Correlators of giant gravitons

Edoardo Vescovi

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1 Introduction

We work with gauge group $U(N)$.

$$\mathcal{O}_1 = \det Z(x_1) = \frac{1}{N!}\varepsilon_{i_1\dots i_N}\varepsilon_{j_1\dots j_N} Z_{i_1 j_1}\dots Z_{i_N j_N} \tag{1.1}$$

$$\mathcal{O}_2 = \det \bar{Z}(x_2) = \frac{1}{N!}\varepsilon_{k_1\dots k_N}\varepsilon_{l_1\dots l_N} \bar{Z}_{k_1 l_1}\dots \bar{Z}_{k_N l_N} \tag{1.2}$$

$$\mathcal{O}_3, \mathcal{O}_4 = \text{single-trace operators to specify later} \tag{1.3}$$

2-pt function of 2 GGs

We start with the 2-point function of giant gravitons (p.29 [1], p.7 [2]). We choose the N pairs made of one Z and one $\bar{Z}$ to contract ($N!$ possibilities) with the same result.

$$\langle \mathcal{O}_1\mathcal{O}_2\rangle_{\text{LO}} = \frac{N!}{(N!)^2}\varepsilon_{i_1\dots i_N}\varepsilon_{j_1\dots j_N}\varepsilon_{i_1\dots i_N}\varepsilon_{j_1\dots j_N}\left(\frac{1}{2}\frac{g^2}{4\pi^2(x_1-x_2)^2}u_{Z\bar{Z}}\right)^N = \frac{N!}{2^N(4\pi^2)^N N^N}\frac{\lambda^N}{(x_1-x_2)^{2N}}u^N_{Z\bar{Z}} \tag{1.4}$$

The derivation is thoroughly discussed in the toy-model section, now dressed up by coupling and spacetime factors.

3-pt function of 2 GGs and 1 operator & partially-contracted giant gravitons

We move to the 3-point function of 2 giant gravitons and 1 local operator [1].

$$\begin{aligned}&\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_{\text{LO}} \\ =&\frac{1}{(N!)^2}\varepsilon_{i_1\dots i_N}\varepsilon_{j_1\dots j_N}\varepsilon_{k_1\dots k_N}\varepsilon_{l_1\dots l_N}\left\langle\left(Z_{i_1j_1}\dots Z_{i_nj_n}\right)\left(\bar{Z}_{k_1l_1}\dots\bar{Z}_{k_Nl_N}\right)\mathcal{O}_3\right\rangle_{\text{LO}}.\end{aligned} \tag{1.5}$$

Binomial coefficients count the number of ways of splitting the scalars in $\mathcal{O}_1$ into 2 sets and those in $\mathcal{O}_2$ into 2 sets. Red/black separate the sets.

$$\begin{aligned}=&\frac{1}{(N!)^2}\binom{N}{\frac{L}{2}}^2\varepsilon_{i_1\dots i_N}\varepsilon_{j_1\dots j_N}\varepsilon_{k_1\dots k_N}\varepsilon_{l_1\dots l_N}\\ &\left\langle\left(Z_{i_1j_1}\dots Z_{i_{N-L/2}j_{N-L/2}}Z_{i_{N-L/2+1}j_{N-L/2+1}}\dots Z_{i_Nj_N}\right)\right.\\ &\left.\left(\bar{Z}_{k_1l_1}\dots\bar{Z}_{k_{N-L/2}l_{N-L/2}}\bar{Z}_{k_{N-L/2+1}l_{N-L/2+1}}\dots\bar{Z}_{k_Nl_N}\right)\mathcal{O}_3\right\rangle_{\text{LO}}.\end{aligned} \tag{1.6}$$

Yunfeng's partially-contracted giant gravitons (PCGG) is a non-local operator that results from the $\left(N-\frac{L}{2}\right)!$ distinct free contractions of red scalars.

$$\begin{aligned}=&\frac{1}{(N!)^2}\binom{N}{\frac{L}{2}}^2\left(N-\frac{L}{2}\right)!\left(\frac{1}{2}\frac{\lambda}{N}I_{12}u_{Z\bar{Z}}\right)^{N-L/2}\left(\left(N-\frac{L}{2}\right)!\right)^2\delta^{i_{N-L/2+1}\dots i_N}_{l_{N-L/2+1}\dots l_N}\delta^{j_{N-L/2+1}\dots j_N}_{k_{N-L/2+1}\dots k_N}\\ &\left\langle\left(Z_{i_{N-L/2+1}j_{N-L/2+1}}\dots Z_{i_Nj_N}\right)\left(\bar{Z}_{k_{N-L/2+1}l_{N-L/2+1}}\dots\bar{Z}_{k_Nl_N}\right)\mathcal{O}_3\right\rangle_{\text{LO}}\\ \equiv&\left\langle\mathcal{G}^{\text{LO}}_L\mathcal{O}_3\right\rangle_{\text{LO}}\end{aligned} \tag{1.7}$$

The PCGG admits a multitrace expansion.

$$\begin{aligned}\mathcal{G}^{\text{LO}}_L \overset{L\neq0,\,N\gg1}{=}&\frac{1}{(N!)^2}\binom{N}{\frac{L}{2}}^2\left(N-\frac{L}{2}\right)!\left(\frac{1}{2}\frac{\lambda}{N}I_{12}u_{Z\bar{Z}}\right)^{N-L/2}\left(\left(N-\frac{L}{2}\right)!\right)^2\\ &(-)^{L/2}\frac{L}{2}!\left(\frac{L}{2}-1\right)!\left[-\text{tr}\left(Z\bar{Z}\right)^{L/2}+\frac{L}{2}\sum_{k=1}^{\lfloor L/4\rfloor}\frac{\ell_{k,L}}{k\left(\frac{L}{2}-k\right)}\text{tr}\left(Z\bar{Z}\right)^k\text{tr}\left(Z\bar{Z}\right)^{L/2-k}+(\geq3\text{ trace terms})\right]\\ =&\frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N-L/2-1}L}\frac{\lambda^{N-L/2}}{N^{N-L/2}}I_{12}^{N-L/2}u_{Z\bar{Z}}^{N-L/2}\\ &\left[-\text{tr}\left(Z\bar{Z}\right)^{L/2}+\frac{L}{2}\sum_{k=1}^{\lfloor L/4\rfloor}\frac{\ell_{k,L/2}}{k\left(\frac{L}{2}-k\right)}\text{tr}\left(Z\bar{Z}\right)^k\text{tr}\left(Z\bar{Z}\right)^{L/2-k}+(\geq3\text{ trace terms})\right]\end{aligned} \tag{1.8}$$

[1] The tree-level of $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle$ looks like the one loop of $\langle\mathcal{O}_1\mathcal{O}_2\rangle$.

Partially-contracted giant gravitons at 1 loop

The contributions to the 1-loop PCGG of length L come from contracting the 2-legged O vertex and 4-legged X+H vertex to the tree-level PCGG of lengths $L+2$ and $L+4$ respectively.

$$\langle \mathcal{G}_L^{\mathrm{NLO}} \dots \rangle_{\mathrm{LO}} = \langle \mathcal{G}_{L+2}^{\mathrm{LO}} V_O \dots \rangle_{\mathrm{LO}} + \langle \mathcal{G}_{L+4}^{\mathrm{LO}} V_{X+H} \dots \rangle_{\mathrm{LO}} \tag{1.9}$$

Only here $\mathcal{G}_L^{\mathrm{LO}}$ denotes what has been called $\mathcal{G}_L$ in the previous section. Since the 1-loop PCGG is defined as a partial contraction, the dots here stand for the operators that remain spectators and will contract later.

Let us begin with basic matrix identities. The $\delta\delta$ with 2 pairs of cross-contracted indices

$$\delta^{i_1\dots i_{L/2} i}_{l_1 \dots l_{L/2} l} \delta^{j_1 \dots j_{L/2} i}_{k_1 \dots k_{L/2} l} \equiv N \delta^{i_1 \dots i_{L/2}}_{l_1 \dots l_{L/2}} \delta^{j_1 \dots j_{L/2}}_{k_1 \dots k_{L/2}} + M^{i_1 \dots i_{L/2}; j_1 \dots j_{L/2}}_{l_1 \dots l_{L/2}; k_1 \dots k_{L/2}} \tag{1.10}$$

contains an order-N term proportional to the $\delta\delta$ that appears in $\mathcal{G}_L^{\mathrm{LO}}$ and an order-N^0 that is implicitly defined by subtraction. The definition implies

$$\left(M^{i_1 \dots i_{L/2}; j_1 \dots j_{L/2}}_{l_1 \dots l_{L/2}; k_1 \dots k_{L/2}} \right)_{L=0} = 0 \tag{1.11}$$

with this being useful for the end of the section. For now the definition disentangles powers of N in

$$\begin{aligned}
&\delta^{i_1\dots i_{L/2} i}_{l_1 \dots l_{L/2} l} \delta^{j_1 \dots j_{L/2} j}_{k_1 \dots k_{L/2} k} \left(\delta_{il}\delta_{jk} - \frac{1}{N} \delta_{ij}\delta_{kl} \right) \\
&= \left[\left(N - \frac{L}{2} \right)^2 - 1 \right] \delta^{i_1 \dots i_{L/2}}_{l_1 \dots l_{L/2}} \delta^{j_1 \dots j_{L/2}}_{k_1 \dots k_{L/2}} - \frac{1}{N} M^{i_1 \dots i_{L/2}; j_1 \dots j_{L/2}}_{l_1 \dots l_{L/2}; k_1 \dots k_{L/2}}
\end{aligned} \tag{1.12}$$

$$\begin{aligned}
&\delta^{i_1\dots i_{L/2} i i'}_{l_1 \dots l_{L/2} l l'} \delta^{j_1 \dots j_{L/2} j j'}_{k_1 \dots k_{L/2} k k'} \left(-2\delta_{il}\delta_{i'l'}\delta_{jk'}\delta_{j'k} - 2\delta_{il'}\delta_{i'l}\delta_{jk}\delta_{j'k'} + \delta_{ij'}\delta_{i'l}\delta_{jk'}\delta_{kl'} + \delta_{il}\delta_{i'j}\delta_{j'k'}\delta_{kl'} + \delta_{ij'}\delta_{i'l'}\delta_{jk}\delta_{k'l} + \delta_{il'}\delta_{i'j}\delta_{j'k}\delta_{k'l} \right) \\
=&4 \left[\left(N - \frac{L}{2} \right)^2 + N \right] \left(N - \frac{L}{2} - 1 \right)^2 \delta^{i_1 \dots i_{L/2}}_{l_1 \dots l_{L/2}} \delta^{j_1 \dots j_{L/2}}_{k_1 \dots k_{L/2}} + 4 \left(N - \frac{L}{2} - 1 \right)^2 M^{i_1 \dots i_{L/2}; j_1 \dots j_{L/2}}_{l_1 \dots l_{L/2}; k_1 \dots k_{L/2}} .
\end{aligned} \tag{1.13}$$

Next we derive building blocks for Wick contractions involving the O vertex

$$\begin{aligned}
&\left\langle Z^{a_1}_{i_1 j_1}(x_1) \bar{Z}^{a_2}_{k_1 l_1}(x_2) V_O \right\rangle_{\mathrm{LO}} \\
&= -\frac{\lambda^2}{2N} u_{Z\bar{Z}} \left(\delta_{i_1 l_1}\delta_{j_1 k_1} - \frac{1}{N}\delta_{i_1 j_1}\delta_{k_1 l_1} \right) (Y_{112} + Y_{122})
\end{aligned} \tag{1.14}$$

$$\begin{aligned}
&\delta^{i_1\dots i_{L/2} i}_{l_1 \dots l_{L/2} l} \delta^{j_1 \dots j_{L/2} j}_{k_1 \dots k_{L/2} k} \left(\delta_{il}\delta_{jk} - \frac{1}{N} \delta_{ij}\delta_{kl} \right) \\
&= \left[\left(N - \frac{L}{2} \right)^2 - 1 \right] \delta^{i_1 \dots i_{L/2}}_{l_1 \dots l_{L/2}} \delta^{j_1 \dots j_{L/2}}_{k_1 \dots k_{L/2}} - \frac{1}{N} M^{i_1 \dots i_{L/2}; j_1 \dots j_{L/2}}_{l_1 \dots l_{L/2}; k_1 \dots k_{L/2}}
\end{aligned} \tag{1.15}$$

and the X+H vertex

$$\begin{aligned}
&\left\langle Z^{a_1}_{i_1 j_1}(x_1) Z^{a_2}_{i_2 j_2}(x_1) \bar{Z}^{a_3}_{k_1 l_1}(x_2) \bar{Z}^{a_4}_{k_2 l_2}(x_2) V_{X+H} \right\rangle_{\mathrm{LO}} \\
=&\frac{\lambda^3}{N^3} u_{Z\bar{Z}}^2 \left(f^{aa_1a_4} f^{aa_2a_3} T^{a_1}_{i_1j_1} T^{a_4}_{k_2l_2} T^{a_2}_{i_2j_2} T^{a_3}_{k_1l_1} + f^{aa_1a_3} f^{aa_2a_4} T^{a_1}_{i_1j_1} T^{a_3}_{k_1l_1} T^{a_2}_{i_2j_2} T^{a_4}_{k_2l_2} \right) \left(X_{1122} + I_{12}^2 F_{12,12} \right) \\
&= \frac{\lambda^3}{8N^3} u_{Z\bar{Z}}^2 \left(-2\delta_{i_1l_1}\delta_{i_2l_2}\delta_{j_1k_2}\delta_{j_2k_1} - 2\delta_{i_1l_2}\delta_{i_2l_1}\delta_{j_1k_1}\delta_{j_2k_2} + \delta_{i_1l_2}\delta_{i_2j_1}\delta_{j_2k_1}\delta_{k_2l_1} \right. \\
&\qquad \left. + \delta_{i_1j_2}\delta_{i_2l_1}\delta_{j_1k_2}\delta_{k_1l_2} + \delta_{i_1l_1}\delta_{i_2j_1}\delta_{j_2k_2}\delta_{k_1l_2} + \delta_{i_1j_2}\delta_{i_2l_2}\delta_{j_1k_1}\delta_{k_2l_1} \right) \left(X_{1122} + I_{12}^2 F_{12,12} \right)
\end{aligned} \tag{1.16}$$

$$\begin{aligned}
&\delta^{i_1\dots i_{L/2} i i'}_{l_1 \dots l_{L/2} l l'} \delta^{j_1 \dots j_{L/2} j j'}_{k_1 \dots k_{L/2} k k'} \left(-2\delta_{il}\delta_{i'l'}\delta_{jk'}\delta_{j'k} - 2\delta_{il'}\delta_{i'l}\delta_{jk}\delta_{j'k'} + \delta_{il'}\delta_{i'j}\delta_{j'k}\delta_{k'l} + \delta_{ij'}\delta_{i'l}\delta_{jk'}\delta_{kl'} + \delta_{il}\delta_{i'j}\delta_{j'k'}\delta_{kl'} + \delta_{ij'}\delta_{i'l'}\delta_{jk}\delta_{k'l} \right) \\
=&4 \left[\left(N - \frac{L}{2} \right)^2 + N \right] \left(N - \frac{L}{2} - 1 \right)^2 \delta^{i_1 \dots i_{L/2}}_{l_1 \dots l_{L/2}} \delta^{j_1 \dots j_{L/2}}_{k_1 \dots k_{L/2}} + 4 \left(N - \frac{L}{2} - 1 \right)^2 M^{i_1 \dots i_{L/2}; j_1 \dots j_{L/2}}_{l_1 \dots l_{L/2}; k_1 \dots k_{L/2}} .
\end{aligned} \tag{1.17}$$

Now we are able to compute the diagrams involving the O and X+H vertex.

$$\langle \mathcal{G}_{L+2}^{\mathrm{LO}} V_O \dots \rangle_{\mathrm{LO}} \tag{1.18}$$

$$
\begin{aligned}
=&\frac{1}{(N!)^2}\binom{N}{\frac{L}{2}+1}^2\left(N-\frac{L}{2}-1\right)!\left(\frac{1}{2}\frac{\lambda}{N}I_{12}u_{Z\bar{Z}}\right)^{N-L/2-1}\left(\left(N-\frac{L}{2}-1\right)!\right)^2\delta^{i_{N-L/2}\dots i_N}_{l_{N-L/2}\dots l_N}\delta^{j_{N-L/2}\dots j_N}_{k_{N-L/2}\dots k_N}\\
&\left\langle\left(Z_{i_{N-L/2}j_{N-L/2}}\dots Z_{i_Nj_N}\right)\left(\bar{Z}_{k_{N-L/2}l_{N-L/2}}\dots\bar{Z}_{k_Nl_N}\right)V_O\dots\right\rangle_{\mathrm{LO}}\\
=&\frac{1}{(N!)^2}\binom{N}{\frac{L}{2}+1}^2\left(N-\frac{L}{2}-1\right)!\left(\frac{1}{2}\frac{\lambda}{N}I_{12}u_{Z\bar{Z}}\right)^{N-L/2-1}\left(\left(N-\frac{L}{2}-1\right)!\right)^2\\
&\underline{\left(\frac{L}{2}+1\right)^2}\left[-\frac{\lambda^2}{2N}u_{Z\bar{Z}}\left(Y_{112}+Y_{122}\right)\right]\left\{\left[\left(N-\frac{L}{2}\right)^2-1\right]\delta^{i_1\dots i_{L/2}}_{l_1\dots l_{L/2}}\delta^{j_1\dots j_{L/2}}_{k_1\dots k_{L/2}}-\frac{1}{N}M^{i_1\dots i_{L/2};j_1\dots j_{L/2}}_{l_1\dots l_{L/2};k_1\dots k_{L/2}}\right\}\\
&\left\langle\left(Z_{i_1j_1}\dots Z_{i_{L/2}j_{L/2}}\right)\left(\bar{Z}_{k_1l_1}\dots\bar{Z}_{k_{L/2}l_{L/2}}\right)\dots\right\rangle_{\mathrm{LO}}\\
=&-\frac{\left(N-\frac{L}{2}-1\right)!}{2^{N-L/2}\left(\frac{L}{2}!\right)^2}\frac{\lambda^{N-L/2+1}}{N^{N-L/2}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-L/2}I_{12}^{-1}\left(Y_{112}+Y_{122}\right)\\
&\left\{\left[\left(N-\frac{L}{2}\right)^2-1\right]\delta^{i_1\dots i_{L/2}}_{l_1\dots l_{L/2}}\delta^{j_1\dots j_{L/2}}_{k_1\dots k_{L/2}}-\frac{1}{N}M^{i_1\dots i_{L/2};j_1\dots j_{L/2}}_{l_1\dots l_{L/2};k_1\dots k_{L/2}}\right\}\left\langle\left(Z_{i_1j_1}\dots Z_{i_{L/2}j_{L/2}}\right)\left(\bar{Z}_{k_1l_1}\dots\bar{Z}_{k_{L/2}l_{L/2}}\right)\dots\right\rangle_{\mathrm{LO}}
\end{aligned}
$$

$$
\begin{aligned}
&\left\langle\mathcal{G}^{\mathrm{LO}}_{L+4}V_{X+H}\dots\right\rangle_{\mathrm{LO}}\\
=&\frac{1}{(N!)^2}\binom{N}{\frac{L}{2}+2}^2\left(N-\frac{L}{2}-2\right)!\left(\frac{1}{2}\frac{\lambda}{N}I_{12}u_{Z\bar{Z}}\right)^{N-L/2-2}\left(\left(N-\frac{L}{2}-2\right)!\right)^2\delta^{i_{N-L/2-1}\dots i_N}_{l_{N-L/2-1}\dots l_N}\delta^{j_{N-L/2-1}\dots j_N}_{k_{N-L/2-1}\dots k_N}\\
&\left\langle\left(Z_{i_{N-L/2-1}j_{N-L/2-1}}\dots Z_{i_Nj_N}\right)\left(\bar{Z}_{k_{N-L/2-1}l_{N-L/2-1}}\dots\bar{Z}_{k_Nl_N}\right)V_{X+H}\dots\right\rangle_{\mathrm{LO}}\\
=&\frac{1}{(N!)^2}\binom{N}{\frac{L}{2}+2}^2\left(N-\frac{L}{2}-2\right)!\left(\frac{1}{2}\frac{\lambda}{N}I_{12}u_{Z\bar{Z}}\right)^{N-L/2-2}\left(\left(N-\frac{L}{2}-2\right)!\right)^2\\
&\underline{\frac{1}{4}\left(\frac{L}{2}+2\right)^2\left(\frac{L}{2}+1\right)^2}\left[\frac{\lambda^3}{8N^3}u_{Z\bar{Z}}^2\left(X_{1122}+I_{12}^2F_{12,12}\right)\right]\\
&\left\{4\left[\left(N-\frac{L}{2}\right)^2+N\right]\left(N-\frac{L}{2}-1\right)^2\delta^{i_1\dots i_{L/2}}_{l_1\dots l_{L/2}}\delta^{j_1\dots j_{L/2}}_{k_1\dots k_{L/2}}+4\left(N-\frac{L}{2}-1\right)^2M^{i_1\dots i_{L/2};j_1\dots j_{L/2}}_{l_1\dots l_{L/2};k_1\dots k_{L/2}}\right\}\\
&\left\langle\left(Z_{i_1j_1}\dots Z_{i_{L/2}j_{L/2}}\right)\left(\bar{Z}_{k_1l_1}\dots\bar{Z}_{k_{L/2}l_{L/2}}\right)\dots\right\rangle_{\mathrm{LO}}\\
=&\frac{\left(N-\frac{L}{2}-2\right)!}{2^{N-L/2+1}\left(\frac{L}{2}!\right)^2}\frac{\lambda^{N-L/2+1}}{N^{N-L/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-L/2}I_{12}^{-2}\left(X_{1122}+I_{12}^2F_{12,12}\right)\\
&\left\{\left[\left(N-\frac{L}{2}\right)^2+N\right]\left(N-\frac{L}{2}-1\right)^2\delta^{i_1\dots i_{L/2}}_{l_1\dots l_{L/2}}\delta^{j_1\dots j_{L/2}}_{k_1\dots k_{L/2}}+\left(N-\frac{L}{2}-1\right)^2M^{i_1\dots i_{L/2};j_1\dots j_{L/2}}_{l_1\dots l_{L/2};k_1\dots k_{L/2}}\right\}\\
&\left\langle\left(Z_{i_1j_1}\dots Z_{i_{L/2}j_{L/2}}\right)\left(\bar{Z}_{k_1l_1}\dots\bar{Z}_{k_{L/2}l_{L/2}}\right)\dots\right\rangle_{\mathrm{LO}}
\end{aligned}
\tag{1.19}
$$

Using the relation $X_{1122}+I_{12}^2F_{12,12}-2I_{12}\left(Y_{112}+Y_{122}\right)=0$ to eliminate X_{1122} from the second diagram, the diagram sum yields the 1-loop finite-N PCGG.

$$
\begin{aligned}
&\left\langle\mathcal{G}^{\mathrm{NLO}}_{L}\dots\right\rangle_{\mathrm{LO}}\\
=&\frac{\left(N-\frac{L}{2}-1\right)!}{2^{N-L/2}\left(\frac{L}{2}!\right)^2}\frac{\lambda^{N-L/2+1}}{N^{N-L/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-L/2}I_{12}^{-1}\left(Y_{112}+Y_{122}\right)\\
&\left\{-\frac{L}{2}\left(N-\frac{L}{2}\right)\left(N-\frac{L}{2}-1\right)\delta^{i_1\dots i_{L/2}}_{l_1\dots l_{L/2}}\delta^{j_1\dots j_{L/2}}_{k_1\dots k_{L/2}}+\left(N-\frac{L}{2}\right)M^{i_1\dots i_{L/2};j_1\dots j_{L/2}}_{l_1\dots l_{L/2};k_1\dots k_{L/2}}\right\}\\
&\left\langle\left(Z_{i_1j_1}\dots Z_{i_{L/2}j_{L/2}}\right)\left(\bar{Z}_{k_1l_1}\dots\bar{Z}_{k_{L/2}l_{L/2}}\right)\dots\right\rangle_{\mathrm{LO}}\\
=&-\frac{L\left(N-\frac{L}{2}-1\right)}{2N}\lambda I_{12}^{-1}\left(Y_{112}+Y_{122}\right)\left\langle\mathcal{G}^{\mathrm{LO}}_{L}\dots\right\rangle_{\mathrm{LO}}\\
&+\frac{\left(N-\frac{L}{2}\right)!}{2^{N-L/2}\left(\frac{L}{2}!\right)^2}\frac{\lambda^{N-L/2+1}}{N^{N-L/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-L/2}I_{12}^{-1}\left(Y_{112}+Y_{122}\right)M^{i_1\dots i_{L/2};j_1\dots j_{L/2}}_{l_1\dots l_{L/2};k_1\dots k_{L/2}}\left\langle\left(Z_{i_1j_1}\dots Z_{i_{L/2}j_{L/2}}\right)\left(\bar{Z}_{k_1l_1}\dots\bar{Z}_{k_{L/2}l_{L/2}}\right)\dots\right\rangle_{\mathrm{LO}}
\end{aligned}
\tag{1.20}
$$

The leading powers N^3 cancel in curly brackets. The leftover with $\delta\delta$ is basically the tree-level PCGG, plus the less famous matrix M which carries a power of N less. We conclude with few remarks.

- The 1-loop PCGG has a multitrace expansion. The part $\delta\delta$ has the well-known multitrace expansion in appendix. To derive the expansion of the part M, one takes the same formula but with one index more

$$\begin{aligned} &\delta^{i_1,\dots i_{L/2+1}}_{l_1,\dots l_{L/2+1}}\delta^{j_1,\dots j_{L/2+1}}_{k_1,\dots k_{L/2+1}} Z_{i_1k_1}\dots Z_{i_{L/2+1}k_{L/2+1}}\bar{Z}_{j_1l_1}\dots\bar{Z}_{j_{L/2+1}l_{L/2+1}} \\ &= -\sum_{k_1,k_2,\dots k_{L/2+1}=0}^{L/2+1} \frac{(-)^{L/2}\left(\left(\frac{L}{2}+1\right)!\right)^2}{k_1!\dots k_{L/2+1}!}\left(-\mathrm{tr}\left(Z\bar{Z}\right)\right)^{k_1}\dots\left(-\frac{1}{L/2+1}\mathrm{tr}\left(Z\bar{Z}\right)^{L/2+1}\right)^{k_{L/2+1}} \\ &\qquad k_1+2k_2+\dots\left(\frac{L}{2}+1\right)k_{L/2+1}=\frac{L}{2}+1 \end{aligned} \tag{1.21}$$

and applies the substitution rule

$$Z_{ij}\bar{Z}_{kl}\to\delta_{ij}\delta_{kl} \tag{1.22}$$

to all possible pairs of Z and $\bar{Z}$ such that they do not belong to the same trace $\mathrm{tr}\left(Z\bar{Z}\right)$. The replacement puts an identity matrix in the place of one Z and one $\bar{Z}$ while preventing that the operation generates a power of N in the move $\mathrm{tr}\left(Z\bar{Z}\right)$ (other traces) $\to$ N (other traces). Remember the definition of M to understand the exclusion. The effect of replacements is to generate M in the lhs

$$\left(\frac{L}{2}+1\right)^2 M^{i_1,\dots i_{L/2};j_1,\dots j_{L/2}}_{l_1,\dots l_{L/2};k_1,\dots k_{L/2}} Z_{i_1k_1}\dots Z_{i_{L/2}k_{L/2}}\bar{Z}_{j_1l_1}\dots\bar{Z}_{j_{L/2}l_{L/2}} \tag{1.23}$$

$$\begin{aligned} &= \left[-\sum_{k_1,k_2,\dots k_{L/2+1}=0}^{L/2+1} \frac{(-)^{L/2}\left(\left(\frac{L}{2}+1\right)!\right)^2}{k_1!\dots k_{L/2+1}!}\left(-\mathrm{tr}\left(Z\bar{Z}\right)\right)^{k_1}\dots\left(-\frac{1}{L/2+1}\mathrm{tr}\left(Z\bar{Z}\right)^{L/2+1}\right)^{k_{L/2+1}}\right]_{\text{replacement rule}} \\ &\qquad k_1+2k_2+\dots\left(\frac{L}{2}+1\right)k_{L/2+1}=\frac{L}{2}+1\,. \end{aligned} \tag{1.24}$$

Note that the replacement rule creates non-alternating sequences $\dots Z\bar{Z}ZZ\bar{Z}\dots$ and $\dots Z\bar{Z}\bar{Z}Z\bar{Z}\dots$. We will use this approach in diagrams 1+3 in the section about the 4-pt function.

- Taking the large-N limit of prefactors only [2]

$$\left\langle\mathcal{G}^{\text{NLO}}_L\dots\right\rangle_{\text{LO}} \overset{N\gg1}{=} -\frac{\lambda L}{2}I^{-1}_{12}\left(Y_{112}+Y_{122}\right)\left\langle\mathcal{G}^{\text{LO}}_L\dots\right\rangle_{\text{LO}} \tag{1.25}$$

may be incorrect as long as the operators in dots remain unspecified. For instance, it wrongly assumes that diagrams from the contraction of M with dots never contribute to the leading N-power. To clarify why the formula potentially fails, remember that the piece with M contains uncontracted scalars that generate powers of N when Wick contractions with dots are performed. These may greater of equal than the the powers of N generated from the contraction of $\mathcal{G}^{\text{NLO}}_L$ with dots. Therefore, the correct procedure for taking the planar limit of the correlator $\left\langle\mathcal{G}^{\text{NLO}}_L\dots\right\rangle_{\text{LO}}$ with any choice of dots is plugging the operators in dots, Wick contract and only then take N large. Let us stress again though that for some specific choice of dots one can instead apply the formula, plug the operators in dots and Wick contract [3].

- The 1-loop PCGG correctly predicts that the 2-pt function of giant gravitons (= 2 fully-contracted giant gravitons = the PCGG with $L=0$ scalars remaining uncontracted) vanishes at 1 loop and finite N:

$$\left\langle\mathcal{O}_1\mathcal{O}_2\right\rangle_{\text{NLO}} = \left\langle\mathcal{G}^{\text{NLO}}_{L=0}\right\rangle_{\text{LO}} = 0\,. \tag{1.26}$$

Wick contractions of determinants

A consistency check

We take 2 fields with propagator $\left\langle A_{ik}B_{jl}\right\rangle_{\text{LO}}=\delta_{il}\delta_{jk}/2$. We want to show that the 2-pt function of 2 determinants

$$\left\langle\det A\,\det B\right\rangle_{\text{LO}} = \frac{1}{(N!)^2}\varepsilon_{i_1,\dots i_N}\varepsilon_{k_1,\dots k_N}\varepsilon_{j_1,\dots j_N}\varepsilon_{l_1,\dots l_N}\left\langle A_{i_1k_1}\dots A_{i_Nk_N}B_{j_1l_1}\dots B_{j_Nl_N}\right\rangle_{\text{LO}} \tag{1.27}$$

is the same doing N contractions all at once or in two rounds of $N-n$ and n contractions.

[2] Regardless of validity, the formula is referred to as "approximated planar formula" when diagrams 1+3 involving the 1-loop PCGG are discussed in sections to come.

[3] An instance of the formula delivering an incorrect planar correlator is the 4-pt function in a section below, where diagrams 1+3 involving the 1-loop PCGG get contributions from both $\mathcal{G}^{\text{LO}}_L$ and M. On the contrary, an instance of the formula delivering a correct planar correlator is the 3-pt functions in $SO(6)$ sector, where diagrams 1+3 involving the 1-loop PCGG get contributions from $\mathcal{G}^{\text{LO}}_L$ only and M can be harmlessly dropped.

- In the first way of proceeding, there are $N!$ ways of contracting pairs AB.

$$\begin{aligned}\langle \det A \det B\rangle_{\text{LO}} &= \frac{1}{(N!)^2} N! \left(\varepsilon_{i_1,\dots i_N}\varepsilon_{i_1,\dots i_N}\right)^2 \frac{1}{2^N} \\ &= \frac{N!}{2^N}\end{aligned} \tag{1.28}$$

This answer is certainly correct in the lower cases $N = 2, 3$ by Sarrus's rule and sum of $2N$-point functions.

- In the second way of proceeding, there are $\binom{N}{n}$ ways of choosing fields [4] and $(N-n)!$ ways of contracting pairs AB.

$$\begin{aligned}\langle \det A \det B\rangle_{\text{LO}} =& \frac{1}{(N!)^2}\varepsilon_{i_1,\dots i_N}\varepsilon_{k_1,\dots k_N}\varepsilon_{j_1,\dots j_N}\varepsilon_{l_1,\dots l_N}\left\langle A_{i_1k_1}\dots A_{i_Nk_N}B_{j_1l_1}\dots B_{j_Nl_N}\right\rangle_{\text{LO}} \\ =& \frac{1}{(N!)^2}\binom{N}{n}(N-n)!\varepsilon\varepsilon^{i_1,\dots i_n}_{l_1,\dots l_n}\varepsilon\varepsilon^{k_1,\dots k_n}_{j_1,\dots j_n}\left\langle A_{i_1k_1}\dots A_{i_nk_n}B_{j_1l_1}\dots B_{j_nl_n}\right\rangle_{\text{LO}}\frac{1}{2^{N-n}} \\ =& \frac{1}{n!N!}\varepsilon\varepsilon^{i_1,\dots i_n}_{l_1,\dots l_n}\varepsilon\varepsilon^{k_1,\dots k_n}_{j_1,\dots j_n}\left\langle A_{i_1k_1}\dots A_{i_nk_n}B_{j_1l_1}\dots B_{j_nl_n}\right\rangle_{\text{LO}}\frac{1}{2^{N-n}}\end{aligned} \tag{1.29}$$

This exhausts the first round. There are now $n!$ ways to contract pairs AB.

$$\begin{aligned}&= \frac{1}{n!N!} n!\varepsilon\varepsilon^{i_1,\dots i_n}_{i_1,\dots i_n}\varepsilon\varepsilon^{j_1,\dots j_n}_{j_1,\dots j_n}\frac{1}{2^N} \\ &= \frac{1}{n!N!} n!\,(N!)^2\,\frac{1}{2^N} \\ &= \frac{N!}{2^N}\end{aligned} \tag{1.30}$$

This exhausts the second round and the result agrees with the previous bullet.

Conclusion: one feels confident in doing two-step contractions. For instance, one can construct the (tree-level/1-loop) PCGG first and deal with (1-loop/tree-level) contractions with $\mathcal{O}_3$ then. The two-step procedure at 1 loop should not have symmetry factors if the first step is doing all free contractions and the second step is putting vertices, because this separation has a physical meaning and is not artificial as at tree-level.

Note that our 3pt functions should not include the symmetry factor because the two rounds of contractions are physically distinct (first round of contractions among the GG, the second one among the uncontracted scalars and the operator) [5].

The PCGG for $L = 0$ is the 2-pt function discussed above and gets the symmetry factor.

Another consistency check

Now we test if a truncated multitrace expansion equals the result above for $n \ll N \to \infty$. We restart from the second bullet and expand in traces.

$$\begin{aligned}\langle \det A \det B\rangle_{\text{LO}} &= \frac{1}{n!N!}\varepsilon\varepsilon^{i_1,\dots i_n}_{l_1,\dots l_n}\varepsilon\varepsilon^{k_1,\dots k_n}_{j_1,\dots j_n}\left\langle A_{i_1k_1}\dots A_{i_nk_n}B_{j_1l_1}\dots B_{j_nl_n}\right\rangle_{\text{LO}}\frac{1}{2^{N-n}} \\ &= \frac{1}{n!N!}\left((N-n)!\right)^2\delta^{i_1,\dots i_n}_{l_1,\dots l_n}\delta^{k_1,\dots k_n}_{j_1,\dots j_n}\left\langle A_{i_1k_1}\dots A_{i_nk_n}B_{j_1l_1}\dots B_{j_nl_n}\right\rangle_{\text{LO}}\frac{1}{2^{N-n}} \\ &= \frac{1}{n!N!}\left((N-n)!\right)^2 n!\delta^{i_1,\dots i_n}_{j_1,\dots j_n}\left\langle (AB)_{i_1j_1}\dots (AB)_{i_nj_n}\right\rangle_{\text{LO}}\frac{1}{2^{N-n}} \\ &= \frac{1}{n!N!}\left((N-n)!\right)^2\frac{(-)^n (n!)^2}{n!}\left(-\text{tr}\,(AB)\right)^n\frac{1}{2^{N-n}} + \dots \\ &= \frac{\left(N^n (N-n)!\right)^2}{2^N N!} + \dots \\ &\sim \frac{N!}{2^N}\end{aligned} \tag{1.31}$$

[4] The naive counting $\binom{N}{n}^2$ of splitting A's and B's among themselves is corrected by a symmetry factor $\binom{N}{n}$. The reason is that the splitting of the first set (say A's) is fictitious (because one field component like A_{12} can still appear in both subsets while indices run, hence no $\binom{N}{n}$ to put) while the splitting of the second set (say B's) is meaningful (because it is relative to the splitting of A's , hence one $\binom{N}{n}$ to put). The simplest non-trivial example $N = 3, L = 4$ is better than any words.

[5] The best proof of this statement is that the normalized 3-pt function at LO is of order N^0.

Conclusion: besides the right result, this example shows a subtle point of determinants in planar limit. A large number of disjoint traces $\left(\langle \mathrm{tr}\,(AB)\rangle_{\mathrm{LO}}\right)^n = \frac{N^{2n}}{2^n}$ dominates over the unique single-trace $\langle \mathrm{tr}\,(AB)^n\rangle_{\mathrm{LO}} = \frac{(2n)!}{(n+1)!n!}\frac{N^{n+1}}{2^n}$ [3] in 2-pt function of determinants [6]. Note also that the number of fields inside a trace is bounded by $n \ll N$, so only matrix identities produce N-dependence.

All above happens in our planar $SU\left(N\right)$ theory, as commented at the beginning.

A quick sketch in double notation shows that the single-trace contribution dominates in our 3-pt function, where we create a PCGG in the first round of contractions and a 3-pt function in the second one. The difference is the presence of the local operator in $\left\langle (AB)_{i_1 j_1} \ldots (AB)_{i_n j_n} \right\rangle$ here above, which forces contractions to happen between A, B's and the operator instead of among A, B's, with the effect that the single-trace contribution dominates.

[6] A quick argument to discard the single-trace contribution is that it does not bring $(-1)^n$ to compensate for that in front of the multitrace sum in order to get a positive result.

2 $SO(6)$ sector

$$\mathcal{O}_3^{I_1 \ldots I_L} = \operatorname{tr}\left(\phi^{I_1} \ldots \phi^{I_L}\right)(x_3) = \operatorname{tr}\left(\tilde{Z}^{N_1} \tilde{Y}^{M_1} \ldots \tilde{Z}^{N_n} \tilde{Y}^{M_n}\right)(x_3) \tag{2.1}$$

The first is for $SO(6)$ sector, the second for $SU(2)$. The vacuum is all ϕ^I (with $u_{II}=0$) and all $\tilde{Z}$ (with $\underline{u_{\tilde{Z}\bar{\tilde{Z}}}=0}$) respectively.

2.1 Tree level with vacuum

The factor L originates from the L distinct contractions between the 2 traces.

$$\begin{aligned}
&\left\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3^{I_1 \ldots I_L} \right\rangle_{\mathrm{LO}} \\
&= \left\langle \mathcal{G}_L \mathcal{O}_3^{I_1 \ldots I_L} \right\rangle_{\mathrm{LO}} \\
&\overset{N \gg 1}{=} \frac{(-)^{L/2+1}}{\frac{L}{2}!} \left(\frac{L}{2}-1\right)! \left(N-\frac{L}{2}\right)! \left(\frac{1}{2}\frac{\lambda}{N} I_{12} u_{Z\bar{Z}}\right)^{N-L/2} \\
&\quad L \left(\frac{\lambda}{N} I_{13} u_{Z\tilde{Z}}\right)^{L/2} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}\tilde{Z}}\right)^{L/2} \left(\frac{N}{2}\right)^{L} \\
&= \frac{(-)^{L/2+1}\left(N-\frac{L}{2}\right)!}{2^{N+L/2-1} N^{N-L/2}} \lambda^{N+L/2} I_{12}^{N-L/2} I_{13}^{L/2} I_{23}^{L/2} u_{Z\bar{Z}}^{N-L/2} u_{Z\tilde{Z}}^{L/2} u_{\bar{Z}\tilde{Z}}^{L/2}
\end{aligned} \tag{2.2}$$

We multiply by the correct normalization to have a vacuum with unit-normalized two-point function. The normalization factor is computed at the beginning of the section.

$$\begin{aligned}
&\left(\frac{L}{2^L\left(4\pi^2\right)^L} \lambda^L u_{\tilde{Z}\bar{\tilde{Z}}}^L\right)^{-1/2} \left(\frac{N!}{2^N\left(4\pi^2\right)^N N^N} \lambda^N u_{Z\bar{Z}}^N\right)^{-1} \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_{\mathrm{LO}} \\
&= \frac{2(-)^{L/2+1}\left(4\pi^2\right)^{N+L/2}}{L^{1/2}} \frac{N^{L/2}\left(N-\frac{L}{2}\right)!}{N!} I_{12}^{N-L/2} I_{13}^{L/2} I_{23}^{L/2} \frac{u_{Z\bar{Z}}^{N-L/2} u_{Z\tilde{Z}}^{L/2} u_{\bar{Z}\tilde{Z}}^{L/2}}{u_{Z\bar{Z}}^N u_{\tilde{Z}\bar{\tilde{Z}}}^{L/2}}
\end{aligned} \tag{2.3}$$

Yunfeng notices the right scaling

$$\frac{N^{L/2}\left(N-\frac{L}{2}\right)!}{N!} = \prod_{i=1}^{L/2-1} \frac{N}{N-j} = O\left(N^0\right) \qquad L \ll N,\, N \to \infty\,. \tag{2.4}$$

2.2 One loop with vacuum

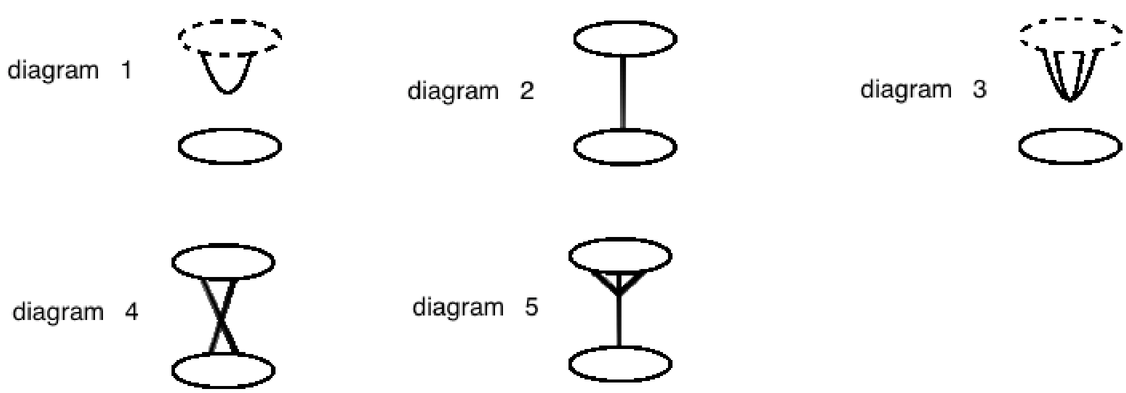

Figure 1: In each drawing, the top solid circle is the single-trace in the PCGG, the top dashed circle is the full PCGG, the bottom circle is the operator. Free contractions are omitted, the line is the O vertex (= 1-loop propagator), the cross is the X+H vertex. Diagrams 2,4 have $\mathcal{G}_L$, diagrams 1,5 have $\mathcal{G}_{L+2}$, diagrams 3 have $\mathcal{G}_{L+4}$.

Diagram 1

The case $L=0$ for finite N is straightforward from the PCGG before the multitrace expansion.

$$\mathrm{diagram}_1 \tag{2.5}$$

$$
\begin{aligned}
&\overset{L=0}{=} \left\langle \mathcal{G}_2 V_O \right\rangle_{\mathrm{LO}} \\
=&\frac{1}{(N!)^2}\binom{N}{1}^2 (N-1)!\left(\frac{1}{2}\frac{\lambda}{N} I_{12} u_{Z\tilde{Z}}\right)^{N-1} \left((N-1)!\right)^2 \delta^{i_N}_{l_N}\delta^{j_N}_{k_N} \left\langle Z_{i_N j_N} \bar{Z}_{k_N l_N} V_O \right\rangle_{\mathrm{LO}} \\
=&\frac{1}{(N!)^2}\binom{N}{1}^2 (N-1)!\left(\frac{1}{2}\frac{\lambda}{N} I_{12} u_{Z\tilde{Z}}\right)^{N-1} \left((N-1)!\right)^2 \left[-\frac{\lambda^2}{2N} u_{Z\bar{Z}} \left(N^2-1\right)\left(Y_{112}+Y_{122}\right)\right] \\
=&-\frac{\left(N^2-1\right)N!}{2^N N^{N+1}} \lambda^{N+1} I_{12}^{N-1} \left(Y_{112}+Y_{122}\right) u_{Z\bar{Z}}^N
\end{aligned}
$$

We used the following.

$$
\left\langle Z^{a_1}(x_1)\bar{Z}^{a_2}(x_2) V_O \right\rangle_{\mathrm{LO}} T^{a_1}_{ij} T^{a_2}_{ji} = -\frac{\lambda^2}{2N} u_{Z\bar{Z}} \left(N^2-1\right)\left(Y_{112}+Y_{122}\right) \tag{2.6}
$$

In the case $L \neq 0$ we begin with the single-trace part of the PCGG.

$$
\begin{aligned}
&\mathrm{diagram}_{1a} \\
&= \left\langle \mathcal{G}_{L+2} \mathcal{O}_3 V_O \right\rangle_{\mathrm{LO}} \\
&\overset{N\geqq 1}{=} \frac{(-)^{L/2}\left(N-\frac{L}{2}-1\right)!}{2^{N-L/2-2}(L+2)} \frac{\lambda^{N-L/2-1}}{N^{N-L/2-1}} I_{12}^{N-L/2-1} u_{Z\bar{Z}}^{N-L/2-1} \left\langle \mathrm{tr}\left(Z\bar{Z}\right)^{L/2+1} \mathcal{O}_3 V_O \right\rangle_{\mathrm{LO}}
\end{aligned} \tag{2.7}
$$

We underline the number $(L+2)\times L$ of ways to choose the scalars to contract.

$$
\begin{aligned}
=&\frac{(-)^{L/2}\left(N-\frac{L}{2}-1\right)!}{2^{N-L/2-2}(L+2)} \frac{\lambda^{N-L/2-1}}{N^{N-L/2-1}} I_{12}^{N-L/2-1} u_{Z\bar{Z}}^{N-L/2-1} \\
&\underline{(L+2)}\left[-\frac{\lambda^2}{2} u_{Z\bar{Z}}\left(Y_{112}+Y_{122}\right)\right]\left[\underline{L}\left(\frac{\lambda}{N} I_{13} u_{Z\tilde{Z}}\right)^{L/2}\left(\frac{\lambda}{N} I_{23} u_{\bar{Z}\tilde{Z}}\right)^{L/2}\left(\frac{N}{2}\right)^L\right] \\
=&\frac{(-)^{L/2}\left(N-\frac{L}{2}-1\right)!}{2^{N-L/2-2}(L+2)} \frac{\lambda^{N-L/2-1}}{N^{N-L/2-1}} I_{12}^{N-L/2-1} u_{Z\bar{Z}}^{N-L/2-1} \\
&\frac{-L(L+2)}{2^{L+1}} \lambda^{L+2} I_{13}^{L/2} I_{23}^{L/2} \left(Y_{112}+Y_{122}\right) u_{Z\bar{Z}} u_{Z\tilde{Z}}^{L/2} u_{\bar{Z}\tilde{Z}}^{L/2} \\
=&-\frac{(-)^{L/2} L\left(N-\frac{L}{2}-1\right)!}{2^{N+L/2-1} N^{N-L/2-1}} \lambda^{N+L/2+1} I_{12}^{N-L/2-1} I_{13}^{L/2} I_{23}^{L/2} \left(Y_{112}+Y_{122}\right) u_{Z\bar{Z}}^{N-L/2} u_{Z\tilde{Z}}^{L/2} u_{\bar{Z}\tilde{Z}}^{L/2}
\end{aligned} \tag{2.8}
$$

We used the following.

$$
\left\langle Z^{a_1}(x_1)\bar{Z}^{a_2}(x_2) V_O \right\rangle_{\mathrm{LO}} T^{a_1}_{ij} T^{a_2}_{jl} \overset{N\geqq 1}{=} -\frac{\lambda^2}{2} u_{Z\bar{Z}} \delta_{il}\delta_{jk} \left(Y_{112}+Y_{122}\right) \tag{2.9}
$$

We don't need to compute the other diagrams.

Diagram 2

We underline the number $L\left(\frac{L}{2}+\frac{L}{2}\right)$ of ways to choose the scalars to contract.

$$
\begin{aligned}
&\mathrm{diagram}_2 \\
&= \left\langle \mathcal{G}_L \mathcal{O}_3 V_O \right\rangle_{\mathrm{LO}} \\
&\overset{N\geqq 1}{=} -\frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N-L/2-1}L} \frac{\lambda^{N-L/2}}{N^{N-L/2}} I_{12}^{N-L/2} u_{Z\bar{Z}}^{N-L/2} \left\langle \mathrm{tr}\left(Z\bar{Z}\right)^{L/2} \mathcal{O}_3 V_O \right\rangle_{\mathrm{LO}} \\
&= -\frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N-L/2-1}L} \frac{\lambda^{N-L/2}}{N^{N-L/2}} I_{12}^{N-L/2} u_{Z\bar{Z}}^{N-L/2} \\
&\quad \underline{L}\left[\underline{\frac{L}{2}}\left(\frac{\lambda}{N} I_{13} u_{Z\tilde{Z}}\right)^{L/2-1}\left(\frac{\lambda}{N} I_{23} u_{\bar{Z}\tilde{Z}}\right)^{L/2}\left(-\frac{N\lambda^2}{2}\left(Y_{113}+Y_{133}\right) u_{Z\tilde{Z}}\right)\right. \\
&\quad \left. +\underline{\frac{L}{2}}\left(\frac{\lambda}{N} I_{13} u_{Z\tilde{Z}}\right)^{L/2}\left(\frac{\lambda}{N} I_{23} u_{\bar{Z}\tilde{Z}}\right)^{L/2-1}\left(-\frac{N\lambda^2}{2}\left(Y_{223}+Y_{233}\right) u_{\bar{Z}\tilde{Z}}\right)\right] \frac{N^{L-2}}{2^{L-1}} \\
&= -\frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N-L/2-1}L} \frac{\lambda^{N-L/2}}{N^{N-L/2}} I_{12}^{N-L/2} u_{Z\bar{Z}}^{N-L/2}
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
&\frac{-L^2\lambda^{L+1}}{2^{L+1}}\left[I_{13}^{L/2-1}I_{23}^{L/2}\left(Y_{113}+Y_{133}\right)+I_{13}^{L/2}I_{23}^{L/2-1}\left(Y_{223}+Y_{233}\right)\right]u_{Z\tilde{Z}}^{L/2}u_{\bar{Z}\tilde{Z}}^{L/2}\\
=&\frac{(-)^{L/2}L\left(N-\frac{L}{2}\right)!}{2^{N+L/2}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2}I_{13}^{L/2}I_{23}^{L/2}\\
&\left[I_{13}^{-1}\left(Y_{113}+Y_{133}\right)+I_{23}^{-1}\left(Y_{223}+Y_{233}\right)\right]u_{Z\bar{Z}}^{N-L/2}u_{Z\tilde{Z}}^{L/2}u_{\bar{Z}\tilde{Z}}^{L/2}
\end{aligned}
$$

We used the following.

$$
\left\langle \mathrm{tr}\left[Z\left(x_1\right)\tilde{Z}\left(x_3\right)\right]V_O\right\rangle_{\mathrm{LO}}\overset{N\gg 1}{=}-\frac{N\lambda^2}{2}\left(Y_{113}+Y_{133}\right)u_{Z\tilde{Z}} \tag{2.11}
$$

$$
\left\langle \mathrm{tr}\left[\bar{Z}\left(x_2\right)\tilde{Z}\left(x_3\right)\right]V_O\right\rangle_{\mathrm{LO}}\overset{N\gg 1}{=}-\frac{N\lambda^2}{2}\left(Y_{223}+Y_{233}\right)u_{\bar{Z}\tilde{Z}} \tag{2.12}
$$

Diagram 3

The case $L=0$ for finite N is straightforward from the PCGG before the multitrace expansion.

$$
\begin{aligned}
&\mathrm{diagram}_3 \\
\overset{L=0}{=}&\left\langle \mathcal{G}_4 V_{X+H}\right\rangle_{\mathrm{LO}}\\
=&\frac{1}{(N!)^2}\binom{N}{2}^2(N-2)!\left(\frac{1}{2}\frac{\lambda}{N}I_{12}u_{Z\bar{Z}}\right)^{N-2}((N-2)!)^2\,\delta^{i_{N-1}i_N}_{l_{N-1}l_N}\delta^{j_{N-1}j_N}_{k_{N-1}k_N}\left\langle Z_{i_{N-1}j_{N-1}}Z_{i_Nj_N}\bar{Z}_{k_{N-1}l_{N-1}}\bar{Z}_{k_Nl_N}V_{X+H}\right\rangle_{\mathrm{LO}}\\
=&\frac{1}{(N!)^2}\binom{N}{2}^2(N-2)!\left(\frac{1}{2}\frac{\lambda}{N}I_{12}u_{Z\bar{Z}}\right)^{N-2}((N-2)!)^2\left[\frac{\lambda^3}{N^3}u_{Z\bar{Z}}^2\left(X_{1122}+I_{12}^2F_{12,12}\right)\frac{N\left(N-1\right)\left(N^2-1\right)}{2}\right]\\
=&\frac{\left(N^2-1\right)N!}{2^{N-1}N^{N+1}}\lambda^{N+1}I_{12}^{N-2}\left(X_{1122}+I_{12}^2F_{12,12}\right)u_{Z\bar{Z}}^N
\end{aligned} \tag{2.13}
$$

We used the following.

$$
\begin{aligned}
&\delta^{i_{N-1}i_N}_{l_{N-1}l_N}\delta^{j_{N-1}j_N}_{k_{N-1}k_N}\left\langle Z_{i_{N-1}j_{N-1}}Z_{i_Nj_N}\bar{Z}_{k_{N-1}l_{N-1}}\bar{Z}_{k_Nl_N}V_{X+H}\right\rangle_{\mathrm{LO}}\\
=&\delta^{i_{N-1}i_N}_{l_{N-1}l_N}\delta^{j_{N-1}j_N}_{k_{N-1}k_N}\frac{\lambda^3}{N^3}u_{Z\bar{Z}}^{N-L/2}\left(X_{1122}+I_{12}^2F_{12,12}\right)\left(f^{aa_1a_4}f^{aa_2a_3}+f^{aa_1a_3}f^{aa_2a_4}\right)T^{a_1}_{i_{N-1}j_{N-1}}T^{a_2}_{i_Nj_N}T^{a_3}_{k_{N-1}l_{N-1}}T^{a_4}_{k_Nl_N}\\
=&\frac{\lambda^3}{N^3}u_{Z\bar{Z}}^{N-L/2}\left(X_{1122}+I_{12}^2F_{12,12}\right)\left(f^{aa_1a_4}f^{aa_2a_3}+f^{aa_1a_3}f^{aa_2a_4}\right)\\
&\left[\mathrm{tr}\left(T^{a_1}T^{a_3}\right)\mathrm{tr}\left(T^{a_2}T^{a_4}\right)-\mathrm{tr}\left(T^{a_1}T^{a_4}T^{a_2}T^{a_3}\right)-\mathrm{tr}\left(T^{a_1}T^{a_3}T^{a_2}T^{a_4}\right)+\mathrm{tr}\left(T^{a_1}T^{a_4}\right)\mathrm{tr}\left(T^{a_2}T^{a_3}\right)\right]\\
=&\frac{\lambda^3}{N^3}u_{Z\bar{Z}}^{N-L/2}\left(X_{1122}+I_{12}^2F_{12,12}\right)\\
&\left[\left(-\frac{1}{2^2}N\left(N^2-1\right)+0\right)-\left(-N^2\frac{N^2-1}{8}-N^2\frac{N^2-1}{8}\right)-\left(-N^2\frac{N^2-1}{8}-N^2\frac{N^2-1}{8}\right)+\left(0-\frac{1}{2^2}N\left(N^2-1\right)\right)\right]\\
=&\frac{\lambda^3}{N^3}u_{Z\bar{Z}}^{N-L/2}\left(X_{1122}+I_{12}^2F_{12,12}\right)\frac{N\left(N-1\right)\left(N^2-1\right)}{2}
\end{aligned} \tag{2.14}
$$

Note that 1-loop of the 2-point function of PCGG is zero: $\sum_{i=1,3}\mathrm{diagram}_i\overset{L=0}{\propto}X_{1122}+I_{12}^2F_{12,12}-2I_{12}\left(Y_{112}+Y_{122}\right)=0$ at finite N.

We distinguish 6 types of diagrams in the case $L\neq 0$.

$$
\begin{aligned}
&\mathrm{diagram}_{3a}\\
=&\left\langle \mathcal{G}_{L+4}\mathcal{O}_3V_{X+H}\right\rangle_{\mathrm{LO}}\\
\overset{N\gg 1}{=}&-\frac{(-)^{L/2}\left(N-\frac{L}{2}-2\right)!}{2^{N-L/2-3}\left(L+4\right)}\frac{\lambda^{N-L/2-2}}{N^{N-L/2-2}}I_{12}^{N-L/2-2}u_{Z\bar{Z}}^{N-L/2-2}\left\langle \mathrm{tr}\left(Z\bar{Z}\right)^{L/2+2}\mathcal{O}_3V_{X+H}\right\rangle_{\mathrm{LO}}
\end{aligned} \tag{2.15}
$$

We underline the number $(L+4)\times L$ of ways to choose the scalars to contract.

$$
\begin{aligned}
=&-\frac{(-)^{L/2}\left(N-\frac{L}{2}-2\right)!}{2^{N-L/2-3}\left(L+4\right)}\frac{\lambda^{N-L/2-2}}{N^{N-L/2-2}}I_{12}^{N-L/2-2}u_{Z\bar{Z}}^{N-L/2-2}\\
&\underline{(L+4)}\left[\frac{\lambda^3N}{4}u_{Z\bar{Z}}^2\left(-X_{1122}-I_{12}^2F_{12,12}\right)\right]\left[\underline{L}\left(\frac{\lambda}{N}I_{13}u_{Z\tilde{Z}}\right)^{L/2}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}\tilde{Z}}\right)^{L/2}\left(\frac{N}{2}\right)^L\right]
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
=& -\frac{(-)^{L/2}\left(N-\frac{L}{2}-2\right)!}{2^{N-L/2-3}\left(L+4\right)}\frac{\lambda^{N-L/2-2}}{N^{N-L/2-2}}I_{12}^{N-L/2-2}u_{Z\bar{Z}}^{N-L/2-2}\\
&\frac{L\left(L+4\right)}{2^{L+2}}\lambda^{L+3}\left(-X_{1122}-I_{12}^{2}F_{12,12}\right)I_{13}^{L/2}I_{23}^{L/2}u_{Z\bar{Z}}^{2}u_{Z\tilde{Z}}^{L/2}u_{\bar{Z}\tilde{Z}}^{L/2}\\
=&\frac{(-)^{L/2}L\left(N-\frac{L}{2}-2\right)!}{2^{N+L/2-1}N^{N-L/2-2}}\lambda^{N+L/2+1}I_{12}^{N-L/2-2}I_{13}^{L/2}I_{23}^{L/2}\left(X_{1122}+I_{12}^{2}F_{12,12}\right)u_{Z\bar{Z}}^{N-L/2}u_{Z\tilde{Z}}^{L/2}u_{\bar{Z}\tilde{Z}}^{L/2}
\end{aligned}
$$

We used the following (see diagram 5).

$$
\left\langle \mathrm{tr}\left[Z\left(x_{1}\right)\bar{Z}\left(x_{2}\right)Z\left(x_{1}\right)\bar{Z}\left(x_{2}\right)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\overset{N\gg1}{=}\frac{\lambda^{3}N}{4}u_{Z\bar{Z}}^{2}\left(-X_{1122}-I_{12}^{2}F_{12,12}\right) \tag{2.17}
$$

We don't need to compute the other diagrams.

Diagram 1+3

We calculate the sum of diagrams 1,3 at given L and fit. The pieces $\sim N^{-N+1}\,N!$ cancel and the pieces $\sim N^{-N}\,N!$ deliver the result

$$
\begin{aligned}
&\sum_{i=1,3}\mathrm{diagram}_{i} \qquad (2.18)\\
=&\frac{1}{(N!)^{2}}\binom{N}{\frac{L}{2}+1}^{2}\left(N-\frac{L}{2}-1\right)!\left(\frac{1}{2}\frac{\lambda}{N}I_{12}u_{Z\bar{Z}}\right)^{N-L/2-1}\left(\left(N-\frac{L}{2}-1\right)!\right)^{2}\\
&\frac{(-)^{L/2}L\left(\left(\frac{L}{2}+1\right)!\right)^{2}}{2^{L+1}}\lambda^{L+2}I_{13}^{L/2}I_{23}^{L/2}\left(Y_{112}+Y_{122}\right)u_{Z\bar{Z}}u_{Z\tilde{Z}}^{L/2}u_{\bar{Z}\tilde{Z}}^{L/2}\\
=&\frac{(-)^{L/2}L\left(N-\frac{L}{2}-1\right)!}{2^{N+L/2}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2-1}}I_{12}^{N-L/2-1}I_{13}^{L/2}I_{23}^{L/2}\left(Y_{112}+Y_{122}\right)u_{Z\bar{Z}}^{N-L/2}u_{Z\tilde{Z}}^{L/2}u_{\bar{Z}\tilde{Z}}^{L/2}\\
=&\frac{(-)^{L/2}L\left(N-\frac{L}{2}\right)!}{2^{N+L/2}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2-1}I_{13}^{L/2}I_{23}^{L/2}\left(Y_{112}+Y_{122}\right)u_{Z\bar{Z}}^{N-L/2}u_{Z\tilde{Z}}^{L/2}u_{\bar{Z}\tilde{Z}}^{L/2}\,.
\end{aligned}
$$

We used the relation $X_{1122}+I_{12}^{2}F_{12,12}-2I_{12}\left(Y_{112}+Y_{122}\right)=0$ to eliminate X_{1122}. In turn $F_{12,12}$ disappears although it could be here. I incidentally notice that

$$
0=\mathrm{diagram}_{3a}+2\mathrm{diagram}_{1a} \tag{2.19}
$$

$$
0=\mathrm{diagram}_{1a}+\mathrm{diagram}_{3a}+2\left(\mathrm{diagram}_{1\neq a}+\mathrm{diagram}_{3\neq a}\right) \tag{2.20}
$$

$$
\sum_{i=1,3}\mathrm{diagram}_{i}=\frac{1}{2}\left(\mathrm{diagram}_{1a}+\mathrm{diagram}_{3a}\right)=-\frac{1}{2}\mathrm{diagram}_{1a}\,. \tag{2.21}
$$

Instead of these empirical observations at low L combined with an extrapolation at any L, we can use the knowledge of the 1-loop PCGG and calculate analytically

$$
\begin{aligned}
&\sum_{i=1,3}\mathrm{diagram}_{i} \qquad (2.22)\\
=&\left\langle \mathcal{G}_{L}^{\mathrm{NLO}}\mathcal{O}_{3}\right\rangle_{\mathrm{LO}}\\
\overset{N\gg1}{=}&-\frac{\lambda L}{2}I_{12}^{-1}\left(Y_{112}+Y_{122}\right)\left\langle \mathcal{G}_{L}^{\mathrm{LO}}\mathcal{O}_{3}\right\rangle_{\mathrm{LO}}\\
=&\frac{(-)^{L/2}L\left(N-\frac{L}{2}\right)!}{2^{N+L/2}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2-1}I_{13}^{L/2}I_{23}^{L/2}\left(Y_{112}+Y_{122}\right)u_{Z\bar{Z}}^{N-L/2}u_{Z\tilde{Z}}^{L/2}u_{\bar{Z}\tilde{Z}}^{L/2}\,.
\end{aligned}
$$

Note that the equality with "$N\gg1$" uses the "approximated planar formula" for the 1-loop PCGG because the matrix M here does not contribute. The two approaches nicely agree.

Diagram 4

We underline the number $L\times L$ of ways to choose the scalars to contract.

$$
\begin{aligned}
&\mathrm{diagram}_{4} \qquad (2.23)\\
=&\left\langle \mathcal{G}_{L}\mathcal{O}_{3}V_{X+H}\right\rangle_{\mathrm{LO}}\\
\overset{N\gg1}{=}&-\frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N-L/2-1}L}\frac{\lambda^{N-L/2}}{N^{N-L/2}}I_{12}^{N-L/2}u_{Z\bar{Z}}^{N-L/2}\left\langle \mathrm{tr}\left(Z\bar{Z}\right)^{L/2}\mathcal{O}_{3}V_{X+H}\right\rangle_{\mathrm{LO}}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N-L/2-1}L}\frac{\lambda^{N-L/2}}{N^{N-L/2}}I_{12}^{N-L/2}u_{Z\bar{Z}}^{N-L/2}\\
&\quad \underline{L^2}\left(\frac{\lambda}{N}I_{13}u_{Z\tilde{Z}}\right)^{L/2-1}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}\tilde{Z}}\right)^{L/2-1}\frac{N^{L-3}}{2^{L-2}}\\
&\quad \frac{\lambda^3 N}{8}u_{Z\tilde{Z}}u_{\bar{Z}\tilde{Z}}\left(X_{1233}+I_{13}I_{23}F_{13,23}\right)\\
&= -\frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N-L/2-1}L}\frac{\lambda^{N-L/2}}{N^{N-L/2}}I_{12}^{N-L/2}u_{Z\bar{Z}}^{N-L/2}\\
&\quad L^2\frac{\lambda^{L+1}}{2^{L+1}}I_{13}^{L/2-1}I_{23}^{L/2-1}\left(X_{1233}+I_{13}I_{23}F_{13,23}\right)u_{Z\tilde{Z}}^{L/2}u_{\bar{Z}\tilde{Z}}^{L/2}\\
&= -\frac{(-)^{L/2}L\left(N-\frac{L}{2}\right)!}{2^{N+L/2}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2}I_{13}^{L/2-1}I_{23}^{L/2-1}\left(X_{1233}+I_{13}I_{23}F_{13,23}\right)u_{Z\bar{Z}}^{N-L/2}u_{Z\tilde{Z}}^{L/2}u_{\bar{Z}\tilde{Z}}^{L/2}
\end{aligned}
$$

We used the following.

$$
\begin{aligned}
&\left\langle \mathrm{tr}\left[Z(x_1)\bar{Z}(x_2)\tilde{Z}(x_3)\tilde{Z}(x_3)\right]V_{X+H}\right\rangle_{\mathrm{LO}} \\
&=\frac{\lambda^3}{N^3}u_{Z\tilde{Z}}u_{\bar{Z}\tilde{Z}}\mathrm{tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right)\\
&\quad\left[X_{1233}\left(f^{aa_1a_4}f^{aa_2a_3}+f^{aa_1a_3}f^{aa_2a_4}\right)+f^{ba_1a_3}f^{ba_2a_4}I_{13}I_{23}F_{13,23}+f^{ba_1a_4}f^{ba_2a_3}I_{13}I_{23}F_{13,23}\right]\\
&=\frac{\lambda^3}{N^3}u_{Z\tilde{Z}}u_{\bar{Z}\tilde{Z}}N^2\frac{N^2-1}{8}\left(X_{1233}+I_{13}I_{23}F_{13,23}\right)\\
&\overset{N\gg1}{=}\frac{N\lambda^3}{8}u_{Z\tilde{Z}}u_{\bar{Z}\tilde{Z}}\left(X_{1233}+I_{13}I_{23}F_{13,23}\right)
\end{aligned}
\tag{2.24}
$$

Diagram 5

The vertex can join either $Z\bar{Z}Z\tilde{Z}$ or $\bar{Z}Z\bar{Z}\tilde{Z}$. We underline the number $L\left(\frac{L+2}{2}+\frac{L+2}{2}\right)$ of ways to choose the scalars to contract.

$$
\begin{aligned}
&\mathrm{diagram}_5\\
&=\left\langle \mathcal{G}_{L+2}\mathcal{O}_3V_{X+H}\right\rangle_{\mathrm{LO}}\\
&\overset{N\gg1}{=}\frac{(-)^{L/2}\left(N-\frac{L}{2}-1\right)!}{2^{N-L/2-2}(L+2)}\frac{\lambda^{N-L/2-1}}{N^{N-L/2-1}}I_{12}^{N-L/2-1}u_{Z\bar{Z}}^{N-L/2-1}\left\langle \mathrm{tr}\left(Z\bar{Z}\right)^{L/2+1}\mathcal{O}_3V_{X+H}\right\rangle_{\mathrm{LO}}\\
&=\frac{(-)^{L/2}\left(N-\frac{L}{2}-1\right)!}{2^{N-L/2-2}(L+2)}\frac{\lambda^{N-L/2-1}}{N^{N-L/2-1}}I_{12}^{N-L/2-1}u_{Z\bar{Z}}^{N-L/2-1}\\
&\quad\underline{L}\left[\underline{\frac{L+2}{2}}\left(\frac{\lambda}{N}I_{13}u_{Z\tilde{Z}}\right)^{L/2-1}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}\tilde{Z}}\right)^{L/2}\frac{N^{L-2}}{2^{L-1}}\frac{\lambda^3N}{4}u_{Z\bar{Z}}u_{Z\tilde{Z}}\left(-X_{1123}-I_{12}I_{13}F_{12,13}\right)\right.\\
&\quad\left.+\underline{\frac{L+2}{2}}\left(\frac{\lambda}{N}I_{13}u_{Z\tilde{Z}}\right)^{L/2}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}\tilde{Z}}\right)^{L/2-1}\frac{N^{L-2}}{2^{L-1}}\frac{\lambda^3N}{4}u_{Z\bar{Z}}u_{\bar{Z}\tilde{Z}}\left(-X_{1223}-I_{12}I_{23}F_{21,23}\right)\right]\\
&=\frac{(-)^{L/2}\left(N-\frac{L}{2}-1\right)!}{2^{N-L/2-2}(L+2)}\frac{\lambda^{N-L/2-1}}{N^{N-L/2-1}}I_{12}^{N-L/2-1}u_{Z\bar{Z}}^{N-L/2-1}\\
&\quad\frac{L(L+2)\lambda^{L+2}}{2^{L+2}}u_{Z\tilde{Z}}^{L/2}u_{\bar{Z}\tilde{Z}}^{L/2}u_{Z\bar{Z}}\left[I_{13}^{L/2-1}I_{23}^{L/2}\left(-X_{1123}-I_{12}I_{13}F_{12,13}\right)+(1\leftrightarrow2)\right]\\
&=\frac{(-)^{L/2}L\left(N-\frac{L}{2}-1\right)!}{2^{N+L/2}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2-1}}I_{12}^{N-L/2-1}u_{Z\bar{Z}}^{N-L/2}u_{Z\tilde{Z}}^{L/2}u_{\bar{Z}\tilde{Z}}^{L/2}\\
&\quad\left[I_{13}^{L/2-1}I_{23}^{L/2}\left(-X_{1123}-I_{12}I_{13}F_{12,13}\right)+(1\leftrightarrow2)\right]\\
&=\frac{(-)^{L/2}L\left(N-\frac{L}{2}\right)!}{2^{N+L/2}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2-1}u_{Z\bar{Z}}^{N-L/2}u_{Z\tilde{Z}}^{L/2}u_{\bar{Z}\tilde{Z}}^{L/2}\\
&\quad\left[I_{13}^{L/2-1}I_{23}^{L/2}\left(-X_{1123}-I_{12}I_{13}F_{12,13}\right)+(1\leftrightarrow2)\right]
\end{aligned}
\tag{2.25}
$$

We used the following.

$$
\left\langle \mathrm{tr}\left[Z(x_1)\bar{Z}(x_2)Z(x_1)\tilde{Z}(x_3)\right]V_{X+H}\right\rangle_{\mathrm{LO}}
\tag{2.26}
$$

$$
\begin{aligned}
=&\frac{\lambda^3}{N^3}u_{Z\bar{Z}}u_{Z\tilde{Z}}\mathrm{tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right)\\
&\left[X_{1123}\left(-f^{aa_1a_4}f^{aa_2a_3}+f^{aa_1a_2}f^{aa_3a_4}\right)+f^{aa_1a_2}f^{aa_3a_4}I_{12}I_{13}F_{12,13}+f^{aa_1a_4}f^{aa_2a_3}I_{12}I_{13}F_{13,21}\right]\\
=&\frac{\lambda^3}{N^3}u_{Z\bar{Z}}u_{Z\tilde{Z}}N^2\frac{N^2-1}{8}\left(-2X_{1123}-I_{12}I_{13}F_{12,13}+I_{12}I_{13}F_{13,21}\right)\\
\overset{N\gg1}{=}&\frac{\lambda^3N}{8}u_{Z\bar{Z}}u_{Z\tilde{Z}}\left(-2X_{1123}-I_{12}I_{13}F_{12,13}+I_{12}I_{13}F_{13,21}\right)\\
=&\frac{\lambda^3N}{4}u_{Z\bar{Z}}u_{Z\tilde{Z}}\left(-X_{1123}-I_{12}I_{13}F_{12,13}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\left\langle\mathrm{tr}\left[\bar{Z}(x_2)\,Z(x_1)\,\bar{Z}(x_2)\,\tilde{Z}(x_3)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\
\overset{N\gg1}{=}&\frac{\lambda^3N}{4}u_{Z\bar{Z}}u_{Z\tilde{Z}}\left(-X_{1223}-I_{12}I_{23}F_{21,23}\right)
\end{aligned}
\tag{2.27}
$$

Sum diagrams

I put a weight to keep diagrams apart, then set to 1.

$$
\begin{aligned}
&\sum_{i=1}^{5}D_i\,\mathrm{diagram}_i\\
=&\frac{(-)^{L/2}L\left(N-\frac{L}{2}\right)!}{2^{N+L/2}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2}I_{13}^{L/2}I_{23}^{L/2}u_{Z\bar{Z}}^{N-L/2}u_{Z\tilde{Z}}^{L/2}u_{\bar{Z}\tilde{Z}}^{L/2}\\
&\left\{D_{1+3}I_{12}^{-1}\left(Y_{112}+Y_{122}\right)+D_2\left[I_{13}^{-1}\left(Y_{113}+Y_{133}\right)+I_{23}^{-1}\left(Y_{223}+Y_{233}\right)\right]\right.\\
&\left.-D_4I_{13}^{-1}I_{23}^{-1}\left(X_{1233}+I_{13}I_{23}F_{13,23}\right)+D_5I_{12}^{-1}\left[I_{13}^{-1}\left(-X_{1123}-I_{12}I_{13}F_{12,13}\right)+I_{23}^{-1}\left(-X_{1223}-I_{12}I_{23}F_{21,23}\right)\right]\right\}
\end{aligned}
\tag{2.28}
$$

$$
\begin{aligned}
&\sum_{i=1}^{5}\mathrm{diagram}_i\\
=&0
\end{aligned}
\tag{2.29}
$$

2.3 Tree level with excited state

$$
\begin{aligned}
=&\left\langle\mathcal{G}_L\mathcal{O}_3^{I_1\ldots I_L}\right\rangle_{\mathrm{LO}}\\
\overset{N\gg1}{=}&\frac{(-)^{L/2+1}}{\frac{L}{2}!}\left(\frac{L}{2}-1\right)!\left(N-\frac{L}{2}\right)!\left(\frac{1}{2}\frac{\lambda}{N}I_{12}u_{Z\bar{Z}}\right)^{N-L/2}\\
&\left(\frac{\lambda}{N}I_{13}\right)^{L/2}\left(\frac{\lambda}{N}I_{23}\right)^{L/2}\left(\frac{N}{2}\right)^{L}\left(\frac{L}{2}u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+\frac{L}{2}u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
=&\frac{(-)^{L/2+1}\left(N-\frac{L}{2}\right)!}{2^{N+L/2}N^{N-L/2}}\lambda^{N+L/2}I_{12}^{N-L/2}I_{13}^{L/2}I_{23}^{L/2}u_{Z\bar{Z}}^{N-L/2}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)
\end{aligned}
\tag{2.30}
$$

2.4 One loop with excited state

Diagram 1

We calculate only the diagram involving the single-trace part of the PCGG.

$$
\begin{aligned}
&\mathrm{diagram}_{1a}\\
=&\left\langle\mathcal{G}_{L+2}\mathcal{O}_3V_O\right\rangle_{\mathrm{LO}}\\
\overset{N\gg1}{=}&\frac{(-)^{L/2}\left(N-\frac{L}{2}-1\right)!}{2^{N-L/2-2}(L+2)}\frac{\lambda^{N-L/2-1}}{N^{N-L/2-1}}I_{12}^{N-L/2-1}u_{Z\bar{Z}}^{N-L/2-1}\left\langle\mathrm{tr}\left(Z\bar{Z}\right)^{L/2+1}\mathcal{O}_3V_O\right\rangle_{\mathrm{LO}}\\
=&\frac{(-)^{L/2}\left(N-\frac{L}{2}-1\right)!}{2^{N-L/2-2}(L+2)}\frac{\lambda^{N-L/2-1}}{N^{N-L/2-1}}I_{12}^{N-L/2-1}u_{Z\bar{Z}}^{N-L/2-1}\\
&\left[(L+2)\left(-\frac{N\lambda^2}{2}\left(Y_{112}+Y_{122}\right)u_{Z\bar{Z}}\right)\left(\frac{\lambda}{N}I_{13}\right)^{L/2}\left(\frac{\lambda}{N}I_{23}\right)^{L/2}\left(\frac{L}{2}u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+\frac{L}{2}u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\frac{N^{L-1}}{2^L}\right]
\end{aligned}
\tag{2.31}
$$

$$
\begin{aligned}
=&\frac{(-)^{L/2}\left(N-\frac{L}{2}-1\right)!}{2^{N-L/2-2}\,(L+2)}\frac{\lambda^{N-L/2-1}}{N^{N-L/2-1}}I_{12}^{N-L/2-1}u_{Z\bar{Z}}^{N-L/2-1}\\
&\frac{-L\,(L+2)}{2^{L+2}}\lambda^{L+2}I_{13}^{L/2}I_{23}^{L/2}\left(Y_{112}+Y_{122}\right)u_{Z\bar{Z}}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
=&-\frac{(-)^{L/2}\,L\left(N-\frac{L}{2}-1\right)!}{2^{N+L/2}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2-1}}I_{12}^{N-L/2-1}I_{13}^{L/2}I_{23}^{L/2}\left(Y_{112}+Y_{122}\right)u_{Z\bar{Z}}^{N-L/2}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
=&-\frac{(-)^{L/2}\,L\left(N-\frac{L}{2}\right)!}{2^{N+L/2}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2-1}I_{13}^{L/2}I_{23}^{L/2}\left(Y_{112}+Y_{122}\right)u_{Z\bar{Z}}^{N-L/2}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)
\end{aligned}
$$

We used the following.

$$
\left\langle Z^{a_1}(x_1)\,\bar{Z}^{a_2}(x_2)\,V_O\right\rangle_{\mathrm{LO}}\mathrm{tr}\left(T^{a_1}T^{a_2}\right)\overset{N\gg1}{=}-\frac{N\lambda^2}{2}u_{Z\bar{Z}}\left(Y_{112}+Y_{122}\right) \tag{2.32}
$$

Diagram 2

$$
\begin{aligned}
&\mathrm{diagram}_2\\
=&\left\langle\mathcal{G}_L\mathcal{O}_3^{I_1\ldots I_L}V_O\right\rangle_{\mathrm{LO}}\\
\overset{N\gg1}{=}&-\frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N-L/2-1}L}\frac{\lambda^{N-L/2}}{N^{N-L/2}}I_{12}^{N-L/2}u_{Z\bar{Z}}^{N-L/2}\left\langle\mathrm{tr}\left(Z\bar{Z}\right)^{L/2}\mathcal{O}_3V_O\right\rangle_{\mathrm{LO}}\\
=&-\frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N-L/2-1}L}\frac{\lambda^{N-L/2}}{N^{N-L/2}}I_{12}^{N-L/2}u_{Z\bar{Z}}^{N-L/2}\\
&\Bigg[\sum_{k\ \mathrm{odd}}\left(\frac{\lambda}{N}I_{13}\right)^{L/2-1}\left(\frac{\lambda}{N}I_{23}\right)^{L/2}\frac{L}{2}\left(-\frac{N\lambda^2}{2}\left(Y_{113}+Y_{133}\right)\right)\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_k}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}\right)\\
&+\sum_{k\ \mathrm{even}}\left(\frac{\lambda}{N}I_{13}\right)^{L/2}\left(\frac{\lambda}{N}I_{23}\right)^{L/2-1}\frac{L}{2}\left(-\frac{N\lambda^2}{2}\left(Y_{223}+Y_{233}\right)\right)\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_{\bar{Z}}^{I_k}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}\right)\\
&+\sum_{k\ \mathrm{odd}}\left(\frac{\lambda}{N}I_{13}\right)^{L/2-1}\left(\frac{\lambda}{N}I_{23}\right)^{L/2}\frac{L}{2}\left(-\frac{N\lambda^2}{2}\left(Y_{113}+Y_{133}\right)\right)\left(u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_Z^{I_k}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
&+\sum_{k\ \mathrm{even}}\left(\frac{\lambda}{N}I_{13}\right)^{L/2}\left(\frac{\lambda}{N}I_{23}\right)^{L/2-1}\frac{L}{2}\left(-\frac{N\lambda^2}{2}\left(Y_{223}+Y_{233}\right)\right)\left(u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_k}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\Bigg]\frac{N^{L-2}}{2^{L-1}}\\
=&-\frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N-L/2-1}L}\frac{\lambda^{N-L/2}}{N^{N-L/2}}I_{12}^{N-L/2}u_{Z\bar{Z}}^{N-L/2}\\
&\frac{-L^2\lambda^{L+1}}{2^{L+2}}I_{13}^{L/2}I_{23}^{L/2}\left[I_{13}^{-1}\left(Y_{113}+Y_{133}\right)+I_{23}^{-1}\left(Y_{223}+Y_{233}\right)\right]\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
=&\frac{(-)^{L/2}\,L\left(N-\frac{L}{2}\right)!}{2^{N+L/2+1}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2}I_{13}^{L/2}I_{23}^{L/2}u_{Z\bar{Z}}^{N-L/2}\\
&\left[I_{13}^{-1}\left(Y_{113}+Y_{133}\right)+I_{23}^{-1}\left(Y_{223}+Y_{233}\right)\right]\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)
\end{aligned} \tag{2.33}
$$

Diagram 3

We calculate only the diagram involving the single-trace part of the PCGG.

$$
\begin{aligned}
&\mathrm{diagram}_{3a}\\
=&\left\langle\mathcal{G}_{L+4}\mathcal{O}_3^{I_1\ldots I_L}V_{X+H}\right\rangle_{\mathrm{LO}}\\
\overset{N\gg1}{=}&-\frac{(-)^{L/2}\left(N-\frac{L}{2}-2\right)!}{2^{N-L/2-3}\,(L+4)}\frac{\lambda^{N-L/2-2}}{N^{N-L/2-2}}I_{12}^{N-L/2-2}u_{Z\bar{Z}}^{N-L/2-2}\left\langle\mathrm{tr}\left(Z\bar{Z}\right)^{L/2+2}\mathcal{O}_3V_{X+H}\right\rangle_{\mathrm{LO}}\\
=&-\frac{(-)^{L/2}\left(N-\frac{L}{2}-2\right)!}{2^{N-L/2-3}\,(L+4)}\frac{\lambda^{N-L/2-2}}{N^{N-L/2-2}}I_{12}^{N-L/2-2}u_{Z\bar{Z}}^{N-L/2-2}\\
&\left[(L+4)\left(\frac{\lambda^3N}{4}u_{Z\bar{Z}}^2\left(-X_{1122}-I_{12}^2F_{12,12}\right)\right)\left(\frac{\lambda}{N}I_{13}\right)^{L/2}\left(\frac{\lambda}{N}I_{23}\right)^{L/2}\left(\frac{L}{2}u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+\frac{L}{2}u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\frac{N^{L-1}}{2^L}\right]
\end{aligned} \tag{2.34}
$$

$$
\begin{aligned}
&= -\frac{(-)^{L/2}\left(N-\frac{L}{2}-2\right)!}{2^{N-L/2-3}(L+4)}\frac{\lambda^{N-L/2-2}}{N^{N-L/2-2}}I_{12}^{N-L/2-2}u_{Z\bar{Z}}^{N-L/2-2}\\
&\quad \frac{-L(L+4)}{2^{L+3}}\lambda^{L+3}I_{13}^{L/2}I_{23}^{L/2}u_{Z\bar{Z}}^{2}\left(X_{1122}+I_{12}^{2}F_{12,12}\right)\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
&=\frac{(-)^{L/2}L\left(N-\frac{L}{2}-2\right)!}{2^{N+L/2}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2-2}}I_{12}^{N-L/2-2}I_{13}^{L/2}I_{23}^{L/2}\left(X_{1122}+I_{12}^{2}F_{12,12}\right)u_{Z\bar{Z}}^{N-L/2}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
&=\frac{(-)^{L/2}L\left(N-\frac{L}{2}\right)!}{2^{N+L/2}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2-2}I_{13}^{L/2}I_{23}^{L/2}\left(X_{1122}+I_{12}^{2}F_{12,12}\right)u_{Z\bar{Z}}^{N-L/2}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)
\end{aligned}
$$

We used the following.

$$
\begin{aligned}
&\left\langle \mathrm{tr}\left[Z(x_1)\bar{Z}(x_2)Z(x_1)\bar{Z}(x_2)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\
=&\frac{\lambda^3}{N^3}u_{Z\bar{Z}}^{2}\mathrm{tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right)\left[-X_{1122}\left(f^{aa_1a_3}f^{aa_2a_4}+f^{aa_1a_4}f^{aa_2a_3}\right)\right.\\
&\left.+X_{1122}\left(f^{aa_1a_2}f^{aa_3a_4}+f^{aa_1a_3}f^{aa_2a_4}\right)+f^{aa_1a_2}f^{aa_3a_4}I_{12}^{2}F_{12,12}+f^{aa_1a_4}f^{aa_2a_3}I_{12}^{2}F_{12,21}\right]\\
=&\frac{\lambda^3}{N^3}u_{Z\bar{Z}}^{2}N^2\frac{N^2-1}{8}\left(-2X_{1122}-I_{12}^{2}F_{12,12}-I_{12}^{2}F_{12,12}\right)\\
\overset{N\gg 1}{=}&\frac{\lambda^3 N}{4}u_{Z\bar{Z}}^{2}\left(-X_{1122}-I_{12}^{2}F_{12,12}\right)
\end{aligned}
\tag{2.35}
$$

Diagram 1+3

We calculate the sum of diagrams 1,3 at given L,M and fit. The result is similar to that with $M=0$ up to a 1/2 and R-symmetry factors.

$$
\begin{aligned}
&\sum_{i=1,3}\mathrm{diagram}_i\\
=&\frac{1}{(N!)^2}\binom{N}{\frac{L}{2}+1}^2\left(N-\frac{L}{2}-1\right)!\left(\frac{1}{2}\frac{\lambda}{N}I_{12}u_{Z\bar{Z}}\right)^{N-L/2-1}\left(\left(N-\frac{L}{2}-1\right)!\right)^2\\
&\frac{(-)^{L/2}L\left(\left(\frac{L}{2}+1\right)!\right)^2}{2^{L+2}}\lambda^{L+2}I_{13}^{L/2}I_{23}^{L/2}\left(Y_{112}+Y_{122}\right)u_{Z\bar{Z}}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
=&\frac{(-)^{L/2}L\left(N-\frac{L}{2}-1\right)!}{2^{N+L/2+1}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2-1}}I_{12}^{N-L/2-1}I_{13}^{L/2}I_{23}^{L/2}\left(Y_{112}+Y_{122}\right)u_{Z\bar{Z}}^{N-L/2}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
=&\frac{(-)^{L/2}L\left(N-\frac{L}{2}\right)!}{2^{N+L/2+1}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2-1}I_{13}^{L/2}I_{23}^{L/2}\left(Y_{112}+Y_{122}\right)u_{Z\bar{Z}}^{N-L/2}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)
\end{aligned}
\tag{2.36}
$$

I incidentally notice that the relations valid for the vacuum still hold:

$$
0=\mathrm{diagram}_{3a}+2\mathrm{diagram}_{1a} \tag{2.37}
$$

$$
0=2\left(\mathrm{diagram}_1+\mathrm{diagram}_3\right)-\left(\mathrm{diagram}_{1a}+\mathrm{diagram}_{3a}\right) \tag{2.38}
$$

$$
\sum_{i=1,3}\mathrm{diagram}_i=\frac{1}{2}\left(\mathrm{diagram}_{1a}+\mathrm{diagram}_{3a}\right)=-\frac{1}{2}\mathrm{diagram}_{1a}\,. \tag{2.39}
$$

Instead of these empirical observations at low L combined with an extrapolation at any L, we can use the knowledge of the 1-loop PCGG and calculate analytically

$$
\begin{aligned}
&\sum_{i=1,3}\mathrm{diagram}_i\\
&=\left\langle \mathcal{G}_L^{\mathrm{NLO}}\mathcal{O}_3^{I_1\ldots I_L}\right\rangle_{\mathrm{LO}}\\
&\overset{N\gg 1}{=}-\frac{\lambda L}{2}I_{12}^{-1}\left(Y_{112}+Y_{122}\right)\left\langle \mathcal{G}_L^{\mathrm{LO}}\mathcal{O}_3^{I_1\ldots I_L}\right\rangle_{\mathrm{LO}}\\
&=\frac{(-)^{L/2}L\left(N-\frac{L}{2}\right)!}{2^{N+L/2+1}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2-1}I_{13}^{L/2}I_{23}^{L/2}\left(Y_{112}+Y_{122}\right)u_{Z\bar{Z}}^{N-L/2}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right).
\end{aligned}
\tag{2.40}
$$

Note that the equality with "$N\gg 1$" uses the "approximated planar formula" for the 1-loop PCGG because the matrix M here does not contribute. The two approaches nicely agree.

Diagram 4

We start using the periodic identification $I_{L+n} = I_n$.

$$
\begin{aligned}
&\text{diagram}_4 \qquad (2.41)\\
&= \left\langle \mathcal{G}_L \mathcal{O}_3^{I_1\ldots I_L} V_{X+H} \right\rangle_{\text{LO}}\\
&\overset{N\gg 1}{=} - \frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N-L/2-1}L} \frac{\lambda^{N-L/2}}{N^{N-L/2}} I_{12}^{N-L/2} u_{Z\bar{Z}}^{N-L/2} \left\langle \operatorname{tr}\left(Z\bar{Z}\right)^{L/2} \mathcal{O}_3 V_{X+H} \right\rangle_{\text{LO}}\\
&= - \frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N-L/2-1}L} \frac{\lambda^{N-L/2}}{N^{N-L/2}} I_{12}^{N-L/2} u_{Z\bar{Z}}^{N-L/2}\\
&\quad \left[\sum_{k \text{ odd}} \left(\frac{\lambda}{N} I_{13}\right)^{L/2-1} \left(\frac{\lambda}{N} I_{23}\right)^{L/2-1} \frac{N^{L-3}}{2^{L-2}} \frac{L}{\underline{2}} \left(u_Z^{I_1} u_{\bar{Z}}^{I_2} \ldots u_{\bar{Z}}^{I_{k-1}} u_Z^{I_{k+2}} \ldots u_Z^{I_{L-1}} u_{\bar{Z}}^{I_L}\right)\right.\\
&\quad \frac{\lambda^3 N}{8} \left(-X_{1233} u_{Z\bar{Z}} u_{I_k I_{k+1}} + 2X_{1233} u_{ZI_{k+1}} u_{\bar{Z}I_k} - X_{1233} u_{ZI_k} u_{\bar{Z}I_{k+1}} + u_{ZI_k} u_{\bar{Z}I_{k+1}} I_{13} I_{23} F_{13,23}\right)\\
&\quad + \sum_{k \text{ even}} \left(\frac{\lambda}{N} I_{13}\right)^{L/2-1} \left(\frac{\lambda}{N} I_{23}\right)^{L/2-1} \frac{N^{L-3}}{2^{L-2}} \frac{L}{\underline{2}} \left(u_Z^{I_1} u_{\bar{Z}}^{I_2} \ldots u_Z^{I_{k-1}} u_{\bar{Z}}^{I_{k+2}} \ldots u_Z^{I_{L-1}} u_{\bar{Z}}^{I_L}\right)\\
&\quad \frac{\lambda^3 N}{8} \left(-X_{1233} u_{Z\bar{Z}} u_{I_k I_{k+1}} + 2X_{1233} u_{ZI_k} u_{\bar{Z}I_{k+1}} - X_{1233} u_{ZI_{k+1}} u_{\bar{Z}I_k} + u_{ZI_{k+1}} u_{\bar{Z}I_k} I_{13} I_{23} F_{13,23}\right)\\
&\quad + \sum_{k \text{ odd}} \left(\frac{\lambda}{N} I_{13}\right)^{L/2-1} \left(\frac{\lambda}{N} I_{23}\right)^{L/2-1} \frac{N^{L-3}}{2^{L-2}} \frac{L}{\underline{2}} \left(u_{\bar{Z}}^{I_1} u_Z^{I_2} \ldots u_Z^{I_{k-1}} u_{\bar{Z}}^{I_{k+2}} \ldots u_{\bar{Z}}^{I_{L-1}} u_Z^{I_L}\right)\\
&\quad \frac{\lambda^3 N}{8} \left(-X_{1233} u_{Z\bar{Z}} u_{I_k I_{k+1}} + 2X_{1233} u_{ZI_k} u_{\bar{Z}I_{k+1}} - X_{1233} u_{ZI_{k+1}} u_{\bar{Z}I_k} + u_{ZI_{k+1}} u_{\bar{Z}I_k} I_{13} I_{23} F_{13,23}\right)\\
&\quad + \sum_{k \text{ even}} \left(\frac{\lambda}{N} I_{13}\right)^{L/2-1} \left(\frac{\lambda}{N} I_{23}\right)^{L/2-1} \frac{N^{L-3}}{2^{L-2}} \frac{L}{\underline{2}} \left(u_{\bar{Z}}^{I_1} u_Z^{I_2} \ldots u_{\bar{Z}}^{I_{k-1}} u_Z^{I_{k+2}} \ldots u_{\bar{Z}}^{I_{L-1}} u_Z^{I_L}\right)\\
&\quad \left.\frac{\lambda^3 N}{8} \left(-X_{1233} u_{Z\bar{Z}} u_{I_k I_{k+1}} + 2X_{1233} u_{ZI_{k+1}} u_{\bar{Z}I_k} - X_{1233} u_{ZI_k} u_{\bar{Z}I_{k+1}} + u_{ZI_k} u_{\bar{Z}I_{k+1}} I_{13} I_{23} F_{13,23}\right)\right]\\
&= - \frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N-L/2-1}L} \frac{\lambda^{N-L/2}}{N^{N-L/2}} I_{12}^{N-L/2} u_{Z\bar{Z}}^{N-L/2}\\
&\quad \frac{L}{2^{L+2}} \lambda^{L+1} I_{13}^{L/2-1} I_{23}^{L/2-1}\\
&\quad \left[\sum_{k \text{ odd}} \left(u_Z^{I_1} u_{\bar{Z}}^{I_2} \ldots u_{\bar{Z}}^{I_{k-1}} u_Z^{I_{k+2}} \ldots u_Z^{I_{L-1}} u_{\bar{Z}}^{I_L}\right)\right.\\
&\quad \left(-X_{1233} u_{Z\bar{Z}} u_{I_k I_{k+1}} + 2X_{1233} u_{ZI_{k+1}} u_{\bar{Z}I_k} - X_{1233} u_{ZI_k} u_{\bar{Z}I_{k+1}} + u_{ZI_k} u_{\bar{Z}I_{k+1}} I_{13} I_{23} F_{13,23}\right)\\
&\quad + \sum_{k \text{ even}} \left(u_Z^{I_1} u_{\bar{Z}}^{I_2} \ldots u_Z^{I_{k-1}} u_{\bar{Z}}^{I_{k+2}} \ldots u_Z^{I_{L-1}} u_{\bar{Z}}^{I_L}\right)\\
&\quad \left(-X_{1233} u_{Z\bar{Z}} u_{I_k I_{k+1}} + 2X_{1233} u_{ZI_k} u_{\bar{Z}I_{k+1}} - X_{1233} u_{ZI_{k+1}} u_{\bar{Z}I_k} + u_{ZI_{k+1}} u_{\bar{Z}I_k} I_{13} I_{23} F_{13,23}\right)\\
&\quad + \sum_{k \text{ odd}} \left(u_{\bar{Z}}^{I_1} u_Z^{I_2} \ldots u_Z^{I_{k-1}} u_{\bar{Z}}^{I_{k+2}} \ldots u_{\bar{Z}}^{I_{L-1}} u_Z^{I_L}\right)\\
&\quad \left(-X_{1233} u_{Z\bar{Z}} u_{I_k I_{k+1}} + 2X_{1233} u_{ZI_k} u_{\bar{Z}I_{k+1}} - X_{1233} u_{ZI_{k+1}} u_{\bar{Z}I_k} + u_{ZI_{k+1}} u_{\bar{Z}I_k} I_{13} I_{23} F_{13,23}\right)\\
&\quad + \sum_{k \text{ even}} \left(u_{\bar{Z}}^{I_1} u_Z^{I_2} \ldots u_{\bar{Z}}^{I_{k-1}} u_Z^{I_{k+2}} \ldots u_{\bar{Z}}^{I_{L-1}} u_Z^{I_L}\right)\\
&\quad \left.\left(-X_{1233} u_{Z\bar{Z}} u_{I_k I_{k+1}} + 2X_{1233} u_{ZI_{k+1}} u_{\bar{Z}I_k} - X_{1233} u_{ZI_k} u_{\bar{Z}I_{k+1}} + u_{ZI_k} u_{\bar{Z}I_{k+1}} I_{13} I_{23} F_{13,23}\right)\right]\\
&= - \frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N+L/2+1}} \frac{\lambda^{N+L/2+1}}{N^{N-L/2}} I_{12}^{N-L/2} I_{13}^{L/2-1} I_{23}^{L/2-1} u_{Z\bar{Z}}^{N-L/2}\\
&\quad \left[\sum_{k \text{ odd}} \left(u_Z^{I_1} u_{\bar{Z}}^{I_2} \ldots u_{\bar{Z}}^{I_{k-1}} u_Z^{I_{k+2}} \ldots u_Z^{I_{L-1}} u_{\bar{Z}}^{I_L}\right)\right.\\
&\quad \left(-X_{1233} u_{Z\bar{Z}} u_{I_k I_{k+1}} + 2X_{1233} u_{ZI_{k+1}} u_{\bar{Z}I_k} - X_{1233} u_{ZI_k} u_{\bar{Z}I_{k+1}} + u_{ZI_k} u_{\bar{Z}I_{k+1}} I_{13} I_{23} F_{13,23}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k\text{ even}}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{k-1}}u_{\bar{Z}}^{I_{k+2}}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}\right)\\
&\left(-X_{1233}u_{Z\bar{Z}}u_{I_kI_{k+1}}+2X_{1233}u_{ZI_k}u_{\bar{Z}I_{k+1}}-X_{1233}u_{ZI_{k+1}}u_{\bar{Z}I_k}+u_{ZI_{k+1}}u_{\bar{Z}I_k}I_{13}I_{23}F_{13,23}\right)\\
&+\sum_{k\text{ odd}}\left(u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_Z^{I_{k-1}}u_{\bar{Z}}^{I_{k+2}}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
&\left(-X_{1233}u_{Z\bar{Z}}u_{I_kI_{k+1}}+2X_{1233}u_{ZI_k}u_{\bar{Z}I_{k+1}}-X_{1233}u_{ZI_{k+1}}u_{\bar{Z}I_k}+u_{ZI_{k+1}}u_{\bar{Z}I_k}I_{13}I_{23}F_{13,23}\right)\\
&+\sum_{k\text{ even}}\left(u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{k-1}}u_Z^{I_{k+2}}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
&\left.\left(-X_{1233}u_{Z\bar{Z}}u_{I_kI_{k+1}}+2X_{1233}u_{ZI_{k+1}}u_{\bar{Z}I_k}-X_{1233}u_{ZI_k}u_{\bar{Z}I_{k+1}}+u_{ZI_k}u_{\bar{Z}I_{k+1}}I_{13}I_{23}F_{13,23}\right)\right]
\end{aligned}
$$

We used the following.

$$
\begin{aligned}
&\left\langle \mathrm{tr}\left[Z\left(x_1\right)\bar{Z}\left(x_2\right)\phi_{I_{k+1}}\left(x_3\right)\phi_{I_k}\left(x_3\right)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\
=&\frac{\lambda^3}{N^3}\mathrm{tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right)\left[-X_{1233}\left(f^{aa_1a_3}f^{aa_2a_4}+f^{aa_1a_4}f^{aa_2a_3}\right)u_{Z\bar{Z}}u_{I_kI_{k+1}}+X_{1233}\left(f^{aa_1a_4}f^{aa_2a_3}-f^{aa_1a_2}f^{aa_3a_4}\right)u_{ZI_{k+1}}u_{\bar{Z}I_k}\right.\\
&\left.+X_{1233}\left(f^{aa_1a_2}f^{aa_3a_4}+f^{aa_1a_3}f^{aa_2a_4}\right)u_{ZI_k}u_{\bar{Z}I_{k+1}}+f^{aa_1a_3}f^{aa_2a_4}u_{ZI_{k+1}}u_{\bar{Z}I_k}I_{13}I_{23}F_{13,23}+f^{aa_1a_4}f^{aa_2a_3}u_{ZI_k}u_{\bar{Z}I_{k+1}}I_{13}I_{23}F_{13,23}\right]\\
=&\frac{\lambda^3}{N^3}N^2\frac{N^2-1}{8}\left(-X_{1233}u_{Z\bar{Z}}u_{I_kI_{k+1}}+2X_{1233}u_{ZI_{k+1}}u_{\bar{Z}I_k}-X_{1233}u_{ZI_k}u_{\bar{Z}I_{k+1}}+u_{ZI_k}u_{\bar{Z}I_{k+1}}I_{13}I_{23}F_{13,23}\right)\\
\overset{N\gg1}{=}&\frac{\lambda^3N}{8}\left(-X_{1233}u_{Z\bar{Z}}u_{I_kI_{k+1}}+2X_{1233}u_{ZI_{k+1}}u_{\bar{Z}I_k}-X_{1233}u_{ZI_k}u_{\bar{Z}I_{k+1}}+u_{ZI_k}u_{\bar{Z}I_{k+1}}I_{13}I_{23}F_{13,23}\right)
\end{aligned}
\tag{2.42}
$$

$$
\begin{aligned}
&\left\langle \mathrm{tr}\left[\bar{Z}\left(x_2\right)Z\left(x_1\right)\phi_{I_{k+1}}\left(x_3\right)\phi_{I_k}\left(x_3\right)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\
=&\frac{\lambda^3N}{8}\left(-X_{1233}u_{Z\bar{Z}}u_{I_kI_{k+1}}+2X_{1233}u_{ZI_k}u_{\bar{Z}I_{k+1}}-X_{1233}u_{ZI_{k+1}}u_{\bar{Z}I_k}+u_{ZI_{k+1}}u_{\bar{Z}I_k}I_{13}I_{23}F_{13,23}\right)
\end{aligned}
\tag{2.43}
$$

Diagram 5

$$
\begin{aligned}
&\mathrm{diagram}_5\\
=&\left\langle\mathcal{G}_{L+2}\mathcal{O}_3^{I_1\ldots I_L}V_{X+H}\right\rangle_{\mathrm{LO}}\\
\overset{N\gg1}{=}&\frac{(-)^{L/2}\left(N-\frac{L}{2}-1\right)!}{2^{N-L/2-2}\left(L+2\right)}\frac{\lambda^{N-L/2-1}}{N^{N-L/2-1}}I_{12}^{N-L/2-1}u_{Z\bar{Z}}^{N-L/2-1}\left\langle\mathrm{tr}\left(Z\bar{Z}\right)^{L/2+1}\mathcal{O}_3V_{X+H}\right\rangle_{\mathrm{LO}}\\
=&\frac{(-)^{L/2}\left(N-\frac{L}{2}-1\right)!}{2^{N-L/2-2}\left(L+2\right)}\frac{\lambda^{N-L/2-1}}{N^{N-L/2-1}}I_{12}^{N-L/2-1}u_{Z\bar{Z}}^{N-L/2-1}\\
&\left[\sum_{k\text{ odd}}\left(\frac{\lambda}{N}I_{13}\right)^{L/2-1}\left(\frac{\lambda}{N}I_{23}\right)^{L/2}\frac{N^{L-2}}{2^{L-1}}\frac{L+2}{\underline{2}}\frac{\lambda^3N}{4}u_{Z\bar{Z}}u_{ZI_k}\left(-X_{1123}-I_{12}I_{13}F_{12,13}\right)\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_{\bar{Z}}^{I_{k-1}}u_{\bar{Z}}^{I_{k+1}}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}\right)\right.\\
&+\sum_{k\text{ even}}\left(\frac{\lambda}{N}I_{13}\right)^{L/2}\left(\frac{\lambda}{N}I_{23}\right)^{L/2-1}\frac{N^{L-2}}{2^{L-1}}\frac{L+2}{\underline{2}}\frac{\lambda^3N}{4}u_{Z\bar{Z}}u_{\bar{Z}I_k}\left(-X_{1223}-I_{12}I_{23}F_{21,23}\right)\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{k-1}}u_Z^{I_{k+1}}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}\right)\\
&+\sum_{k\text{ odd}}\left(\frac{\lambda}{N}I_{13}\right)^{L/2-1}\left(\frac{\lambda}{N}I_{23}\right)^{L/2}\frac{N^{L-2}}{2^{L-1}}\frac{L+2}{\underline{2}}\frac{\lambda^3N}{4}u_{Z\bar{Z}}u_{\bar{Z}I_k}\left(-X_{1123}-I_{12}I_{13}F_{12,13}\right)\left(u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{k-1}}u_{\bar{Z}}^{I_{k+1}}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
&\left.+\sum_{k\text{ even}}\left(\frac{\lambda}{N}I_{13}\right)^{L/2}\left(\frac{\lambda}{N}I_{23}\right)^{L/2-1}\frac{N^{L-2}}{2^{L-1}}\frac{L+2}{\underline{2}}\frac{\lambda^3N}{4}u_{Z\bar{Z}}u_{ZI_k}\left(-X_{1223}-I_{12}I_{23}F_{21,23}\right)\left(u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_Z^{I_{k-1}}u_Z^{I_{k+1}}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\right]\\
=&\frac{(-)^{L/2}\left(N-\frac{L}{2}-1\right)!}{2^{N-L/2-2}\left(L+2\right)}\frac{\lambda^{N-L/2-1}}{N^{N-L/2-1}}I_{12}^{N-L/2-1}u_{Z\bar{Z}}^{N-L/2-1}\\
&\frac{-L\left(L+2\right)}{2^{L+3}}\lambda^{L+2}I_{13}^{L/2}I_{23}^{L/2}\left[I_{13}^{-1}\left(X_{1123}+I_{12}I_{13}F_{12,13}\right)+I_{23}^{-1}\left(X_{1223}+I_{12}I_{23}F_{21,23}\right)\right]\\
&u_{Z\bar{Z}}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}....u_Z^{I_{L-2}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
=&-\frac{(-)^{L/2}L\left(N-\frac{L}{2}-1\right)!}{2^{N+L/2+1}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2-1}}I_{12}^{N-L/2-1}I_{13}^{L/2}I_{23}^{L/2}
\end{aligned}
\tag{2.44}
$$

$$
\begin{aligned}
&\left[I_{13}^{-1}\left(X_{1123}+I_{12}I_{13}F_{12,13}\right)+I_{23}^{-1}\left(X_{1223}+I_{12}I_{23}F_{21,23}\right)\right]\\
&u_{Z\bar{Z}}^{N-L/2}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}....u_Z^{I_{L-2}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
=-&\frac{(-)^{L/2}L\left(N-\frac{L}{2}\right)!}{2^{N+L/2+1}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2-1}I_{13}^{L/2}I_{23}^{L/2}\\
&\left[I_{13}^{-1}\left(X_{1123}+I_{12}I_{13}F_{12,13}\right)+I_{23}^{-1}\left(X_{1223}+I_{12}I_{23}F_{21,23}\right)\right]\\
&u_{Z\bar{Z}}^{N-L/2}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}....u_Z^{I_{L-2}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)
\end{aligned}
$$

We used the following.

$$
\begin{aligned}
&\left\langle \operatorname{tr}\left[Z(x_1)\bar{Z}(x_2)Z(x_1)\phi_{I_k}(x_3)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\
=&\frac{\lambda^3}{N^3}u_{Z\bar{Z}}u_{ZI_k}\operatorname{tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right)\left[-X_{1123}\left(f^{aa_1a_3}f^{aa_2a_4}+f^{aa_1a_4}f^{aa_2a_3}\right)\right.\\
&\left.+X_{1123}\left(f^{aa_1a_2}f^{aa_3a_4}+f^{aa_1a_3}f^{aa_2a_4}\right)+f^{aa_1a_2}f^{aa_3a_4}I_{12}I_{13}F_{12,13}+f^{aa_1a_4}f^{aa_2a_3}I_{13}I_{12}F_{13,21}\right]\\
=&\frac{\lambda^3}{N^3}u_{Z\bar{Z}}u_{ZI_k}N^2\frac{N^2-1}{8}\left(-2X_{1123}-I_{12}I_{13}F_{12,13}-I_{13}I_{21}F_{13,12}\right)\\
\overset{N\gg 1}{=}&\frac{\lambda^3 N}{4}u_{Z\bar{Z}}u_{ZI_k}\left(-X_{1123}-I_{12}I_{13}F_{12,13}\right)
\end{aligned}
\tag{2.45}
$$

$$
\begin{aligned}
&\left\langle \operatorname{tr}\left[\bar{Z}(x_2)Z(x_1)\bar{Z}(x_2)\phi_{I_k}(x_3)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\
\overset{N\gg 1}{=}&\frac{\lambda^3 N}{4}u_{Z\bar{Z}}u_{\bar{Z}I_k}\left(-X_{1223}-I_{12}I_{23}F_{21,23}\right)
\end{aligned}
\tag{2.46}
$$

Sum diagrams

I put a weight to keep diagrams apart.

$$
\begin{aligned}
&\sum_{i=1}^{5}D_i\,\mathrm{diagram}_i\\
&\frac{(-)^{L/2}L\left(N-\frac{L}{2}\right)!}{2^{N+L/2+1}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2}I_{13}^{L/2}I_{23}^{L/2}u_{Z\bar{Z}}^{N-L/2}\left\{\left[D_{1+3}I_{12}^{-1}\left(Y_{112}+Y_{122}\right)+D_2I_{13}^{-1}\left(Y_{113}+Y_{133}\right)+D_2I_{23}^{-1}\left(Y_{223}+Y_{233}\right)\right.\right.\\
&\left.+D_4\left(I_{13}^{-1}I_{23}^{-1}X_{1233}-F_{13,23}\right)-D_5\left(I_{12}^{-1}I_{13}^{-1}X_{1123}+I_{12}^{-1}I_{23}^{-1}X_{1223}+F_{12,13}+F_{21,23}\right)\right]\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\\
&+\frac{D_4}{L}I_{13}^{-1}I_{23}^{-1}X_{1233}\left[\sum_{k\text{ odd}}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_{\bar{Z}}^{I_{k-1}}u_Z^{I_{k+2}}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}\right)\left(u_{Z\bar{Z}}u_{I_kI_{k+1}}-2u_{ZI_{k+1}}u_{\bar{Z}I_k}\right)\right.\\
&+\sum_{k\text{ even}}\left(u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{k-1}}u_{\bar{Z}}^{I_{k+2}}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}\right)\left(u_{Z\bar{Z}}u_{I_kI_{k+1}}-2u_{ZI_k}u_{\bar{Z}I_{k+1}}\right)\\
&+\sum_{k\text{ odd}}\left(u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_Z^{I_{k-1}}u_{\bar{Z}}^{I_{k+2}}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\left(u_{Z\bar{Z}}u_{I_kI_{k+1}}-2u_{ZI_k}u_{\bar{Z}I_{k+1}}\right)\\
&\left.\left.+\sum_{k\text{ even}}\left(u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{k-1}}u_Z^{I_{k+2}}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right)\left(u_{Z\bar{Z}}u_{I_kI_{k+1}}-2u_{ZI_{k+1}}u_{\bar{Z}I_k}\right)\right]\right\}
\end{aligned}
\tag{2.47}
$$

I set weights to 1. The identity that made the 1-loop 3-pt function with vacuum vanish

$$
\begin{aligned}
0=&I_{12}^{-1}\left(Y_{112}+Y_{122}\right)+I_{13}^{-1}\left(Y_{113}+Y_{133}\right)+I_{23}^{-1}\left(Y_{223}+Y_{233}\right)-I_{13}^{-1}I_{23}^{-1}\left(X_{1233}+I_{13}I_{23}F_{13,23}\right)\\
&+I_{12}^{-1}I_{13}^{-1}\left(-X_{1123}-I_{12}I_{13}F_{12,13}\right)+I_{12}^{-1}I_{23}^{-1}\left(-X_{1223}-I_{12}I_{23}F_{21,23}\right)
\end{aligned}
\tag{2.48}
$$

allows to express the 1-loop 3-pt function with a general state in terms of X only.

$$
\begin{aligned}
&\sum_{i=1}^{5}\mathrm{diagram}_i\\
=&\frac{(-)^{L/2}L\left(N-\frac{L}{2}\right)!}{2^{N+L/2+1}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2}I_{13}^{L/2-1}I_{23}^{L/2-1}X_{1233}u_{Z\bar{Z}}^{N-L/2}\\
&\left[2u_Z^{I_1}u_{\bar{Z}}^{I_2}...u_Z^{I_{L-1}}u_{\bar{Z}}^{I_L}+2u_{\bar{Z}}^{I_1}u_Z^{I_2}...u_{\bar{Z}}^{I_{L-1}}u_Z^{I_L}\right.
\end{aligned}
\tag{2.49}
$$

$$+\frac{1}{L}\sum_{k\text{ odd}}\left(u_Z^{I_1}u_{\bar Z}^{I_2}...u_{\bar Z}^{I_{k-1}}u_Z^{I_{k+2}}...u_Z^{L_{L-1}}u_{\bar Z}^{I_L}\right)\left(u_{Z\bar Z}u_{I_kI_{k+1}}-2u_{ZI_{k+1}}u_{\bar ZI_k}\right)$$
$$+\frac{1}{L}\sum_{k\text{ even}}\left(u_Z^{I_1}u_{\bar Z}^{I_2}...u_Z^{I_{k-1}}u_{\bar Z}^{I_{k+2}}...u_Z^{I_{L-1}}u_{\bar Z}^{I_L}\right)\left(u_{Z\bar Z}u_{I_kI_{k+1}}-2u_{ZI_k}u_{\bar ZI_{k+1}}\right)$$
$$+\frac{1}{L}\sum_{k\text{ odd}}\left(u_{\bar Z}^{I_1}u_Z^{I_2}...u_Z^{I_{k-1}}u_{\bar Z}^{I_{k+2}}...u_{\bar Z}^{I_{L-1}}u_Z^{I_L}\right)\left(u_{Z\bar Z}u_{I_kI_{k+1}}-2u_{ZI_k}u_{\bar ZI_{k+1}}\right)$$
$$\left.+\frac{1}{L}\sum_{k\text{ even}}\left(u_{\bar Z}^{I_1}u_Z^{I_2}...u_{\bar Z}^{I_{k-1}}u_Z^{I_{k+2}}...u_{\bar Z}^{I_{L-1}}u_Z^{I_L}\right)\left(u_{Z\bar Z}u_{I_kI_{k+1}}-2u_{ZI_{k+1}}u_{\bar ZI_k}\right)\right]$$
$$=\frac{(-)^{L/2}\left(N-\frac{L}{2}\right)!}{2^{N+L/2+1}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2}}I_{12}^{N-L/2}I_{13}^{L/2-1}I_{23}^{L/2-1}X_{1233}u_{Z\bar Z}^{N-L/2}$$
$$\sum_{k=1}^{L}\left(-2P_{k,k+1}+K_{k,k+1}+2I_{k,k+1}\right)\left(u_Z^{I_1}u_{\bar Z}^{I_2}...u_Z^{I_{L-1}}u_{\bar Z}^{I_L}+u_{\bar Z}^{I_1}u_{\bar Z}^{I_2}...u_{\bar Z}^{I_{L-1}}u_Z^{I_L}\right)$$

The final line is akin to the 1-loop Hamiltonian of the $SO(6)$ spin-chain, but referred to two states $Z\bar ZZ\bar Z...$ and $\bar ZZ\bar ZZ....$ Identity I, exchange P and trace K operators are in [4]. The diagrammatical computation did not make assumptions on R-symmetry indices, hence these are $SO(6)$ indices. We can restrict to $SU(2)$ operators made of $\tilde Z$ and $\tilde Y$. Now let us complete the 1 loop with tree level and make log-divergences manifest.

$$\begin{aligned}\left\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3^{I_1...I_L}\right\rangle &= \frac{(-)^{L/2+1}\left(N-\frac{L}{2}\right)!}{2^{N+L/2}N^{N-L/2}}\lambda^{N+L/2}I_{12}^{N-L/2}I_{13}^{L/2}I_{23}^{L/2}u_{Z\bar Z}^{N-L/2}\\ &\left[u_Z^{I_1}u_{\bar Z}^{I_2}...u_Z^{I_{L-1}}u_{\bar Z}^{I_L}+u_{\bar Z}^{I_1}u_Z^{I_2}...u_{\bar Z}^{I_{L-1}}u_Z^{I_L}+\frac{\lambda}{32\pi^2}\left(\log\frac{x_{13}^2x_{23}^2}{\epsilon^2x_{12}^2}+2\right)\right.\\ &\left.\times\sum_{k=1}^{L}\left(2P_{k,k+1}-K_{k,k+1}-2I_{k,k+1}\right)\left(u_Z^{I_1}u_{\bar Z}^{I_2}...u_Z^{I_{L-1}}u_{\bar Z}^{I_L}+u_{\bar Z}^{I_1}u_Z^{I_2}...u_{\bar Z}^{I_{L-1}}u_Z^{I_L}\right)+O\left(\lambda^2\right)\right]\end{aligned}\tag{2.50}$$

We recover zero at one loop for the vacuum $\mathcal{O}_3^{I_1...I_L}=\text{tr}\tilde Z^L$ as seen in previous section.

2.5 Structure constant as Hamiltonian insertion

I report (2.1) and (2.4) [5].

$$\left\langle\mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\right\rangle=\frac{h_{\mathcal{O}_i\mathcal{O}_j}+g_{\mathcal{O}_i\mathcal{O}_j}-\gamma_{\mathcal{O}_i\mathcal{O}_j}\log\frac{x_{ij}^2}{\epsilon^2}}{x_{ij}^{\Delta_i+\Delta_j}}\tag{2.51}$$

$$\left\langle\mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\mathcal{O}_k(x_k)\right\rangle=\frac{C^{(0)}_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k}-\frac{1}{2}\left(C^{(0)}\gamma\right)_{\mathcal{O}_i\mathcal{O}_j}\log\frac{x_{ik}^2x_{jk}^2}{\epsilon^2x_{ij}^2}-\frac{1}{2}\left(C^{(0)}\gamma\right)_{\mathcal{O}_i\mathcal{O}_k}\log\frac{x_{ij}^2x_{jk}^2}{\epsilon^2x_{ik}^2}-\frac{1}{2}\left(C^{(0)}\gamma\right)_{\mathcal{O}_j\mathcal{O}_k}\log\frac{x_{ij}^2x_{ik}^2}{\epsilon^2x_{jk}^2}+\tilde C^{(1)}_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k}}{x_{ij}^{\Delta_i+\Delta_j-\Delta_k}x_{ik}^{\Delta_i+\Delta_k-\Delta_j}x_{jk}^{\Delta_j+\Delta_k-\Delta_i}}\tag{2.52}$$

In this section we associate $I=1,...6$ to $Z,\bar Z,Y,\bar Y,X,\bar X$ respectively and make index positions meaningful

$$u_{\bar Z}^{Z}\equiv u_Z\cdot(u_Z)^*=u_Z\cdot u_{\bar Z}\equiv u_{Z\bar Z}\tag{2.53}$$
$$u_{\bar Z}^{Z}\equiv u_Z\cdot(u_{\bar Z})^*=u_Z\cdot u_Z\equiv u_{ZZ}=0\tag{2.54}$$

so that the fundamental matrix of R-symmetry products becomes diagonal

$$u_J^I=\begin{pmatrix}u_{\bar Z}^{Z}&0&0&0&0&0\\0&u_{\bar Z}^{Z}&0&0&0&0\\0&0&u_{\bar Y}^{Y}&0&0&0\\0&0&0&u_{\bar Y}^{Y}&0&0\\0&0&0&0&u_{\bar X}^{X}&0\\0&0&0&0&0&u_{\bar X}^{X}\end{pmatrix}=\delta_J^Iu_I^I\,,\tag{2.55}$$

the completeness relation becomes that of an orthogonal basis

$$\delta_J^I=\sum_K\frac{u_K^Iu_J^K}{u_K^K\sqrt{u_I^Iu_J^J}}=\sum_K\frac{u_K^Iu_J^K}{u_K^Ku_I^I}=\sum_K\frac{u_K^Iu_J^K}{u_K^Ku_J^J}\,,\tag{2.56}$$

and the product of 2 u's becomes simpler

$$u^I_J = \sum_K \frac{u^I_K u^K_J}{\sqrt{u^I_I u^J_J}} = \sum_K \frac{u^I_K u^K_J}{u^I_I} = \sum_K \frac{u^I_K u^K_J}{u^J_J}\,. \tag{2.57}$$

Nothing changes if $I = 1,\dots 6$ labels the twisted-translated scalars because twist-translation preserves orthogonality. In what follows, the bar conjugates fields and reverses their order:

$$\mathrm{tr}\left(\dots \underbrace{Y}_{2}\underbrace{\bar{Z}}_{1}\right) = \bar{\mathcal{O}}_{I_1\dots I_L} = \left(\mathcal{O}^{I_1\dots I_L}\right)^\dagger = \left(\mathrm{tr}\left(\underbrace{Z}_{1}\underbrace{\bar{Y}}_{2}\dots\right)\right)^\dagger\,. \tag{2.58}$$

2-pt function of single-traces

We compute the 2-pt function of generic $SO(6)$ operators, which we set temporarily in the origin and in x. "c.p." stands for cyclic permutations and acts last, for example after P, K, I. The symmetry factor $C_{I_1\dots I_L}$ is n if its indices are invariant when shifting by L/n [4] and satisfies $C_{I_1\dots I_L} u^{I_1}_{I_1}\dots u^{I_L}_{I_L} = u^{I_1}_{I_1}\dots u^{I_L}_{I_L} + \text{c.p.}$.

$$\left\langle \mathcal{O}_3^{I_1\dots I_L}\bar{\mathcal{O}}^3_{J_1\dots J_L}\right\rangle_{\text{LO}} = \left(\frac{N}{2}\right)^L \left(\frac{\lambda}{N} I_{0x}\right)^L \left(u^{I_1}_{J_1}\dots u^{I_L}_{J_L} + \text{c.p.}\right) \tag{2.59}$$

$$\left\langle \mathcal{O}_3^{I_1\dots I_L}\bar{\mathcal{O}}^3_{I_1\dots I_L}\right\rangle_{\text{LO}} = C_{I_1\dots I_L}\left(\frac{N}{2}\right)^L \left(\frac{\lambda}{N} I_{0x}\right)^L u^{I_1}_{I_1}\dots u^{I_L}_{I_L} \tag{2.60}$$

$$\begin{aligned}\left\langle \mathcal{O}_3^{I_1\dots I_L}\bar{\mathcal{O}}^3_{J_1\dots J_L}\right\rangle_{\text{NLO}} &= \frac{\lambda^{L+1}}{2^{L+1}} I_{0x}^{L-2}\sum_{k=1}^{L}\left(u^{I_1}_{J_1}\dots u^{I_{k-1}}_{J_{k-1}} u^{I_{k+2}}_{J_{k+2}}\dots u^{I_L}_{J_L}\right)\Big[-2I_{0x}\left(Y_{00x}+Y_{0xx}\right)\left(u^{I_k}_{J_k} u^{I_{k+1}}_{J_{k+1}}\right) \\ &\quad + X_{00xx}\left(-u^{I_k I_{k+1}} u_{J_k J_{k+1}} + 2u^{I_k}_{J_{k+1}} u^{I_{k+1}}_{J_k} - u^{I_k}_{J_k} u^{I_{k+1}}_{J_{k+1}}\right) + u^{I_k}_{J_k} u^{I_{k+1}}_{J_{k+1}} I_{0x}^2 F_{0x,0x}\Big] + \text{c.p.} \\ &= \frac{\lambda^{L+1}}{2^{L+4}\pi^2} I_{0x}^L \sum_{k=1}^{L}\left[\left(\log\frac{x^2}{\epsilon^2}+1\right)\left(2P_{k,k+1} - K_{k,k+1} - 2I_{k,k+1}\right)\left(u^{I_1}_{J_1}\dots u^{I_L}_{J_L}\right) + \text{c.p.}\right]\end{aligned} \tag{2.61}$$

The last expression is the sum of interacting diagrams with vertex O and X+H respectively:

$$\sum_{k=1}^{L}\left(u^{I_1}_{J_1}\dots u^{I_k}_{J_k}\dots u^{I_L}_{J_L} + \text{c.p.}\right)\left[-\frac{N\lambda^2}{2}\left(Y_{00x}+Y_{0xx}\right)\right]\left[\left(\frac{\lambda}{N} I_{0x}\right)^{L-1}\frac{N^{L-2}}{2^{L-1}}\right] \tag{2.62}$$

and

$$\begin{aligned}&\sum_{k=1}^{L}\left(u^{I_1}_{J_1}\dots u^{I_{k-1}}_{J_{k-1}} u^{I_{k+2}}_{J_{k+2}}\dots u^{I_L}_{J_L}\right)\left[\left(\frac{\lambda}{N} I_{0x}\right)^{L-2}\frac{N^{L-3}}{2^{L-2}}\right] \\ &\times \frac{\lambda^3 N}{8}\left[X_{00xx}\left(-u^{I_k I_{k+1}} u_{J_k J_{k+1}} + 2u^{I_k}_{J_{k+1}} u^{I_{k+1}}_{J_k} - u^{I_k}_{J_k} u^{I_{k+1}}_{J_{k+1}}\right) + u^{I_k}_{J_k} u^{I_{k+1}}_{J_{k+1}} I_{0x}^2 F_{0x,0x}\right] + \text{c.p.}\,.\end{aligned} \tag{2.63}$$

The generic 2-pt function (2.1) [5] for unit-normalized operators is essentially (5.18) [4]

$$\begin{aligned}&\left(C_{I_1\dots I_L}\left(\frac{\lambda}{8\pi^2}\right)^L u^{I_1}_{I_1}\dots u^{I_L}_{I_L}\right)^{-1/2}\left(C_{J_1\dots J_L}\left(\frac{\lambda}{8\pi^2}\right)^L u^{J_1}_{J_1}\dots u^{J_L}_{J_L}\right)^{-1/2}\left\langle \mathcal{O}_3^{I_1\dots I_L}\bar{\mathcal{O}}^3_{J_1\dots J_L}\right\rangle \\ &= \left(C_{I_1\dots I_L} u^{I_1}_{I_1}\dots u^{I_L}_{I_L}\right)^{-1/2}\left(C_{J_1\dots J_L} u^{J_1}_{J_1}\dots u^{J_L}_{J_L}\right)^{-1/2}\frac{1}{x^{2L}}\Big\{\left(u^{I_1}_{J_1}\dots u^{I_L}_{J_L} + \text{c.p.}\right) \\ &\quad - \left(\log\frac{x^2}{\epsilon^2}+1\right)\left[\sum_{k=1}^{L} H_{k,k+1}\left(u^{I_1}_{J_1}\dots u^{I_L}_{J_L}\right) + \text{c.p.}\right]\Big\} + O\left(\lambda^2\right)\end{aligned} \tag{2.64}$$

with 1-loop Hamiltonian density operator in (7) [6], (73) [7] – or Hamiltonian from now on –

$$H_{i,j} = \frac{\lambda}{8\pi^2}\left(I_{i,j} + \frac{1}{2}K_{i,j} - P_{i,j}\right)\,. \tag{2.65}$$

We read off the tree-level mixing matrix, the mixing matrix at 1 loop and the anomalous dimension matrix at 1 loop

$$h_{\mathcal{O}_3^{I_1\dots I_L}\bar{\mathcal{O}}^3_{J_1\dots J_L}} = \left(C_{I_1\dots I_L} u^{I_1}_{I_1}\dots u^{I_L}_{I_L}\right)^{-1/2}\left(C_{J_1\dots J_L} u^{J_1}_{J_1}\dots u^{J_L}_{J_L}\right)^{-1/2}\left(u^{I_1}_{J_1}\dots u^{I_L}_{J_L} + \text{c.p.}\right) \tag{2.66}$$

$$
\begin{aligned}
h^{\mathcal{O}_3^{I_1\dots I_L}\bar{\mathcal{O}}^3_{J_1\dots J_L}} &= \frac{1}{L^2}\left(\frac{C_{I_1\dots I_L}}{u^{I_1}_{I_1}\dots u^{I_L}_{I_L}}\right)^{1/2}\left(\frac{C_{J_1\dots J_L}}{u^{J_1}_{J_1}\dots u^{J_L}_{J_L}}\right)^{1/2}\left(u^{I_1}_{J_1}\dots u^{I_L}_{J_L}+\text{c.p.}\right) && (2.67)\\
h_{\mathcal{O}_3^{I_1\dots I_L}\bar{\mathcal{O}}^3_{J_1\dots J_L}}h^{\mathcal{O}_3^{J_1\dots J_L}\bar{\mathcal{O}}^3_{K_1\dots K_L}} &= \frac{\delta^{I_1}_{K_1}\dots\delta^{I_L}_{K_L}+\text{c.p.}}{L} && (2.68)\\
g_{\mathcal{O}_3^{I_1\dots I_L}\bar{\mathcal{O}}^3_{J_1\dots J_L}} &= -\left(C_{I_1\dots I_L}u^{I_1}_{I_1}\dots u^{I_L}_{I_L}\right)^{-1/2}\left(C_{J_1\dots J_L}u^{J_1}_{J_1}\dots u^{J_L}_{J_L}\right)^{-1/2}\left[\sum_{k=1}^{L}H_{k,k+1}\left(u^{I_1}_{J_1}\dots u^{I_L}_{J_L}\right)+\text{c.p.}\right] && (2.69)\\
\gamma_{\mathcal{O}_3^{I_1\dots I_L}\bar{\mathcal{O}}^3_{J_1\dots J_L}} &= -g_{\mathcal{O}_3^{I_1\dots I_L}\bar{\mathcal{O}}^3_{J_1\dots J_L}}\,. && (2.70)
\end{aligned}
$$

Note that the product of h and its inverse correctly gives the identity in the space of tensors with cyclic indices, which has the defining property of squaring to itself

$$
\frac{\delta^{I_1}_{K_1}\dots\delta^{I_L}_{K_L}+\text{c.p.}}{L}\,\frac{\delta^{K_1}_{J_1}\dots\delta^{K_L}_{J_L}+\text{c.p.}}{L}=\frac{\delta^{I_1}_{J_1}\dots\delta^{I_L}_{J_L}+\text{c.p.}}{L}\,. \tag{2.71}
$$

2-pt function of determinants

$$
\left\langle\mathcal{O}_1\bar{\mathcal{O}}_1\right\rangle=\left\langle\mathcal{O}_2\bar{\mathcal{O}}_2\right\rangle=\mathcal{G}_{L=0}=\frac{N!}{2^N\left(4\pi^2\right)^N N^N}\frac{\lambda^N}{x^{2N}}u^N_{Z\bar{Z}} \tag{2.72}
$$

Our 3-pt function

The 3-pt function (2.4) [5] of 2 unit-normalized GG and 1 unit-normalized single-trace operator reads [7]

$$
\begin{aligned}
&\left(\frac{N!}{2^N\left(4\pi^2\right)^N N^N}\lambda^N u^N_{Z\bar{Z}}\right)^{-1}\left(C_{I_1\dots I_L}\left(\frac{\lambda}{8\pi^2}\right)^L u^{I_1}_{I_1}\dots u^{I_L}_{I_L}\right)^{-1/2}\left\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3^{I_1\dots I_L}\right\rangle && (2.73)\\
=&\frac{(-)^{L/2+1}}{u^{L/2}_{Z\bar{Z}}\left(C_{I_1\dots I_L}u^{I_1}_{I_1}\dots u^{I_L}_{I_L}\right)^{1/2}}\frac{1}{(x_1-x_2)^{2N-L}(x_2-x_3)^L(x_1-x_3)^L}\left[u^{I_1}_Z u^{I_2}_{\bar{Z}}\dots u^{I_{L-1}}_Z u^{I_L}_{\bar{Z}}+u^{I_1}_{\bar{Z}}u^{I_2}_Z\dots u^{I_{L-1}}_{\bar{Z}}u^{I_L}_Z\right.\\
&\left.-\frac{1}{2}\left(\log\frac{x^2_{13}x^2_{23}}{\epsilon^2x^2_{12}}+2\right)\sum_{k=1}^{L}H_{k,k+1}\left(u^{I_1}_Z u^{I_2}_{\bar{Z}}\dots u^{I_{L-1}}_Z u^{I_L}_{\bar{Z}}+u^{I_1}_{\bar{Z}}u^{I_2}_Z\dots u^{I_{L-1}}_{\bar{Z}}u^{I_L}_Z\right)+O\left(\lambda^2\right)\right]
\end{aligned}
$$

which leads to classical and anomalous dimensions, tree-level and 1-loop structure constant

$$
\Delta_{\mathcal{O}_1}=\Delta_{\mathcal{O}_2}=N\qquad\Delta_{\mathcal{O}_3^{I_1\dots I_L}}=L\qquad\left(C^{(0)}\gamma\right)_{\mathcal{O}_1\mathcal{O}_3^{I_1\dots I_L}}=\left(C^{(0)}\gamma\right)_{\mathcal{O}_2\mathcal{O}_3^{I_1\dots I_L}}=0 \tag{2.74}
$$

$$
\begin{aligned}
C^{(0)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3^{I_1\dots I_L}} &= \frac{(-)^{L/2+1}}{u^{L/2}_{Z\bar{Z}}\left(C_{I_1\dots I_L}u^{I_1}_{I_1}\dots u^{I_L}_{I_L}\right)^{1/2}}\left(u^{I_1}_Z u^{I_2}_{\bar{Z}}\dots u^{I_{L-1}}_Z u^{I_L}_{\bar{Z}}+u^{I_1}_{\bar{Z}}u^{I_2}_Z\dots u^{I_{L-1}}_{\bar{Z}}u^{I_L}_Z\right) && (2.75)\\
\left(C^{(0)}\gamma\right)_{\mathcal{O}_1\mathcal{O}_2} &= \frac{(-)^{L/2+1}}{u^{L/2}_{Z\bar{Z}}\left(C_{I_1\dots I_L}u^{I_1}_{I_1}\dots u^{I_L}_{I_L}\right)^{1/2}}\sum_{k=1}^{L}H_{k,k+1}\left(u^{I_1}_Z u^{I_2}_{\bar{Z}}\dots u^{I_{L-1}}_Z u^{I_L}_{\bar{Z}}+u^{I_1}_{\bar{Z}}u^{I_2}_Z\dots u^{I_{L-1}}_{\bar{Z}}u^{I_L}_Z\right) && (2.76)\\
\tilde{C}^{(1)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3^{I_1\dots I_L}} &= -\left(C^{(0)}\gamma\right)_{\mathcal{O}_1\mathcal{O}_2}\,. && (2.77)
\end{aligned}
$$

We will need the tree-level structure constant with the last index up.

$$
\begin{aligned}
C^{(0)\mathcal{O}_3^{I_1\dots I_L}}_{\mathcal{O}_1\mathcal{O}_2} &= C^{(0)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3^{J_1\dots J_L}}h^{\mathcal{O}_3^{I_1\dots I_L}\bar{\mathcal{O}}^3_{J_1\dots J_L}} && (2.78)\\
&= \frac{(-)^{L/2+1}}{Lu^{L/2}_{Z\bar{Z}}}\left(\frac{C_{I_1\dots I_L}}{u^{I_1}_{I_1}\dots u^{I_L}_{I_L}}\right)^{1/2}\left(u^{I_1}_Z u^{I_2}_{\bar{Z}}\dots u^{I_{L-1}}_Z u^{I_L}_{\bar{Z}}+u^{I_1}_{\bar{Z}}u^{I_2}_Z\dots u^{I_{L-1}}_{\bar{Z}}u^{I_L}_Z\right)\\
&= \frac{C_{I_1\dots I_L}}{L}C^{(0)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3^{I_1\dots I_L}}
\end{aligned}
$$

[7] Unit-normalized at all loop for the GG and unit-normalized at tree-level for the single-trace.

What changes for determinants in [5]

The objective is finding a way to calculate the definition of renormalization-invariant structure constant using (2.6) [5] [8]

$$C^{(1)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3} = \tilde{C}^{(1)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3} - \frac{1}{2} g_{\mathcal{O}_1\alpha'} C^{(0)\alpha'}_{\mathcal{O}_2\mathcal{O}_3} - \frac{1}{2} g_{\mathcal{O}_2\beta'} C^{(0)\beta'}_{\mathcal{O}_1\mathcal{O}_3} - \frac{1}{2} g_{\mathcal{O}_3\gamma'} C^{(0)\gamma'}_{\mathcal{O}_1\mathcal{O}_2} \,. \tag{2.79}$$

The sums run over single-trace operators of any length and made of 6 scalars, so we can write explicitly

$$C^{(1)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3^{I_1\ldots I_L}} = \tilde{C}^{(1)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3^{I_1\ldots I_L}} - \frac{1}{2} g_{\mathcal{O}_1\bar{\mathcal{O}}^3_{J_1\ldots}} C^{(0)\mathcal{O}_3^{J_1\cdots}}_{\mathcal{O}_2\mathcal{O}_3^{I_1\ldots I_L}} - \frac{1}{2} g_{\mathcal{O}_2\bar{\mathcal{O}}^3_{J_1\ldots}} C^{(0)\mathcal{O}_3^{J_1\cdots}}_{\mathcal{O}_1\mathcal{O}_3^{I_1\ldots I_L}} - \frac{1}{2} g_{\mathcal{O}_3^{I_1\ldots I_L}\bar{\mathcal{O}}^3_{J_1\ldots}} C^{(0)\mathcal{O}_3^{J_1\cdots}}_{\mathcal{O}_1\mathcal{O}_2} \tag{2.80}$$

We specialized $\alpha,\beta,\gamma \to \mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3^{I_1\ldots I_L}$ in the general notation of [5]. This is found scattered around its Figure 5:

- $\tilde{C}^{(1)}_{\alpha\beta\gamma}$ is the constant piece of the 1-loop structure constant,
- $U^{\alpha}_{\beta\gamma}$ is the constant piece of genuine 3-body interactions,
- $U_{\alpha\beta}$ is the constant piece of the diagram with same operators, but now thought of one with the insertion of a complete basis of single-trace operators.

"Constant" means independent of spacetime points and cutoff. Only this matters because the 1-loop pieces g and $\tilde{C}^{(1)}$ in formula can be defined as the constant part of the corresponding full 1-loop objects in (2.1) and (2.4). Few conclusions follow when 3-pt functions involve determinants.

- The assumption in definition (2.6) is the locality of the operators and the structure of their 2-,3-pt functions (2.1) and (2.4), so it stays untouched.
- The insertion of a complete basis is a general procedure that involves the middle part of diagrams and not the operators themselves. The interpretation of the products "$g\, C^{(0)}$" as insertions holds.
- The assumption of locality in 1-loop interactions (= 4-leg vertices join only neighboring fields in traces) in planar limit fails because the notion of neighborhood ceases to exist for fields in determinants.
- I argue that metric subtractions cancel 2-body terms because this cancellation does not base on the locality assumption.

Calculate renormalization-invariant structure constant

Let us continue with the final expression for $C^{(1)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3^{I_1\ldots I_L}}$. The first 2 sums reduces to only one term $\mathcal{O}_3^{J_1\cdots} = \mathrm{tr}\left(\bar{Z}^N\right)$ and $\mathcal{O}_3^{J_1\cdots} = \mathrm{tr}\left(Z^N\right)$ respectively. Here the 2 g's are the constant part of the mixed correlators

$$\left\langle \mathcal{O}_1 \mathrm{tr}\left(\bar{Z}^N\right)\right\rangle_{\mathrm{NLO}} \qquad\qquad \left\langle \mathcal{O}_2 \mathrm{tr}\left(Z^N\right)\right\rangle_{\mathrm{NLO}} \,, \tag{2.81}$$

but these vanishes completely because of supersymmetry and also our check for $N = 1, \ldots 5$. What is left is

$$\begin{aligned}
&C^{(1)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3^{I_1\ldots I_L}} \\
=&\tilde{C}^{(1)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3^{I_1\ldots I_L}} - \frac{1}{2} g_{\mathcal{O}_3^{I_1\ldots I_L}\bar{\mathcal{O}}^3_{J_1\ldots J_L}} C^{(0)\mathcal{O}_3^{J_1\ldots J_L}}_{\mathcal{O}_1\mathcal{O}_2} \\
=&\frac{(-)^{L/2}}{2u_{Z\bar{Z}}^{L/2}\left(C_{I_1\ldots I_L} u^{I_1}_{I_1}\ldots u^{I_L}_{I_L}\right)^{1/2}} \sum_{k=1}^{L} H_{k,k+1}\left(u^{I_1}_{Z} u^{I_2}_{\bar{Z}} \ldots u^{I_{L-1}}_{Z} u^{I_L}_{\bar{Z}} + u^{I_1}_{\bar{Z}} u^{I_2}_{Z} \ldots u^{I_{L-1}}_{\bar{Z}} u^{I_L}_{Z}\right) .
\end{aligned} \tag{2.82}$$

This is the conclusive line to compare with (3.2) [5]: the renormalization-invariant structure constant is the sum of the 1-loop Hamiltonian of the $SO(6)$ spin-chain over all splitting points, which can sit between each pair of scalars in $\mathcal{O}_3$ for us. Let us stress again that both the renormalization-dependent $\tilde{C}^{(1)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3^{I_1\ldots I_L}}$ and the non-vanishing metric subtraction are proportional to the Hamiltonian, hence their difference $C^{(1)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3^{I_1\ldots I_L}}$ must be too.

[8] More pedagogically in [8]. Here operators are diagonal under the action the 1-loop dilatation operator, i.e. $g_{ij} \propto \delta_{ij}$, so the slicing argument selects only one term in sums over primaries. I compared the parametrization of 2-,3-pt functions to check that metric subtractions are the same of those in [5].

3 Structure constant of 3 single-traces as Hamiltonian insertion

We consider "non-length-preserving" single-traces (3.1) [5]

$$\tilde{\mathcal{O}}_1 = \operatorname{tr}\left(Z^{L_1}\right)(x_1) \qquad \tilde{\mathcal{O}}_2 = \operatorname{tr}\left(\bar{Z}^{L_2}\right)(x_2) \qquad \tilde{\mathcal{O}}_3^{I_1 \ldots I_L} = \operatorname{tr}\left(\phi^{I_1} \ldots \phi^{I_L}\right)(x_3) \tag{3.1}$$

and repeat its section 3 independently. Our result must be (3.2) in a more explicit fashion.

2-pt function of single-traces

We borrow formulas from previous section for computing the 2-pt coefficient of each operator, which are temporarily set in the origin and in x.

$$\left\langle \tilde{\mathcal{O}}_i \bar{\tilde{\mathcal{O}}}_i \right\rangle = L_i \left(\frac{N}{2}\right)^{L_i} \left(\frac{\lambda}{N} I_{0x}\right)^{L_i} u_{Z\bar{Z}}^{L_i} \qquad i = 1, 2 \tag{3.2}$$

$$\left\langle \tilde{\mathcal{O}}_3^{I_1 \ldots I_L} \left(\bar{\tilde{\mathcal{O}}}_3\right)_{I_1 \ldots I_L} \right\rangle_{LO} = C_{I_1 \ldots I_L} \left(\frac{N}{2}\right)^{L} \left(\frac{\lambda}{N} I_{0x}\right)^{L} u_{I_1}^{I_1} \ldots u_{I_L}^{I_L} \tag{3.3}$$

$$\left\langle \tilde{\mathcal{O}}_3^{I_1 \ldots I_L} \left(\bar{\tilde{\mathcal{O}}}_3\right)_{J_1 \ldots J_L} \right\rangle_{NLO} = -\frac{\lambda^L}{2^L} I_{0x}^L \left(\log \frac{x^2}{\epsilon^2} + 1\right) \left[\sum_{k=1}^{L} H_{k,k+1} \left(u_{J_1}^{I_1} \ldots u_{J_L}^{I_L}\right) + c.p.\right] \tag{3.4}$$

Again one copies

$$h_{\tilde{\mathcal{O}}_3^{I_1 \ldots I_L} \bar{\tilde{\mathcal{O}}}^3_{J_1 \ldots J_L}} = \left(C_{I_1 \ldots I_L} u_{I_1}^{I_1} \ldots u_{I_L}^{I_L}\right)^{-1/2} \left(C_{J_1 \ldots J_L} u_{J_1}^{J_1} \ldots u_{J_L}^{J_L}\right)^{-1/2} \left(u_{J_1}^{I_1} \ldots u_{J_L}^{I_L} + \text{c.p.}\right) \tag{3.5}$$

$$h^{\tilde{\mathcal{O}}_3^{I_1 \ldots I_L} \bar{\tilde{\mathcal{O}}}^3_{J_1 \ldots J_L}} = \frac{1}{L^2} \left(\frac{C_{I_1 \ldots I_L}}{u_{I_1}^{I_1} \ldots u_{I_L}^{I_L}}\right)^{1/2} \left(\frac{C_{J_1 \ldots J_L}}{u_{J_1}^{J_1} \ldots u_{J_L}^{J_L}}\right)^{1/2} \left(u_{J_1}^{I_1} \ldots u_{J_L}^{I_L} + \text{c.p.}\right) \tag{3.6}$$

$$g_{\tilde{\mathcal{O}}_3^{I_1 \ldots I_L} \bar{\tilde{\mathcal{O}}}^3_{J_1 \ldots J_L}} = -\left(C_{I_1 \ldots I_L} u_{I_1}^{I_1} \ldots u_{I_L}^{I_L}\right)^{-1/2} \left(C_{J_1 \ldots J_L} u_{J_1}^{J_1} \ldots u_{J_L}^{J_L}\right)^{-1/2} \left[\sum_{k=1}^{L} H_{k,k+1} \left(u_{J_1}^{I_1} \ldots u_{J_L}^{I_L}\right) + \text{c.p.}\right] \tag{3.7}$$

$$\gamma_{\tilde{\mathcal{O}}_3^{I_1 \ldots I_L} \bar{\tilde{\mathcal{O}}}^3_{J_1 \ldots J_L}} = -g_{\tilde{\mathcal{O}}_3^{I_1 \ldots I_L} \tilde{\mathcal{O}}_3^{J_1 \ldots J_L}} \,. \tag{3.8}$$

3-pt function of single-traces

The tree level has three bundles of propagators of bridge lengths

$$L_{13} = (L_1 - L_2 + L)/2 > 0 \qquad L_{23} = (-L_1 + L_2 + L)/2 > 0 \qquad L_{12} = (L_1 + L_2 - L)/2 > 0\,. \tag{3.9}$$

$L_1 \times L_2$ accounts for cyclic permutations of scalars in $\tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}_2$, while the sum over k implements cyclic permutations of scalars in $\tilde{\mathcal{O}}_3^{I_1 \ldots I_L}$. We pose $I_{k+L} = I_k$ for any $k \geq 0$.

$$\begin{aligned}
&\left\langle \tilde{\mathcal{O}}_1 \tilde{\mathcal{O}}_2 \tilde{\mathcal{O}}_3^{I_1 \ldots I_L} \right\rangle_{\text{LO}} \\
&\overset{N \gg 1}{=} \left(\frac{\lambda}{N} I_{13}\right)^{L_{13}} \left(\frac{\lambda}{N} I_{23}\right)^{L_{23}} \left(\frac{\lambda}{N} I_{12} u_{Z\bar{Z}}\right)^{L_{12}} \left(L_1 L_2 \sum_{k=0}^{L-1} u_Z^{I_{k+1}} \ldots u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+1}} \ldots u_{\bar{Z}}^{I_{k+L}}\right) \frac{N^{(L_1+L_2+L)/2-1}}{2^{(L_1+L_2+L)/2}} \\
&= \frac{L_1 L_2}{2^{(L_1+L_2+L)/2}} \frac{\lambda^{(L_1+L_2+L)/2}}{N} \left(I_{12} u_{Z\bar{Z}}\right)^{L_{12}} I_{13}^{L_{13}} I_{23}^{L_{23}} \sum_{k=0}^{L-1} u_Z^{I_{k+1}} \ldots u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+1}} \ldots u_{\bar{Z}}^{I_{k+L}} \\
&= \frac{L_1 L_2}{2^{(L_1+L_2+L)/2}} \frac{\lambda^{(L_1+L_2+L)/2}}{N} \left(I_{12} u_{Z\bar{Z}}\right)^{L_{12}} I_{13}^{L_{13}} I_{23}^{L_{23}} \left(u_Z^{I_1} \ldots u_Z^{I_{L_{13}}} u_{\bar{Z}}^{I_{L_{13}+1}} \ldots u_{\bar{Z}}^{I_L} + \text{c.p.}\right)
\end{aligned} \tag{3.10}$$

The 1-loop order is the tree-level dressed up with vertices. One diagram accounts for vertices O.

$$\begin{aligned}
&\text{diagram}_1 \\
&\overset{N \gg 1}{=} \left(\frac{\lambda}{N} I_{13}\right)^{L_{13}-1} \left(\frac{\lambda}{N} I_{23}\right)^{L_{23}} \left(\frac{\lambda}{N} I_{12} u_{Z\bar{Z}}\right)^{L_{12}} L_{13} \left(-\frac{N\lambda^2}{2} \left(Y_{113} + Y_{133}\right)\right) \left(L_1 L_2 \sum_{k=0}^{L-1} u_Z^{I_{k+1}} \ldots u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+1}} \ldots u_{\bar{Z}}^{I_{k+L}}\right) \frac{N^{(L_1+L_2+L)/2-3}}{2^{(L_1+L_2+L)/2-1}} \\
&+ \left(\frac{\lambda}{N} I_{13}\right)^{L_{13}} \left(\frac{\lambda}{N} I_{23}\right)^{L_{23}-1} \left(\frac{\lambda}{N} I_{12} u_{Z\bar{Z}}\right)^{L_{12}} L_{23} \left(-\frac{N\lambda^2}{2} \left(Y_{223} + Y_{223}\right)\right) \left(L_1 L_2 \sum_{k=0}^{L-1} u_Z^{I_{k+1}} \ldots u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+1}} \ldots u_{\bar{Z}}^{I_{k+L}}\right) \frac{N^{(L_1+L_2+L)/2-3}}{2^{(L_1+L_2+L)/2-1}} \\
&+ \left(\frac{\lambda}{N} I_{13}\right)^{L_{13}} \left(\frac{\lambda}{N} I_{23}\right)^{L_{23}-1} \left(\frac{\lambda}{N} I_{12} u_{Z\bar{Z}}\right)^{L_{12}-1} L_{12} \left(-\frac{N\lambda^2}{2} \left(Y_{112} + Y_{221}\right) u_{Z\bar{Z}}\right) \left(L_1 L_2 \sum_{k=0}^{L-1} u_Z^{I_{k+1}} \ldots u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+1}} \ldots u_{\bar{Z}}^{I_{k+L}}\right) \frac{N^{(L_1+L_2+L)/2-3}}{2^{(L_1+L_2+L)/2-1}}
\end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&= - \frac{L_1 L_2}{2^{(L_1+L_2+L)/2}} \frac{\lambda^{(L_1+L_2+L)/2+1}}{N} \left(I_{12} u_{Z\bar{Z}}\right)^{L_{12}} I_{13}^{L_{13}} I_{23}^{L_{23}} \\
&\quad \left[L_{12} I_{12}^{-1} \left(Y_{112}+Y_{221}\right) + L_{13} I_{13}^{-1} \left(Y_{113}+Y_{133}\right) + L_{23} I_{23}^{-1} \left(Y_{223}+Y_{223}\right)\right] \sum_{k=0}^{L-1} u_Z^{I_{k+1}} ... u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+1}} ... u_{\bar{Z}}^{I_{k+L}}
\end{aligned}
$$

The next for X+H vertices joining propagators in the same bundle

$$
\begin{aligned}
&\text{diagram}_2 \\
&\stackrel{N \gg 1}{=} \sum_{k=0}^{L-1} \sum_{l=1}^{L_{13}-1} \left(\frac{\lambda}{N} I_{13}\right)^{L_{13}-2} \left(\frac{\lambda}{N} I_{23}\right)^{L_{23}} \left(\frac{\lambda}{N} I_{12} u_{Z\bar{Z}}\right)^{L_{12}} c_{ZZI_{k+l}I_{k+l+1}} \left(L_1 L_2 u_Z^{I_{k+1}} ... u_Z^{I_{k+l-1}} u_Z^{I_{k+l+2}} ... u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+1}} ... u_{\bar{Z}}^{I_{k+L}}\right) \frac{N^{(L_1+L_2+L)/2-4}}{2^{(L_1+L_2+L)/2-2}} \\
&\quad + \sum_{k=0}^{L-1} \sum_{l=L_{13}+1}^{L-1} \left(\frac{\lambda}{N} I_{13}\right)^{L_{13}} \left(\frac{\lambda}{N} I_{23}\right)^{L_{23}-2} \left(\frac{\lambda}{N} I_{12} u_{Z\bar{Z}}\right)^{L_{12}} c_{\bar{Z}\bar{Z}I_{k+l}I_{k+l+1}} \left(L_1 L_2 u_Z^{I_{k+1}} ... u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+1}} ... u_{\bar{Z}}^{I_{k+l-1}} u_{\bar{Z}}^{I_{k+l+1}} ... u_{\bar{Z}}^{I_{k+L}}\right) \frac{N^{(L_1+L_2+L)/2-4}}{2^{(L_1+L_2+L)/2-2}} \\
&\quad + \sum_{k=0}^{L-1} \left(L_{12}-1\right) \left(\frac{\lambda}{N} I_{13}\right)^{L_{13}} \left(\frac{\lambda}{N} I_{23}\right)^{L_{23}} \left(\frac{\lambda}{N} I_{12} u_{Z\bar{Z}}\right)^{L_{12}-2} c_{ZZ\bar{Z}\bar{Z}} \left(L_1 L_2 u_Z^{I_{k+1}} ... u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+1}} ... u_{\bar{Z}}^{I_{k+L}}\right) \frac{N^{(L_1+L_2+L)/2-4}}{2^{(L_1+L_2+L)/2-2}} \\
&= \frac{L_1 L_2}{2^{(L_1+L_2+L)/2+1}} \frac{\lambda^{(L_1+L_2+L)2+1}}{N} \left(I_{12} u_{Z\bar{Z}}\right)^{L_{12}} I_{13}^{L_{13}} I_{23}^{L_{23}} \\
&\quad \left[\left(L_{12}-1\right)\left(I_{12}^{-2} X_{1122} + F_{12,12}\right) + \left(L_{13}-1\right)\left(I_{13}^{-2} X_{1133} + F_{13,13}\right) + \left(L_{23}-1\right)\left(I_{23}^{-2} X_{2233} + F_{23,23}\right)\right] \sum_{k=0}^{L-1} u_Z^{I_{k+1}} ... u_Z^{I_{k+L/2}} u_{\bar{Z}}^{I_{k+L/2+1}} ... u_{\bar{Z}}^{I_{k+L}}
\end{aligned} \tag{3.12}
$$

where we used

$$
\begin{aligned}
c_{ZZI_{k+l}I_{k+l+1}} &= \left\langle \operatorname{tr}\left[Z\left(x_1\right) Z\left(x_1\right) \phi_{I_k}\left(x_3\right) \phi_{I_{k+1}}\left(x_3\right)\right] V_{X+H} \right\rangle_{\mathrm{LO}} \\
&\stackrel{N \gg 1}{=} \frac{\lambda^3 N}{8} \left(X_{1133} + I_{13}^2 F_{13,13}\right) u_{ZI_{k+l}} u_{ZI_{k+l+1}}
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
c_{\bar{Z}\bar{Z}I_{k+l}I_{k+l+1}} &= \left\langle \operatorname{tr}\left[\bar{Z}\left(x_2\right) \bar{Z}\left(x_2\right) \phi_{I_{k+l}}\left(x_3\right) \phi_{I_{k+l+1}}\left(x_3\right)\right] V_{X+H} \right\rangle_{\mathrm{LO}} \\
&\stackrel{N \gg 1}{=} \frac{\lambda^3 N}{8} \left(X_{2233} + I_{23}^2 F_{23,23}\right) u_{\bar{Z}I_{k+l}} u_{\bar{Z}I_{k+l+1}}
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
c_{ZZ\bar{Z}\bar{Z}} &= \left\langle \operatorname{tr}\left[Z\left(x_1\right) Z\left(x_1\right) \bar{Z}\left(x_2\right) \bar{Z}\left(x_2\right)\right] V_{X+H} \right\rangle_{\mathrm{LO}} \\
&\stackrel{N \gg 1}{=} \frac{\lambda^3 N}{8} \left(X_{1122} + I_{12}^2 F_{12,12}\right) u_{Z\bar{Z}}^2 .
\end{aligned} \tag{3.15}
$$

The last diagram accounts for X+H vertices joining propagators in different bundles

$$
\begin{aligned}
&\text{diagram}_3 \\
&\stackrel{N \gg 1}{=} \sum_{k=0}^{L-1} \left(\frac{\lambda}{N} I_{13}\right)^{L_{13}-1} \left(\frac{\lambda}{N} I_{23}\right)^{L_{23}} \left(\frac{\lambda}{N} I_{12} u_{Z\bar{Z}}\right)^{L_{12}-1} c_{\bar{Z}I_{k+1}ZZ} \left(L_1 L_2 u_Z^{I_{k+2}} ... u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+1}} ... u_{\bar{Z}}^{I_{k+L}}\right) \frac{N^{(L_1+L_2+L)/2-4}}{2^{(L_1+L_2+L)/2-2}} \\
&\quad + \sum_{k=0}^{L-1} \left(\frac{\lambda}{N} I_{13}\right)^{L_{13}-1} \left(\frac{\lambda}{N} I_{23}\right)^{L_{23}} \left(\frac{\lambda}{N} I_{12} u_{Z\bar{Z}}\right)^{L_{12}-1} c_{I_{k+L/2}\bar{Z}ZZ} \left(L_1 L_2 u_Z^{I_{k+1}} ... u_Z^{I_{k+L_{13}-1}} u_{\bar{Z}}^{I_{k+L_{13}+1}} ... u_{\bar{Z}}^{I_{k+L}}\right) \frac{N^{(L_1+L_2+L)/2-4}}{2^{(L_1+L_2+L)/2-2}} \\
&\quad + \sum_{k=0}^{L-1} \left(\frac{\lambda}{N} I_{13}\right)^{L_{13}} \left(\frac{\lambda}{N} I_{23}\right)^{L_{23}-1} \left(\frac{\lambda}{N} I_{12} u_{Z\bar{Z}}\right)^{L_{12}-1} c_{I_{k+L}Z\bar{Z}\bar{Z}} \left(L_1 L_2 u_Z^{I_{k+1}} ... u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+1}} ... u_{\bar{Z}}^{I_{k+L-1}}\right) \frac{N^{(L_1+L_2+L)/2-4}}{2^{(L_1+L_2+L)/2-2}} \\
&\quad + \sum_{k=0}^{L-1} \left(\frac{\lambda}{N} I_{13}\right)^{L_{13}} \left(\frac{\lambda}{N} I_{23}\right)^{L_{23}-1} \left(\frac{\lambda}{N} I_{12} u_{Z\bar{Z}}\right)^{L_{12}-1} c_{ZI_{k+L_{13}+1}\bar{Z}\bar{Z}} \left(L_1 L_2 u_Z^{I_{k+1}} ... u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+2}} ... u_{\bar{Z}}^{I_{k+L}}\right) \frac{N^{(L_1+L_2+L)/2-4}}{2^{(L_1+L_2+L)/2-2}} \\
&\quad + \sum_{k=0}^{L-1} \left(\frac{\lambda}{N} I_{13}\right)^{L_{13}-1} \left(\frac{\lambda}{N} I_{23}\right)^{L_{23}-1} \left(\frac{\lambda}{N} I_{12} u_{Z\bar{Z}}\right)^{L_{12}} c_{I_{k+L}I_{k+1}Z\bar{Z}} \left(L_1 L_2 u_Z^{I_{k+2}} ... u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+1}} ... u_{\bar{Z}}^{I_{k+L-1}}\right) \frac{N^{(L_1+L_2+L)/2-4}}{2^{(L_1+L_2+L)/2-2}} \\
&\quad + \sum_{k=0}^{L-1} \left(\frac{\lambda}{N} I_{13}\right)^{L_{13}-1} \left(\frac{\lambda}{N} I_{23}\right)^{L_{23}-1} \left(\frac{\lambda}{N} I_{12} u_{Z\bar{Z}}\right)^{L_{12}} c_{I_{k+L/2}I_{k+L_{13}+1}\bar{Z}Z} \left(L_1 L_2 u_Z^{I_{k+1}} ... u_Z^{I_{k+L_{13}-1}} u_{\bar{Z}}^{I_{k+L_{13}+2}} ... u_{\bar{Z}}^{I_{k+L}}\right) \frac{N^{(L_1+L_2+L)/2-4}}{2^{(L_1+L_2+L)/2-2}} \\
&= \frac{L_1 L_2}{2^{(L_1+L_2+L)/2+1}} \frac{\lambda^{(L_1+L_2+L)/2+1}}{N} \left(I_{12} u_{Z\bar{Z}}\right)^{L_{12}} I_{13}^{L_{13}} I_{23}^{L_{23}} \sum_{k=0}^{L-1} \Big\{ 2\left(I_{12}^{-1} I_{13}^{-1} X_{1123} + F_{12,13} + I_{12}^{-1} I_{23}^{-1} X_{1223} - F_{12,23}\right) u_Z^{I_{k+1}} ... u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+1}} ... u_{\bar{Z}}^{I_{k+L}} \\
&\quad + \left[I_{13}^{-1} I_{23}^{-1} X_{1233} \left(-u_{I_{k+L}I_{k+1}} u_{Z\bar{Z}} + 2 u_{ZI_{k+L}} u_{\bar{Z}I_{k+1}} - u_{\bar{Z}I_{k+L}} u_{ZI_{k+1}}\right) + F_{13,23} u_{\bar{Z}I_{k+L}} u_{ZI_{k+1}}\right] u_Z^{I_{k+2}} ... u_Z^{I_{k+L_{13}}} u_{\bar{Z}}^{I_{k+L_{13}+1}} ... u_{\bar{Z}}^{I_{k+L-1}} \\
&\quad + \left[I_{13}^{-1} I_{23}^{-1} X_{1233} \left(-u_{I_{k+L_{13}}I_{k+L_{13}+1}} u_{Z\bar{Z}} + 2 u_{\bar{Z}I_{k+L_{13}}} u_{ZI_{k+L_{13}+1}} - u_{ZI_{k+L_{13}}} u_{\bar{Z}I_{k+L_{13}+1}}\right) + F_{13,23} u_{ZI_{k+L_{13}}} u_{\bar{Z}I_{k+L_{13}+1}}\right] \\
&\quad u_Z^{I_{k+1}} ... u_Z^{I_{k+L_{13}-1}} u_{\bar{Z}}^{I_{k+L_{13}+2}} ... u_{\bar{Z}}^{I_{k+L}} \Big\}
\end{aligned} \tag{3.16}
$$

where we used

$$
\begin{aligned}
c_{\bar{Z}I_{k+1}ZZ} &= \left\langle \operatorname{tr}\left[Z\left(x_1\right) Z\left(x_1\right) \bar{Z}\left(x_2\right) \phi_{I_{k+1}}\left(x_3\right)\right] V_{X+H} \right\rangle_{\mathrm{LO}} \\
&\stackrel{N \gg 1}{=} \frac{\lambda^3 N}{8} \left(X_{1123} + I_{12} I_{13} F_{12,13}\right) u_{Z\bar{Z}} u_{ZI_{k+1}}
\end{aligned} \tag{3.17}
$$

$$
c_{I_{k+L_{13}}\bar{Z}ZZ} = \left\langle \operatorname{tr}\left[\bar{Z}\left(x_2\right) Z\left(x_1\right) Z\left(x_1\right) \phi_{I_{k+L_{13}}}\left(x_3\right)\right] V_{X+H} \right\rangle_{\mathrm{LO}} \tag{3.18}
$$

$$\overset{N\geqslant 1}{=}\frac{\lambda^3 N}{8}\left(X_{1123}+I_{12}I_{13}F_{12,13}\right)u_{Z\bar{Z}}u_{ZI_{k+L_{13}}}$$

$$c_{I_{k+L}Z\bar{Z}\bar{Z}}=\left\langle \mathrm{tr}\left[Z\left(x_1\right)\bar{Z}\left(x_2\right)\bar{Z}\left(x_2\right)\phi_{I_{k+L}}\left(x_3\right)\right]V_{X+H}\right\rangle_{\mathrm{LO}} \tag{3.19}$$

$$\overset{N\geqslant 1}{=}\frac{\lambda^3 N}{8}\left(X_{1223}-I_{12}I_{23}F_{12,23}\right)u_{Z\bar{Z}}u_{\bar{Z}I_{k+L}}$$

$$c_{ZI_{k+L_{13}+1}\bar{Z}\bar{Z}}=\left\langle \mathrm{tr}\left[\bar{Z}\left(x_2\right)\bar{Z}\left(x_2\right)Z\left(x_1\right)\phi_{I_{k+L_{13}+1}}\left(x_3\right)\right]V_{X+H}\right\rangle_{\mathrm{LO}} \tag{3.20}$$

$$\overset{N\geqslant 1}{=}\frac{\lambda^3 N}{8}\left(X_{1223}-I_{12}I_{23}F_{12,23}\right)u_{Z\bar{Z}}u_{\bar{Z}I_{k+L_{13}+1}}$$

$$c_{I_{k+L}I_{k+1}Z\bar{Z}}=\left\langle \mathrm{tr}\left[Z\left(x_1\right)\bar{Z}\left(x_2\right)\phi_{I_{k+L}}\left(x_3\right)\phi_{I_{k+1}}\left(x_3\right)\right]V_{X+H}\right\rangle_{\mathrm{LO}} \tag{3.21}$$

$$\overset{N\geqslant 1}{=}\frac{\lambda^3 N}{8}\left[X_{1233}\left(-u_{I_{k+L}I_{k+1}}u_{Z\bar{Z}}+2u_{ZI_{k+L}}u_{\bar{Z}I_{k+1}}-u_{\bar{Z}I_{k+L}}u_{ZI_{k+1}}\right)+I_{13}I_{23}F_{13,23}u_{\bar{Z}I_{k+L}}u_{ZI_{k+1}}\right]$$

$$c_{I_{k+L_{13}}I_{k+L_{13}+1}\bar{Z}Z}=\left\langle \mathrm{tr}\left[\bar{Z}\left(x_2\right)Z\left(x_1\right)\phi_{I_{k+L_{13}}}\left(x_3\right)\phi_{I_{k+L_{13}+1}}\left(x_3\right)\right]V_{X+H}\right\rangle_{\mathrm{LO}} \tag{3.22}$$

$$\overset{N\geqslant 1}{=}\frac{\lambda^3 N}{8}\left[X_{1233}\left(-u_{I_{k+L_{13}}I_{k+L_{13}+1}}u_{Z\bar{Z}}+2u_{\bar{Z}I_{k+L_{13}}}u_{ZI_{k+L_{13}+1}}-u_{ZI_{k+L_{13}}}u_{\bar{Z}I_{k+L_{13}+1}}\right)+I_{13}I_{23}F_{13,23}u_{ZI_{k+L_{13}}}u_{\bar{Z}I_{k+L_{13}+1}}\right].$$

The 3-pt function at 1 loop is proportional to the Hamiltonian inserted at the two splitting points (in positions $1, L$ and $L_{13}, L_{13}+1$, up to permutations) in $\mathcal{O}_3$.

$$\begin{aligned}
&\sum_{i=1}^{3}\mathrm{diagram}_i \\
&=-\frac{L_1L_2}{2^{(L_1+L_2+L)/2+1}}\frac{\lambda^{(L_1+L_2+L)/2}}{N}\left(I_{12}u_{Z\bar{Z}}\right)^{L_{12}}I_{13}^{L_{13}}I_{23}^{L_{23}}\left(\log\frac{x_{13}^2x_{23}^2}{\epsilon^2x_{12}^2}+2\right)\\
&\sum_{k=0}^{L-1}\left(H_{k+1,k+L}+H_{k+L_{13},k+L_{13}+1}\right)\left(u_Z^{I_{k+1}}...u_Z^{I_{k+L_{13}}}u_{\bar{Z}}^{I_{k+L_{13}+1}}...u_{\bar{Z}}^{I_{k+L}}\right)\\
&=-\frac{L_1L_2}{2^{(L_1+L_2+L)/2+1}}\frac{\lambda^{(L_1+L_2+L)/2}}{N}\left(I_{12}u_{Z\bar{Z}}\right)^{L_{12}}I_{13}^{L_{13}}I_{23}^{L_{23}}\left(\log\frac{x_{13}^2x_{23}^2}{\epsilon^2x_{12}^2}+2\right)\\
&\left[\left(H_{1,L}+H_{L_{13},L_{13}+1}\right)\left(u_Z^{I_1}...u_Z^{I_{L_{13}}}u_{\bar{Z}}^{I_{L_{13}+1}}...u_{\bar{Z}}^{I_L}\right)+\mathrm{c.p.}\right]
\end{aligned} \tag{3.23}$$

We remind that operators P, I, K act first and cyclic permutations afterwards. The 3-pt function of the unit-normalized three single-traces reads

$$\begin{aligned}
&\left(L_1\left(\frac{\lambda}{8\pi^2}\right)^{L_1}u_{Z\bar{Z}}^{L_1}\right)^{-1/2}\left(L_2\left(\frac{\lambda}{8\pi^2}\right)^{L_2}u_{Z\bar{Z}}^{L_2}\right)^{-1/2}\left(C_{I_1\dots I_L}\left(\frac{\lambda}{8\pi^2}\right)^{L}u_{I_1}^{I_1}...u_{I_L}^{I_L}\right)^{-1/2}\left\langle\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_3^{I_1\dots I_L}\right\rangle\\
=&\frac{\sqrt{L_1L_2}}{Nu_{Z\bar{Z}}^{L/2}\left(C_{I_1\dots I_L}u_{I_1}^{I_1}...u_{I_L}^{I_L}\right)^{1/2}}\frac{1}{\left(x_1-x_2\right)^{2L_{12}}\left(x_1-x_3\right)^{2L_{13}}\left(x_2-x_3\right)^{2L_{23}}}\left[u_Z^{I_1}...u_Z^{I_{L_{13}}}u_{\bar{Z}}^{I_{L_{13}+1}}...u_{\bar{Z}}^{I_L}\right.\\
&\left.-\frac{1}{2}\left(\log\frac{x_{13}^2x_{23}^2}{\epsilon^2x_{12}^2}+2\right)\left(H_{1,L}+H_{L_{13},L_{13}+1}\right)\left(u_Z^{I_1}...u_Z^{I_{L_{13}}}u_{\bar{Z}}^{I_{L_{13}+1}}...u_{\bar{Z}}^{I_L}\right)+\mathrm{c.p.}+O\left(\lambda^2\right)\right]
\end{aligned} \tag{3.24}$$

which leads to

$$\Delta_{\mathcal{O}_1}=L_1\qquad\Delta_{\mathcal{O}_2}=L_2\qquad\Delta_{\mathcal{O}_3^{I_1\dots I_L}}=L\qquad\left(C^{(0)}\gamma\right)_{\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_3^{I_1\dots I_L}}=\left(C^{(0)}\gamma\right)_{\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_3^{I_1\dots I_L}}=0 \tag{3.25}$$

$$C^{(0)}_{\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_3^{I_1\dots I_L}}=\frac{\sqrt{L_1L_2}}{Nu_{Z\bar{Z}}^{L/2}\left(C_{I_1\dots I_L}u_{I_1}^{I_1}...u_{I_L}^{I_L}\right)^{1/2}}\left(u_Z^{I_1}...u_Z^{I_{L_{13}}}u_{\bar{Z}}^{I_{L_{13}+1}}...u_{\bar{Z}}^{I_L}+\mathrm{c.p.}\right) \tag{3.26}$$

$$\begin{aligned}
\left(C^{(0)}\gamma\right)_{\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2}=&\frac{2\sqrt{L_1L_2}}{Nu_{Z\bar{Z}}^{L/2}\left(C_{I_1\dots I_L}u_{I_1}^{I_1}...u_{I_L}^{I_L}\right)^{1/2}}\\
&\left[\left(H_{1,L}+H_{L_{13},L_{13}+1}\right)\left(u_Z^{I_1}...u_Z^{I_{L_{13}}}u_{\bar{Z}}^{I_{L_{13}+1}}...u_{\bar{Z}}^{I_L}\right)+\mathrm{c.p.}\right]
\end{aligned} \tag{3.27}$$

$$\tilde{C}^{(1)}_{\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_3^{I_1\dots I_L}}=-\frac{1}{2}\left(C^{(0)}\gamma\right)_{\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2}. \tag{3.28}$$

We will need the tree-level structure constant with the last index up.

$$C^{(0)\tilde{\mathcal{O}}_3^{I_1\dots I_L}}_{\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2}=C^{(0)}_{\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_3^{J_1\dots J_L}}h^{\tilde{\mathcal{O}}_3^{I_1\dots I_L}\tilde{\mathcal{O}}^3_{J_1\dots J_L}} \tag{3.29}$$

$$
\begin{aligned}
=&\frac{\sqrt{L_1L_2}}{LNu_{Z\bar{Z}}^{L/2}}\left(\frac{C_{I_1\dots I_L}}{u_{I_1}^{I_1}\dots u_{I_L}^{I_L}}\right)^{1/2}\left(u_Z^{I_1}\dots u_Z^{I_{L_{13}}}u_{\bar{Z}}^{I_{L_{13}+1}}\dots u_{\bar{Z}}^{I_L}+\text{c.p.}\right)\\
=&\frac{C_{I_1\dots I_L}}{L}C^{(0)}_{\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_3^{J_1\dots J_L}}
\end{aligned}
$$

Calculate renormalization-invariant structure constant

The aim is expanding the renormalization-invariant structure constant

$$
C^{(1)}_{\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_3^{I_1\cdots I_L}}=\tilde{C}^{(1)}_{\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_3^{I_1\cdots I_L}}-\frac{1}{2}g_{\tilde{\mathcal{O}}_1\bar{\tilde{\mathcal{O}}}^3_{J_1\cdots}}C^{(0)\tilde{\mathcal{O}}_3^{J_1\cdots}}_{\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_3^{I_1\cdots I_L}}-\frac{1}{2}g_{\tilde{\mathcal{O}}_2\bar{\tilde{\mathcal{O}}}^3_{J_1\cdots}}C^{(0)\tilde{\mathcal{O}}_3^{J_1\cdots}}_{\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_3^{I_1\cdots I_L}}-\frac{1}{2}g_{\tilde{\mathcal{O}}_3^{I_1\cdots I_L}\bar{\mathcal{O}}^3_{J_1\cdots}}C^{(0)\tilde{\mathcal{O}}_3^{J_1\cdots}}_{\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2} \tag{3.30}
$$

Two operators are BPS, so what is left is

$$
\begin{aligned}
=&\tilde{C}^{(1)}_{\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_3^{I_1\cdots I_L}}-\frac{1}{2}g_{\tilde{\mathcal{O}}_1\bar{\tilde{\mathcal{O}}}^3_{J_1\cdots}}C^{(0)\tilde{\mathcal{O}}_3^{J_1\cdots}}_{\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_3^{I_1\cdots I_L}} \\
=&-\frac{\sqrt{L_1L_2}}{2Nu_{Z\bar{Z}}^{L/2}\left(C_{I_1\dots I_L}u_{I_1}^{I_1}\dots u_{I_L}^{I_L}\right)^{1/2}}\left[(H_{1,L}+H_{L_{13},L_{13}+1})\left(u_Z^{I_1}\dots u_Z^{I_{L_{13}}}u_{\bar{Z}}^{I_{L_{13}+1}}\dots u_{\bar{Z}}^{I_L}\right)+\text{c.p.}\right].
\end{aligned} \tag{3.31}
$$

We do tricks done in the analogue computation for GG. The Hamiltonian on the sequence $u_Z^{I_1}\dots u_Z^{I_{L_{13}}}u_{\bar{Z}}^{I_{L_{13}+1}}\dots u_{\bar{Z}}^{I_L}$ truncates to the splitting points $k=L_{13},\,L$.

4 $SL(2)$ sector

We approach the 3-pt function through the connected 4-pt function of 2 GG and 2 vacua

$$\mathcal{O}_3 = \mathrm{tr}\left(\phi^I\right)^{L/2}(x_3) \qquad\qquad \mathcal{O}_4 = \mathrm{tr}\left(\phi^J\right)^{L/2}(x_4) \tag{4.1}$$

followed by OPE. We assume $L = 2, 4, 6, ... \ll N$. Note that when $L = 4, 8, 12, ...$ there is the contribution of the term $k = L/4$ in the 2-trace (tree-level and one-loop) PCGG. At tree level, it gives something $\propto \left\langle \mathrm{tr}\left(Z\bar{Z}\right)^{L/2}\mathcal{O}_3\right\rangle_{\mathrm{LO}}\left\langle \mathrm{tr}\left(Z\bar{Z}\right)^{L/2}\mathcal{O}_4\right\rangle_{\mathrm{LO}}$ like the product of 3-pt functions of GGs and the vacuum; at one-loop the net contribution is zero. We are going to neglect these all together.

We focus on the connected part of the correlator

$$\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle = \langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle^{\mathrm{connected}} + \langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle^{\mathrm{disconnected}} .$$

The disconnected piece is the product of 2 protected 2-pt functions.

$$\begin{aligned}
&\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle^{\mathrm{disconnected}}\\
\equiv&\langle\mathcal{O}_1\mathcal{O}_2\rangle\langle\mathcal{O}_3\mathcal{O}_4\rangle\\
=&\langle\mathcal{G}_{L=0}\rangle_{\mathrm{LO}}\langle\mathcal{O}_3\mathcal{O}_4\rangle_{\mathrm{LO}}\\
=&\frac{N!}{2^N}\frac{\lambda^N}{N^N}\left(I_{12}u_{Z\bar{Z}}\right)^N \times \frac{L}{2^{L/2+1}}\lambda^{L/2}\left(I_{34}u_{IJ}\right)^{L/2}\\
=&\frac{L\,N!}{2^{N+L/2+1}}\frac{\lambda^{N+L/2}}{N^N}\left(I_{12}u_{Z\bar{Z}}\right)^N\left(I_{34}u_{IJ}\right)^{L/2}
\end{aligned}\tag{4.2}$$

4.1 Tree level

When the PCGG has length $\ell = 2, 4, ...L$, the bridge lengths between PCGG and operators are $\ell/2$ and the bridge length between the operators is $(L-\ell)/2$. The underlined factors originate from the $\frac{\ell}{2}\times\frac{L}{2}\times\frac{L}{2}$ and $\ell\times\frac{L}{2}\times\frac{L}{2}$ distinct contractions between the 3 traces. I use a formally incorrect but compact notation that collects common factors before/after sums and not in sums.

$$a(\ell)\left(\sum_{\ell\in I}b(\ell)+\sum_{\ell\in J}c(\ell)\right)d(\ell)\equiv\sum_{\ell\in I}a(\ell)\,b(\ell)\,d(\ell)+\sum_{\ell\in J}a(\ell)\,c(\ell)\,d(\ell) \tag{4.3}$$

$$\begin{aligned}
&\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle^{\mathrm{connected}}_{\mathrm{LO}}\\
=&\sum_{\ell/2=1}^{L/2}\langle\mathcal{G}_\ell\mathcal{O}_3\mathcal{O}_4\rangle^{\mathrm{connected}}_{\mathrm{LO}}\\
\overset{N\gg1}{=}&-\frac{(-)^{\ell/2}\left(N-\frac{\ell}{2}\right)!}{2^{N-\ell/2-1}\ell}\frac{\lambda^{N-\ell/2}}{N^{N-\ell/2}}I_{12}^{N-\ell/2}u_{Z\bar{Z}}^{N-\ell/2}\\
&\left\{\sum_{\ell/2=1,3,...}\left[\underline{\frac{\ell L^2}{8}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell+2)/4}\right.\right.\\
&\left.+\underline{\frac{\ell L^2}{8}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell-2)/4}\right]\\
&\left.+\sum_{\ell/2=2,4,...}\left[\underline{\frac{\ell L^2}{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4}\right]\right\}\left(\frac{\lambda}{N}I_{34}u_{IJ}\right)^{(L-\ell)/2}\frac{N^{(L+\ell)/2-1}}{2^{(L+\ell)/2}}\\
=&-\frac{(-)^{\ell/2}L^2\left(N-\frac{\ell}{2}\right)!}{2^{N+L/2+2}}\frac{\lambda^{N+L/2}}{N^{N-\ell/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-\ell/2}\left(I_{34}u_{IJ}\right)^{(L-\ell)/2}\left(I_{13}I_{14}I_{23}I_{24}u_{ZI}u_{\bar{Z}I}u_{ZJ}u_{\bar{Z}J}\right)^{\ell/4}\\
&\left\{\sum_{\ell/2=1,3,...}\left[\left(\frac{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}\right)^{1/2}+\left(\frac{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}\right)^{1/2}\right]+\sum_{\ell/2=2,4,...}2\right\}
\end{aligned}\tag{4.4}$$

Nothing is lost in the approximation $\left(N-\frac{\ell}{2}\right)!\sim N^{-\ell/2}N!$ at fixed ℓ because sums have finitely-many terms.

4.2 One loop

It is similar to 3-pt function in $SO(6)$ sector. The exceptions are:

- bridge lengths $\ell/2$ are variable and take even/odd values,
- non-nearest neighbor interactions are possible in "length-conserving" diagrams $\ell = L$ where the bridge length between $\mathcal{O}_3\mathcal{O}_4$ is zero.

Kronecker deltas account for these exceptions while keeping the same summation ranges. $c_{\phi_1\phi_2\phi_3\phi_4} \equiv \langle \mathrm{tr}(\phi_1\phi_2\phi_3\phi_4) V_{X+H}\rangle_{\mathrm{LO}}$ denotes the usual tree-level contraction between a 4-scalar single-trace and the X+H vertex, namely the X+H contribution to the 1-loop of a 4-scalar single-trace.

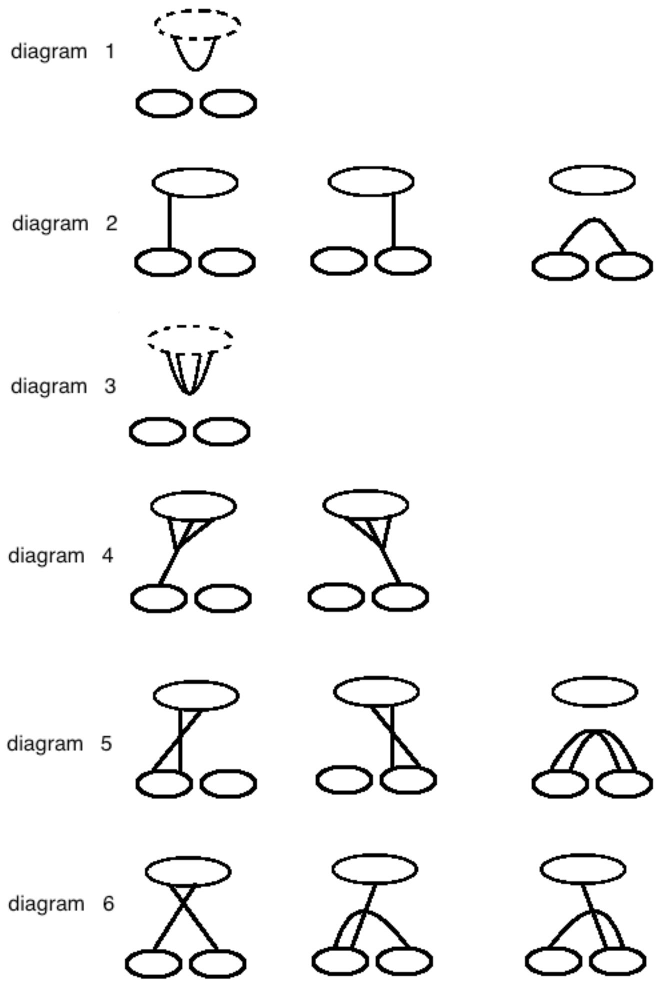

Figure 2: Diagrams with the same number are labeled by $a, b, \ldots$, for example 2a, 2b, 2c. In each drawing, the top solid circle is the single-trace in the PCGG, the top dashed circle is the full PCGG, the bottom circles are the operators. Free contractions are omitted, the line is the O vertex (= 1-loop propagator), the cross is the X+H vertex. Diagrams 2,5,6 have $\mathcal{G}_L$, diagrams 1,4 have $\mathcal{G}_{L+2}$, diagrams 3 have $\mathcal{G}_{L+4}$.

Diagram 2

It has the topology of LO, thus no exception shows up. When the operators have length $L/2 = 1$ the combination of diagram 2a and 2b vanishes. We include the special case with a Kronecker delta.

$$
\begin{aligned}
&\sum_{i=a,b} \mathrm{diagram}_{2i} \qquad (4.5)\\
&\overset{N\gg 1}{=} \sum_{\ell/2=1}^{L/2} \langle \mathcal{G}_\ell \mathcal{O}_3\mathcal{O}_4 V_O\rangle_{\mathrm{LO}}^{\mathrm{connected}}\\
&= -\frac{(-)^{\ell/2}\left(N-\frac{\ell}{2}\right)!}{2^{N-\ell/2-1}\ell}\frac{\lambda^{N-\ell/2}}{N^{N-\ell/2}} I_{12}^{N-\ell/2} u_{Z\bar{Z}}^{N-\ell/2}\left(1-\delta_{L,2}\right)\Bigg\{\sum_{\ell/2=1,3,\ldots}
\end{aligned}
$$

$$
\left[\underline{\frac{\ell L^2}{8}\frac{\ell+2}{4}}\left(-\frac{\lambda^2}{2}u_{ZI}\left(Y_{113}+Y_{133}\right)\right)\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell+2)/4-1}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell+2)/4}\right.
$$

$$
+\underline{\frac{\ell L^2}{8}\frac{\ell-2}{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell+2)/4}\left(-\frac{\lambda^2}{2}u_{\bar{Z}I}\left(Y_{223}+Y_{233}\right)\right)\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell-2)/4-1}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell+2)/4}
$$

$$
+\underline{\frac{\ell L^2}{8}\frac{\ell-2}{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell-2)/4}\left(-\frac{\lambda^2}{2}u_{ZJ}\left(Y_{114}+Y_{144}\right)\right)\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell-2)/4-1}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell+2)/4}
$$

$$
+\underline{\frac{\ell L^2}{8}\frac{\ell+2}{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell-2)/4}\left(-\frac{\lambda^2}{2}u_{\bar{Z}J}\left(Y_{224}+Y_{244}\right)\right)\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell+2)/4-1}
$$

$$
+\underline{\frac{\ell L^2}{8}\frac{\ell-2}{4}}\left(-\frac{\lambda^2}{2}u_{ZI}\left(Y_{113}+Y_{133}\right)\right)\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell-2)/4-1}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell-2)/4}
$$

$$
+\underline{\frac{\ell L^2}{8}\frac{\ell+2}{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell-2)/4}\left(-\frac{\lambda^2}{2}u_{\bar{Z}I}\left(Y_{223}+Y_{233}\right)\right)\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell+2)/4-1}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell-2)/4}
$$

$$
+\underline{\frac{\ell L^2}{8}\frac{\ell+2}{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell+2)/4}\left(-\frac{\lambda^2}{2}u_{ZJ}\left(Y_{114}+Y_{144}\right)\right)\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell+2)/4-1}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell-2)/4}
$$

$$
\left.+\underline{\frac{\ell L^2}{8}\frac{\ell-2}{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell+2)/4}\left(-\frac{\lambda^2}{2}u_{\bar{Z}J}\left(Y_{224}+Y_{244}\right)\right)\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell-2)/4-1}\right]
$$

$$
+\sum_{\ell/2=2,4,\ldots}\left[\underline{\frac{\ell L^2}{4}\frac{\ell}{4}}\left(-\frac{\lambda^2}{2}u_{ZI}\left(Y_{113}+Y_{133}\right)\right)\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4-1}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4}\right.
$$

$$
+\underline{\frac{\ell L^2}{4}\frac{\ell}{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4}\left(-\frac{\lambda^2}{2}u_{\bar{Z}I}\left(Y_{223}+Y_{233}\right)\right)\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4-1}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4}
$$

$$
+\underline{\frac{\ell L^2}{4}\frac{\ell}{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4}\left(-\frac{\lambda^2}{2}u_{ZJ}\left(Y_{114}+Y_{144}\right)\right)\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4-1}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4}
$$

$$
\left.\left.+\underline{\frac{\ell L^2}{4}\frac{\ell}{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4}\left(-\frac{\lambda^2}{2}u_{\bar{Z}J}\left(Y_{224}+Y_{244}\right)\right)\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4-1}\right]\right\}
$$

$$
\left(\frac{\lambda}{N}I_{34}u_{IJ}\right)^{(L-\ell)/2}\frac{N^{(L+\ell)/2-2}}{2^{(L+\ell)/2-1}}
$$

$$
=\frac{(-)^{\ell/2}L^2\left(N-\frac{\ell}{2}\right)!}{2^{N+L/2+4}}\frac{\lambda^{N+L/2+1}}{N^{N-\ell/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-\ell/2}\left(I_{34}u_{IJ}\right)^{(L-\ell)/2}\left(I_{13}I_{23}I_{14}I_{24}u_{ZI}u_{\bar{Z}I}u_{ZJ}u_{\bar{Z}J}\right)^{\ell/4}
$$

$$
(1-\delta_{L,2})\left\{\sum_{\ell/2=1,3,\ldots}\left[(\ell+2)\,I_{13}^{-1}\left(Y_{113}+Y_{133}\right)\left(\frac{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}\right)^{1/2}+(\ell-2)\,I_{23}^{-1}\left(Y_{223}+Y_{233}\right)\left(\frac{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}{I_{14}I_{23}u_{ZJ}u_{\bar{Z}I}}\right)^{1/2}\right.\right.
$$

$$
+(\ell-2)\,I_{14}^{-1}\left(Y_{114}+Y_{144}\right)\left(\frac{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}\right)^{1/2}+(\ell+2)\,I_{24}^{-1}\left(Y_{224}+Y_{244}\right)\left(\frac{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}\right)^{1/2}
$$

$$
+(\ell-2)\,I_{13}^{-1}\left(Y_{113}+Y_{133}\right)\left(\frac{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}{I_{24}I_{13}u_{\bar{Z}J}u_{ZI}}\right)^{1/2}+(\ell+2)\,I_{23}^{-1}\left(Y_{223}+Y_{233}\right)\left(\frac{I_{14}I_{23}u_{ZJ}u_{\bar{Z}I}}{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}\right)^{1/2}
$$

$$
\left.+(\ell+2)\,I_{14}^{-1}\left(Y_{114}+Y_{144}\right)\left(\frac{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}\right)^{1/2}+(\ell-2)\,I_{24}^{-1}\left(Y_{224}+Y_{244}\right)\left(\frac{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}\right)^{1/2}\right]
$$

$$
\left.+\sum_{\ell/2=2,4,\ldots}\left[2\ell I_{13}^{-1}\left(Y_{113}+Y_{133}\right)+2\ell I_{23}^{-1}\left(Y_{223}+Y_{233}\right)+2\ell I_{14}^{-1}\left(Y_{114}+Y_{144}\right)+2\ell I_{24}^{-1}\left(Y_{224}+Y_{244}\right)\right]\right\}
$$

Operators can have length $L/2=1$, which implies $\ell=L=2$, in diagram 2c. It vanishes automatically for the special value, so no concern about a possible correction.

$$
\begin{aligned}
&\text{diagram}_{2c} \qquad (4.6)\\
&=\sum_{\ell/2=1}^{L/2}\langle\mathcal{G}_\ell\mathcal{O}_3\mathcal{O}_4\rangle^{\text{connected}}_{\text{LO}}\\
&\overset{N\gg1}{=}-\frac{(-)^{\ell/2}\left(N-\frac{\ell}{2}\right)!}{2^{N-\ell/2-1}\ell}\frac{\lambda^{N-\ell/2}}{N^{N-\ell/2}}I_{12}^{N-\ell/2}u_{Z\bar{Z}}^{N-\ell/2}
\end{aligned}
$$

$$
\begin{aligned}
&\left\{\sum_{\ell/2=1,3,\dots}\left[\frac{\ell L^2}{\underline{8}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell+2)/4}\right.\right.\\
&\left.+\frac{\ell L^2}{\underline{8}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell-2)/4}\right]\\
&\left.+\sum_{\ell/2=2,4,\dots}\left[\frac{\ell L^2}{\underline{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4}\right]\right\}\\
&\frac{L-\ell}{\underline{2}}\left(-\frac{\lambda^2}{2}u_{IJ}\left(Y_{334}+Y_{344}\right)\right)\left(\frac{\lambda}{N}I_{34}u_{IJ}\right)^{(L-\ell)/2-1}\frac{N^{(L+\ell)/2-2}}{2^{(L+\ell)/2-1}}\\
=&\frac{(-)^{\ell/2}L^2(L-\ell)\left(N-\frac{\ell}{2}\right)!}{2^{N+L/2+3}}\frac{\lambda^{N+L/2+1}}{N^{N-\ell/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-\ell/2}\left(I_{34}u_{IJ}\right)^{(L-\ell)/2}\left(I_{13}I_{23}I_{14}I_{24}u_{ZI}u_{\bar{Z}I}u_{ZJ}u_{\bar{Z}J}\right)^{\ell/4}\\
&I_{34}^{-1}\left(Y_{334}+Y_{344}\right)\left\{\sum_{\ell/2=1,3,\dots}\left[\left(\frac{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}\right)^{1/2}+\left(\frac{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}\right)^{1/2}\right]+\sum_{\ell/2=2,4,\dots}2\right\}
\end{aligned}
$$

Diagram 1+3

Unlike what was done for the 3-pt functions, we don't have enough computational power to look for relations between diagram$_1$, diagram$_3$ and their components diagram$_{1a}$, diagram$_{3a}$ at given low L,ℓ, followed by an extrapolation at any L,ℓ. We use instead the knowledge of the 1-loop PCGG and calculate analytically

$$
\begin{aligned}
&\sum_{i=1,3}\text{diagram}_i \qquad (4.7)\\
=&\sum_{\ell/2=1}^{L/2}\left\langle\mathcal{G}_\ell^{\text{NLO}}\mathcal{O}_3\mathcal{O}_4\right\rangle_{\text{LO}}\\
\overset{N\gg 1}{=}&\sum_{\ell/2=1}^{L/2}\left\{-\frac{\lambda\ell}{2}I_{12}^{-1}\left(Y_{112}+Y_{122}\right)\left\langle\mathcal{G}_\ell^{\text{LO}}\mathcal{O}_3\mathcal{O}_4\right\rangle_{\text{LO}}\right\}\\
&+\delta_{L/2}^{\text{odd}}\frac{\left(N-\frac{L}{2}\right)!}{2^{N-L/2}\left(\frac{L}{2}!\right)^2}\frac{\lambda^{N-L/2+1}}{N^{N-L/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-L/2}I_{12}^{-1}\left(Y_{112}+Y_{122}\right)\left(\frac{L}{2}!\right)^2\left\langle\text{tr}\left[Z\left(Z\bar{Z}\right)^{\frac{L-2}{4}}\right]\text{tr}\left[\bar{Z}\left(Z\bar{Z}\right)^{\frac{L-2}{4}}\right]\mathcal{O}_3\mathcal{O}_4\right\rangle_{\text{LO}}\\
&-\delta_{L/2}^{\text{even}}\frac{\left(N-\frac{L}{2}\right)!}{2^{N-L/2}\left(\frac{L}{2}!\right)^2}\frac{\lambda^{N-L/2+1}}{N^{N-L/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-L/2}I_{12}^{-1}\left(Y_{112}+Y_{122}\right)\frac{L}{2}!\left(\frac{L}{2}-1\right)!\\
&\left\langle\left\{4\text{tr}\left(Z\bar{Z}\right)^{L/4}\text{tr}\left(Z\bar{Z}\right)^{L/4}+\left(\frac{L}{2}-2\right)\sum_{i=0}^{L/4-2}\text{tr}\left(Z\bar{Z}\right)^{L/4}\text{tr}\left[\left(Z\bar{Z}\right)^iZZ\left(Z\bar{Z}\right)^{L/4-2-i}\bar{Z}\bar{Z}\right]\right\}\mathcal{O}_3\mathcal{O}_4\right\rangle_{\text{LO}}\\
=&\frac{(-)^{\ell/2}\ell\left(N-\frac{\ell}{2}\right)!}{2^{N-\ell/2}\ell}\frac{\lambda^{N-\ell/2+1}}{N^{N-\ell/2}}I_{12}^{N-\ell/2-1}\left(Y_{112}+Y_{122}\right)u_{Z\bar{Z}}^{N-\ell/2}\\
&\left\{\sum_{\ell/2=1,3,\dots}\left[\frac{\ell L^2}{\underline{8}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell+2)/4}\right.\right.\\
&\left.+\frac{\ell L^2}{\underline{8}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell-2)/4}\right]\\
&\left.+\sum_{\ell/2=2,4,\dots}\left[\frac{\ell L^2}{\underline{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4}\right]\right\}\left(\frac{\lambda}{N}I_{34}u_{IJ}\right)^{(L-\ell)/2}\frac{N^{(L+\ell)/2-1}}{2^{(L+\ell)/2}}\\
&+\delta_{L/2}^{\text{odd}}\frac{\left(N-\frac{L}{2}\right)!}{2^{N-L/2}\left(\frac{L}{2}!\right)^2}\frac{\lambda^{N-L/2+1}}{N^{N-L/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-L/2}I_{12}^{-1}\left(Y_{112}+Y_{122}\right)\left(\frac{L}{2}!\right)^2\\
&\left[\frac{L^2}{\underline{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(L+2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(L-2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(L-2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(L+2)/4}\right.
\end{aligned}
$$

$$+\frac{L^2}{\underline{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(L-2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(L+2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(L+2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(L-2)/4}\Bigg]\frac{N^{L/2}}{2^{L/2}}\frac{N^{L/2}}{2^{L/2}}$$

$$-\delta^{\text{even}}_{L/2}\frac{\left(N-\frac{L}{2}\right)!}{2^{N-L/2}\left(\frac{L}{2}!\right)^2}\frac{\lambda^{N-L/2+1}}{N^{N-L/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-L/2}I_{12}^{-1}\left(Y_{112}+Y_{122}\right)\frac{L}{2}!\left(\frac{L}{2}-1\right)!$$

$$\left[2\frac{L^2}{\underline{4}}\left(\frac{L}{2}+2\right)\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{L/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{L/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{L/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{L/4}\right]\frac{N^{L/2}}{2^{L/2}}\frac{N^{L/2}}{2^{L/2}}$$

$$=\frac{(-)^{\ell/2}L^2\ell\left(N-\frac{\ell}{2}\right)!}{2^{N+L/2+3}}\frac{\lambda^{N+L/2+1}}{N^{N-\ell/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-\ell/2}\left(I_{34}u_{IJ}\right)^{(L-\ell)/2}\left(I_{13}I_{23}I_{14}I_{24}u_{ZI}u_{\bar{Z}I}u_{ZJ}u_{\bar{Z}J}\right)^{\ell/4}$$

$$I_{12}^{-1}\left(Y_{112}+Y_{122}\right)\left\{\sum_{\ell/2=1,3,\dots L/2}\left[\left(\frac{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}\right)^{1/2}+\left(\frac{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}\right)^{1/2}\right]+\sum_{\ell/2=2,4,\dots L/2-1}2\right\}$$

$$+\delta^{\text{odd}}_{L/2}\frac{L^2\left(N-\frac{L}{2}\right)!}{2^{N+L/2+2}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-L/2}\left(I_{13}I_{23}I_{14}I_{24}u_{ZI}u_{\bar{Z}I}u_{ZJ}u_{\bar{Z}J}\right)^{L/4}$$

$$I_{12}^{-1}\left(Y_{112}+Y_{122}\right)\left[\left(\frac{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}\right)^{1/2}+\left(\frac{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}\right)^{1/2}\right]$$

$$-\delta^{\text{even}}_{L/2}\frac{L\left(L+4\right)\left(N-\frac{L}{2}\right)!}{2^{N+L/2+1}}\frac{\lambda^{N+L/2+1}}{N^{N-L/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-L/2}\left(I_{13}I_{23}I_{14}I_{24}u_{ZI}u_{\bar{Z}I}u_{ZJ}u_{\bar{Z}J}\right)^{L/4}I_{12}^{-1}\left(Y_{112}+Y_{122}\right)$$

Many comments on derivation.

- The equality with "$N\gg 1$" does not use the "approximated planar formula" for the 1-loop PCGG because the M part contributes with the product of two equal-length traces. Once such $\mathrm{tr}\left[Z\left(Z\bar{Z}\right)^{\frac{L-2}{4}}\right]\mathrm{tr}\left[\bar{Z}\left(Z\bar{Z}\right)^{\frac{L-2}{4}}\right]$ gets contracted with the operators, it gives rise to

$$\left\langle\mathrm{tr}\left[Z\left(Z\bar{Z}\right)^{\frac{L-2}{4}}\right]\mathcal{O}_3\right\rangle_{\text{LO}}\left\langle\mathrm{tr}\left[\bar{Z}\left(Z\bar{Z}\right)^{\frac{L-2}{4}}\right]\mathcal{O}_4\right\rangle_{\text{LO}}+(3\leftrightarrow 4)\tag{4.8}$$

  that competes with $\left\langle\mathcal{G}^{\text{LO}}_L\mathcal{O}_3\mathcal{O}_4\right\rangle_{\text{LO}}$ in powers of N. Note also that it corresponds to diagrams with connected topology too, thanks to Z and $\bar{Z}$ in number $\sim N$ acting like a bridge between $\mathcal{O}_3$ and $\mathcal{O}_4$, hence no reason exists to put it aside.

- To derive the two equal-length traces from the multitrace expansion of M, we follow the strategy laid out in the section about the 1-loop PCGG. Let us present $L/2=2,6,\dots$ (a) and $L/2=4,8,\dots$ (b).
  a) Start from $\delta\delta$ contracted with $\frac{L}{2}+1$ pairs of $Z\bar{Z}$

$$\begin{aligned}&\delta^{i_1,\dots i_{L/2+1}}_{l_1,\dots l_{L/2+1}}\delta^{j_1,\dots j_{L/2+1}}_{k_1,\dots k_{L/2+1}}Z_{i_1k_1}\dots Z_{i_{L/2+1}k_{L/2+1}}\bar{Z}_{j_1l_1}\dots\bar{Z}_{j_{L/2+1}l_{L/2+1}}\\&=\left(\left(\frac{L}{2}+1\right)!\right)^2\sum_{k=1}^{(L+2)/4}\frac{\ell_{k,L/2+1}}{k\left(\frac{L}{2}+1-k\right)}\mathrm{tr}\left(Z\bar{Z}\right)^k\mathrm{tr}\left(Z\bar{Z}\right)^{L/2+1-k}+\dots\end{aligned}\tag{4.9}$$

  where all non-double-trace terms are dumped into dots. To contract 2 pairs of indices in the lhs and read off the wanted result in the rhs, apply the substitution rule

$$Z_{ij}\bar{Z}_{kl}\to\delta_{ij}\delta_{kl}\,.\tag{4.10}$$

  There are no forbidden replacements [9]. Replacements generate M in the lhs, while in the rhs we want to do only those replacements that generate two traces of equal length $L/2$. This means focusing on the term $k=(L+2)/4$ and "remove" $Z_{ij}\to\delta_{ij}$ from one trace and $\bar{Z}_{kl}\to\delta_{kl}$ from the other one.

$$\begin{aligned}&\underline{\left(\frac{L}{2}+1\right)^2}M^{i_1,\dots i_{L/2};j_1,\dots j_{L/2}}_{l_1,\dots l_{L/2};k_1,\dots k_{L/2}}Z_{i_1k_1}\dots Z_{i_{L/2}k_{L/2}}\bar{Z}_{j_1l_1}\dots\bar{Z}_{j_{L/2}l_{L/2}}\\&=\frac{\left(\left(\frac{L}{2}+1\right)!\right)^2}{2\left(\frac{L+2}{4}\right)^2}\underline{2\left(\frac{L+2}{4}\right)^2}\mathrm{tr}\left[Z\left(Z\bar{Z}\right)^{\frac{L-2}{4}}\right]\mathrm{tr}\left[\bar{Z}\left(Z\bar{Z}\right)^{\frac{L-2}{4}}\right]+\dots\end{aligned}\tag{4.11}$$

  Dots collect more junk we are not interested in and combinatorial factors from replacements are underlined. We obtain the product of two equal-length traces in the multitrace expansion of M contracted with scalars

$$M^{i_1,\dots i_{L/2};j_1,\dots j_{L/2}}_{l_1,\dots l_{L/2};k_1,\dots k_{L/2}}Z_{i_1k_1}\dots Z_{i_{L/2}k_{L/2}}\bar{Z}_{j_1l_1}\dots\bar{Z}_{j_{L/2}l_{L/2}}=\left(\frac{L}{2}!\right)^2\mathrm{tr}\left[Z\left(Z\bar{Z}\right)^{\frac{L-2}{4}}\right]\mathrm{tr}\left[\bar{Z}\left(Z\bar{Z}\right)^{\frac{L-2}{4}}\right]+\dots\,.\tag{4.12}$$

[9] There would be some in applying the rule to $\mathrm{tr}\left(Z\bar{Z}\right)\mathrm{tr}\left(\dots\right)\mathrm{tr}\left(\dots\right)$, but these terms got dumped already.

b) Let us repeat:

$$
\begin{aligned}
&\delta^{i_1,\dots i_{L/2+1}}_{l_1,\dots l_{L/2+1}}\delta^{j_1,\dots j_{L/2+1}}_{k_1,\dots k_{L/2+1}} Z_{i_1k_1}\dots Z_{i_{L/2+1}k_{L/2+1}}\bar{Z}_{j_1l_1}\dots\bar{Z}_{j_{L/2+1}l_{L/2+1}} \\
&= -\left(\left(\frac{L}{2}+1\right)!\right)^2 \sum_{k=1}^{L/4} \frac{\ell_{k,L/2+1}}{k\left(\frac{L}{2}+1-k\right)} \mathrm{tr}\left(Z\bar{Z}\right)^k \mathrm{tr}\left(Z\bar{Z}\right)^{L/2+1-k} + \dots \\
&= -\frac{\left(\left(\frac{L}{2}+1\right)!\right)^2}{\frac{L}{4}\left(\frac{L}{4}+1\right)} \mathrm{tr}\left(Z\bar{Z}\right)^{L/4} \mathrm{tr}\left(Z\bar{Z}\right)^{L/4+1} + \dots \\
&\downarrow \\
&\underline{\left(\frac{L}{2}+1\right)^2} M^{i_1,\dots i_{L/2};j_1,\dots j_{L/2}}_{l_1,\dots l_{L/2};k_1,\dots k_{L/2}} Z_{i_1k_1}\dots Z_{i_{L/2}k_{L/2}}\bar{Z}_{j_1l_1}\dots\bar{Z}_{j_{L/2}l_{L/2}} \\
&= -\frac{\left(\left(\frac{L}{2}+1\right)!\right)^2}{\frac{L}{4}\left(\frac{L}{4}+1\right)} \left\{\underline{2\left(\frac{L}{4}+1\right)} \mathrm{tr}\left(Z\bar{Z}\right)^{L/4} \mathrm{tr}\left(Z\bar{Z}\right)^{L/4} + \underline{\left(\frac{L}{4}+1\right)\sum_{i=0}^{L/4-2}} \mathrm{tr}\left(Z\bar{Z}\right)^{L/4} \mathrm{tr}\left[\left(Z\bar{Z}\right)^i ZZ\left(Z\bar{Z}\right)^{L/4-2-i}\bar{Z}\bar{Z}\right]\right\} + \dots
\end{aligned}
\tag{4.13}
$$

so

$$
\begin{aligned}
&M^{i_1,\dots i_{L/2};j_1,\dots j_{L/2}}_{l_1,\dots l_{L/2};k_1,\dots k_{L/2}} Z_{i_1k_1}\dots Z_{i_{L/2}k_{L/2}}\bar{Z}_{j_1l_1}\dots\bar{Z}_{j_{L/2}l_{L/2}} \\
&= -\frac{L}{2}!\left(\frac{L}{2}-1\right)! \left\{4\mathrm{tr}\left(Z\bar{Z}\right)^{L/4} \mathrm{tr}\left(Z\bar{Z}\right)^{L/4} + 2\sum_{i=0}^{L/4-2} \mathrm{tr}\left(Z\bar{Z}\right)^{L/4} \mathrm{tr}\left[\left(Z\bar{Z}\right)^i ZZ\left(Z\bar{Z}\right)^{L/4-2-i}\bar{Z}\bar{Z}\right]\right\} + \dots .
\end{aligned}
\tag{4.14}
$$

Diagram 4

The X+H vertex connects neighboring scalars in PCGG, so the rest of the diagram looks like LO. When $\ell = L$, it can connect also

- Z and $\bar{Z}Z$ with $(\ell-2)/2$ scalars in between,
- Z and $Z\bar{Z}$ with $\ell/2$ scalars in between,
- $\bar{Z}$ and $Z\bar{Z}$ with $(\ell-2)/2$ scalars in between,
- $\bar{Z}$ and $\bar{Z}Z$ with $\ell/2$ scalars in between.

The case $L = 2$ is again to exclude with a delta.

$$
\begin{aligned}
&\mathrm{diagram}_4 \\
&= \sum_{\ell/2=1}^{L/2} \langle \mathcal{G}_{\ell+2}\mathcal{O}_3\mathcal{O}_4 V_{X+H}\rangle_{\mathrm{LO}} \\
&\overset{N\geqq 1}{=} \frac{(-)^{\ell/2}\left(N-\frac{\ell}{2}-1\right)!}{2^{N-\ell/2-2}(\ell+2)} \frac{\lambda^{N-\ell/2-1}}{N^{N-\ell/2-1}} I_{12}^{N-\ell/2-1} u_{Z\bar{Z}}^{N-\ell/2-1}\left(1-\delta_{L,2}\right)\Bigg\{\sum_{\ell/2=1,3,\dots} \\
&\Bigg[\left(\underline{\frac{\ell+2}{2}\frac{\ell+2}{4}\frac{L}{2}\frac{L}{2}} c_{\phi_I Z\bar{Z}Z} + \underline{\delta_{\ell,L}\frac{\ell+2}{2}\frac{L}{2}\frac{L}{2}}\left(c_{\phi_I ZZ\bar{Z}} + c_{\phi_I \bar{Z}ZZ}\right)\right)\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell+2)/4-1}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell+2)/4} \\
&+\left(\underline{\frac{\ell+2}{2}\frac{\ell-2}{4}\frac{L}{2}\frac{L}{2}} c_{\phi_I \bar{Z}Z\bar{Z}}\right)\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell-2)/4-1}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell+2)/4} \\
&+\left(\underline{\frac{\ell+2}{2}\frac{\ell-2}{4}\frac{L}{2}\frac{L}{2}} c_{\phi_J Z\bar{Z}Z}\right)\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell-2)/4-1}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell+2)/4} \\
&+\left(\underline{\frac{\ell+2}{2}\frac{\ell+2}{4}\frac{L}{2}\frac{L}{2}} c_{\phi_J \bar{Z}Z\bar{Z}} + \underline{\delta_{\ell,L}\frac{\ell+2}{2}\frac{L}{2}\frac{L}{2}}\left(c_{\phi_J \bar{Z}\bar{Z}Z} + c_{\phi_J Z\bar{Z}\bar{Z}}\right)\right)\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell+2)/4-1} \\
&+\left(\underline{\frac{\ell+2}{2}\frac{\ell-2}{4}\frac{L}{2}\frac{L}{2}} c_{\phi_I Z\bar{Z}Z}\right)\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell-2)/4-1}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell-2)/4} \\
&+\left(\underline{\frac{\ell+2}{2}\frac{\ell+2}{4}\frac{L}{2}\frac{L}{2}} c_{\phi_I \bar{Z}Z\bar{Z}} + \underline{\delta_{\ell,L}\frac{\ell+2}{2}\frac{L}{2}\frac{L}{2}}\left(c_{\phi_I \bar{Z}\bar{Z}Z} + c_{\phi_I Z\bar{Z}\bar{Z}}\right)\right)\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell+2)/4-1}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell-2)/4} \\
&+\left(\underline{\frac{\ell+2}{2}\frac{\ell+2}{4}\frac{L}{2}\frac{L}{2}} c_{\phi_J Z\bar{Z}Z} + \underline{\delta_{\ell,L}\frac{\ell+2}{2}\frac{L}{2}\frac{L}{2}}\left(c_{\phi_J ZZ\bar{Z}} + c_{\phi_J \bar{Z}ZZ}\right)\right)\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell+2)/4-1}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell-2)/4} \\
&+\left(\underline{\frac{\ell+2}{2}\frac{\ell-2}{4}\frac{L}{2}\frac{L}{2}} c_{\phi_J \bar{Z}Z\bar{Z}}\right)\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell-2)/4-1}\Bigg] \\
&+\sum_{\ell/2=2,4,\dots}\Bigg[\underline{\frac{\ell+2}{2}\frac{\ell}{2}\frac{L}{2}\frac{L}{2}} c_{\phi_I Z\bar{Z}Z}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4-1}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4} \\
&+\underline{\frac{\ell+2}{2}\frac{\ell}{2}\frac{L}{2}\frac{L}{2}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4} c_{\phi_I \bar{Z}Z\bar{Z}}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4-1}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4}
\end{aligned}
\tag{4.15}
$$

$$+\underline{\frac{\ell+2}{2}\frac{\ell}{2}\frac{L}{2}\frac{L}{2}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4}c_{\phi_J Z\bar{Z}Z}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4-1}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4}$$

$$+\underline{\frac{\ell+2}{2}\frac{\ell}{2}\frac{L}{2}\frac{L}{2}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4}c_{\phi_J \bar{Z}Z\bar{Z}}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4-1}\Bigg]\Bigg\}$$

$$\left(\frac{\lambda}{N}I_{34}u_{IJ}\right)^{(L-\ell)/2}\frac{N^{(L+\ell)/2-3}}{2^{(L+\ell)/2-1}}$$

$$=\frac{(-)^{\ell/2}L^2\left(N-\frac{\ell}{2}-1\right)!}{2^{N+L/2+4}}\frac{\lambda^{N+L/2+1}}{N^{N-\ell/2}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-\ell/2}\left(I_{34}u_{IJ}\right)^{(L-\ell)/2}\left(I_{13}I_{23}I_{14}I_{24}u_{ZI}u_{\bar{Z}I}u_{ZJ}u_{\bar{Z}J}\right)^{\ell/4}$$

$$(1-\delta_{L,2})\,I_{12}^{-1}\Bigg\{\sum_{\ell/2=1,3,\ldots}\Bigg[(\ell+2-4\delta_{\ell,L})\,I_{13}^{-1}\left(-X_{1123}-I_{12}I_{13}F_{12,13}\right)\left(\frac{I_{24}I_{13}u_{\bar{Z}J}u_{ZI}}{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}\right)^{1/2}+(\ell-2)\,I_{23}^{-1}\left(-X_{1223}+I_{12}I_{23}F_{12,23}\right)\left(\frac{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}{I_{14}I_{23}u_{ZJ}u_{\bar{Z}I}}\right)^{1/2}$$

$$+(\ell-2)\,I_{14}^{-1}\left(-X_{1124}-I_{12}I_{14}F_{12,14}\right)\left(\frac{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}\right)^{1/2}+(\ell+2-4\delta_{\ell,L})\,I_{24}^{-1}\left(-X_{1224}+I_{12}I_{24}F_{12,24}\right)\left(\frac{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}\right)^{1/2}$$

$$+(\ell-2)\,I_{13}^{-1}\left(-X_{1123}-I_{12}I_{13}F_{12,13}\right)\left(\frac{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}{I_{24}I_{13}u_{\bar{Z}J}u_{ZI}}\right)^{1/2}+(\ell+2-4\delta_{\ell,L})\,I_{23}^{-1}\left(-X_{1223}+I_{12}I_{23}F_{12,23}\right)\left(\frac{I_{14}I_{23}u_{ZJ}u_{\bar{Z}I}}{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}\right)^{1/2}$$

$$+(\ell+2-4\delta_{\ell,L})\,I_{14}^{-1}\left(-X_{1124}-I_{12}I_{14}F_{12,14}\right)\left(\frac{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}\right)^{1/2}+(\ell-2)\,I_{24}^{-1}\left(-X_{1224}+I_{12}I_{24}F_{12,24}\right)\left(\frac{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}\right)^{1/2}\Bigg]$$

$$+\sum_{\ell/2=2,4,\ldots}\Big[2\ell I_{13}^{-1}\left(-X_{1123}-I_{12}I_{13}F_{12,13}\right)+2\ell I_{23}^{-1}\left(-X_{1223}+I_{12}I_{23}F_{12,23}\right)$$

$$+2\ell I_{14}^{-1}\left(-X_{1124}-I_{12}I_{14}F_{12,14}\right)+2\ell I_{24}^{-1}\left(-X_{1224}+I_{12}I_{24}F_{12,24}\right)\Big]\Bigg\}$$

We used the following

$$\begin{aligned}c_{\phi_I Z\bar{Z}Z}&=\left\langle \mathrm{tr}\left[\phi_I(x_3)\,Z(x_1)\,\bar{Z}(x_2)\,Z(x_1)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\ N&\underset{=}{\gg}1\,\frac{\lambda^3 N}{4}u_{Z\bar{Z}}u_{ZI}\left(-X_{1123}-I_{12}I_{13}F_{12,13}\right)\end{aligned}\tag{4.16}$$

$$\begin{aligned}c_{\phi_I \bar{Z}Z\bar{Z}}&=\left\langle \mathrm{tr}\left[\phi_I(x_3)\,\bar{Z}(x_2)\,Z(x_1)\,\bar{Z}(x_2)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\ N&\underset{=}{\gg}1\,\frac{\lambda^3 N}{4}u_{Z\bar{Z}}u_{\bar{Z}I}\left(-X_{1223}+I_{12}I_{23}F_{12,23}\right)\end{aligned}\tag{4.17}$$

$$\begin{aligned}c_{\phi_J Z\bar{Z}Z}&=\left\langle \mathrm{tr}\left[\phi_J(x_4)\,Z(x_1)\,\bar{Z}(x_2)\,Z(x_1)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\ N&\underset{=}{\gg}1\,\frac{\lambda^3 N}{4}u_{Z\bar{Z}}u_{ZJ}\left(-X_{1124}-I_{12}I_{14}F_{12,14}\right)\end{aligned}\tag{4.18}$$

$$\begin{aligned}c_{\phi_J \bar{Z}Z\bar{Z}}&=\left\langle \mathrm{tr}\left[\phi_J(x_4)\,\bar{Z}(x_2)\,Z(x_1)\,\bar{Z}(x_2)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\ N&\underset{=}{\gg}1\,\frac{\lambda^3 N}{4}u_{Z\bar{Z}}u_{\bar{Z}J}\left(-X_{1224}+I_{12}I_{24}F_{12,24}\right)\end{aligned}\tag{4.19}$$

and for those with deltas

$$\begin{aligned}c_{\phi_I ZZ\bar{Z}}&=\left\langle \mathrm{tr}\left[\phi_I(x_3)\,Z(x_1)\,Z(x_1)\,\bar{Z}(x_2)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\ =c_{\phi_I \bar{Z}ZZ}&=\left\langle \mathrm{tr}\left[\phi_I(x_3)\,\bar{Z}(x_2)\,Z(x_1)\,Z(x_1)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\ N&\underset{=}{\gg}1\,\frac{\lambda^3 N}{8}u_{Z\bar{Z}}u_{ZI}\left(X_{1123}+I_{12}I_{13}F_{12,13}\right)\end{aligned}\tag{4.20}$$

$$\begin{aligned}c_{\phi_I \bar{Z}\bar{Z}Z}&=\left\langle \mathrm{tr}\left[\phi_I(x_3)\,\bar{Z}(x_2)\,\bar{Z}(x_2)\,Z(x_1)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\ =c_{\phi_I Z\bar{Z}\bar{Z}}&=\left\langle \mathrm{tr}\left[\phi_I(x_3)\,Z(x_1)\,\bar{Z}(x_2)\,\bar{Z}(x_2)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\ N&\underset{=}{\gg}1\,\frac{\lambda^3 N}{8}u_{Z\bar{Z}}u_{\bar{Z}I}\left(X_{1223}-I_{12}I_{23}F_{12,23}\right)\end{aligned}\tag{4.21}$$

$$\begin{aligned}c_{\phi_J ZZ\bar{Z}}&=\left\langle \mathrm{tr}\left[\phi_J(x_4)\,Z(x_1)\,Z(x_1)\,\bar{Z}(x_2)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\ =c_{\phi_J \bar{Z}ZZ}&=\left\langle \mathrm{tr}\left[\phi_J(x_3)\,\bar{Z}(x_2)\,Z(x_1)\,Z(x_1)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\ N&\underset{=}{\gg}1\,\frac{\lambda^3 N}{8}u_{Z\bar{Z}}u_{ZJ}\left(X_{1124}+I_{12}I_{14}F_{12,14}\right)\end{aligned}\tag{4.22}$$

$$c_{\phi_J \bar{Z}\bar{Z}Z}=\left\langle \mathrm{tr}\left[\phi_J(x_4)\,\bar{Z}(x_2)\,\bar{Z}(x_2)\,Z(x_1)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\tag{4.23}$$

$$
\begin{aligned}
&= c_{\phi_J Z \bar{Z} \bar{Z}} = \left\langle \mathrm{tr}\left[\phi_J(x_3)\,\bar{Z}(x_2)\,Z(x_1)\,Z(x_1)\right] V_{X+H}\right\rangle_{\mathrm{LO}} \\
&\overset{N\gg 1}{=} \frac{\lambda^3 N}{8} u_{Z\bar{Z}} u_{\bar{Z}J}\left(X_{1224} - I_{12} I_{24} F_{12,24}\right) .
\end{aligned}
$$

Diagram 5

In diagram 5a and 5b the X+H vertex connects neighboring scalars in PCGG, so the rest of the diagram looks like LO. When $\ell = L$, it can connect also

- Z and Z with $(\ell-4)/2$ scalars in between,
- $\bar{Z}$ and $\bar{Z}$ with $(\ell-4)/2$ scalars in between.

$$
\begin{aligned}
&\sum_{i=a,b} \mathrm{diagram}_{5i} \\
&= \sum_{\ell/2=1}^{L/2} \left\langle \mathcal{G}_\ell \mathcal{O}_3 \mathcal{O}_4 V_{X+H}\right\rangle_{\mathrm{LO}} \\
&\overset{N\gg 1}{=} - \frac{(-)^{\ell/2}\left(N-\frac{\ell}{2}\right)!}{2^{N-\ell/2-1}\ell} \frac{\lambda^{N-\ell/2}}{N^{N-\ell/2}} I_{12}^{N-\ell/2} u_{Z\bar{Z}}^{N-\ell/2} \\
&\quad (1-\delta_{L,2}) \Bigg\{ \sum_{\ell/2=1,3,\ldots} \Bigg[\underline{\left(\frac{\ell}{2}-1\right)\frac{\ell L^2}{8}} c_{Z\bar{Z}\phi_I\phi_I} \left(\frac{\lambda}{N} I_{13} u_{ZI}\right)^{(\ell+2)/4-1} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}I}\right)^{(\ell-2)/4-1} \left(\frac{\lambda}{N} I_{14} u_{ZJ}\right)^{(\ell-2)/4} \left(\frac{\lambda}{N} I_{24} u_{\bar{Z}J}\right)^{(\ell+2)/4} \\
&\quad + \underline{\left(\frac{\ell}{2}-1\right)\frac{\ell L^2}{8}} \left(\frac{\lambda}{N} I_{13} u_{ZI}\right)^{(\ell+2)/4} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}I}\right)^{(\ell-2)/4} c_{Z\bar{Z}\phi_J\phi_J} \left(\frac{\lambda}{N} I_{14} u_{ZJ}\right)^{(\ell-2)/4-1} \left(\frac{\lambda}{N} I_{24} u_{\bar{Z}J}\right)^{(\ell+2)/4-1} \\
&\quad + \underline{\left(\frac{\ell}{2}-1\right)\frac{\ell L^2}{8}} c_{Z\bar{Z}\phi_I\phi_I} \left(\frac{\lambda}{N} I_{13} u_{ZI}\right)^{(\ell-2)/4-1} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}I}\right)^{(\ell+2)/4-1} \left(\frac{\lambda}{N} I_{14} u_{ZJ}\right)^{(\ell+2)/4} \left(\frac{\lambda}{N} I_{24} u_{\bar{Z}J}\right)^{(\ell-2)/4} \\
&\quad + \underline{\left(\frac{\ell}{2}-1\right)\frac{\ell L^2}{8}} \left(\frac{\lambda}{N} I_{13} u_{ZI}\right)^{(\ell-2)/4} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}I}\right)^{(\ell+2)/4} c_{Z\bar{Z}\phi_J\phi_J} \left(\frac{\lambda}{N} I_{14} u_{ZJ}\right)^{(\ell+2)/4-1} \left(\frac{\lambda}{N} I_{24} u_{\bar{Z}J}\right)^{(\ell-2)/4-1} \\
&\quad \underline{+\delta_{\ell,L}\frac{\ell}{2}\frac{L}{2}\frac{L}{2}} c_{ZZ\phi_I\phi_I} \left(\frac{\lambda}{N} I_{13} u_{ZI}\right)^{(\ell+2)/4-2} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}I}\right)^{(\ell-2)/4} \left(\frac{\lambda}{N} I_{14} u_{ZJ}\right)^{(\ell-2)/4} \left(\frac{\lambda}{N} I_{24} u_{\bar{Z}J}\right)^{(\ell+2)/4} \\
&\quad \underline{+\delta_{\ell,L}\frac{\ell}{2}\frac{L}{2}\frac{L}{2}} c_{\bar{Z}\bar{Z}\phi_J\phi_J} \left(\frac{\lambda}{N} I_{13} u_{ZI}\right)^{(\ell+2)/4} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}I}\right)^{(\ell-2)/4} \left(\frac{\lambda}{N} I_{14} u_{ZJ}\right)^{(\ell-2)/4} \left(\frac{\lambda}{N} I_{24} u_{\bar{Z}J}\right)^{(\ell+2)/4-2} \\
&\quad \underline{+\delta_{\ell,L}\frac{\ell}{2}\frac{L}{2}\frac{L}{2}} c_{\bar{Z}\bar{Z}\phi_I\phi_I} \left(\frac{\lambda}{N} I_{13} u_{ZI}\right)^{(\ell-2)/4} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}I}\right)^{(\ell+2)/4-2} \left(\frac{\lambda}{N} I_{14} u_{ZJ}\right)^{(\ell+2)/4} \left(\frac{\lambda}{N} I_{24} u_{\bar{Z}J}\right)^{(\ell-2)/4} \\
&\quad \underline{+\delta_{\ell,L}\frac{\ell}{2}\frac{L}{2}\frac{L}{2}} c_{ZZ\phi_J\phi_J} \left(\frac{\lambda}{N} I_{13} u_{ZI}\right)^{(\ell-2)/4} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}I}\right)^{(\ell+2)/4} \left(\frac{\lambda}{N} I_{14} u_{ZJ}\right)^{(\ell+2)/4-2} \left(\frac{\lambda}{N} I_{24} u_{\bar{Z}J}\right)^{(\ell-2)/4} \Bigg] \\
&\quad + \sum_{\ell/2=2,4,\ldots} \Bigg[\underline{\left(\frac{\ell}{2}-1\right)\frac{\ell L^2}{4}} c_{Z\bar{Z}\phi_I\phi_I} \left(\frac{\lambda}{N} I_{13} u_{ZI}\right)^{\ell/4-1} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}I}\right)^{\ell/4-1} \left(\frac{\lambda}{N} I_{14} u_{ZJ}\right)^{\ell/4} \left(\frac{\lambda}{N} I_{24} u_{\bar{Z}J}\right)^{\ell/4} \\
&\quad + \underline{\left(\frac{\ell}{2}-1\right)\frac{\ell L^2}{4}} \left(\frac{\lambda}{N} I_{13} u_{ZI}\right)^{\ell/4} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}I}\right)^{\ell/4} c_{Z\bar{Z}\phi_J\phi_J} \left(\frac{\lambda}{N} I_{14} u_{ZJ}\right)^{\ell/4-1} \left(\frac{\lambda}{N} I_{24} u_{\bar{Z}J}\right)^{\ell/4-1} \Bigg] \Bigg\} \\
&\quad \left(\frac{\lambda}{N} I_{34} u_{IJ}\right)^{(L-\ell)/2} \frac{N^{(L+\ell)/2-4}}{2^{(L+\ell)/2-2}} \\
&= - \frac{(-)^{\ell/2} L^2 \left(N-\frac{\ell}{2}\right)!}{2^{N+L/2+4}} \frac{\lambda^{N+L/2+1}}{N^{N-\ell/2+1}} \left(I_{12} u_{Z\bar{Z}}\right)^{N-\ell/2} \left(I_{34} u_{IJ}\right)^{(L-\ell)/2} \left(I_{13} I_{23} I_{14} I_{24} u_{ZI} u_{\bar{Z}I} u_{ZJ} u_{\bar{Z}J}\right)^{\ell/4} \\
&\quad (1-\delta_{L,2}) \Bigg\{ \sum_{\ell/2=1,3,\ldots} \Bigg[\left[(\ell-2) I_{13}^{-1} I_{23}^{-1}\left(X_{1233} + I_{13} I_{23} F_{13,23}\right) + 2\delta_{\ell,L} I_{13}^{-2}\left(X_{1133} + I_{13}^2 F_{13,13}\right)\right] \left(\frac{I_{13} I_{24} u_{ZI} u_{\bar{Z}J}}{I_{14} I_{23} u_{ZJ} u_{\bar{Z}I}}\right)^{1/2} \\
&\quad + \left[(\ell-2) I_{14}^{-1} I_{24}^{-1}\left(X_{1244} + I_{14} I_{24} F_{14,24}\right) + 2\delta_{\ell,L} I_{24}^{-2}\left(X_{2244} + I_{24}^2 F_{24,24}\right)\right] \left(\frac{I_{24} I_{13} u_{\bar{Z}J} u_{ZI}}{I_{23} I_{14} u_{\bar{Z}I} u_{ZJ}}\right)^{1/2} \\
&\quad + \left[(\ell-2) I_{13}^{-1} I_{23}^{-1}\left(X_{1233} + I_{13} I_{23} F_{13,23}\right) + 2\delta_{\ell,L} I_{23}^{-2}\left(X_{2233} + I_{23}^2 F_{23,23}\right)\right] \left(\frac{I_{23} I_{14} u_{\bar{Z}I} u_{ZJ}}{I_{24} I_{13} u_{\bar{Z}J} u_{ZI}}\right)^{1/2} \\
&\quad + \left[(\ell-2) I_{14}^{-1} I_{24}^{-1}\left(X_{1244} + I_{14} I_{24} F_{14,24}\right) + 2\delta_{\ell,L} I_{14}^{-2}\left(X_{1144} + I_{14}^2 F_{14,14}\right)\right] \left(\frac{I_{14} I_{23} u_{ZJ} u_{\bar{Z}I}}{I_{13} I_{24} u_{ZI} u_{\bar{Z}J}}\right)^{1/2} \Bigg] \\
&\quad + \sum_{\ell/2=2,4,\ldots} \left[(2\ell-4) I_{13}^{-1} I_{23}^{-1}\left(X_{1233} + I_{13} I_{23} F_{13,23}\right) + (2\ell-4) I_{14}^{-1} I_{24}^{-1}\left(X_{1244} + I_{14} I_{24} F_{14,24}\right)\right] \Bigg\}
\end{aligned}
\tag{4.24}
$$

We used the following

$$
\begin{aligned}
c_{Z\bar{Z}\phi_I\phi_I} &= \left\langle \mathrm{tr}\left[Z(x_1)\,\bar{Z}(x_2)\,\phi_I(x_3)\,\phi_I(x_3)\right] V_{X+H}\right\rangle_{\mathrm{LO}} \\
= c_{\bar{Z}Z\phi_I\phi_I} &= \left\langle \mathrm{tr}\left[\bar{Z}(x_2)\,Z(x_1)\,\phi_I(x_3)\,\phi_I(x_3)\right] V_{X+H}\right\rangle_{\mathrm{LO}} \\
&\overset{N\gg 1}{=} \frac{\lambda^3 N}{8} u_{ZI} u_{\bar{Z}I}\left(X_{1233} + I_{13} I_{23} F_{13,23}\right)
\end{aligned}
\tag{4.25}
$$

$$
\begin{aligned}
c_{Z\bar{Z}\phi_J\phi_J} &= \left\langle \mathrm{tr}\left[Z(x_1)\,\bar{Z}(x_2)\,\phi_J(x_4)\,\phi_J(x_4)\right]V_{X+H}\right\rangle_{\mathrm{LO}} \\
= c_{\bar{Z}Z\phi_J\phi_J} &= \left\langle \mathrm{tr}\left[\bar{Z}(x_2)\,Z(x_1)\,\phi_J(x_4)\,\phi_J(x_4)\right]V_{X+H}\right\rangle_{\mathrm{LO}} \\
&= N \underset{=}{\gg} 1 \frac{\lambda^3 N}{8} u_{ZJ} u_{\bar{Z}J}\left(X_{1244} + I_{14}I_{24}F_{14,24}\right)
\end{aligned}
\tag{4.26}
$$

and for those with deltas

$$
\begin{aligned}
c_{ZZ\phi_I\phi_I} &= \left\langle \mathrm{tr}\left[Z(x_1)\,Z(x_1)\,\phi_I(x_3)\,\phi_I(x_3)\right]V_{X+H}\right\rangle_{\mathrm{LO}} \\
N &\underset{=}{\gg} 1 \frac{\lambda^3 N}{8} u_{\bar{Z}I}^2\left(X_{1133} + I_{13}^2 F_{13,13}\right)
\end{aligned}
\tag{4.27}
$$

$$
\begin{aligned}
c_{ZZ\phi_J\phi_J} &= \left\langle \mathrm{tr}\left[Z(x_1)\,Z(x_1)\,\phi_J(x_4)\,\phi_J(x_4)\right]V_{X+H}\right\rangle_{\mathrm{LO}} \\
N &\underset{=}{\gg} 1 \frac{\lambda^3 N}{8} u_{\bar{Z}J}^2\left(X_{1144} + I_{14}^2 F_{14,14}\right)
\end{aligned}
\tag{4.28}
$$

$$
\begin{aligned}
c_{\bar{Z}\bar{Z}\phi_I\phi_I} &= \left\langle \mathrm{tr}\left[\bar{Z}(x_2)\,\bar{Z}(x_2)\,\phi_I(x_3)\,\phi_I(x_3)\right]V_{X+H}\right\rangle_{\mathrm{LO}} \\
N &\underset{=}{\gg} 1 \frac{\lambda^3 N}{8} u_{\bar{Z}I}^2\left(X_{2233} + I_{23}^2 F_{23,23}\right)
\end{aligned}
\tag{4.29}
$$

$$
\begin{aligned}
c_{\bar{Z}\bar{Z}\phi_J\phi_J} &= \left\langle \mathrm{tr}\left[\bar{Z}(x_2)\,\bar{Z}(x_2)\,\phi_J(x_4)\,\phi_J(x_4)\right]V_{X+H}\right\rangle_{\mathrm{LO}} \\
N &\underset{=}{\gg} 1 \frac{\lambda^3 N}{8} u_{\bar{Z}J}^2\left(X_{2244} + I_{24}^2 F_{24,24}\right)\,.
\end{aligned}
\tag{4.30}
$$

Diagram 5c has no exceptional case. It has the bridge length $(L-\ell)/2 \geq 2$ by construction, hence the Kronecker delta by hand to enforce it.

$$
\begin{aligned}
&\mathrm{diagram}_{5c} \\
&\overset{N\gg 1}{=} \sum_{\ell/2=1}^{L/2} \langle \mathcal{G}_\ell \mathcal{O}_3 \mathcal{O}_4 V_{X+H}\rangle_{\mathrm{LO}} \\
&= -\frac{(-)^{\ell/2}\left(N-\frac{\ell}{2}\right)!}{2^{N-\ell/2-1}\ell}\frac{\lambda^{N-\ell/2}}{N^{N-\ell/2}} I_{12}^{N-\ell/2} u_{Z\bar{Z}}^{N-\ell/2} \\
&\left\{\sum_{\ell/2=1,3,\dots}\left[\underline{\frac{\ell L^2}{8}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell+2)/4}\right.\right. \\
&\left.+\underline{\frac{\ell L^2}{8}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell-2)/4}\right] \\
&\left.+\sum_{\ell/2=2,4,\dots}\left[\underline{\frac{\ell L^2}{4}}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4}\right]\right\} \\
&(1-\delta_{L,l})\underline{\left(\frac{L-\ell}{2}-1\right)}c_{IIJJ}\left(\frac{\lambda}{N}I_{34}u_{IJ}\right)^{(L-\ell)/2-2}\frac{N^{(L+\ell)/2-4}}{2^{(L+\ell)/2-2}} \\
&= -\frac{(-)^{\ell/2}L^2(L-\ell-2)\left(N-\frac{\ell}{2}\right)!}{2^{N+L/2+4}}(1-\delta_{L,l})\frac{\lambda^{N+L/2+1}}{N^{N-\ell/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-\ell/2}\left(I_{34}u_{IJ}\right)^{(L-\ell)/2}\left(I_{13}I_{23}I_{14}I_{24}u_{ZI}u_{\bar{Z}I}u_{ZJ}u_{\bar{Z}J}\right)^{\ell/4} \\
&I_{34}^{-2}\left(X_{3434}+I_{34}^2F_{34,34}\right)\left\{\sum_{\ell/2=1,3,\dots}\left[\left(\frac{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}\right)^{1/2}+\left(\frac{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}\right)^{1/2}\right]+\sum_{\ell/2=2,4,\dots}2\right\}
\end{aligned}
\tag{4.31}
$$

We used the following.

$$
\begin{aligned}
c_{\phi_I\phi_I\phi_J\phi_J} &= \left\langle \mathrm{tr}\left[\phi_I(x_3)\,\phi_I(x_3)\,\phi_J(x_4)\,\phi_J(x_4)\right]V_{X+H}\right\rangle_{\mathrm{LO}} \\
N &\underset{=}{\gg} 1 \frac{\lambda^3 N}{8} u_{IJ}^2\left(X_{3434} + I_{34}^2 F_{34,34}\right)
\end{aligned}
\tag{4.32}
$$

Diagram 6

In diagram 6a the X+H vertex connects neighboring scalars in PCGG, so the rest of the diagram looks like LO. When $\ell = L$, it can connect also

- Z and $\bar{Z}$ with $(\ell-2)/2$ scalars in between,
- $\bar{Z}$ and Z with $(\ell-2)/2$ scalars in between

like in figure 6 right [5].

$$
\begin{aligned}
&\text{diagram}_{6a} \\
&= \sum_{\ell/2=1}^{L/2} \langle \mathcal{G}_\ell \mathcal{O}_3 \mathcal{O}_4 V_{X+H} \rangle_{\text{LO}} \\
&\overset{N\gg1}{=} -\frac{(-)^{\ell/2}\left(N-\frac{\ell}{2}\right)!}{2^{N-\ell/2-1}\ell} \frac{\lambda^{N-\ell/2}}{N^{N-\ell/2}} I_{12}^{N-\ell/2} u_{Z\bar{Z}}^{N-\ell/2} \\
&\left\{ \sum_{\ell/2=1,3,\ldots} \left[\underline{\frac{L}{2}\frac{L}{2}} \left(\underline{\frac{\ell}{4}} c_{\bar{Z}Z\phi_I\phi_J} + \underline{\frac{\ell}{4}} c_{Z\bar{Z}\phi_J\phi_I} + \delta_{\ell,L} \underline{\frac{\ell}{8}} \left(c_{Z\phi_I\bar{Z}\phi_J} + c_{Z\phi_J\bar{Z}\phi_I} \right) \right) \right. \right. \\
&\left(\frac{\lambda}{N} I_{13} u_{ZI}\right)^{(\ell+2)/4-1} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}I}\right)^{(\ell-2)/4} \left(\frac{\lambda}{N} I_{14} u_{ZJ}\right)^{(\ell-2)/4} \left(\frac{\lambda}{N} I_{24} u_{\bar{Z}J}\right)^{(\ell+2)/4-1} \\
&\left. + \underline{\frac{L}{2}\frac{L}{2}} \left(\underline{\frac{\ell}{4}} c_{Z\bar{Z}\phi_I\phi_J} + \underline{\frac{\ell}{4}} c_{\bar{Z}Z\phi_J\phi_I} + \delta_{\ell,L} \frac{\ell}{8} \left(c_{Z\phi_I\bar{Z}\phi_J} + c_{Z\phi_J\bar{Z}\phi_I} \right) \right) \left(\frac{\lambda}{N} I_{13} u_{ZI}\right)^{(\ell-2)/4} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}I}\right)^{(\ell+2)/4-1} \left(\frac{\lambda}{N} I_{14} u_{ZJ}\right)^{(\ell+2)/4-1} \left(\frac{\lambda}{N} I_{24} u_{\bar{Z}J}\right)^{(\ell-2)/4} \right] \\
&+ \sum_{\ell/2=2,4,\ldots} \left[\underline{\frac{L}{2}\frac{L}{2}} \left(\underline{\frac{\ell}{4}} c_{\bar{Z}Z\phi_I\phi_J} + \underline{\frac{\ell}{4}} c_{Z\bar{Z}\phi_J\phi_I} \right) \left(\frac{\lambda}{N} I_{13} u_{ZI}\right)^{\ell/4-1} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}I}\right)^{\ell/4} \left(\frac{\lambda}{N} I_{14} u_{ZJ}\right)^{\ell/4} \left(\frac{\lambda}{N} I_{24} u_{\bar{Z}J}\right)^{\ell/4-1} \right. \\
&\left.\left. + \underline{\frac{L}{2}\frac{L}{2}} \left(\underline{\frac{\ell}{4}} c_{Z\bar{Z}\phi_I\phi_J} + \underline{\frac{\ell}{4}} c_{\bar{Z}Z\phi_J\phi_I} \right) \left(\frac{\lambda}{N} I_{13} u_{ZI}\right)^{\ell/4} \left(\frac{\lambda}{N} I_{23} u_{\bar{Z}I}\right)^{\ell/4-1} \left(\frac{\lambda}{N} I_{14} u_{ZJ}\right)^{\ell/4-1} \left(\frac{\lambda}{N} I_{24} u_{\bar{Z}J}\right)^{\ell/4} \right] \right\} \left(\frac{\lambda}{N} I_{34} u_{IJ}\right)^{(L-\ell)/2} \frac{N^{(L+\ell)/2-4}}{2^{(L+\ell)/2-3}} \\
&= \ldots
\end{aligned}
\tag{4.33}
$$

We used the following

$$
\begin{aligned}
c_{\bar{Z}Z\phi_I\phi_J} &= \left\langle \operatorname{tr}\left[\bar{Z}(x_2) Z(x_1) \phi_I(x_3) \phi_J(x_4)\right] V_{X+H} \right\rangle_{\text{LO}} \\
= c_{Z\bar{Z}\phi_J\phi_I} &= \left\langle \operatorname{tr}\left[Z(x_1) \bar{Z}(x_2) \phi_I(x_3) \phi_J(x_4)\right] V_{X+H} \right\rangle_{\text{LO}} \\
N &\underset{=}{\gg} 1 \frac{\lambda^3 N}{8} \left[X_{1234} \left(-u_{Z\bar{Z}} u_{IJ} + 2u_{\bar{Z}I} u_{ZJ} - u_{\bar{Z}J} u_{ZI} \right) + u_{Z\bar{Z}} u_{IJ} I_{12} I_{34} F_{12,34} + u_{\bar{Z}J} u_{ZI} I_{13} I_{24} F_{13,24} \right]
\end{aligned}
\tag{4.34}
$$

$$
\begin{aligned}
c_{Z\bar{Z}\phi_I\phi_J} &= \left\langle \operatorname{tr}\left[Z(x_1) \bar{Z}(x_2) \phi_I(x_3) \phi_J(x_4)\right] V_{X+H} \right\rangle_{\text{LO}} \\
= c_{\bar{Z}Z\phi_J\phi_I} &= \left\langle \operatorname{tr}\left[\bar{Z}(x_2) Z(x_1) \phi_J(x_4) \phi_I(x_3)\right] V_{X+H} \right\rangle_{\text{LO}} \\
N &\underset{=}{\gg} 1 \frac{\lambda^3 N}{8} \left[X_{1234} \left(-u_{Z\bar{Z}} u_{IJ} + 2u_{\bar{Z}J} u_{ZI} - u_{\bar{Z}I} u_{ZJ} \right) - u_{Z\bar{Z}} u_{IJ} I_{12} I_{34} F_{12,34} + u_{\bar{Z}I} u_{ZJ} I_{14} I_{23} F_{14,23} \right]
\end{aligned}
\tag{4.35}
$$

and for those with deltas

$$
\begin{aligned}
c_{Z\phi_I\bar{Z}\phi_J} &= \left\langle \operatorname{tr}\left[Z(x_1) \phi_I(x_3) \bar{Z}(x_2) \phi_J(x_4)\right] V_{X+H} \right\rangle_{\text{LO}} \\
N &\underset{=}{\gg} 1 \frac{\lambda^3 N}{8} \left[X_{1234} \left(-u_{ZI} u_{\bar{Z}J} + 2u_{Z\bar{Z}} u_{IJ} - u_{ZJ} u_{I\bar{Z}} \right) - u_{ZI} u_{\bar{Z}J} I_{13} I_{24} F_{13,24} - u_{ZJ} u_{I\bar{Z}} I_{14} I_{23} F_{14,23} \right]
\end{aligned}
\tag{4.36}
$$

$$
\begin{aligned}
c_{Z\phi_J\bar{Z}\phi_I} &= \left\langle \operatorname{tr}\left[Z(x_1) \phi_I(x_3) \bar{Z}(x_2) \phi_J(x_4)\right] V_{X+H} \right\rangle_{\text{LO}} \\
N &\underset{=}{\gg} 1 \frac{\lambda^3 N}{8} \left[X_{1234} \left(-u_{ZJ} u_{\bar{Z}I} + 2u_{Z\bar{Z}} u_{IJ} - u_{ZI} u_{J\bar{Z}} \right) - u_{ZJ} u_{\bar{Z}I} I_{14} I_{23} F_{14,23} - u_{ZI} u_{J\bar{Z}} I_{13} I_{24} F_{13,24} \right] .
\end{aligned}
\tag{4.37}
$$

Diagrams 6b and 6c have no exceptional case. They have the bridge length $(L-\ell)/2 \geq 2$ by construction, hence the Kronecker delta by hand to enforce it.

$$
\begin{aligned}
&\sum_{i=b,c} \text{diagram}_{6i} \\
&= \sum_{\ell/2=1}^{L/2} \langle \mathcal{G}_\ell \mathcal{O}_3 \mathcal{O}_4 V_{X+H} \rangle_{\text{LO}} \\
&\overset{N\gg1}{=} -\frac{(-)^{\ell/2}\left(N-\frac{\ell}{2}\right)!}{2^{N-\ell/2-1}\ell} \frac{\lambda^{N-\ell/2}}{N^{N-\ell/2}} I_{12}^{N-\ell/2} u_{Z\bar{Z}}^{N-\ell/2}
\end{aligned}
\tag{4.38}
$$

$$
\begin{aligned}
&\Bigg\{\sum_{\ell/2=1,3,\ldots}\Bigg[\underline{\frac{L}{2}\frac{L}{2}\frac{\ell}{2}}c_{Z\phi_I\phi_I\phi_J}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell+2)/4-1}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell+2)/4}\\
&+\underline{\frac{L}{2}\frac{L}{2}\frac{\ell}{2}}c_{\bar{Z}\phi_J\phi_J\phi_I}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell+2)/4-1}\\
&+\underline{\frac{L}{2}\frac{L}{2}\frac{\ell}{2}}c_{Z\phi_J\phi_J\phi_I}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell+2)/4-1}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell-2)/4}\\
&+\underline{\frac{L}{2}\frac{L}{2}\frac{\ell}{2}}c_{\bar{Z}\phi_I\phi_I\phi_J}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{(\ell-2)/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{(\ell+2)/4-1}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{(\ell+2)/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{(\ell-2)/4}\Bigg]\\
&+\sum_{\ell/2=2,4,\ldots}\Bigg[\underline{\frac{L}{2}\frac{L}{2}\frac{\ell}{2}}c_{Z\phi_I\phi_I\phi_J}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4-1}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4}\\
&+\underline{\frac{L}{2}\frac{L}{2}\frac{\ell}{2}}c_{Z\phi_J\phi_J\phi_I}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4-1}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4}\\
&+\underline{\frac{L}{2}\frac{L}{2}\frac{\ell}{2}}c_{\bar{Z}\phi_I\phi_I\phi_J}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4-1}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4}\\
&+\underline{\frac{L}{2}\frac{L}{2}\frac{\ell}{2}}c_{\bar{Z}\phi_J\phi_J\phi_I}\left(\frac{\lambda}{N}I_{13}u_{ZI}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{23}u_{\bar{Z}I}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{14}u_{ZJ}\right)^{\ell/4}\left(\frac{\lambda}{N}I_{24}u_{\bar{Z}J}\right)^{\ell/4-1}\Bigg]\Bigg\}\\
&(1-\delta_{\ell,L})\left(\frac{\lambda}{N}I_{34}u_{IJ}\right)^{(L-\ell)/2-1}\frac{N^{(L+\ell)/2-4}}{2^{(L+\ell)/2-3}}\\
=&-\frac{(-)^{\ell/2}L^2\left(N-\frac{\ell}{2}\right)!}{2^{N+L/2+2}}(1-\delta_{\ell,L})\frac{\lambda^{N+L/2+1}}{N^{N-\ell/2+1}}\left(I_{12}u_{Z\bar{Z}}\right)^{N-\ell/2}\left(I_{34}u_{IJ}\right)^{(L-\ell)/2}\left(I_{13}I_{23}I_{14}I_{24}u_{ZI}u_{\bar{Z}I}u_{ZJ}u_{\bar{Z}J}\right)^{\ell/4}\\
&\Bigg\{\sum_{\ell/2=1,3,\ldots}\Bigg[I_{13}^{-1}I_{34}^{-1}\left(X_{1334}-I_{13}I_{34}F_{13,34}\right)\left(\frac{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}{I_{14}I_{23}u_{\bar{Z}I}u_{ZJ}}\right)^{1/2}+I_{24}^{-1}I_{34}^{-1}\left(X_{2344}+I_{24}I_{34}F_{24,34}\right)\left(\frac{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}{I_{23}I_{14}u_{\bar{Z}I}u_{ZJ}}\right)^{1/2}\\
&+I_{14}^{-1}I_{34}^{-1}\left(X_{1344}+I_{14}I_{34}F_{14,34}\right)\left(\frac{I_{14}I_{23}u_{ZJ}u_{\bar{Z}I}}{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}\right)^{1/2}+I_{23}^{-1}I_{34}^{-1}\left(X_{2334}-I_{23}I_{34}F_{23,34}\right)\left(\frac{I_{14}I_{23}u_{ZJ}u_{\bar{Z}I}}{I_{13}I_{24}u_{ZI}u_{\bar{Z}J}}\right)^{1/2}\Bigg]\\
&+\sum_{\ell/2=2,4,\ldots}\Big[I_{13}^{-1}I_{34}^{-1}\left(X_{1334}-I_{13}I_{34}F_{13,34}\right)+I_{14}^{-1}I_{34}^{-1}\left(X_{1344}+I_{14}I_{34}F_{14,34}\right)\\
&+I_{23}^{-1}I_{34}^{-1}\left(X_{2334}-I_{23}I_{34}F_{23,34}\right)+I_{24}^{-1}I_{34}^{-1}\left(X_{2344}+I_{24}I_{34}F_{24,34}\right)\Big]\Bigg\}
\end{aligned}
$$

We used the following.

$$
\begin{aligned}
c_{Z\phi_I\phi_I\phi_J}&=\left\langle \mathrm{tr}\left[Z(x_1)\,\phi_I(x_3)\,\phi_I(x_3)\,\phi_J(x_4)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\
N&\underset{=}{\gg}1\,\frac{\lambda^3N}{8}\left(X_{1334}-I_{13}I_{34}F_{13,34}\right)u_{ZI}u_{IJ}
\end{aligned}
\tag{4.39}
$$

$$
\begin{aligned}
c_{\bar{Z}\phi_J\phi_J\phi_I}&=\left\langle \mathrm{tr}\left[\bar{Z}(x_2)\,\phi_J(x_4)\,\phi_J(x_4)\,\phi_I(x_3)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\
N&\underset{=}{\gg}1\,\frac{\lambda^3N}{8}\left(X_{2344}+I_{24}I_{34}F_{24,34}\right)u_{\bar{Z}J}u_{IJ}
\end{aligned}
\tag{4.40}
$$

$$
\begin{aligned}
c_{Z\phi_J\phi_J\phi_I}&=\left\langle \mathrm{tr}\left[Z(x_1)\,\phi_J(x_4)\,\phi_J(x_4)\,\phi_I(x_3)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\
N&\underset{=}{\gg}1\,\frac{\lambda^3N}{8}\left(X_{1344}+I_{14}I_{34}F_{14,34}\right)u_{ZJ}u_{IJ}
\end{aligned}
\tag{4.41}
$$

$$
\begin{aligned}
c_{\bar{Z}\phi_I\phi_I\phi_J}&=\left\langle \mathrm{tr}\left[\bar{Z}(x_2)\,\phi_J(x_4)\,\phi_J(x_4)\,\phi_I(x_3)\right]V_{X+H}\right\rangle_{\mathrm{LO}}\\
&=\frac{\lambda^3N}{8}\left(X_{2334}-I_{23}I_{34}F_{23,34}\right)u_{\bar{Z}I}u_{IJ}
\end{aligned}
\tag{4.42}
$$

Sum diagrams

The normalized correlator

$$\begin{aligned}&\mathcal{N}\left\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\right\rangle \\ =&\mathcal{N}\left\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\right\rangle^{\text{disconnected}}+\mathcal{N}\left\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\right\rangle^{\text{connected}}_{\text{LO}}+\mathcal{N}\left\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\right\rangle^{\text{connected}}_{\text{NLO}}+O\left(\lambda^2\right)\end{aligned}\tag{4.43}$$

with

$$\begin{aligned}\mathcal{N}=&\left(\frac{N!}{2^N\left(4\pi^2\right)^N N^N}\lambda^N u^N_{Z\bar{Z}}\right)^{-1}\left(C_{I_1\dots I_{L/2}}\left(\frac{\lambda}{8\pi^2}\right)^{L/2}u^{I_1}_{I_1}\dots u^{I_{L/2}}_{I_{L/2}}\right)^{-1/2}\left(C_{J_1\dots J_{L/2}}\left(\frac{\lambda}{8\pi^2}\right)^{L/2}u^{J_1}_{J_1}\dots u^{J_{L/2}}_{J_{L/2}}\right)^{-1/2} \\ =&\left(\frac{L\,N!}{2\left(8\pi^2\right)^{N+L/2}N^N}\lambda^{N+L/2}u^N_{Z\bar{Z}}\left(u^I_I u^J_J\right)^{L/4}\right)^{-1}\end{aligned}\tag{4.44}$$

$$\left\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\right\rangle^{\text{connected}}_{\text{NLO}}=\sum_{i=1}^{6}\text{diagram}_i\tag{4.45}$$

is a factor $x_{12}^{-2N}x_{34}^{-L}u_{IJ}^{L/2}\left(u^I_I u^J_J\right)^{-L/4}$ (see below) times a nice conformal-invariant quantity. The latter is function of the $SO\left(2,4\right)$ and $SO\left(6\right)$ cross-ratios [9]

$$z\bar{z}=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\qquad\qquad\left(1-z\right)\left(1-\bar{z}\right)=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}\tag{4.46}$$

$$\alpha\bar{\alpha}=\frac{u_{Z\bar{Z}}u_{IJ}}{u_{ZI}u_{\bar{Z}J}}\qquad\qquad\left(1-\alpha\right)\left(1-\bar{\alpha}\right)=\frac{u_{ZJ}u_{\bar{Z}I}}{u_{ZI}u_{\bar{Z}J}}\,.\tag{4.47}$$

Remember the special functions from the appendix [10]

$$X_{1234}=\frac{\pi^2F^{(1)}\left(z,\bar{z}\right)}{\left(2\pi\right)^8x_{13}^2x_{24}^2}\,,\qquad\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}=z\bar{z}\,,\qquad\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}=\left(1-z\right)\left(1-\bar{z}\right)\tag{4.48}$$

$$Y_{123}=\frac{\pi^2F^{(1)}\left(w_{123},\bar{w}_{123}\right)}{\left(2\pi\right)^6x_{13}^2}\,,\qquad\frac{x_{12}^2}{x_{13}^2}=w_{123}\bar{w}_{123}\,,\qquad\frac{x_{23}^2}{x_{13}^2}=\left(1-w_{123}\right)\left(1-\bar{w}_{123}\right)\tag{4.49}$$

$$F^{(1)}\left(z,\bar{z}\right)=\frac{2\text{Li}_2\left(z\right)-2\text{Li}_2\left(\bar{z}\right)+\log\left(z\bar{z}\right)\log\frac{1-z}{1-\bar{z}}}{z-\bar{z}}\,.\tag{4.50}$$

The tree level reads in these variables

$$\begin{aligned}&\frac{\mathcal{N}\left\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\right\rangle^{\text{connected}}_{\text{LO}}}{\frac{1}{x_{12}^{2N}x_{34}^{L}}\left(\frac{u_{IJ}^2}{u^I_I u^J_J}\right)^{L/4}} \\ =&-\frac{L}{2N}\sum_{\ell/2=1,3,\dots}\left(-\frac{z\bar{z}}{\alpha\bar{\alpha}}\right)^{\ell/2}\left[\left(\frac{\left(1-\alpha\right)\left(1-\bar{\alpha}\right)}{\left(1-z\right)\left(1-\bar{z}\right)}\right)^{\ell/4+1/2}+\left(\frac{\left(1-\alpha\right)\left(1-\bar{\alpha}\right)}{\left(1-z\right)\left(1-\bar{z}\right)}\right)^{\ell/4-1/2}\right] \\ &-\frac{L}{N}\sum_{\ell/2=2,4,\dots}\left(-\frac{z\bar{z}}{\alpha\bar{\alpha}}\right)^{\ell/2}\left(\frac{\left(1-\alpha\right)\left(1-\bar{\alpha}\right)}{\left(1-z\right)\left(1-\bar{z}\right)}\right)^{\ell/4} \\ =&\frac{L}{2N}\frac{z\bar{z}}{z^2\bar{z}^2\left(1-\alpha\right)\left(1-\bar{\alpha}\right)-\left(1-z\right)\left(1-\bar{z}\right)\alpha^2\bar{\alpha}^2}\left[-\alpha\bar{\alpha}\left(1-\alpha\right)\left(1-\bar{\alpha}\right)+2z\bar{z}\left(1-\alpha\right)\left(1-\bar{\alpha}\right)-\left(1-z\right)\left(1-\bar{z}\right)\alpha\bar{\alpha}\right. \\ &+\left[\left(1-z\right)\left(1-\bar{z}\right)+\left(1-\alpha\right)\left(1-\bar{\alpha}\right)\right]\alpha\bar{\alpha}\left(\frac{z^2\bar{z}^2\left(1-\alpha\right)\left(1-\bar{\alpha}\right)}{\left(1-z\right)\left(1-\bar{z}\right)\alpha^2\bar{\alpha}^2}\right)^{\lfloor(L+2)/4\rfloor} \\ &\left.-2z\bar{z}\left(1-\alpha\right)\left(1-\bar{\alpha}\right)\left(\frac{z^2\bar{z}^2\left(1-\alpha\right)\left(1-\bar{\alpha}\right)}{\left(1-z\right)\left(1-\bar{z}\right)\alpha^2\bar{\alpha}^2}\right)^{\lfloor L/4\rfloor}\right]\end{aligned}\tag{4.51}$$

The 1 loop distinguishes values of L.

- Case $L=2$. It implies $\ell=2$ and makes each 1-loop diagram vanish, so

$$\left\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\right\rangle^{\text{connected},\ L=2}_{\text{NLO}}=0\,.\tag{4.52}$$

[10] The dependence on spacetime points comes from the inversion of cross-ratio definitions. Note that sign of Imz is undetermined but X_{1234} is insensitive to $z\leftrightarrow\bar{z}$. The same goes for $w_{123},\bar{w}_{123}$ with respect to Y_{123}.

- Case $L = 4, 6, \dots$. It is convenient to create two big sums over $\ell/2$ odd and even, plus the terms corresponding to $\ell = L$.

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle_{\text{NLO}}^{\text{connected}, L\neq 2} = \sum_{\ell/2=1,3,\dots} (\dots) + \sum_{\ell/2=2,4,\dots} (\dots) + (\dots)_{\ell=L} \tag{4.53}$$

Terms $(\dots)_{\ell=L}$ happen to be zero [11]. The splitting allows to set the values of $\delta_{L,l}$. After approximating $(N-n)! \sim N^{-n} N!$, the ℓ-dependence disappears from factorials and sums become either trivial $\sum_\ell 1$ or geometric $\sum_\ell x^\ell$, but we decide not to perform them for now. Substitute the Y, X, F with coincident points with their values in point-splitting regularization. In what rests, substitute the F with non-coincident points with its definition in terms of Y, X. The operations in this order deliver

$$\left. \frac{\mathcal{N} \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle_{\text{NLO}}^{\text{connected}}}{\frac{1}{x_{12}^{2N} x_{34}^L} \left(\frac{u_{IJ}^2}{u_I^I u_J^J} \right)^{L/4}} \right|_{L\neq 2} \tag{4.54}$$

$$\begin{aligned}
=& \sum_{\ell/2=1,3,\dots L/2-2} \frac{L\lambda}{32\pi^2 N} \left(-\frac{z\bar{z}}{\alpha\bar{\alpha}} \right)^{\ell/2} \left(\frac{(1-\alpha)(1-\bar{\alpha})}{(1-z)(1-\bar{z})} \right)^{\ell/4} \Bigg\{ -\sqrt{(1-z)(1-\bar{z})(1-\alpha)(1-\bar{\alpha})} F^{(1)}(z,\bar{z}) \\
&+ \sqrt{\frac{(1-\alpha)(1-\bar{\alpha})}{(1-z)(1-\bar{z})}} (1 - z\bar{z}) F^{(1)}(z,\bar{z}) + \sqrt{\frac{(1-z)(1-\bar{z})}{(1-\alpha)(1-\bar{\alpha})}} (-1 - z\bar{z} + 2\alpha\bar{\alpha} + (1-z)(1-\bar{z})) F^{(1)}(z,\bar{z}) \\
&+ \left(\sqrt{\frac{(1-\alpha)(1-\bar{\alpha})}{(1-z)(1-\bar{z})}} - \sqrt{\frac{(1-z)(1-\bar{z})}{(1-\alpha)(1-\bar{\alpha})}} \right) \Bigg[\left(1 - \frac{x_{23}^2}{x_{13}^2} \right) F^{(1)}(w_{123}, \bar{w}_{123}) + \left(\frac{x_{24}^2}{x_{14}^2} - 1 \right) F^{(1)}(w_{124}, \bar{w}_{124}) \\
&+ \left(\frac{x_{13}^2}{x_{14}^2} - 1 \right) F^{(1)}(w_{134}, \bar{w}_{134}) + \left(1 - \frac{x_{23}^2}{x_{24}^2} \right) F^{(1)}(w_{234}, \bar{w}_{234}) \Bigg] \Bigg\} \\
&+ \sum_{\ell/2=2,4,\dots L/2-1} \frac{L\lambda}{32\pi^2 N} \left(-\frac{z\bar{z}}{\alpha\bar{\alpha}} \right)^{\ell/2} \left(\frac{(1-\alpha)(1-\bar{\alpha})}{(1-z)(1-\bar{z})} \right)^{\ell/4} \Big\{ (2 - 2z\bar{z} + \alpha\bar{\alpha} + 2(1-z)(1-\bar{z}) - 2(1-\alpha)(1-\bar{\alpha})) F^{(1)}(z,\bar{z}) \\
&+ \frac{((1-z)(1-\bar{z}) - 1)\alpha\bar{\alpha}}{z\bar{z}} F^{(1)}(z,\bar{z}) + \frac{(1-z)(1-\bar{z})(\alpha\bar{\alpha} - 2)}{(1-\alpha)(1-\bar{\alpha})} F^{(1)}(z,\bar{z}) + \frac{(1-z)(1-\bar{z})(1-(1-z)(1-\bar{z}))\alpha\bar{\alpha}}{(1-\alpha)(1-\bar{\alpha}) z\bar{z}} F^{(1)}(z,\bar{z}) \\
&+ \left(\frac{\alpha\bar{\alpha}}{z\bar{z}} - \frac{(1-\alpha)(1-\bar{\alpha})}{(1-z)(1-\bar{z})} \right) \Bigg[\left(1 - \frac{x_{23}^2}{x_{13}^2} \right) F^{(1)}(w_{123}, \bar{w}_{123}) + \left(\frac{x_{24}^2}{x_{14}^2} - 1 \right) F^{(1)}(w_{124}, \bar{w}_{124}) \\
&+ \left(\frac{x_{13}^2}{x_{14}^2} - 1 \right) F^{(1)}(w_{134}, \bar{w}_{134}) + \left(1 - \frac{x_{23}^2}{x_{24}^2} \right) F^{(1)}(w_{234}, \bar{w}_{234}) \Bigg] \Big\} \\
=& \frac{L\lambda}{16\pi^2 N} \frac{z\bar{z}(z-\alpha)(z-\bar{\alpha})(\bar{z}-\alpha)(\bar{z}-\bar{\alpha}) F^{(1)}(z,\bar{z})}{z^2\bar{z}^2(1-\alpha)(1-\bar{\alpha}) - (1-z)(1-\bar{z})\alpha^2\bar{\alpha}^2} \left[1 - \left(\frac{z^2\bar{z}^2(1-\alpha)(1-\bar{\alpha})}{\alpha^2\bar{\alpha}^2(1-z)(1-\bar{z})} \right)^{(L-2)/4} \right]
\end{aligned}$$

4.3 Structure constants from perturbation theory

We pose $\phi_I = u_I(\beta_3) \cdot \phi$ and $\phi_J = u_J(\beta_4) \cdot \phi$ to compute [9]

$$\begin{aligned}
&\left\langle \mathcal{O}_1 \mathcal{O}_2 \text{tr} \left((\phi_I(0))^{L'/2} \left(\phi_I'(0) \right)^{(L-L')/2} \right) \text{tr} \left((\phi_J(0))^{L'/2} \left(\phi_J'(0) \right)^{(L-L')/2} \right) \right\rangle \\
=& \frac{1}{\left(\frac{L}{2}\right)^2 \left(\frac{L-L'}{2}!\right)^2} \left[\left(\frac{\partial}{\partial\beta_3} \frac{\partial}{\partial\beta_4} \right)^{(L-L')/2} \left\langle \mathcal{O}_1 \mathcal{O}_2 \text{tr}(\phi_I(\beta_3))^{L/2} \text{tr}(\phi_J(\beta_4))^{L/2} \right\rangle \right]_{\beta_3=\beta_4=0} .
\end{aligned} \tag{4.55}$$

We stripped off $\mathcal{N}$ in the rhs because the trick works for the unnormalized correlator. We derive the normalization of the lhs later. The choice of parametrization $L'/2 = 1, \dots (L-2)/2$ will be clear in the OPE, as L' is the length of the exchanged $SL(2)$ operator.

Polarization are linear functions, hence each encapsulates two types of scalars at $\beta_3 = \beta_4 = 0$:

- ϕ_I defines $u_I(0) \cdot \phi$ and $u_I'(0) \cdot \phi$,
- ϕ_J defines $u_J(0) \cdot \phi$ and $u_J'(0) \cdot \phi$.

These scalars

[11] There may be a reason for the sum of all 1-loop diagrams with fixed $\ell = L$ being zero.

- have null polarizations[12]

$$u_I(0)\cdot u_I(0)=0 \tag{4.56}$$
$$u_J(0)\cdot u_J(0)=0 \tag{4.57}$$
$$u'_I(0)\cdot u'_I(0)=0 \tag{4.58}$$
$$u'_J(0)\cdot u'_J(0)=0 \tag{4.59}$$

- generate a unique scalar (acted upon derivates, i.e. $SL(2)$ sector) when OPEing later

$$u_I(0)=u_J(0) \tag{4.60}$$

- have an overall phase that makes these quantities positive in order to avoid sqrt branch cuts in $\mathcal{P}$'s (see below)

$$\frac{u_{ZI}(0)\,u_{\bar{Z}I}(0)\,u_{ZJ}(0)\,u_{\bar{Z}J}(0)}{u^2_{Z\bar{Z}}u^I_I(0,0)\,u^J_J(0,0)}>0 \qquad \frac{\left(\frac{\partial}{\partial\beta_3}\frac{\partial}{\partial\beta_4}u_{IJ}(0,0)\right)^2}{u^{I'}_{I'}(0,0)\,u^{J'}_{J'}(0,0)}>0 \qquad u_{ZI}(0)\,u_{\bar{Z}I}(0)\,u_{ZJ}(0)\,u_{\bar{Z}J}(0)>0 \tag{4.61}$$

- obey

$$u_I(0)\cdot u_J(0)=0 \tag{4.62}$$
$$u'_I(0)\cdot u'_J(0)\neq 0 \tag{4.63}$$

$$u_I(0)\cdot u_Z\neq 0 \tag{4.64}$$
$$u_I(0)\cdot u_{\bar{Z}}\neq 0 \tag{4.65}$$
$$u_J(0)\cdot u_Z\neq 0 \tag{4.66}$$
$$u_J(0)\cdot u_{\bar{Z}}\neq 0 \tag{4.67}$$
$$u'_I(0)\cdot u_Z=0 \tag{4.68}$$
$$u'_I(0)\cdot u_{\bar{Z}}=0 \tag{4.69}$$
$$u'_J(0)\cdot u_Z=0 \tag{4.70}$$
$$u'_J(0)\cdot u_{\bar{Z}}=0 \tag{4.71}$$

  in order to make any tree-level and one-loop diagram with bridge length $(L-\ell)/2$ proportional to $(u'_I(0)\cdot u'_J(0))^{(L-\ell)/2}$ [13]. This one-to-one correspondence is important because the conformally-invariant object is the full correlator (which is a sum over ℓ) and in it we will tell apart contributions with a certain bridge length by reading off their R-symmetry dependence.

A solution exists

$$\phi_I=\sqrt{2}\,(-1,-1,i\beta_3,-\beta_3,i,i)\cdot\phi=(1+i)\left(Z+i\bar{Z}+iX-\bar{X}\right)+2i\beta_3 Y \tag{4.72}$$
$$\phi_J=\sqrt{2}\,(-1,-1,i\beta_4,\beta_4,i,i)\cdot\phi=(1+i)\left(Z+i\bar{Z}+iX-\bar{X}\right)+2i\beta_4 \bar{Y} \tag{4.73}$$

with

$$X=(\phi_1+i\phi_2)/\sqrt{2} \qquad Y=(\phi_3+i\phi_4)/\sqrt{2} \qquad Z=(\phi_5+i\phi_6)/\sqrt{2}\,. \tag{4.74}$$

We come back to the normalization of the "differentiated" operator: we call the operator

$$\mathcal{N}'\left\langle \mathcal{O}_1\mathcal{O}_2\mathrm{tr}\left((\phi_I(0))^{L'/2}\left(\phi'_I(0)\right)^{(L-L')/2}\right)\mathrm{tr}\left((\phi_J(0))^{L'/2}\left(\phi'_J(0)\right)^{(L-L')/2}\right)\right\rangle \tag{4.75}$$

and its normalization is[14]

$$\mathcal{N}'=\left(\frac{N!}{2^N\left(4\pi^2\right)^N N^N}\lambda^N u^N_{Z\bar{Z}}\right)^{-1}\left(\left(\frac{\lambda}{8\pi^2}\right)^{L/2}\left(u^I_I(0,0)\right)^{L'/2}\left(\frac{\partial}{\partial\beta_3}u_I(0)\cdot\frac{\partial}{\partial\beta_3}u^*_I(0)\right)^{(L-L')/2}\right)^{-1/2}$$
$$\times\left(\left(\frac{\lambda}{8\pi^2}\right)^{L/2}\left(u^J_J(0,0)\right)^{L'/2}\left(\frac{\partial}{\partial\beta_4}u_J(0)\cdot\frac{\partial}{\partial\beta_4}u^*_J(0)\right)^{(L-L')/2}\right)^{-1/2} \tag{4.76}$$

[12] They can be non-null for $\beta_3,\beta_4\neq 0$.
[13] This is indeed not generally true at one loop in the previous section: a given bridge length doesn't generates a unique R-symmetry dependence.
[14] I'm sloppy with R-symmetry products, but clear in Mathematica.

$$= \left(\frac{N!}{(8\pi^2)^{N+L/2} N^N} \lambda^{N+L/2} u_{Z\bar{Z}}^N \left(u_I^I(0,0)\, u_J^J(0,0) \right)^{L'/4} \left[\left(\frac{\partial}{\partial \beta_3} u_I(0) \cdot \frac{\partial}{\partial \beta_3} u_I^*(0) \right) \left(\frac{\partial}{\partial \beta_4} u_J(0) \cdot \frac{\partial}{\partial \beta_4} u_J^*(0) \right) \right]^{(L-L')/4} \right)^{-1} .$$

Now we describe such 4-pt function through its infinite OPE series governing what flows from the GGs to the single-traces. The path to follow is below (22) [10] and in section 4 [9]. The operators in $SL(2)$ sector are those with the smallest possible twist, which means leading power of $\bar{z}$ in the double expansion $z, \bar{z} \to 0$. The choice of scalars guarantees that $f_{L,L'}(z,\tau) \propto \left(\frac{\partial}{\partial \beta_3} \frac{\partial}{\partial \beta_4} u_{IJ}(0,0) \right)^{(L-L')/2}$.

$$\mathcal{N}' \langle \mathcal{O}_1 \mathcal{O}_2 \mathrm{tr}(\dots)\, \mathrm{tr}(\dots) \rangle^{\text{connected}} = \frac{1}{x_{12}^{2N} x_{34}^{L}} \frac{1}{N} \bar{z}^{L'/2} f_{L,L'}(z,\tau) + O\left(\bar{z}^{L'/2+1} \right) \qquad \tau = \frac{1}{2} \log(z\bar{z}) \tag{4.77}$$

$$f_{L,L'}(z,\tau) = \sum_{n=0}^{\infty} \left(\frac{\lambda}{16\pi^2} \right)^n \sum_{m=0}^{n} \sum_{S=0}^{\infty} \mathcal{P}_{L,L',S}^{n,m} f_{L',S}^{(m)}(z,\tau) \tag{4.78}$$

$$= \sum_{S=0}^{\infty} \left[\mathcal{P}_{L,L',S}^{0,0} f_{L',S}^{(0)}(z,\tau) + \frac{\lambda}{16\pi^2} \left(\mathcal{P}_{L,L',S}^{1,0} f_{L',S}^{(0)}(z,\tau) + \mathcal{P}_{L,L',S}^{1,1} f_{L',S}^{(1)}(z,\tau) \right) \right] + O\left(\lambda^2\right)$$

$$f_{L',S}^{(m)}(z,\tau) = z^{\frac{L'+2S}{2}} \frac{\partial^m}{\partial \gamma^m} \left[e^{\tau\gamma} {}_2F_1 \left(\frac{L'+2S+\gamma}{2}, \frac{L'+2S+\gamma}{2}; L'+2S+\gamma; z \right) \right]_{\gamma=0} \tag{4.79}$$

The OPE of $\mathcal{O}_3$ and $\mathcal{O}_4$ produces operators with L' units of R-charge in the $\phi_I(0)$ $(= \phi_J(0))$ direction. The operators with the smallest possible twist in the OPE are of the form (4) [10] with twist L'. The contribution of the leading twist operators to $\mathcal{G}(z,\bar{z})$ is governed by $\mathcal{P}$'s. The previous section measures the lhs above to 1-loop precision, so the comparison delivers $\mathcal{P}$'s: expand the correlator in lhs to leading $\bar{z} \to 0$ and as many powers in $z \to 0$ as wanted (from $z^{L'/2}$ to $z^{L'/2+S_{\max}}$), truncate the series in rhs (from $S = 0$ to $S = S_{\max}$) and expand $f_{L',S}^{(m)}$ in $z \to 0$ (from $z^{L'/2+S}$ to $z^{L'/2+S_{\max}}$). The comparison at tree level

$$\mathcal{N}' \langle \mathcal{O}_1 \mathcal{O}_2 \mathrm{tr}(\dots)\, \mathrm{tr}(\dots) \rangle_{\text{LO}}^{\text{connected}} = \frac{1}{x_{12}^{2N} x_{34}^{L}} \frac{1}{N} \bar{z}^{L'/2} \sum_{S=0}^{\infty} \mathcal{P}_{L,L',S}^{0,0} f_{L',S}^{(0)}(z,\tau) + O\left(\bar{z}^{L'/2+1} \right) \tag{4.80}$$

delivers the numbers #

$$\mathcal{P}_{L,L',S}^{0,0} = \# \left(\frac{u_{ZI}(0)\, u_{\bar{Z}I}(0)\, u_{ZJ}(0)\, u_{\bar{Z}J}(0)}{u_{Z\bar{Z}}^2 u_I^I(0,0)\, u_J^J(0,0)} \right)^{L'/4} \left[\frac{\left(\frac{\partial}{\partial \beta_3} \frac{\partial}{\partial \beta_4} u_{IJ}(0,0) \right)^2}{\left(\frac{\partial}{\partial \beta_3} u_I(0) \cdot \frac{\partial}{\partial \beta_3} u_I^*(0) \right) \left(\frac{\partial}{\partial \beta_4} u_J(0) \cdot \frac{\partial}{\partial \beta_4} u_J^*(0) \right)} \right]^{(L-L')/4} \times \begin{cases} \frac{(-)^S u_{ZI}(0) u_{\bar{Z}J}(0) + u_{\bar{Z}I}(0) u_{ZJ}(0)}{\sqrt{u_{ZI}(0) u_{\bar{Z}I}(0) u_{ZJ}(0) u_{\bar{Z}J}(0)}} & L'/2 = 1, 3, \dots L/2 \\ 1 & L'/2 = 2, 4, \dots L/2 - 1 \end{cases} \tag{4.81}$$

while the comparison at one loop

$$\mathcal{N}' \langle \mathcal{O}_1 \mathcal{O}_2 \mathrm{tr}(\dots)\, \mathrm{tr}(\dots) \rangle_{\text{NLO}}^{\text{connected}} = \frac{\lambda}{16\pi^2} \frac{1}{x_{12}^{2N} x_{34}^{L}} \frac{1}{N} \bar{z}^{L'/2} \sum_{S=0}^{\infty} \left[\mathcal{P}_{L,L',S}^{1,0} f_{L',S}^{(0)}(z,\tau) + \mathcal{P}_{L,L',S}^{(1,1)} f_{L',S}^{(1)}(z,\tau) \right] + O\left(\bar{z}^{L'/2+1} \right) \tag{4.82}$$

delivers the numbers #

$$\mathcal{P}_{L,L',S}^{1,0}, \mathcal{P}_{L,L',S}^{1,1} = \# \left(\frac{u_{ZI}(0)\, u_{\bar{Z}I}(0)\, u_{ZJ}(0)\, u_{\bar{Z}J}(0)}{u_{Z\bar{Z}}^2 u_I^I(0,0)\, u_J^J(0,0)} \right)^{L'/4} \left[\frac{\left(\frac{\partial}{\partial \beta_3} \frac{\partial}{\partial \beta_4} u_{IJ}(0,0) \right)^2}{\left(\frac{\partial}{\partial \beta_3} u_I(0) \cdot \frac{\partial}{\partial \beta_3} u_I^*(0) \right) \left(\frac{\partial}{\partial \beta_4} u_J(0) \cdot \frac{\partial}{\partial \beta_4} u_J^*(0) \right)} \right]^{(L-L')/4} \times \frac{-(-)^{L'} u_{ZI}(0)\, u_{\bar{Z}J}(0) + (-)^S u_{\bar{Z}I}(0)\, u_{ZJ}(0)}{\sqrt{u_{ZI}(0)\, u_{\bar{Z}I}(0)\, u_{ZJ}(0)\, u_{\bar{Z}J}(0)}} \begin{cases} 1 & L'/2 = 1, 3, \dots L/2 - 2 \\ 0 & L'/2 = L/2 \\ \frac{-u_{ZI}(0) u_{\bar{Z}J}(0) + u_{\bar{Z}I}(0) u_{ZJ}(0)}{\sqrt{u_{ZI}(0) u_{\bar{Z}I}(0) u_{ZJ}(0) u_{\bar{Z}J}(0)}} & L'/2 = 2, 4, \dots L/2 - 1 . \end{cases} \tag{4.83}$$

4.4 Structure constants from integrability

The conjecture for $SL(2)$ operators of length L and spin S is the lhs (27) in Shota's "SL2 and OPE". Consistency with our notation forces to rename $L \to L'$ and $\ell \to L'/2$. The conjecture is as it is for positive even S, it gives 1 for $S = 0$ and 0 for odd S.

4.5 Test conjecture

Consider the generating function (29) [10][15] in the auxiliary y

$$\frac{1}{N}\sum_{n=0}^{\infty}\left(\frac{\lambda}{16\pi^2}\right)^n\sum_{m=0}^{n}y^m\mathcal{P}^{n,m}_{L,L',S}=\sum_{\mathbf{u}}C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}(L',S)}C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(L',S)}e^{\gamma y}\,. \tag{4.84}$$

The sum is over the paired solutions $\mathbf{u}$ of the Bethe equations with given L', S. Perturbation theory computes the lhs, Shota's integrability formula predicts $C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}(L',S)}C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(L',S)}$ in the rhs. Let's write the series in λ (not $\lambda/\left(16\pi^2\right)$ as common in integrability)

$$C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}(L',S)}C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(L',S)}\equiv\left[C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}(L',S)}C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(L',S)}\right]_{(0)}+\left[C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}(L',S)}C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(L',S)}\right]_{(1)}+O\left(\lambda^2\right) \tag{4.85}$$

$$\gamma\equiv\gamma_{(1)}+O\left(\lambda^2\right) \tag{4.86}$$

and equate powers of y,λ. We observe at tree level

$$\frac{1}{N}\mathcal{P}^{0,0}_{L,L',S}=\sum_{\mathbf{u}}\frac{(-)^{1+S/2}}{2^{S-1}N}\left(\frac{u_{ZI}(0)\,u_{\bar{Z}I}(0)\,u_{ZJ}(0)\,u_{\bar{Z}J}(0)}{u^2_{Z\bar{Z}}u^I_I(0,0)\,u^J_J(0,0)}\right)^{L'/4}\left[\frac{\left(\frac{\partial}{\partial\beta_3}\frac{\partial}{\partial\beta_4}u_{IJ}(0,0)\right)^2}{\left(\frac{\partial}{\partial\beta_3}u_I(0)\cdot\frac{\partial}{\partial\beta_3}u^*_I(0)\right)\left(\frac{\partial}{\partial\beta_4}u_J(0)\cdot\frac{\partial}{\partial\beta_4}u^*_J(0)\right)}\right]^{(L-L')/4}$$
$$\times\left[C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}(L',S)}C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(L',S)}\right]_{(0)} \tag{4.87}$$

and one loop

$$\frac{1}{16\pi^2}\frac{1}{N}\mathcal{P}^{1,0}_{L,L',S}=\sum_{\mathbf{u}}\frac{(-)^{1+S/2}}{2^{S-1}N}\left(\frac{u_{ZI}(0)\,u_{\bar{Z}I}(0)\,u_{ZJ}(0)\,u_{\bar{Z}J}(0)}{u^2_{Z\bar{Z}}u^I_I(0,0)\,u^J_J(0,0)}\right)^{L'/4}\left[\frac{\left(\frac{\partial}{\partial\beta_3}\frac{\partial}{\partial\beta_4}u_{IJ}(0,0)\right)^2}{\left(\frac{\partial}{\partial\beta_3}u_I(0)\cdot\frac{\partial}{\partial\beta_3}u^*_I(0)\right)\left(\frac{\partial}{\partial\beta_4}u_J(0)\cdot\frac{\partial}{\partial\beta_4}u^*_J(0)\right)}\right]^{(L-L')/4}$$
$$\times\left[C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}(L',S)}C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(L',S)}\right]_{(1)} \tag{4.88}$$

$$\frac{1}{16\pi^2}\frac{1}{N}\mathcal{P}^{1,1}_{L,L',S}=\sum_{\mathbf{u}}\frac{(-)^{1+S/2}}{2^{S-1}N}\left(\frac{u_{ZI}(0)\,u_{\bar{Z}I}(0)\,u_{ZJ}(0)\,u_{\bar{Z}J}(0)}{u^2_{Z\bar{Z}}u^I_I(0,0)\,u^J_J(0,0)}\right)^{L'/4}\left[\frac{\left(\frac{\partial}{\partial\beta_3}\frac{\partial}{\partial\beta_4}u_{IJ}(0,0)\right)^2}{\left(\frac{\partial}{\partial\beta_3}u_I(0)\cdot\frac{\partial}{\partial\beta_3}u^*_I(0)\right)\left(\frac{\partial}{\partial\beta_4}u_J(0)\cdot\frac{\partial}{\partial\beta_4}u^*_J(0)\right)}\right]^{(L-L')/4}$$
$$\times\left[C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}(L',S)}C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}(L',S)}\right]_{(0)}\gamma_{(1)} \tag{4.89}$$

with the factor being $\frac{(-)^{1+S/2}}{2^{S+L'-1}N}$.

[15] Typo: x should read y.

5 Fermionic approach

The usual perturbation theory is difficult because of combinatorics problems in Feynman diagrams. We attempt to absorb the two determinants in the path-integral measure, then to evaluate 3-pt functions in $\mathcal{N}=4$ SYM as 1-pt functions in a modified theory with extra fields and interactions. The objective is to compute this object at planar 1 loop, not only at tree-level as Shota does.

I start isolating the kinetic term of Z in the action. The action was in the Wilson loop file's appendix.

$$-\int_x \mathcal{L}_2 = -\frac{1}{g^2}\int_x \frac{1}{2}\partial_\mu\phi_i^a\partial_\mu\phi_i^a + ... \equiv -\frac{1}{g^2}\int_x \operatorname{tr}\left(\partial_\mu Z\partial_\mu \bar{Z}\right) - \int_x \mathcal{L}_2^{\text{rest}} \tag{5.1}$$

I do not normalize complex combinations of scalars by $2^{-1/2}$, so some 2 disappear compared to Shota. We repeat some Shota's steps

$$\begin{aligned}
\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle &= \mathcal{N}\int\mathcal{D}\left[Y,\bar{Y},Z,\bar{Z}\right]\,\det Z\left(x_1\right)\det\bar{Z}\left(x_2\right)\mathcal{O}_3\left(x_3\right)\exp\left[-\frac{1}{g^2}\int_x \operatorname{tr}\left(\partial_\mu Z\partial_\mu\bar{Z}\right)_x - \int_x\left(\mathcal{L}_2^{\text{rest}}+\mathcal{L}_3+\mathcal{L}_4\right)\right] \qquad (5.2)\\
&= \mathcal{N}\int\mathcal{D}\left[Y,\bar{Y},Z,\bar{Z},\chi,\bar{\chi},\eta,\bar{\eta}\right]\,\mathcal{O}_3\left(x_3\right)\exp\left[-\int_x\left(\mathcal{L}_2^{\text{rest}}+\mathcal{L}_3+\mathcal{L}_4\right)\right]\\
&\quad\times\exp\operatorname{tr}\left[-\frac{1}{g^2}\int_x\left(\partial_\mu Z\partial_\mu\bar{Z}\right)_x+\bar{\chi}Z\left(x_1\right)\chi+\bar{\eta}\bar{Z}\left(x_2\right)\eta\right]\\
&= \mathcal{N}\int\mathcal{D}\left[Y,\bar{Y},Z,\bar{Z},\chi,\bar{\chi},\eta,\bar{\eta}\right]\,\mathcal{O}_3\left(x_3\right)\exp\left[-\int_x\left(\mathcal{L}_2^{\text{rest}}+\mathcal{L}_3+\mathcal{L}_4\right)\right]\\
&\quad\times\exp\int_x\operatorname{tr}\left[-\frac{1}{g^2}\left(\partial_\mu\left(Z-F\right)\partial_\mu\left(\bar{Z}-\bar{F}\right)-\partial_\mu F\partial_\mu\bar{F}\right)_x\right]
\end{aligned}$$

with

$$F=\frac{g^2\eta\bar{\eta}}{4\pi^2\left(x-x_2\right)^2}=\Delta_{xx_2}\eta\bar{\eta} \qquad \bar{F}=\frac{g^2\chi\bar{\chi}}{4\pi^2\left(x-x_1\right)^2}=\Delta_{xx_1}\chi\bar{\chi} \tag{5.3}$$

$$\partial_x^2 F=-g^2\eta\bar{\eta}\delta^4\left(x-x_2\right) \qquad \partial_x^2\bar{F}=-g^2\chi\bar{\chi}\delta\left(x-x_1\right)\,. \tag{5.4}$$

The Berezin integral is

$$1=\int d\left[\chi,\bar{\chi},\eta,\bar{\eta}\right]\chi^1\bar{\chi}_1...\chi^N\bar{\chi}_N\eta^1\bar{\eta}_1...\eta^N\bar{\eta}_N\,. \tag{5.5}$$

Giving a spacetime dependence to fermions does not change anything because they don't have a propagator and the $\eta^2\chi^2$-interaction preserves their number. We shift some scalars

$$Z\to Z+F \qquad \bar{Z}\to\bar{Z}+\bar{F} \qquad \phi_1\to\phi_1+\frac{F+\bar{F}}{2} \qquad \phi_2\to\phi_2+\frac{F-\bar{F}}{2i} \tag{5.6}$$

and "reconstruct" the action plus quartic fermions

$$\begin{aligned}
\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_{\text{NLO}} &= \mathcal{N}\int\mathcal{D}\left[Y,\bar{Y},Z,\bar{Z}\right]d\left[\chi,\bar{\chi},\eta,\bar{\eta}\right]\left(\mathcal{O}_3\left(x_3\right)\right)_{Z\to Z+F,\,\bar{Z}\to\bar{Z}+\bar{F}} \qquad (5.7)\\
&\quad\times\exp\left[-\int_x\left[\mathcal{L}_2+\left(\mathcal{L}_3+\mathcal{L}_4\right)_{Z\to Z+F,\,\bar{Z}\to\bar{Z}+\bar{F}}\right]+\frac{g^2\left(\bar{\chi}\eta\right)\left(\bar{\eta}\chi\right)}{4\pi^2\left(x_1-x_2\right)^2}\right]
\end{aligned}$$

Let us ignore $\mathcal{O}_3$, the interacting Lagrangian and the fermionic integration: this is the textbook generating functional for the fields $Z,\bar{Z}$ of source $\eta\bar{\eta},\chi\bar{\chi}$. Indeed the quartic term is the product of two sources and the free scalar propagator.

Then we do a Hubbard-Stratonovich transformation on the quartic fermions that brings in two complex scalars.

$$\begin{aligned}
\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_{\text{NLO}} &= \mathcal{N}\int\mathcal{D}\left[Y,\bar{Y},Z,\bar{Z}\right]d\left[\chi,\bar{\chi},\eta,\bar{\eta},\rho,\bar{\rho}\right]\left(\mathcal{O}_3\left(x_3\right)\right)_{Z\to Z+F,\,\bar{Z}\to\bar{Z}+\bar{F}} \qquad (5.8)\\
&\quad\times\exp\left[-\int_x\left[\mathcal{L}_2+\left(\mathcal{L}_3+\mathcal{L}_4\right)_{Z\to Z+F,\,\bar{Z}\to\bar{Z}+\bar{F}}\right]-\frac{4\pi^2\left(x_1-x_2\right)^2}{g^2}\rho\bar{\rho}+\rho\left(\bar{\chi}\eta\right)+\bar{\rho}\left(\bar{\eta}\chi\right)\right]
\end{aligned}$$

The effect of absorbing the determinants into the action is:

- an interaction Lagrangian evaluated on a non-constant vacuum (= plethora of new vertices), but the same free propagators of $\mathcal{N}=4$ SYM,

- free complex fermions (in total $4N$ real elementary fields) with mixed propagators,
- a complex scalar (in total 2 real elementary fields), which is a background for now and integrated in the last step.

Notes

- ρ is not a worthless complication: it gives propagators to fermions, so we do Feynman diagrams instead of Berezin integrations (= tons of combinatorics).
- It is tempting to "bosonize" $\eta, \chi \to F, \bar{F}$ and see F as fundamental field, but I think the Jacobian would lead back to a determinant in the path-integral or similar.

[...]

6 Feynman rules

Integrals [11, 5, 12, 13, 14]

$$I_{12} = \frac{1}{(2\pi)^2} \frac{1}{(x_1 - x_2)^2} \tag{6.1}$$

$$Y_{123} = \int d^4w \, I_{1w} I_{2w} I_{3w} \tag{6.2}$$

$$X_{1234} = \int d^4w \, I_{1w} I_{2w} I_{3w} I_{4w} \tag{6.3}$$

$$G_{1,23} = Y_{123} \left(\frac{1}{I_{13}} - \frac{1}{I_{12}} \right) \tag{6.4}$$

$$F_{12,34} = \frac{X_{1234}}{I_{13} I_{24}} - \frac{X_{1234}}{I_{14} I_{23}} + G_{1,34} - G_{2,34} + G_{3,12} - G_{4,12} \tag{6.5}$$

I, Y, X are totally symmetric and

$$G_{1,23} = -G_{1,32} \qquad\qquad F_{12,34} = -F_{21,34} = -F_{12,43} = F_{34,12} \,. \tag{6.6}$$

Functional form for non-singular points [16]

$$X_{1234} = \frac{\pi^2 F^{(1)}(z, \bar{z})}{(2\pi)^8 \, x_{13}^2 x_{24}^2} \,, \qquad \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z} \,, \qquad \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1 - z)(1 - \bar{z}) \tag{6.7}$$

$$F^{(1)}(z, \bar{z}) = \frac{2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log(z\bar{z}) \log \frac{1-z}{1-\bar{z}}}{z - \bar{z}} \tag{6.8}$$

$$Y_{123} = \lim_{x_4 \to \infty} (2\pi)^2 x_4^2 X_{1234} = \frac{\pi^2 F^{(1)}(w_{123}, \bar{w}_{123})}{(2\pi)^6 \, x_{13}^2} \,, \qquad \frac{x_{12}^2}{x_{13}^2} = w_{123}\bar{w}_{123} \,, \qquad \frac{x_{23}^2}{x_{13}^2} = (1 - w_{123})(1 - \bar{w}_{123}) \tag{6.9}$$

Properties of one-loop conformal integral

$$F^{(1)}(\bar{z}, z) = F^{(1)}(z, \bar{z}) \tag{6.10}$$

$$F^{(1)}(1 - z, 1 - \bar{z}) = F^{(1)}(z, \bar{z}) \tag{6.11}$$

$$F^{(1)}\left(\frac{1}{z}, \frac{1}{\bar{z}} \right) = z\bar{z} F^{(1)}(z, \bar{z}) \tag{6.12}$$

$$F^{(1)}\left(\frac{z}{z - 1}, \frac{\bar{z}}{\bar{z} - 1} \right) = (1 - z)(1 - \bar{z}) F^{(1)}(z, \bar{z}) \tag{6.13}$$

Point-splitting regularization for coincident points [17]

$$Y_{112} = Y_{122} = -\frac{I_{12}}{16\pi^2} \left(\log \frac{\epsilon^2}{x_{12}^2} - 2 \right) \tag{6.14}$$

$$X_{1123} = -\frac{I_{12} I_{13}}{16\pi^2} \left(\log \frac{\epsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2 \right) \tag{6.15}$$

$$X_{1122} = -\frac{I_{12}^2}{8\pi^2} \left(\log \frac{\epsilon^2}{x_{12}^2} - 1 \right) \tag{6.16}$$

$$\begin{aligned} F_{12,13} &= -I_{12}^{-1} I_{13}^{-1} X_{1123} + G_{1,12} + G_{1,13} - G_{2,13} - G_{3,12} \\ &= -I_{12}^{-1} I_{13}^{-1} X_{1123} + I_{12}^{-1} Y_{112} + I_{13}^{-1} Y_{113} + Y_{123} \left(\frac{1}{I_{12}} + \frac{1}{I_{13}} - \frac{2}{I_{23}} \right) \\ &= -\frac{1}{16\pi^2} \left(\log \frac{\epsilon^2}{x_{23}^2} - 2 \right) + Y_{123} \left(\frac{1}{I_{12}} + \frac{1}{I_{13}} - \frac{2}{I_{23}} \right) \end{aligned}$$

$$F_{12,12} = -I_{12}^{-2} X_{1122} + 4 I_{12}^{-1} Y_{112} = -\frac{1}{8\pi^2} \left(\log \frac{\epsilon^2}{x_{12}^2} - 3 \right) \tag{6.17}$$

Scalar propagator

$$\left\langle \phi_{I_1}^{a_1}(\bar{x}_1) \, \phi_{I_2}^{a_2}(\bar{x}_2) \right\rangle_{\mathrm{LO}} = \frac{\lambda}{N} \delta_{I_1 I_2} \delta^{a_1 a_2} I_{12} \tag{6.18}$$

[16] The dependence on spacetime points comes from the inversion of cross-ratio definitions. Note that sign of $\mathrm{Im}z$ is undetermined but X_{1234} is insensitive to $z \leftrightarrow \bar{z}$. The same goes for $w_{123}, \bar{w}_{123}$ with respect to Y_{123}.

[17] The last formula was corrected by taking the limit of $F_{12,34}$.

$$\left\langle \phi_{I_1}^{a_1}\left(\bar{x}_1\right)\phi_{I_2}^{a_2}\left(\bar{x}_2\right)\right\rangle_{\mathrm{LO}} T_{ij}^{a_1}T_{kl}^{a_2} = \frac{\lambda}{2N}\delta_{I_1I_2}\delta_{il}\delta_{jk}I_{12} \tag{6.19}$$

"O vertex" = one-loop correction to scalar propagator

$$\left\langle \phi_{I_1}^{a_1}\left(x_1\right)\phi_{I_2}^{a_2}\left(x_2\right)V_O\right\rangle_{\mathrm{LO}} = -\frac{\lambda^2}{N}\delta_{I_1I_2}\left(\delta^{a_1a_2}-\delta^{a_1N^2}\delta^{a_2N^2}\right)\left(Y_{112}+Y_{122}\right) \tag{6.20}$$

$$\left\langle \phi_{I_1}^{a_1}\left(x_1\right)\phi_{I_2}^{a_1}\left(x_2\right)V_O\right\rangle_{\mathrm{LO}} = -\lambda^2\frac{N^2-1}{N}\delta_{I_1I_2}\left(Y_{112}+Y_{122}\right) \tag{6.21}$$

$$\left\langle \phi_{I_1}^{a_1}\left(x_1\right)\phi_{I_2}^{a_2}\left(x_2\right)V_O\right\rangle_{\mathrm{LO}} T_{ij}^{a_1}T_{kl}^{a_2} = -\frac{\lambda^2}{2N}\delta_{I_1I_2}\left(\delta_{il}\delta_{jk}-\frac{1}{N}\delta_{ij}\delta_{kl}\right)\left(Y_{112}+Y_{122}\right) \tag{6.22}$$

$$\left\langle \phi_{I_1}^{a_1}\left(x_1\right)\phi_{I_2}^{a_2}\left(x_2\right)V_O\right\rangle_{\mathrm{LO}} T_{ij}^{a_1}T_{jl}^{a_2} = -\frac{\left(N^2-1\right)\lambda^2}{2N^2}\delta_{I_1I_2}\delta_{il}\left(Y_{112}+Y_{122}\right) \tag{6.23}$$

$$\left\langle \phi_{I_1}^{a_1}\left(x_1\right)\phi_{I_2}^{a_2}\left(x_2\right)V_O\right\rangle_{\mathrm{LO}} T_{ij}^{a_1}T_{ji}^{a_2} = -\frac{\left(N^2-1\right)\lambda^2}{2N}\delta_{I_1I_2}\left(Y_{112}+Y_{122}\right) \tag{6.24}$$

"X vertex" made of the Lagrangian vertex ϕ^4

$$\begin{aligned}
&\left\langle \phi_{I_1}^{a_1}\left(x_1\right)\phi_{I_2}^{a_2}\left(x_2\right)\phi_{I_3}^{a_3}\left(x_3\right)\phi_{I_4}^{a_4}\left(x_4\right)V_X\right\rangle_{\mathrm{LO}} \\
=&\int_y\left\langle \phi_{I_1}^{a_1}\left(x_1\right)\phi_{I_2}^{a_2}\left(x_2\right)\left(-\frac{1}{4g^2}f^{abc}f^{ade}\phi_I^b\left(y\right)\phi_J^c\left(y\right)\phi_I^d\left(y\right)\phi_J^e\left(y\right)\right)\phi_{I_3}^{a_3}\left(x_3\right)\phi_{I_4}^{a_4}\left(x_4\right)\right\rangle_{\mathrm{LO}} \\
=&\frac{\lambda^3}{N^3}\left[-\left(f^{aa_1a_3}f^{aa_2a_4}+f^{aa_1a_4}f^{aa_2a_3}\right)\delta_{I_1I_2}\delta_{I_3I_4}+\left(f^{aa_1a_4}f^{aa_2a_3}-f^{aa_1a_2}f^{aa_3a_4}\right)\delta_{I_1I_3}\delta_{I_2I_4}\right. \\
&\left.+\left(f^{aa_1a_2}f^{aa_3a_4}+f^{aa_1a_3}f^{aa_2a_4}\right)\delta_{I_1I_4}\delta_{I_2I_3}\right]X_{1234}
\end{aligned} \tag{6.25}$$

"H vertex" made of 2 Lagrangian vertices $A\phi^2$ and a gauge propagator

$$\begin{aligned}
&\left\langle \phi_{I_1}^{a_1}\left(x_1\right)\phi_{I_2}^{a_2}\left(x_2\right)\phi_{I_3}^{a_3}\left(x_3\right)\phi_{I_4}^{a_4}\left(x_4\right)V_H\right\rangle_{\mathrm{LO}} \\
=&\int_{y,z}\left\langle \phi_{I_1}^{a_1}\left(x_1\right)\phi_{I_2}^{a_2}\left(x_2\right)\frac{1}{2!}\left(-\frac{1}{g^2}f^{abc}\partial_{y^\mu}\phi_I^a\left(y\right)A_\mu^b\left(y\right)\phi_I^c\left(y\right)\right)\left(-\frac{1}{g^2}f^{def}\partial_{z^\nu}\phi_J^d\left(z\right)A_\nu^e\left(z\right)\phi_J^f\left(z\right)\right)\phi_{I_3}^{a_3}\left(x_3\right)\phi_{I_4}^{a_4}\left(x_4\right)\right\rangle_{\mathrm{LO}} \\
=&\frac{\lambda^3}{N^3}f^{aa_1a_2}f^{aa_3a_4}\delta_{I_1I_2}\delta_{I_3I_4}I_{12}I_{34}F_{12,34}+\frac{\lambda^3}{N^3}f^{aa_1a_3}f^{aa_2a_4}\delta_{I_1I_3}\delta_{I_2I_4}I_{13}I_{24}F_{13,24}+\frac{\lambda^3}{N^3}f^{aa_1a_4}f^{aa_2a_3}\delta_{I_1I_4}\delta_{I_2I_3}I_{14}I_{23}F_{14,23}
\end{aligned} \tag{6.26}$$

Vertices dress up tree-level structures in main text with the combinations

$$\begin{aligned}
c_{I_1I_2} &= \left\langle \mathrm{tr}\left[\phi_{I_1}\left(x_1\right)\phi_{I_2}\left(x_2\right)\right]V_O\right\rangle_{\mathrm{LO}} \\
&= -\frac{N^2-1}{2N}\lambda^2\left(Y_{112}+Y_{122}\right)\delta_{I_1I_2}
\end{aligned} \tag{6.27}$$

$$\begin{aligned}
c_{I_1I_2I_3I_4} &= \left\langle \mathrm{tr}\left[\phi_{I_1}\left(x_1\right)\phi_{I_2}\left(x_2\right)\phi_{I_3}\left(x_3\right)\phi_{I_4}\left(x_4\right)\right]V_{X+H}\right\rangle_{\mathrm{LO}} \\
&=\frac{N^2-1}{8N}\lambda^3\left[X_{1234}\left(-\delta_{I_1I_2}\delta_{I_3I_4}+2\delta_{I_1I_3}\delta_{I_2I_4}-\delta_{I_1I_4}\delta_{I_2I_3}\right)-I_{12}I_{34}F_{12,34}\delta_{I_1I_2}\delta_{I_3I_4}+I_{14}I_{23}F_{14,23}\delta_{I_1I_4}\delta_{I_2I_3}\right]
\end{aligned} \tag{6.28}$$

with $c_{I_1I_2I_3I_4}=c_{I_2I_1I_4I_3}$ and cyclic periodicity. We use just once

$$\begin{aligned}
\tilde{c}_{I_1I_2I_3I_4} &= \left\langle \mathrm{tr}\left[\phi_{I_1}\left(x_1\right)\phi_{I_2}\left(x_2\right)\right]\mathrm{tr}\left[\phi_{I_3}\left(x_3\right)\phi_{I_4}\left(x_4\right)\right]V_{X+H}\right\rangle_{\mathrm{LO}} \\
&=\frac{N^2-1}{4N^2}\lambda^3\left[\left(\delta_{I_1I_3}\delta_{I_2I_4}+\delta_{I_1I_4}\delta_{I_2I_3}\right)X_{1234}+\delta_{I_1I_3}\delta_{I_2I_4}I_{13}I_{24}F_{13,24}+\delta_{I_1I_4}\delta_{I_2I_3}I_{14}I_{23}F_{14,23}\right] .
\end{aligned} \tag{6.29}$$

7 $U(N)$ identities

The $\mathfrak{u}(N)$ generators T^a $(a = 1, \ldots N^2)$ are $N \times N$ matrices with real structure constants [3, 15, 16].

$$T^{N^2} = \frac{\mathbb{I}_N}{\sqrt{2N}} \qquad \mathrm{tr}\left(T^a\right) = \delta^{aN^2}\sqrt{\frac{N}{2}} \qquad f^{abc} = \begin{cases} \ldots & a, b, c \neq N^2 \\ 0 & \text{otherwise} \end{cases} . \tag{7.1}$$

$$\mathrm{tr}\left(\mathbb{I}\right) = \delta_{ii} = N \tag{7.2}$$

$$\delta^{aa} = N^2 \tag{7.3}$$

$$\left[T^a, T^b\right] = i f^{abc} T^c \tag{7.4}$$

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\delta_{il}\delta_{kj} \tag{7.5}$$

$$T^a T^a = \frac{N}{2}\mathbb{I}_N \tag{7.6}$$

$$\mathrm{tr}\left(T^a T^b\right) = \frac{\delta^{ab}}{2} \tag{7.7}$$

$$\mathrm{tr}\left(T^a T^a\right) = \frac{N^2}{2} \tag{7.8}$$

$$f^{ba_2a_3} f^{ba_1c} + f^{ba_3a_1} f^{ba_2c} + f^{ba_1a_2} f^{ba_3c} = 0 \tag{7.9}$$

$$f^{acd} f^{bcd} = \begin{cases} N\delta^{ab} & a, b \neq N^2 \\ 0 & \text{otherwise} \end{cases} = N\left(\delta^{ab} - \delta^{aN^2}\delta^{bN^2}\right) \tag{7.10}$$

$$f^{abc} f^{abc} = N\left(N^2 - 1\right) \tag{7.11}$$

$$f^{abc} T^b T^c = \begin{cases} \frac{iN}{2} T^a & a \neq N^2 \\ 0 & \text{otherwise} \end{cases} \tag{7.12}$$

$$f^{abc}\mathrm{tr}\left(T^a T^b T^c\right) = \frac{iN\left(N^2 - 1\right)}{4} \tag{7.13}$$

$$T^a T^b T^a = \begin{cases} 0 & b \neq N^2 \\ \sqrt{\frac{N}{8}}\mathbb{I}_N & \text{otherwise} \end{cases} \tag{7.14}$$

$$\begin{aligned} f^{aa_1a_2} f^{aa_3a_4} T^{a_1}_{i_1j_1} T^{a_2}_{i_2j_2} T^{a_3}_{i_3j_3} T^{a_4}_{i_4j_4} = \frac{1}{8}&\left(-\delta_{i_1j_2}\delta_{i_2j_3}\delta_{i_3j_4}\delta_{i_4j_1} + \delta_{i_1j_2}\delta_{i_2j_4}\delta_{i_3j_1}\delta_{i_4j_3}\right. \\ &\left. + \delta_{i_1j_3}\delta_{i_2j_1}\delta_{i_3j_4}\delta_{i_4j_2} - \delta_{i_1j_4}\delta_{i_2j_1}\delta_{i_3j_2}\delta_{i_4j_3}\right) \end{aligned} \tag{7.15}$$

$$f^{aa_1a_2} f^{aa_3a_4} T^{a_1}_{i_1j_1}\left(T^{a_2}T^{a_3}T^{a_4}\right)_{i_2j_2} = -\frac{N^2}{8}\left(\delta_{i_1j_2}\delta_{i_2j_1} - \frac{1}{N}\delta_{i_1j_1}\delta_{i_2j_2}\right) \tag{7.16}$$

$$f^{aa_1a_2} f^{aa_3a_4}\left(T^{a_1}T^{a_2}\right)_{i_1j_1}\left(T^{a_3}T^{a_4}\right)_{i_2j_2} = \frac{N^2}{8}\left(\delta_{i_1j_1}\delta_{i_2j_2} - \delta_{i_1j_2}\delta_{i_2j_1}\right) \tag{7.17}$$

$$f^{aa_1a_2} f^{aa_3a_4}\left(T^{a_1}T^{a_3}\right)_{i_1j_1}\left(T^{a_2}T^{a_4}\right)_{i_2j_2} = \frac{N}{4}\left(\delta_{i_1j_1}\delta_{i_2j_2} - \frac{1}{N}\delta_{i_1j_2}\delta_{i_2j_1}\right) \tag{7.18}$$

$$f^{aa_1a_2} f^{aa_3a_4}\left(T^{a_1}T^{a_2}T^{a_3}\right)_{i_1j_1} T^{a_4}_{i_2j_2} = -\frac{N^2}{8}\left(\delta_{i_1j_2}\delta_{i_2j_1} - \frac{1}{N}\delta_{i_1j_1}\delta_{i_2j_2}\right) \tag{7.19}$$

$$\mathrm{tr}\left(T^a T^a T^b T^b\right) = \frac{N^3}{4} \tag{7.20}$$

$$\mathrm{tr}\left(T^a T^b T^a T^b\right) = \frac{N}{4} \tag{7.21}$$

$$f^{aa_1a_2} f^{aa_3a_4} T^{a_1}T^{a_2}T^{a_3}T^{a_4} = -N\frac{N^2 - 1}{8}\mathbb{I}_N \tag{7.22}$$

$$f^{aa_1a_3} f^{aa_2a_4} T^{a_1}T^{a_2}T^{a_3}T^{a_4} = 0 \tag{7.23}$$

$$f^{aa_1a_4} f^{aa_2a_3} T^{a_1}T^{a_2}T^{a_3}T^{a_4} = N\frac{N^2 - 1}{8}\mathbb{I}_N \tag{7.24}$$

$$f^{aa_1a_2} f^{aa_3a_4}\mathrm{tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right) = -N^2\frac{N^2 - 1}{8} \tag{7.25}$$

$$f^{aa_1a_3} f^{aa_2a_4}\mathrm{tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right) = 0 \tag{7.26}$$

$$f^{aa_1a_4} f^{aa_2a_3}\mathrm{tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right) = N^2\frac{N^2 - 1}{8} . \tag{7.27}$$

8 Combinatorics

Appendix A [2], Wikipedia. Index ranges from 1 to N.

Formulas with one δ

$$\delta^{i_1,\dots i_n}_{j_1,\dots j_n} \equiv \Sigma_\sigma \left(-\right)^\sigma \delta^{i_1}_{j_{\sigma(1)}} \dots \delta^{i_n}_{j_{\sigma(n)}} \tag{8.1}$$

$$\delta^{i_1,\dots i_n}_{j_2,\dots j_n,j_1} = \left(-\right)^{n+1} \delta^{i_1,\dots i_n}_{j_1,\dots j_n} \tag{8.2}$$

$$\delta^{i_1,\dots i_p,i_{p+1},\dots i_{p+n}}_{l_1,\dots l_p,i_{p+1},\dots i_{p+n}} = \frac{(N-p)!}{(N-p-n)!} \delta^{i_1,\dots i_p}_{l_1,\dots l_p} \tag{8.3}$$

$$\delta^{i_1,\dots i_n}_{i_1,\dots i_n} = \frac{N!}{(N-n)!} = n! \binom{N}{n} \tag{8.4}$$

$$\delta^{i_1,\dots i_N}_{i_1,\dots i_N} = N! \tag{8.5}$$

Formulas with two δ

$$\delta^{i_1,\dots i_n}_{j_1,\dots j_n} \delta^{j_1,\dots j_n}_{l_1,\dots l_n} = n! \delta^{i_1,\dots i_n}_{l_1,\dots l_n} \tag{8.6}$$

$$\begin{aligned} \delta^{i_1,\dots i_n}_{j_1,\dots j_n} \delta^{i_1,\dots i_n}_{j_1,\dots j_n} &= \frac{N! n!}{(N-n)!} = (n!)^2 \binom{N}{n} \\ &= n! N^n \prod_{j=1}^{n-1} \left(1 - \frac{j}{N}\right) \\ &= O\left(N^n\right) \qquad\qquad n \ll N,\, N \to \infty \end{aligned} \tag{8.7}$$

Formulas with one $\varepsilon\varepsilon$

$$\varepsilon\varepsilon^{i_1,\dots i_n}_{j_1,\dots j_n} \equiv \varepsilon_{k_1,\dots k_{N-n} i_1,\dots i_n} \varepsilon_{k_1 \dots k_{N-n} j_1 \dots j_n} = (N-n)! \delta^{i_1,\dots i_n}_{j_1,\dots j_n} \tag{8.8}$$

$$\varepsilon\varepsilon^{i_1,\dots i_n}_{j_2,\dots j_n,j_1} = \left(-\right)^{n+1} \varepsilon\varepsilon^{i_1,\dots i_n}_{j_1,\dots j_n} \tag{8.9}$$

$$\varepsilon\varepsilon^{i_1,\dots i_n}_{i_1,\dots i_n} = \varepsilon_{k_1 \dots k_N} \varepsilon_{k_1 \dots k_N} = N! \tag{8.10}$$

$$\varepsilon\varepsilon^{i_1,\dots i_N}_{j_1,\dots j_N} = \varepsilon_{i_1,\dots i_N} \varepsilon_{j_1,\dots j_N} = \delta^{i_1,\dots i_N}_{j_1,\dots j_N} \tag{8.11}$$

Formulas with two $\varepsilon\varepsilon$

$$\varepsilon\varepsilon^{i_1,\dots i_n}_{j_1,\dots j_n} \varepsilon\varepsilon^{i_1,\dots i_n}_{j_1,\dots j_n} = \left((N-n)!\right)^2 \delta^{i_1,\dots i_n}_{j_1,\dots j_n} \delta^{i_1,\dots i_n}_{j_1,\dots j_n} = N! \, (N-n)! n! \tag{8.12}$$

$$\varepsilon\varepsilon^{i_1,\dots i_N}_{j_1,\dots j_N} \varepsilon\varepsilon^{i_1,\dots i_N}_{j_1,\dots j_N} = (N!)^2 \tag{8.13}$$

Determinants and one δ or $\varepsilon\varepsilon$

$$\begin{aligned} \det A &= \frac{1}{N!} \varepsilon_{i_1 \dots i_N} \varepsilon_{j_1 \dots j_N} A_{i_1 j_1} \dots A_{i_N j_N} \\ &= \frac{1}{N!} \delta^{i_1,\dots i_N}_{j_1,\dots j_N} A_{i_1 j_1} \dots A_{i_N j_N} \\ &= \frac{1}{N!} \varepsilon\varepsilon^{i_1,\dots i_N}_{j_1,\dots j_N} A_{i_1 j_1} \dots A_{i_N j_N} \\ &= \sum_{k_1,k_2,\dots k_N=0}^{N} \frac{(-)^N}{k_1! \dots k_N!} \left(-\mathrm{tr} A\right)^{k_1} \dots \left(-\frac{1}{N} \mathrm{tr} A^N\right)^{k_N} \qquad k_1 + 2k_2 + \dots N k_N = N \end{aligned} \tag{8.14}$$

$$\begin{aligned} \delta^{i_1,\dots i_n}_{j_1,\dots j_n} A^{(1)}_{i_1 j_1} \dots A^{(n)}_{i_n j_n} &= \Sigma_\sigma \left(-\right)^\sigma \delta^{i_1}_{j_{\sigma(1)}} \dots \delta^{i_n}_{j_{\sigma(n)}} A^{(1)}_{i_1 j_1} \dots A^{(n)}_{i_n j_n} \\ &= \left(-\right)^{n+1} \mathrm{tr} \left(A^{(1)} \left(A^{(2)} \dots A^{(n)} + (\text{permutations } 2, \dots n)\right)\right) + \text{multitrace terms} \end{aligned} \tag{8.15}$$

$$\begin{aligned} \delta^{i_1,\dots i_n}_{j_1,\dots j_n} A_{i_1 j_1} \dots A_{i_n j_n} &= \sum_{k_1,k_2,\dots k_n=0}^{n} \frac{(-)^n \, n!}{k_1! \dots k_n!} \left(-\mathrm{tr} A\right)^{k_1} \dots \left(-\frac{1}{n} \mathrm{tr} A^n\right)^{k_n} \qquad k_1 + 2k_2 + \dots n k_n = n \\ &= \left(-\right)^{n+1} (n-1)! \mathrm{tr} A^n + \text{multitrace terms} \end{aligned} \tag{8.16}$$

Determinants and two δ or $\varepsilon\varepsilon$

$$\delta^{i_1,\dots i_n}_{l_1,\dots l_n} \delta^{j_1,\dots j_n}_{k_1,\dots k_n} A^{(1)}_{i_1 k_1} \dots A^{(n)}_{i_n k_n} B^{(1)}_{j_1 l_1} \dots B^{(n)}_{j_n l_n} = \Sigma_\sigma \delta^{i_1,\dots i_n}_{j_1,\dots j_n} \left(A^{(1)} B^{(\sigma(1))}\right)_{i_1 j_1} \dots \left(A^{(n)} B^{(\sigma(n))}\right)_{i_n j_n} \tag{8.17}$$

$$
\begin{aligned}
&= \Sigma_\sigma \Sigma_{\sigma'} (-)^{\sigma'} \delta^{i_1}_{j_{\sigma'(1)}} ... \delta^{i_n}_{j_{\sigma'(n)}} \left(A^{(1)} B^{(\sigma(1))}\right)_{i_1 j_1} ... \left(A^{(n)} B^{(\sigma(n))}\right)_{i_n j_n} \\
&= (-)^{n+1} \Sigma_\sigma \mathrm{tr} \left(A^{(1)} B^{(\sigma(1))} \left(A^{(2)} B^{(\sigma(2))} ... A^{(n)} B^{(\sigma(n))} + (\text{permutations } 2, ...n)\right)\right) \\
&+ (-)^n \sum_{k=1}^{\lfloor n/2 \rfloor} \ell_k \Sigma_{\sigma,\sigma'} \mathrm{tr} \left(A^{(\sigma'(l_1))} B^{(\sigma(l_1))} \left(A^{(\sigma'(l_2))} B^{(\sigma(l_2))} ... A^{(\sigma'(l_k))} B^{(\sigma(l_k))}\right) + (\text{permutations } 2, ...k)\right) \\
&\quad \times \mathrm{tr} \left(A^{(\sigma'(l_{k+1}))} B^{(\sigma(l_{k+1}))} \left(A^{(\sigma'(l_{k+2}))} B^{(\sigma(l_{k+2}))} ... A^{(\sigma'(l_n))} B^{(\sigma(l_n))} + (\text{permutations } k+2, ...n)\right)\right) \\
&+ (\geq 3 \text{ trace terms})
\end{aligned}
$$

$$
\begin{aligned}
\delta^{i_1,...i_n}_{l_1,...l_n} \delta^{j_1,...j_n}_{k_1,...k_n} A_{i_1 k_1} ... A_{i_n k_n} B_{j_1 l_1} ... B_{j_n l_n} &= n! \delta^{i_1,...i_n}_{j_1,...j_n} (AB)_{i_1 j_1} ... (AB)_{i_n j_n} \\
&= \sum_{k_1,k_2,...k_n=0}^{n} \frac{(-)^n (n!)^2}{k_1! ... k_n!} (-\mathrm{tr}(AB))^{k_1} ... \left(-\frac{1}{n} \mathrm{tr}(AB)^n\right)^{k_n} \qquad k_1 + 2k_2 + ... nk_n = n \\
&= (-)^{n+1} n! (n-1)! \mathrm{tr}(AB)^n + (-)^n (n!)^2 \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{\ell_{k,n}}{k(n-k)} \mathrm{tr}(AB)^k \, \mathrm{tr}(AB)^{n-k} \\
&+ \left[(-)^{n+1} (n!)^2 \sum_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{\tilde{\ell}_{k,n}}{k(n-k-1)} \mathrm{tr}(AB) \, \mathrm{tr}(AB)^k \, \mathrm{tr}(AB)^{n-k-1} + (\text{other 3-trace terms})\right] + (\geq 4\text{-trace terms})
\end{aligned}
\tag{8.18}
$$

with "symmetry factors"

$$
\ell_{k,n} = \begin{cases} 1 & k \neq n/2 \quad n \geq 2 \\ 1/2 & k = n/2 \quad n \geq 2 \end{cases} \qquad \tilde{\ell}_{k.n} = \begin{cases} 1 & k \neq 1, \frac{n-1}{2} \quad n \geq 4 \\ 1/2 & k = 1, \frac{n-1}{2} \quad n \geq 4 \\ 1/12 & k = 1 \quad n = 3 \end{cases} \tag{8.19}
$$

not to overcount equal-length traces. The formula where these appear reproduces Yunfeng's examples $n = 2, 3$. I never used three- and higher multi-traces.

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