Heat kernel spectroscopy of latitude Wilson loops at strong coupling

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Motivations

- The precise match between perturbation theory for the string sigma-model at large 't Hooft coupling λ and the corresponding (expansion of) exact gauge-theory results in the $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality is a **delicate issue** not fully sorted out yet for some supersymmetric observables. The one-loop corrections to the expectation values of the **1/2-BPS circular** [1, 2] and **1/4-BPS latitude Wilson loops** [3, 4] (and poster session at IGST2015) still defy an agreement with supersymmetric localization [5, 6].
- Finding one-loop sigma-model corrections implies solving the eigenspectra of 2d differential operators and relies on a **regularization scheme** that must be **compatible with the symmetries** of the problem. We make steps in this direction by developing a fully 2d approach based on the **heat kernel method**.

A novel heat kernel approach to nearly-H^2 classical surfaces

In AdS/CFT we measure the vev of supersymmetric Wilson loop operators $\mathcal{W}[\mathcal{C}]$ as the path-integral

$$\mathcal{W}[\mathcal{C}] = Z_{\text{string}}[\mathcal{C}] \equiv \int \mathcal{D}X\,\mathcal{D}\Psi\, e^{-S_{\text{string}}[X,\Psi]}$$

over all worldsheets ending on the path $\mathcal{C}$ located at the AdS boundary where the dual gauge theory lives. In semiclassical approximation $\lambda \gg 1$ we are allowed to approximate **the effective action**

$$\Gamma \equiv -\log Z_{\text{string}}[\mathcal{C}] = \Gamma^{(0)} + \Gamma^{(1)} + \dots\,, \qquad \Gamma^{(0)} = S_{\text{string}}[X = X_{\text{cl}}, \Psi = 0]\,, \qquad \Gamma^{(1)} = -\frac{1}{2}\log\frac{\mathrm{Det}\,\mathcal{O}_F}{\mathrm{Det}\,\mathcal{O}_B}\,.$$

The leading term is the area of the classical surface $X_{\text{cl}}(\tau,\sigma)$, while the next-to-leading correction comprises the determinants of the bosonic and fermionic operators of the fluctuation fields over the classical solution.

- For operators translationally invariant in one variable, say τ, we can Fourier-transform $\partial_\tau \to i\omega$ and solve the infinitely-many 1d determinants in σ using the corollaries of the **Gel'fand-Yaglom theorem** [7]

$$\log \mathrm{Det}\,(\mathcal{O}(\tau,\sigma)) = \begin{cases} \left(\int_0^{2\pi}\frac{d\tau}{2\pi}\right)\sum_\omega \log\mathrm{Det}\,(\mathcal{O}(\omega,\sigma)) & \text{if } \tau\in[0,2\pi) \\ \left(\int_{-\infty}^{+\infty}\frac{d\tau}{2\pi}\right)\int_{-\infty}^{+\infty} d\omega\,\log\mathrm{Det}\,(\mathcal{O}(\omega,\sigma)) & \text{if } \tau\in\mathbb{R}\,. \end{cases}$$

A fully 2d definition of determinant exploits the notion of **heat kernel propagator** $K_{\mathcal{O}}$, *i.e.* the solution of the **heat equation** on the worldsheet and supplemented by a normalization à la Dirac delta

$$(\partial_t + \mathcal{O}(\tau,\sigma))\,K_{\mathcal{O}}(\tau,\sigma;\tau',\sigma';t) = 0\,, \qquad \lim_{t\to 0^+} K_{\mathcal{O}}(\tau,\sigma;\tau',\sigma';t) = \frac{1}{\sqrt{g}}\,\delta(\tau-\tau')\,\delta(\sigma-\sigma')\,\mathbb{I}\,.$$

The Mellin transform of the traced heat kernel defines the determinant in zeta-function regularization

$$\log\mathrm{Det}\,(\mathcal{O}(\tau,\sigma)) \equiv -\frac{d}{ds}\zeta_{\mathcal{O}}(s)\Big|_{s=0} \quad \text{with} \quad \zeta_{\mathcal{O}}(s) \equiv \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left[\int d\tau d\sigma\sqrt{g}\,K_{\mathcal{O}}(\tau,\sigma;\tau,\sigma;t)\right]dt\,.$$

A vast literature has covered heat kernels of Laplace and Dirac operators on maximally symmetric Euclidean spaces (flat spaces $\mathbb{R}^d$, spheres S^d and hyperboloids H^d). **Here we propose to evaluate one-loop determinants over a worldsheet, controlled by a small parameter α, that becomes equivalent to the hyperbolic space H^2 at $\alpha = 0$.** The limit allows to approximate the given operators

$$\mathcal{O} = \bar{\mathcal{O}} + \alpha\,\tilde{\mathcal{O}} + \dots$$

and then **solve the heat equation at first order in** α in terms of the known heat kernels at zero deformation

$$K_{\mathcal{O}} = \bar{K}_{\mathcal{O}} + \alpha\,\tilde{K}_{\mathcal{O}} + \dots \quad \to \quad \zeta_{\mathcal{O}} = \bar{\zeta}_{\mathcal{O}} + \alpha\,\tilde{\zeta}_{\mathcal{O}} + \dots\,.$$

Testing the method with the one-loop Bremsstrahlung function

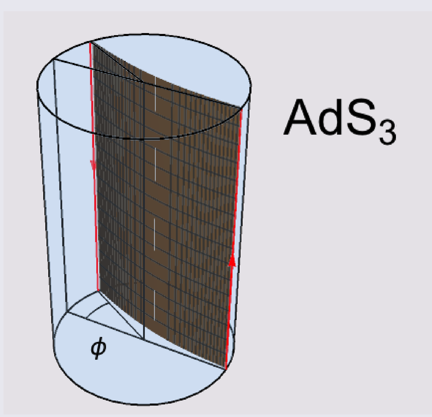
The **Bremsstrahlung function** [8] is an ubiquitous quantity in $\mathcal{N}=4$ SYM and a beautiful common ground where supersymmetric localization and integrability (TBA and QSC method) are in agreement. It measures the near-BPS behavior of the cusp anomaly $\Gamma_{\text{cusp}}(\lambda,\phi)$ of a cusped Wilson line in $\mathbb{R}^4$. This configuration is conformally mapped to two antiparallel lines (red lines) sitting at two points on S^3 at a relative angle $\pi-\phi$, where we have the relation (T is the time cutoff)

$$-\frac{1}{T}\log\langle\mathcal{W}_{\text{cusp}}(\lambda,\phi)\rangle = \Gamma_{\text{cusp}}(\lambda,\phi) \overset{\phi\ll 1}{=} -\phi^2 B(\lambda) + O(\phi^4)\,.$$

For planar $SU(N)$ SYM the Bremsstrahlung function is a combination [8] of modified Bessel functions of the 't Hooft coupling λ

$$B(\lambda) = \frac{\sqrt{\lambda}}{4\pi^2}\frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \overset{\lambda\gg 1}{=} \frac{\sqrt{\lambda}}{4\pi^2} - \frac{3}{8\pi^2} + O(\lambda^{-1/2})\,.$$

We test our method by reproducing the one-loop constant coefficient of the Bremsstrahlung function at strong coupling.



We consider the classical string solution (brown surface) [9] in $AdS_3 \subset AdS_5$ (blue cylinder) dual to the Wilson loop (red lines) in $\mathbb{R}\times S^3$. The angle $\phi\in[0,\pi)$, here given by $k\in\left[0,\frac{1}{\sqrt{2}}\right)$ via Jacobi elliptic functions

$$\phi = \pi - 2k\sqrt{\frac{2k^2-1}{k^2-1}}\left[\Pi\left(1-k^2|k^2\right) - \mathbb{K}\left(k^2\right)\right]\,,$$

parametrizes a (Wick-rotated) surface that becomes the (infinite-strip) H^2 at the "antipodal" configuration $\phi = 0$

$$ds^2 = \frac{1-k^2}{\mathrm{cn}^2(\sigma|k^2)}\left(d\sigma^2 + d\tau^2\right) = \frac{d\tau^2 + d\sigma^2}{\cos^2\sigma} + O(k^2)\,.$$

The one-loop determinants of the fluctuation operators around this solution

$$\mathcal{O}_0(k) \equiv -\frac{\mathrm{cn}^2(\sigma|k^2)}{1-k^2}\left(\partial_\tau^2 + \partial_\sigma^2\right)\,, \qquad \mathcal{O}_1(k) \equiv \mathcal{O}_0(k) + 2\,, \qquad \mathcal{O}_2(k) \equiv \mathcal{O}_0(k) + 2 - \frac{2k^2\mathrm{cn}^4(\sigma|k^2)}{1-k^2}\,,$$

$$\mathcal{O}_F(k) \equiv -i\frac{\mathrm{cn}(\sigma|k^2)}{\sqrt{1-k^2}}\left(\partial_\sigma + \frac{\mathrm{sn}(\sigma|k^2)\,\mathrm{dn}(\sigma|k^2)}{2\mathrm{cn}(\sigma|k^2)}\right)\sigma_1 - i\frac{\mathrm{cn}(\sigma|k^2)}{\sqrt{1-k^2}}\sigma_2\partial_\tau + \sigma_3$$

were evaluated [9] at finite ϕ with the Gel'fand-Yaglom method and gave the one-loop $-\frac{3}{8\pi^2}$ for small ϕ. In the **heat kernel approach** we expand the operators in $k^2 \approx \frac{\phi^2}{\pi^2}$, solve their heat equations, plug into

$$\Gamma^{(1)}(k) = \frac{5}{2}\log\mathrm{Det}\,\mathcal{O}_0 + \frac{2}{2}\log\mathrm{Det}\,\mathcal{O}_1 + \frac{1}{2}\log\mathrm{Det}\,\mathcal{O}_2 - \frac{8}{4}\log\mathrm{Det}\,\mathcal{O}_F^2$$

and zeta-function regularization **returns the same one-loop coefficient $-\frac{3}{8\pi^2}$ in** [9, 8]

$$\frac{1}{T}\left[\Gamma^{(1)}(k=0) - \Gamma^{(1)}(k)\right] \approx \frac{1}{T}\frac{d}{ds}\left(\frac{5}{2}\tilde{\zeta}_{\mathcal{O}_0}(s) + \frac{2}{2}\tilde{\zeta}_{\mathcal{O}_1}(s) + \frac{1}{2}\tilde{\zeta}_{\mathcal{O}_2}(s) - \frac{8}{4}\tilde{\zeta}_{\mathcal{O}_F^2}(s)\right)\Big|_{s=0} k^2 = -\frac{3}{8}k^2 \approx -\frac{3\phi^2}{8\pi^2}\,.$$

Reconciling string fluctuations with localization for latitude loops

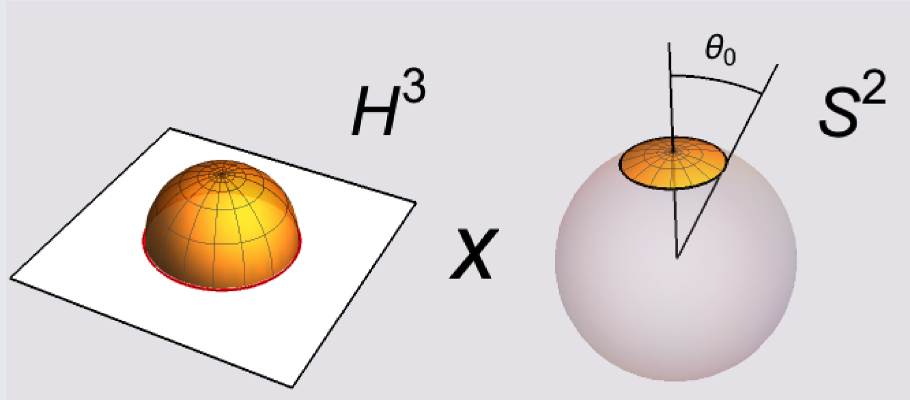
The **Wilson loops** of [10] on S^2 are 1/8-BPS operators in $SU(N)$ $\mathcal{N}=4$ SYM that localize to a Gaussian matrix model for any value of the Yang-Mills coupling and N [6]. A notable subclass comprises the **one-parameter family of** 1/4**-BPS latitude loops** labelled by the angle $\theta_0\in[0,\pi]$. The 1/2-BPS circular loop is the limiting case $\theta_0 = 0$.

In the planar limit supersymmetric localization returns a modified Bessel function of the 't Hooft coupling λ and the angle θ_0

$$\langle\mathcal{W}_{\text{latitude}}(\lambda,\theta_0)\rangle = \frac{2}{\sqrt{\lambda}\cos\theta_0}I_1\left(\sqrt{\lambda}\cos\theta_0\right)\,.$$

This translates into a prediction for the one-loop string effective action at $\lambda\gg 1$ (and here also at small angle)

$$\Gamma^{(1)}(\theta_0 = 0) - \Gamma^{(1)}(\theta_0) = -\frac{3}{2}\log\cos\theta_0 = \frac{3}{4}\theta_0^2 + O(\theta_0^4)\,.$$



Latitude loops are holographically mapped to fundamental strings in $AdS_5\times S^5$. The dual minimal surface (orange surface) is composed of a dome-like surface in Poincaré $H^3\subset AdS_5$, ending on the Wilson loop (red circle), and of a cup-like embedding wrapping the north pole of $S^2\subset S^5$. The worldsheet is a 2d Euclidean space which approximates H^2 when the dual latitude loop approaches the maximal circle for small θ_0

$$ds^2 = \Omega^2(\sigma)\left(d\tau^2 + d\sigma^2\right)\,, \qquad \Omega^2(\sigma) \equiv \frac{1}{\sinh^2\sigma} + \frac{1}{\cosh^2(\sigma+\sigma_0)} = \frac{1}{\sinh^2\sigma} + O\left(\theta_0^2\right)\,, \qquad \cos\theta_0 \equiv \tanh\sigma_0\,.$$

The one-loop operators [3, 4] for generic θ_0 read

$$\mathcal{O}_1 \equiv \frac{1}{\Omega^2}\left(-\partial_\tau^2 - \partial_\sigma^2 + \frac{2}{\sinh^2\sigma}\right)\,, \qquad \mathcal{O}_2(\theta_0) \equiv \frac{1}{\Omega^2}\left(-\partial_\tau^2 - \partial_\sigma^2 - \frac{2}{\cosh^2(\sigma+\sigma_0)}\right)\,,$$

$$\mathcal{O}_{3\pm}(\theta_0) \equiv \frac{1}{\Omega^2}\left(-\partial_\tau^2 - \partial_\sigma^2 \pm 2i\left(\tanh(2\sigma+\sigma_0) - 1\right)\partial_\tau - 1 - 2\tanh(2\sigma+\sigma_0) + 3\tanh^2(2\sigma+\sigma_0)\right)\,,$$

$$\mathcal{O}_{p_{12},p_{56}}(\theta_0) \equiv \frac{1}{\Omega}\left(-i\partial_\tau + p_{56}\frac{1-\tanh(2\sigma+\sigma_0)}{2}\right)\sigma_2 + \frac{i}{\Omega}\left(\partial_\sigma + \frac{\Omega'}{2\Omega}\right)\sigma_1 - \frac{p_{12}p_{56}}{\Omega^2\cosh^2(\sigma+\sigma_0)}\mathbb{I}_{2\times 2} + \frac{p_{12}}{\Omega^2\sinh^2\sigma}\sigma_3\,.$$

The one-loop effective action receives contributions from all fluctuations fields weighted by their multiplicities

$$\Gamma^{(1)}(\theta_0) = \frac{3}{2}\log\mathrm{Det}\,\mathcal{O}_1 + \frac{3}{2}\log\mathrm{Det}\,\mathcal{O}_2 + \frac{1}{2}\log\mathrm{Det}\,\mathcal{O}_{3+} + \frac{1}{2}\log\mathrm{Det}\,\mathcal{O}_{3-} - \frac{1}{2}\sum_{p_{12},p_{56}=\pm 1}\log\mathrm{Det}\,\left(\mathcal{O}^2_{p_{12},p_{56}}\right)\,.$$

To avoid ambiguities due to the absolute normalization of the string partition function [1] (that our method does not attempt to solve), **we must consider the ratio** between a latitude ($\theta_0\neq 0$) and the circular loop ($\theta_0 = 0$).

- In [3] two of us evaluated the normalized one-loop effective action by means of the **Gel'fand-Yaglom method**, paired up with cutoff regularization in $\sigma\in[0,\infty)\to\sigma\in[\epsilon_0,R]$ and followed by the summation over the Fourier modes (see left column). The output matched the one-loop localization prediction up to a factor of unclear origin

$$\Gamma^{(1)}(\theta_0 = 0) - \Gamma^{(1)}(\theta_0) = -\frac{3}{2}\log\cos\theta_0 + \log\cos\frac{\theta_0}{2}\,, \qquad \theta_0\in\left[0,\frac{\pi}{2}\right)\,.$$

  Subsequent work [4] computed instead the determinant $\mathrm{Det}\,(\mathcal{O}_{p_{12},p_{56}})$ of the not-squared fermionic operator and elucidated the role of the symmetry group $SU(2|2)\subset PSU(2,2|4)$ of the Wilson loop in organizing the Fourier summation, but the same spurious remnant $\log\cos\frac{\theta_0}{2}$ persisted.
- **In the configuration when the latitude almost coincides with the circular loop**, our heat kernel approach provides a neat way to **reproduce the one-loop correction of the ratio latitude/circle from string theory**

$$\Gamma^{(1)}(\theta_0 = 0) - \Gamma^{(1)}(\theta_0) \approx \frac{d}{ds}\left(\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_1}(s) + \frac{3}{2}\tilde{\zeta}_{\mathcal{O}_2}(s) + \frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3+}}(s) + \frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3-}}(s) - \frac{1}{2}\sum_{p_{12},p_{56}=\pm 1}\tilde{\zeta}_{\mathcal{O}^2_{p_{12},p_{56}}}(s)\right)\Big|_{s=0}\theta_0^2 = \frac{3}{4}\theta_0^2$$

$$\text{with} \quad \frac{3}{2}\tilde{\zeta}_{\mathcal{O}_1}(s) = \frac{3s}{4}\int_0^\infty dv\frac{v\tanh(\pi v)}{\left(v^2+\frac{9}{4}\right)^s}\,, \qquad \frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3+}}(s) + \frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3-}}(s) = \frac{s}{2}\int_0^\infty dv\frac{\left(v^2+\frac{5}{4}\right)v\tanh(\pi v)}{\left(v^2+\frac{1}{4}\right)^{s+1}}\,,$$

$$\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_2}(s) = \frac{3s}{4}\int_0^\infty dv\frac{\left(v^2+\frac{9}{4}\right)v\tanh(\pi v)}{\left(v^2+\frac{1}{4}\right)^{s+1}}\,, \qquad -\frac{1}{2}\sum_{p_{12},p_{56}=\pm 1}\tilde{\zeta}_{\mathcal{O}^2_{p_{12},p_{56}}}(s) = -2s\int_0^\infty dv\frac{\left(v^2+2\right)v\coth(\pi v)}{\left(v^2+1\right)^{s+1}}\,.$$

Conclusions

- We developed heat kernel techniques to evaluate determinants of one-loop fluctuations around string minimal surfaces when these possess a worldsheet geometry "close" to the one of a maximally symmetric space, *e.g.* here H^2.
- This computational setup for fluctuation determinants **solves the mismatch localization/string theory** [3, 4] for the 1/4-BPS latitude Wilson loop normalized to the 1/2-BPS circular loop **at least in the limit** $\theta_0\to 0$.
- Our algorithm is **highly versatile** and ready to be applied beyond its original scope of evaluating Wilson loops.

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