Exact structure constants of determinant operators

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Summary

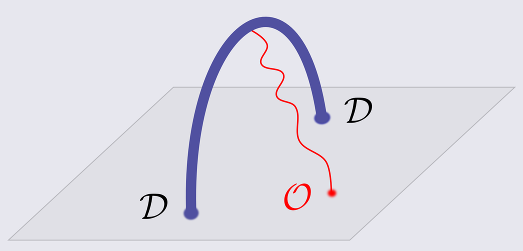
First result for a class of structure constants in $\mathcal{N}=4$ super Yang-Mills at large N and finite 't Hooft coupling λ.

$$\frac{\langle \mathcal{D}_1\mathcal{D}_2\mathcal{O}(0)\rangle}{\langle \mathcal{D}_1\mathcal{D}_2\rangle\sqrt{(\text{2-point coefficient of }\mathcal{O})}} = \underbrace{\left(\frac{a_1-a_2}{a_1 a_2}\right)^{\Delta-J}}_{\substack{\text{superconformal symmetry}\\ \text{fixes spacetime dependence}}} \times \underbrace{\mathfrak{D}_{\mathcal{O}}}_{\text{structure constant }=\,?}$$

$\mathcal{D}_{1,2}=\det \mathfrak{Z}(a_{1,2})$ are 1/2-BPS operators of scalar fields $\mathfrak{Z}(a)=\left[(1+a^2)\Phi^1+i(1-a^2)\Phi^2+2ia\Phi^4\right]/\sqrt{2}\big|_{x^\mu=(0,a,0,0)}$.
$\mathcal{O}$ is a non-BPS single-trace operator with conformal dimension Δ, length L and $U(1)$ R-charge J.

Why? Super Yang-Mills is the ideal playground (superconformal symmetry, integrability, large-N expansion and gauge/gravity duality). Here determinants have been less studied than single-trace operators. The integrability description of the former turns out to be simpler than the hexagon formalism [1] for the latter, as the sum over "intermediate states" is automatic. Holographically, determinants provide better (point-like) probes of the local bulk physics in AdS space. They can also tell about qualitative features of baryons in large-N QCD.

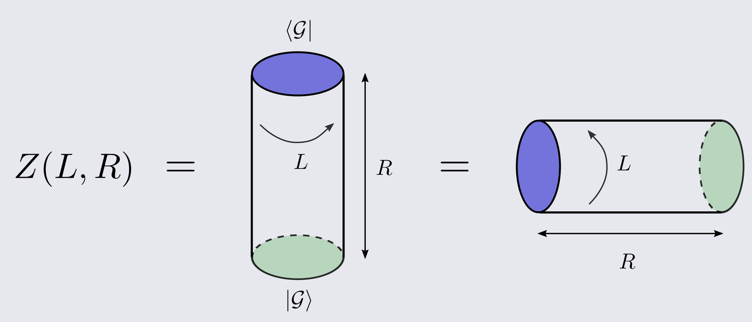
Main idea. In gauge/gravity duality, determinants create a **maximal giant graviton D3-brane** along a geodesics in AdS_5. The single-trace operator creates a **closed string** that attaches to the D-brane. On the string worldsheet, we view the 3-point function as the overlap $\langle \mathcal{G}|\mathcal{O}\rangle$ between the boundary state dual the determinants and the state dual to $\mathcal{O}$.



How to evaluate the overlap at finite λ? Consider the partition function Z of a cylinder worldsheet whose ends are capped off by the **boundary states**. For $R\to\infty$, in the closed-string channel Z is captured by the ground state

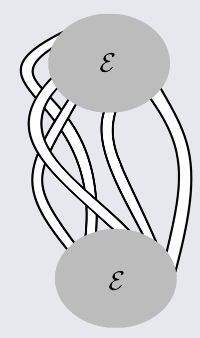
$$Z(L,R)=\sum_{\psi_c}\langle\mathcal{G}|\psi_c\rangle e^{-E_{\psi_c}R}\langle\psi_c|\mathcal{G}\rangle \to |\langle\mathcal{G}|\Omega\rangle|^2 e^{-E_\Omega R}\,,$$

while in the open-string channel Z is the thermal free energy in infinite volume, which the thermodynamic Bethe ansatz (TBA) method computes if the 2d theory is integrable. We generalize to excited states $\langle\mathcal{G}|\Omega\rangle\to\langle\mathcal{G}|\mathcal{O}\rangle$.



Is the boundary state integrable? Selection rules and a determinant-ratio formula for the overlap at zero coupling are a strong positive evidence.

How to compute the overlap at $\lambda \ll 1$? Determinants $\det Z \propto \epsilon^{a_1,\dots a_N}\epsilon^{b_1,\dots b_N}Z_{a_1b_1}\dots Z_{a_Nb_N}$ differ from single-traces in the large number N of fields and the ϵ-symbol. Feynman diagrams proliferate and those non-planar contribute too. We develop new methods to perform many Wick contractions at once.

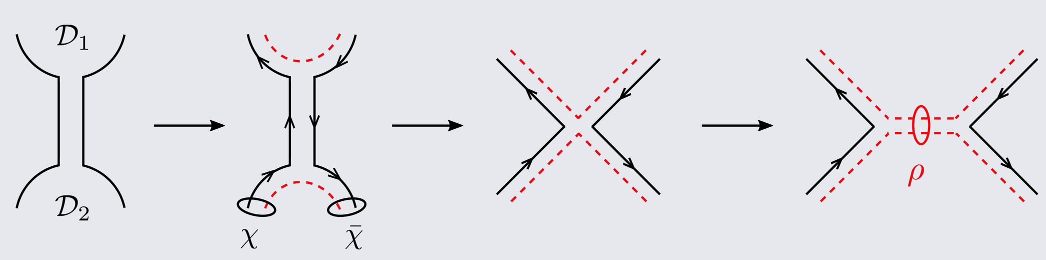


Tree level: effective theory

$$\langle\mathcal{D}_1\dots\mathcal{D}_m\,\mathcal{O}\rangle_{\text{tree level}}=?\qquad\text{with}\qquad \mathcal{D}_k=\det(Y_k\cdot\Phi(x_k))\,,\qquad \mathcal{O}=\mathrm{tr}\left(\Phi^{I_1}\Phi^{I_2}\cdots\right)(y)$$

Through a series of integrating fields in and out, the correlator becomes an integral over the auxiliary $m\times m$ matrix ρ.

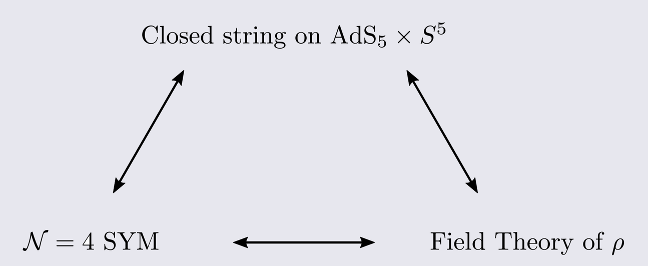
$$\frac{1}{Z_\rho}\int d\rho\,\langle\mathcal{O}^S\rangle_\chi\, e^{N\,S_{\text{eff}}[\rho]}$$



It gets evaluated on an emergent classical background at large N, cf. 1-point functions in defect SYM [2].

$$\sum_{\rho^*:\ \delta S[\rho^*]=0}\langle\mathcal{O}^S\rangle_\chi\Big|_{\rho=\rho^*}\exp\left(N\,S_{\text{eff}}[\rho^*]\right)\qquad\text{with}\qquad \langle\mathcal{O}^S\rangle_\chi\Big|_{\rho=\rho^*}=-\mathrm{tr}_m\left(M^{I_1}\dots M^{I_m}\right)(\{x_k\},\{Y_k\})$$

Rewriting the path-integral as a matrix integral is not a technical trick, but a graph duality that swaps faces and vertices. It realizes an example of open-closed-open string triality [3] in the AdS_5/CFT_4 system.



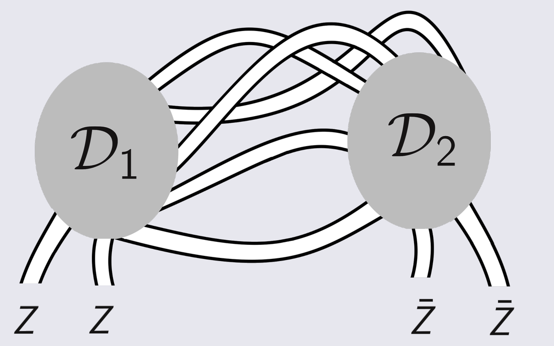
Tree level and beyond: partial contraction

$$\langle\mathcal{D}_1\,\mathcal{D}_2\,\mathrm{tr}(\dots)\,\mathrm{tr}(\dots)\,\dots\rangle_{\text{tree level, 1 loop,...}}=?\qquad\text{with}\qquad \mathcal{D}_1=\det Z(x_1)\,,\qquad \mathcal{D}_2=\det\bar{Z}(x_2)$$

We "divide and conquer" numerous Wick contractions.

Step 1 Contractions between determinants $\det Z\,\det Z\sim\left(Z\bar{Z}\right)^N$ produce a sum of multi-traces $\sim\left(Z\bar{Z}\right)^l$. This is a bilocal operator that we dub "partially-contracted giant gravitons" (PCGG).

$$\langle\mathcal{D}_1\mathcal{D}_2\rangle\Big|_{\text{contractions of }(Z\bar{Z})^{N-l}\text{ only}}=\mathrm{PCGG}_{\text{tree level}}+\mathrm{PCGG}_{1\text{ loop}}+\dots$$



The leading part arises from free contractions.

$$\mathrm{PCGG}_{\text{tree level}}=\frac{(N-l)!}{(l\,!)^2}\left(\frac{\lambda}{8\pi^2N\,(x_1-x_2)^2}\right)^{N-l}\delta^{a_1\dots a_l}_{d_1\dots d_l}\,\delta^{b_1\dots b_l}_{c_1\dots c_l}Z_{a_1b_1}\dots Z_{a_lb_l}\bar{Z}_{c_1d_1}\dots\bar{Z}_{c_ld_l}\ \sim\ \mathrm{tr}\left(Z\bar{Z}\right)^l+\text{multi-traces}$$

Multi-traces do not have extra N-powers and planar expansion becomes possible.

Step 2 Contractions among PCGG and external single-traces.

$$\langle\mathcal{D}_1\,\mathcal{D}_2\,\mathrm{tr}(\dots)\,\mathrm{tr}(\dots)\,\dots\rangle=\langle\mathrm{PCGG}\,\mathrm{tr}(\dots)\,\mathrm{tr}(\dots)\,\dots\rangle$$

Bootstrapping the boundary state

In the free theory the overlap is expressible as an overlap (cf. [2]) between $|\mathcal{O}\rangle$ and a generalized...

... matrix product state, using effective theory, e.g.

$$\langle M|\Big|_{\substack{SU(2),\\ a_2=-a_1}}=\mathrm{tr}_{m=2}\left[\prod_{s=1}^{L}\left(\langle\uparrow|_s\otimes t_1-\frac{i}{a_1}\langle\downarrow|_s\otimes t_2\right)\right]$$

... Néel state, using partial contraction, e.g.

$$\langle\text{Néel}|\Big|_{SU(2)}=\sum_M\sum_{\substack{n_1<\dots<n_M\\ |n_i-n_j|=\text{ even}}}\langle\uparrow\cdots\underset{n_1}{\downarrow}\cdots\underset{n_M}{\downarrow}\cdots\uparrow|\,.$$

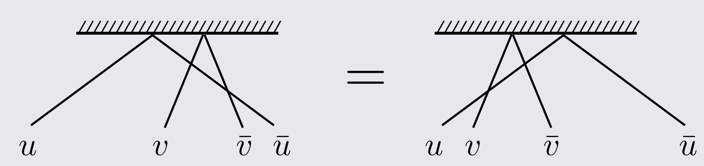
The overlap
– is non-zero when Bethe roots $\mathbf{u}$ are parity symmetric (while others obey $\mathbf{w}=-\mathbf{v}$) and L,M are even,
– is the same determinant ratio in the sectors $SU(2)$, $SL(2)$ and $SO(6)$.

$$\mathfrak{D}_{\mathcal{O}}|_{\text{tree level}}=-\frac{i^J+(-i)^J}{2^M\sqrt{L}}\sqrt{\left(\prod_{1\le s\le\frac{M}{2}}\frac{u_s^2+\frac14}{u_s^2}\right)\frac{\det G_+}{\det G_-}}\qquad\text{with }G_\pm=\text{ sector-dependent matrices}$$

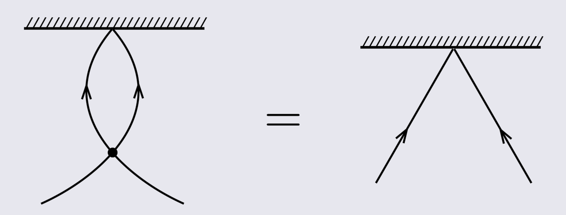
We argue the boundary state $\langle\mathcal{G}|$ is integrable. Integrable bootstrap determines $\langle\mathcal{G}|\mathcal{X}_{A\dot{A}}(u)\mathcal{X}_{B\dot{B}}(\bar{u})\rangle$.

Symmetry The invariance of the 3-point function under centrally-extended $PSU(2|2)_D$ fixes $\langle\mathcal{G}|\mathcal{X}_{A\dot{A}}(u)\mathcal{X}_{B\dot{B}}(\bar{u})\rangle=g_0(u)(-1)^{|\dot{A}||B|}M\left[\mathcal{X}_{A\dot{A}}\mathcal{X}_{B\dot{B}}\right](u)$ up to a scalar function $g_0(u)$ and, unlike the hexagon form factor [1], a constant z.

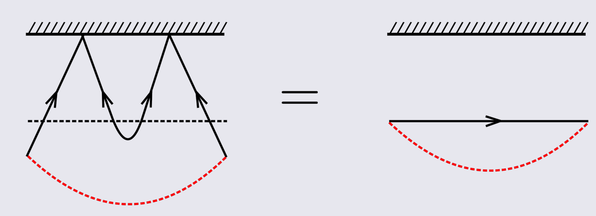
Boundary Yang-Baxter equation The 4-particle overlap is computable in two different ways. This selects $z=\pm i$.



Watson equation The form factor does not change if one acts the S-matrix and then computes the form factor. We solve with the ansatz $g_0(u)=\frac{x^+}{x^-}\frac{u-\frac{i}{2}}{u}\frac{\sigma_B(u)}{\sigma(u,\bar{u})}$ and the boundary dressing phase obeying $\sigma_B(u)=\sigma_B(\bar{u})$.



Decoupling equation The form factor does not change if one adds a particle-antiparticle pair. This finds $\sigma_B(u)$.



Result: large-length structure constants

If $\mathcal{O}$ corresponds to the Bethe roots $\{\mathbf{w},\mathbf{u},\mathbf{v}\}$ obeying the asymptotic Bethe equation, we conjecture

$$\mathfrak{D}_{\mathcal{O}}|_{L\gg1}=\underbrace{-\frac{i^J+(-i)^J}{\sqrt{L}}}_{\substack{\text{tree level analysis,}\\ \langle\mathcal{G}|\text{ with }z=\pm i}}\times\sqrt{\prod_{1\le s\le\frac{M}{2}}\underbrace{\frac{u_s^2+\frac14}{u_s^2}\sigma_B^2(u_s)}_{g_0(u_s)g_0(\bar{u}_s)}}\times\underbrace{\sqrt{\frac{\det G_+}{\det G_-}}}_{\text{generalization of tree level}}\quad\underbrace{\text{with }\mathbf{u}\text{ parity symmetric, }\mathbf{w}_{1,2}=\bar{\mathbf{v}}_{1,2}}_{\text{tree level analysis}}\,.$$

Result: finite-length structure constants

The overlap $\langle\mathcal{G}|\Omega\rangle$ is an example of g-function. In literature it measures the boundary degrees of freedom and plays the role of the central charge in theories without boundaries.
States in the open-string channel are described by rapidities that obey the asymptotic Bethe equation (mirror version of that in [4]). This use of TBA is new in the context of $\mathcal{N}=4$ SYM. Take the thermodynamic limit of the Bethe equation and write a path-integral representation over the rapidity density ρ and the auxiliary η

$$Z(L,R\gg1)\sim\int\mathcal{D}\rho\mathcal{D}\eta\;\mathcal{N}\times e^{-R\,S[\rho,\eta]}\,.$$

Unlike the TBA for the spectrum, we integrate fluctuations to quantify the subleading factor in $R\gg1$, which is $\langle\mathcal{G}|\Omega\rangle$.
We apply the analytic continuation trick [5] to derive $\langle\mathcal{G}|\mathcal{O}\rangle$ in the $SL(2)$ sector and conjecture

$$\mathfrak{D}_{\mathcal{O}}=-\frac{i^J+(-i)^J}{\sqrt{J}}\exp\left[\sum_{a=1}^{\infty}\int_0^\infty\frac{du}{2\pi}\Theta_a(u)\log(1+Y_{a,0}(u))\right]\sqrt{\left(\prod_{1\le s\le\frac{M}{2}}\frac{u_s^2+\frac14}{u_s^2}\sigma_B^2(u_s)\right)\frac{\det\left[1-\hat{G}_-^\bullet\right]}{\det\left[1-\hat{G}_+^\bullet\right]}}\,.$$

This is the first fully non-perturbative proposal for structure constants of finite-length operators. It passes extensive tests against perturbative calculations at small/large λ.

Future directions

- The simple large-spin scaling $\log\left[\mathfrak{D}_{\mathcal{O}}/\mathfrak{D}_{\mathcal{O}}|_{\text{tree level}}\right]=f_1(\lambda)\log S+f_2(\lambda)+O(1/S)$ for length-2 operators in $SL(2)$ sector hints at functions analogue to the cusp anomalous dimension and computable with integrability.
- We have to numerically evaluate $\mathfrak{D}_{\mathcal{O}}$ in $SL(2)$ sector and connect to the formula at large L.
- The cylinder partition function can also compute four-point functions of determinants.
- The same strategy is applicable to Schur polynomial operators.
- g-functions compute correlators of a Wilson loop and a single-trace operator too [6].

References

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