Appendices

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Contents

1 The geometry of H^2

1.1 Polar coordinates

We work in geodesic polar coordinates $x=(\rho,\tau)$

$$ds^2 = d\rho^2 + \sinh^2\rho d\tau^2 \qquad \rho>0 \qquad \tau\in[0,2\pi)\,. \tag{1.1}$$

The geodesic distance between two points x, x' is

$$\cosh d\left(x,x'\right) = \cosh\rho\cosh\rho^{'} - \sinh\rho\sinh\rho^{'}\cos\left(\tau-\tau'\right)\,. \tag{1.2}$$

<u>Scalar operators</u>
The scalar covariant Laplacian is hermitian and has spectrum $\lambda\in\left(-\infty,-\frac{1}{4}\right]$.

$$\begin{aligned}\Delta_{\rho,\tau} &= \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right)\\ &= \frac{1}{\sinh\rho}\partial_\rho\left(\sinh\rho\partial_\rho\right) + \frac{1}{\sinh^2\rho}\partial_\tau^2\\ &= \partial_\rho^2 + \frac{\cosh\rho}{\sinh\rho}\partial_\rho + \frac{1}{\sinh^2\rho}\partial_\tau^2\end{aligned} \tag{1.3}$$

<u>Spinor operators</u>
The Dirac operator is anti-hermitian and reads [1]

$$\begin{aligned}\nabla\!\!\!/_{\rho,\tau} &= \Gamma^i\nabla_i\\ &= \Gamma^i\left(\partial_i + \frac{1}{4}\omega^{jk}{}_i\Gamma_{jk}\right)\\ &= \Gamma^1\partial_\rho + \frac{1}{\sinh\rho}\Gamma^2\left(\partial_\tau - \frac{\cosh\rho}{2}\Gamma_{12}\right)\\ &= \Gamma^1\left(\partial_\rho + \frac{\cosh\rho}{2\sinh\rho}\right) + \frac{1}{\sinh\rho}\Gamma^2\partial_\tau\,.\end{aligned} \tag{1.4}$$

and has purely imaginary spectrum $i\lambda$, $\lambda\in\mathbb{R}$. We also write its square

$$\nabla\!\!\!/^2_{\rho,\tau} = \frac{1}{\sinh\rho}\partial_\rho\left(\sinh\rho\partial_\rho\right) + \frac{1}{\sinh^2\rho}\partial_\tau^2 - \frac{\cosh\rho}{\sinh^2\rho}\Gamma^{12}\partial_\tau - \frac{1}{4\sinh^2\rho} + \frac{1}{4}\,. \tag{1.5}$$

We use the matrix representation

$$\Gamma^1=\Gamma^\rho=\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad \Gamma^2=\Gamma^\tau=\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix} \qquad \Gamma^3=-i\Gamma^{12}=\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\,, \tag{1.6}$$

Compared to [3], the first direction is the radial one and we choose a different representation.

[1] We follow the index conventions in Appendix A [2].

1.2 Infinite-strip coordinates

The geodesic distance between two points x, x' can be derived from the distance formula in polar coordinates via the chain of change of coordinates [2].

$$ds^2 = d\rho^2 + \sinh^2\rho d\tau^2 \qquad \rho > 0 \qquad \tau \in [0, 2\pi) \qquad \text{polar coordinates} \tag{1.7}$$

$$\cosh d(x_1, x_2) = \cosh\rho_1 \cosh\rho_2 - \sinh\rho_1 \sinh\rho_2 \cos(\tau_1 - \tau_2)$$

$$\updownarrow$$

$$\begin{cases} \sin\tau &= \frac{4x}{\sqrt{(4+x^2+z^2)^2-16z^2}} \\ \cos\tau &= \frac{4-x^2-z^2}{\sqrt{(4+x^2+z^2)^2-16z^2}} \\ \cosh\rho &= \frac{4+x^2+z^2}{4z} \end{cases} \qquad \begin{cases} x &= \frac{2\sin\tau\sinh\rho(\cosh\rho-\cos\tau\sinh\rho)}{1+\sin^2\tau\sinh^2\rho} \\ z &= \frac{2}{\cosh\rho+\cos\tau\sinh\rho} \end{cases}$$

$$\updownarrow$$

$$ds^2 = \frac{dz^2 + dx^2}{z^2} \qquad z > 0 \qquad x \in \mathbb{R} \qquad \text{Poincare' half-plane model} \tag{1.8}$$

$$\cosh d(x_1, x_2) = 1 + \frac{(z_1 - z_2)^2 + (x_1 - x_2)^2}{2z_1 z_2}$$

$$\updownarrow$$

$$\begin{cases} x &= e^w \tanh\rho \\ z &= \frac{e^w}{\cosh\rho} \end{cases} \qquad \begin{cases} \sinh\rho = \frac{x}{z} \\ \tanh w = \frac{z^2+x^2-1}{z^2+x^2+1} \end{cases}$$

$$\updownarrow$$

$$ds^2 = d\rho^2 + \cosh^2\rho dw^2 \qquad \rho \in \mathbb{R} \qquad w \in \mathbb{R} \qquad \text{unnamed global coords} \tag{1.9}$$

$$\cosh d(x_1, x_2) = -\sinh\rho_1 \sinh\rho_2 + \cosh\rho_1 \cosh\rho_2 \cosh(w_1 - w_2)$$

$$\updownarrow$$

$$\sinh\rho = \tan r$$

$$\updownarrow$$

$$ds^2 = \frac{dr^2 + dw^2}{\cos^2 r} \qquad r \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \qquad w \in \mathbb{R} \qquad \text{infinite strip} \tag{1.10}$$

$$\cosh d(x_1, x_2) = -\tan r_1 \tan r_2 + \frac{\cosh(w_1 - w_2)}{\cos r_1 \cos r_2}$$

We read off the direct transformation from end to end $(\rho, \tau) \to (r, w)$

$$\begin{cases} \cosh\rho &= \frac{5\cosh w - 3\sinh w}{4\cos r} = \frac{4e^{-w}+e^w}{4\cos r} = \frac{\cosh(w-\log 2)}{\cos r} \\ \sin\tau &= \frac{4\sin r}{\sqrt{(4e^{-w}+e^w)^2-16\cos^2 r}} = \frac{\sin r}{\sqrt{\sin^2 r+\sinh^2(w-\log 2)}} \\ \cos\tau &= \frac{4e^{-w}-e^w}{\sqrt{(4e^{-w}+e^w)^2-16\cos^2 r}} = \frac{-\sinh(w-\log 2)}{\sqrt{\sin^2 r+\sinh^2(w-\log 2)}} \end{cases} \qquad \begin{cases} \tan r &= \sinh\rho\sin\tau \\ e^w &= 2\frac{\cosh\rho-\sinh\rho\cos\tau}{\sqrt{1+\sinh^2\rho\sin^2\tau}} = 2\sqrt{\frac{\cosh\rho-\sinh\rho\cos\tau}{\cosh\rho+\sinh\rho\cos\tau}} \\ e^{-w} &= \frac{\cosh\rho+\cos\tau\sinh\rho}{2\sqrt{1+\sinh^2\rho\sin^2\tau}} = \frac{1}{2}\sqrt{\frac{\cosh\rho+\sinh\rho\cos\tau}{\cosh\rho-\sinh\rho\cos\tau}} \\ \cosh(w-\log 2) &= \frac{\cosh\rho}{\sqrt{1+\sinh^2\rho\sin^2\tau}} \\ \sinh(w-\log 2) &= -\frac{\sinh\rho\cos\tau}{\sqrt{1+\sinh^2\rho\sin^2\tau}} \\ \tanh(w-\log 2) &= -\frac{\sinh\rho\cos\tau}{\cosh\rho} \end{cases} . \tag{1.11}$$

In the paper we simplified these expressions posing $\log 2 - w_{\text{notes}} = w_{\text{paper}}$.

Scalar operators

The scalar covariant Laplacian becomes in these coordinates

$$\begin{aligned} \Delta_{r,w} &= \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right) \\ &= \cos^2 r\left(\partial_r^2 + \partial_w^2\right) \end{aligned} \tag{1.12}$$

[2]The first change comes from the figure below, the second one from footnote 7 in [4] and the third one from pg. 22 of [5]. Do not confuse the homonymous coordinates ρ, τ in the list.

Spinor operators

The Dirac operator is anti-hermitian and reads

$$
\begin{aligned}
\nabla\!\!\!\!/\,_{r,w} &= \Gamma^i \nabla_i \\
&= \Gamma^i \left(\partial_i + \frac{1}{4}\omega^{jk}{}_i \Gamma_{jk}\right) \\
&= \cos^2 r \Gamma^1 \partial_r + \cos^2 r \Gamma^2 \left(\partial_w - \frac{\tan r}{2}\Gamma_{12}\right) \\
&= \Gamma^1 \left(\cos r \partial_r + \frac{\sin r}{2}\right) + \cos r \Gamma^2 \partial_w \,.
\end{aligned}
\tag{1.13}
$$

Heat kernels are 2d diffeomorphism scalars, functions of the geodesic distance, so stay unchanged. The spinors one carry *tangent-space* indices, so this is true as well.

However, we made a local rotation in the tangent space, under which the spinor heat kernel covariantly transforms. Indeed we did not just changed coordinates, we also decided to choose a new set of vielbein (the standard “square root” of the new metric $g_{i'j'}(r,w)$)

$$
e^{a'}{}_{i'}(r,w) = \Lambda^{a'}{}_b(\rho,\tau) \frac{\partial x^i}{\partial x^{i'}} e^b{}_i(\rho,\tau) \,. \tag{1.14}
$$

The matrix Λ transforms in the vector representation of $SO(2)$ [3] and it is the local Lorentz transformation rotating the complicated vielbein $\frac{\partial x^i}{\partial x^{i'}} e^b{}_i$ into the simple, chosen vielbein $e^{a'}{}_{i'}$. Λ can be found by solving the relation above

$$
\begin{aligned}
\Lambda^a{}_b &\equiv \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix} \\
&= \frac{1}{\sqrt{(5\cosh w - 3\sinh w)^2 - 16\cos^2 r}} \begin{pmatrix} \sin r\,(5\cosh w - 3\sinh w) & \cos r\,(3\cosh w - 5\sinh w) \\ \cos r\,(-3\cosh w + 5\sinh w) & \sin r\,(5\cosh w - 3\sinh w) \end{pmatrix} \\
&= \frac{1}{\sqrt{\sinh^2(w-\log 2) + \sinh^2 r}} \begin{pmatrix} \sin r\cosh(w-\log 2) & -\cos r\sinh(w-\log 2) \\ \cos r\sinh(w-\log 2) & \sin r\cosh(w-\log 2) \end{pmatrix} .
\end{aligned}
\tag{1.15}
$$

The corresponding unitary matrix acting on flat spinor indices is

$$
\begin{aligned}
&S \equiv \exp\left(\frac{1}{4}\omega_{ab}\Gamma^{ab}\right) = \exp\left(\frac{i\delta}{2}\Gamma^3\right) = \cos\frac{\delta}{2}\mathbb{I}_2 + i\Gamma^3 \sin\frac{\delta}{2} \\
&\omega_{ab} \equiv \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}_{ab} \qquad S^\dagger \Gamma^a S = \Lambda^a{}_b \Gamma^b \qquad S^\dagger \Gamma^3 S = \Gamma^3 \,.
\end{aligned}
\tag{1.16}
$$

This rotation $S \in \mathrm{Spin}(2)$ induces the transformation laws for the parallel spinor propagator (and similarly for the spinor heat kernel)

$$
U(r,w,r',w') = S(\rho,\tau)\, U(\rho,\tau,\rho',\tau')\, S^\dagger(\rho',\tau') \,. \tag{1.17}
$$

[3] It has positive determinant because no parity transformation occurred in the chain of change of coordinates. Indeed, the Jacobian matrices associated to them had positive determinant.

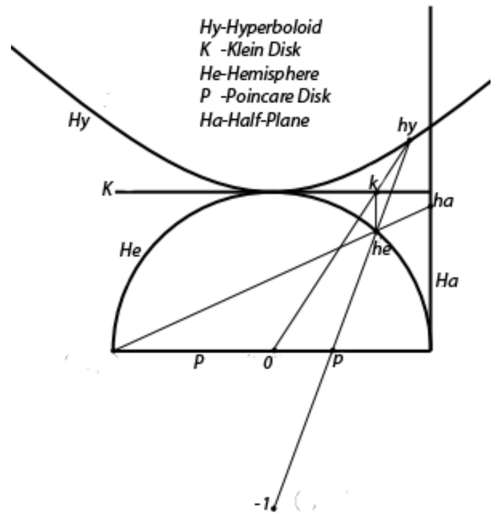

Figure 1: Connection between models of hyperbolic geometry (taken from `https://en.wikipedia.org/wiki/Hyperbolic_geometry`) .

2 Basic introduction to the heat kernel method

A nice introduction can be found in [6, 7, 8].

Let $\mathcal{O}(x,y) = \frac{\delta^{(d)}(x-y)}{\sqrt{g}}\mathcal{O}(x)$ be a local (matrix-valued) differential operator on a compact manifold $\mathcal{M}$ [4], with discrete eigenspectrum $\{\lambda_n\}$ and the set of orthonormal (vector-valued) eigenfunctions $f_n(x)$ [5]

$$\begin{gathered}\frac{\delta^{(d)}(x-y)}{\sqrt{g}}\mathcal{O}_x = \sum_n \lambda_n f_n(x) f_n^\dagger(y) = \delta^{(d)}(x-y)\sum_n \lambda_n f_n(x) f_n^\dagger(x) \\ \delta_{mn} = \int_x f_m^\dagger(x) f_n(x) \qquad\qquad \mathbb{I} = \sum_n \int_x f_n(x) f_n^\dagger(x) \\ \mathcal{O}_x f_n(x) = \lambda_n f_n(x)\,. \end{gathered} \tag{2.2}$$

We allow an infinite number of negative eigenvalues when the operator is not positive definite [6]. The $\zeta-$regularized functional determinant

$$\log \mathrm{Det}\mathcal{O} \equiv -\zeta_{\mathcal{O}}^{'}(0)\,. \tag{2.3}$$

is defined through the associated *zeta function* $\zeta_{\mathcal{O}}(s)$, which is the meromorphic extension on the s-complex plane, analytic in $s=0$, of the infinite sum

$$\zeta_{\mathcal{O}}(s) \equiv \mathrm{Tr}\mathcal{O}^{-s} = \sum_n \lambda_n^{-s}. \tag{2.4}$$

Similarly we introduce the eta-function characterizing the spectral asymmetry of the operator

$$\begin{aligned}\eta_{\mathcal{O}}(s) &= \sum_{\lambda_n>0}\lambda_n^{-s} - \sum_{\lambda_n<0}(-\lambda_n)^{-s} \\ &= \frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty dt\, t^{\frac{s+1}{2}-1}\mathrm{Tr}\left(\mathcal{O}e^{-t\mathcal{O}^2}\right)\,.\end{aligned} \tag{2.5}$$

The *heat kernel method* is an efficient tool to provide an integral representation of $\zeta_{\mathcal{O}}(s)$ and of the Green's function $G(x,y) = G(x-y)$ of the operator $\mathcal{O}$

$$\int_y \mathcal{O}(x,y)\, G(y,z) = \mathcal{O}(x)\, G(x-z) = \frac{\delta^{(d)}(x-z)}{\sqrt{g}}\mathbb{I}\,. \tag{2.6}$$

We define the *heat operator* $e^{-t\mathcal{O}}$, the *heat kernel propagator* $K(x,y;t)$ and its trace $K(t)$ as

$$\begin{aligned} K(x,y;t) &\equiv e^{-t\mathcal{O}(x)}\frac{\delta^{(d)}(x-y)}{\sqrt{g}} = \sum_n e^{-t\lambda_n} f_n(x) f_n^\dagger(y) \\ K(x,x;t) &= \sum_n e^{-t\lambda_n} f_n(x) f_n^\dagger(x) \\ K(t) &\equiv \mathrm{Tr}K(x,y;t) = V\sum_n e^{-t\lambda_n}\,. \end{aligned} \tag{2.7}$$

---

[4] The following statements admit a generalization to non-compact spaces (e.g. H^2).

[5] Integration over $\mathcal{M}$ is accompanied by the proper volume element. The Dirac delta is accordingly normalized as $\int_x \sqrt{g}\frac{\delta^{(d)}(x)}{\sqrt{g}} = 1$. The functional trace Tr involves the trace over matrix indices (tr) and over $\mathcal{M}$ ($\int_x \sqrt{g}\int_y\sqrt{g}$)

$$\mathrm{Tr}\mathcal{O} \equiv \int_x \sqrt{g}\int_y \sqrt{g}\,\mathrm{tr}\mathcal{O}(x,y) = \int_x \sqrt{g}\,\mathrm{tr}\mathcal{O}(x)\,. \tag{2.1}$$

We denote by $\mathbb{I}$ the unit matrix of the internal space, whose indices are always implicit.

[6] In the case of interest we do not have vanishing eigenvalues (zero modes) and complex eigenvalues either.

such that [7]

$$
\begin{aligned}
\zeta_{\mathcal{O}}(s) = \mathrm{Tr}\mathcal{O}^{-s} &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K(t)\,dt = \frac{V}{\Gamma(s)}\int_0^\infty t^{s-1}\sum_n e^{-t\lambda_n}dt \\
\log\mathrm{Det}\mathcal{O} = \mathrm{Tr}\log\mathcal{O} &= -\zeta_{\mathcal{O}}^{'}(0) = -\int_0^\infty \frac{dt}{t}K(t) = -V\int_0^\infty \frac{dt}{t}\sum_n e^{-t\lambda_n} \\
G(x,y) = \mathcal{O}^{-1}(x,y) &= \int_0^\infty K(x,y;t)\,dt = \sum_n \frac{1}{\lambda_n}f_n(x)\,f_n^\dagger(y) \\
G(x,x) &= \int_0^\infty K(x,x;t)\,dt = \sum_n \frac{1}{\lambda_n}f_n(x)\,f_n^\dagger(x) \\
\int_{\mathcal{M}} G(x,x) = \mathrm{Tr}G(x,y) &= V\zeta_{\mathcal{O}}(1) = \int_0^\infty K(t)\,dt = V\sum_n\frac{1}{\lambda_n}\,.
\end{aligned}
\tag{2.8}
$$

The advantage of this formalism is the *heat equation*, or better the Euclidean version of the evolutionary Schrödinger-type equation in the *proper time* t along the geodesic connecting x, y

$$
\left(\partial_t + \mathcal{O}_x\right)K(x,y;t) = 0 \qquad\qquad K(x,y;0) = \frac{\delta^{(d)}(x-y)}{\sqrt{g}}\mathbb{I}\,, \tag{2.9}
$$

and the (exact for H^2 and approximate for near-H^2 geometry) methods for its solution for generic metric and operators.

[7]Notice the heat kernel can be understood as the integral kernel of the Green's function.

3 Heat kernel for massless scalar in $\mathbb{R}^2$

Cartesian coordinates

$$
\begin{aligned}
\mathcal{O} &= -\left(\partial_x^2 + \partial_y^2\right) \\
f_{p_x,p_y}\left(x,y\right) &= \frac{1}{2\pi} e^{i\left(p_x x + p_y y\right)} \\
\lambda &= p_x^2 + p_y^2 \\
K_{\mathcal{O}}\left(x,y,x^{'},y^{'};t\right) &= \frac{1}{2\pi t}\exp\left[-\frac{\left(x-x^{'}\right)^2+\left(y-y^{'}\right)^2}{4t^2}\right] \\
K\left(t\right) &= \frac{V_{\mathbb{R}^2}}{2\pi t} \\
G_{\mathcal{O}}\left(x,y,x^{'},y^{'}\right) &= \frac{1}{4\pi}\log\left[\left(x-x^{'}\right)^2+\left(y-y^{'}\right)^2\right] \\
&= \int_0^\infty \frac{1}{2\pi t}\exp\left[-\frac{\left(x-x^{'}\right)^2+\left(y-y^{'}\right)^2}{4t^2}\right]dt \\
G_{\mathcal{O}}\left(x,y,x,y\right) &= \int_0^\infty \frac{dt}{2\pi t} \\
\delta\left(x-x^{'}\right)\delta\left(y-y^{'}\right) &= \mathcal{O}G\left(x,y,x^{'},y^{'}\right) \\
\delta\left(0\right)\delta\left(0\right) &= \int_0^\infty \frac{dt}{2\pi t}
\end{aligned}
\tag{3.1}
$$

Polar coordinates

$$
\begin{aligned}
\mathcal{O} &= -\left(\frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho\right) + \frac{1}{\rho^2}\partial_\tau^2\right) \\
f_{p_x,p_y}\left(\rho,\tau\right) &= \frac{1}{2\pi} e^{i\left(p_x\rho\cos\tau + p_y\rho\sin\tau\right)} \\
\lambda &= p_x^2 + p_y^2 \\
K_{\mathcal{O}}\left(\rho,\tau,\rho^{'},\tau^{'};t\right) &= \frac{1}{2\pi t}\exp\left(-\frac{\rho^2+\rho^{'2}-2\rho\rho^{'}\cos\left(\tau-\tau^{'}\right)}{4t^2}\right) \\
K\left(t\right) &= \frac{V_{\mathbb{R}^2}}{2\pi t} \\
G_{\mathcal{O}}\left(x,y,x^{'},y^{'}\right) &= \frac{1}{4\pi}\log\left[\rho^2+\rho^{'2}-2\rho\rho^{'}\cos\left(\tau-\tau^{'}\right)\right] \\
&= \int_0^\infty \frac{1}{2\pi t}\exp\left[-\frac{\rho^2+\rho^{'2}-2\rho\rho^{'}\cos\left(\tau-\tau^{'}\right)}{4t^2}\right]dt \\
G_{\mathcal{O}}\left(x,y,x,y\right) &= \int_0^\infty \frac{dt}{2\pi t} \\
\frac{1}{\rho}\delta\left(x-x^{'}\right)\delta\left(y-y^{'}\right) &= \mathcal{O}G\left(x,y,x^{'},y^{'}\right) \\
\frac{1}{\rho}\delta\left(0\right)\delta\left(0\right) &= \int_0^\infty \frac{dt}{2\pi t}
\end{aligned}
\tag{3.2}
$$

4 Heat kernels propagators in H^2: massive scalar

The heat kernel for massless scalars with Dirichlet b.c. on ∂H^2 is in [9] and a derivation can be found in [10]. Here we generalize the derivation presented in Sec. 3.3 [11] to construct the heat kernel of the massive scalar Laplacian

$$\mathcal{O} = -\Delta + m^2 = -\left(\partial_\rho^2 + \frac{\cosh\rho}{\sinh\rho}\partial_\rho + \frac{1}{\sinh^2\rho}\partial_\tau^2\right) + m^2\,. \tag{4.1}$$

On an homogeneous space the heat kernel depends on x, x' only through their geodesic distance $d(x, x')$. This happens because the heat kernel is invariant under isometry and a rotation about any point is an isometry. Thus we fix one of the points at the origin and observe the heat kernel is rotationally symmetric, i.e. it does not depend on the angular variable. Without loss in generality, we assume $x = (\rho, 0)$ and $x' = (0, 0)$, so that $d(x, x') = \rho$.

The heat kernel and its boundary conditions are rotationally symmetric, so we focus on the eigenfunctions with radial symmetry $f(\rho, \tau) = f(\rho)$

$$\left(-\partial_\rho^2 - \frac{\cosh\rho}{\sinh\rho}\partial_\rho + m^2\right) f(\rho) = \lambda f(\rho) \tag{4.2}$$

with $\lambda \geq m^2 + \frac{1}{4}$ [9]. The only regular solution is

$$\begin{aligned} f_{\bar\rho}(\rho) &= P_{-\frac{1}{2}+i\bar\rho}(\cosh\rho) \\ \bar\rho &= \sqrt{\lambda - \frac{1}{4} - m^2}\,. \end{aligned} \tag{4.3}$$

This turns out to be a real function as well. We define the eigenfunction transform and its inverse as spacetime inner products

$$\begin{aligned} \hat\alpha(\bar\rho) &= \int_0^\infty \sinh\rho d\rho\, \alpha(\rho) f_{\bar\rho}(\rho) = \langle\alpha,\, f_{\bar\rho}\rangle_{\sinh\rho} \\ \alpha(\bar\rho) &= \int_0^\infty \bar\rho\tanh(\pi\bar\rho)\, d\bar\rho\, \hat\alpha(\bar\rho) f_{\bar\rho}(\rho) \end{aligned}$$

and transform the heat equation operator (acting on functions with radial symmetry)

$$\begin{aligned} &(\partial_t + \mathcal{O}_x)\,\alpha(\rho;\, t) = 0 \\ \to &\int_0^\infty \sinh\rho d\rho\, f_{\bar\rho}(\rho)\,(\partial_t + \mathcal{O}_x)\,\alpha(\rho;\, t) = 0 \\ \to &\lambda\hat\alpha(\rho;\, t) + \partial_t\hat\alpha(\rho; t) = 0 \\ \to &\hat\alpha(\rho;\, t) = \hat f_0(\rho)\, e^{-\left(\bar\rho^2 + \frac{1}{4} + m^2\right)t} \end{aligned}$$

the solution satisfies the boundary condition $\hat\alpha(\rho;\, t = 0) = \hat f_0(\rho)$.

One repeats the arguments on pg. 48 [11] to prove that [8]

$$\begin{aligned} K_{-\Delta+m^2}(x, x'; t) &= \frac{1}{2\pi}\int_0^\infty dv\, v\tanh(\pi v)\, P_{-\frac{1}{2}+iv}(\cosh d(x, x'))\, e^{-t\left(v^2+\frac{1}{4}+m^2\right)} \\ &= \frac{\sqrt{2}}{(4\pi t)^{3/2}}\int_{d(x,x')}^\infty dv\, \frac{v}{\sqrt{\cosh v - \cosh d(x, x')}} e^{-\left(\frac{v^2}{4t}+\frac{t}{4}+m^2 t\right)}\,. \end{aligned} \tag{4.4}$$

The heat kernel for massive scalar is proportional to the heat kernel at zero mass, and an overall factor accounts for the mass parameter. Heat equation and boundary condition in the case $m \neq 0$ descend trivially from the ones at $m = 0$. We

[8] The second form uses the integral representation for the Legendre polynomial on pg. 246 [10].

also report the Green function and some identities

$$\begin{aligned}
G_{-\Delta+m^2}(x,x') &= \frac{1}{2\pi}\int_0^\infty dv\,\frac{v\tanh(\pi v)}{v^2+\frac{1}{4}+m^2}P_{-\frac{1}{2}+iv}(\cosh d(x,x')) \\
G_{-\Delta+m^2}(x,x) &= \frac{1}{2\pi}\int_0^\infty dv\,\frac{v\tanh(\pi v)}{v^2+\frac{1}{4}+m^2} \\
\frac{\delta_B^{(2)}(x-x')}{\sqrt{g}} &= \left(-\Delta+m^2\right)G_{-\Delta+m^2}(x,x') \\
\frac{\delta_B^{(2)}(0)}{\sqrt{g}} &= \frac{1}{2\pi}\int_0^\infty dv\,v\tanh(\pi v)
\end{aligned} \tag{4.5}$$

useful for the diagrammatical evaluations in the main text. We notice the Dirac delta has a quite interesting integral representation in terms of the spectral density (see below).

For completeness, we report here the normalized eigenfunctions ($l\in\mathbb{Z}$, $\lambda\equiv\bar{\rho}^2+m^2+\frac{1}{4}$, $\bar{\rho}>0$) with eigenvalues λ [12, 13]:

$$\begin{aligned}
f_{\bar{\rho}l}(\rho,\tau) &= \frac{1}{\sqrt{2\pi}2^{|l|}}\left|\frac{\Gamma\left(\frac{1}{2}+|l|+i\bar{\rho}\right)}{\Gamma(1+|l|)\,\Gamma(i\bar{\rho})}\right|(\sinh\rho)^{|l|} \\
&\times {}_2F_1\left(\frac{1}{2}+|l|+i\bar{\rho},\,\frac{1}{2}+|l|-i\bar{\rho},\,1+|l|\,,\,-\sinh^2\frac{\rho}{2}\right)e^{il\tau}\,.
\end{aligned} \tag{4.6}$$

This can be written in terms of an associated Legendre function as above [9]. The solutions are real at $\tau=0$, regular over the whole radial line and non-zero in the origin only for the angular mode $l=0$:

$$f_{\bar{\rho}l}(0,\tau) = \begin{cases} \frac{1}{\sqrt{2\pi}}\left|\frac{\Gamma\left(\frac{1}{2}+i\bar{\rho}\right)}{\Gamma(i\bar{\rho})}\right| & l=0 \\ 0 & l=\pm1,\pm2,\ldots \end{cases}\,. \tag{4.7}$$

The spectral density of the eigenvalues reads

$$\begin{aligned}
\mu(\bar{\rho}) &\equiv \pi^2\sum_l f^\dagger_{\bar{\rho}l}f_{\bar{\rho}l}\Big|_{\rho=0} \\
&= \pi\left|\frac{\Gamma\left(\frac{1}{2}+i\bar{\rho}\right)}{\Gamma(i\bar{\rho})}\right|^2 \\
&= \pi\bar{\rho}\tanh(\pi\bar{\rho})\,.
\end{aligned} \tag{4.8}$$

Here we appreciate that the asymptotic behaviour at $\rho\to\infty$ fixed the normalization constant and eventually determined the spectral density.

The asymptotic regime $\rho\to\infty$ is [10]

$$\begin{aligned}
&(\sinh\rho)^{|l|}\,{}_2F_1\left(\frac{1}{2}+|l|+i\bar{\rho},\,\frac{1}{2}+|l|-i\bar{\rho},\,1+|l|\,,\,-\sinh^2\frac{\rho}{2}\right) \\
\sim&\frac{2^{|l|}}{\sqrt{\pi}}\frac{\Gamma(1+|l|)\,\Gamma(i\bar{\rho})}{\Gamma\left(\frac{1}{2}+|l|+i\bar{\rho}\right)}e^{\left(-\frac{1}{2}+i\bar{\rho}\right)\rho}+\text{c.c.}\,.
\end{aligned} \tag{4.9}$$

It is also crucial to check that the eigenfunctions were properly normalized [11]. Let us start with the overlap

$$\begin{aligned}
&\int_\tau\int_\rho \sinh\rho\, f^\dagger_{\bar{\rho}l}(\rho,\tau)\, f_{\bar{\rho}'l'}(\rho,\tau) \\
=&\frac{1}{\bar{\rho}-\bar{\rho}'}\int_\tau\int_\rho \sinh\rho\,\nabla_i\left[f^\dagger_{\bar{\rho}l}(\rho,\tau)\,\nabla_i f_{\bar{\rho}'l'}(\rho,\tau)-\left(\nabla_i f^\dagger_{\bar{\rho}l}(\rho,\tau)\right)f_{\bar{\rho}'l'}(\rho,\tau)\right] \\
=&\frac{1}{\bar{\rho}-\bar{\rho}'}\lim_{\Lambda\to\infty}\int_\tau \sinh\Lambda\left[f^\dagger_{\bar{\rho}l}(\Lambda,\tau)\,\partial_\rho f_{\bar{\rho}'l'}(\Lambda,\tau)-\left(\partial_\rho f^\dagger_{\bar{\rho}l}(\Lambda,\tau)\right)f_{\bar{\rho}'l'}(\Lambda,\tau)\right].
\end{aligned}$$

[9] See (8.702) and the transformation formula (9.131.1) [14].
[10] See formula (9.132.2) [14].
[11] We use the trick in (2.81) [12] and the divergence theorem.

This expression gets contribution only on a S^1 of infinite radius (H^2 boundary). The angular integration forces $l = l^{'}$.

$$
\begin{aligned}
&\int_\tau \int_\rho \sinh\rho \, f^\dagger_{\bar\rho l}(\rho,\tau)\, f_{\bar\rho' l'}(\rho,\tau) \\
=&\frac{1}{\bar\rho-\bar\rho'} \lim_{\Lambda\to\infty} \delta_{ll'} \int_\tau \sinh\Lambda \left[f^\dagger_{\bar\rho l}(\Lambda)\, \partial_\rho f_{\bar\rho' l}(\Lambda) - \left(\partial_\rho f^\dagger_{\bar\rho l}(\Lambda)\right) f_{\bar\rho' l}(\Lambda)\right] \\
=&\frac{1}{\bar\rho-\bar\rho'} \lim_{\Lambda\to\infty} \left[\frac{2}{\pi}\delta_{ll'}\Lambda\left(\bar\rho-\bar\rho'\right) + O\left(\bar\rho-\bar\rho'\right)^2\right] \\
=&2\delta_{ll'}\left[\delta\left(\bar\rho-\bar\rho'\right) + O\left(\lambda-\lambda'\right)^1\right]
\end{aligned}
$$

We proved orthonormality

$$
\int_\tau \int_\rho \sinh\rho \, f^*_{\bar\rho l}(\rho,\tau)\, f_{\bar\rho' l'}(\rho,\tau) = \delta_{ll'}\delta\left(\bar\rho-\bar\rho'\right) \; . \tag{4.10}
$$

From the knowledge of the spectrum we can confirm the form of the heat kernel, previously obtained by solving the heat equation.

$$
\begin{aligned}
K_{-\Delta+m^2}(\rho,\tau,0,0;t) &= \int_0^\infty d\bar\rho \sum_{l\in\mathbb{Z}} f_{\bar\rho l}(\rho,\tau)\, f^*_{\bar\rho l}(0,0)\, e^{-t\left(\bar\rho^2+m^2+\frac{1}{2}\right)} \\
&= \frac{1}{2\pi}\int_0^\infty d\bar\rho\, \bar\rho \tanh(\pi\bar\rho)\, P_{-\frac{1}{2}+i\bar\rho}(\cosh\rho)\, e^{-t\left(\bar\rho+m^2+\frac{1}{4}\right)} \\
K_{-\Delta+m^2}\left(x,x^{'};t\right) &= \frac{1}{2\pi}\int_0^\infty d\bar\rho\, \bar\rho \tanh(\pi\bar\rho)\, P_{-\frac{1}{2}+i\bar\rho}(\cosh d(x,x'))\, e^{-t\left(\bar\rho+m^2+\frac{1}{4}\right)}
\end{aligned} \tag{4.11}
$$

5 Heat kernels propagators in H^2: massive spinor

We generalize the heat kernel for massless spinors with Dirichlet b.c. [3, 15] to the massive case ($m \in \mathbb{R}$) with different boundary conditions.

$$\mathcal{O} = -i\left[\Gamma^1\left(\partial_\rho + \frac{\cosh\rho}{2\sinh\rho}\right) + \frac{1}{\sinh\rho}\Gamma^2\partial_\tau + im\Gamma^3\right] \tag{5.1}$$

Although our spinors satisfy the Majorana-Weyl conditions, we shall compute their heat kernel by first computing the result for a Dirac spinor and then taking appropriate square roots. Here we do not include the fermionic statistics minus sign in the definition of the fermionic heat kernel, accounting for this sign explicitly when combining the bosonic and fermionic contributions.

We begin with of the eigenproblem for the 2D spinor $\psi = (\phi_+,\, \phi_-)$

$$\left(\nabla\!\!\!\!/ + im\Gamma^3\right)\psi = i\lambda\psi \tag{5.2}$$

which reads in our matrix representation

$$\begin{cases} 0 &= \left[\left(\partial_\rho + \frac{\cosh\rho}{2\sinh\rho}\right) - \frac{i}{\sinh\rho}\partial_\tau\right]\phi_- + i\left(m-\lambda\right)\phi_+ \\ 0 &= \left[\left(\partial_\rho + \frac{\cosh\rho}{2\sinh\rho}\right) + \frac{i}{\sinh\rho}\partial_\tau\right]\phi_+ - i\left(m+\lambda\right)\phi_- \end{cases}$$

to get the decoupled equations for the two components

$$\begin{cases} 0 &= \left[\left(\partial_\rho + \frac{\cosh\rho}{2\sinh\rho}\right) - \frac{i}{\sinh\rho}\partial_\tau\right]\left[\left(\partial_\rho + \frac{\cosh\rho}{2\sinh\rho}\right) + \frac{i}{\sinh\rho}\partial_\tau\right]\phi_+ - \left(m^2-\lambda^2\right)\phi_+ \\ 0 &= \left[\left(\partial_\rho + \frac{\cosh\rho}{2\sinh\rho}\right) + \frac{i}{\sinh\rho}\partial_\tau\right]\left[\left(\partial_\rho + \frac{\cosh\rho}{2\sinh\rho}\right) - \frac{i}{\sinh\rho}\partial_\tau\right]\phi_- - \left(m^2-\lambda^2\right)\phi_- \end{cases}$$

i.e.

$$\left(\partial_\rho + \frac{\cosh\rho}{2\sinh\rho}\right)^2\phi_\pm + \frac{1}{\sinh^2\rho}\partial_\tau^2\phi_\pm \mp i\frac{\cosh\rho}{\sinh^2\rho}\partial_\tau\phi_\pm = \left(m^2-\lambda^2\right)\phi_\pm\,. \tag{5.3}$$

Notice that the eigenproblem for the *massive* Dirac operator is equivalent to the one for the *squared massless* Dirac operator with *shifted* eigenvalues

$$-\nabla\!\!\!\!/^2\psi = \left(\lambda^2 - m^2\right)\psi\,.$$

Now we solve the angular and radial part of the component equations separately after posing

$$\begin{aligned} {}^{(1)}\phi_{\pm\lambda l}\left(\rho,\tau\right) &\equiv \phi_{\lambda l}\left(\rho\right)\chi_l^-\left(\tau\right) \\ {}^{(2)}\phi_{\pm\lambda l}\left(\rho,\tau\right) &\equiv \psi_{\lambda l}\left(\rho\right)\chi_l^+\left(\tau\right)\,. \end{aligned} \tag{5.4}$$

- It's easy to solve the angular part on S^1

  $$\partial_\tau\chi_l^{(\pm)} = \mp i\left(l+\frac{1}{2}\right)\chi_l^{(\pm)} \qquad\qquad l = 0,\,1,\ldots$$

  with antiperiodic boundary conditions:

  $$\chi_l^{(\pm)}\left(\tau\right) \propto \exp\left[\mp i\left(l+\frac{1}{2}\right)\tau\right]\,. \tag{5.5}$$

- For the eigenfunction ${}^{(1)}\phi_{\pm\lambda l}\left(\rho,\tau\right)$ the radial equation becomes

$$\begin{aligned} \left(m^2-\lambda^2\right)\phi_{\lambda l} &= \left(\partial_\rho + \frac{\cosh\rho}{2\sinh\rho}\right)^2\phi_{\lambda l} - \frac{1}{\sinh^2\rho}\left(l+\frac{1}{2}\right)^2\phi_{\lambda l} + \left(l+\frac{1}{2}\right)\frac{\cosh\rho}{\sinh^2\rho}\phi_{\lambda l} \\ &= \partial_\rho^2\phi_{\lambda l} + \frac{\cosh\rho}{\sinh\rho}\partial_\rho\phi_{\lambda l} - \frac{\left(l+\frac{1}{2}\right)^2 + \frac{1}{4}}{\sinh^2\rho}\phi_{\lambda l} + \left(l+\frac{1}{2}\right)\frac{\cosh\rho}{\sinh^2\rho}\phi_{\lambda l} + \frac{1}{4}\phi_{\lambda l} \\ &= \left(\cosh\frac{\rho}{2}\right)^{l+1}\left(\sinh\frac{\rho}{2}\right)^l\left[D^{(l,\,l+1)}_{\cosh\rho} + (l+1)^2\right]\left[\left(\cosh\frac{\rho}{2}\right)^{-l-1}\left(\sinh\frac{\rho}{2}\right)^{-l}\phi_{\lambda l}\right] \end{aligned}$$

with $D_x^{(l,\,l+1)}$

$$D_x^{(l,\,l+1)} \equiv -\left(1-x^2\right)\partial_x^2 - \left[1-(2l+3)\,x\right]\partial_x$$
$$= -\left[\frac{1-x}{2}\left(1-\frac{1-x}{2}\right)\partial^2_{\frac{1-x}{2}} + \left[l+1-(2l+3)\,\frac{1-x}{2}\right]\partial_{\frac{1-x}{2}}\right].$$

The *regular* eigenfunction of this operator is [12]

$${}_2F_1\left(l+1+i\sqrt{\lambda^2-m^2},\, l+1-i\sqrt{\lambda^2-m^2};\, l+1;\, \frac{1-x}{2}\right)$$

with eigenvalue $(l+1)^2+\lambda^2-m^2$, so that

$$\phi_{\lambda l}\left(\rho\right) \propto \left(\cosh\frac{\rho}{2}\right)^{l+1}\left(\sinh\frac{\rho}{2}\right)^{l}{}_2F_1\left(l+1+i\sqrt{\lambda^2-m^2},\, l+1-i\sqrt{\lambda^2-m^2};\, l+1;\, -\sinh^2\frac{\rho}{2}\right) \qquad \lambda^2 \geq m^2\,. \tag{5.6}$$

- For the eigenfunction ${}^{(2)}\phi_{\pm\lambda l}\left(\rho,\tau\right)$ the radial equation gives

$$\psi_{\lambda l}\left(\rho\right) \propto \left(\cosh\frac{\rho}{2}\right)^{l}\left(\sinh\frac{\rho}{2}\right)^{l+1}{}_2F_1\left(l+1+i\sqrt{\lambda^2-m^2},\, l+1-i\sqrt{\lambda^2-m^2};\, l+2;\, -\sinh^2\frac{\rho}{2}\right) \qquad \lambda^2 \geq m^2\,. \tag{5.7}$$

The massive Dirac operator has normalized eigenvalues ($\lambda^2 \geq m^2$, $l = 0, 1, \ldots$) [13]

$$\begin{aligned}
\psi^{(-)}_{\lambda l}\left(\rho,\tau\right) &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\lambda+m}{2\lambda}}\left|\frac{\Gamma\left(1+l+i\sqrt{\lambda^2-m^2}\right)}{\Gamma\left(1+l\right)\Gamma\left(\frac{1}{2}+i\sqrt{\lambda^2-m^2}\right)}\right| \\
&\times \begin{pmatrix} \left(\cosh\frac{\rho}{2}\right)^{l+1}\left(\sinh\frac{\rho}{2}\right)^{l}{}_2F_1\left(l+1+i\sqrt{\lambda^2-m^2},\, l+1-i\sqrt{\lambda^2-m^2};\, l+1;\, -\sinh^2\frac{\rho}{2}\right) \\ i\frac{\lambda-m}{1+l}\left(\cosh\frac{\rho}{2}\right)^{l}\left(\sinh\frac{\rho}{2}\right)^{l+1}{}_2F_1\left(l+1+i\sqrt{\lambda^2-m^2},\, l+1-i\sqrt{\lambda^2-m^2};\, l+2;\, -\sinh^2\frac{\rho}{2}\right)\end{pmatrix} e^{i\left(l+\frac{1}{2}\right)\tau} \\
\psi^{(+)}_{\lambda l}\left(\rho,\tau\right) &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\lambda-m}{2\lambda}}\left|\frac{\Gamma\left(1+l+i\sqrt{\lambda^2-m^2}\right)}{\Gamma\left(1+l\right)\Gamma\left(\frac{1}{2}+i\sqrt{\lambda^2-m^2}\right)}\right| C_{-m} \\
&\times \begin{pmatrix} \frac{\lambda+m}{1+l}\left(\cosh\frac{\rho}{2}\right)^{l}\left(\sinh\frac{\rho}{2}\right)^{l+1}{}_2F_1\left(l+1+i\sqrt{\lambda^2-m^2},\, l+1-i\sqrt{\lambda^2-m^2};\, l+2;\, -\sinh^2\frac{\rho}{2}\right) \\ -i\left(\cosh\frac{\rho}{2}\right)^{l+1}\left(\sinh\frac{\rho}{2}\right)^{l}{}_2F_1\left(l+1+i\sqrt{\lambda^2-m^2},\, l+1-i\sqrt{\lambda^2-m^2};\, l+1;\, -\sinh^2\frac{\rho}{2}\right)\end{pmatrix} e^{-i\left(l+\frac{1}{2}\right)\tau}\,.
\end{aligned} \tag{5.8}$$

They are regular and asymptotic zero for large ρ. The upper component is real and the lower is purely imaginary, up to the overall angular phase in τ. They vanish in the origin only if $l \neq 0$.

$$\begin{aligned}
\psi^{(-)}_{\lambda l}\left(0,\tau\right) &= \begin{cases} \begin{pmatrix} \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\lambda+m}{2\lambda}}\left|\frac{\Gamma\left(1+i\sqrt{\lambda^2-m^2}\right)}{\Gamma\left(\frac{1}{2}+i\sqrt{\lambda^2-m^2}\right)}\right| \\ 0 \end{pmatrix} e^{\frac{i}{2}\tau} & l=0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & l=1,2,\ldots \end{cases} \\
\psi^{(+)}_{\lambda l}\left(0,\tau\right) &= \begin{cases} \begin{pmatrix} 0 \\ -i\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\lambda-m}{2\lambda}}\left|\frac{\Gamma\left(1+i\sqrt{\lambda^2-m^2}\right)}{\Gamma\left(\frac{1}{2}+i\sqrt{\lambda^2-m^2}\right)}\right| \end{pmatrix} e^{-\frac{i}{2}\tau} & l=0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & l=1,2,\ldots \end{cases}
\end{aligned} \tag{5.9}$$

[12] Discussion of the independent solutions of the hypergeometric functions can be found in sec. 9.151 [14].

[13] They agree with (3.3.12) [13] in the zero-mass limit. There are instead some 2π factors missing comparing to [3] in the same limit. We checked that the former reference gives the right normalization factor.

The spectral distribution of the eigenvalues is given by the spectral function (Plancherel measure), which is completely determined by the values of the eigenfunctions in the origin. Only the term with $l=0$ survives in the sum [14]

$$
\begin{aligned}
\mu\left(\lambda\right) &\equiv \pi^2 \sum_{s,l} \psi^{(s)\dagger}_{\lambda l}\psi^{(s)}_{\lambda l}\Big|_{\rho=0} \\
&= \frac{\pi}{2}\left|\frac{\Gamma\left(1+i\sqrt{\lambda^2-m^2}\right)}{\Gamma\left(\frac{1}{2}+i\sqrt{\lambda^2-m^2}\right)}\right|^2 \\
&= \frac{\pi}{2}\sqrt{\lambda^2-m^2}\coth^2\left(\pi\sqrt{\lambda^2-m^2}\right) .
\end{aligned} \tag{5.10}
$$

In the derivations above we used the asymptotics $\rho\to\infty$ of the radial functions [15]

$$
\begin{aligned}
&\left(\cosh\frac{\rho}{2}\right)^{l+1}\left(\sinh\frac{\rho}{2}\right)^{l} {}_2F_1\left(l+1+i\sqrt{\lambda^2-m^2},\, l+1-i\sqrt{\lambda^2-m^2};\, l+1;\, -\sinh^2\frac{\rho}{2}\right) \\
\sim&\frac{\Gamma\left(l+1\right)\Gamma\left(\frac{1}{2}+i\sqrt{\lambda^2-m^2}\right)}{\sqrt{\pi}\Gamma\left(l+1+i\sqrt{\lambda^2-m^2}\right)}e^{\left(-\frac{1}{2}+i\sqrt{\lambda^2-m^2}\right)\rho}+\text{c.c.}
\end{aligned} \tag{5.11}
$$

$$
\begin{aligned}
&\frac{\lambda-m}{1+l}\left(\cosh\frac{\rho}{2}\right)^{l}\left(\sinh\frac{\rho}{2}\right)^{l+1} {}_2F_1\left(l+1+i\sqrt{\lambda^2-m^2},\, l+1-i\sqrt{\lambda^2-m^2};\, l+2;\, -\sinh^2\frac{\rho}{2}\right) \\
\sim&-i\frac{\lambda-m}{\sqrt{\lambda^2-m^2}}\frac{\Gamma\left(l+1\right)\Gamma\left(\frac{1}{2}+i\sqrt{\lambda^2-m^2}\right)}{\sqrt{\pi}\Gamma\left(l+1+i\sqrt{\lambda^2-m^2}\right)}e^{\left(-\frac{1}{2}+i\sqrt{\lambda^2-m^2}\right)\rho}+\text{c.c.}\,.
\end{aligned} \tag{5.12}
$$

It is easy to check they are properly normalized [16]. We start with the inner product

$$
\begin{aligned}
&\int_\tau\int_\rho \sinh\rho\, \psi^{(s)\dagger}_{\lambda l}\left(\rho,\tau\right)\psi^{(s)}_{\lambda' l'}\left(\rho,\tau\right) \\
=&\frac{i}{\lambda-\lambda'}\int_\tau\int_\rho \sinh\rho\,\nabla_i\left[\psi^{(s)\dagger}_{\lambda l}\left(\rho,\tau\right)\Gamma_i\psi^{(s)}_{\lambda' l'}\left(\rho,\tau\right)\right] \\
=&\frac{i}{\lambda-\lambda'}\lim_{\Lambda\to\infty}\int_\tau \sinh\Lambda\left[\psi^{(s)\dagger}_{\lambda l}\left(\Lambda,\tau\right)\Gamma_1\psi^{(s)}_{\lambda' l'}\left(\Lambda,\tau\right)\right] .
\end{aligned}
$$

This expression gets contribution only on a S^1 of infinite radius (H^2 boundary). The angular integration forces $l=l^{'}$ and then we can separately analyze the four combinations of the labels s,s'. The result is the one expected:

$$
\int_\tau\int_\rho \sinh\rho\, \psi^{(s)\dagger}_{\lambda l}\psi^{(s')}_{\lambda' l'} = \delta_{ll'}\delta_{ss'}\delta\left(\lambda-\lambda^{'}\right) . \tag{5.13}
$$

We construct the heat kernel for the *squared* massive Dirac operator $\mathcal{O}^2=\mathcal{O}^\dagger\mathcal{O}$ [15, 3, 16] [17]

$$
\begin{aligned}
K_{-\nabla\!\!\!/^2+m^2}\left(x,x';t\right) &= U\left(x,x'\right)\frac{1}{2\pi}\int_0^\infty dv\, v\coth\pi v\,\cosh\frac{d\left(x,x^{'}\right)}{2}\,{}_2F_1\left(1+iv,\,1-iv,\,1,\,\frac{1-\cosh d\left(x,x^{'}\right)}{2}\right)e^{-t\left(v^2+m^2\right)} \\
&= U\left(x,x'\right)\frac{\sqrt{2}}{\left(4\pi t\right)^{3/2}\cosh\frac{d\left(x,x'\right)}{2}}\int_{d\left(x,x'\right)}^\infty dv\,\frac{v\cosh\frac{v}{2}}{\sqrt{\cosh v-\cosh d\left(x,x'\right)}}e^{-\frac{v^2}{4t}-m^2 t}
\end{aligned} \tag{5.14}
$$

[14] Notice that the second component of $\psi^{(-)}_{\lambda l}$ and the first one of $\psi^{(+)}_{\lambda l}$ vanish in the origin, so they do not contribute to the sum with a factor linear in $\lambda\pm m$. As a consequence, the spectral distribution for the massive case is equal to the massless one with the replacement $\lambda\to\sqrt{\lambda^2-m^2}$.

[15] See formula (9.132.2) [14].

[16] We adapt the trick in (2.81) [12] and use then the divergence theorem.

[17] We have not incorporated the factor of $\frac{1}{2}$ for the Majorana fields at variance with [17, 16]. Indeed, it is absent in the standard mathematical literature for the Dirac case [15, 3].

so that the propagator reads [18]

$$
\begin{aligned}
G_{-\nabla\!\!\!/^2+m^2}\left(x,x'\right) &= U\left(x,x'\right)\frac{1}{2\pi}\int_0^\infty dv\,\frac{v\coth\pi v}{v^2+m^2}\cosh\frac{d\left(x,x'\right)}{2}\,{}_2F_1\left(1+iv,\,1-iv,\,1,\,\frac{1-\cosh d\left(x,x'\right)}{2}\right)\\
G_{-\nabla\!\!\!/^2+m^2}\left(x,x\right) &= \frac{1}{2\pi}\mathbb{I}_2\int_0^\infty dv\,\frac{v\coth\pi v}{v^2+m^2}\\
\frac{\delta_F^{(2)}\left(x-x'\right)}{\sqrt{g}}\mathbb{I}_2 &= \left(-\nabla\!\!\!/^2+m^2\right)G_{-\nabla\!\!\!/^2+m^2}\left(x,x'\right)\\
\frac{\delta_F^{(2)}\left(0\right)}{\sqrt{g}} &= \frac{1}{2\pi}\int_0^\infty dv\,v\coth\pi v\,.
\end{aligned}
\tag{5.15}
$$

We notice that the "sum" of all the eigenvalues diverges as the Dirac delta on H^2 for the Dirac operator too. We can deduce the Green function of the not-squared operator as

$$
\begin{aligned}
G_{-i\nabla\!\!\!/+m}\left(x,x'\right) &= \left(-i\nabla\!\!\!/+m\Gamma^3\right)G_{-\nabla\!\!\!/^2+m^2}\left(x,x'\right)\\
&= \frac{1}{2\pi}\int_0^\infty dv\,\frac{v\coth\pi v}{v^2+m^2}\left(-i\nabla\!\!\!/_x+m\Gamma^3\right)\left[U\left(x,x'\right)\cosh\frac{d\left(x,x'\right)}{2}\,{}_2F_1\left(1+iv,\,1-iv,\,1,\,\frac{1-\cosh d\left(x,x'\right)}{2}\right)\right]\\
G_{-i\nabla\!\!\!/+m}\left(x,x\right) &= \frac{m}{2\pi}\Gamma_3\int_0^\infty dv\,\frac{v\coth\pi v}{v^2+m^2}\\
\frac{\delta_F^{(2)}\left(x-x'\right)}{\sqrt{g}}\mathbb{I}_2 &= \left(-i\nabla\!\!\!/+m\right)G_{-i\nabla\!\!\!/+m}\left(x,x'\right)\\
\frac{\delta_F^{(2)}\left(0\right)}{\sqrt{g}} &= \frac{1}{2\pi}\int_0^\infty dv\,v\coth\pi v\,.
\end{aligned}
\tag{5.16}
$$

[18] The Green function diverges in the massless case because of the presence of zero eigenvalues in the operator spectrum.

6 Parallel spinor propagator

To this end one considers the parallel spinor propagator $U\left(x_0, x_0'\right)$, a matrix in the spinor indices which parallel transports a (two-dimensional) spinor along a geodesic connecting two given points x_0, $x_0^{'}$. This matrix satisfies satisfies the *parallel transport equation* [15]

$$\begin{cases} t^i\left(u\right)\nabla_i U\left(x\left(u\right), x_0\right) = 0 \\ U\left(x_0, x_0\right) = \mathbb{I}_2 \end{cases} \tag{6.1}$$

where $t_i\left(u\right) = \gamma_i'\left(u\right) = \partial_i d\left(x_0, x\left(u\right)\right)$ is the unit tangent vector to the shortest geodesic $\gamma\left(u\right)$ between x_0, x. That is $U\left(x_0, x\right)$ is covariantly constant along the geodesic of parallel transport: given a spinor $\psi\left(x_0\right)$ in x_0, $U\left(x_0, x_0^{'}\right)\psi\left(x_0\right)$ is a spinor at the point $x_0^{'}$ obtained by parallel transport along the geodesic $\gamma\left(u\right)$.

To solve this equation, notice that the spinor covariant derivative acts only on the upper index of the matrix U^α_β because this is just a shorthand of

$$\begin{gathered} \psi^\alpha\left(x\right) = U^\alpha_\beta\left(x, x_0\right)\psi_\beta\left(x_0\right) \\ \downarrow \\ 0 = t^i\nabla_i\psi^\alpha\left(x\right) = t^i\nabla_i\left[U^\alpha_\beta\left(x, x_0\right)\right]\psi_\beta\left(x_0\right)\ . \end{gathered}$$

In components, for any initial spinor and geodesic path we must have

$$0 = t^i\left(\partial_i + \frac{1}{4}\omega^{jk}{}_i\Gamma_{jk}\right)U$$

Although a formal solution exists (making use of the path-ordering exponential, [15]), we look for an explicit formula in polar [16] and infinite-strip coordinates:

$$\begin{aligned} U\left(\rho, \tau, \rho', \tau'\right) &\equiv \mathbb{I}_2\cos\theta\left(\rho, \tau, \rho', \tau'\right) + i\Gamma_3\sin\theta\left(\rho, \tau, \rho', \tau'\right) \\ &= \mathbb{I}_2\cos\left(\operatorname{atan}\frac{\cosh\frac{\rho+\rho'}{2}\tan\frac{\tau-\tau'}{2}}{\cosh\frac{\rho-\rho'}{2}}\right) + i\Gamma_3\sin\left(\operatorname{atan}\frac{\cosh\frac{\rho+\rho'}{2}\tan\frac{\tau-\tau'}{2}}{\cosh\frac{\rho-\rho'}{2}}\right) \\ U\left(r, w, r', w'\right) &= \mathbb{I}_2\cos\left(\theta\left(\rho, \tau, \rho', \tau'\right) + \frac{\delta\left(r, w\right) - \delta\left(r', w'\right)}{2}\right) + i\Gamma_3\sin\left(\theta\left(\rho, \tau, \rho', \tau'\right) + \frac{\delta\left(r, w\right) - \delta\left(r', w'\right)}{2}\right)\ . \end{aligned} \tag{6.2}$$

It is the unique *regular* solution of the parallel transport equation above [15].

7 Heat kernels propagators in H^2: summary

Let us consider the propagation of the worldsheet excitations on the hyperbolic background geometry with Dirichlet conditions on the S^1 boundary at infinity. The scalars enter the Lagrangian with the scalar Laplace operator $-\Delta + m^2$ and the spinors with the Dirac operator $i\nabla\!\!\!/ - m$, where m is the mass of the relevant field. Their traced heat kernels can be cast into the compact formula

$$K_{\mathcal{O}}(t) = \frac{V_{H^2}}{2\pi} \int_0^\infty \mu(v)\, e^{-t\left(v^2+M\right)} dv. \tag{7.1}$$

The eigenspectra are continuous due to the non-compactness of the hyperbolic space. The degeneracy of the spectral parameter $v > 0$ enters the spectral density $\mu(v)$. Mass parameter and spectral density for scalar fields are

$$\mathcal{O} = -\Delta + m^2 \qquad\qquad M = \frac{1}{4} + m^2 \qquad\qquad \mu(v) = v \tanh \pi v \tag{7.2}$$

and for Dirac spinors [19]

$$\mathcal{O} = -i\nabla\!\!\!/ + m \qquad\qquad M = m^2 \qquad\qquad \mu(v) = 2v \coth \pi v \tag{7.3}$$

where m is the mass of the relevant field. The corresponding ζ-functions are

$$\begin{aligned} \zeta_{\mathcal{O}}(s) &= \frac{V_{H^2}}{2\pi} \frac{1}{\Gamma(s)} \int_0^\infty dv \mu(v) \int_0^\infty dt\, t^{s-1} e^{-t\left(v^2+M\right)} \\ &= -\int_0^\infty dv \frac{\mu(v)}{\left(v^2+M\right)^s} \;. \end{aligned} \tag{7.4}$$

[19] At variance with [17], we work with a positive-definite spinor spectral density and we will take into account the Grassmann-odd statistics of spinors in summing over the zeta-functions. Moreover, we present the result for *Dirac* spinors, so no Majorana factor of $\frac{1}{2}$ in the traced heat kernel.

8 Remark on the fermionic Lagrangian for the latitude Wilson loops

We want to check that the quadratic fermionic Lagrangian in the string sigma-model coincides with the natural Dirac operator in the space with metric

$$ds^2 = \Omega^2(\sigma)\left(d\sigma^2 + d\tau^2\right) . \tag{8.1}$$

From here we extract metric and vielbein

$$g_{ij} = \Omega^2 \delta_{ij} \qquad\qquad e_i^a = \Omega \delta_i^a \tag{8.2}$$

in order to compute the spin connection ${\omega^{ab}}_i$

$${\omega^{12}}_1 = \frac{\Omega^{'}}{\Omega} . \tag{8.3}$$

Then massive Dirac operator on this geometry is easily computed from its definition

$$\begin{aligned} & -i\left(\nabla\!\!\!\!/ + im\right) \\ =& -i\left[\Gamma^i\left(\partial_i + \frac{1}{4}{\omega^{jk}}_i \Gamma_{jk}\right) + im\right] \\ =& -\frac{i}{\Omega}\Gamma^1\left(\partial_\sigma + \frac{\Omega^{'}}{2\Omega}\right) - \frac{i}{\Omega}\Gamma^2\partial_\tau + m \end{aligned} \tag{8.4}$$

and has to be compared with our fermionic operator

$$\mathcal{O}_{p_{12},p_{56},p_{89}} \equiv \frac{i}{\Omega}\left(\partial_\sigma + \frac{\Omega^{'}}{2\Omega}\right)\sigma_1 + \frac{1}{\Omega}\left(-i\partial_\tau + p_{56}\frac{1-\tanh\left(2\sigma+\sigma_0\right)}{2}\right)\sigma_2 - \frac{p_{12}p_{56}}{\Omega^2\cosh^2\left(\sigma+\sigma_0\right)}\mathbb{I}_2 + \frac{p_{12}}{\Omega^2\sinh^2\sigma}\sigma_3 . \tag{8.5}$$

We match the two expressions by identifying the "chiral" mass term as (identify $\Gamma_1 = -\sigma_1$, $\Gamma_2 = \sigma_2$)

$$m = \frac{\Omega^{'}}{2\Omega}\sigma_1 + \frac{p_{56}}{2\Omega}\left(1-\tanh\left(2\sigma+\sigma_0\right)\right)\sigma_2 + \frac{p_{12}}{\Omega^2\sinh^2\sigma}\sigma_3 - \frac{p_{12}p_{56}}{\Omega^2\cosh^2\left(\sigma+\sigma_0\right)}\mathbb{I}_2 . \tag{8.6}$$

It is important to emphasize that the mass is self-adjoint, so the Dirac operator has real spectrum for *any* θ_0 and $\mathcal{O}^\dagger\mathcal{O} = \mathcal{O}^2$.

9 Gel'fand-Yaglom vs heat kernel for latitude loops

9.1 Determinants and vacuum energy

Let us consider a manifold $\mathcal{M}$ of dimension d and equipped with metric g_{ij}. The computation of the one-loop effective action involves calculating the determinant of $\mathcal{O} = -\nabla^2 + m^2$ with the heat kernel method

$$(\partial_t + \mathcal{O}_x) K(x, y; t) = 0 \qquad\qquad K(x, y; 0) = \frac{\delta^{(d)}(x-y)}{\sqrt{g}}\mathbb{I} \tag{9.1}$$

$$\begin{aligned} \zeta_{\mathcal{O}}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\int_x \sqrt{g}K(x, x; t) \\ \log \mathrm{Det}\mathcal{O} &= -\zeta'_{\mathcal{O}}(0) \end{aligned} \tag{9.2}$$

The determinant is the (zeta-function regularized) product $\log \mathrm{Det}\mathcal{O} = \prod_n \lambda_n$ of the eigenvalues λ_n of $\mathcal{O}$

$$\left(-\nabla^2 + m^2\right) f_n = \lambda_n f_n \,. \tag{9.3}$$

If the metric is static

$$ds^2 = g_{11}\left(x^k\right)\left(dx^1\right)^2 + g_{ij}\left(x^k\right)dx^i dx^j \,, \qquad i\,, j\,, k = 2\,, ..., d \tag{9.4}$$

the relevant operator reads

$$\mathcal{O}_x = -\nabla_x^2 + m^2 = \frac{1}{g_{11}}\partial_{x^1}^2 + \frac{1}{\sqrt{\det g_{ij}}}\partial_i\left(\sqrt{\det g_{ij}}g^{ij}\partial_j\right) + m^2 \,, \qquad i\,, j = 2\,, ..., d \tag{9.5}$$

and for it we can also compute the vacuum energy (with respect to the time x^1) $E_{\mathrm{vacuum}} = \frac{1}{2}\sum_m \omega_m$ as the product of the *on-shell frequencies* ω_m of the rescaled operator $g_{11}\mathcal{O}$

$$g_{11}\left(-\frac{1}{g_{11}}\omega_m^2 + \frac{1}{\sqrt{\det g_{ij}}}\partial_i\left(\sqrt{\det g_{ij}}g^{ij}\partial_j\right) + m^2\right) f_m = 0\,. \tag{9.6}$$

If we further assume that $g_{11} = 1$, the value of the effective action coincides with the vacuum energy $\frac{1}{2}\log \mathrm{Det}\mathcal{O} = E_{\mathrm{vacuum}}$ [18].

9.2 Amendment to Sakai Tanii paper

Page 3 [19]

$$\begin{aligned} f(v) &= C_1 {}_2F_1\left(\frac{\lambda+\omega}{2}, \frac{\lambda-\omega}{2}; \frac{1}{2}; v\right) + C_2\sqrt{v}\,{}_2F_1\left(\frac{1+\lambda+\omega}{2}; \frac{1+\lambda-\omega}{2}, \frac{3}{2}; v\right) \\ \phi(\rho) &= \cos^\lambda\rho\left[C_1 {}_2F_1\left(\frac{\lambda+\omega}{2}, \frac{\lambda-\omega}{2}; \frac{1}{2}; \sin^2\rho\right) + C_2\sin\rho\, {}_2F_1\left(\frac{1+\lambda+\omega}{2}, \frac{1+\lambda-\omega}{2}; \frac{3}{2}; \sin^2\rho\right)\right] \end{aligned} \tag{9.7}$$

9.3 Vacuum energy for latitude loops

9.3.1 Leading order: scalar

The action of a massive scalar in H^2 is

$$S = \int_x \sqrt{\bar{g}}\left(\bar{g}^{ij}\partial_i\phi\partial_j\phi + m^2\phi^2\right) = \int_x\left[-\frac{\phi}{\sqrt{\bar{g}}}\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\right)\phi + m^2\phi^2\right] = \int_x\left[-\phi\left(\partial_\rho^2 + \frac{\cosh\rho}{\sinh\rho}\partial_\rho + \frac{1}{\sinh^2\rho}\partial_\tau^2\right)\phi + m^2\phi^2\right] \tag{9.8}$$

and can be used to write the equation of motion

$$-\left(\partial_\rho^2 + \frac{\cosh\rho}{\sinh\rho}\partial_\rho + \frac{1}{\sinh^2\rho}\partial_\tau^2\right)\phi + m^2\phi = 0\,. \tag{9.9}$$

and the expression of the energy-momentum tensor [20]

$$T_{ij} \equiv -\frac{1}{\sqrt{\bar{g}}}\frac{\delta S}{\delta \bar{g}^{ij}} + \text{(divergenceless) improvement term} \tag{9.10}$$
$$= -\partial_i\phi\partial_j\phi + \frac{1}{2}\bar{g}_{ij}\left(\bar{g}^{kl}\partial_k\phi\partial_l\phi + m^2\phi^2\right) - \frac{\beta}{2}\left(\bar{g}_{ij}\Delta - \nabla_i\nabla_j + \bar{R}_{ij}\right)\phi^2 .$$

It is easy to prove that the improvement term has vanishing covariant derivative and β can be taken to be an arbitrary real parameter. The total stress-energy tensor remains divergenceless and symmetric for $\beta \neq 0$. The energy of the scalar field is defined as an integral over a 1d spacelike manifold

$$E = \int \sqrt{\bar{g}} T^2_i \xi^i \tag{9.11}$$

with respect to a Killing vector ξ^i that serves as a "time" direction [21]. We pick $\xi^i = (0,1)$ as the temporal direction, out of the three Killing vectors of H^2, because we want to consider τ as the "time" coordinate [22]. Therefore we pose

$$E = \int_\rho \sqrt{\bar{g}} T^2_2 \tag{9.12}$$

and find the solutions to the eom with constant energy. The energy-momentum density current is divergenceless $\nabla_i T^i_2 = 0$ by construction, but we need also that the current has vanishing flux at the endpoints ($\rho = 0, \infty$) of a timeslice at fixed τ [23]

$$0 = \frac{dE}{dt} = \int_\rho \sqrt{\bar{g}}\nabla_2\left(T^2_2\right) = -\int_\rho \sqrt{\bar{g}}\nabla_1\left(T^1_2\right) = \left(\sqrt{\bar{g}}T_{12}\right)_{\rho=0} - \left(\sqrt{\bar{g}}T_{12}\right)_{\rho=\infty} . \tag{9.13}$$

In the following we require the strong condition

$$\left(\sqrt{\bar{g}}T_{12}\right)_{\rho=0} = \left(\sqrt{\bar{g}}T_{12}\right)_{\rho=\infty} = 0 . \tag{9.14}$$

The vanishing-flux condition in $\rho = 0$ looks too restrictive, because $\rho = 0$ is the center of H^2 and not a real boundary of H^2, therefore it will be not imposed.

The solution to the problem above follows the lines of [20, 21]. The time-indipendence of the metric and the operators allows to look only for solutions decomposed in modes as $\phi(\rho,\tau) = \sum_\omega \phi_\omega(\rho)\, e^{i\omega\tau}$. The parity in τ of the Lagrangian and of the eom implies that the Fourier modes ϕ_ω depends on $|\omega|$. We go to the covering space of H^2 by relaxing the condition $\tau \in [0,\, 2\pi) \to \tau \in \mathbb{R}$ to allow for non-integer frequencies. If we then pose

$$x = \tanh^2\rho\,, \qquad \lambda = \lambda_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + m^2}\,, \qquad \phi_\omega(\rho) = \frac{\tanh^{|\omega|}\rho}{\cosh^\lambda \rho} f_\omega(x) \tag{9.15}$$

the eom becomes the hypergeometric differential equation

$$x(1-x)\, f^{''}_\omega(x) + \left[c - (a+b+1)\,x\right] f^{'}_\omega(x) - ab f(x) = 0\,, \tag{9.16}$$
$$a = \frac{\lambda + |\omega|}{2}\,, \qquad b = \frac{\lambda + |\omega| + 1}{2}\,, \qquad c = 1 + |\omega|\ .$$

We limit the following discussion to the only two relevant cases for the spectrum of worldsheet excitations:

$$\begin{aligned} m^2 = 0, \qquad \lambda_+ = 1\,, \qquad \lambda_- = 0 \\ m^2 = 2, \qquad \lambda_+ = 2\,, \qquad \lambda_- = -1\,. \end{aligned} \tag{9.17}$$

[20] We use $\delta\bar{g}^{ij} = \bar{g}^{ik}\bar{g}^{jl}\delta\bar{g}_{kl}$ and $\delta\bar{g} = \bar{g}\bar{g}^{ij}\delta\bar{g}_{ij}$, and define $R_{ij} \equiv g^{kl}R_{ikjl}$.

[21] Remember that H^2 is a Euclidean space and has only spacelike Killing vectors.

[22] In this we differ from [19] because our temporal and spatial directions are non-trivially related to the ones of the reference by a change of coordinates. Foliating the space with different coordinates has deep consequences: the spatial boundary at infinity is defined as a curve of constant time and different time coordinates lead to different global properties of H^2, namely "global" H^2 for us and "infinite-strip" H^2 in [19].

[23] The formula comes from the divergence theorem $\int_\mathcal{M}\sqrt{\bar{g}}\nabla_i T^i_2 = \int_\mathcal{M}\sqrt{\bar{g}}n^i T_{i2}$, where $\nabla_1 T^1_2 + \nabla_2 T^2_2 = 0$ and we integrate over a surface of constant τ instead of the entire manifold $\mathcal{M}$ spanned by ρ, τ. The orthonormal vector along the radial ρ is trivially $n_i = (\pm 1, 0)$.

The general solutions are

$$\phi_\omega(\rho) = C_1^{|\omega|} (\tanh\rho)^{|\omega|} {}_2F_1\left(\frac{|\omega|}{2}, \frac{1+|\omega|}{2}, 1+|\omega|, \tanh^2\rho\right) \tag{9.18}$$

$$+ C_2^{|\omega|} (\tanh\rho)^{-|\omega|} \rho_2F_1\left(-\frac{|\omega|}{2}, \frac{1-|\omega|}{2}, 1-|\omega|, \tanh^2\rho\right) \qquad \text{for } m^2=0\,, \quad \omega\in\mathbb{R}\setminus\mathbb{Z}$$

$$\phi_\omega(\rho) = C_3^{|\omega|}\left(\tanh\frac{\rho}{2}\right)^{|\omega|} + C_4^{|\omega|}\left(\tanh\frac{\rho}{2}\right)^{-|\omega|} \qquad \text{for } m^2=0\,, \quad \omega\in\mathbb{Z} \tag{9.19}$$

$$\phi_\omega(\rho) = C_1^{|\omega|}\cosh\rho\,(\tanh\rho)^{|\omega|} {}_2F_1\left(\frac{|\omega|}{2}, \frac{|\omega|-1}{2}, 1+|\omega|, \tanh^2\rho\right) \tag{9.20}$$

$$+ C_2^{|\omega|}\cosh\rho\,(\tanh\rho)^{-|\omega|} \rho_2F_1\left(-\frac{|\omega|}{2}, -\frac{|\omega|+1}{2}, 1-|\omega|, \tanh^2\rho\right) \qquad \text{for } m^2=2\,, \quad \omega\in\mathbb{R}\setminus\{\pm1\}$$

$$\phi_\omega(\rho) = \sinh\rho\left(C_3^{|\omega|} + C_4^{|\omega|}\text{atanh}\frac{1}{\cosh\rho} - \frac{\coth\rho}{\sinh\rho}\right) \qquad \text{for } m^2=2\,, \quad \omega=\pm1\,. \tag{9.21}$$

The set of allowed frequencies is to be determined by the energy-conservation constraint at $\rho=\infty$, which become for the Fourier modes

$$\omega\left(\left(\sinh\rho\,(2\beta-1)\,\partial_\rho + 2\beta\cosh\rho\right)\phi_\omega^2\right)_{\rho=\infty} = 0\,. \tag{9.22}$$

We find the spectrum of frequencies compatible with the energy conservation for each of the independent solutions above. To study the limit of large ρ, it is convenient to pass to the new variable $\rho' = 1-\tanh\rho \to 0$ and require that the coefficient of the Laurent series with non-positive powers vanish. We have for the massless scalar

- no frequency from $(\tanh\rho)^{|\omega|} {}_2F_1\left(\frac{|\omega|}{2}, \frac{1+|\omega|}{2}, 1+|\omega|, \tanh^2\rho\right)$
- no frequency from $(\tanh\rho)^{-|\omega|} \rho_2F_1\left(-\frac{|\omega|}{2}, \frac{1-|\omega|}{2}, 1-|\omega|, \tanh^2\rho\right)$
- $\omega=0$ from $\left(\tanh\frac{\rho}{2}\right)^{|\omega|}$
- $\omega=0$ from $\left(\tanh\frac{\rho}{2}\right)^{-|\omega|}$

and the one with mass squared equal to 2

- no frequency from $\cosh\rho\,(\tanh\rho)^{|\omega|} {}_2F_1\left(\frac{|\omega|}{2}, \frac{|\omega|-1}{2}, 1+|\omega|, \tanh^2\rho\right)$
- no frequency from $\cosh\rho\,(\tanh\rho)^{-|\omega|} \rho_2F_1\left(-\frac{|\omega|}{2}, -\frac{|\omega|+1}{2}, 1-|\omega|, \tanh^2\rho\right)$
- $\omega=\pm1$ with $\beta=\frac{1}{3}$ from $\sinh\rho$
- $\omega=\pm1$ from $\sinh\rho\left(\text{atanh}\frac{1}{\cosh\rho} - \frac{\coth\rho}{\sinh\rho}\right)$

This is very different from the spectrum of [19] because the number of allowed frequencies is finite.

9.4 Redo fermionic determinants using Drukker Forini's rescaling and GY theorem

Suppose we want to quantify the one-loop string correction in the form

$$\Gamma_1(\theta_0) \equiv \sum_{\ell\in\mathbb{Z}}\left[\frac{3}{2}\log\left[\text{Det}_\ell\mathcal{O}_1(\theta_0)\right] + \frac{3}{2}\log\left[\text{Det}_\ell\mathcal{O}_2(\theta_0)\right] + \frac{1}{2}\log\left[\text{Det}_\ell\mathcal{O}_{3+}(\theta_0)\right] + \frac{1}{2}\log\left[\text{Det}_\ell\mathcal{O}_{3-}(\theta_0)\right]\right]$$
$$- \sum_{s\in\mathbb{Z}+\frac{1}{2}}\left[\frac{4}{2}\log\left[\text{Det}_s\mathcal{O}^2_{1,1,1}(\theta_0)\right] + \frac{4}{2}\log\left[\text{Det}_{-s}\mathcal{O}^2_{1,1,1}(\theta_0)\right]\right]$$

using the standard Fourier expansion followed by the GY theorem. In [2] the operators are first rescaled as

$$\mathcal{O}_i \quad\to\quad \Omega^2\,\mathcal{O}_i \qquad i=1,\,2,\,3+,\,3- \tag{9.23}$$

$$\mathcal{O}^2_{p_{12},p_{56},p_{89}} \quad\to\quad \Omega^2\,\mathcal{O}^2_{p_{12},p_{56},p_{89}} \tag{9.24}$$

with $ds^2 = \Omega^2(\sigma)\left(d\tau^2 + d\sigma^2\right)$. In this section we perform a different rescaling of the fermionic operators [4]. The motivation for this change lies in the observation that the same computational setup (GY theorem, Fourier summation and cutoff regularization) in [4, 2] leads to a result in agreement with localization (for the Bremsstrahlung function at one-loop) in the former and one not (for the latitude Wilson loops at one loop) in the latter. The different outcome may be due to an unknown issue of this method with the topology of the classical string (infinite-strip in the former and disk in the latter), but implementing the rescaling procedure of [4] (e.g. D.14) is the only modification that we can borrow from this paper. The difference consists in solving the spectral problems for

$$\mathcal{O}^{\text{new}}_{p_{12},p_{56},p_{89}} = \frac{i}{\Omega}\left(-i\sigma_2\omega + \sigma_1\partial_\sigma - a_{34}\sigma_1 + ip_{56}a_{56}\sigma_2\right) + \frac{1}{\Omega^2}\left(-p_{12}p_{56}\sin^2\theta\mathbb{I}_2 + p_{12}\sinh^2\rho\sigma_3\right) \tag{9.25}$$

which are first found from a tedious exercise in Mathematica

$$\left(\Omega\mathcal{O}_{p_{12},p_{56},p_{89}}\right)^2 = \Omega^{-1/2}\mathcal{O}^{\text{new}}_{p_{12},p_{56},p_{89}}\Omega^{1/2} \tag{9.26}$$

with the input

$$\mathcal{O}_{p_{12},p_{56},p_{89}} = \frac{1}{\Omega}\left(-i\partial_\tau + p_{56}\frac{1-\tanh\left(2\sigma+\sigma_0\right)}{2}\right)\sigma_2 + \frac{i}{\Omega}\left(\partial_\sigma + \frac{\Omega'}{2\Omega}\right)\sigma_1 - \frac{p_{12}p_{56}}{\Omega^2\cosh^2\left(\sigma+\sigma_0\right)}\mathbb{I}_2 + \frac{p_{12}}{\Omega^2\sinh^2\sigma}\sigma_3. \tag{9.27}$$

The rescaled operator in the circular case

$$\mathcal{O}^{\text{new}}_{p_{12},p_{56},p_{89}}\left(\theta_0=0\right) = \begin{pmatrix} -\partial_\tau^2 - \partial_\sigma^2 + \frac{1}{\sinh^2\sigma} & ip_{12}\frac{\cosh\sigma}{\sinh^2\sigma} \\ -ip_{12}\frac{\cosh\sigma}{\sinh^2\sigma} & -\partial_\tau^2 - \partial_\sigma^2 + \frac{1}{\sinh^2\sigma} \end{pmatrix} \tag{9.28}$$

can be diagonalized

$$\begin{aligned}\mathcal{O}^{\text{new, diag}}_{p_{12},p_{56},p_{89}}\left(\theta_0=0\right) &= \begin{pmatrix} -\partial_\tau^2 - \partial_\sigma^2 - p_{12}\frac{\cosh\sigma}{\sinh^2\sigma} + \frac{1}{\sinh^2\sigma} & 0 \\ 0 & -\partial_\tau^2 - \partial_\sigma^2 + p_{12}\frac{\cosh\sigma}{\sinh^2\sigma} + \frac{1}{\sinh^2\sigma} \end{pmatrix} \\ &= M^\dagger\mathcal{O}^{\text{new}}_{p_{12},p_{56},p_{89}}\left(\theta_0=0\right)M\,, \qquad M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1\end{pmatrix}.\end{aligned} \tag{9.29}$$

It is easy to construct the zero-eigenfunctions of $\mathcal{O}^{\text{new}}_{p_{12},p_{56},p_{89}} = \Omega^{1/2}\left(\Omega\mathcal{O}_{p_{12},p_{56},p_{89}}\right)^2\Omega^{-1/2}$ from the ones of $\mathcal{O}_{p_{12},p_{56},p_{89}}$ computed in [2], following the lines of appendix B.3 therein. Let us drop the triplet of labels and work with a generic $r\times r$ matrix-valued first-order operator $\mathcal{O} = P_0\partial_\sigma + P_1$ and the "rescaled" second-order operator

$$\begin{aligned}\mathcal{O}^{\text{new}} &= \Omega^{1/2}\left(\Omega\mathcal{O}\right)^2\Omega^{-1/2} \\ &= \Omega^2P_0^2\partial_\sigma^2 + \left[2\Omega^{5/2}\partial_\sigma\left(\Omega^{-1/2}\right)P_0^2 + \Omega P_0\left(\Omega P_0\right)' + \Omega^2\left\{P_0,P_1\right\}\right]\partial_\sigma \\ &+ \Omega^{5/2}\left(\Omega^{-1/2}\right)''P_0^2 + \Omega^{1/2}\left(\Omega^{-1/2}\right)'\left[\Omega P_0\left(\Omega P_0\right)' + \Omega^2\left\{P_0,P_1\right\}\right] \\ &+ \Omega^2P_1^2 + \Omega P_0\left(\Omega P_1\right)'\,.\end{aligned} \tag{9.30}$$

From the matrix of fundamental solutions with vanishing eigenvalue of $\mathcal{O}$

$$\mathcal{O}Y_{\mathcal{O}} = 0\,, \qquad Y_{\mathcal{O}}\left(\epsilon_0\right) = \mathbb{I}_r \tag{9.31}$$

we can construct the one of the latter as follows. There are two functions that live in the kernel of $\Omega\mathcal{O}\Omega^{-1/2}$

$$\Omega\mathcal{O}\left(\Omega^{-1/2}Y_{\mathcal{O}^{\text{new}}}\right) = 0\,, \qquad Y_{\mathcal{O}^{\text{new}}}\left(\sigma\right) = \sqrt{\frac{\Omega\left(\sigma\right)}{\Omega\left(\epsilon_0\right)}}Y_{\mathcal{O}}\left(\sigma\right)\,, \qquad Y_{\mathcal{O}^{\text{new}}}\left(\epsilon_0\right) = \mathbb{I}_r \tag{9.32}$$

and two more in the kernel of $\left(\Omega\mathcal{O}\right)^2\Omega^{-1/2}$

$$\Omega\mathcal{O}\left(\Omega^{-1/2}Z_{\mathcal{O}^{\text{new}}}\right) = \Omega^{-1/2}Y_{\mathcal{O}^{\text{new}}}C\,, \qquad Z_{\mathcal{O}}\left(\epsilon_0\right) = 0_r\,, \quad Z'_{\mathcal{O}}\left(\epsilon_0\right) = \mathbb{I}_r\,, \quad C = \text{constant matrix} \tag{9.33}$$

$$\begin{aligned}Z_{\mathcal{O}^{\text{new}}}\left(\sigma\right) &= \Omega\left(\epsilon_0\right)Y_{\mathcal{O}^{\text{new}}}\left(\sigma\right)\int_{\epsilon_0}^{\sigma}\frac{ds}{\Omega\left(s\right)}Y^{-1}_{\mathcal{O}^{\text{new}}}\left(s\right)P_0^{-1}\left(s\right)Y_{\mathcal{O}^{\text{new}}}\left(s\right)P_0\left(\epsilon_0\right) \\ &= \sqrt{\Omega\left(\epsilon_0\right)\Omega\left(\sigma\right)}Y_{\mathcal{O}}\left(\sigma\right)\int_{\epsilon_0}^{\sigma}\frac{ds}{\Omega\left(s\right)}Y^{-1}_{\mathcal{O}}\left(s\right)P_0^{-1}\left(s\right)Y_{\mathcal{O}}\left(s\right)P_0\left(\epsilon_0\right)\,,\end{aligned} \tag{9.34}$$

so the matrix of fundamental solutions with vanishing eigenvalue of $\mathcal{O}^{\rm new}$ is (cf. B.22)

$$Y_{\mathcal{O}^{\rm new}}(\sigma)=\begin{pmatrix} Y_{\mathcal{O}^{\rm new}}(\sigma)-Z_{\mathcal{O}^{\rm new}}(\sigma)\,Y'_{\mathcal{O}^{\rm new}}(\epsilon_0) & Z_{\mathcal{O}^{\rm new}}(\sigma) \\ Y'_{\mathcal{O}^{\rm new}}(\sigma)-Z'_{\mathcal{O}^{\rm new}}(\sigma)\,Y'_{\mathcal{O}^{\rm new}}(\epsilon_0) & Z'_{\mathcal{O}^{\rm new}}(\sigma)\end{pmatrix}\,,\qquad Y_{\mathcal{O}^{\rm new}}(\epsilon_0)=\mathbb{I}_{2r}\,. \tag{9.35}$$

The GY theorem delivers the expression of the functional determinant

$$\mathrm{Det}\mathcal{O}^{\rm new}=\sqrt{\frac{\det P_0(R)}{\det P_0(\epsilon_0)}}\,\frac{\det\left[M_{\mathcal{O}^{\rm new}}+N_{\mathcal{O}^{\rm new}}Y_{\mathcal{O}^{\rm new}}(R)\right]}{\det Y_{\mathcal{O}}(R)} \tag{9.36}$$

which is for Dirichlet bc

$$\left(\mathrm{Det}\mathcal{O}^{\rm new}\right)_{\rm Dirichlet}=\left(\Omega(\epsilon_0)\,\Omega(R)\right)^{r/2}\sqrt{\det P_0(\epsilon_0)\det P_0(R)}\det\left[\int_{\epsilon_0}^{R}\frac{ds}{\Omega(\sigma)}Y_{\mathcal{O}}^{-1}(\sigma)\,P_0^{-1}(\sigma)\,Y_{\mathcal{O}}(\sigma)\right]\,. \tag{9.37}$$

Setting $r=2$, we use this formula in

$$\begin{aligned}\Gamma_1(\theta_0)\equiv&\sum_{\ell\in\mathbb{Z}}\left[\frac{3}{2}\log\left[\mathrm{Det}_\ell\mathcal{O}_1(\theta_0)\right]+\frac{3}{2}\log\left[\mathrm{Det}_\ell\mathcal{O}_2(\theta_0)\right]+\frac{1}{2}\log\left[\mathrm{Det}_\ell\mathcal{O}_{3+}(\theta_0)\right]+\frac{1}{2}\log\left[\mathrm{Det}_\ell\mathcal{O}_{3-}(\theta_0)\right]\right]\\ &-\sum_{s\in\mathbb{Z}+\frac{1}{2}}\left[\frac{4}{2}\log\left[\mathrm{Det}_s\mathcal{O}^{\rm new}_{1,1,1}(\theta_0)\right]+\frac{4}{2}\log\left[\mathrm{Det}_{-s}\mathcal{O}^{\rm new}_{1,1,1}(\theta_0)\right]\right]\end{aligned}$$

and calculate the determinants with GY

$$\Gamma_1(\theta_0)-\Gamma_1(0)=\frac{5}{8}\theta_0^2+O\left(\theta_0^2\right)\,, \tag{9.38}$$

which the limit of the wrong expression [2]

$$\Gamma_1(\theta_0)-\Gamma_1(0)=-\frac{3}{2}\log\cos\theta_0+\log\cos\frac{\theta_0}{2}\,. \tag{9.39}$$

9.5 Zeta-function regularization of the Fourier summation: Dunne Kirsten's paper

The evaluation of the fluctuation determinants with the Gel'fand-Yaglom theorem involves a summation over the angular modes that is divergent and requires regularization. We are motivated by [22] to check that the zeta-function regularization over the angular modes coincides with our standard cutoff regularization $|\omega|\leq\Lambda$ and removal of the $\log\Lambda$-divergence by hand. We follow the lines of [22] for radially separable partial differential operators in $\mathbb{R}^{d\geq 2}$ and translate it into $H^{d\geq 2}$ with the needed modifications.

The metric of the hyperbolic space splits into a radial and a spherical part, parametrized by $d-1$ coordinates $\{\theta\}$,

$$ds^2_{H^d}=d\rho^2+\sinh^2\rho\,ds^2_{S^{d-1}}\left(\{\theta\}\right)\,. \tag{9.40}$$

The determinant ratio in question involves an "interacting" and a "free" operator

$$\frac{\mathrm{Det}\left(-\Delta+V(\rho)+m^2\right)}{\mathrm{Det}\left(-\Delta+m^2\right)} \tag{9.41}$$

with the Laplacian $\Delta=\frac{\partial^2}{\partial\rho^2}+(d-1)\coth\rho\frac{\partial}{\partial\rho}+\frac{\tilde\Delta_{d-1}}{\sinh^2\rho}$ in H^d , the Laplacian $\tilde\Delta_{d-1}$ in S^{d-1}, the radial potential $V(\rho)$ and the constant mass m. We can express a generic function as linear combinations of basis functions of the form

$$\phi_{\lambda l\{\sigma\}}(\rho,\{\theta\})\equiv q_{\lambda l}(\rho)\,\tilde Y_{l\{\sigma\}}(\{\theta\})\equiv(\sinh\rho)^{\frac{1-d}{2}}\,\psi_{\lambda l}(\rho)\,\tilde Y_{l\{\sigma\}}(\{\theta\}) \tag{9.42}$$

with the hyperspherical harmonics labeled by $l\in\mathbb{N}$ and possibly other quantum numbers $\{\sigma\}$

$$\tilde\Delta_{d-1}\tilde Y_{l\{\sigma\}}(\{\theta\})=-l\,(l+d-2)\,\tilde Y_{l\{\sigma\}}(\{\theta\})\,,\qquad \deg(l;d)=\frac{(2l+d-2)\,(l+d-3)!}{l!\,(d-2)!}\,. \tag{9.43}$$

Further manipulations on the radial part to eliminate the $\frac{\partial}{\partial\rho}$ in the operators (which would complicate the expression of the GY theorem) yield

$$-\Delta_{(l)} \equiv -\frac{\partial^2}{\partial\rho^2} - (d-1)\coth\rho\frac{\partial}{\partial\rho} + \frac{l\,(l+d-2)}{\sinh^2\rho} \tag{9.44}$$

$$\mathcal{M}^{\text{free}}_{(l)} \equiv -r^{\frac{d-1}{2}}\Delta_{(l)}r^{\frac{1-d}{2}} = -\frac{\partial^2}{\partial\rho^2} + \left(l+\frac{d-3}{2}\right)\left(l+\frac{d-1}{2}\right)\frac{1}{\sinh^2\rho} + \left(\frac{d-1}{2}\right)^2 \tag{9.45}$$

$$\mathcal{M}_{(l)} \equiv \mathcal{M}^{\text{free}}_{(l)} + V(\rho)\,. \tag{9.46}$$

Formally, for separable operators the logarithm of the determinant ratio can be written as a sum of the logarithm of one-dimensional determinant ratios over l, weighted with the degeneracy factor of the hyperspherical harmonics,

$$\log\frac{\text{Det}\left(-\Delta+V(\rho)+m^2\right)}{\text{Det}\left(-\Delta+m^2\right)} = \log\frac{\text{Det}\left(\mathcal{M}+m^2\right)}{\text{Det}\left(\mathcal{M}^{\text{free}}+m^2\right)} \tag{9.47}$$

$$= \sum_{l\geq 0}\deg(l;d)\log\frac{\text{Det}\left(-\Delta_{(l)}+V(\rho)+m^2\right)}{\text{Det}\left(-\Delta_{(l)}+m^2\right)} = \sum_{l\geq 0}\deg(l,d)\log\frac{\text{Det}\left(\mathcal{M}_{(l)}+m^2\right)}{\text{Det}\left(\mathcal{M}^{\text{free}}_{(l)}+m^2\right)}\,.$$

Each term in the sum can be computed using the Gel'fand-Yaglom result. However, the sum is divergent. The authors of [22] show how to define a finite and renormalized determinant ratio for the radially separable partial differential operators in flat space. We derive the analogous expressions in hyperbolic space. The sum over l is made convergent once an appropriate subtractions is made.

From the eigenfunctions of the Laplacian in hyperbolic space (p. 4222 [12])

$$\left(-\Delta_{(l)}+m^2\right)\left((\sinh\rho)^l\,{}_2F_1\left(i\lambda+\frac{d-1}{2}+l,\,-i\lambda+\frac{d-1}{2}+l,\,l+\frac{d}{2},\,-\sinh^2\frac{\rho}{2}\right)\right) \tag{9.48}$$

$$= \left(\lambda^2+\left(\frac{d-1}{2}\right)^2+m^2\right)(\sinh\rho)^l\,{}_2F_1\left(i\lambda+\frac{d-1}{2}+l,\,-i\lambda+\frac{d-1}{2}+l,\,l+\frac{d}{2},\,-\sinh^2\frac{\rho}{2}\right),$$

the results of Gel'fand and Yaglom lead to the expression for the 1d determinant ratio with Dirichlet b.c. in $\rho=0$, R with $R\to\infty$

$$\left(\mathcal{M}^{\text{free}}_{(l)}+m^2\right)\psi^{\text{free}}_{\lambda l}(\rho) = 0\,, \qquad \psi^{\text{free}}_{\lambda l}(0)=0,\ \psi^{\text{free}'}_{\lambda l}(0)=\text{constant} \tag{9.49}$$

$$\longrightarrow\quad \psi^{\text{free}}_{\lambda l}(\rho) \propto (\sinh\rho)^{l+\frac{d-1}{2}}\,{}_2F_1\left(i\lambda+\frac{d-1}{2}+l,\,-i\lambda+\frac{d-1}{2}+l,\,l+\frac{d}{2},\,-\sinh^2\frac{\rho}{2}\right), \qquad \lambda = i\sqrt{\left(\frac{d-1}{2}\right)^2+m^2}$$

$$\left(\mathcal{M}_{(l)}+m^2\right)\psi_{\lambda l}(\rho)=0 \qquad \psi_{\lambda l}(0)=0,\ \psi'_{\lambda l}(0)=\text{constant}\,. \tag{9.50}$$

Specifically, for the choice $V(\rho)=M^2-m^2$ we have

$$\psi_{\lambda l}(\rho) \propto (\sinh\rho)^{l+\frac{d-1}{2}}\,{}_2F_1\left(i\lambda+\frac{d-1}{2}+l,\,-i\lambda+\frac{d-1}{2}+l,\,l+\frac{d}{2},\,-\sinh^2\frac{\rho}{2}\right), \qquad \lambda = i\sqrt{\left(\frac{d-1}{2}\right)^2+M^2}\,. \tag{9.51}$$

The determinant ratio can be defined in terms of a spectral zeta function difference

$$\zeta(s) \equiv \zeta_{-\Delta+V+m^2}(s) - \zeta_{-\Delta+m^2}(s) \tag{9.52}$$

$$\begin{aligned}\zeta(0) &= a_{\frac{d}{2},-\Delta+V+m^2} - a_{\frac{d}{2},-\Delta+m^2} \\ &= a_{1,-\Delta+V+m^2} - a_{1,-\Delta+m^2} \qquad \text{for } d=2 \\ &= \frac{1}{4\pi}\int d\rho d\tau\sinh\rho\left(\frac{R}{2}-V-m^2\right) - \frac{1}{4\pi}\int d\rho d\tau\sinh\rho\left(\frac{R}{2}-m^2\right) \\ &= -\frac{1}{2}\int d\rho\sinh\rho\,V\,.\end{aligned} \tag{9.53}$$

In flat space, the derivative of the zeta function at $s=0$ could be evaluated using the relation to the Jost functions of scattering theory. Following the same procedure, but without explicit reference to scattering theory in hyperbolic space, we consider the radial eigenvalue equation for the *regular solution*, defined to have the same behaviour in the origin as the solution without potential

$$\left(\mathcal{M}_{(l)}^{\text{free}}-p^2\right)\psi_{p,\,l}^{\text{free, reg}}(\rho)=0\,,\qquad \psi_{p,\,l}^{\text{free, reg}}(\rho)\sim\rho^{l+\frac{d-1}{2}}\quad\text{for }\rho\to0 \tag{9.54}$$

$$\longrightarrow\quad \psi_{p,\,l}^{\text{free, reg}}(\rho)\propto(\sinh\rho)^{l+\frac{d-1}{2}}\;{}_2F_1\left(i\lambda+\frac{d-1}{2}+l,\;-i\lambda+\frac{d-1}{2}+l,\;l+\frac{d}{2},\;-\sinh^2\frac{\rho}{2}\right),\qquad \lambda=i\sqrt{\left(\frac{d-1}{2}\right)^2-p^2}$$

$$\left(\mathcal{M}_{(l)}-p^2\right)\psi_{p,\,l}^{\text{reg}}(\rho)=0\,,\qquad \psi_{p,\,l}^{\text{reg}}(\rho)\sim\rho^{l+\frac{d-1}{2}}\quad\text{for }\rho\to0 \tag{9.55}$$

and if $V(\rho)=M^2-m^2$ we have also

$$\psi_{p,\,l}^{\text{reg}}(\rho)\propto(\sinh\rho)^{l+\frac{d-1}{2}}\;{}_2F_1\left(i\lambda+\frac{d-1}{2}+l,\;-i\lambda+\frac{d-1}{2}+l,\;l+\frac{d}{2},\;-\sinh^2\frac{\rho}{2}\right),\qquad \lambda=i\sqrt{\left(\frac{d-1}{2}\right)^2+M^2-p^2}\,. \tag{9.56}$$

By standard contour manipulations, the zeta function can be expressed in terms of

$$\zeta(s)=\frac{\sin\pi s}{\pi}\sum_{l=0}^{\infty}\deg(l;d)\int_{\sqrt{m^2+\left(\frac{d-1}{2}\right)^2}}^{\infty}dk\,\left(k^2-m^2-\left(\frac{d-1}{2}\right)^2\right)^{-s}\frac{\partial}{\partial k}\left[\log f_{(l)}(ik)\right] \tag{9.57}$$

$$=\frac{\sin\pi s}{\pi}\sum_{l=0}^{\infty}\deg(l;d)\int_{m}^{\infty}dk\,\left(k^2-m^2\right)^{-s}\frac{\partial}{\partial k}\left[\log f_{(l)}\left(i\sqrt{k^2+\left(\frac{d-1}{2}\right)^2}\right)\right],\qquad f_{(l)}(ik)=\frac{\psi_{p=ik,l}^{\text{reg}}(\rho=\infty)}{\psi_{p=ik,l}^{\text{free, reg}}(\rho=\infty)}\,. \tag{9.58}$$

The technical problem is the construction of its analytic continuation to a neighborhood about $s=0$. If this spectral zeta function were analytic at $s=0$, then we would deduce that the determinant ratio

$$\log\frac{\text{Det}\left(-\Delta+V+m^2\right)}{\text{Det}\left(-\Delta+m^2\right)}=-\zeta'(0)=\sum_{l=0}^{\infty}\deg(l;d)\log f_{(l)}\left(i\sqrt{m^2+\left(\frac{d-1}{2}\right)^2}\right) \tag{9.59}$$

$$\log f_{(l)}\left(i\sqrt{m^2+\left(\frac{d-1}{2}\right)^2}\right)=\frac{\psi_{p=ik,l}^{\text{reg}}(\rho=\infty)}{\psi_{p=ik,l}^{\text{free, reg}}(\rho=\infty)}=\frac{\psi_{\lambda=im,l}(\rho=\infty)}{\psi_{\lambda=im,l}^{\text{free}}(\rho=\infty)} \tag{9.60}$$

coincides with the formal partial wave expansion using the GY theorem for each l. However, the zeta function is not analytic in $s=0$ because the partial-wave series is divergent. The analytic continuation is achieved by adding and subtracting the leading asymptotic terms of $f_{(l)}(ik)$ at large k

$$\begin{aligned}\zeta(s)&=\zeta_f(s)+\zeta_{as}(s) \qquad (9.61)\\ &=\frac{\sin\pi s}{\pi}\sum_{l=0}^{\infty}\deg(l;d)\int_{\sqrt{m^2+\left(\frac{d-1}{2}\right)^2}}^{\infty}dk\,\left(k^2-m^2-\left(\frac{d-1}{2}\right)^2\right)^{-s}\frac{\partial}{\partial k}\left[\log f_{(l)}(ik)-\log f_{(l)}(i\infty)\right]\\ &+\frac{\sin\pi s}{\pi}\sum_{l=0}^{\infty}\deg(l;d)\int_{\sqrt{m^2+\left(\frac{d-1}{2}\right)^2}}^{\infty}dk\,\left(k^2-m^2-\left(\frac{d-1}{2}\right)^2\right)^{-s}\frac{\partial}{\partial k}\left[\log f_{(l)}(i\infty)\right]\,.\end{aligned}$$

so that $\zeta_f(s)$ is analytic in $s=0$ and $\zeta_{as}(s)$ can be zeta-function regularized separately.

Lacking standard results in scattering theory in hyperbolic space, we focus to the determinat ratio of two Laplacians with mass m and M by setting $V(\rho)=M^2-m^2$. In complete analogy with the flat-space case

$$\begin{aligned}\log f_{(l)}(ik)&=\log\frac{\Gamma\left(\sqrt{k^2+\left(\frac{d-1}{2}\right)^2}+l+\frac{d-1}{2}\right)\Gamma\left(\sqrt{k^2+\left(\frac{d-1}{2}\right)^2+M^2}\right)\exp\left(\sqrt{M^2+k^2+\left(\frac{d-1}{2}\right)^2}R\right)}{\Gamma\left(\sqrt{k^2+\left(\frac{d-1}{2}\right)^2+M^2}+l+\frac{d-1}{2}\right)\Gamma\left(\sqrt{k^2+\left(\frac{d-1}{2}\right)^2}\right)\exp\sqrt{k^2+\left(\frac{d-1}{2}\right)^2}}\\ &\sim\frac{M^2R\nu}{2kl}+O\left(l^{-2}\right)\qquad\text{for }k,\,\nu\equiv l+\frac{d}{2}-1\to\infty\text{ and }\frac{k}{\nu}=\text{constant}\,.\end{aligned} \tag{9.62}$$

the limit on k brings along a precise limite on l too. In flat space, the quantity $l+\frac{d}{2}-1$ is the index of the Bessel function in $\psi^{\text{reg}}_{p=ik,l}(\rho)$. In hyperbolic space, it should retain the same role in the limit because the two spaces (and so the regular solutions in them) "look alike" $ds^2_{H^2}\to ds^2_{\mathbb{R}^2}$ in a neighboorhood of the origin $\rho\to 0$.
It is possible to find an even simpler expression because it turns out that the subtraction above may be an over-subtraction. To see this in $d=2$ dimensions, we expand the subtraction term in its large l behavior and *define*

$$\log f_{(l)}(i\infty)=\frac{M^2R\nu}{2kl}\,. \tag{9.63}$$

Then

$$\zeta_{as}(s)=\frac{\sin\pi s}{\pi}\sum_{l=0}^{\infty}\deg(l;d)\int_{\sqrt{m^2+\left(\frac{d-1}{2}\right)^2}}^{\infty}dk\left(k^2-m^2-\left(\frac{d-1}{2}\right)^2\right)^{-s}\frac{\partial}{\partial k}\left(\frac{M^2R\nu}{2kl}\right) \tag{9.64}$$

[to be completed]

9.6 Cutoff regularization of the Fourier summation, but without "supersymmetric regularization"

In formula 4.4 of [2] we used the so-called "supersymmetric regularization scheme" in order to shift the fermionic Fourier modes such that the summation will be over integer numbers as for the boson. Here we show that this (somewhat arbitrary) manipulation of the fermionic infinite sum could be avoided, hence avoiding potential ambiguities in the regularized part. The upshot is that it returns the same (wrong) result of the one-loop effective action of [2].

The formal expression of the sum over the angular modes defining a single partition function

$$\begin{aligned}\Gamma_1(\theta_0)&\equiv\sum_{\ell\in\mathbb{Z}}\Omega^B_\ell(\theta_0)-\sum_{s\in\mathbb{Z}+\frac{1}{2}}\Omega^F_s(\theta_0) \\ &\equiv\sum_{\ell\in\mathbb{Z}}\left[\frac{3}{2}\log\left[\text{Det}_\ell\mathcal{O}_1(\theta_0)\right]+\frac{3}{2}\log\left[\text{Det}_\ell\mathcal{O}_2(\theta_0)\right]+\frac{1}{2}\log\left[\text{Det}_\ell\mathcal{O}_{3+}(\theta_0)\right]+\frac{1}{2}\log\left[\text{Det}_\ell\mathcal{O}_{3-}(\theta_0)\right]\right] \\ &-\sum_{s\in\mathbb{Z}+\frac{1}{2}}\left[\frac{4}{2}\log\left[\text{Det}_s\mathcal{O}^2_{1,1,1}(\theta_0)\right]+\frac{4}{2}\log\left[\text{Det}_{-s}\mathcal{O}^2_{1,1,1}(\theta_0)\right]\right]\end{aligned} \tag{9.65}$$

was regularized in [2] with cutoffs Λ, $\mu^{-1}\gg 1$

$$\begin{aligned}\Gamma_1(\theta_0)&=\sum_{\ell\in\mathbb{Z}}e^{-|\mu|\ell}\left[\Omega^B_\ell(\theta_0)-\frac{\Omega^F_{\ell+\frac{1}{2}}(\theta_0)+\Omega^F_{\ell-\frac{1}{2}}(\theta_0)}{2}\right] \\ &=\frac{\mu}{2}\Omega^F_{\frac{1}{2}}(\theta_0)+\frac{\mu}{2}\sum_{\ell\geq 1}e^{-|\mu|\ell}\left[\Omega^F_{\ell+\frac{1}{2}}(\theta_0)-\Omega^F_{\ell-\frac{1}{2}}(\theta_0)\right]+\chi_v(\theta_0)\log\Lambda\end{aligned} \tag{9.66}$$

and by the addition of the (volume part of the) Euler number to eliminate the logarithmic divergence $\sim\sum_\ell\frac{1}{\ell}$ of the angular summation. We remarked that one could remove the AdS regulator $\epsilon_0\to 0$ in the determinants from the beginning, in order to simplify the summands and avoid numerics. A crucial observation in favor of this scheme was that the unphysical cutoff R in the determinants already canceled in $\Gamma_1(\theta_0)$ for a single latitude due to non-trivial cross-cancellations between the first and second line above.

Now, instead of shifting s by an half-integer and encapsulate the fermionic contribution into the sum over the integer ℓ, we introduce the *same* integer UV cutoff Λ in *both* sums as

$$\begin{aligned}\Gamma_1(\theta_0)-\Gamma_1(0)\equiv&\sum_{\ell=-\Lambda}^{\Lambda}\left[\frac{3}{2}\log\frac{\text{Det}_\ell\left[\mathcal{O}_1(\theta_0)\right]}{\text{Det}_\ell\left[\mathcal{O}_1(0)\right]}+\frac{3}{2}\log\frac{\text{Det}_\ell\left[\mathcal{O}_2(\theta_0)\right]}{\text{Det}_\ell\left[\mathcal{O}_2(0)\right]}+\frac{1}{2}\log\frac{\text{Det}_\ell\left[\mathcal{O}_{3+}(\theta_0)\right]}{\text{Det}_\ell\left[\mathcal{O}_{3+}(0)\right]}+\frac{1}{2}\log\frac{\text{Det}_\ell\left[\mathcal{O}_{3-}(\theta_0)\right]}{\text{Det}_\ell\left[\mathcal{O}_{3-}(0)\right]}\right] \\ &-\sum_{s=-\Lambda-\frac{1}{2}-c}^{\Lambda+\frac{1}{2}+c}\left[\frac{4}{2}\log\frac{\text{Det}_s\left[\mathcal{O}^2_{1,1,1}(\theta_0)\right]}{\text{Det}_s\left[\mathcal{O}^2_{1,1,1}(0)\right]}+\frac{4}{2}\log\frac{\text{Det}_{-s}\left[\mathcal{O}^2_{1,1,1}(\theta_0)\right]}{\text{Det}_{-s}\left[\mathcal{O}^2_{1,1,1}(0)\right]}\right]\,.\end{aligned} \tag{9.67}$$

The constant c can be kept arbitrary because it leaves the regularized result unaffected when $\Lambda\to\infty$

$$\Gamma_1(\theta_0)-\Gamma_1(0)=\frac{3}{2}\log\cos\theta_0-\log\cos\frac{\theta_0}{2}+O\left(\Lambda^{-1}\right)\,. \tag{9.68}$$

The divergence $\Gamma_1(\theta_0) - \Gamma_1(0) \sim -(\chi_v(\theta_0) - \chi_v(0)) \log \Lambda$ does not appear because $\chi_v(\theta_0) \to \chi_v(0)$ when $\epsilon_0 \to 0$, as we considered in each summand from the beginning. Altought the cutoff R drops out only in $\Gamma_1(\theta_0) - \Gamma_1(0)$ for the ratio latitude/circle, it is nice to see that we recover the same (wrong) result with less manipulations of the infinite sums.

10 Diagrammatical method for latitude Wilson loops

$\Gamma_1(\theta_0)$ can be obtained from the path-integral [24]

$$Z_1(\theta_0) = \int \prod_{i=1}^{8} \mathcal{D}\xi_i \exp\left[-\int_{\tau,\sigma} \Omega^2 \left(\sum_{i=1,2,3} \xi_i \mathcal{O}_1 \xi_i + \sum_{i=4,5,6} \xi_i \mathcal{O}_2 \xi_i + \xi_7 \mathcal{O}_{3+} \xi_7 + \xi_8 \mathcal{O}_{3-} \xi_8\right)\right]$$
$$\times \int \prod_{p_{12},p_{56},p_{89}=\pm 1} \mathcal{D}\Psi_{p_{12},p_{56},p_{89}} \left[\exp - \int_{\tau,\sigma} \Omega^2 \left(\sum_{p_{12},p_{56},p_{89}=\pm1} \bar{\Psi}_{p_{12},p_{56},p_{89}} \mathcal{O}_{p_{12},p_{56},p_{89}} \Psi_{p_{12},p_{56},p_{89}}\right)\right]. \tag{10.1}$$

For small angle, we expand the operators as shown in the previous sections and we perform a sort of perturbation theory for small "coupling constant" θ_0 via diagrammatical methods.

We split the Lagrangian into the "free" and "interacting" part

$$Z_1(\theta_0) \approx \int \prod_{i=1}^{8} \mathcal{D}\xi_i \exp\left[-\int_{\tau,\rho} \sinh\rho \left(\sum_{i=1,2,3} \xi_i \bar{\mathcal{O}}_1 \xi_i + \sum_{i=4,5,6} \xi_i \bar{\mathcal{O}}_2 \xi_i + \xi_7 \bar{\mathcal{O}}_{3-} \xi_7 + \xi_8 \bar{\mathcal{O}}_{3+} \xi_8\right)\right]$$
$$\times \exp\left\{-\theta_0^2 \int_{\tau,\rho} \sinh\rho \left[\sum_{i=2,3} \xi_i \left(\tilde{\mathcal{O}}_1 + f(\rho)\bar{\mathcal{O}}_1\right)\xi_i + \sum_{i=4,5,6} \xi_i \left(\tilde{\mathcal{O}}_2 + f(\rho)\bar{\mathcal{O}}_2\right)\xi_i\right]\right\}$$
$$\times \exp\left\{-\theta_0^2 \int_{\tau,\rho} \sinh\rho \left[\xi_7 \left(\tilde{\mathcal{O}}_{3+} + f(\rho)\bar{\mathcal{O}}_{3+}\right)\xi_7 + \xi_8 \left(\tilde{\mathcal{O}}_{3-} + f(\rho)\bar{\mathcal{O}}_{3-}\right)\xi_8\right]\right\} \tag{10.2}$$
$$\times \int \prod_{p_{12},p_{56},p_{89}=\pm1} \mathcal{D}\Psi_i \exp\left[-\int_{\tau,\rho} \sinh\rho \left(\sum_{p_{12},p_{56},p_{89}=\pm1} \bar{\Psi}_{p_{12},p_{56},p_{89}} \bar{\mathcal{O}}_{p_{12},p_{56},p_{89}} \Psi_{p_{12},p_{56},p_{89}}\right)\right]$$
$$\times \exp\left\{-\theta_0^2 \int_{\tau,\rho} \sinh\rho \left[\sum_{p_{12},p_{56},p_{89}=\pm1} \bar{\Psi}_{p_{12},p_{56},p_{89}} \left(\tilde{\mathcal{O}}_{p_{12},p_{56},p_{89}} + f(\rho)\bar{\mathcal{O}}_{p_{12},p_{56},p_{89}}\right)\Psi_{p_{12},p_{56},p_{89}}\right]\right\}$$

where we defined the function $f(\rho)$ in the interaction vertices

$$f(\rho) \equiv \frac{\left(\Omega^2 \left|\frac{d\sigma}{d\rho}\right|\right)_{\text{order } \theta_0^2}}{\left(\Omega^2 \left|\frac{d\sigma}{d\rho}\right|\right)_{\text{order } \theta_0^0}} = \frac{(1-\cosh\rho)^2}{\sinh^4\rho}\,. \tag{10.3}$$

The expansion of the metric around H^2 and the change of coordinates $\sigma \to \rho$ is responsible for $f(\rho)$ not being zero.

Feynman rules are easy to derive for the action in the main file using path-integral formalism. The starting point is the path-integral coupled to sources for the $8_B + 8_F$ fields [25].

$$Z(\lambda, \theta_0;\, J_1,\, J_2,\, \dots)$$
$$= \sum_{V=0}^{\infty} \frac{1}{V!} \sum_{V_1,V_2,\dots} \begin{pmatrix} V \\ V_1\, V_2 \dots \end{pmatrix} \prod_{i=1}^{8} \left(-\int_{\rho,\tau} \sinh\rho\, \mathcal{L}^i_{\text{int}}\right)^{V_i} \prod_{p_{12},p_{56},p_{89}=\pm1} \left(-\int_{\rho,\tau} \sinh\rho\, \mathcal{L}^{p_{12},p_{56},p_{89}}_{\text{int}}\right)^{V_{p_{12},p_{56},p_{89}}} \dots \tag{10.4}$$
$$\times \sum_{P=0}^{\infty} \frac{1}{P!} \sum_{P_1,P_2,\dots} \begin{pmatrix} P \\ P_1\, P_2 \dots \end{pmatrix} \prod_{i=1}^{8} \left(-\frac{1}{2} J_i \cdot G_i \cdot J_i\right)^{P_i} \prod_{p_{12},p_{56},p_{89}=\pm1} \left(-J_{p_{12},p_{56},p_{89}} \cdot G_{p_{12},p_{56},p_{89}} \cdot J_{p_{12},p_{56},p_{89}}\right)^{P_{p_{12},p_{56},p_{89}}} \dots$$

The interaction Lagrangians in the formula above are understood to be functions of functional derivatives, upon the replacement

$$\xi_i \to \frac{\delta}{\delta J_i} \qquad i = 1, \dots 8$$
$$\Psi_{p_{12},p_{56},p_{89}} \to \frac{\delta}{\delta \bar{J}_{p_{12},p_{56},p_{89}}} \tag{10.5}$$
$$\bar{\Psi}_{p_{12},p_{56},p_{89}} \to -\frac{\delta}{\delta J_{p_{12},p_{56},p_{89}}} \qquad p_{12}, p_{56}, p_{89} = \pm 1\,.$$

[24] We do not integrate over the adjoints of $\Psi_{p_{12},p_{56},p_{89}}$ since they are Majorana fields.
[25] We leave out the classical value of the action.

Correlation functions are functional derivatives of this generating functional at vanishing sources. In particular, the vacuum bubbles relevant to us are diagrams with one propagator ($P = 1$) and one insertion ($V = 1$) in the generating functional of connected diagrams [26]:

$$\begin{aligned}\Gamma_1(\theta_0) &= \sum_{i=1}^{8}\left(\int_{\rho,\tau}\sinh\rho\,\mathcal{L}^1_{\text{int}}\right)\left(\frac{1}{2}J_i\cdot G_i\cdot J_i\right)\\ &\quad+\sum_{p_{12},p_{56},p_{89}=\pm1}\left(\int_{\rho,\tau}\sinh\rho\,\mathcal{L}^{p_{12},p_{56},p_{89}}_{\text{int}}\right)\left(\bar{J}_{p_{12},p_{56},p_{89}}\cdot G_{p_{12},p_{56},p_{89}}\cdot J_{p_{12},p_{56},p_{89}}\right)\\ &= \sum_{i=1}^{8}\left(\int_{\rho,\tau}\sinh\rho\,\mathcal{L}^1_{\text{int}}\right)G_i(x,x') - \sum_{p_{12},p_{56},p_{89}=\pm1}\left(\int_{\rho,\tau}\sinh\rho\,\mathcal{L}^{p_{12},p_{56},p_{89}}_{\text{int}}\right)G_{p_{12},p_{56},p_{89}}(x,x')\Bigg|_{x=x'}\end{aligned} \tag{10.6}$$

In the last line, the interaction Lagrangian is has been stripped off of its fields, and so it is a differential operator. We report these vertex interactions

$$\begin{aligned}\tilde{\mathcal{O}}_1+f(\rho)\,\bar{\mathcal{O}}_1 &= 0\\ \tilde{\mathcal{O}}_2+f(\rho)\,\bar{\mathcal{O}}_2 &= -\frac{2}{(1+\cosh\rho)^2}\\ \tilde{\mathcal{O}}_{3\pm}+f(\rho)\,\bar{\mathcal{O}}_{3\pm} &= -\left[\frac{1-\cosh\rho}{\sinh\rho\,(1+\cosh\rho)}\right]^2(2\pm i\partial_\tau)\\ \tilde{\mathcal{O}}_{p_{12},p_{56},p_{89}}+f(\rho)\,\bar{\mathcal{O}}_{p_{12},p_{56},p_{89}} &= -\frac{i}{2(1+\cosh\rho)^2}\not{\nabla}_{\rho,\tau}-\frac{i(1-\cosh\rho)}{2\sinh\rho\,(1+\cosh\rho)^2}\sigma_1\\ &\quad+\frac{p_{56}\sinh^3\rho}{4(1+\cosh\rho)^4}\sigma_2-\frac{p_{12}p_{56}\mathbb{I}_2}{(1+\cosh\rho)^2}\,.\end{aligned} \tag{10.7}$$

We presented the general approach capable of catching all diagrammatical contributions (if we went beyond the leading approximation $\sim\theta_0^2$, or even interactions of the initial GS action). We notice that the same quantity can be obtained by expanding the effective action

$$\Gamma_1(\theta_0)\approx\bar{\Gamma}_1+\theta_0^2\tilde{\Gamma}_1\,. \tag{10.8}$$

[26] In $\bar{J}_{p_{12},p_{56},p_{89}}\cdot G_{p_{12},p_{56},p_{89}}\cdot J_{p_{12},p_{56},p_{89}}$, we incorporated the $\frac{1}{2}$ of the Majorana Ψ fields into the propagator $G_{p_{12},p_{56},p_{89}}$.

11 Notes on [1]

We recall that the - albeit formal - solution of the inhomogeneous equation in $H^2 \times \mathbb{R}$ with source term $s(x)$ and initial data $u(x)$

$$L_{t,x} u(x,t) = s(x) \qquad\qquad u(x,0) = u(x) \tag{11.1}$$

is

$$u(t,x) = \int_{x''} \sqrt{h(x'')} u^0 \left(x, x'', t\right) s\left(x''\right) \tag{11.2}$$

provided the fundamental solution

$$L_{t,x} u^0 \left(x, x'', t\right) = \frac{\delta\left(x - x''\right)}{\sqrt{h(x)}} \qquad u^0\left(x, x'', 0\right) = \frac{u(x)}{s(x)} \frac{\delta\left(x - x''\right)}{\sqrt{h(x)}} . \tag{11.3}$$

The fundamental solution of $-\Delta_{\rho,\tau} + m^2$

$$\left(-\Delta_{\rho,\tau} + m^2\right) f\left(x, x'\right) = \frac{1}{\sinh\rho} \delta\left(x - x'\right) \tag{11.4}$$

is

$$f\left(x, x'\right) = \begin{cases} \frac{1}{2\pi} \operatorname{atanh} \frac{1}{\cosh d\left(x,x'\right)} = -\frac{1}{2\pi} \log\tanh \frac{d\left(x,x'\right)}{2} & m^2 = 0 \\ \frac{-1 - \cosh\rho \log\tanh \frac{d\left(x,x'\right)}{2}}{2\pi} = \frac{-1 + \cosh\rho \operatorname{atanh} \frac{1}{\cosh d\left(x,x'\right)}}{2\pi} & m^2 = 2 \\ \frac{i}{4} P_{-1/2 + \sqrt{1/4 + M^2}} \left(\cosh d\left(x, x'\right)\right) + \frac{1}{2\pi} Q_{-1/2 + \sqrt{1/4 + M^2}} \left(\cosh d\left(x, x'\right)\right) & \forall\, m^2 \end{cases} . \tag{11.5}$$

12 Perturbation theory for the heat kernel

12.1 General definitions

Let us consider two d-dimensional spaces $\bar{\mathcal{M}}$ and $\mathcal{M}$ with infinitesimally-close metrics $\bar{g}_{ij}$ and $g_{ij} = \bar{g}_{ij} + \alpha\tilde{g}_{ij} + O\left(\alpha^2\right)$. The boundaries have one dimension less and characterized by the metrics $\bar{\gamma}_{ij}$ and $\gamma_{ij} = \bar{\gamma}_{ij} + \alpha\tilde{\gamma}_{ij} + O\left(\alpha^2\right)$ with $x \in \partial\mathcal{M}$. The determinant of the metric g is an expansion in small α

$$g = \bar{g} + \alpha\tilde{g} + O\left(\alpha^2\right) \tag{12.1}$$

and similarly for γ. We shall use the notation

$$\int_x f \equiv \int_{\mathcal{M}} f \equiv \int_{x\in\mathcal{M}} d^d x\, f(x) \qquad\qquad \int_{\partial\mathcal{M}} f \equiv \int_{x\in\partial\mathcal{M}} d^{d-1}x\, f(x) \tag{12.2}$$

for volume integrals and the integrations restricted to the boundary, with no volume factors included in the definition.

We define the "flat" Dirac delta $\delta^{(d)}(x)$ on $\mathcal{M}$

$$\int_{\mathcal{M}} \delta^{(d)}(x) = 1 \qquad\qquad \frac{1}{\sqrt{g}}\delta^{(d)}(x) = \left(\frac{1}{\sqrt{\bar{g}}} - \alpha\frac{\tilde{g}}{2\bar{g}^{3/2}} + O\left(\alpha^2\right)\right)\delta^{(d)}(x) \tag{12.3}$$

and the analogue one $\delta^{(d-1)}(x)$ with support on the boundary

$$\int_{\mathcal{M}} \delta^{(d-1)}(x) = 1 \qquad\qquad \frac{1}{\sqrt{\gamma}}\delta^{(d-1)}(x) = \left(\frac{1}{\sqrt{\bar{\gamma}}} - \alpha\frac{\tilde{\gamma}}{2\bar{\gamma}^{3/2}} + O\left(\alpha^2\right)\right)\delta^{(d-1)}(x)\,. \tag{12.4}$$

We shall consider an operator $\mathcal{O}$ on $\mathcal{M}$ that is the perturbation of an operator $\bar{\mathcal{O}}$ on $\bar{\mathcal{M}}$

$$\mathcal{O} = \bar{\mathcal{O}} + \alpha\tilde{\mathcal{O}} + O\left(\alpha^2\right)\,, \tag{12.5}$$

and suppose that its heat kernel propagator and zeta-function have the following expansions

$$K_{\mathcal{O}}\left(x, x^{'}; t\right) = \bar{K}_{\mathcal{O}}\left(x, x^{'}; t\right) + \alpha\tilde{K}_{\mathcal{O}}\left(x, x^{'}; t\right) + O\left(\alpha^2\right) \tag{12.6}$$

$$K_{\mathcal{O}}(t) = \bar{K}_{\mathcal{O}}(t) + \alpha\tilde{K}_{\mathcal{O}}(t) + O\left(\alpha^2\right) \tag{12.7}$$

$$\zeta_{\mathcal{O}}(s) = \bar{\zeta}_{\mathcal{O}}(s) + \alpha\tilde{\zeta}_{\mathcal{O}}(s) + O\left(\alpha^2\right)\,. \tag{12.8}$$

The heat equation [27]

$$\left(\partial_t + \mathcal{O}_x\right) K_{\mathcal{O}}\left(x, x^{'}; t\right) = 0 \qquad\qquad K_{\mathcal{O}}\left(x, x^{'}; 0\right) = \frac{1}{\sqrt{g}}\delta^{(d)}\left(x - x^{'}\right)\mathbb{I} \tag{12.9}$$

splits into the order α^0 and α^1 equations

$$\begin{aligned} &\left(\partial_t + \bar{\mathcal{O}}_x\right)\bar{K}_{\mathcal{O}}\left(x, x^{'}; t\right) = 0 && \bar{K}_{\mathcal{O}}\left(x, x^{'}; 0\right) = \frac{1}{\sqrt{\bar{g}}}\delta^{(d)}\left(x - x^{'}\right)\mathbb{I} \\ &\left(\partial_t + \bar{\mathcal{O}}_x\right)\tilde{K}_{\mathcal{O}}\left(x, x^{'}; t\right) + \tilde{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x, x^{'}; t\right) = 0 && \tilde{K}_{\mathcal{O}}\left(x, x^{'}; 0\right) = -\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x - x^{'}\right)\mathbb{I}\,. \end{aligned} \tag{12.10}$$

We assume that the unperturbed heat kernel propagator in $\bar{\mathcal{M}}$ is known in literature. One can now develop a perturbative expansion, e.g. following [23, 24].

12.2 First correction

The equation

$$\left(\partial_t + \bar{\mathcal{O}}_x\right)\tilde{K}_{\mathcal{O}}\left(x, x^{'}; t\right) + \tilde{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x, x^{'}; t\right) = 0 \qquad\qquad \tilde{K}_{\mathcal{O}}\left(x, x^{'}; 0\right) = -\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x - x^{'}\right)\mathbb{I} \tag{12.11}$$

can be solved using the standard method of variation of constants. Let us set

$$\tilde{K}_{\mathcal{O}}\left(x, x^{'}; t\right) \equiv -\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x - x^{'}\right)\mathbb{I} + \int_{x^{''}}\sqrt{\bar{g}}\bar{K}_{\mathcal{O}}\left(x, x^{''}; t\right) C_{\mathcal{O}}\left(x^{''}, x^{'}; t\right) \qquad\qquad C_{\mathcal{O}}\left(x^{''}, x^{'}; 0\right) = 0 \tag{12.12}$$

[27] $\mathbb{I}$ is the unit matrix of the internal space (1×1 for scalars and $2^n \times 2^n$ for spinors in $d = 2n$, $2n+1$ dimensions).

where $C_{\mathcal{O}}$ is function to be determined. Substituting the expression into the equation, we arrive to the expression

$$\begin{aligned}C_{\mathcal{O}}\left(x^{''},x^{'};t\right) &= \int_0^t dt^{'}\int_{x^{'''}}\sqrt{\bar{g}}\bar{K}_{\mathcal{O}}\left(x^{''},x^{'''};-t'\right)\bar{\mathcal{O}}_{x^{'''}}\left(\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x^{'''}-x^{'}\right)\right)\\ &-\int_0^t dt^{'}\int_{x^{'''}}\sqrt{\bar{g}}\bar{K}_{\mathcal{O}}\left(x^{''},x^{'''};-t'\right)\tilde{\mathcal{O}}_{x^{'''}}\bar{K}_{\mathcal{O}}\left(x^{'''},x^{'};t^{'}\right)\end{aligned}\tag{12.13}$$

so that we have for the heat kernel propagator

$$\begin{aligned}\tilde{K}_{\mathcal{O}}\left(x,x^{'};t\right) &= -\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x-x^{'}\right)\mathbb{I}+\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\bar{K}_{\mathcal{O}}\left(x,x^{''};t-t'\right)\bar{\mathcal{O}}_{x^{''}}\left(\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x^{''}-x^{'}\right)\right)\\ &-\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\bar{K}_{\mathcal{O}}\left(x,x^{''};t-t^{'}\right)\tilde{\mathcal{O}}_{x^{''}}\bar{K}_{\mathcal{O}}\left(x^{''},x^{'};t^{'}\right).\end{aligned}\tag{12.14}$$

As a double check, we can directly prove that it is a solution by plugging into the first-order heat equation

$$\begin{aligned}&\left(\partial_t+\bar{\mathcal{O}}_x\right)\tilde{K}_{\mathcal{O}}\left(x,x^{'};t\right)+\tilde{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x^{'};t\right)\\ &\int_{x^{''}}\sqrt{\bar{g}}\bar{K}_{\mathcal{O}}\left(x,x^{''};0\right)\bar{\mathcal{O}}_{x^{''}}\left(\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x^{''}-x^{'}\right)\right)+\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\partial_t\bar{K}_{\mathcal{O}}\left(x,x^{''};t-t'\right)\bar{\mathcal{O}}_{x^{''}}\left(\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x^{''}-x^{'}\right)\right)\\ &-\int_{x^{''}}\sqrt{\bar{g}}\bar{K}_{\mathcal{O}}\left(x,x^{''};0\right)\left[\tilde{\mathcal{O}}_{x^{''}}\bar{K}_{\mathcal{O}}\left(x^{''},x^{'};t^{'}\right)\right]-\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\partial_t\bar{K}_{\mathcal{O}}\left(x,x^{''};t-t^{'}\right)\left[\tilde{\mathcal{O}}_{x^{''}}\bar{K}_{\mathcal{O}}\left(x^{''},x^{'};t^{'}\right)\right]\\ &-\bar{\mathcal{O}}_x\left(\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x-x^{'}\right)\right)+\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\left[\bar{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x^{''};t-t'\right)\right]\bar{\mathcal{O}}_{x^{''}}\left(\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x^{''}-x^{'}\right)\right)\\ &-\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\left[\bar{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x^{''};t-t^{'}\right)\right]\left[\tilde{\mathcal{O}}_{x^{''}}\bar{K}_{\mathcal{O}}\left(x^{''},x^{'};t^{'}\right)\right]+\tilde{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x^{'};t\right),\end{aligned}$$

use the property $\int_{x'}\sqrt{\bar{g}}\bar{K}_{\mathcal{O}}\left(x,x^{'};t_1\right)\bar{K}_{\mathcal{O}}\left(x^{'},x^{''};t_2\right)=\bar{K}_{\mathcal{O}}\left(x,x^{''};t_1+t_2\right)$

$$\begin{aligned}=&\int_{x^{''}}\sqrt{\bar{g}}\bar{K}_{\mathcal{O}}\left(x,x^{''};0\right)\bar{\mathcal{O}}_{x^{''}}\left(\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x^{''}-x^{'}\right)\right)+\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\int_{x^{'''}}\sqrt{\bar{g}}\partial_t\bar{K}_{\mathcal{O}}\left(x,x^{'''};t\right)\bar{K}_{\mathcal{O}}\left(x^{'''},x^{''};-t'\right)\bar{\mathcal{O}}_{x^{''}}\left(\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x^{''}-\right.\right.\\ &-\int_{x^{''}}\sqrt{\bar{g}}\bar{K}_{\mathcal{O}}\left(x,x^{''};0\right)\left[\tilde{\mathcal{O}}_{x^{''}}\bar{K}_{\mathcal{O}}\left(x^{''},x^{'};t\right)\right]-\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\int_{x^{'''}}\sqrt{\bar{g}}\partial_t\bar{K}_{\mathcal{O}}\left(x,x^{'''};t\right)\bar{K}_{\mathcal{O}}\left(x^{'''},x^{''};-t^{'}\right)\left[\tilde{\mathcal{O}}_{x^{''}}\bar{K}_{\mathcal{O}}\left(x^{''},x^{'};t^{'}\right)\right]\\ &-\bar{\mathcal{O}}_x\left(\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x-x^{'}\right)\right)+\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\left[\bar{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x^{''};t-t'\right)\right]\bar{\mathcal{O}}_{x^{''}}\left(\frac{\tilde{\sqrt{g}}}{\bar{g}}\delta^{(d)}\left(x^{''}-x^{'}\right)\right)\\ &-\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\left[\bar{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x^{''};t-t^{'}\right)\right]\left[\tilde{\mathcal{O}}_{x^{''}}\bar{K}_{\mathcal{O}}\left(x^{''},x^{'};t^{'}\right)\right]+\tilde{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x^{'};t\right),\end{aligned}$$

use the heat equation for $\bar{K}_{\mathcal{O}}\left(x,x^{'};t\right)$ (underlined terms cancels)

$$\begin{aligned}=&\underline{\int_{x^{''}}\delta\left(x-x^{''}\right)\bar{\mathcal{O}}_{x^{''}}\left(\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x^{''}-x^{'}\right)\right)}-\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\int_{x^{'''}}\sqrt{\bar{g}}\left[\bar{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x^{'''};t\right)\right]\bar{K}_{\mathcal{O}}\left(x^{'''},x^{''};-t'\right)\bar{\mathcal{O}}_{x^{''}}\left(\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x^{''}-x\right.\right.\\ &-\underline{\int_{x^{''}}\delta^{(d)}\left(x-x^{''}\right)\left[\tilde{\mathcal{O}}_{x^{''}}\bar{K}_{\mathcal{O}}\left(x^{''},x^{'};t\right)\right]}+\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\int_{x^{'''}}\sqrt{\bar{g}}\left[\bar{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x^{'''};t\right)\right]\bar{K}_{\mathcal{O}}\left(x^{'''},x^{''};-t^{'}\right)\left[\tilde{\mathcal{O}}_{x^{''}}\bar{K}_{\mathcal{O}}\left(x^{''},x^{'};t^{'}\right)\right]\\ &-\underline{\bar{\mathcal{O}}_x\left(\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x-x^{'}\right)\right)}+\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\left[\bar{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x^{''};t-t'\right)\right]\bar{\mathcal{O}}_{x^{''}}\left(\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta^{(d)}\left(x^{''}-x^{'}\right)\right)\\ &-\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\left[\bar{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x^{''};t-t^{'}\right)\right]\left[\tilde{\mathcal{O}}_{x^{''}}\bar{K}_{\mathcal{O}}\left(x^{''},x^{'};t^{'}\right)\right]+\underline{\tilde{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x^{'};t\right)},\end{aligned}$$

use the property $\int_{x'} \sqrt{\bar{g}} \bar{K}_{\mathcal{O}} \left(x, x^{'}; t_1 \right) \bar{K}_{\mathcal{O}} \left(x^{'}, x^{''}; t_2 \right) = \bar{K}_{\mathcal{O}} \left(x, x^{''}; t_1 + t_2 \right)$

$$
\begin{aligned}
=& - \int_0^t dt^{'} \int_{x^{''}} \sqrt{\bar{g}} \left[\bar{\mathcal{O}}_x \bar{K}_{\mathcal{O}} \left(x, x^{''}; t - t^{'} \right) \right] \bar{\mathcal{O}}_{x^{''}} \left(\frac{\tilde{g}}{2\bar{g}^{3/2}} \delta^{(d)} \left(x^{''} - x^{'} \right) \right) \\
& + \int_0^t dt^{'} \int_{x^{''}} \sqrt{\bar{g}} \left[\bar{\mathcal{O}}_x \bar{K}_{\mathcal{O}} \left(x, x^{''}; t - t^{'} \right) \right] \left[\tilde{\mathcal{O}}_{x^{''}} \bar{K}_{\mathcal{O}} \left(x^{''}, x^{'}; t^{'} \right) \right] \\
& + \int_0^t dt^{'} \int_{x^{''}} \sqrt{\bar{g}} \left[\bar{\mathcal{O}}_x \bar{K}_{\mathcal{O}} \left(x, x^{''}; t - t' \right) \right] \bar{\mathcal{O}}_{x^{''}} \left(\frac{\tilde{g}}{2\bar{g}^{3/2}} \delta^{(d)} \left(x^{''} - x^{'} \right) \right) \\
& - \int_0^t dt^{'} \int_{x^{''}} \sqrt{\bar{g}} \left[\bar{\mathcal{O}}_x \bar{K}_{\mathcal{O}} \left(x, x^{''}; t - t^{'} \right) \right] \left[\tilde{\mathcal{O}}_{x^{''}} \bar{K}_{\mathcal{O}} \left(x^{''}, x^{'}; t^{'} \right) \right] \\
=& 0 \, .
\end{aligned}
$$

Next we compute the functional trace of $K_{\mathcal{O}} \left(x, x^{'}; t \right)$

$$
\begin{aligned}
K_{\mathcal{O}}(t) &= \int_x \sqrt{g} \mathrm{tr} K_{\mathcal{O}}(x, x; t) \\
&= \int_x \frac{\tilde{g}}{2\sqrt{\bar{g}}} \mathrm{tr} \bar{K}_{\mathcal{O}}(x, x; t) + \int_x \sqrt{\bar{g}} \mathrm{tr} \tilde{K}_{\mathcal{O}}(x, x; t) \\
&= \int_x \frac{\tilde{g}}{2\sqrt{\bar{g}}} \mathrm{tr} \bar{K}_{\mathcal{O}}(x, x; t) - \delta^{(d)}(0) \int_x \frac{\tilde{g}}{2\bar{g}} \mathrm{tr}\mathbb{I} + \int_0^t dt^{'} \int_x \sqrt{\bar{g}} \int_{x^{''}} \sqrt{\bar{g}} \mathrm{tr} \left[\bar{K}_{\mathcal{O}} \left(x, x^{''}; t - t^{'} \right) \bar{\mathcal{O}}_{x^{''}} \left(\frac{\tilde{g}}{2\bar{g}^{3/2}} \delta^{(d)} \left(x^{''} - x \right) \right) \right] \\
&- \int_x \sqrt{\bar{g}} \int_0^t dt^{'} \int_{x^{''}} \sqrt{\bar{g}} \mathrm{tr} \left[\bar{K}_{\mathcal{O}} \left(x, x^{''}; t - t^{'} \right) \tilde{\mathcal{O}}_{x^{''}} \bar{K}_{\mathcal{O}} \left(x^{''}, x; t^{'} \right) \right] \, ,
\end{aligned}
$$

recall that $\bar{\mathcal{O}}_x$ is self-adjoint and the heat equation for $\bar{K}_{\mathcal{O}} \left(x, x^{'}; t \right)$, definining also $t^{''} = t - t'$,

$$
\begin{aligned}
&= \int_x \frac{\tilde{g}}{2\sqrt{\bar{g}}} \mathrm{tr} \bar{K}_{\mathcal{O}}(x, x; t) - \delta^{(d)}(0) \int_x \frac{\tilde{g}}{2\bar{g}} \mathrm{tr}\mathbb{I} - \int_x \sqrt{\bar{g}} \int_0^t dt^{''} \int_{x^{''}} \frac{\tilde{g}}{2\bar{g}} \delta^{(d)} \left(x^{''} - x \right) \mathrm{tr} \left[\partial_{t^{''}} \bar{K}_{\mathcal{O}} \left(x, x^{''}; t^{''} \right) \right] \\
&- \int_0^t dt^{'} \int_x \sqrt{\bar{g}} \int_{x^{''}} \sqrt{\bar{g}} \mathrm{tr} \left[\bar{K}_{\mathcal{O}} \left(x, x^{''}; t - t^{'} \right) \tilde{\mathcal{O}}_{x^{''}} \bar{K}_{\mathcal{O}} \left(x^{''}, x; t^{'} \right) \right] \, ,
\end{aligned}
$$

integrate in $t^{''}$

$$
\begin{aligned}
&= \underline{\int_x \frac{\tilde{g}}{2\sqrt{\bar{g}}} \mathrm{tr} \bar{K}_{\mathcal{O}}(x, x; t)} - \underline{\delta^{(d)}(0) \int_x \frac{\tilde{g}}{2\bar{g}} \mathrm{tr}\mathbb{I}} - \underline{\int_x \sqrt{\bar{g}} \mathrm{tr} \left[\int_{x^{''}} \frac{\tilde{g}}{2\bar{g}^{3/2}} \delta^{(d)} \left(x - x^{''} \right) \delta^{(d)} \left(x^{''} - x \right) \mathbb{I} \right]} - \underline{\int_x \sqrt{\bar{g}} \int_{x^{''}} \frac{\tilde{g}}{2\bar{g}} \delta^{(d)} \left(x^{''} - x \right) \mathrm{tr} \bar{K}_{\mathcal{O}} \left(\right.} \\
&- \int_0^t dt^{'} \int_x \sqrt{\bar{g}} \int_{x^{''}} \sqrt{\bar{g}} \mathrm{tr} \left[\bar{K}_{\mathcal{O}} \left(x, x^{''}; t - t^{'} \right) \tilde{\mathcal{O}}_{x^{''}} \bar{K}_{\mathcal{O}} \left(x^{''}, x; t^{'} \right) \right] \, ,
\end{aligned}
$$

use the trace ciclicity

$$
= - \int_0^t dt^{'} \int_x \sqrt{\bar{g}} \int_{x^{''}} \sqrt{\bar{g}} \mathrm{tr} \left[\left(\tilde{\mathcal{O}}_{x^{''}} \bar{K}_{\mathcal{O}} \left(x^{''}, x; t^{'} \right) \right) \bar{K}_{\mathcal{O}} \left(x, x^{''}; t - t^{'} \right) \right] \, ,
$$

bring $\bar{K}_{\mathcal{O}} \left(x, x^{''}; t - t^{'} \right)$ under the action of $\tilde{\mathcal{O}}_{x^{''}}$ [28] and use $\int_{x'} \sqrt{\bar{g}} \bar{K}_{\mathcal{O}} \left(x, x^{'}; t_1 \right) \bar{K}_{\mathcal{O}} \left(x^{'}, x^{''}; t_2 \right) = \bar{K}_{\mathcal{O}} \left(x, x^{''}; t_1 + t_2 \right)$ to finally arrive to

$$
= -t \int_x \sqrt{\bar{g}} \mathrm{tr} \left[\tilde{\mathcal{O}}_x \bar{K}_{\mathcal{O}} \left(x, x^{'}; t \right) \right]_{x = x^{'}} \, .
$$

We showed that

$$
\tilde{K}_{\mathcal{O}}(t) = -t \int_x \sqrt{\bar{g}} \mathrm{tr} \left[\tilde{\mathcal{O}}_x \bar{K}_{\mathcal{O}} \left(x, x^{'}; t \right) \right]_{x = x^{'}} \tag{12.15}
$$

$$
\tilde{\zeta}_{\mathcal{O}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \tilde{K}_{\mathcal{O}}(t) \, . \tag{12.16}
$$

[28] This step requires some care because we cannot *directly* make $\tilde{\mathcal{O}}_x$ act on $\bar{K}_{\mathcal{O}} \left(x, x^{'}; t - t' \right)$ because this depends on x. In the formula in the text, the operator acts first and then the coincident limit is taken. I do not see inconsistencies with the fact that the reverse order of operations should have be taken. One can easily show that the formula in the text is indeed a correct form of tracing, using the bra-ket notation for the operator and the heat kernel propagator.

12.3 Second correction

The equation to consider at the next perturbative order is

$$\left(\partial_t + \bar{\mathcal{O}}_x\right) \tilde{\tilde{K}}_{\mathcal{O}}\left(x, x^{'}; t\right) + \tilde{\mathcal{O}}_x \tilde{K}_{\mathcal{O}}\left(x, x^{'}; t\right) + \tilde{\tilde{\mathcal{O}}}_x \bar{K}_{\mathcal{O}}\left(x, x^{'}; t\right) = 0 \qquad \tilde{\tilde{K}}_{\mathcal{O}}\left(x, x^{'}; 0\right) = \frac{3\tilde{g}^2 - 4\bar{g}\tilde{\tilde{g}}}{8\bar{g}^{5/2}} \delta^{(d)}\left(x - x^{'}\right) \tag{12.17}$$

and it can be solved along the lines of the NLO [29]. Let us pose again

$$\tilde{\tilde{K}}_{\mathcal{O}}\left(x, x^{'}; t\right) \equiv \frac{3\tilde{g}^2 - 4\bar{g}\tilde{\tilde{g}}}{8\bar{g}^{5/2}} \delta^{(d)}\left(x - x^{'}\right) + \int_{x''} \sqrt{\bar{g}} \bar{K}_{\mathcal{O}}\left(x, x^{''}; t\right) C_{\mathcal{O}}\left(x^{''}, x^{'}; t\right) \qquad C_{\mathcal{O}}\left(x^{''}, x^{'}; 0\right) = 0 \tag{12.18}$$

where $C_{\mathcal{O}}$ is a new function to be determined. We eventually arrive to

$$\begin{aligned}\tilde{\tilde{K}}_{\mathcal{O}}\left(x, x^{'}; t\right) = {} & \frac{3\tilde{g}^2 - 4\bar{g}\tilde{\tilde{g}}}{8\bar{g}^{5/2}} \delta^{(d)}\left(x - x^{'}\right) \mathbb{I} - \int_0^t dt^{'} \int_{x''} \sqrt{\bar{g}} \bar{K}_{\mathcal{O}}\left(x, x^{''}; t - t'\right) \bar{\mathcal{O}}_{x''} \left[\frac{3\tilde{g}^2 - 4\bar{g}\tilde{\tilde{g}}}{8\bar{g}^{5/2}} \delta^{(d)}\left(x^{'} - x^{''}\right)\right] \\ & - \int_0^t dt^{'} \int_{x''} \sqrt{\bar{g}} \bar{K}_{\mathcal{O}}\left(x, x^{''}; t - t'\right) \tilde{\mathcal{O}}_{x''} \tilde{K}_{\mathcal{O}}\left(x^{''}, x^{'}; t^{'}\right) - \int_0^t dt^{'} \int_{x''} \sqrt{\bar{g}} \bar{K}_{\mathcal{O}}\left(x, x^{''}; t - t'\right) \tilde{\tilde{\mathcal{O}}}_{x''} \bar{K}_{\mathcal{O}}\left(x^{''}, x^{'}; t^{'}\right) .\end{aligned} \tag{12.19}$$

[29] The covariant perturbation expansion of the heat kernel up to n order is nicely presented in [24]. The main differences between the present notes and this standard literature are that the "potential" $\mathcal{O} - \bar{\mathcal{O}}$ is an infinite series in the expansion parameter (not just proportional to it) and the metric depends on the expansion parameter. The former says that at order n the equation has n terms and the latter implies that the initial condition is highly non-trivial. These are the reasons why we refrain from (even formally) solving the equation at perturbative order $(\theta_0)^{2n}$ with $n \geq 3$.

13 First correction to Seeley coefficients

13.1 General definitions

Let us consider two d-dimensional spaces $\bar{\mathcal{M}}$ and $\mathcal{M}$ with conformally equivalent metrics $\bar{g}_{ij}$ and $g_{ij} = e^{2\alpha\Omega(x)}\bar{g}_{ij}$ and without boundary. The determinant of the metric g is an expansion in small α

$$
\begin{aligned}
g &= e^{2\alpha d\Omega}\bar{g} \qquad (13.1)\\
&= \bar{g} + \alpha\tilde{g} + O\left(\alpha^2\right)\\
&= \bar{g}\left(1 + 2\alpha d\Omega + O\left(\alpha^2\right)\right)\\
\sqrt{g} &= \sqrt{\bar{g}}\left(1 + \alpha d\Omega + O\left(\alpha^2\right)\right) . \qquad (13.2)
\end{aligned}
$$

We compute the difference of Ricci scalars [30]

$$
\begin{aligned}
R &= e^{-2\alpha\Omega}\left[\bar{R} - \frac{2\alpha(d-1)}{\sqrt{\bar{g}}}\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\Omega\right) - \alpha^2(d-1)(d-2)\,\partial_i\Omega\partial_j\Omega\right] \qquad (13.3)\\
&= \bar{R} + \alpha\tilde{R} + O\left(\alpha^2\right)\\
&= \bar{R} - 2\left[\Omega\bar{R} + \frac{d-1}{\sqrt{\bar{g}}}\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\Omega\right)\right]\alpha + O\left(\alpha^2\right)\\
\sqrt{g}R &= \sqrt{\bar{g}}\bar{R} + \left[(d-2)\sqrt{\bar{g}}\Omega\bar{R} - 2(d-1)\,\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\Omega\right)\right]\alpha + O\left(\alpha^2\right) . \qquad (13.4)
\end{aligned}
$$

For spinors we need the vielbein and its inverse

$$
\begin{aligned}
e_i^a &= e^{-\alpha\Omega}\bar{e}_i^a\\
&= \bar{e}_i^a + \alpha\tilde{e}_i^a + O\left(\alpha^2\right)\\
&= \bar{e}_i^a\left(1 - \alpha\Omega + O\left(\alpha^2\right)\right)\\
E_a^i &= e^{\alpha\Omega}\bar{E}_a^i\\
&= \bar{E}_a^i + \alpha\tilde{E}_a^i + O\left(\alpha^2\right)\\
&= \bar{E}_a^i\left(1 + \alpha\Omega + O\left(\alpha^2\right)\right)
\end{aligned}
$$

and the (Riemann and spin) connection

$$
\begin{aligned}
\Gamma^i_{kl} &= \frac{g^{im}}{2}\left(g_{mk,l} + g_{ml,k} - g_{kl,m}\right) \qquad (13.5)\\
&= \bar{\Gamma}^i_{kl} + \alpha\tilde{\Gamma}^i_{kl}\\
&= \bar{\Gamma}^i_{kl} + \frac{\alpha\Omega'}{2}\left(\delta^i_k\partial_l\Omega + \delta^i_l\partial_k\Omega - \bar{g}_{kl}\bar{g}^{im}\partial_m\Omega\right)
\end{aligned}
$$

$$
\begin{aligned}
\xi_{abc} &= \delta_{ad}E^m_bE^n_c\left(\partial_me^d_n - \partial_ne^d_m\right) \qquad (13.6)\\
&= e^{\alpha\Omega}\left[\bar{\xi}_{abc} + \alpha\partial_i\Omega\left(\delta_{ab}\bar{E}^i_c - \delta_{ac}\bar{E}^i_b\right)\right]\\
&= \bar{\xi}_{abc} + \alpha\tilde{\xi}_{abc} + O\left(\alpha^2\right)\\
&= \bar{\xi}_{abc} + \alpha\left[\Omega\bar{\xi}_{abc} + \partial_i\Omega\left(\delta_{ab}\bar{E}^i_c - \delta_{ac}\bar{E}^i_b\right)\right] + O\left(\alpha^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\omega_{abc} &= \frac{1}{2}\left(\xi_{abc} + \xi_{bca} - \xi_{cab}\right) \qquad (13.7)\\
&= e^{\alpha\Omega}\left[\bar{\omega}_{abc} + \alpha\partial_i\Omega\left(\delta_{bc}\bar{E}^i_a - \delta_{ac}\bar{E}^i_b\right)\right]\\
&= \bar{\omega}_{abc} + \alpha\tilde{\omega}_{abc} + O\left(\alpha^2\right)\\
&= \bar{\omega}_{abc} + \alpha\left[\Omega\bar{\omega}_{abc} + \partial_i\Omega\left(\delta_{bc}\bar{E}^i_a - \delta_{ac}\bar{E}^i_b\right)\right] + O\left(\alpha^2\right) .
\end{aligned}
$$

We also recall the statement of the divergence theorem on a boundaryless space

$$
\int_x \partial_iV^i = 0\,. \qquad (13.8)
$$

[30] The Ricci scalar under a conformal transformation is (8.118) [8] and agrees with (4.51) [8] for small α.

In the next two sections we focus on the Seeley-de Witt coefficients a_0, a_2 for the d-dimensional scalar Laplacian and then for the Dirac operator on $\mathcal{M}$. Let us refer to them as $\mathcal{O}$. We will study the corrections $\tilde{a}_i$

$$a_i = \bar{a}_i + \alpha\tilde{a}_i + O\left(\alpha^2\right) \qquad\qquad i = 0,2 \tag{13.9}$$

in two independent ways:

- we write the a_i [25] and expand in α,
- we take our expression for the heat kernel of $\mathcal{O}$ (in terms of the heat kernel for $\bar{\mathcal{O}}$)

$$\begin{aligned} K_{\mathcal{O}}\left(x,x^{'};t\right) &= \bar{K}_{\mathcal{O}}\left(x,x^{'};t\right) + \alpha\tilde{K}_{\mathcal{O}}\left(x,x^{'};t\right) + O\left(\alpha^2\right) & (13.10)\\ &= \left(1-\alpha\frac{\tilde{g}}{2\bar{g}}\right)\bar{K}_{\mathcal{O}}\left(x,x^{'};t\right) + \alpha\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\bar{K}_{\mathcal{O}}\left(x,x^{''};t-t^{'}\right)\tilde{\mathcal{O}}_{x^{''}}\bar{K}_{\mathcal{O}}\left(x^{''},x^{'};t^{'}\right) + O\left(\alpha^2\right) \\ K_{\mathcal{O}}\left(t\right) &= \bar{K}_{\mathcal{O}}\left(t\right) + \alpha\tilde{K}_{\mathcal{O}}\left(t\right) + O\left(\alpha^2\right) & (13.11)\\ &= \int_x\sqrt{\bar{g}}\left[\bar{K}_{\mathcal{O}}\left(x,x;t\right) - \alpha t\left(\tilde{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x^{'};t\right)\right)_{x=x^{'}}\right] + O\left(\alpha^2\right) \end{aligned}$$

  and, given the $\bar{a}_i$ of $\bar{\mathcal{O}}$ in [25], we recover the same corrections $\tilde{a}_i$.

The former is a mere manipulation of literature results, whereas the latter shows the ability of our method to capture the correction to the heat kernel (so to the Seeley coefficients) using *only* on the unperturbed one.

The finite part of the determinant is not captured by the small-t expansion, as we would need the heat kernel for any finite $t > 0$. The information on the divergence of $\mathcal{O}$ in d dimensions is carried by the $d+1$ coefficients a_i with $i = 0, \ldots d$. Although the check above should be extended to all the infinitely-many a_i ($i = 0, 1, \ldots$), we chose to restrict to $i = 0, 2$ because

- their expression is particularly compact to handle,
- only a_0, a_2 are relevant for divergences on two-dimensional worldsheets without boundary.

13.2 Massive Laplacian: literature in Gilkey

Let us focus on the Laplacian with an arbitrary non-constant mass term

$$\begin{aligned} \mathcal{O} &= -\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right) + X & (13.12)\\ &= -\frac{e^{-\alpha d\Omega}}{\sqrt{\bar{g}}}\partial_i\left(e^{\alpha(d-2)\Omega}\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\right) + X\,. \end{aligned}$$

The small-t expansion of the heat kernel of the operator $\mathcal{O}$ on $\mathcal{M}$

$$K_{\mathcal{O}}\left(x,x;t\right) = \frac{1}{\left(4\pi\right)^{d/2}}\sum_{k=0,2,\ldots} t^{(k-d)/2}b_{k/2} \tag{13.13}$$

$$K_{\mathcal{O}}\left(t\right) = \int_x\sqrt{g}K_{\mathcal{O}}\left(x,x;t\right) = \sum_{k=0,2,\ldots} t^{(k-d)/2}a_k \tag{13.14}$$

is given by the coefficients [31]

$$\begin{aligned} b_0 &= 1 & (13.15)\\ b_1 &= \frac{1}{6}R - X & (13.16)\\ &\ldots \\ a_0 &= \frac{1}{\left(4\pi\right)^{d/2}}\int_x\sqrt{g} & (13.17)\\ a_2 &= \frac{1}{\left(4\pi\right)^{d/2}}\int_x\sqrt{g}\left(\frac{1}{6}R - X\right) & (13.18)\\ &\ldots \end{aligned}$$

[31] See theorem Theorem 4.5.1 and 4.5.2 [25]. We try to collect the relevant notation in the book: the small-t expansion is (4.5.4), τ is the Ricci scalar (4.0.3), L_{aa} is the (covariant) trace of the second fundamental form (4.0.5), E is (minus) the Laplacian mass term (4.1.13), I_V is our identity matrix in the internal space $\mathbb{I}_n$. For boundaryless manifolds and massless scalar, it is (5.74) [8] with renamed UV cutoff $\delta = \Lambda^{-2}$ and with a_2 and a_0 given by (4.28).

The determinant inherits the divergencies from the first $d+1$ Seeley coefficients [32]

$$(\log \mathrm{Det}\mathcal{O})_\infty = -\int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} \left[K_{\mathcal{O}}(t)\right]_{\text{non-positive powers of } t} \tag{13.19}$$
$$= \sum_{k=0,2,\ldots d-2} \frac{2a_k}{k-d} \Lambda^{d-k} - a_d \log \Lambda^2 \,.$$

We think of the expressions above as series in small α, for the mass

$$X = \bar{X} + \alpha \tilde{X} + O\left(\alpha^2\right) \,, \tag{13.20}$$

the operator

$$\mathcal{O} = \bar{\mathcal{O}} + \alpha \tilde{\mathcal{O}} + O\left(\alpha^2\right) \tag{13.21}$$
$$= \left(-\frac{1}{\sqrt{\bar{g}}}\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\right) + \bar{X}\right) + \left[-(d-2)\,\bar{g}^{ij}\partial_i\Omega\partial_j + \frac{2\Omega}{\sqrt{\bar{g}}}\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\right) + \tilde{X}\right]\alpha + O\left(\alpha^2\right)$$

and for the Seeley-de Witt coefficients

$$\bar{b}_0 = 1 \tag{13.22}$$
$$\bar{b}_1 = \frac{1}{6}\bar{R} - \bar{X} \tag{13.23}$$
$$\ldots$$
$$\tilde{b}_0 = 0 \tag{13.24}$$
$$\tilde{b}_1 = \frac{1}{6}\tilde{R} - \tilde{X} = -\frac{1}{3}\Omega\bar{R} - \frac{d-1}{3\sqrt{\bar{g}}}\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\Omega\right) - \tilde{X} \tag{13.25}$$
$$\ldots$$

$$\bar{a}_0 = \frac{1}{(4\pi)^{d/2}}\int_x \sqrt{\bar{g}} \tag{13.26}$$
$$\bar{a}_2 = \frac{1}{(4\pi)^{d/2}}\int_x \sqrt{\bar{g}}\left(\frac{1}{6}\bar{R} - \bar{X}\right) \tag{13.27}$$
$$\ldots$$
$$\tilde{a}_0 = \frac{d}{(4\pi)^{d/2}}\int_x \sqrt{\bar{g}}\Omega \tag{13.28}$$
$$\tilde{a}_2 = \frac{1}{(4\pi)^{d/2}}\int_x \sqrt{\bar{g}}\left(\frac{d-2}{6}\Omega\bar{R} - d\Omega\bar{X} - \tilde{X}\right) \tag{13.29}$$
$$\ldots$$

[32] To single out the divergencies we cannot employ the usual analytic continuation of the zeta-function in s, but the definition in Section 5.8 [8].

13.3 Massive Laplacian: our formula

The perturbative heat kernel yields for any operator yields

$$
\begin{aligned}
K_{\mathcal{O}}(t) &= \int_x \sqrt{\bar{g}}\bar{K}_{\mathcal{O}}(x,x;t) - \alpha t \int_x \sqrt{\bar{g}}\left[\tilde{\mathcal{O}}_x \bar{K}_{\mathcal{O}}\left(x,x';t\right)\right]_{x=x'} + O\left(\alpha^2\right) \\
&= \bar{K}_{\mathcal{O}}(t) + \alpha t \int_x \left\{\left[(d-2)\sqrt{\bar{g}}\bar{g}^{ij}\partial_i\Omega\partial_j - 2\Omega\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\right) - \sqrt{\bar{g}}\tilde{X}\right]\bar{K}_{\mathcal{O}}\left(x,x';t\right)\right\}_{x=x'} + O\left(\alpha^2\right)
\end{aligned}
\tag{13.30}
$$

integrate first term by parts

$$
\begin{aligned}
&= \bar{K}_{\mathcal{O}}(t) + \alpha t \int_x \Big[(d-2)\partial_j\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_i\Omega\bar{K}_{\mathcal{O}}\left(x,x';t\right)\right) - (d-2)\partial_j\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_i\Omega\right)\bar{K}_{\mathcal{O}}\left(x,x';t\right) \\
&\qquad -2\Omega\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\bar{K}_{\mathcal{O}}\left(x,x';t\right)\right) - \sqrt{\bar{g}}\tilde{X}\bar{K}_{\mathcal{O}}\left(x,x';t\right)\Big]_{x=x'} + O\left(\alpha^2\right)
\end{aligned}
$$

use heat equation for $\bar{\mathcal{O}}$ and divergence theorem

$$
\begin{aligned}
&= \bar{K}_{\mathcal{O}}(t) + \alpha t\left[-(d-2)\int_x \partial_j\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_i\Omega\right)\bar{K}_{\mathcal{O}}(x,x;t) - \int_x \sqrt{\bar{g}}\left(2\Omega\bar{X} + \tilde{X}\right)\bar{K}_{\mathcal{O}}(x,x;t) - 2\int_x \sqrt{\bar{g}}\Omega\partial_t\bar{K}_{\mathcal{O}}(x,x;t)\right] \\
&\qquad + O\left(\alpha^2\right)
\end{aligned}
$$

plug time-expansion for $\bar{K}_{\mathcal{O}}(x,x;t)$ and use divergence theorem

$$
= \bar{K}_{\mathcal{O}}(t) + \frac{\alpha}{(4\pi)^{d/2}}\left[dt^{-d/2}\int_x \sqrt{\bar{g}}\Omega + t^{(2-d)/2}\int_x \sqrt{\bar{g}}\left(\frac{d-2}{6}\Omega\bar{R} - d\Omega\bar{X} - \tilde{X}\right) + O\left(t^{(3-d)/2}\right)\right] + O\left(\alpha^2\right) .
$$

From here we extract the coefficients

$$
\tilde{a}_0 = \frac{d}{(4\pi)^{d/2}}\int_{\mathcal{M}} \sqrt{\bar{g}}\Omega \tag{13.31}
$$

$$
\tilde{a}_2 = \frac{1}{(4\pi)^{d/2}}\int_{\mathcal{M}} \sqrt{\bar{g}}\left(\frac{d-2}{6}\Omega\bar{R} - d\Omega\bar{X} - \tilde{X}\right) \tag{13.32}
$$

that match the ones in literature at the end of the previous section.

13.4 Massive Dirac operator: literature in Gilkey

Let us focus on the Dirac operator with an arbitrary non-constant hermitian mass term

$$
\begin{aligned}
\mathcal{O} &= -i\Gamma^a E_a^i\left(\partial_i + \frac{1}{4}\omega^{bc}{}_d e_i^d\Gamma_{bc}\right) + X \\
&= e^{\alpha\Omega}\left[-i\Gamma^a\bar{E}_a^i\left(\partial_i + \frac{1}{4}\bar{\omega}^{bc}{}_d\bar{e}_i^d\Gamma_{bc}\right)\right] + i\alpha(d-1)e^{\alpha\Omega}\partial_i\Omega\bar{E}^{ia}\Gamma_a + X .
\end{aligned}
\tag{13.33}
$$

The small-t expansion of the heat kernel of the operator $\mathcal{O}^2$ on $\mathcal{M}$

$$
K_{\mathcal{O}^2}(x,x;t) = \frac{1}{(4\pi)^{d/2}}\sum_{k=0,2,\ldots} t^{(k-d)/2} b_{k/2} \tag{13.34}
$$

$$
K_{\mathcal{O}^2}(t) = \int_x \sqrt{g}\,\mathrm{tr}\left[K_{\mathcal{O}^2}(x,x;t)\right] = \sum_{k=0,2,\ldots} t^{(k-d)/2} a_k \tag{13.35}
$$

is given by[33]

$$b_0 = \mathbb{I} \tag{13.36}$$

$$b_1 = \left(-\frac{R}{12}\mathbb{I} + \frac{d-2}{2}X^2 + \frac{1}{2}\Gamma^a X \Gamma_a X\right) \tag{13.37}$$

$$\dots$$

$$a_0 = \frac{1}{(4\pi)^{d/2}} \int_x \sqrt{g}\, \mathrm{tr}\mathbb{I} \tag{13.38}$$

$$a_2 = \frac{1}{(4\pi)^{d/2}} \int_x \sqrt{g}\, \mathrm{tr}\left(-\frac{R}{12}\mathbb{I} + \left(\frac{d}{2}-1\right)X^2 + \frac{1}{2}\Gamma^a X \Gamma_a X\right) \tag{13.39}$$

$$\dots$$

We think of the expressions above as series in small α, for the mass

$$X = \bar{X} + \alpha \tilde{X} + O\left(\alpha^2\right) , \tag{13.40}$$

the operator

$$\begin{aligned} \mathcal{O} &= \bar{\mathcal{O}} + \alpha \tilde{\mathcal{O}} + O\left(\alpha^2\right) \\ &= \left[-i\Gamma^a \bar{E}^i_a \left(\partial_i + \frac{1}{4}\bar{\omega}^{bc}{}_d \bar{e}^d_i \Gamma_{bc}\right) + \bar{X}\right] + \left[-i\Omega\Gamma^a \bar{E}^i_a \left(\partial_i + \frac{1}{4}\bar{\omega}^{bc}{}_d \bar{e}^d_i \Gamma_{bc}\right) + i\left(d-1\right)\partial_i \Omega \bar{E}^i_a \Gamma^a + \tilde{X}\right]\alpha + O\left(\alpha^2\right) \end{aligned} \tag{13.41}$$

and for the Seeley-de Witt coefficients

$$\bar{b}_0 = \mathrm{tr}\mathbb{I} \tag{13.42}$$

$$\bar{b}_1 = -\frac{\bar{R}}{12}\mathbb{I} + \frac{d-2}{2}\bar{X}^2 + \frac{1}{2}\Gamma^a \bar{X}\Gamma_a \bar{X} \tag{13.43}$$

$$\dots$$

$$\tilde{b}_0 = 0 \tag{13.44}$$

$$\begin{aligned} \tilde{b}_1 &= -\frac{\tilde{R}}{12}\mathbb{I} + \frac{d-2}{2}\left\{\bar{X}, \tilde{X}\right\} + \frac{1}{2}\left\{\Gamma^a \bar{X}, \Gamma_a \tilde{X}\right\} \\ &= \left(\frac{1}{6}\Omega \bar{R} + \frac{d-1}{6\sqrt{\bar{g}}}\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j \Omega\right)\right)\mathbb{I} + \frac{d-2}{2}\left\{\bar{X}, \tilde{X}\right\} + \frac{1}{2}\left\{\Gamma^a \bar{X}, \Gamma_a \tilde{X}\right\} \end{aligned} \tag{13.45}$$

$$\dots$$

$$\bar{a}_0 = \frac{1}{(4\pi)^{d/2}} \int_x \sqrt{\bar{g}}\, \mathrm{tr}\mathbb{I} \tag{13.46}$$

$$\bar{a}_1 = \frac{1}{(4\pi)^{d/2}} \int_x \sqrt{\bar{g}}\, \mathrm{tr}\left[-\frac{\bar{R}}{12}\mathbb{I} + \frac{d-2}{2}\bar{X}^2 + \frac{1}{2}\Gamma^a \bar{X}\Gamma_a \bar{X}\right] \tag{13.47}$$

$$\dots$$

$$\tilde{a}_0 = \frac{d}{(4\pi)^{d/2}} \int_x \sqrt{\bar{g}}\, \Omega \mathrm{tr}\mathbb{I} \tag{13.48}$$

$$\tilde{a}_1 = \frac{1}{(4\pi)^{d/2}} \int_x \sqrt{\bar{g}}\, \mathrm{tr}\left[-\frac{d-2}{12}\Omega \bar{R}\mathbb{I} + (d-2)\bar{X}\tilde{X} + \frac{d-2}{2}d\Omega \bar{X}^2 + \Gamma^a \bar{X}\Gamma_a \tilde{X} + \frac{d\Omega}{2}\Gamma^a \bar{X}\Gamma_a \bar{X}\right] \tag{13.49}$$

$$\dots$$

[33] See theorem Theorem 4.4.1 [25]: the small-t expansion is (4.4.3), τ is the Ricci scalar (4.0.3), Ψ is the combination (4.4.12) from (4.4.9), 1 is our identity matrix in the internal space $\mathbb{I}_n$.

13.5 Massive Dirac operator: our formula

We calculate the square of the reference operator

$$\bar{\mathcal{O}}^2 = \left[-i\Gamma^a \bar{E}_a^i\left(\partial_i + \frac{1}{4}\bar{\omega}^{bc}{}_d \bar{e}_i^d \Gamma_{bc}\right) + \bar{X}\right]\left[-i\Gamma^a \bar{E}_a^i\left(\partial_i + \frac{1}{4}\bar{\omega}^{bc}{}_d \bar{e}_i^d \Gamma_{bc}\right) + \bar{X}\right] \tag{13.50}$$

$$= \left(-i\bar{\nabla\!\!\!/}\right)^2 - i\bar{E}_a^i\Gamma^a\partial_i\bar{X} - i\bar{E}_a^i\left\{\Gamma^a, \bar{X}\right\}\partial_i - \frac{i}{4}\bar{\omega}^{bc}{}_a\left\{\Gamma^a\Gamma_{bc}, \bar{X}\right\} + \bar{X}^2$$

$$2\Omega\sqrt{\bar{g}}\bar{\mathcal{O}}^2 = 2\sqrt{\bar{g}}\Omega\left(-i\bar{\nabla\!\!\!/}\right)^2 - 2i\sqrt{\bar{g}}\Omega\bar{E}_a^i\Gamma^a\partial_i\bar{X} - 2i\sqrt{\bar{g}}\Omega\bar{E}_a^i\left\{\Gamma^a, \bar{X}\right\}\partial_i - \frac{i}{2}\sqrt{\bar{g}}\Omega\bar{\omega}^{bc}{}_a\left\{\Gamma^a\Gamma_{bc}, \bar{X}\right\} + 2\sqrt{\bar{g}}\Omega\bar{X}^2 \tag{13.51}$$

then the anticommutator on a test function $f(x)$ and feel free to integrate by parts

$$\begin{aligned}
\sqrt{\bar{g}}\left\{\bar{\mathcal{O}}, \tilde{\mathcal{O}}\right\}f &= 2\sqrt{\bar{g}}\Omega\left(-i\bar{\nabla\!\!\!/}\right)^2 f + \partial_j\left(\sqrt{\bar{g}}\partial_i\Omega\bar{E}_a^i\bar{E}_b^j\right)\Gamma^a\Gamma^b f - \frac{1}{4}\sqrt{\bar{g}}\partial_i\Omega\bar{\omega}^{bc}{}_d\bar{E}_a^i\Gamma^a\Gamma^d\Gamma_{bc}f \\
&- (d-1)\,\partial_i\left(\sqrt{\bar{g}}\bar{E}_a^i\right)\left(\partial_j\Omega\bar{E}_b^j\Gamma^a\Gamma^b f + \Gamma^a\tilde{X}f\right) + \frac{d-1}{4}\sqrt{\bar{g}}\partial_i\Omega\bar{\omega}^{bc}{}_a\bar{E}_d^i\left\{\Gamma^a\Gamma_{bc}, \Gamma^d\right\}f - \frac{i}{4}\sqrt{\bar{g}}\bar{\omega}^{bc}{}_a\left\{\Gamma^a\Gamma_{bc}, \tilde{X}\right\}f \\
&+ i\partial_i\left(\sqrt{\bar{g}}\Omega\bar{E}_a^i\bar{X}\right)\Gamma^a f - \frac{i}{4}\sqrt{\bar{g}}\Omega\bar{\omega}^{bc}{}_a\left\{\Gamma^a\Gamma_{bc}, \bar{X}\right\}f + i\,(d-1)\sqrt{\bar{g}}\partial_i\Omega\bar{E}_a^i\left\{\bar{X}, \Gamma^a\right\}f + \sqrt{\bar{g}}\left\{\bar{X}, \tilde{X}\right\}f \\
&+ i\partial_i\left(\sqrt{\bar{g}}\Omega\bar{E}_a^i\right)\Gamma^a\bar{X}f - (d-1)\,\partial_i\left(\sqrt{\bar{g}}\partial_j\Omega\bar{E}_b^j\bar{E}_a^i\right)\Gamma^b\Gamma^a f + i\partial_i\left(\sqrt{\bar{g}}\bar{E}_a^i\tilde{X}\right)\Gamma^a f \\
&= 2\sqrt{\bar{g}}\Omega\bar{\mathcal{O}}^2 f + 2i\sqrt{\bar{g}}\Omega\bar{E}_a^i\Gamma^a\partial_i\bar{X}f - 2i\partial_i\left(\sqrt{\bar{g}}\Omega\bar{E}_a^i\Gamma^a\bar{X}\right)f - 2\sqrt{\bar{g}}\Omega\bar{X}^2 f \\
&+ \partial_j\left(\sqrt{\bar{g}}\partial_i\Omega\bar{E}_a^i\bar{E}_b^j\right)\Gamma^a\Gamma^b f - \frac{1}{4}\sqrt{\bar{g}}\partial_i\Omega\bar{\omega}^{bc}{}_d\bar{E}_a^i\Gamma^a\Gamma^d\Gamma_{bc}f \\
&- (d-1)\,\partial_i\left(\sqrt{\bar{g}}\bar{E}_a^i\right)\left(\partial_j\Omega\bar{E}_b^j\Gamma^a\Gamma^b f + \Gamma^a\tilde{X}f\right) + \frac{d-1}{4}\sqrt{\bar{g}}\partial_i\Omega\bar{\omega}^{bc}{}_a\bar{E}_d^i\left\{\Gamma^a\Gamma_{bc}, \Gamma^d\right\}f - \frac{i}{4}\sqrt{\bar{g}}\bar{\omega}^{bc}{}_a\left\{\Gamma^a\Gamma_{bc}, \tilde{X}\right\}f \\
&- i\partial_i\left(\sqrt{\bar{g}}\Omega\bar{E}_a^i\bar{X}\right)\Gamma^a f + \frac{i}{4}\sqrt{\bar{g}}\Omega\bar{\omega}^{bc}{}_a\left\{\Gamma^a\Gamma_{bc}, \bar{X}\right\}f + i\,(d-1)\sqrt{\bar{g}}\partial_i\Omega\bar{E}_a^i\left\{\Gamma^a, \bar{X}\right\}f + \sqrt{\bar{g}}\left\{\bar{X}, \tilde{X}\right\}f \\
&+ i\partial_i\left(\sqrt{\bar{g}}\Omega\bar{E}_a^i\right)\Gamma^a\bar{X}f - (d-1)\,\partial_i\left(\sqrt{\bar{g}}\partial_j\Omega\bar{E}_b^j\bar{E}_a^i\right)\Gamma^b\Gamma^a f + i\partial_i\left(\sqrt{\bar{g}}\bar{E}_a^i\tilde{X}\right)\Gamma^a f
\end{aligned}$$

We are ready to compute

$K_{\mathcal{O}^2}(t) =$ use heat equation for $\bar{\mathcal{O}}^2$

$$\bar{K}_{\mathcal{O}^2}(t) - \frac{\alpha t}{(4\pi)^{d/2}}\int_x \mathrm{tr}\,[...$$

$$...]\,\bar{K}_{\mathcal{O}^2}(x, x; t) + O\left(\alpha^2\right)$$

plug time-expansion for $\bar{K}_{\mathcal{O}^2}(x, x; t)$

$$= \bar{K}_{\mathcal{O}^2}(t) - \alpha\int_x \sqrt{\bar{g}}\Omega\,\mathrm{tr}\left[dt^{-d/2} + (2-d)\,t^{(2-d)/2}\left(-\frac{\bar{R}}{12} + \frac{d-2}{2}\bar{X}^2 + \frac{1}{2}\Gamma^a\bar{X}\Gamma_a\bar{X}\right) + O\left(t^{(4-d)/2}\right)\right]$$

$$- \alpha\int_x \mathrm{tr}\,[...$$

$$...]\left(t^{(2-d)/2} + O\left(t^{(4-d)/2}\right)\right) + O\left(\alpha^2\right)$$

use divergence theorem and trace ciclicity

$$= \bar{K}_{\mathcal{O}^2}(t) - \alpha\int_x \sqrt{\bar{g}}\Omega\,\mathrm{tr}\,[...]$$

$$...] + O\left(\alpha^2\right)$$

The trace identies were

$$\mathrm{tr}\left(\Gamma^a\Gamma^b\right) = 2\delta^{ab} \tag{13.52}$$

$$\mathrm{tr}\left(\Gamma^a\Gamma^b\Gamma^c\Gamma^d\right) = d\left(\delta^{ab}\delta^{cd} - \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right) \tag{13.53}$$

14 Latitude heat kernel

14.1 Leading order

For the scalars we have

$$\bar{K}_{-\Delta+m^2}(t)=\frac{V_{H^2}}{2\pi}\int_0^\infty dv\,v\tanh(\pi v)\,e^{-t\left(v^2+\frac{1}{4}+m^2\right)} \tag{14.1}$$

$$\begin{aligned}\bar{\zeta}_{-\Delta+m^2}(s)&=\frac{V_{H^2}}{2\pi}\int_0^\infty dv\frac{v\tanh\pi v}{\left(v^2+m^2+\frac{1}{4}\right)^s} \\ &=\frac{V_{H^2}}{2\pi}\int_0^\infty dv\left[\frac{v}{\left(v^2+m^2+\frac{1}{4}\right)^s}-\frac{2v}{\left(e^{2\pi v}+1\right)\left(v^2+m^2+\frac{1}{4}\right)^s}\right] \\ &=\frac{V_{H^2}}{2\pi}\left[\frac{\left(m^2+\frac{1}{4}\right)^{1-s}}{2(s-1)}-2\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+m^2+\frac{1}{4}\right)^s}\right]\end{aligned} \tag{14.2}$$

$$\begin{aligned}\bar{\zeta}'_{-\Delta+m^2}(0)&=\frac{V_{H^2}}{2\pi}\left\{\frac{m^2+\frac{1}{4}}{2}\left[\log\left(m^2+\frac{1}{4}\right)-1\right]+2\int_0^\infty dv\frac{v\log\left(v^2+m^2+\frac{1}{4}\right)}{e^{2\pi v}+1}\right\} \\ &=\frac{V_{H^2}}{2\pi}\left[\frac{1+\log 2}{12}-\log A+\int_0^{m^2+1/4}dx\psi\left(\sqrt{x}+\frac{1}{2}\right)\right]\end{aligned} \tag{14.3}$$

while for each spinor

$$\bar{K}_{-\nabla\!\!\!/^2+m^2}(t)=\frac{V_{H^2}}{\pi}\int_0^\infty dv\,v\coth(\pi v)\,e^{-t\left(v^2+m^2\right)} \tag{14.4}$$

$$\begin{aligned}\bar{\zeta}_{-\nabla\!\!\!/^2+m^2}(s)&=\frac{V_{H^2}}{\pi}\int_0^\infty dv\frac{v\coth\pi v}{\left(v^2+m^2\right)^s} \\ &=\frac{V_{H^2}}{\pi}\int_0^\infty dv\left[\frac{v}{\left(v^2+m^2\right)^s}+\frac{2v}{\left(e^{2\pi v}-1\right)\left(v^2+m^2\right)^s}\right] \\ &=\frac{V_{H^2}}{\pi}\left[\frac{\left(m^2\right)^{1-s}}{2(s-1)}+2\int_0^\infty dv\frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+m^2\right)^s}\right]\end{aligned} \tag{14.5}$$

$$\begin{aligned}\bar{\zeta}'_{-\nabla\!\!\!/^2+m^2}(0)&=\frac{V_{H^2}}{\pi}\left[\frac{m^2}{2}\left(\log m^2-1\right)-2\int_0^\infty dv\frac{v\log\left(v^2+m^2\right)}{e^{2\pi v}-1}\right] \\ &=\frac{V_{H^2}}{\pi}\left[-\frac{1}{6}+2\log A+\sqrt{m^2}+\int_0^{m^2}dx\psi\left(\sqrt{x}\right)\right]\end{aligned} \tag{14.6}$$

where $A\approx 1.28243$ is the Glaisher constant and $\psi(x)\equiv\frac{d}{dx}\log\Gamma(x)$ is the digamma function. The total zeta function is additive:

$$\begin{aligned}\bar{\Gamma}_1&=\frac{d}{ds}\left(-\frac{3}{2}\bar{\zeta}_{\mathcal{O}_1}(s)-\frac{3}{2}\bar{\zeta}_{\mathcal{O}_2}(s)-\frac{1}{2}\bar{\zeta}_{\mathcal{O}_{3+}}(s)-\frac{1}{2}\bar{\zeta}_{\mathcal{O}_{3-}}(s)+\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\bar{\zeta}_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(s)\right)\Bigg|_{s=0} \\ &=-\frac{3}{2}\left(-\frac{25}{12}+\frac{3\log 2}{2}-2\log A+\frac{3\log\pi}{2}\right)-\frac{5}{2}\left(-\frac{1}{12}+\frac{\log 2}{2}-2\log A+\frac{\log\pi}{2}\right)+\frac{8}{4}\left(-\frac{5}{3}+2\log 2-4\log A+2\log\pi\right) \\ &=\frac{1}{2}\log 2\pi.\end{aligned} \tag{14.7}$$

We can also check that the small-t asymptotics of the traced heat kernel contains the known Seeley coefficients

$$\begin{aligned}\bar{K}_{-\Delta+m^2}(t) &= \frac{V_{H^2}}{2\pi}\int_0^\infty dv\, v\tanh(\pi v)\, e^{-t\left(v^2+\frac{1}{4}+m^2\right)}\\ &= \frac{V_{H^2}}{2\pi}\int_0^\infty dv\left(v-\frac{2v}{e^{2\pi v}+1}\right)e^{-t\left(v^2+\frac{1}{4}+m^2\right)}\\ &= \frac{V_{H^2}}{2\pi}\left[\frac{e^{-\left(\frac{1}{4}+m^2\right)t}}{2t}-\int_0^\infty dv\,\frac{2v}{e^{2\pi v}+1}e^{-t\left(v^2+\frac{1}{4}+m^2\right)}\right]\\ &= \frac{V_{H^2}}{2\pi}\left[\left(\frac{1}{2t}-\left(\frac{1}{8}+\frac{m^2}{2}\right)+O(t)\right)-\int_0^\infty dv\,\frac{2v}{e^{2\pi v}+1}\left(1+O(t)\right)\right]\\ &= \frac{V_{H^2}}{4\pi}\left[\frac{1}{t}+\left(\frac{-2}{6}-m^2\right)+O(t)\right]\end{aligned} \tag{14.8}$$

$$\begin{aligned}\bar{K}_{-\nabla\!\!\!/^2+m^2}(t) &= \frac{V_{H^2}}{\pi}\int_0^\infty dv\, v\coth(\pi v)\, e^{-t\left(v^2+m^2\right)}\\ &= \frac{V_{H^2}}{\pi}\int_0^\infty dv\left(v+\frac{2v}{e^{2\pi v}-1}\right)e^{-t\left(v^2+m^2\right)}\\ &= \frac{V_{H^2}}{\pi}\left[\frac{e^{-m^2t}}{2t}+\int_0^\infty dv\frac{2v}{e^{2\pi v}-1}e^{-t\left(v^2+m^2\right)}\right]\\ &= \frac{V_{H^2}}{\pi}\left[\left(\frac{1}{2t}-\frac{m^2}{2}+O(t)\right)+\int_0^\infty dv\frac{2v}{e^{2\pi v}-1}\left(1+O(t)\right)\right]\\ &= \frac{V_{H^2}}{4\pi}\left[\frac{2}{t}+\left(\frac{1}{3}-2m^2\right)+O(t)\right]\\ &= \frac{V_{H^2}}{4\pi}\left[\frac{\mathrm{tr}\left(\mathbb{I}_2\right)}{t}+\mathrm{tr}\left(-\frac{-2\mathbb{I}_2}{12}+\frac{\Gamma^a\left(m\Gamma^3\right)\Gamma^a\left(m\Gamma^3\right)}{2}\right)+O(t)\right].\end{aligned} \tag{14.9}$$

14.2 First correction

Upon the change of coordinates $\sigma\to\rho$ with $\sinh\sigma\sinh\rho=1$,

$$g_{ij}=\begin{pmatrix}\left(\frac{d\sigma}{d\rho}\right)^2\Omega^2(\sigma) & 0\\ 0 & \Omega^2(\sigma)\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & \sinh^2\rho\end{pmatrix}+\theta_0^2\begin{pmatrix}\frac{1}{(1+\cosh\rho)^2} & 0\\ 0 & \frac{\cosh\rho-1}{\cosh\rho+1}\end{pmatrix}+\dots \tag{14.10}$$

$$\bar{g}=\sinh^2\rho \qquad \tilde{g}=2\left(\frac{1-\cosh\rho}{\sinh\rho}\right)^2=2\left(\frac{\sinh\rho}{1+\cosh\rho}\right)^2.$$

We use the observation that the coincident heat kernel $\bar{K}(x,x;t)$ is independent of the manifold position x on homogeneous spaces. Each individual zeta-function is not well-defined for $s=0$, but the divergences at $s=0$ are renormalized by the analytic continuation in s using the gamma function.

$$\frac{3}{2}\tilde{K}_{\mathcal{O}_1}(t) = -\frac{3t}{2}\left(\int_x \frac{\sinh\rho}{(1+\cosh\rho)^2}\right)\left[(\Delta_x - 2)\bar{K}_{-\Delta+2}\left(x,x^{'};t\right)\right]_{x=x'} \tag{14.11}$$
$$= \frac{3t}{4}\int_0^\infty dv\, v\tanh(\pi v)\left(v^2+\frac{9}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}$$

$$\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_1}(s) = \frac{3s}{4}\int_0^\infty dv \frac{v\tanh(\pi v)}{\left(v^2+\frac{9}{4}\right)^s} \tag{14.12}$$
$$= \frac{3s}{4}\int_0^\infty dv \frac{1}{\left(v^2+\frac{9}{4}\right)^s}\left(v - \frac{2v}{e^{2\pi v}+1}\right)$$
$$= \frac{3s}{8(s-1)}\left(\frac{9}{4}\right)^{1-s} - \frac{3s}{2}\int_0^\infty dv \frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^s}$$

$$\frac{3}{2}\tilde{\zeta}'_{\mathcal{O}_1}(0) = \frac{d}{ds}\left[\frac{3s}{8(s-1)}\left(\frac{9}{4}\right)^{1-s}\right]_{s=0} - \frac{3}{2}\frac{1}{48} \tag{14.13}$$
$$= -\frac{7}{8}$$

$$\frac{3}{2}\tilde{K}_{\mathcal{O}_2}(t) = -\frac{3t}{2}\left(\int_x \frac{\sinh\rho}{(1+\cosh\rho)^2}\right)\left[(\Delta_x - 2)\bar{K}_{-\Delta}\left(x,x^{'};t\right)\right]_{x=x'} \tag{14.14}$$
$$= \frac{3t}{4}\int_0^\infty dv\, v\tanh(\pi v)\left(v^2+\frac{9}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}$$

$$\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_2}(s) = \frac{3s}{4}\int_0^\infty dv \frac{v^2+\frac{9}{4}}{\left(v^2+\frac{1}{4}\right)^{s+1}} v\tanh(\pi v) \tag{14.15}$$
$$= \int_0^\infty dv \frac{1}{\left(v^2+\frac{1}{4}\right)^s}\left[\frac{3s}{4} + \frac{3s}{2\left(v^2+\frac{1}{4}\right)}\right]\left(v - \frac{2v}{e^{2\pi v}+1}\right)$$
$$= \frac{3s}{8(s-1)}\left(\frac{1}{4}\right)^{1-s} + \frac{3}{4}\left(\frac{1}{4}\right)^{-s}$$
$$- \frac{3s}{2}\int_0^\infty dv \frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^s} - 3s\int_0^\infty dv \frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^{s+1}}$$

$$\frac{3}{2}\tilde{\zeta}'_{\mathcal{O}_2}(0) = \frac{d}{ds}\left[\frac{3s}{8(s-1)}\left(\frac{1}{4}\right)^{1-s} + \frac{3}{4}\left(\frac{1}{4}\right)^{-s}\right]_{s=0} - \frac{3}{2}\frac{1}{48} - 3\left(\frac{\log 2}{2} - \frac{\gamma}{2}\right) \tag{14.16}$$
$$= -\frac{1}{8} + \frac{3\gamma}{2}$$

$$\begin{aligned}
\frac{1}{2}\tilde{K}_{\mathcal{O}_{3+}}(t)+\frac{1}{2}\tilde{K}_{\mathcal{O}_{3-}}(t) &= -t\int_x \frac{\sinh\rho}{(1+\cosh\rho)^2}\left[\left(\Delta_x-2\left(\frac{1-\cosh\rho}{\sinh\rho}\right)^2\right)\bar{K}_{-\Delta}\left(x,x^{'};t\right)\right]_{x=x'} \\
&= \frac{t}{2}\int_0^\infty dv\, v\tanh(\pi v)\left(v^2+\frac{5}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}
\end{aligned} \tag{14.17}$$

$$\begin{aligned}
\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3+}}(s)+\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3-}}(s) &= \frac{s}{2}\int_0^\infty dv\frac{v^2+\frac{5}{4}}{\left(v^2+\frac{1}{4}\right)^{s+1}}v\tanh(\pi v) \\
&= \int_0^\infty dv\frac{1}{\left(v^2+\frac{1}{4}\right)^s}\left[\frac{s}{2}+\frac{s}{2\left(v^2+\frac{1}{4}\right)}\right]\left(v-\frac{2v}{e^{2\pi v}+1}\right) \\
&= \frac{s}{4(s-1)}\left(\frac{1}{4}\right)^{1-s}+\frac{1}{4}\left(\frac{1}{4}\right)^{-s} \\
&\quad -s\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^s}-s\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^{s+1}}
\end{aligned} \tag{14.18}$$

$$\begin{aligned}
\frac{1}{2}\tilde{\zeta}'_{\mathcal{O}_{3+}}(0)+\frac{1}{2}\tilde{\zeta}'_{\mathcal{O}_{3-}}(0) &= \frac{d}{ds}\left[\frac{s}{4(s-1)}\left(\frac{1}{4}\right)^{1-s}+\frac{1}{4}\left(\frac{1}{4}\right)^{-s}\right]_{s=0}-\frac{1}{48}-\left(\frac{\log 2}{2}-\frac{\gamma}{2}\right) \\
&= -\frac{1}{12}+\frac{\gamma}{2}
\end{aligned} \tag{14.19}$$

$$\begin{aligned}
-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\tilde{K}_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(t) &= \frac{t}{4}\sum_{p_{12},p_{56},p_{89}}\int_x \sinh\rho\,\mathrm{tr}\left[\left\{\bar{\mathcal{O}}^x_{p_{12},p_{56},p_{89}},\tilde{\mathcal{O}}^x_{p_{12},p_{56},p_{89}}\right\}\bar{K}_{-\nabla^2+1}\left(x,x^{'};t\right)\right]_{x=x'} \\
&= -2t\int_0^\infty dv\, v\coth(\pi v)\left(v^2+2\right)e^{-t\left(v^2+1\right)}
\end{aligned} \tag{14.20}$$

$$\begin{aligned}
-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\tilde{\zeta}_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(s) &= -2s\int_0^\infty dv\frac{v^2+2}{v^2+1}\frac{v\coth(\pi v)}{\left(v^2+1\right)^s} \\
&= \int_0^\infty dv\frac{1}{\left(v^2+1\right)^s}\left(-2s-\frac{2s}{v^2+1}\right)\left(v+\frac{2v}{e^{2\pi v}-1}\right) \\
&= -\frac{s}{s-1}-1-4s\int_0^\infty dv\frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)^s}-4s\int_0^\infty dv\frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)^{s+1}}
\end{aligned} \tag{14.21}$$

$$\begin{aligned}
-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\tilde{\zeta}'_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(0) &= \frac{d}{ds}\left(-\frac{s}{s-1}\right)_{s=0}-4\frac{1}{24}-4\left(-\frac{1}{4}+\frac{\gamma}{2}\right) \\
&= \frac{11}{6}-2\gamma
\end{aligned} \tag{14.22}$$

Summing over the field content of theory, we obtain the localization prediction

$$\tilde{\Gamma} = -\frac{d}{ds}\left(\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_1}(s)+\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_2}(s)+\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3+}}(s)+\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3-}}(s)-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\tilde{\zeta}_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(s)\right)\Bigg|_{s=0} = -\frac{3}{4}\,. \tag{14.23}$$

14.3 First correction with rescaled operators

We can start with the rescaled operators

$$\begin{aligned}
\mathcal{O}_i^{\mathrm{res}} &\equiv \Omega^2\mathcal{O}_i^{\mathrm{res}} \qquad\qquad i=1,2,3\pm \\
\mathcal{O}^{\mathrm{res}}_{p_{12},p_{56},p_{89}} &\equiv \Omega\mathcal{O}^{\mathrm{res}}_{p_{12},p_{56},p_{89}}
\end{aligned} \tag{14.24}$$

for which we have the expansions

$$\begin{aligned}
\bar{\mathcal{O}}_i^{\mathrm{res}} &= \sinh^2\rho\,\bar{\mathcal{O}}_i^{\mathrm{res}} \qquad\qquad i=1,2,3\pm \\
\bar{\mathcal{O}}_{p_{12},p_{56},p_{89}}^{\mathrm{res}} &= \sinh\rho\,\bar{\mathcal{O}}_{p_{12},p_{56},p_{89}}^{\mathrm{res}}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{O}}_1^{\mathrm{res}} &= 0 \\
\tilde{\mathcal{O}}_2^{\mathrm{res}} &= -\frac{2\sinh^2\rho}{(1+\cosh\rho)^2} \\
\tilde{\mathcal{O}}_{3\pm}^{\mathrm{res}} &= -\left(\frac{1-\cosh\rho}{1+\cosh\rho}\right)^2(2\pm i\partial_\tau) \\
\tilde{\mathcal{O}}_{p_{12},p_{56},p_{89}} &= \frac{i\sinh^2\rho}{2(1+\cosh\rho)^3}\sigma_1+\frac{p_{56}}{4}\left(\frac{\cosh\rho-1}{\cosh\rho+1}\right)^2\sigma_2-p_{12}\frac{(\sigma_3+2p_{56}\mathbb{I}_2)\sinh\rho}{2(1+\cosh\rho)^2}
\end{aligned} \tag{14.25}$$

We keep the heat kernels of the unrescaled operators and take the new interaction vertices above.

$$\frac{3}{2}\tilde{K}_{\mathcal{O}_1}(t)=0 \tag{14.26}$$

$$\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_1}(s)=0 \tag{14.27}$$

$$\frac{3}{2}\tilde{\zeta}'_{\mathcal{O}_1}(0)=0 \tag{14.28}$$

$$\begin{aligned}
\frac{3}{2}\tilde{K}_{\mathcal{O}_2}(t) &= 3t\left(\int_x\frac{\sinh^3\rho}{(1+\cosh\rho)^2}\right)\bar{K}_{-\Delta}(x,x;t) \\
&= \frac{3t}{2\pi}\left(\frac{2\pi}{\epsilon}+4\pi\log\epsilon+2\pi(2\log 2-1)\right)\int_0^\infty dv\,v\tanh(\pi v)\,e^{-t\left(v^2+\frac{1}{4}\right)}
\end{aligned} \tag{14.29}$$

$$\begin{aligned}
\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_2}(s) &= \frac{3s}{2\pi}\left(\frac{2\pi}{\epsilon}+4\pi\log\epsilon+2\pi(2\log 2-1)\right)\int_0^\infty dv\frac{v\tanh(\pi v)}{\left(v^2+\frac{1}{4}\right)^{s+1}} \\
&= \frac{3s}{2\pi}\left(\frac{2\pi}{\epsilon}+4\pi\log\epsilon+2\pi(2\log 2-1)\right)\int_0^\infty dv\frac{1}{\left(v^2+\frac{1}{4}\right)^{s+1}}\left(v-\frac{2v}{e^{2\pi v}+1}\right) \\
&= \frac{3}{4\pi}\left(\frac{2\pi}{\epsilon}+4\pi\log\epsilon+2\pi(2\log 2-1)\right)\left(\frac{1}{4}\right)^{-s} \\
&\quad -\frac{3s}{\pi}\left(\frac{2\pi}{\epsilon}+4\pi\log\epsilon+2\pi(2\log 2-1)\right)\int_0^\infty dv\frac{v}{(e^{2\pi v}+1)\left(v^2+\frac{1}{4}\right)^{s+1}}
\end{aligned} \tag{14.30}$$

$$\begin{aligned}
\frac{3}{2}\tilde{\zeta}'_{\mathcal{O}_2}(0) &= \frac{d}{ds}\left[\frac{3}{4\pi}\left(\frac{2\pi}{\epsilon}+4\pi\log\epsilon+2\pi(2\log 2-1)\right)\left(\frac{1}{4}\right)^{-s}\right]_{s=0} \\
&= -\frac{3}{\pi}\left(\frac{\log 2}{2}-\frac{\gamma}{2}\right)\left(\frac{2\pi}{\epsilon}+4\pi\log\epsilon+2\pi(2\log 2-1)\right)
\end{aligned} \tag{14.31}$$

$$\frac{1}{2}\tilde{K}_{\mathcal{O}_{3+}}(t)+\frac{1}{2}\tilde{K}_{\mathcal{O}_{3-}}(t)=2t\left(\int_x \sinh\rho\left(\frac{1-\cosh\rho}{1+\cosh\rho}\right)^2\right)\bar{K}_{-\Delta}(x,x;t) \tag{14.32}$$
$$=\frac{t}{\pi}\left(\frac{2\pi}{\epsilon}+8\pi\log\epsilon+2\pi(4\log 2+1)\right)\int_0^\infty dv\, v\tanh(\pi v)\, e^{-t\left(v^2+\frac{1}{4}\right)}$$

$$\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3+}}(s)+\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3-}}(s)=\frac{s}{\pi}\left(\frac{2\pi}{\epsilon}+8\pi\log\epsilon+2\pi(4\log 2+1)\right)\int_0^\infty dv\frac{v\tanh(\pi v)}{\left(v^2+\frac{1}{4}\right)^{s+1}} \tag{14.33}$$
$$=\frac{s}{\pi}\left(\frac{2\pi}{\epsilon}+8\pi\log\epsilon+2\pi(4\log 2+1)\right)\int_0^\infty dv\frac{1}{\left(v^2+\frac{1}{4}\right)^{s+1}}\left(v-\frac{2v}{e^{2\pi v}+1}\right)$$
$$=\frac{1}{2\pi}\left(\frac{2\pi}{\epsilon}+8\pi\log\epsilon+2\pi(4\log 2+1)\right)\left(\frac{1}{4}\right)^{-s}$$
$$-\frac{2s}{\pi}\left(\frac{2\pi}{\epsilon}+8\pi\log\epsilon+2\pi(4\log 2+1)\right)\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^{s+1}}$$

$$\frac{1}{2}\tilde{\zeta}'_{\mathcal{O}_{3+}}(0)+\frac{1}{2}\tilde{\zeta}'_{\mathcal{O}_{3-}}(0)=\frac{d}{ds}\left[\frac{1}{2\pi}\left(\frac{2\pi}{\epsilon}+8\pi\log\epsilon+2\pi(4\log 2+1)\right)\left(\frac{1}{4}\right)^{-s}\right]_{s=0} \tag{14.34}$$
$$=-\frac{2}{\pi}\left(\frac{\log 2}{2}-\frac{\gamma}{2}\right)\left(\frac{2\pi}{\epsilon}+8\pi\log\epsilon+2\pi(4\log 2+1)\right)$$

$$-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\tilde{K}_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(t)=\frac{t}{4}\sum_{p_{12},p_{56},p_{89}}\int_x \sinh\rho\,\mathrm{tr}\left[\left\{\bar{\mathcal{O}}^{\mathrm{res}}_{p_{12},p_{56},p_{89}},\tilde{\mathcal{O}}^{\mathrm{res}}_{p_{12},p_{56},p_{89}}\right\}\bar{K}_{-\nabla\!\!\!/^2+1}\left(x,x';t\right)\right]_{x=x'} \tag{14.35}$$
$$=\frac{t}{\pi}\left(-\frac{4\pi}{\epsilon}-14\pi\log\epsilon-2\pi(7\log 2+1)\right)\int_0^\infty dv\, v\coth\pi v e^{-t\left(v^2+1\right)}$$

$$-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\tilde{\zeta}_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(s)=\frac{s}{\pi}\left(-\frac{4\pi}{\epsilon}-14\pi\log\epsilon-2\pi(7\log 2+1)\right)\int_0^\infty dv\frac{1}{\left(v^2+1\right)}\frac{v\coth\pi v}{\left(v^2+1\right)^s} \tag{14.36}$$
$$=\frac{s}{\pi}\left(-\frac{4\pi}{\epsilon}-14\pi\log\epsilon-2\pi(7\log 2+1)\right)\int_0^\infty dv\frac{1}{\left(v^2+1\right)^{s+1}}\left(v+\frac{2v}{e^{2\pi v}-1}\right)$$
$$=\frac{1}{2\pi}\left(-\frac{4\pi}{\epsilon}-14\pi\log\epsilon-2\pi(7\log 2+1)\right)$$
$$+\frac{2s}{\pi}\left(-\frac{4\pi}{\epsilon}-14\pi\log\epsilon-2\pi(7\log 2+1)\right)\int_0^\infty dv\frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)^{s+1}}$$

$$-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\tilde{\zeta}'_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(0)=\frac{2}{\pi}\left(-\frac{1}{4}+\frac{\gamma}{2}\right)\left(-\frac{4\pi}{\epsilon}-14\pi\log\epsilon-2\pi(7\log 2+1)\right) \tag{14.37}$$

Summing over the field content of theory, we do not obtain the localization prediction (no cancellation of $\log 2$ and γ at all).

Now we use the heat kernel of the unrescaled operators:

$$\bar{K}_{-\Delta+m^2}\left(\rho,\tau,\rho',\tau';t\right)=\frac{1}{4\pi^2}\int dp_x\int dp_y\, e^{i\left(p_x\rho\left(\cos\tau-\cos\tau'\right)+p_y\rho\left(\sin\tau-\sin\tau'\right)\right)-t\left(p_x^2+p_y^2\right)-m^2t} \tag{14.38}$$
$$=\frac{1}{2\pi t}\exp\left(-\frac{\rho^2+\rho'^2-2\rho\rho'\cos\left(\tau-\tau'\right)}{4t^2}-m^2t\right).$$

$$\frac{3}{2}\tilde{K}_{\mathcal{O}_1}(t)=0 \tag{14.39}$$
$$\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_1}(s)=0 \tag{14.40}$$
$$\frac{3}{2}\tilde{\zeta}'_{\mathcal{O}_1}(0)=0 \tag{14.41}$$

$$
\begin{aligned}
\frac{3}{2}\tilde{K}_{\mathcal{O}_2}(t) &= 3t\left(\int_x \frac{\sinh^3\rho}{(1+\cosh\rho)^2}\right)\bar{K}_{-\Delta}(x,x;t) \\
&= \frac{3}{2\pi}\left(\int_x \frac{\sinh^3\rho}{(1+\cosh\rho)^2}\right) \\
&= \frac{1}{\epsilon} - 2\log\left(1+\frac{1}{\epsilon}\right)
\end{aligned}
\tag{14.42}
$$

15 k-wound circle heat kernel

We make the basic definitions and prove the same formulas of the latitude upon the replacement $\theta_0^2 \to k-1$.

15.1 First correction

Upon the change of coordinates $\sigma \to \rho$ with $\sinh\sigma \sinh\rho = 1$,

$$g_{ij} = \begin{pmatrix} \left(\frac{d\sigma}{d\rho}\right)^2 \frac{k^2}{\sinh^2 k\sigma} & 0 \\ 0 & \frac{k^2}{\sinh^2 k\sigma} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\rho \end{pmatrix} + (k-1)\begin{pmatrix} 0 & 0 \\ 0 & 2\sinh^2\rho \end{pmatrix} + \dots \tag{15.1}$$

$$\bar{g} = \sinh^2\rho \qquad \tilde{g} = 2\sinh^2\rho\,, \tag{15.2}$$

we proceed to the computation of the integrated heat kernels.

$$\begin{aligned} \frac{5}{2}\tilde{K}_{\mathcal{O}_0}(t) &= -\frac{5t}{2}\int_x \frac{2}{\sinh\rho}\left[\partial_\tau^2 \bar{K}_{-\Delta}\left(x, x^{'}; t\right)\right]_{x=x'} \\ &= -\frac{5\left(-\frac{1}{\epsilon}+1\right)t}{2}\int_0^\infty dv\, v\tanh(\pi v)\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)} \end{aligned} \tag{15.3}$$

$$\begin{aligned} \frac{5}{2}\tilde{\zeta}_{\mathcal{O}_0}(s) &= -\frac{5\left(-\frac{1}{\epsilon}+1\right)s}{2}\int_0^\infty dv \frac{v\tanh(\pi v)}{\left(v^2+\frac{1}{4}\right)^s} \\ &= -\frac{5\left(-\frac{1}{\epsilon}+1\right)s}{2}\int_0^\infty dv \frac{1}{\left(v^2+\frac{1}{4}\right)^s}\left(v-\frac{2v}{e^{2\pi v}+1}\right) \\ &= -\frac{5\left(-\frac{1}{\epsilon}+1\right)s}{4(s-1)}\left(\frac{1}{4}\right)^{1-s} + 5\left(-\frac{1}{\epsilon}+1\right)s\int_0^\infty dv \frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^s} \end{aligned} \tag{15.4}$$

$$\begin{aligned} \frac{5}{2}\tilde{\zeta}'_{\mathcal{O}_0}(s) &= \left(-\frac{1}{\epsilon}+1\right)\frac{d}{ds}\left[-\frac{5s}{4(s-1)}\left(\frac{1}{4}\right)^{1-s}\right]_{s=0} + 5\left(-\frac{1}{\epsilon}+1\right)\frac{1}{48} \\ &= \frac{5}{12}\left(-\frac{1}{\epsilon}+1\right) \end{aligned} \tag{15.5}$$

$$\begin{aligned} &\frac{3}{2}\tilde{K}_{\mathcal{O}_1}(t) - \frac{3t}{2}\int_x \frac{2}{\sinh\rho}\left[\partial_\tau^2 \bar{K}_{-\Delta+2}\left(x, x^{'}; t\right)\right]_{x=x'} \\ &= -\frac{3\left(-\frac{1}{\epsilon}+1\right)t}{2}\int_0^\infty dv\, v\tanh(\pi v)\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)} \end{aligned} \tag{15.6}$$

$$\begin{aligned} \frac{3}{2}\tilde{\zeta}_{\mathcal{O}_1}(s) &= -\frac{3\left(-\frac{1}{\epsilon}+1\right)s}{2}\int_0^\infty dv\, v\tanh(\pi v)\,\frac{v^2+\frac{1}{4}}{\left(v^2+\frac{9}{4}\right)^s} \\ &= -\frac{3\left(-\frac{1}{\epsilon}+1\right)s}{2}\int_0^\infty dv \frac{1}{\left(v^2+\frac{9}{4}\right)^s}\left(1-\frac{2}{v^2+\frac{9}{4}}\right)\left(v-\frac{2v}{e^{2\pi v}+1}\right) \\ &= -\frac{3\left(-\frac{1}{\epsilon}+1\right)s}{4(s-1)}\left(\frac{9}{4}\right)^{1-s} + \frac{3\left(-\frac{1}{\epsilon}+1\right)}{2}\left(\frac{9}{4}\right)^{-s} \\ &+ 3\left(-\frac{1}{\epsilon}+1\right)s\int_0^\infty dv \frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^s} - 6\left(-\frac{1}{\epsilon}+1\right)s\int_0^\infty dv \frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^{s+1}} \end{aligned} \tag{15.7}$$

$$\begin{aligned} \frac{3}{2}\tilde{\zeta}'_{\mathcal{O}_1}(s) &= \left(-\frac{1}{\epsilon}+1\right)\frac{d}{ds}\left[-\frac{3s}{4(s-1)}\left(\frac{9}{4}\right)^{1-s} + \frac{3}{2}\left(\frac{9}{4}\right)^{-s}\right]_{s=0} + 3\left(-\frac{1}{\epsilon}+1\right)\frac{1}{48} - 6\left(-\frac{1}{\epsilon}+1\right)\left(-\frac{\log\frac{3}{2}}{2}+\frac{1}{2}-\frac{\gamma}{2}\right) \\ &= \left(-\frac{5}{4}+3\gamma\right)\left(-\frac{1}{\epsilon}+1\right) \end{aligned} \tag{15.8}$$

$$-\frac{8}{4}\tilde{K}_{\mathcal{O}_F^2}(t) = \frac{8t}{4}\int_x \sinh\rho \mathrm{tr}\left[\left\{\bar{\mathcal{O}}_F^x, \tilde{\mathcal{O}}_F^x\right\}\bar{K}_{-\nabla^2+1}\left(x, x^{'}; t\right)\right]_{x=x'} \tag{15.9}$$
$$= 2\left(-\frac{1}{\epsilon}+1\right) t \int_0^\infty dv\, v \coth(\pi v)\left(2v^2+1\right) e^{-t\left(v^2+1\right)}$$

$$-\frac{8}{4}\tilde{\zeta}_{\mathcal{O}_F^2}(s) = 2\left(-\frac{1}{\epsilon}+1\right) s \int_0^\infty dv\, \frac{2v^2+1}{\left(v^2+1\right)^{s+1}} v \coth(\pi v) \tag{15.10}$$
$$= 2\left(-\frac{1}{\epsilon}+1\right) s \int_0^\infty dv\, \frac{1}{\left(v^2+1\right)^s}\left(2-\frac{1}{v^2+1}\right)\left(v+\frac{2v}{e^{2\pi v}-1}\right)$$
$$= \frac{2\left(-\frac{1}{\epsilon}+1\right)s}{s-1} - \left(-\frac{1}{\epsilon}+1\right) + 8\left(-\frac{1}{\epsilon}+1\right) s \int_0^\infty dv\, \frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)^s} - 4\left(-\frac{1}{\epsilon}+1\right) s \int_0^\infty dv\, \frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)^{s+1}}$$

$$-\frac{8}{4}\tilde{\zeta}_{\mathcal{O}_F^2}(s) = \left(-\frac{1}{\epsilon}+1\right)\frac{d}{ds}\left(\frac{2s}{s-1}-1\right)_{s=0} + 8\left(-\frac{1}{\epsilon}+1\right)\frac{1}{24} - 4\left(\frac{1}{\epsilon}-1\right)\left(-\frac{1}{4}+\frac{\gamma}{2}\right) \tag{15.11}$$
$$= \left(-\frac{2}{3}-2\gamma\right)\left(-\frac{1}{\epsilon}+1\right)$$

We drop the divergences. Summing over the field content of theory and dropping the e^{ρ_0} divergencies, we get

$$\tilde{\Gamma} = -\frac{d}{ds}\left(\frac{5}{2}\tilde{\zeta}_{\mathcal{O}_0}(s) + \frac{3}{2}\tilde{\zeta}_{\mathcal{O}_1}(s) - \frac{8}{4}\tilde{\zeta}_{\mathcal{O}_F^2}(s)\right)\Bigg|_{s=0} = \frac{3}{2} - \gamma\,. \tag{15.12}$$

16 Cusp heat kernel

16.1 First correction

Upon the change of coordinates $(\sigma,\tau)\to(r,w)$ with $\sigma=\frac{2}{\pi}\mathbb{K}\left(k^2\right)r$ and $\tau=\frac{\sqrt{b^4+p^2}}{bp}w$,

$$g_{ij}=\frac{1-k^2}{\operatorname{cn}^2\left(\sigma|k^2\right)}\begin{pmatrix}\left(\frac{d\sigma}{dr}\right)^2 & 0\\ 0 & \left(\frac{d\tau}{dw}\right)^2\end{pmatrix}=\frac{1}{\cos^2 r}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+k^2\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & \frac{3}{2\cos^2 r}-\frac{1}{2}\end{pmatrix}+\dots \tag{16.1}$$

$$\bar{g}=\frac{1}{\cos^4 r}\qquad \tilde{g}=\frac{1}{\cos^2 r}\left(\frac{3}{2\cos^2 r}-1\right), \tag{16.2}$$

we compute the following expressions.

$$\begin{aligned}\frac{5}{2}\tilde{K}_{\mathcal{O}_0}(t)&=-\frac{5t}{2}\int_x\frac{1}{\cos^2 r}\left[\tilde{\mathcal{O}}_0\bar{K}_{-\Delta}\left(x,x^{'};t\right)\right]_{x=x'} \\ &=\frac{5t}{16\pi}\int_x\left(\frac{3}{\cos^2 r}-2\right)\int_0^\infty dv\,v\tanh\pi v\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)} \\ &=\frac{5\left(3e^{\rho_0}-2\pi\right)tT}{16\pi}\int_0^\infty dv\,v\tanh\pi v\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}\end{aligned} \tag{16.3}$$

$$\begin{aligned}\frac{5}{2}\tilde{\zeta}_{\mathcal{O}_0}(s)&=\frac{5\left(3e^{\rho_0}-2\pi\right)sT}{16\pi}\int_0^\infty dv\frac{v\tanh\pi v}{\left(v^2+\frac{1}{4}\right)^s} \\ &=\frac{5\left(3e^{\rho_0}-2\pi\right)sT}{16\pi}\int_0^\infty dv\frac{1}{\left(v^2+\frac{1}{4}\right)^s}\left(v-\frac{2v}{e^{2\pi v}+1}\right) \\ &=\frac{5\left(3e^{\rho_0}-2\pi\right)sT}{32\pi\left(s-1\right)}\left(\frac{1}{4}\right)^{1-s}-\frac{5\left(3e^{\rho_0}-2\pi\right)sT}{8\pi}\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^s}\end{aligned} \tag{16.4}$$

$$\begin{aligned}\frac{5}{2}\tilde{\zeta}'_{\mathcal{O}_0}(0)&=\frac{d}{ds}\left[\frac{5\left(3e^{\rho_0}-2\pi\right)sT}{32\pi\left(s-1\right)}\left(\frac{1}{4}\right)^{1-s}\right]_{s=0}-\frac{5\left(3e^{\rho_0}-2\pi\right)T}{8\pi}\frac{1}{48} \\ &=\frac{5T}{48}-\frac{5e^{\rho_0}T}{32\pi}\end{aligned} \tag{16.5}$$

$$\begin{aligned}\frac{2}{2}\tilde{K}_{\mathcal{O}_1}(t)&=-\frac{2t}{2}\int_x\frac{1}{\cos^2 r}\left[\tilde{\mathcal{O}}_1\bar{K}_{-\Delta+2}\left(x,x^{'};t\right)\right]_{x=x'} \\ &=\frac{t}{8\pi}\int_x\left(\frac{3}{\cos^2 r}-2\right)\int_0^\infty dv\,v\tanh\pi v\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)} \\ &=\frac{\left(3e^{\rho_0}-2\pi\right)tT}{8\pi}\int_0^\infty dv\,v\tanh\pi v\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}\end{aligned} \tag{16.6}$$

$$\begin{aligned}\frac{2}{2}\tilde{\zeta}_{\mathcal{O}_1}(s)&=\frac{\left(3e^{\rho_0}-2\pi\right)sT}{8\pi}\int_0^\infty dv\frac{v^2+\frac{1}{4}}{\left(v^2+\frac{9}{4}\right)^{s+1}}v\tanh\pi v \\ &=\frac{\left(3e^{\rho_0}-2\pi\right)sT}{8\pi}\int_0^\infty dv\frac{1}{\left(v^2+\frac{9}{4}\right)^s}\left(1-\frac{2}{v^2+\frac{9}{4}}\right)\left(v-\frac{2v}{e^{2\pi v}+1}\right) \\ &=\frac{\left(3e^{\rho_0}-2\pi\right)sT}{16\pi\left(s-1\right)}\left(\frac{9}{4}\right)^{1-s}-\frac{\left(3e^{\rho_0}-2\pi\right)T}{8\pi}\left(\frac{9}{4}\right)^{-s} \\ &\quad-\frac{\left(3e^{\rho_0}-2\pi\right)sT}{4\pi}\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^s}+\frac{\left(3e^{\rho_0}-2\pi\right)sT}{2\pi}\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^{s+1}}\end{aligned} \tag{16.7}$$

$$\begin{aligned}\frac{2}{2}\tilde{\zeta}'_{\mathcal{O}_1}(0)&=\frac{d}{ds}\left[\frac{\left(3e^{\rho_0}-2\pi\right)sT}{16\pi\left(s-1\right)}\left(\frac{9}{4}\right)^{1-s}-\frac{\left(3e^{\rho_0}-2\pi\right)T}{8\pi}\left(\frac{9}{4}\right)^{-s}\right]_{s=0}-\frac{\left(3e^{\rho_0}-2\pi\right)T}{4\pi}\frac{1}{48}+\frac{\left(3e^{\rho_0}-2\pi\right)T}{2\pi}\left(-\frac{\log\frac{3}{2}}{2}+\frac{1}{2}-\frac{\gamma}{2}\right) \\ &=\frac{\left(-5+12\gamma\right)T}{24}+\frac{e^{\rho_0}\left(5-12\gamma\right)T}{16\pi}\end{aligned} \tag{16.8}$$

$$
\begin{aligned}
\frac{1}{2}\tilde{K}_{\mathcal{O}_2}(t) &= -\frac{t}{2}\int_x \frac{1}{\cos^2 r}\left[\tilde{\mathcal{O}}_2\bar{K}_{-\Delta+2}\left(x,x^{'};t\right)\right]_{x=x'} \\
&= \frac{t}{16\pi}\int_x\int_0^\infty dv\, v\tanh\pi v\left[\left(\frac{3}{\cos^2 r}-2\right)\left(v^2+\frac{1}{4}\right)+8\cos^2 r\right]e^{-t\left(v^2+\frac{9}{4}\right)} \\
&= \frac{tT}{16\pi}\int_0^\infty dv\, v\tanh\pi v\left[\left(3e^{\rho_0}-2\pi\right)\left(v^2+\frac{1}{4}\right)+4\pi\right]e^{-t\left(v^2+\frac{9}{4}\right)}
\end{aligned}
\tag{16.9}
$$

$$
\begin{aligned}
\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_2}(s) &= \frac{sT}{16\pi}\int_0^\infty dv\frac{\left(3e^{\rho_0}-2\pi\right)\left(v^2+\frac{1}{4}\right)+4\pi}{\left(v^2+\frac{9}{4}\right)^{s+1}}v\tanh\pi v \\
&= \frac{sT}{16\pi}\int_0^\infty dv\frac{1}{\left(v^2+\frac{9}{4}\right)^s}\left(3e^{\rho_0}-2\pi+\frac{8\pi-6e^{\rho_0}}{v^2+\frac{9}{4}}\right)\left(v-\frac{2v}{e^{2\pi v}+1}\right) \\
&= \frac{\left(3e^{\rho_0}-2\pi\right)sT}{32\pi\left(s-1\right)}\left(\frac{9}{4}\right)^{1-s}+\frac{\left(4\pi-3e^{\rho_0}\right)T}{16\pi}\left(\frac{9}{4}\right)^{-s} \\
&\quad -\frac{\left(3e^{\rho_0}-2\pi\right)sT}{8\pi}\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^s}-\frac{\left(4\pi-3e^{\rho_0}\right)sT}{4\pi}\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^{s+1}}
\end{aligned}
\tag{16.10}
$$

$$
\begin{aligned}
\frac{1}{2}\tilde{\zeta}'_{\mathcal{O}_2}(0) &= \frac{d}{ds}\left[\frac{\left(3e^{\rho_0}-2\pi\right)sT}{32\pi\left(s-1\right)}\left(\frac{9}{4}\right)^{1-s}+\frac{\left(4\pi-3e^{\rho_0}\right)T}{16\pi}\left(\frac{9}{4}\right)^{-s}\right]_{s=0} \\
&\quad -\frac{\left(3e^{\rho_0}-2\pi\right)T}{8\pi}\frac{1}{48}-\frac{\left(4\pi-3e^{\rho_0}\right)T}{4\pi}\left(-\frac{\log\frac{3}{2}}{2}+\frac{1}{2}-\frac{\gamma}{2}\right) \\
&= \frac{\left(-17+24\gamma\right)T}{48}+\frac{e^{\rho_0}\left(5-12\gamma\right)T}{32\pi}
\end{aligned}
\tag{16.11}
$$

$$
\begin{aligned}
-\frac{8}{4}\tilde{K}_{\mathcal{O}_F}(t) &= 2t\int_x\frac{1}{\cos^2 r}\mathrm{tr}\left[\left\{\bar{\mathcal{O}}_F^x,\tilde{\mathcal{O}}_F^x\right\}\bar{K}_{-\nabla\!\!\!/^2+1}\left(x,x^{'};t\right)\right]_{x=x'} \\
&= \frac{t}{\pi}\int_x\int_0^\infty dv\, v\coth\pi v\left[\left(1-\frac{3}{2\cos^2 r}\right)\left(v^2+1\right)-1-\frac{\cos 2r}{2}+\frac{3}{4\cos^2 r}\right]e^{-t\left(v^2+1\right)} \\
&= \frac{tT}{\pi}\int_0^\infty dv\, v\coth\pi v\left[\left(\pi-\frac{3e^{\rho_0}}{2}\right)\left(v^2+1\right)+\frac{3e^{\rho_0}}{4}-\pi\right]e^{-t\left(v^2+1\right)}
\end{aligned}
\tag{16.12}
$$

$$
\begin{aligned}
-\frac{8}{4}\tilde{\zeta}_{\mathcal{O}_F^2}(s) &= \frac{sT}{\pi}\int_0^\infty dv\frac{1}{\left(v^2+1\right)^s}\left(\pi-\frac{3e^{\rho_0}}{2}+\frac{\frac{3e^{\rho_0}}{4}-\pi}{v^2+1}\right)v\coth\pi v \\
&= \frac{sT}{\pi}\int_0^\infty dv\frac{1}{\left(v^2+1\right)^s}\left(\pi-\frac{3}{2}e^{\rho_0}+\frac{\frac{3}{4}e^{\rho_0}-\pi}{v^2+1}\right)\left(v+\frac{2v}{e^{2\pi v}-1}\right) \\
&= \frac{sT}{2\pi\left(s-1\right)}\left(\pi-\frac{3}{2}e^{\rho_0}\right)+\frac{T}{2\pi}\left(\frac{3}{4}e^{\rho_0}-\pi\right) \\
&\quad +\frac{2sT}{\pi}\left(\pi-\frac{3}{2}e^{\rho_0}\right)\int_0^\infty dv\frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)^s}+\frac{2sT}{\pi}\left(\frac{3}{4}e^{\rho_0}-\pi\right)\int_0^\infty dv\frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)^{s+1}}
\end{aligned}
\tag{16.13}
$$

$$
\begin{aligned}
-\frac{8}{4}\tilde{\zeta}'_{\mathcal{O}_F^2}(0) &= \frac{d}{ds}\left[\frac{sT}{2\pi\left(s-1\right)}\left(\pi-\frac{3}{2}e^{\rho_0}\right)\right]_{s=0} \\
&\quad +\frac{2T}{\pi}\left(\pi-\frac{3}{2}e^{\rho_0}\right)\frac{1}{24}+\frac{2T}{\pi}\left(\frac{3}{4}e^{\rho_0}-\pi\right)\left(-\frac{1}{4}+\frac{\gamma}{2}\right) \\
&= \left(\frac{1}{12}-\gamma\right)T+\frac{e^{\rho_0}\left(3+\gamma\right)T}{4\pi}
\end{aligned}
\tag{16.14}
$$

Summing over the field content of theory and dropping the e^{ρ_0} divergencies, we obtain the Drukker-Forini result

$$
\tilde{\Gamma} = -\frac{d}{ds}\left(\frac{5}{2}\tilde{\zeta}_{\mathcal{O}_0}(s)+\frac{2}{2}\tilde{\zeta}_{\mathcal{O}_1}(s)+\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_2}(s)-\frac{8}{4}\tilde{\zeta}_{\mathcal{O}_F^2}(s)\right)\Bigg|_{s=0} = \frac{3T}{8}\,.
\tag{16.15}
$$

17 Redo Bergamin Tseytlin

17.1 Introduction

(1.5)

$$\xi = -\log\tanh\frac{k\sigma}{2} \tag{17.1}$$

17.2 Section 2

above (2.11)

$$\sinh\frac{d(\xi,z)}{2} = \sinh\xi\sin\frac{z}{2} \tag{17.2}$$

(2.11)

$$\begin{aligned}
K(z;t) &= \int_0^{2\pi\kappa} d\phi\int_0^\infty d\xi\sinh\xi\,\frac{\sqrt{2}e^{-\left(m^2+\frac{1}{4}\right)t}}{(4\pi t)^{3/2}}\int_{d(\xi,z)}^\infty\frac{dy\,ye^{-\frac{y^2}{4t}}}{\sqrt{\cosh y-\cosh d(\xi,z)}} \\
&= 2\pi\kappa\int_0^\infty\frac{dv}{\sqrt{1+v^2}}v\frac{\sqrt{2}e^{-\left(m^2+\frac{1}{4}\right)t}}{(4\pi t)^{3/2}}\int_{d(v,z)}^\infty\frac{dy\,ye^{-\frac{y^2}{4t}}}{\sqrt{2}\sin\frac{z}{2}\sqrt{\frac{\sinh^2\frac{y}{2}}{\sin^2\frac{z}{2}}-v^2}} \\
&= \frac{\kappa e^{-\left(m^2+\frac{1}{4}\right)t}}{2(4\pi)^{1/2}t^{3/2}\sin\frac{z}{2}}\int_0^\infty\frac{v\,dv}{\sqrt{1+v^2}}\int_{d(v,z)}^\infty\frac{dy\,ye^{-\frac{y^2}{4t}}}{\sqrt{\frac{\sinh^2\frac{y}{2}}{\sin^2\frac{z}{2}}-v^2}}
\end{aligned} \tag{17.3}$$

(2.12)

$$\begin{aligned}
K(z;t) &= \frac{\kappa e^{-\left(m^2+\frac{1}{4}\right)t}}{2(4\pi)^{1/2}t^{3/2}\sin\frac{z}{2}}\int_0^\infty dy\,ye^{-\frac{y^2}{4t}}\int_0^{\sinh\frac{y}{2}/\sin\frac{z}{2}}dv\frac{v}{\sqrt{1+v^2}\sqrt{\frac{\sinh^2\frac{y}{2}}{\sin^2\frac{z}{2}}-v^2}} \\
&= \frac{\kappa e^{-\left(m^2+\frac{1}{4}\right)t}}{2(4\pi)^{1/2}t^{3/2}\sin\frac{z}{2}}\int_0^\infty dy\,ye^{-\frac{y^2}{4t}}\left(\frac{\pi}{2}-\arctan\left(\frac{\sin\frac{z}{2}}{\sinh\frac{y}{2}}\right)\right) \\
&= \frac{\kappa e^{-\left(m^2+\frac{1}{4}\right)t}}{2(4\pi)^{1/2}t^{3/2}\sin\frac{z}{2}}\int_0^\infty dy\left(2te^{-\frac{y^2}{4t}}\frac{\cosh\frac{y}{2}\sin\frac{z}{2}}{\cosh y-\cos z}\right) \\
&= \frac{\kappa e^{-\left(m^2+\frac{1}{4}\right)t}}{(4\pi t)^{1/2}\sin\frac{z}{2}}\int_0^\infty dy\,e^{-\frac{y^2}{4t}}\frac{\cosh\frac{y}{2}\sin\frac{z}{2}}{\cosh y-\cos z}
\end{aligned} \tag{17.4}$$

(2.13)

$$K(z;t) = \frac{\kappa e^{-\left(m^2+\frac{1}{4}\right)t}}{(4\pi t)^{1/2}}\int_0^\infty dy\,e^{-\frac{y^2}{t}}\frac{\cosh y}{\sinh^2 y+\sin^2\frac{z}{2}} \tag{17.5}$$

(2.14)-(2.19)

$$\begin{aligned}
\mathrm{K}_\kappa(t) &= \kappa\mathrm{K}_{AdS_2} + \frac{\kappa e^{-\left(m^2+\frac{1}{4}\right)t}}{(4\pi t)^{1/2}}\int_0^\infty dy\,\cosh y\,e^{-\frac{y^2}{t}}\frac{i}{4\pi\kappa}\int_\Gamma\underbrace{\frac{dz}{\sinh^2 y+\sin^2\frac{z}{2}}\cot\frac{z}{2\kappa}}_{F(z;y,\kappa)} \\
&= \kappa\mathrm{K}_{AdS_2} + \frac{\kappa e^{-\left(m^2+\frac{1}{4}\right)t}}{(4\pi t)^{1/2}}\int_0^\infty dy\,\cosh y\,e^{-\frac{y^2}{t}}\frac{i}{4\pi\kappa}2\pi i\left(\frac{2\kappa}{\sinh^2 y}-\frac{2\coth\frac{y}{\kappa}}{\sinh y\cosh y}\right) \\
&= \kappa\mathrm{K}_{AdS_2} + \frac{\kappa e^{-\left(m^2+\frac{1}{4}\right)t}}{(4\pi t)^{1/2}}\int_0^\infty dy\,\cosh y\,e^{-\frac{y^2}{t}}\frac{1}{\sinh^2 y}\left(\frac{\tanh y}{\kappa\tanh\frac{y}{\kappa}}-1\right) \\
&= \kappa\mathrm{K}_{AdS_2} + \frac{e^{-\left(m^2+\frac{1}{4}\right)t}}{(4\pi t)^{1/2}}\int_0^\infty dy\,e^{-\frac{y^2}{t}}\frac{1}{\sinh y}\left(\coth\frac{y}{\kappa}-\kappa\coth y\right)
\end{aligned} \tag{17.6}$$

$$\operatorname{Res}\left(\frac{1}{\sinh^2 y+\sin^2\frac{z}{2}}\cot\frac{z}{2\kappa},\ z=0\right)=\frac{2\kappa}{\sinh^2 y} \tag{17.7}$$

$$\operatorname{Res}\left(\frac{1}{\sinh^2 y+\sin^2\frac{z}{2}}\cot\frac{z}{2\kappa},\ z=\pm 2iy\right)=-\frac{\coth\frac{y}{\kappa}}{\sinh y\cosh y} \tag{17.8}$$

17.3 Section 3

We use the hk (3.1) for the Dirac operator squared on Majorana spinors.
(3.9)

$$\begin{aligned}
\operatorname{Tr}K(z;t) &= \int_0^{2\pi\kappa} d\phi\int_0^\infty d\xi\sinh\xi\,\frac{2}{\sqrt{1+\cosh^2\xi\tan^2\frac{z}{2}}}\,\frac{e^{-m^2t}}{\sqrt{2}\,(4\pi t)^{3/2}}\,\frac{1}{\cosh\frac{d(\xi,z)}{2}}\int_{d(\xi,z)}^\infty\frac{dy\,y\cosh\frac{y}{2}e^{-\frac{y^2}{4t}}}{\sqrt{\cosh y-\cosh d(\xi,z)}} \\
&= 2\pi\kappa\int_0^\infty\frac{dv}{\sqrt{1+v^2}}v\frac{2}{\sqrt{1+(1+v^2)\tan^2\frac{z}{2}}}\,\frac{e^{-m^2t}}{\sqrt{2}\,(4\pi t)^{3/2}}\,\frac{1}{\sqrt{1+v^2\sin^2\frac{z}{2}}}\int_{d(v,z)}^\infty\frac{dy\,y\cosh\frac{y}{2}e^{-\frac{y^2}{4t}}}{\sqrt{2}\sqrt{\sinh^2\frac{y}{2}-v^2\sin^2\frac{z}{2}}} \\
&= \frac{\kappa e^{-m^2t}}{{\color{red}2}\,(4\pi)^{1/2}t^{3/2}}\int_0^\infty\frac{dv\,v}{\sqrt{1+v^2}}\,\frac{\cos\frac{z}{2}}{1+v^2\sin^2\frac{z}{2}}\int_{d(v,z)}^\infty\frac{dy\,y\cosh\frac{y}{2}e^{-\frac{y^2}{4t}}}{\sqrt{\sinh^2\frac{y}{2}-v^2\sin^2\frac{z}{2}}}
\end{aligned} \tag{17.9}$$

(3.10)-(3.13)

$$\begin{aligned}
\operatorname{Tr}K(z;t) &= \frac{\kappa e^{-m^2t}}{{\color{red}2}\,(4\pi)^{1/2}t^{3/2}\sin\frac{z}{2}}\int_0^\infty dy\,y\cosh\frac{y}{2}e^{-\frac{y^2}{4t}}\underbrace{\int_0^{\sinh\frac{y}{2}/\sin\frac{z}{2}}\frac{dv\,v}{\sqrt{1+v^2}}\,\frac{\cos\frac{z}{2}}{\sqrt{\frac{\sinh^2\frac{y}{2}}{\sin^2\frac{z}{2}}-v^2}\left(1+v^2\sin^2\frac{z}{2}\right)}}_{G(y,z)} \\
&= \frac{\kappa e^{-m^2t}}{{\color{red}2}\,(4\pi)^{1/2}t^{3/2}\sin\frac{z}{2}}\int_0^\infty dy\,y\cosh\frac{y}{2}e^{-\frac{y^2}{4t}}\frac{1}{\cosh\frac{y}{2}}\left(\frac{\pi}{2}-\arctan\frac{\tan\frac{z}{2}}{\tanh\frac{y}{2}}\right) \\
&= \frac{\kappa e^{-m^2t}}{{\color{red}2}\,(4\pi t)^{1/2}\sin\frac{z}{2}}\int_0^\infty dy\,e^{-\frac{y^2}{4t}}\frac{\sin z}{\cosh y-\cos z} \\
&= \frac{2\kappa e^{-m^2t}}{{\color{red}2}\,(4\pi t)^{1/2}}\int_0^\infty dy\,e^{-\frac{y^2}{t}}\frac{\cos\frac{z}{2}}{\sinh^2 y+\sin^2\frac{z}{2}}
\end{aligned} \tag{17.10}$$

(3.14)-(3.17)

$$\begin{aligned}
\mathrm{K}_\kappa(t) &= \operatorname{Tr}K(0;t)+\frac{i}{4\pi\kappa}\int_\Gamma dz\frac{1}{\sin\frac{z}{2\kappa}}\operatorname{Tr}K(z;t) \\
&= \kappa K_{AdS_2}(t)+\frac{2\kappa e^{-m^2t}}{{\color{red}2}\,(4\pi t)^{1/2}}\int_0^\infty dy\,e^{-\frac{y^2}{t}}\frac{i}{4\pi\kappa}\int_\Gamma dz\underbrace{\frac{1}{\sinh^2 y+\sin^2\frac{z}{2}}\frac{\cos\frac{z}{2}}{\sin\frac{z}{2\kappa}}}_{F_{\frac{1}{2}}(z;y,\kappa)} \\
&= \kappa K_{AdS_2}(t){\color{red}+}\frac{2\kappa e^{-m^2t}}{{\color{red}2}\,(4\pi t)^{1/2}}\int_0^\infty dy\,e^{-\frac{y^2}{t}}{\color{red}\frac{i}{4\pi\kappa}}2\pi i\left({\color{red}\frac{2\kappa}{\sinh^2 y}}-{\color{red}\frac{2}{\sinh\frac{y}{\kappa}\sinh y}}\right) \\
&= \kappa K_{AdS_2}(t){\color{red}+}\frac{2e^{-m^2t}}{{\color{red}2}\,(4\pi t)^{1/2}}\int_0^\infty dy\,e^{-\frac{y^2}{t}}\frac{1}{\sinh y}\left(\frac{1}{\sinh\frac{y}{\kappa}}-\frac{\kappa}{\sinh y}\right)
\end{aligned} \tag{17.11}$$

$$\operatorname{Res}\left(\frac{1}{\sinh^2 y+\sin^2\frac{z}{2}}\frac{\cos\frac{z}{2}}{\sin\frac{z}{2\kappa}},\ z=0\right)=\frac{2\kappa}{\sinh^2 y} \tag{17.12}$$

$$\operatorname{Res}\left(\frac{1}{\sinh^2 y+\sin^2\frac{z}{2}}\frac{\cos\frac{z}{2}}{\sin\frac{z}{2\kappa}},\ z=\pm 2iy\right)=-\frac{1}{\sinh y\sinh\frac{y}{\kappa}} \tag{17.13}$$

(3.18)-(3.19) suffer from the same factor of -2 above.

Note: I have not found out why my exercise disagrees with section 3 of [16].

Considering now the hk given in [16], we can repeat the comparison of Seeley coefficients with those for orbifolds in literature:

- (3.20) is the expansion of hk for the Dirac operator *squared* on *Majorana* fermions.
- (B.6) [26] is the expansion of the hk of the Dirac operator for *Dirac* fermions, according to footnote 23.

Thus they coincide.

17.4 Appendix A.1

Below A.11: the orientation of Γ should be as below (A.8).

17.5 Appendix A.2

The hk is for the Dirac operator squared on Dirac spinors.
Below A.19: the orientation of Γ should be as below (A.8).

18 Expansions

- We need the first k^2-correction to $\mathrm{cn}\left(\frac{2}{\pi}\mathbb{K}\left(k^2\right)r|k^2\right)$. We begin with the integral representation

$$v = \int_{\mathrm{cn}(v,k^2)}^{1} \left(1-t^2\right)^{-1/2}\left(1-k^2+k^2t^2\right)^{-1/2} dt \qquad |v| \leq 1$$

  This is for us

$$\frac{2}{\pi}\mathbb{K}\left(k^2\right)r = \int_{\mathrm{cn}\left(\frac{2}{\pi}\mathbb{K}(k^2)r,k^2\right)}^{1} \left(1-t^2\right)^{-1/2}\left(1-k^2+k^2t^2\right)^{-1/2} dt$$

  and from the Taylor series of both sides we get

$$\mathrm{cn}\left(\frac{2}{\pi}\mathbb{K}\left(k^2\right)r, k^2\right) = \cos r - \frac{\sin^2 r\cos r}{4}k^2 + O\left(k^4\right)\,. \tag{18.1}$$

  It is straightforward then

$$\mathrm{sn}\left(\frac{2}{\pi}\mathbb{K}\left(k^2\right)r, k^2\right) = \sin r + \frac{\sin r\cos^2 r}{4}k^2 + O\left(k^4\right) \tag{18.2}$$

$$\mathrm{dn}\left(\frac{2}{\pi}\mathbb{K}\left(k^2\right)r, k^2\right) = 1 - \frac{\sin^2 r}{2}k^2 + O\left(k^4\right)\,. \tag{18.3}$$

- We have the identity

$$\mathrm{am}\left(\sigma + \mathbb{K}\left(k^2\right)|k^2\right) = \sigma + \frac{\pi}{2}\,. \tag{18.4}$$

- We want to expand $\Pi\left(\frac{b^4}{b^4+p^2}, \sigma+\frac{\pi}{2}|k^2\right) - \Pi\left(\frac{b^4}{b^4+p^2}|k^2\right)$. We use

$$\Pi\left(n,u|m\right) = \int_0^u \left(1-n\sin^2 t\right)^{-1}\left(1-m\sin^2 t\right)^{-1/2} dt \tag{18.5}$$

$$\Pi\left(n,|m\right) = \Pi\left(n,\frac{\pi}{2}|m\right) \tag{18.6}$$

  so that

$$\begin{aligned}\Pi\left(\frac{b^4}{b^4+p^2}, \sigma+\frac{\pi}{2}|k^2\right) - \Pi\left(\frac{b^4}{b^4+p^2}|k^2\right) &= \int_{\frac{\pi}{2}}^{\sigma+\frac{\pi}{2}} \left(1-\frac{b^4}{b^4+p^2}\sin^2 t\right)^{-1}\left(1-k^2\sin^2 t\right)^{-1/2} dt \\ &= \int_0^{\sigma} \left(1-\frac{b^4}{b^4+p^2}\cos^2 t\right)^{-1}\left(1-k^2\cos^2 t\right)^{-1/2} dt \\ &= -\cot\sigma\end{aligned} \tag{18.7}$$

19 Integrals

19.1 Formulas

We report some useful formulas for the perturbative expansion of the heat kernel propagators.

$$\frac{1}{\epsilon} - 2\pi = \int_x \sinh\rho = V_{H^2} \tag{19.1}$$

$$\pi = \int_x \frac{(1-\cosh\rho)^2}{\sinh^3\rho} = \int_x \frac{\sinh\rho}{(1+\cosh\rho)^2} \tag{19.2}$$

$$\frac{\pi}{2} = \int_x \frac{1}{\sinh\rho}\left(\frac{1-\cosh\rho}{1+\cosh\rho}\right)^2 \tag{19.3}$$

$$-\frac{\pi}{6} = \int_x \frac{\sinh\rho\left(-1+\cosh\rho-\cosh^2\rho\right)}{3(1+\cosh\rho)^4} \tag{19.4}$$

$$\frac{\pi}{12} = \int_x \frac{\sinh\rho}{(1+\cosh\rho)^4} \tag{19.5}$$

$$\frac{3\pi}{16} = \int_x \frac{\sinh\rho(\cosh\rho-1)\left(35-38\cosh\rho+11\cosh^2\rho\right)}{12(\cosh\rho+1)^5} \tag{19.6}$$

$$\frac{\pi}{48} = \int_x \frac{(1-\cosh\rho)^2}{\sinh\rho(1+\cosh\rho)^4} \tag{19.7}$$

$$\frac{\pi}{120} = \int_x \frac{1}{\sinh^3\rho}\left(\frac{1-\cosh\rho}{1+\cosh\rho}\right)^4 \tag{19.8}$$

$$\frac{\pi}{4} = \int_x \frac{1}{\sinh\rho}\left(\frac{1-\cosh\rho}{1+\cosh\rho}\right)^4 \tag{19.9}$$

$$-\pi = \int_x \sinh\rho\left(1-\cosh\rho\,\mathrm{asinh}\frac{1}{\sinh\rho}\right) \tag{19.10}$$

$$e^{\rho_0}T = \int_x \frac{1}{\cos^2 r} = V_{H^2} \tag{19.11}$$

$$(3e^{\rho_0}-2\pi)\,T = \int_x \left(\frac{3}{\cos^2 r}-2\right) \tag{19.12}$$

$$\frac{\pi T}{2} = \int_x \cos^2 r \tag{19.13}$$

$$\left(\pi - \frac{3e^{\rho_0}}{2}\right)T = \int_x \left(1-\frac{3}{2\cos^2 r}\right) \tag{19.14}$$

$$\left(\frac{3e^{\rho_0}}{4}-\pi\right)T = \int_x \left(-1-\frac{\cos 2r}{2}+\frac{3}{4\cos^2 r}\right) \tag{19.15}$$

To compute the zeta-functions, we use the following integrals ($a, \alpha > 0$). $A \approx 1.28243$ is the Glaisher constant, $\gamma \approx 0.57721$ is the Euler number and $\psi(x) \equiv \frac{d}{dx} \log \Gamma(x)$ is the digamma function.

$$\tanh \pi v = 1 - \frac{2}{e^{2\pi v} + 1} \tag{19.16}$$

$$\coth \pi v = 1 + \frac{2}{e^{2\pi v} - 1} \tag{19.17}$$

$$\frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} e^{-at} = \frac{1}{a^s} \qquad s > 0 \tag{19.18}$$

$$\frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s} e^{-at} = \frac{s}{a^{s+1}} \qquad s > 0 \tag{19.19}$$

$$\frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s+1} e^{-at} = \frac{s(s+1)}{a^{s+2}} \qquad s > 0 \tag{19.20}$$

$$\int_{\Lambda^{-2}}^\infty \frac{dt}{t} e^{-at} = \log \Lambda^2 - \gamma - \log a \qquad \Lambda \gg 0 \tag{19.21}$$

$$\int_{\Lambda^{-2}}^\infty dt\, e^{-at} = \frac{1}{a} \qquad \Lambda \gg 0 \tag{19.22}$$

$$\int_0^\infty dv \frac{v}{(v^2+\alpha)^s} = \frac{\alpha^{1-s}}{2(s-1)} \qquad s > 0 \tag{19.23}$$

$$\int_0^\infty dv \frac{v}{(v^2+\alpha)^{s+1}} = \frac{\alpha^{-s}}{2s} \qquad s > 0 \tag{19.24}$$

$$\int_0^\infty dv \frac{v}{(v^2+\alpha)^{s+2}} = \frac{\alpha^{-1-s}}{2(s+1)} \qquad s > 0 \tag{19.25}$$

$$\int_0^\infty dv \frac{v}{e^{2\pi v}+1} = \frac{1}{48} \tag{19.26}$$

$$\int_0^\infty dv \frac{v}{e^{2\pi v}-1} = \frac{1}{24} \tag{19.27}$$

$$\int_0^\infty dv \frac{v \log\left(v^2+\alpha\right)}{e^{2\pi v}+1} = \frac{\alpha}{4}(1-\log\alpha) + \frac{1+\log 2}{24} - \frac{\log A}{2} + \frac{1}{2}\int_0^\alpha dx \psi\left(\sqrt{x}+\frac{1}{2}\right) \tag{19.28}$$

$$\int_0^\infty dv \frac{v \log\left(v^2+\frac{1}{4}\right)}{e^{2\pi v}+1} = \frac{5}{48} - \frac{\log 2}{8} + \log A - \frac{\log \pi}{4} \tag{19.29}$$

$$\int_0^\infty dv \frac{v \log\left(v^2+\frac{9}{4}\right)}{e^{2\pi v}+1} = \frac{77}{48} + \frac{3\log 2}{8} - \frac{9\log 3}{8} + \log A - \frac{3\log \pi}{4} \tag{19.30}$$

$$\int_0^\infty dv \frac{v}{(e^{2\pi v}+1)(v^2+\alpha)} = -\frac{\log\alpha}{4} + \frac{1}{2}\psi\left(\sqrt{\alpha}+\frac{1}{2}\right) \tag{19.31}$$

$$\int_0^\infty dv \frac{v}{(e^{2\pi v}+1)\left(v^2+\frac{1}{4}\right)} = \frac{\log 2}{2} - \frac{\gamma}{2} \tag{19.32}$$

$$\int_0^\infty dv \frac{v}{(e^{2\pi v}+1)\left(v^2+\frac{9}{4}\right)} = -\frac{\log\frac{3}{2}}{2} + \frac{1}{2} - \frac{\gamma}{2} \tag{19.33}$$

$$\int_0^\infty dv \frac{v}{(e^{2\pi v}+1)(v^2+\alpha)^2} = \frac{1}{4\alpha} - \frac{1}{4\sqrt{\alpha}}\psi'\left(\sqrt{\alpha}+\frac{1}{2}\right) \tag{19.34}$$

$$\int_0^\infty dv \frac{v}{(e^{2\pi v}+1)\left(v^2+\frac{1}{4}\right)^2} = 1 - \frac{\pi^2}{12} \tag{19.35}$$

$$\int_0^\infty dv \frac{v}{(e^{2\pi v}+1)\left(v^2+\frac{9}{4}\right)^2} = \frac{5}{18} - \frac{\pi^2}{36} \tag{19.36}$$

$$\int_0^\infty dv \frac{v\log\left(v^2+\alpha\right)}{e^{2\pi v}-1} = \frac{\alpha}{4}\left(\log\alpha - 1\right) + \frac{1}{12} - \log A - \frac{\sqrt{\alpha}}{2} - \frac{1}{2}\int_0^\alpha dx \psi\left(\sqrt{x}\right) \tag{19.37}$$

$$\int_0^\infty dv \frac{v\log\left(v^2+1\right)}{e^{2\pi v}-1} = -\frac{2}{3} + \frac{\log 2}{2} - \log A + \frac{\log\pi}{2} \tag{19.38}$$

$$\int_0^\infty dv \frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+\alpha\right)} = \frac{\log\alpha}{4} - \frac{1}{4\sqrt{\alpha}} - \frac{1}{2}\psi\left(\sqrt{\alpha}\right) \tag{19.39}$$

$$\int_0^\infty dv \frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)} = -\frac{1}{4} + \frac{\gamma}{2} \tag{19.40}$$

$$\int_0^\infty dv \frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+\alpha\right)^2} = -\frac{1}{4\alpha} - \frac{1}{8\sqrt{\alpha}^3} + \frac{1}{4\sqrt{\alpha}}\psi^{'}\left(\sqrt{\alpha}\right) \tag{19.41}$$

$$\int_0^\infty dv \frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)^2} = -\frac{3}{8} + \frac{\pi^2}{24} \tag{19.42}$$

19.2 Some proofs

Taken from [17].

$$\begin{aligned}
&\int_0^\infty dv \frac{v\log\left(v^2+\alpha\right)}{e^{2\pi v}+1} \\
=&\int_0^\infty dv \left[\frac{2v\log v}{e^{2\pi v}+1} + \int_0^\alpha dx \frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+x\right)}\right] \\
=&\frac{1+\log 2}{24} - \frac{\log A}{2} + \int_0^\alpha dx \left[-\frac{\log x}{4} + \frac{1}{2}\psi\left(\sqrt{x}+\frac{1}{2}\right)\right] \\
=&\frac{1+\log 2}{24} - \frac{\log A}{2} + \frac{\alpha}{4}\left(1-\log\alpha\right) + \frac{1}{2}\int_0^\alpha dx\, \psi\left(\sqrt{x}+\frac{1}{2}\right)
\end{aligned} \tag{19.43}$$

$$\begin{aligned}
&\int_0^\infty dv \frac{v\log\left(v^2+\alpha\right)}{e^{2\pi v}-1} \\
=&\int_0^\infty dv \left[\frac{2v\log v}{e^{2\pi v}-1} + \int_0^\alpha dx \frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+x\right)}\right] \\
=&\frac{1}{12} - \log A + \int_0^\alpha dx \left[\frac{1}{4}\log x - \frac{1}{2\sqrt{x}} - \frac{1}{2}\psi\left(\sqrt{x}\right)\right] \\
=&\frac{1}{12} - \log A - \frac{\alpha}{4}\left(1-\log\alpha\right) - \frac{1}{2}\sqrt{\alpha} - \frac{1}{2}\int_0^\alpha dx\, \psi\left(\sqrt{x}\right)
\end{aligned} \tag{19.44}$$

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