}\tag{D.5}$$

The sum of (D.2) and (D.4), where (C.8) eliminate X and F in favor of Y, yields (3.1).

E Structure constants in $SO(6)$ sector: details

We reserve the symbol t for the planar limit of (C.13)-(C.14)

$$t_{I,J}\equiv y^Iy^J\left\langle\mathrm{tr}\left(\phi_I\phi_J\right)\right\rangle_O\qquad t_{I,J,K,L}\equiv y^Iy^Jy^Ky^L\left\langle\mathrm{tr}\left(\phi_I\phi_J\phi_K\phi_L\right)\right\rangle_{X+H}\,.\tag{E.1}$$

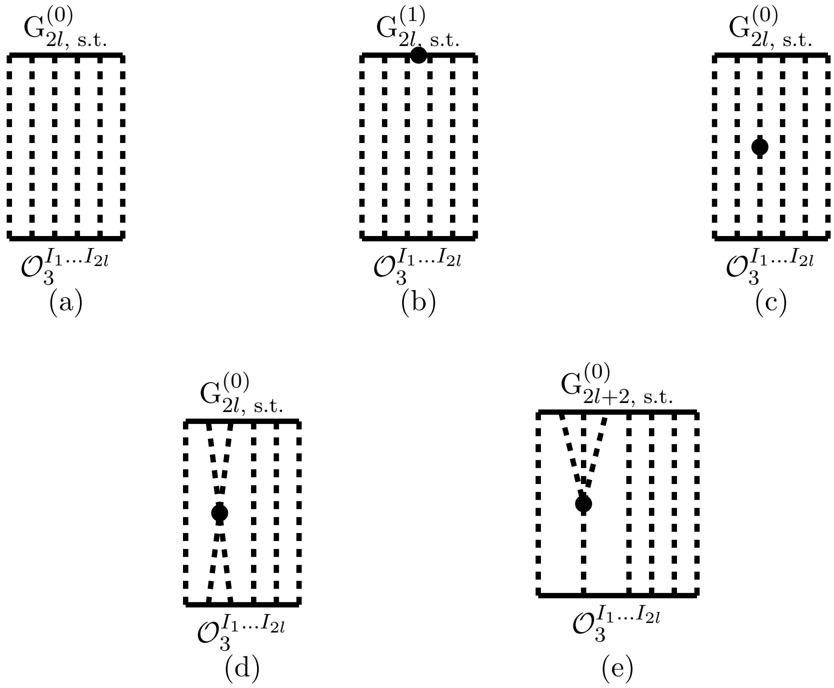

Figure 3: The planar tree-level diagram (a) and one-loop insertions (b)-(e) in the correlator (2.7) of two giant gravitons (1.2) and a single-trace operator $\mathcal{O}_3$ with $2l$ complex scalars (1.3). The figure depicts the situation for $l = 3$ after all contractions between the giant gravitons give rise to the partially-contracted giant graviton operator (2.2), which is here represented by its single-trace part (s.t.) at leading (0) and subleading order (1) in λ. Dotted lines are scalar propagators (C.2). Dots stand for the two-scalar (C.10) and four-scalar interactions (C.11)-(C.12), which occur either within the PGCC in (b) or between the PCGG and $\mathcal{O}_3$ in (c)-(e).

The spacetime position of scalars is inferred from the R-symmetry indices: $Z, \bar{Z}$ and the ϕ's sit in x_1, x_2 and x_3 respectively. For example

$$
\begin{aligned}
t_{ZI_k} &= -\frac{N\lambda^2}{2}\left(Y_{113} + Y_{133}\right) y_{ZI_k} \\
t_{Z\bar{Z}I_{k+1}I_k} &= \frac{N\lambda^3}{8}\Big[\left(-y_{I_kI_{k+1}} + 2y_{ZI_{k+1}}y_{\bar{Z}I_k} - y_{ZI_k}y_{\bar{Z}I_{k+1}}\right) X_{1233} \\
&\quad + y_{ZI_k}y_{\bar{Z}I_{k+1}} I_{13}I_{23}F_{13,23}\Big] \,.
\end{aligned}
\tag{E.2}
$$

The strategy for dealing with determinants in Feynman diagrams is outlined in section (2.2): the three-point functions is equivalent to the two-point function between a PCGG and the local operator. This relieves us from performing Wick contractions between determinants, which are already accounted for by (2.4) and (3.1) to one-loop precision, and let us focus on contractions with the local operator. The tree level of such effective two-point function is build out of the scalar propagators (C.2) between $\mathcal{O}_3$ and the tree-level PCGG (2.4). The latter equals a linear combination of multitrace operators, where the single-trace term (2.6) dominates.

$$\begin{aligned}(a) = \left\langle \mathrm{G}^{(0)}_{2l,\,\text{s.t.}}\, \mathcal{O}_3^{I_1\ldots I_{2l}} \right\rangle_{(0)} &= -(-)^l \frac{1}{l}\,(N-l)!\left(\frac{\lambda}{2N} I_{12}\right)^{N-l} \\ &\times \left(\frac{\lambda}{N} I_{13}\right)^l \left(\frac{\lambda}{N} I_{23}\right)^l l \left(y_Z^{I_1} y_{\bar{Z}}^{I_2} \ldots y_Z^{I_{2l-1}} y_{\bar{Z}}^{I_{2l}} + y_{\bar{Z}}^{I_1} y_Z^{I_2} \ldots y_{\bar{Z}}^{I_{2l-1}} y_Z^{I_{2l}} \right) \frac{N^{2l}}{2^{2l}}\end{aligned} \tag{E.3}$$

The factor in the first line is nothing but that carried by (2.6). Different planar contractions are related to each other by cyclic permutation of the scalars in $\mathcal{O}_3$. Hence, in the second line the factor of l comes from permutations of the I's, while $N^{2l}/2^{2l}$ from repeated application of (A.2). A typical diagram is in figure 3-(a).

Interaction vertices dress up propagators between determinants and those between a determinant and $\mathcal{O}_3$. The first effect is quantified by the one-loop part (3.1) of the PCGG (2.3), so we just need to contract this with $\mathcal{O}_3$ with no further vertex insertions

$$(b) = \left\langle \mathrm{G}^{(1)}_{2l}\, \mathcal{O}_3^{I_1\ldots I_{2l}} \right\rangle_{(0)} = -\lambda l I_{12}^{-1} \left(Y_{112} + Y_{122}\right) \left\langle \mathrm{G}^{(0)}_{2l,\,\text{s.t.}} \mathcal{O}_3^{I_1\ldots I_{2l}} \right\rangle_{(0)} . \tag{E.4}$$

In the last equality we make use again of the observation that only the single-traces in $\mathrm{G}^{(1)}_{2l}$, once expanded in multitraces as described below (3.1), dominate in the free contractions with $\mathcal{O}_3$. While both summands in $\mathrm{G}^{(1)}_{2l}$ give rise to single-traces with coefficients of order 1, they are weighted by different factors of N in (3.1). One takes note of the limit

$$\frac{N!}{N^{2l}\,(N-l)!} = \prod_{n=1}^{l-1}\left(1 - \frac{n}{N}\right) \to 1\,, \qquad l \text{ fixed, } N \to \infty \tag{E.5}$$

to conclude that the single-trace coming from $\mathrm{G}^{(0)}_{2l,\,\text{s.t.}}$ dominates by one power of N over those generated by the tensor M, hence justifying the final result of (E.4) in terms of (E.3). This situation is drawn in figure 3-(b).

What remains is to insert interaction vertices between $\mathcal{O}_3$ and the PCGG. Aiming at a one-loop result at leading order in N, it is sufficient to take the latter in its tree-level form (2.4) and approximate to its single-trace part (2.6). Again, the computation mimics that of a two-point function of single-traces (diagrams (c)-(e) in figure 3), save for the fact that the scalars in (2.6) sit in different points.

One diagram of this type dresses one scalar in $\mathrm{G}^{(0)}_{2l,\,\text{s.t.}}$ (either Z or $\bar{Z}$) and one scalar in $\mathcal{O}_3$ (one I_k among the I's with $k = 1, \ldots 2l$) with their self-energies (C.10), while all free contractions among the other $4l-2$ scalars are fixed by planarity. Below, the hat means that an R-symmetry product y is missing from the free contractions because it is embedded into the self-energy t (E.1). Unlike (E.3), cycling over $Z, \bar{Z}$ as well as I_k produces four different structures (below, one per line), each weighted by a multiplicity factor l.

$$\begin{aligned}(c) &= -(-)^l \frac{1}{l}\,(N-l)!\left(\frac{\lambda}{2N} I_{12}\right)^{N-l} \\ &\times \Bigg[\sum_{k \text{ odd}} \left(\frac{\lambda}{N} I_{13}\right)^{l-1} \left(\frac{\lambda}{N} I_{23}\right)^l l t_{ZI_k} \left(y_{ZI_1} y_{\bar{Z}I_2} \ldots \hat{y}_{ZI_k} \ldots y_{ZI_{2l-1}} y_{\bar{Z}I_{2l}} \right)\end{aligned}$$

$$+\sum_{k\text{ even}}\left(\frac{\lambda}{N}I_{13}\right)^{l}\left(\frac{\lambda}{N}I_{23}\right)^{l-1} l t_{\bar{Z}I_k}\left(y_{ZI_1}y_{\bar{Z}I_2}\dots\hat{y}_{\bar{Z}I_k}\dots y_{ZI_{2l-1}}y_{\bar{Z}I_{2l}}\right) \tag{E.6}$$
$$+\sum_{k\text{ odd}}\left(\frac{\lambda}{N}I_{13}\right)^{l-1}\left(\frac{\lambda}{N}I_{23}\right)^{l} l t_{ZI_k}\left(y_{\bar{Z}I_1}y_{ZI_2}\dots\hat{y}_{ZI_k}\dots y_{\bar{Z}I_{2l-1}}y_{ZI_{2l}}\right)$$
$$\left.+\sum_{k\text{ even}}\left(\frac{\lambda}{N}I_{13}\right)^{l}\left(\frac{\lambda}{N}I_{23}\right)^{l-1} l t_{\bar{Z}I_k}\left(y_{\bar{Z}I_1}y_{ZI_2}\dots\hat{y}_{\bar{Z}I_k}\dots y_{\bar{Z}I_{2l-1}}y_{ZI_{2l}}\right)\right]\frac{N^{2l-2}}{2^{2l-1}}$$

Similar considerations carry over to the diagrams with the four-scalar interaction (E.1). This vertex can involve two neighboring $\phi_{I_k}\phi_{I_{k+1}}$ and two neighboring $Z\bar{Z}$ in a length-$2l$ PCGG

$$(d) = -(-)^l\frac{1}{l}(N-l)!\left(\frac{\lambda}{2N}I_{12}\right)^{N-l} \tag{E.7}$$
$$\left[\sum_{k\text{ odd}}\left(\frac{\lambda}{N}I_{13}\right)^{l-1}\left(\frac{\lambda}{N}I_{23}\right)^{l-1} l t_{Z\bar{Z}I_{k+1}I_k}\left(y_{ZI_1}y_{\bar{Z}I_2}\dots\hat{y}_{ZI_k}\hat{y}_{\bar{Z}I_{k+1}}\dots y_{ZI_{2l-1}}y_{\bar{Z}I_{2l}}\right)\right.$$
$$+\sum_{k\text{ even}}\left(\frac{\lambda}{N}I_{13}\right)^{l-1}\left(\frac{\lambda}{N}I_{23}\right)^{l-1} l t_{\bar{Z}ZI_{k+1}I_k}\left(y_{ZI_1}y_{\bar{Z}I_2}\dots\hat{y}_{\bar{Z}I_k}\hat{y}_{ZI_{k+1}}\dots y_{ZI_{2l-1}}y_{\bar{Z}I_{2l}}\right)$$
$$+\sum_{k\text{ odd}}\left(\frac{\lambda}{N}I_{13}\right)^{l-1}\left(\frac{\lambda}{N}I_{23}\right)^{l-1} l t_{\bar{Z}ZI_{k+1}I_k}\left(y_{\bar{Z}I_1}y_{ZI_2}\dots\hat{y}_{\bar{Z}I_k}\hat{y}_{ZI_{k+1}}\dots y_{\bar{Z}I_{2l-1}}y_{ZI_{2l}}\right)$$
$$\left.+\sum_{k\text{ even}}\left(\frac{\lambda}{N}I_{13}\right)^{l-1}\left(\frac{\lambda}{N}I_{23}\right)^{l-1} l t_{Z\bar{Z}I_{k+1}I_k}\left(y_{\bar{Z}I_1}y_{ZI_2}\dots\hat{y}_{ZI_k}\hat{y}_{\bar{Z}I_{k+1}}\dots y_{\bar{Z}I_{2l-1}}y_{ZI_{2l}}\right)\right]\frac{N^{2l-3}}{2^{2l-2}}$$

as well as a single ϕ_{I_k} and a neighboring triplet of type $Z\bar{Z}Z$ or of type $\bar{Z}Z\bar{Z}$ in a length-$(2l+2)$ PCGG

$$(e) = (-)^l\frac{1}{l+1}(N-l-1)!\left(\frac{\lambda}{2N}I_{12}\right)^{N-l-1}$$
$$\left[\sum_{k\text{ odd}}\left(\frac{\lambda}{N}I_{13}\right)^{l-1}\left(\frac{\lambda}{N}I_{23}\right)^{l}(l+1)t_{Z\bar{Z}ZI_k}\left(y_{ZI_1}y_{\bar{Z}I_2}\dots\hat{y}_{ZI_k}\dots y_{ZI_{2l-1}}y_{\bar{Z}I_{2l}}\right)\right.$$
$$+\sum_{k\text{ even}}\left(\frac{\lambda}{N}I_{13}\right)^{l}\left(\frac{\lambda}{N}I_{23}\right)^{l-1}(l+1)t_{\bar{Z}Z\bar{Z}I_k}\left(y_{ZI_1}y_{\bar{Z}I_2}\dots\hat{y}_{\bar{Z}I_k}\dots y_{ZI_{2l-1}}y_{\bar{Z}I_{2l}}\right) \tag{E.8}$$
$$+\sum_{k\text{ odd}}\left(\frac{\lambda}{N}I_{13}\right)^{l-1}\left(\frac{\lambda}{N}I_{23}\right)^{l}(l+1)t_{Z\bar{Z}ZI_k}\left(y_{\bar{Z}I_1}y_{ZI_2}\dots\hat{y}_{\bar{Z}I_k}\dots y_{\bar{Z}I_{2l-1}}y_{ZI_{2l}}\right)$$
$$\left.+\sum_{k\text{ even}}\left(\frac{\lambda}{N}I_{13}\right)^{l}\left(\frac{\lambda}{N}I_{23}\right)^{l-1}(l+1)t_{\bar{Z}Z\bar{Z}I_k}\left(y_{\bar{Z}I_1}y_{ZI_2}\dots\hat{y}_{ZI_k}\dots y_{\bar{Z}I_{2l-1}}y_{ZI_{2l}}\right)\right]\frac{N^{2l-2}}{2^{2l-1}}.$$

It is easy to use (C.9) on the sum of (E.4)-(E.8) to bring the spacetime-dependence in t's (E.1) in terms of X (C.3). The regularized form of the latter (C.7) gives (3.7).

F Structure constants in $SL(2)$ sector: details

The calculation is very similar to section E once (2.9) is regarded as the correlator between a length-ℓ PCGG and two single traces, with the difference that each planar diagram in figure 4 admit more than one value[8] of $\ell = 2, 4, \ldots 2L'$ at fixed L'. We assume $L' \geq 1$ at tree level, but restrict to $L' = 2n + 3$ with $n \in \mathbb{N}$ at one loop to avoid discussing special cases. Within a diagram, the symbol $\delta^{\text{even}}_{\ell/2} = 1 - \delta^{\text{odd}}_{\ell/2}$, which equals 1 if $\ell/2$ is even and 0 otherwise, tells apart different Wick contractions that depend on the bridge lengths. We also abbreviate

$$\Delta_{IJ} \equiv \frac{\lambda}{N} I_{xy} y_{IJ} \,. \tag{F.1}$$

The points x and y are the positions where the scalars with index I and J are located respectively, e.g. this means $\Delta_{ZI_k} = \frac{\lambda}{N} I_{13} y_{Z,I_k}$.

The tree level consists of $\ell/2$ propagators between $\mathcal{O}_3$ and $\mathcal{O}_4$ and $(2L' - \ell)/2$ propagators between each of the latter and the PCGG

$$(a) = \left\langle \mathrm{G}^{(0)}_{\ell,\,\text{s.t.}} \mathcal{O}_3^{I\ldots I} \mathcal{O}_4^{J\ldots J} \right\rangle_{(0)} \tag{F.2}$$

$$= -\left(-\right)^{\ell/2} \frac{2}{\ell} \left(N - \frac{\ell}{2}\right)! \left(\frac{\Delta_{Z\bar{Z}}}{2}\right)^{N-\ell/2} \left[\delta^{\text{odd}}_{\ell/2} \frac{\ell(2L')^2}{16} \left(\Delta_{ZI}\Delta_{\bar{Z}J}\right)^{(\ell+2)/4} \left(\Delta_{\bar{Z}I}\Delta_{ZJ}\right)^{(\ell-2)/4} \right.$$
$$\left. + \delta^{\text{even}}_{\ell/2} \frac{\ell(2L')^2}{16} \left(\Delta_{ZI}\Delta_{\bar{Z}I}\Delta_{ZJ}\Delta_{\bar{Z}J}\right)^{\ell/4}\right] \Delta_{IJ}^{(2L'-\ell)/2} \frac{N^{(2L'+\ell)/2-1}}{2^{(2L'+\ell)/2}} + (I \leftrightarrow J) + \left(Z \leftrightarrow \bar{Z}\right) , \tag{F.3}$$

where factors placed after the δ's are multiplicities. At next-to-leading order, the one-loop PCGG (3.1) can freely contract with local operators[9]

$$(b) = -\frac{\lambda \ell}{2} I_{12}^{-1} \left(Y_{112} + Y_{122}\right) \left\langle \mathrm{G}^{(0)}_{\ell,\,\text{s.t.}} \mathcal{O}_3^{I\ldots I} \mathcal{O}_4^{J\ldots J} \right\rangle_{(0)} \,. \tag{F.4}$$

The other diagrams place interactions between PCGG and single traces. Self-energies contribute to

$$(c) = -\left(-\right)^{\ell/2} \frac{2}{\ell} \left(N - \frac{\ell}{2}\right)! \left(\frac{\Delta_{Z\bar{Z}}}{2}\right)^{N-\ell/2} \left\{\delta^{\text{odd}}_{\ell/2} \frac{\ell(2L')^2}{32} \left[(\ell+2)\, t_{ZI} \left(\Delta_{ZI}\Delta_{\bar{Z}I}\Delta_{ZJ}\right)^{(\ell-2)/4} \right.\right.$$
$$\left. \times \Delta_{\bar{Z}J}^{(\ell+2)/4} + (\ell-2)\, t_{ZI} \Delta_{ZI}^{(\ell-6)/4} \left(\Delta_{\bar{Z}I}\Delta_{ZJ}\right)^{(\ell+2)/4} \Delta_{\bar{Z}J}^{(\ell-2)/4}\right] + \delta^{\text{even}}_{\ell/2} \frac{\ell^2(2L')^2}{16} t_{ZI}$$
$$\left. \times \left(\Delta_{\bar{Z}I}\Delta_{ZJ}\Delta_{\bar{Z}J}\right)^{\ell/4} \Delta_{ZI}^{\ell/4-1}\right\} \Delta_{IJ}^{(2L'-\ell)/2} \frac{N^{(2L'+\ell)/2-3}}{2^{(2L'+\ell)/2-1}} + (I \leftrightarrow J) + \left(Z \leftrightarrow \bar{Z}\right) \tag{F.5}$$

and

$$(d) = -\left(-\right)^{\ell/2} \frac{2}{\ell} \left(N - \frac{\ell}{2}\right)! \left(\frac{\Delta_{Z\bar{Z}}}{2}\right)^{N-\ell/2} \left(1 - \delta_{\ell,2L'}\right) \left[\delta^{\text{odd}}_{\ell/2} \frac{\ell(2L')^2}{16} \left(\Delta_{ZI}\Delta_{\bar{Z}J}\right)^{(\ell+2)/4}\right. \tag{F.6}$$

[8] The case $\ell = 0$ is the disconnected correlator $\langle \mathcal{G}\bar{\mathcal{G}}\rangle_{(0)} \langle \mathcal{O}_3\mathcal{O}_4\rangle_{(1)}$.

[9] Had we allowed $L' = 2n + 2$ with $n \in \mathbb{N}$, for $\ell = 2L'$ there would have been also a double trace from M to contribute with $\left\langle \mathrm{tr}(Z(Z\bar{Z})^{L'/2-1})\mathcal{O}_3^{I\ldots I}\right\rangle_{(0)} \left\langle \mathrm{tr}(\bar{Z}(Z\bar{Z})^{L'/2-1})\mathcal{O}_4^{J\ldots J}\right\rangle_{(0)}$ and with $\mathcal{O}_3$ and $\mathcal{O}_4$ exchanged.

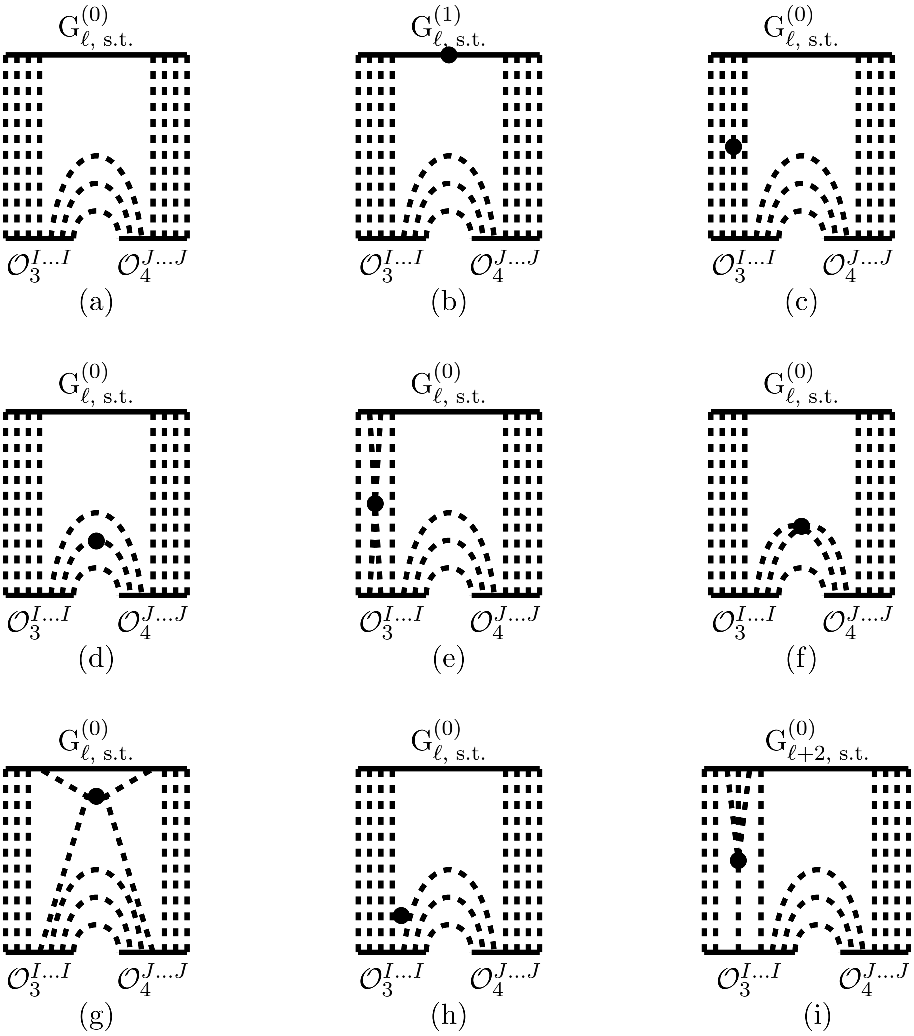

Figure 4: Tree level (a) and one loop (b)-(i) with two giant gravitons and two BPS single traces of length L' (2.9). The PCGG length ℓ is not fixed by the external operators. The figure depicts the case $\ell = 8$ and $2L' = 14$ and omits the "specular" partners of (c),(e),(h) and (i) upon swapping $\mathcal{O}_3$ and $\mathcal{O}_4$. Dots are the interaction vertices (C.10)-(C.12) which can be inserted either in the PGCC in (b) or between the PCGG and the local operators in (c)-(i).

$$
\begin{aligned}
&\times (\Delta_{\bar{Z}I}\Delta_{ZJ})^{(\ell-2)/4} + \delta^{\mathrm{even}}_{\ell/2}\frac{\ell(2L')^2}{16}\left(\Delta_{ZI}\Delta_{\bar{Z}I}\Delta_{ZJ}\Delta_{\bar{Z}J}\right)^{\ell/4}\Big]\,\frac{2L'-\ell}{2}t_{IJ}\Delta_{IJ}^{(2L'-\ell)/2-1} \\
&\times \frac{N^{(2L'+\ell)/2-2}}{2^{(2L'+\ell)/2-1}} + (I \leftrightarrow J) + \left(Z \leftrightarrow \bar{Z}\right) .
\end{aligned}
$$

We place $1 - \delta_{\ell,2L'}$ to enforce the vanishing of (F.6) when no propagator connects $\mathcal{O}_3$ and $\mathcal{O}_4$. The quartic vertices involve a variable number of scalars in the PCGG. The extra terms with $\delta_{\ell,2L'}$ are due to non-nearest-neighbor interactions [6] in the extremal configuration

$\ell = 2L'$.

$$(e) = -\,(-)^{\ell/2}\,\frac{2}{\ell}\left(N-\frac{\ell}{2}\right)!\left(\frac{\Delta_{Z\bar{Z}}}{2}\right)^{N-\ell/2}\frac{\ell(2L')^2}{16}\Big[(\ell-2)\,\delta^{\rm odd}_{\ell/2}t_{IIZ\bar{Z}}\left(\Delta_{ZI}\Delta_{ZJ}\right)^{(\ell-2)/4}$$
$$\times\,\Delta_{\bar{Z}I}^{(\ell-6)/4}\Delta_{\bar{Z}J}^{(\ell+2)/4}+(\ell-2)\,\delta^{\rm even}_{\ell/2}t_{IIZ\bar{Z}}\left(\Delta_{ZI}\Delta_{\bar{Z}I}\right)^{(\ell-4)/4}\left(\Delta_{ZJ}\Delta_{\bar{Z}J}\right)^{\ell/4} \tag{F.7}$$
$$+2\delta_{\ell,2L'}t_{ZZII}\left(\Delta_{\bar{Z}I}\Delta_{ZJ}\right)^{(\ell-2)/4}\Delta_{ZI}^{(\ell-6)/4}\Delta_{\bar{Z}J}^{(\ell+2)/4}\Big]\,\Delta_{IJ}^{(2L'-\ell)/2}\frac{N^{(2L'+\ell)/2-4}}{2^{(2L'+\ell)/2-2}}$$
$$+\,(I\leftrightarrow J)+\big(Z\leftrightarrow\bar{Z}\big)$$

$$(f) = -\,(-)^{\ell/2}\,\frac{2}{\ell}\left(N-\frac{\ell}{2}\right)!\left(\frac{\Delta_{Z\bar{Z}}}{2}\right)^{N-\ell/2}\big(1-\delta_{\ell,2L'}\big)\Big[\delta^{\rm odd}_{\ell/2}\frac{\ell(2L')^2}{16}\left(\Delta_{ZI}\Delta_{\bar{Z}J}\right)^{(\ell+2)/4}\left(\Delta_{\bar{Z}I}\Delta_{ZJ}\right)^{(\ell-2)/4}$$
$$+\delta^{\rm even}_{\ell/2}\frac{\ell(2L')^2}{16}\left(\Delta_{ZI}\Delta_{\bar{Z}I}\Delta_{ZJ}\Delta_{\bar{Z}J}\right)^{\ell/4}\Big]\,\frac{2L'-\ell-2}{2}t_{IIJJ}\Delta_{IJ}^{(2L'-\ell)/2-2}\frac{N^{(2L'+\ell)/2-4}}{2^{(2L'+\ell)/2-2}} \tag{F.8}$$
$$+\,(I\leftrightarrow J)+\big(Z\leftrightarrow\bar{Z}\big)$$

$$(g) = -\,(-)^{\ell/2}\,\frac{2}{\ell}\left(N-\frac{\ell}{2}\right)!\left(\frac{\Delta_{Z\bar{Z}}}{2}\right)^{N-\ell/2}\Big[\delta^{\rm odd}_{\ell/2}\frac{\ell(2L')^2}{32}t_{IJ\bar{Z}Z}\Big(2\left(\Delta_{ZI}\Delta_{\bar{Z}I}\Delta_{ZJ}\Delta_{\bar{Z}J}\right)^{(\ell-2)/4}$$
$$+\delta_{\ell,2L'}\left(t_{ZI\bar{Z}J}+t_{ZJ\bar{Z}I}\right)\Big)+\delta^{\rm even}_{\ell/2}\frac{\ell(2L')^2}{16}t_{IJ\bar{Z}Z}\left(\Delta_{ZI}\Delta_{\bar{Z}J}\right)^{(\ell-4)/4}\left(\Delta_{\bar{Z}I}\Delta_{ZJ}\right)^{\ell/4}\Big] \tag{F.9}$$
$$\times\,\Delta_{IJ}^{(2L'-\ell)/2}\frac{N^{(2L'+\ell)/2-4}}{2^{(2L'+\ell)/2-3}}+(I\leftrightarrow J)+\big(Z\leftrightarrow\bar{Z}\big)$$

$$(h) = -\,(-)^{\ell/2}\,\frac{2}{\ell}\left(N-\frac{\ell}{2}\right)!\left(\frac{\Delta_{Z\bar{Z}}}{2}\right)^{N-\ell/2}\big(1-\delta_{\ell,2L'}\big)\Big[\delta^{\rm odd}_{\ell/2}\frac{\ell(2L')^2}{8}t_{IIJZ}\left(\Delta_{ZI}\Delta_{\bar{Z}I}\Delta_{ZJ}\right)^{(\ell-2)/4}$$
$$\times\Delta_{\bar{Z}J}^{(\ell+2)/4}+\delta^{\rm even}_{\ell/2}\frac{\ell(2L')^2}{8}t_{IIJZ}\Delta_{ZI}^{(\ell-4)/4}\left(\Delta_{\bar{Z}I}\Delta_{ZJ}\Delta_{\bar{Z}J}\right)^{\ell/4}\Big]\,\Delta_{IJ}^{(2L'-\ell)/2-1}\frac{N^{(2L'+\ell)/2-4}}{2^{(2L'+\ell)/2-3}}$$
$$+\,(I\leftrightarrow J)+\big(Z\leftrightarrow\bar{Z}\big) \tag{F.10}$$

$$(i) = (-)^{\ell/2}\,2\left(N-\frac{\ell+2}{2}\right)!\left(\frac{\Delta_{Z\bar{Z}}}{2}\right)^{N-(\ell+2)/2}\Big\{\delta^{\rm odd}_{\ell/2}\frac{(2L')^2}{32}\Big[\Big((\ell+2)\,t_{IZ\bar{Z}Z}+4\delta_{\ell,2L'}\left(t_{IZZ\bar{Z}}+t_{I\bar{Z}ZZ}\right)\Big)$$
$$\times\Delta_{\bar{Z}J}^{(\ell+2)/4}\left(\Delta_{ZI}\Delta_{\bar{Z}I}\Delta_{ZJ}\right)^{(\ell-2)/4}+(\ell-2)\,t_{IZ\bar{Z}Z}\Delta_{ZI}^{(\ell-6)/4}\left(\Delta_{\bar{Z}I}\Delta_{ZJ}\right)^{(\ell+2)/4}\Delta_{\bar{Z}J}^{(\ell-2)/4}\Big]$$
$$+\delta^{\rm even}_{\ell/2}\frac{(\ell+2)\,\ell(2L')^2}{16}t_{IZ\bar{Z}Z}\Delta_{ZI}^{\ell/4-1}\left(\Delta_{\bar{Z}I}\Delta_{ZJ}\Delta_{\bar{Z}J}\right)^{\ell/4}\Big\}\,\Delta_{IJ}^{(2L'-\ell)/2}\frac{N^{(2L'+\ell)/2-3}}{2^{(2L'+\ell)/2-1}} \tag{F.11}$$
$$+\,(I\leftrightarrow J)+\big(Z\leftrightarrow\bar{Z}\big)$$

G Basic correlators

The giant gravitons (1.2) are BPS states. The two-point function is protected from radiative corrections and coincides with (2.4) with no scalars for any N [7, 16].

$$\langle \mathcal{G}(x_1)\,\bar{\mathcal{G}}(x_2)\rangle = \mathrm{G}^{(0)}_{L=0}(x_1,x_2) = \frac{\mathcal{N}_{12}}{x_{12}^{2N}}\,, \qquad \mathcal{N}_{12} \equiv N!\left(\frac{\lambda}{8\pi^2 N}\right)^N \tag{G.1}$$

The vanishing of (3.1) at zero length is consistent with the absence of quantum corrections.

As for single-trace operators, we limit to large N [17]. We define $\bar{\mathcal{O}}_{I_1\ldots I_L}$ as the operator that comprises exactly the same fields of $\mathcal{O}^{I_1\ldots I_L}$ like (1.3), but complex conjugated and in reverse order. We begin with

$$\left\langle \mathcal{O}_3^{I_1\ldots I_L}(x_3)\,\bar{\mathcal{O}}^4_{J_1\ldots J_L}(x_4)\right\rangle_{(0)} = \frac{\mathcal{N}^{I_1,\ldots I_L}_{J_1,\ldots J_L}}{x_{34}^{2L}}\,, \qquad \mathcal{N}^{I_1,\ldots I_L}_{J_1,\ldots J_L} \equiv \left(\frac{\lambda}{8\pi^2}\right)^L \left(y^{I_1}_{J_1}\ldots y^{I_L}_{J_L} + \text{c.p.}\right)\,, \tag{G.2}$$

where cyclic permutations (c.p.) are the $L-1$ cyclic shifts of the indices I's (or J's), and read off the normalization of $\mathcal{O}_3$

$$\mathcal{N}^{I_1,\ldots I_L}_{I_1,\ldots I_L} = C_{I_1\ldots I_L}\left(\frac{\lambda}{8\pi^2}\right)^L y^{I_1}_{I_1}\ldots y^{I_L}_{I_L}\,. \tag{G.3}$$

Repeated indices are not summed. The symmetry factor $C_{I_1\ldots I_L}$ counts the ways of contracting the scalars: it takes the value n if $n \in [1,L]$ is the smallest integer that makes the sequence $\{I_1,\ldots I_L\}$ invariant under cyclic shift $I_i \to I_{i+L/n}$, e.g. $C_{I,\ldots I} = L$ for the vacuum state $\mathcal{O}_3^{I\ldots I}$.

At one-loop order it is easy to dress up the tree-level diagrams with the elementary interactions (C.10)-(C.12), regularize divergences as in (C.7) and obtain [3]

$$\left\langle \mathcal{O}_3^{I_1\ldots I_L}(x_3)\,\bar{\mathcal{O}}^4_{J_1\ldots J_L}(x_4)\right\rangle_{(1)} = -\frac{1}{x^{2L}}\left(\frac{\lambda}{8\pi^2}\right)^L\left(\log\frac{x^2}{\epsilon^2}+1\right)\left(\sum_{k=1}^{L} H_{k,k+1}y^{I_1}_{J_1}\ldots y^{I_L}_{J_L} + \text{c.p.}\right)\,. \tag{G.4}$$

Here H denotes the one-loop Hamiltonian density operator

$$H_{i,j} = \frac{\lambda}{8\pi^2}\left(I_{i,j} + \frac{1}{2}K_{i,j} - P_{i,j}\right) \tag{G.5}$$

while P, K and I are the exchange, trace and identity operator acting on the sites i,j respectively. We remind that operator mixing with other sectors does not affect their anomalous dimensions at one loop [3].

H Renormalization-scheme invariant structure constants

We review the renormalization-group invariant characterization of one-loop corrections to structure constants at large N [6][10]. In a basis of primary operators $\{\mathcal{O}_i(x_i)\}$, conformal

[10] Take note of our change of notation: λ is the true 't Hooft coupling, while i, j and k label the primaries in an arbitrary basis instead of α, β and γ.

symmetry fixes the two- and three-point functions of (unrenormalized) operators as

$$\begin{aligned}\langle\mathcal{O}_i(x_i)\,\mathcal{O}_j(x_j)\rangle &= \frac{h_{ij}+\lambda g_{ij}-\lambda\gamma_{ij}\log\frac{x_{ij}^2}{\epsilon^2}}{x_{ij}^{2\Delta_i}}+O(\lambda^2)\\ \langle\mathcal{O}_i(x_i)\,\mathcal{O}_j(x_j)\,\mathcal{O}_k(x_k)\rangle &= \frac{1}{x_{ij}^{\Delta_i+\Delta_j-\Delta_k}x_{ik}^{\Delta_i+\Delta_k-\Delta_j}x_{jk}^{\Delta_j+\Delta_k-\Delta_i}}\Big[C_{ijk}^{(0)}+\lambda\tilde{C}_{ijk}^{(1)}\\ &-\frac{\lambda}{2}C_{ijk}^{(0)}\gamma_i\log\frac{x_{ij}^2x_{ik}^2}{\epsilon^2x_{jk}^2}-\frac{\lambda}{2}C_{ijk}^{(0)}\gamma_j\log\frac{x_{ij}^2x_{jk}^2}{\epsilon^2x_{ik}^2}-\frac{\lambda}{2}C_{ijk}^{(0)}\gamma_k\log\frac{x_{ik}^2x_{jk}^2}{\epsilon^2x_{ij}^2}+O(\lambda^2)\Big]\,.\end{aligned} \tag{H.1}$$

The scheme-dependent constant ϵ regularizes divergences in loop computations and identifies with the cutoff (C.7) in this paper, Δ's are the classical dimensions, γ's are the anomalous dimension matrices, h and g are the mixing matrix at tree level and at one loop. In an arbitrary basis one does not assume that γ is diagonal and h is a Kronecker delta. $C^{(0)}$ and $\tilde{C}^{(1)}$ are the structure constant at tree level and one loop. The latter is sometimes referred to as "renormalization-scheme dependent" (i.e. ϵ-dependent) in order to distinguish it from the "renormalization-scheme independent" $C^{(1)}$ of renormalized operators. Their relation is

$$C_{ijk}^{(1)}=\tilde{C}_{ijk}^{(1)}-\frac{1}{2}g_{ii'}C^{(0)i'}{}_{jk}-\frac{1}{2}g_{jj'}C_i^{(0)j'}{}_k-\frac{1}{2}g_{kk'}C_{ij}^{(0)\ k'} \tag{H.2}$$

where indices are raised/lowered as

$$C^{(0)i}{}_{jk}=C_{i'jk}^{(0)}h^{ii'}\,,\qquad C_i^{(0)j}{}_k=C_{ij'k}^{(0)}h^{jj'}\,,\qquad C_{ij}^{(0)\ k}=C_{ijk'}^{(0)}h^{kk'} \tag{H.3}$$

by the inverse mixing matrix

$$h^{ii'}h_{i'j}=\delta_j^i\,. \tag{H.4}$$

Summation over primed indices is understood. The last three terms in (H.2) are called metric subtractions. Under a multiplicative scaling of ϵ, they get shifted by an amount that compensates the variation of $\tilde{C}^{(1)}$, with the net effect that $C^{(1)}$ is unchanged.

In the examples of appendix G, one reads off for $(\mathcal{N}_{12})^{-1/2}\,\mathcal{G}$, $(\mathcal{N}_{12})^{-1/2}\,\bar{\mathcal{G}}$ and $\left(\mathcal{N}_{I_1,\ldots I_L}^{I_1,\ldots I_L}\right)^{-1/2}\mathcal{O}_3^{I_1\ldots I_L}$

$$\begin{aligned}&\Delta_{\mathcal{G}}=\Delta_{\bar{\mathcal{G}}}=N\,,\qquad \Delta_{\mathcal{O}_3^{I_1\ldots I_L}}=L\,,\qquad g_{\mathcal{G}\bar{\mathcal{G}}}=\gamma_{\mathcal{G}\bar{\mathcal{G}}}=0\,,\\ h_{\mathcal{O}_3^{I_1\ldots I_L}\bar{\mathcal{O}}^3_{J_1\ldots J_L}}&=\frac{\delta_{J_1}^{I_1}\ldots\delta_{J_L}^{I_L}+\text{c.p.}}{(C_{I_1\ldots I_L}C_{J_1\ldots J_L})^{1/2}}\,,\quad h^{\mathcal{O}_3^{I_1\ldots I_L}\bar{\mathcal{O}}^3_{J_1\ldots J_L}}=\frac{C_{I_1\ldots I_L}C_{J_1\ldots J_L}}{L^2}h_{\mathcal{O}_3^{I_1\ldots I_L}\bar{\mathcal{O}}^3_{J_1\ldots J_L}}\,,\\ g_{\mathcal{O}_3^{I_1\ldots I_L}\bar{\mathcal{O}}^3_{J_1\ldots J_L}}&=-\gamma_{\mathcal{O}_3^{I_1\ldots I_L}\bar{\mathcal{O}}^3_{J_1\ldots J_L}}=-\frac{\sum_{k=1}^L H_{k,k+1}\left(\delta_{J_1}^{I_1}\ldots\delta_{J_L}^{I_L}\right)+\text{c.p.}}{(C_{I_1\ldots I_L}C_{J_1\ldots J_L})^{1/2}}\,.\end{aligned} \tag{H.5}$$

The prefactors (G.3) are for normalization purposes. We remind the R-symmetry conventions (1.1) and note that the product of the mixing matrix with its inverse correctly gives the identity[11] in the space of tensors invariant under cyclic permutations

$$\sum_{K_1,\ldots K_L=1}^{6} h^{\mathcal{O}_3^{I_1\ldots I_L}\bar{\mathcal{O}}^3_{K_1\ldots K_L}} h_{\mathcal{O}_3^{K_1\ldots K_L}\bar{\mathcal{O}}^3_{J_1\ldots J_L}} = \frac{\delta_{I_1}^{J_1}\ldots\delta_{I_L}^{J_L}+\text{c.p.}}{L}\,. \tag{H.7}$$

For the scope of section 3.2, it is instructive to repeat the case of the $SO(6)$ sector [6] for two BPS operators

$$\mathcal{O}_1^{Z\ldots Z}\equiv \text{tr}\,(Z\,(x_1))^{L_1}\,,\qquad \mathcal{O}_2^{\bar{Z}\ldots\bar{Z}}\equiv \text{tr}\,\left(\bar{Z}\,(x_2)\right)^{L_2} \tag{H.8}$$

and the non-BPS one (1.3). Feynman diagrams are depicted in figures 1-3 [6] and lead to

$$\begin{aligned}
&\left(\mathcal{N}^{Z,\ldots Z}_{Z,\ldots Z}\right)^{-1}\left(\mathcal{N}^{I_1,\ldots I_L}_{I_1,\ldots I_L}\right)^{-1/2}\left\langle \mathcal{O}_1^{Z\ldots Z}\,\mathcal{O}_2^{\bar{Z}\ldots\bar{Z}}\,\mathcal{O}_3^{I_1\ldots I_L}\right\rangle\\
=&\frac{\sqrt{L_1L_2}}{N\left(C_{I_1\ldots I_L}\right)^{1/2}}\frac{1}{x_{12}^{2L_{12}}x_{13}^{2L_{13}}x_{23}^{2L_{23}}}\left[\delta_{I_1}^{Z}\ldots\delta_{I_{L_{13}}}^{Z}\delta_{I_{L_{13}+1}}^{\bar{Z}}\ldots\delta_{I_L}^{\bar{Z}}-\frac{1}{2}\left(\log\frac{x_{13}^2x_{23}^2}{\epsilon^2x_{12}^2}+2\right)\right.\\
&\left.\times\left(H_{1,L}+H_{L_{13},L_{13}+1}\right)\left(\delta_{I_1}^{Z}\ldots\delta_{I_{L_{13}}}^{Z}\delta_{I_{L_{13}+1}}^{\bar{Z}}\ldots\delta_{I_L}^{\bar{Z}}\right)+\text{c.p.}+O\left(\lambda^2\right)\right]
\end{aligned} \tag{H.9}$$

in the conventions of appendix C and with the Hamiltonian (G.5). Formula (H.9) is valid for the non-extremal configuration where bridge lengths $L_{12}=(L_1+L_2-L)/2$, $L_{13}=(L_1-L_2+L)/2$ and $L_{23}=(-L_1+L_2+L)/2$ are positive[12]. We still need the data from two-point functions for (H.2). The correlators in appendix G fit into the general structure (H.1) given the quantities labeled by $\mathcal{O}_3$ in (H.5) and

$$\Delta_{\mathcal{O}_1^{Z\ldots Z}}=L_1\,,\qquad \Delta_{\mathcal{O}_2^{\bar{Z}\ldots\bar{Z}}}=L_2\,,\qquad g_{\mathcal{O}_1^{Z\ldots Z}\mathcal{O}_2^{\bar{Z}\ldots\bar{Z}}}=\gamma_{\mathcal{O}_1^{Z\ldots Z}\mathcal{O}_2^{\bar{Z}\ldots\bar{Z}}}=0\,, \tag{H.10}$$

while (H.9) yields

$$\begin{aligned}
C^{(0)\quad\mathcal{O}_3^{I_1\ldots I_L}}_{\mathcal{O}_1^{Z\ldots Z}\mathcal{O}_2^{\bar{Z}\ldots\bar{Z}}}&=\frac{C_{I_1\ldots I_L}}{L}C^{(0)}_{\mathcal{O}_1^{Z\ldots Z}\mathcal{O}_2^{\bar{Z}\ldots\bar{Z}}\mathcal{O}_3^{I_1\ldots I_L}}\\
&=\frac{\sqrt{L_1L_2}}{LN}\left(C_{I_1\ldots I_L}\right)^{1/2}\left(\delta_{Z}^{I_1}\ldots\delta^{Z}_{I_{L_{13}}}\delta^{\bar{Z}}_{I_{L_{13}+1}}\ldots\delta^{\bar{Z}}_{I_L}+\text{c.p.}\right)\,,
\end{aligned} \tag{H.11}$$

$$\tilde{C}^{(1)}_{\mathcal{O}_1^{Z\ldots Z}\mathcal{O}_2^{\bar{Z}\ldots\bar{Z}}\mathcal{O}_3^{I_1\ldots I_L}}=-\frac{\sqrt{L_1L_2}}{N\left(C_{I_1\ldots I_L}\right)^{1/2}}\left[\left(H_{1,L}+H_{L_{13},L_{13}+1}\right)\left(\delta_{I_1}^{Z}\ldots\delta_{I_{L_{13}}}^{Z}\delta_{I_{L_{13}+1}}^{\bar{Z}}\ldots\delta_{I_L}^{\bar{Z}}\right)+\text{c.p.}\right]\,. \tag{H.12}$$

[11] The identity satisfies the "idempotency" property

$$\sum_{K_1,\ldots K_L=1}^{6} L^{-1}\left(\delta_{K_1}^{I_1}\ldots\delta_{K_L}^{I_L}+\text{c.p.}\right)L^{-1}\left(\delta_{J_1}^{K_1}\ldots\delta_{J_L}^{K_L}+\text{c.p.}\right)=L^{-1}\left(\delta_{J_1}^{I_1}\ldots\delta_{J_L}^{I_L}+\text{c.p.}\right)\,. \tag{H.6}$$

[12] Diagramatically it means that only neighboring fields can interact.

This proves that the Hamiltonian density governs the invariant structure constants (H.2) in the $SO(6)$ sector [6][13]

$$
\begin{aligned}
C^{(1)}_{\mathcal{O}_1^{Z\ldots Z}\mathcal{O}_2^{\bar{Z}\ldots\bar{Z}}\mathcal{O}_3^{I_1\ldots I_L}} &= \tilde{C}^{(1)}_{\mathcal{O}_1^{Z\ldots Z}\mathcal{O}_2^{\bar{Z}\ldots\bar{Z}}\mathcal{O}_3^{I_1\ldots I_L}} - \frac{1}{2}\sum_{J_1,\ldots J_L=1}^{6} g_{\mathcal{O}_3^{I_1\ldots I_L}\bar{\mathcal{O}}^3_{J_1\ldots J_L}} C^{(0)\ \mathcal{O}_3^{J_1\ldots J_L}}_{\mathcal{O}_2^{\bar{Z}\ldots\bar{Z}}\mathcal{O}_2^{\bar{Z}\ldots\bar{Z}}} \\
&= -\frac{\sqrt{L_1L_2}}{2N\left(C_{I_1\ldots I_L}\right)^{1/2}}\left[\left(H_{1,L}+H_{L_{13},L_{13}+1}\right)\left(\delta^Z_{I_1}\ldots\delta^Z_{I_{L_{13}}}\delta^{\bar{Z}}_{I_{L_{13}+1}}\ldots\delta^{\bar{Z}}_{I_L}\right)+\text{c.p.}\right].
\end{aligned}
\tag{H.13}
$$

The two allowed positions ($1, L$ and $L_{13}, L_{13}+1$, modulo cyclic permutations) of the scalars of $\mathcal{O}_3$ where the Hamiltonian is inserted are commonly called splitting points. Such name evokes the fact in Feynman diagrams they separate the bundles of free propagators stretching towards $\mathcal{O}_1$ and $\mathcal{O}_2$, see figure 1 (left panel).

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