Four-point function of determinants

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Contents

1 Summary

Sec. 11.1.2 [1] explains the context and footnote 114 suggests the project: determine the order of the phase transition (pt) in the 4-pt function of determinants at changing spacetime position, drawing an analogy to the pt of the thermal partition function $Z(\beta)$ of $SU(N)$ YM on $S^3 \times \mathbb{R}$ at changing temperature [2, 3]. The basic objective is to calculate the 4-pt at weak coupling, up to the lowest order that allows to understand the type of pt.

Details of calculation

It is possible to compute $Z(\beta)$ in two ways at zero coupling: by counting gauge-invariant states and by computing diagrams. The second derivation works for weak coupling too and provides the physical interpretation of $u_1 = \frac{1}{N}\mathrm{tr}(U)$ below as a Polyakov loop operator (= Wilson loop around the time/thermal circle). This is an order parameter for a large-N pt of $Z(\beta)$.

- Fix $\nabla_i A^i = 0$ (Coulomb gauge) and $\partial_t \alpha = 0$ (with $\alpha = g_{\mathrm{YM}} \frac{\int_{S^3} A_0}{\int_{S^3} 1}$, i.e. constant mode of A_0 is constant in time). The Euclidean partition function on $S^3_R \times S^1_\beta$ is

$$Z(\beta) = \int d\alpha\, \Delta_2(\alpha) \underbrace{\int dA\, \Delta_1(A)\, e^{-S_{\mathrm{YM}}(\alpha, A)}}_{e^{-S_{\mathrm{eff}}(\alpha)}} \tag{1.1}$$

  with the two FP determinants Δ's. The change of variable $U = e^{i\beta\alpha}$ delivers

$$Z(\beta) = \int dU\, e^{-S_{\mathrm{eff}}(U)}. \tag{1.2}$$

  α is the zero mode of A_0 and lightest mode of the gauge theory.

- The inner path integral is evaluated diagrammatically, generating an expansion of S_{eff} in powers of the coupling. At 1 loop, the Gaussian integration of A delivers

$$S^{\text{1-loop}}_{\mathrm{eff}} = \frac{11}{120}\beta N^2 - \sum_{n=1}^{\infty} \frac{z_V(x^n)}{n} \mathrm{tr}(U^n)\,\mathrm{tr}\left(U^{-n}\right) \qquad x = e^{-\beta} \qquad R = 1. \tag{1.3}$$

  At higher loop, it is convenient to expand the fields in spherical harmonics: the completeness relation turns integrals of fields into sums of their components. The action S_{eff} is obtained by integrating out the other fields, which is done by computing vacuum diagrams order-by-order in λ.

$$Z(\beta) = e^{-S^{\text{1-loop}}_{\mathrm{eff}}} \left\langle e^{-S^{\mathrm{int}}_{\mathrm{YM}}(U,A,c)} \right\rangle_{\mathrm{free}}. \tag{1.4}$$

- At large N, we exchange U for the phases θ_i of its eigenvalues $e^{i\theta_i}$. We define the density $\rho(\theta) = \frac{1}{N}\sum_{i=1}^N \delta(\theta - \theta_i)$ and its Fourier components $u_n = \int_0^{2\pi} e^{in\theta}\rho(\theta)\, d\theta = \mathrm{tr}(U^n)/N$. The path integral reduces to an effective matrix model.

$$Z(\beta) = \int du_n d\bar{u}_n\, e^{-N^2 S'_{\mathrm{eff}}(u_n)} \tag{1.5}$$

$$S'_{\mathrm{eff}}(u_n) = \frac{11}{120}\beta + \sum_{n=1}^{\infty} \frac{1}{n}\left(1 - z_V(x^n)\right)|u_n|^2 \tag{1.6}$$

  The new action $S'_{\mathrm{eff}}(u_n)$ includes the (logarithm of the) Vandermonde determinant.

- Close to the pt, all $u_{n\geq 1}$ are massive while u_1 is the lightest mode and massless (= a zero mode, because its quadratic action is zero). The former can be integrated out in perturbation theory, the latter cannot and it is treated as a background field. One is led to study:

$$S'_{\mathrm{eff}}(u_1) \sim m_1 |u_1|^2 + b|u_1|^4.$$

- The actual phase pt depends on the sign of b (below and sec. 6.3 [2]). For large N, the saddle-pt configuration is $\delta S'_{\mathrm{eff}} = 0$. When the configuration changes abruptly as a function of temperature, one may have a pt. Confinement occurs when this is $u_n = 0$. When no longer true, the pt to the deconfined phase occurs.

Possible regimes and phase transitions

There are 2 regimes.

- For low T: the theory is confided; $\rho(\theta)$ is constant.
- For high T: the theory is deconfined; $\rho(\theta)$ is sinusoidal and non-zero in a narrow interval.

There are 3 phase transitions.

- First-order: $\rho(\theta)$ jumps suddenly from uniform to sinusoidal.
- Second-order: $\rho(\theta)$ evolves continuously from uniform to non-uniform and nowhere vanishing.
- Third-order: $\rho(\theta)$ develops a gap on which it vanishes.

Transitions occur at infinite N, otherwise a finite number of dof in finite volume smoothens it out.

- For zero coupling: first-order pt at $T_1 = T_H$ (true Hagedorn transition).
- At weak coupling:
  if $b \lesssim 0$ near T_H: first-order pt at $T^{'} < T_H$ (this shields Hagedorn transition at T_H).
  if $b \gtrsim 0$ near T_H: second-order pt at T_H (true Hagedorn transition) and third-order pt at $T^{''} \gtrsim T_H$.
  if $b = 0$ near T_H: further study needed.

At zero coupling, the only pt is first order, but it lies precisely at the border between first- and second-order behavior. To understand the nature of the pt at non-zero coupling, the leading effects of interaction terms must be taken into account via the perturbative calculation.

2 4-pt function

2.1 Effective theory

The aim is twofold: clarify the domain of ρ and include 1-loop interactions perturbatively.

$$G_m = \langle \mathcal{D}_1 \dots \mathcal{D}_m \rangle \tag{2.1}$$

$\langle \dots \rangle$ is weighted by the full SYM action S_{YM}. We will be interested in 1-loop correction of $\langle \mathcal{D}_1 \dots \mathcal{D}_4 \rangle$: we drop the single traces from the interacting part of S_{YM} and then $\langle \dots \rangle$ can be weighted by the free SYM action.

Step 1

$$\begin{aligned} G_m &= \frac{1}{Z} \int DA_\mu \, D\Phi^I \, D\Psi \, Dc \, (\mathcal{D}_1 \dots \mathcal{D}_m) \exp\left(-S_{\text{YM}}\left(A_\mu, \Phi^I, \Psi, c\right)\right) \\ &= \frac{1}{Z} \int DA_\mu \, D\Phi^I \, D\Psi \, Dc \, d\bar{\chi}_k d\chi_k \, \exp\left(-S_{\text{YM}}\left(A_\mu, \Phi^I, \Psi, c\right) + \sum_{i=1}^{m} \int_x \delta^4\left(x - x_i\right) \bar{\chi}_i \left(Y_i \cdot \Phi\right) \chi_i\right) \end{aligned} \tag{2.2}$$

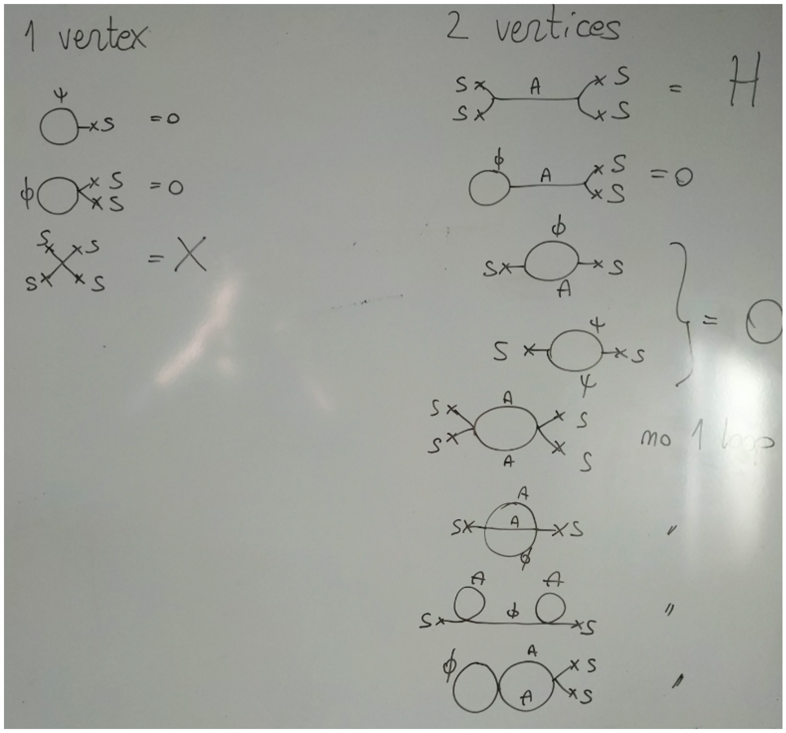
Step 2

$$\begin{aligned} G_m &= \frac{1}{Z} \int DA_\mu \, D\Phi^I \, D\Psi \, Dc \, d\bar{\chi}_k d\chi_k \, \exp\left(-S^{\text{quad}}_{\text{YM}}\left(A_\mu, \Phi^I - S^I, \Psi, c\right) - S^{\text{int}}_{\text{YM}}\left(A_\mu, \Phi^I, \Psi, c\right) - \frac{1}{g^2_{\text{YM}}} \int_x \operatorname{tr}\left(S^I \Box S_I\right)\right) \\ &= \frac{1}{Z} \int DA_\mu \, D\Phi^I \, D\Psi \, Dc \, d\bar{\chi}_k d\chi_k \, \exp\left(-S^{\text{quad}}_{\text{YM}}\left(A_\mu, \Phi^I, \Psi, c\right) - S^{\text{int}}_{\text{YM}}\left(A_\mu, \Phi^I + S^I, \Psi, c\right) - \frac{1}{g^2_{\text{YM}}} \int_x \operatorname{tr}\left(S^I \Box S_I\right)\right) \end{aligned}$$

We completed the square (which affects only the quadratic action) and shifted the scalars (so now it's the interacting part to have extra terms). We have to compute this modified SYM partition function at 1 loop.

- The normalization is the standard SYM partition function and removes S-independent diagrams.
- Propagators and S contribute with g^2_{YM}, vertices with g^{-2}_{YM}.
- 1-loop diagrams $\sim S^n$ are of order g^{2n}_{YM}. Higher powers kick in at higher loops.

The rules select the well-known diagrams O (here split into 2), X and H, but with the external scalars not contracted and replaced by S's.



We take the relevant interaction in the Lagrangian in *Weak coupling*

$$
\begin{aligned}
S^{\text{int}}_{\text{YM}}\left(A_\mu, \Phi^I, \Psi, c\right) &= \int_x \left\{ \frac{1}{g^2_{\text{YM}}} f^{abc} \partial_\mu \Phi^a_I A^b_\mu \Phi^c_I + \frac{i}{2g^2_{\text{YM}}} f^{abc} \left(\bar{\psi}^a \Gamma^I \psi^c\right) \Phi^b_I + \frac{1}{4g^2_{\text{YM}}} f^{abc} f^{ade} \Phi^b_I \Phi^c_J \Phi^d_I \Phi^e_J + \ldots \right\} \\
&= \int_x \left\{ -\frac{2i}{g^2_{\text{YM}}} \text{tr}\left(\partial_\mu \Phi_i \left[A_\mu, \Phi_I\right]\right) - \frac{1}{g^2_{\text{YM}}} \Gamma^I_{\alpha\beta} \text{tr}\left(\bar{\psi}_\alpha \left[\psi_\beta, \Phi_I\right]\right) - \frac{1}{2g^2_{\text{YM}}} \text{tr}\left(\left[\Phi_I, \Phi_J\right]\left[\Phi_I, \Phi_J\right]\right) + \ldots \right\}
\end{aligned}
\tag{2.3}
$$

and use them to write the diagrams

$$
\frac{1}{Z} \int DA_\mu \, D\Phi^I \, D\Psi \, Dc \, d\bar{\chi}_k d\chi_k \, \frac{1}{2!} \left[\frac{2i}{g^2_{\text{YM}}} \int_x \text{tr}\left(\partial_\mu \Phi_I \left[A_\mu, S_I\right]\right) + \frac{2i}{g^2_{\text{YM}}} \int_x \text{tr}\left(\partial_\mu S_I \left[A_\mu, \Phi_I\right]\right) \right]^2 \tag{2.4}
$$

$$
\times \exp\left(-S^{\text{quad}}_{\text{YM}}\left(A_\mu, \Phi^I, \Psi, c\right) - \frac{1}{g^2_{\text{YM}}} \int_x \text{tr}\left(S^I \Box S_I\right) \right)
$$

$$
\frac{1}{Z} \int DA_\mu \, D\Phi^I \, D\Psi \, Dc \, d\bar{\chi}_k d\chi_k \, \frac{1}{2!} \left[\frac{1}{g^2_{\text{YM}}} \Gamma^I_{\alpha\beta} \int_x \text{tr}\left(\bar{\psi}_\alpha \left[\psi_\beta, S_I\right]\right) \right]^2 \exp\left(-S^{\text{quad}}_{\text{YM}}\left(A_\mu, \Phi^I, \Psi, c\right) - \frac{1}{g^2_{\text{YM}}} \int_x \text{tr}\left(S^I \Box S_I\right) \right) \tag{2.5}
$$

$$
\frac{1}{Z} \int DA_\mu \, D\Phi^I \, D\Psi \, Dc \, d\bar{\chi}_k d\chi_k \left[\frac{1}{2g^2_{\text{YM}}} \int_x \text{tr}\left(\left[S_I, S_J\right]\left[S_I, S_J\right]\right) \right] \exp\left(-S^{\text{quad}}_{\text{YM}}\left(A_\mu, \Phi^I, \Psi, c\right) - \frac{1}{g^2_{\text{YM}}} \int_x \text{tr}\left(S^I \Box S_I\right) \right) \tag{2.6}
$$

$$
\frac{1}{Z} \int DA_\mu \, D\Phi^I \, D\Psi \, Dc \, d\bar{\chi}_k d\chi_k \, \frac{1}{2!} \left[\frac{2i}{g^2_{\text{YM}}} \int_x \text{tr}\left(\partial_\mu S_I \left[A_\mu, S_I\right]\right) \right]^2 \exp\left(-S^{\text{quad}}_{\text{YM}}\left(A_\mu, \Phi^I, \Psi, c\right) - \frac{1}{g^2_{\text{YM}}} \int_x \text{tr}\left(S^I \Box S_I\right) \right). \tag{2.7}
$$

We make manifest the scalar propagator I and R-symmetry contraction Y in S^I [1]

$$
S^I(x) = -\frac{g^2_{\text{YM}}}{2} \sum_k I_{xx_k} Y^I_k \chi_k \bar{\chi}_k \,. \tag{2.8}
$$

The diagrams become similar to the original O, X and H (which have the external scalars contracted with the determinants' scalars) in the PCGG approach. This can be made more explicit and lead to a general shortcut. For example, for the diagram X above

$$
\frac{N}{2\lambda} \left(\frac{\lambda}{2N} \right)^4 \sum_{k_1 \neq k_2} \sum_{k_3 \neq k_4} X_{k_1 k_2 k_3 k_4} \left(Y_{k_1} \cdot Y_{k_2}\right) \left(Y_{k_3} \cdot Y_{k_4}\right) \tag{2.9}
$$

[1] A minus is missing in the right side of (3.8) [1].

$$\int d\bar{\chi}_k d\chi_k \,\mathrm{tr}\left(\left[\chi_{k_1}\bar{\chi}_{k_1},\chi_{k_3}\bar{\chi}_{k_3}\right]\left[\chi_{k_2}\bar{\chi}_{k_2},\chi_{k_4}\bar{\chi}_{k_4}\right]\right)\exp\left(-\frac{1}{g_{\mathrm{YM}}^2}\int_x \mathrm{tr}\left(S^I\Box S_I\right)\right)$$

and the diagram H

$$\begin{aligned}
&\frac{1}{2!}\left(\frac{2iN}{\lambda}\right)^2\int_{x,y}\left(\frac{g_{\mathrm{YM}}^2}{2}I_{xy}\right)\int d\bar{\chi}_k d\chi_k \mathrm{tr}\left(\left[S_I,\partial_\mu S_I\right]\left[S_J,\partial_\mu S_J\right]\right)\exp\left(-\frac{1}{g_{\mathrm{YM}}^2}\int_x \mathrm{tr}\left(S^I\Box S_I\right)\right) \\
=&\frac{1}{2!}\left(\frac{2iN}{\lambda}\right)^2\left(\frac{\lambda}{2N}\right)^4\left(\frac{g_{\mathrm{YM}}^2}{2}\right)\sum_{k_1\neq k_2}\sum_{k_3\neq k_4}\left(Y_{k_1}\cdot Y_{k_2}\right)\left(Y_{k_3}\cdot Y_{k_4}\right)\int_{x,y} I_{xx_{k_1}}\partial_{x^\mu}I_{xx_{k_2}}I_{xy}I_{yx_{k_3}}\partial_{x^\mu}I_{yx_{k_4}} \\
&\int d\bar{\chi}_k d\chi_k \mathrm{tr}\left(\left[\chi_{k_1}\bar{\chi}_{k_1},\chi_{k_2}\bar{\chi}_{k_2}\right]\left[\chi_{k_3}\bar{\chi}_{k_3},\chi_{k_4}\bar{\chi}_{k_4}\right]\right)\exp\left(-\frac{1}{g_{\mathrm{YM}}^2}\int_x \mathrm{tr}\left(S^I\Box S_I\right)\right) \\
=&\frac{1}{2!}\left(\frac{2iN}{\lambda}\right)^2\left(\frac{\lambda}{2N}\right)^4\left(\frac{g_{\mathrm{YM}}^2}{2}\right)\frac{1}{4}\sum_{k_1\neq k_2}\sum_{k_3\neq k_4}\left(Y_{k_1}\cdot Y_{k_2}\right)\left(Y_{k_3}\cdot Y_{k_4}\right)I_{k_1k_2}I_{k_3k_4}F_{k_1k_1,k_3k_4} \\
&\int d\bar{\chi}_k d\chi_k \mathrm{tr}\left(\left[\chi_{k_1}\bar{\chi}_{k_1},\chi_{k_2}\bar{\chi}_{k_2}\right]\left[\chi_{k_3}\bar{\chi}_{k_3},\chi_{k_4}\bar{\chi}_{k_4}\right]\right)\exp\left(-\frac{1}{g_{\mathrm{YM}}^2}\int_x \mathrm{tr}\left(S^I\Box S_I\right)\right)
\end{aligned} \tag{2.10}$$

using the identity [4] [2]

$$\begin{aligned}
&\int_{x,y} I_{xx_{k_1}}\partial_{x^\mu}I_{xx_{k_2}}I_{xy}I_{yx_{k_3}}\partial_{x^\mu}I_{yx_{k_4}} \\
=&\frac{1}{4}\left(\partial_{x_1}-\partial_{x_2}\right)\cdot\left(\partial_{x_3}-\partial_{x_4}\right)\mathcal{H}\left(x_{k_1},x_{k_2};x_{k_3},x_{k_4}\right) \\
=&\frac{1}{4}I_{k_1k_2}I_{k_3k_4}F_{k_1k_1,k_3k_4}
\end{aligned} \tag{2.11}$$

From here, one can read off $\mathcal{O}_X^S$ and $\mathcal{O}_H^S$, while $\mathcal{O}_O^S$ would require more work. In general, we can take an approach that recycles the basic building blocks in perturbative SYM in *Weak coupling* 2 and literature. We engineer

$$\begin{aligned}
\mathcal{O}_O &= -\frac{1}{2!}\frac{\lambda^2}{N}\left(\frac{\lambda}{N}\right)^{-2}\delta_{I_1I_2}\left(\delta^{a_1a_2}-\delta^{a_1N^2}\delta^{a_2N^2}\right)\int_{y_1y_2}\left(Y_{112}+Y_{122}\right)\Box_{y_1}\phi_{I_1}^{a_1}\left(y_1\right)\Box_{y_2}\phi_{I_2}^{a_2}\left(y_2\right) \\
&= -\frac{1}{2!}\frac{\lambda^2}{N}\left(\frac{\lambda}{N}\right)^{-2}\int_{y_1y_2}\left(Y_{112}+Y_{122}\right)\Box_{y_1}\Box_{y_2}\left[2\mathrm{tr}\left(\phi_I\left(y_1\right)\phi_I\left(y_2\right)\right)-\frac{2}{N}\mathrm{tr}\phi_I\left(y_1\right)\mathrm{tr}\phi_I\left(y_2\right)\right]
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
\mathcal{O}_X &= \frac{1}{4!}\frac{\lambda^3}{N^3}\left(\frac{\lambda}{N}\right)^{-4}\left[-\left(f^{aa_1a_3}f^{aa_2a_4}+f^{aa_1a_4}f^{aa_2a_3}\right)\delta_{I_1I_2}\delta_{I_3I_4}+\left(f^{aa_1a_4}f^{aa_2a_3}-f^{aa_1a_2}f^{aa_3a_4}\right)\delta_{I_1I_3}\delta_{I_2I_4}\right. \\
&\quad \left.+\left(f^{aa_1a_2}f^{aa_3a_4}+f^{aa_1a_3}f^{aa_2a_4}\right)\delta_{I_1I_4}\delta_{I_2I_3}\right] \\
&\quad \int_{y_1y_2y_3y_4} X_{1234}\,\Box_{y_1}\phi_{I_1}^{a_1}\left(y_1\right)\Box_{y_2}\phi_{I_2}^{a_2}\left(y_2\right)\Box_{y_3}\phi_{I_3}^{a_3}\left(y_3\right)\Box_{y_4}\phi_{I_4}^{a_4}\left(y_4\right) \\
&= -\frac{2}{4!}\frac{\lambda^3}{N^3}\left(\frac{\lambda}{N}\right)^{-4}\int_{y_1y_2y_3y_4} X_{1234}\Box_{y_1}\Box_{y_2}\Box_{y_3}\Box_{y_4} \\
&\quad \left[-\mathrm{tr}\left(\left[\phi_I\left(y_1\right),\phi_J\left(y_3\right)\right]\left[\phi_I\left(y_2\right),\phi_J\left(y_4\right)\right]\right)-\mathrm{tr}\left(\left[\phi_I\left(y_1\right),\phi_J\left(y_4\right)\right]\left[\phi_I\left(y_2\right),\phi_J\left(y_3\right)\right]\right)\right. \\
&\quad +\mathrm{tr}\left(\left[\phi_I\left(y_1\right),\phi_J\left(y_4\right)\right]\left[\phi_J\left(y_2\right),\phi_I\left(y_3\right)\right]\right)-\mathrm{tr}\left(\left[\phi_I\left(y_1\right),\phi_J\left(y_2\right)\right]\left[\phi_I\left(y_3\right),\phi_J\left(y_4\right)\right]\right) \\
&\quad \left.+\mathrm{tr}\left(\left[\phi_I\left(y_1\right),\phi_J\left(y_2\right)\right]\left[\phi_J\left(y_3\right),\phi_I\left(y_4\right)\right]\right)+\mathrm{tr}\left(\left[\phi_I\left(y_1\right),\phi_J\left(y_3\right)\right]\left[\phi_J\left(y_2\right),\phi_I\left(y_4\right)\right]\right)\right]
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
\mathcal{O}_H &= \frac{1}{4!}\frac{\lambda^3}{N^3}\left(\frac{\lambda}{N}\right)^{-4}\int_{y_1y_2y_3y_4}\left[f^{aa_1a_2}f^{aa_3a_4}\delta_{I_1I_2}\delta_{I_3I_4}I_{12}I_{34}F_{12,34}+f^{aa_1a_3}f^{aa_2a_4}\delta_{I_1I_3}\delta_{I_2I_4}I_{13}I_{24}F_{13,24}\right. \\
&\quad \left.+f^{aa_1a_4}f^{aa_2a_3}\delta_{I_1I_4}\delta_{I_2I_3}I_{14}I_{23}F_{14,23}\right]\Box_{y_1}\phi_{I_1}^{a_1}\left(y_1\right)\Box_{y_2}\phi_{I_2}^{a_2}\left(y_2\right)\Box_{y_3}\phi_{I_3}^{a_3}\left(y_3\right)\Box_{y_4}\phi_{I_4}^{a_4}\left(y_4\right) \\
&= -\frac{2}{4!}\frac{\lambda^3}{N^3}\left(\frac{\lambda}{N}\right)^{-4}\int_{y_1y_2y_3y_4}\left[I_{12}I_{34}F_{12,34}\Box_{y_1}\Box_{y_2}\Box_{y_3}\Box_{y_4}\mathrm{tr}\left(\left[\phi_I\left(y_1\right),\phi_I\left(y_2\right)\right]\left[\phi_J\left(y_3\right),\phi_J\left(y_4\right)\right]\right)\right. \\
&\quad +I_{13}I_{24}F_{13,24}\Box_{y_1}\Box_{y_2}\Box_{y_3}\Box_{y_4}\mathrm{tr}\left(\left[\phi_I\left(y_1\right),\phi_I\left(y_3\right)\right]\left[\phi_J\left(y_2\right),\phi_J\left(y_4\right)\right]\right) \\
&\quad \left.+I_{14}I_{23}F_{14,23}\Box_{y_1}\Box_{y_2}\Box_{y_3}\Box_{y_4}\mathrm{tr}\left(\left[\phi_I\left(y_1\right),\phi_I\left(y_4\right)\right]\left[\phi_J\left(y_2\right),\phi_J\left(y_3\right)\right]\right)\right]
\end{aligned} \tag{2.14}$$

such that they reproduce (from first lines) the known SYM correlators in *Weak coupling 2*

$$\left\langle\phi_{I_1}^{a_1}\left(x_1\right)\phi_{I_2}^{a_2}\left(x_2\right)\mathcal{O}_O\right\rangle_{\mathrm{LO}}=\left\langle\phi_{I_1}^{a_1}\left(x_1\right)\phi_{I_2}^{a_2}\left(x_2\right)V_O\right\rangle_{\mathrm{LO}} \tag{2.15}$$

[2] The first equality is valid inside the integrand due to the anti-symmetrization on k's.

$$\left\langle \phi^{a_1}_{I_1}(x_1)\,\phi^{a_2}_{I_2}(x_2)\,\phi^{a_3}_{I_3}(x_3)\,\phi^{a_4}_{I_4}(x_4)\,\mathcal{O}_X\right\rangle_{\rm LO} = \left\langle \phi^{a_1}_{I_1}(x_1)\,\phi^{a_2}_{I_2}(x_2)\,\phi^{a_3}_{I_3}(x_3)\,\phi^{a_4}_{I_4}(x_4)\,V_X\right\rangle_{\rm LO} \tag{2.16}$$

$$\left\langle \phi^{a_1}_{I_1}(x_1)\,\phi^{a_2}_{I_2}(x_2)\,\phi^{a_3}_{I_3}(x_3)\,\phi^{a_4}_{I_4}(x_4)\,\mathcal{O}_H\right\rangle_{\rm LO} = \left\langle \phi^{a_1}_{I_1}(x_1)\,\phi^{a_2}_{I_2}(x_2)\,\phi^{a_3}_{I_3}(x_3)\,\phi^{a_4}_{I_4}(x_4)\,V_H\right\rangle_{\rm LO}\,. \tag{2.17}$$

We work with the second lines, do step 2, reproduce $\mathcal{O}^S_X$ and $\mathcal{O}^S_H$ while find $\mathcal{O}^S_O$ for the first time.

$$\mathcal{O}^S_O = -\frac{1}{2!}\frac{\lambda^2}{N}\left(\frac{\lambda}{N}\right)^{-2}\left(\frac{\lambda}{2N}\right)^{2}\sum_{k_1\neq k_2}\left(Y_{k_1k_1k_2}+Y_{k_1k_2k_2}\right)\left(Y_{k_1}\cdot Y_{k_2}\right)\left[2{\rm tr}\left(\chi_{k_1}\bar\chi_{k_1}\chi_{k_2}\bar\chi_{k_2}\right)-\frac{2}{N}{\rm tr}\left(\chi_{k_1}\bar\chi_{k_1}\right){\rm tr}\left(\chi_{k_2}\bar\chi_{k_2}\right)\right] \tag{2.18}$$

$$\begin{aligned}\mathcal{O}^S_X &= -\frac{4}{4!}\frac{\lambda^3}{N^3}\left(\frac{\lambda}{N}\right)^{-4}\left(\frac{\lambda}{2N}\right)^{4}\sum_{k_1\neq k_2}\sum_{k_3\neq k_4}X_{k_1,k_2,k_3,k_4}\left(Y_{k_1}\cdot Y_{k_2}\right)\left(Y_{k_3}\cdot Y_{k_4}\right)\left[-{\rm tr}\left(\left[\chi_{k_1}\bar\chi_{k_1},\chi_{k_3}\bar\chi_{k_3}\right]\left[\chi_{k_2}\bar\chi_{k_2},\chi_{k_4}\bar\chi_{k_4}\right]\right)\right.\\ &\quad\left.+{\rm tr}\left(\left[\chi_{k_1}\bar\chi_{k_1},\chi_{k_4}\bar\chi_{k_4}\right]\left[\chi_{k_3}\bar\chi_{k_3},\chi_{k_2}\bar\chi_{k_2}\right]\right)+{\rm tr}\left(\left[\chi_{k_1}\bar\chi_{k_1},\chi_{k_3}\bar\chi_{k_3}\right]\left[\chi_{k_4}\bar\chi_{k_4},\chi_{k_2}\bar\chi_{k_2}\right]\right)\right]\\ &= -\frac{24}{4!}\frac{\lambda^3}{N^3}\left(\frac{\lambda}{N}\right)^{-4}\left(\frac{\lambda}{2N}\right)^{4}\sum_{k_1\neq k_2}\sum_{k_3\neq k_4}X_{k_1,k_2,k_3,k_4}\left(Y_{k_1}\cdot Y_{k_2}\right)\left(Y_{k_3}\cdot Y_{k_4}\right){\rm tr}\left(\chi_{k_1}\bar\chi_{k_1}\chi_{k_3}\bar\chi_{k_3}\left[\chi_{k_4}\bar\chi_{k_4},\chi_{k_2}\bar\chi_{k_2}\right]\right)\end{aligned} \tag{2.19}$$

$$\mathcal{O}^S_H = -\frac{24}{4!}\frac{\lambda^3}{N^3}\left(\frac{\lambda}{N}\right)^{-4}\left(\frac{\lambda}{2N}\right)^{4}\sum_{k_1\neq k_2}\sum_{k_3\neq k_4}I_{k_1k_2}I_{k_3k_4}F_{k_1k_2,k_3k_4}\left(Y_{k_1}\cdot Y_{k_2}\right)\left(Y_{k_3}\cdot Y_{k_4}\right){\rm tr}\left(\chi_{k_1}\bar\chi_{k_1}\chi_{k_2}\bar\chi_{k_2}\chi_{k_3}\bar\chi_{k_3}\chi_{k_4}\bar\chi_{k_4}\right) \tag{2.20}$$

The upshot of step 2 is

$$G_m = \int d\bar\chi_k d\chi_k\left(1+\mathcal{O}^S_O+\mathcal{O}^S_X+\mathcal{O}^S_H\right)\,\exp\left[-\frac{g^2}{N}\sum_{i\neq j}d_{ij}\left(\bar\chi_i\chi_j\right)\left(\bar\chi_j\chi_i\right)\right]. \tag{2.21}$$

Step 3

$$\begin{aligned}G_m &= \int d\bar\chi_k d\chi_k\left(1+\mathcal{O}^S_O+\mathcal{O}^S_X+\mathcal{O}^S_H\right)\,\exp\left[-\frac{g^2}{N}\sum_{i\neq j}d_{ij}\left(\bar\chi_i\chi_j\right)\left(\bar\chi_j\chi_i\right)\right]\\ &= \frac{1}{Z_\rho}\int d\rho d\bar\chi_k d\chi_k\left(1+\mathcal{O}^S_O+\mathcal{O}^S_X+\mathcal{O}^S_H\right)\,\exp\left[\sum_{i,j=1}^{m}\left(-\frac{N}{g^2}\rho_{ij}\rho_{ji}+2i\sqrt{d_{ij}}\rho_{ij}\bar\chi_j\chi_i\right)\right]\\ &= \frac{1}{Z_\rho}\int d\rho d\bar\chi_k d\chi_k\left(1+\mathcal{O}^S_O+\mathcal{O}^S_X+\mathcal{O}^S_H\right)\,\exp\left[-\frac{N}{g^2}{\rm Tr}\left(\rho^2\right)+2i{\rm Tr}\left(\hat\rho\bar\chi\chi\right)\right]\end{aligned} \tag{2.22}$$

with

$$\rho=\rho^\dagger\qquad \rho_{ij}=0 \text{ if } i=j\qquad \int d\rho=\int_{-\infty}^{\infty}\prod_{i<j}d{\rm Re}\left(\rho_{ij}\right)\,d{\rm Im}\left(\rho_{ij}\right) \tag{2.23}$$

$$Z_\rho=\int d\rho\,\exp\left(-\frac{N}{g^2}{\rm Tr}\left(\rho^2\right)\right)=\left(\frac{g^2\pi}{2N}\right)^{m(m-1)/2}. \tag{2.24}$$

Step 4

$$G_m=\frac{1}{Z_\rho}\int d\rho\left(1+\left\langle\mathcal{O}^S_O+\mathcal{O}^S_X+\mathcal{O}^S_H\right\rangle_\chi\right)\,\exp\left(-NS_{\rm eff}\left[\rho\right]\right) \tag{2.25}$$

with

$$S_{\rm eff}\left[\rho\right]=\frac{1}{g^2}{\rm Tr}\left(\rho^2\right)-\log{\rm Det}\left(-2i\hat\rho\right) \tag{2.26}$$

$$\left\langle\mathcal{O}^S\right\rangle_\chi=\frac{\int d\bar\chi_k d\chi_k\mathcal{O}^S\,\exp\left[2i{\rm Tr}\left(\hat\rho\bar\chi\chi\right)\right]}{\int d\bar\chi_k d\chi_k\,\exp\left[2i{\rm Tr}\left(\hat\rho\bar\chi\chi\right)\right]} \tag{2.27}$$

$$\left\langle\bar\chi_{i,i_1}\chi^{i_2}_j\right\rangle_\chi=-\frac{i}{2}\delta^{i_2}_{i_1}\left(\hat\rho^{-1}\right)_{ij}. \tag{2.28}$$

To see the propagator, we write the Dirac generating functional

$$Z\left[\eta,\bar\eta\right]=\int d\bar\chi_k d\chi_k\,\exp\left(2i\hat\rho_{ij}\bar\chi_j\chi_i+\bar\eta_i\chi_i+\bar\chi_i\eta_i\right) \tag{2.29}$$

$$= Z[0,0] \exp\left[-\frac{i}{2}\eta_i \left(\hat{\rho}^{-1}\right)_{ij} \bar{\eta}_j\right] ,$$

where the second line comes from shifting the integration field variables

$$\chi_i \to \chi_i + \frac{i}{2}\eta_j \left(\hat{\rho}^{-1}\right)_{ji} \tag{2.30}$$

$$\bar{\chi}_i \to \bar{\chi}_i + \frac{i}{2} \left(\hat{\rho}^{-1}\right)_{ij} \bar{\eta}_j . \tag{2.31}$$

We obtain the Feynman rule by applying $\frac{1}{Z[0,0]} \frac{\delta}{\delta \bar{\eta}_j^{i_2}} \frac{\delta}{\delta \eta_{i,i_1}}$ to both lines and setting sources to zero. Alternatively (9.70) [5].

The saddle-pt equations are

$$\rho_{ij} = \frac{g^2}{2} \left(\hat{\rho}^{-1}\right)_{ij} \sqrt{d_{ij}} . \tag{2.32}$$

2.2 Effective theory - simplest 4-pt - tree level

The simplest 4-pt correlator G_4 has $Y_1 \cdot Y_3 = Y_2 \cdot Y_4 = 0 \to \ell_{13} = 0$. The saddle pt is

$$|\rho_{12}|^2 = |\rho_{34}|^2 = \frac{g^2}{2} \qquad \text{or} \qquad |\rho_{14}|^2 = |\rho_{23}|^2 = \frac{g^2}{2} . \tag{2.33}$$

For the first solution, we change the coordinates to

$$\rho_{12} = r_{12} e^{i\theta_{12}} \tag{2.34}$$

$$\rho_{34} = r_{34} e^{i\theta_{34}} . \tag{2.35}$$

The effect

$$\int d\rho \, \exp\left(-NS_{\text{eff}}\right) = \int d\rho^{'} \, \exp\left(-NS_{\text{eff}}^{'}\right) \tag{2.36}$$

$$= \int_0^\infty dr_{12} dr_{34} \int_0^{2\pi} d\theta_{12} d\theta_{34} \int_{-\infty}^{\infty} \prod_{(i,j)\in\{(1,3),(1,4),(2,3),(2,4)\}} d\mathrm{Re}\left(\rho_{ij}\right) \, d\mathrm{Im}\left(\rho_{ij}\right) \, \exp\left(-NS_{\text{eff}}^{'}\right)$$

$$-NS_{\text{eff}}^{'} = -NS_{\text{eff}} + \log\left(r_{12} r_{34}\right) \tag{2.37}$$

changes the saddle-pt

$$r_{12} = r_{34} = \frac{g}{\sqrt{2}} \sqrt{1 + \frac{1}{2N}} \qquad \forall \theta_{12}, \theta_{34} . \tag{2.38}$$

We expand in ρ_{ij}, r_{12} and r_{34} around the saddle-pt, set the strategy

$$\frac{(G_4)_{\text{LO}}}{\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle} = \frac{1}{\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle} \frac{1}{Z_\rho} \int d\rho^{'} \, \underbrace{\exp\left(-NS_{\text{eff}}^{'(\text{int})}\right)}_{\text{expand in series}} \exp\left(-NS_{\text{eff}}^{'(\text{quad})}\right) \tag{2.39}$$

and explicitate powers of fields and N

$$\frac{1}{\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle} \frac{1}{Z_\rho} \int d\rho^{'} \, \exp\left(-NS_{\text{eff}}^{'(\text{quad})}\right) = N^5 \int d\rho \, \exp\left(N\left(\sum \text{field}^2\right)\right) \tag{2.40}$$

$$\exp\left(-NS_{\text{eff}}^{'(\text{int})}\right) = \exp\left(N \sum_{i=3}^{\infty} \left(\sum \text{field}^i\right)\right) = \prod_{i=3}^{\infty} \sum_{n_i=0}^{\infty} \frac{N^{n_i}}{n_i!} \left(\sum \text{field}^i\right)^{n_i} . \tag{2.41}$$

If we temporarily rescale fields by $N^{-1/2}$ (*)

$$\frac{1}{\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle} \frac{1}{Z_\rho} \int d\rho^{'} \, \exp\left(-NS_{\text{eff}}^{'(\text{quad})}\right) \to \int d\rho^{'} \, \exp\left(\sum \text{field}^2\right) \tag{2.42}$$

$$\exp\left(-NS_{\text{eff}}^{'(\text{int})}\right) \to \prod_{i=3}^{\infty} \sum_{n_i=0}^{\infty} \frac{N^{\frac{1}{2}(2-i)n_i}}{n_i!} \left(\sum \text{field}^i\right)^{n_i} , \tag{2.43}$$

we see the term $(n_3, n_4, \dots)$ produces N^{-n} with $n = \frac{1}{2}\sum_{i=3}^{\infty}(i-2)\,n_i$. If we aim at all powers higher or equal to N^{-n}, we can cut off

$$i = 3, \dots 2n+2 \tag{2.44}$$

$$n_i = 0, \dots 2n\,. \tag{2.45}$$

We check that

$$\frac{(G_4)_{\mathrm{LO}}}{\langle \mathcal{O}_1\mathcal{O}_2\rangle\langle \mathcal{O}_3\mathcal{O}_4\rangle} = \frac{1}{1-r} + O\left(N^{-n}\right) \tag{2.46}$$

for any n, which is consistent with the absence of $1/N$ corrections in the PCGG approach's result. Using a crossing-transformation, it's easy to see that the suppressed term $\sim r^N$ comes from the subdominant saddle-pt.

2.3 Effective theory - simplest 4-pt - tree level (bis)

We can work with the original action without the Jacobian. The saddle-pt is

$$r_{12} = r_{34} = \frac{g}{\sqrt{2}} \qquad\qquad \forall\theta_{12}, \theta_{34}\,. \tag{2.47}$$

The strategy is

$$\frac{(G_4)_{\mathrm{LO}}}{\langle \mathcal{O}_1\mathcal{O}_2\rangle\langle \mathcal{O}_3\mathcal{O}_4\rangle} = \frac{1}{\langle \mathcal{O}_1\mathcal{O}_2\rangle\langle \mathcal{O}_3\mathcal{O}_4\rangle}\frac{1}{Z_\rho}\int d\rho\; \underbrace{r_{12}r_{34}\exp\left(-NS_{\mathrm{eff}}^{(\mathrm{int})}\right)}_{\text{expand in series}}\exp\left(-NS_{\mathrm{eff}}^{(\mathrm{quad})}\right) \tag{2.48}$$

and explicitate powers of fields and N

$$\frac{1}{\langle \mathcal{O}_1\mathcal{O}_2\rangle\langle \mathcal{O}_3\mathcal{O}_4\rangle}\frac{1}{Z_\rho}\int d\rho\,\exp\left(-NS_{\mathrm{eff}}^{(\mathrm{quad})}\right) = N^5\int d\rho\,\exp\left(N\left(\sum \mathrm{field}^2\right)\right) \tag{2.49}$$

$$\exp\left(-NS_{\mathrm{eff}}^{(\mathrm{int})}\right) = \exp\left(N\sum_{i=3}^{\infty}\left(\sum \mathrm{field}^i\right)\right) = \prod_{i=3}^{\infty}\sum_{n_i=0}^{\infty}\frac{N^{n_i}}{n_i!}\left(\sum \mathrm{field}^i\right)^{n_i} \tag{2.50}$$

$$r_{12}r_{34} = \sum_{j=0}^{\infty}\mathrm{field}^j\,. \tag{2.51}$$

The counting argument to obtain N^{-n} is like (**) with $n \to n-1$ in all other formulas.

2.4 Effective theory - simplest 4-pt - beyond tree level

The strategy is to incorporate interactions

$$\begin{aligned} G_4 &= \frac{1}{Z_\rho}\int d\rho'\,\left[1+\sum_{i=1}^{\infty}g^{2i}\left\langle \mathcal{O}^{(i)}\right\rangle_\chi\right]\exp\left(-NS'_{\mathrm{eff}}\right) \\ &= \frac{1}{Z_\rho}\int d\rho'\,\exp\left(-NS'_{\mathrm{eff}} + (g\text{-corrections})\right) + \sum_{i=1}^{\infty}g^{2i}\underbrace{\frac{1}{Z_\rho}\int d\rho'\,\left\langle \mathcal{O}^{(i)}\right\rangle_\chi^{\text{divergent}}\exp\left(-NS'_{\mathrm{eff}}\right)}_{=0} \end{aligned} \tag{2.52}$$

integrate the divergent part around the saddle-pt of S'_{eff}, bring the finite part to the exponent and integrate around the saddle-pt of the g-corrected action. If we stop before the last step, this is the new action to study.

2.5 Effective theory - simplest 4-pt - one loop

We compute

$$\frac{(G_4)_{\mathrm{NLO}}}{\langle \mathcal{O}_1\mathcal{O}_2\rangle\langle \mathcal{O}_3\mathcal{O}_4\rangle} = \frac{1}{\langle \mathcal{O}_1\mathcal{O}_2\rangle\langle \mathcal{O}_3\mathcal{O}_4\rangle}\frac{1}{Z_\rho}\int d\rho'\,\underbrace{\left(\left\langle \mathcal{O}_O^S\right\rangle_\chi + \left\langle \mathcal{O}_X^S\right\rangle_\chi + \left\langle \mathcal{O}_H^S\right\rangle_\chi\right)\exp\left(-NS_{\mathrm{eff}}^{'(\mathrm{int})}\right)}_{\text{expand in series}}\exp\left(-NS_{\mathrm{eff}}^{'(\mathrm{quad})}\right)\,. \tag{2.53}$$

We explicitate powers of fields and N also for (**)

$$\left\langle \mathcal{O}_O^S\right\rangle_\chi + \left\langle \mathcal{O}_X^S\right\rangle_\chi + \left\langle \mathcal{O}_H^S\right\rangle_\chi = N\sum_{j=0}^{\infty}\mathrm{field}^j\,, \tag{2.54}$$

or upon rescaling

$$\left\langle \mathcal{O}_O^S \right\rangle_\chi + \left\langle \mathcal{O}_X^S \right\rangle_\chi + \left\langle \mathcal{O}_H^S \right\rangle_\chi \to \sum_{j=0}^{\infty} N^{\frac{1}{2}(2-j)} \text{field}^j \,. \tag{2.55}$$

We see the term $(j, n_3, n_4, \dots)$ produces N^{-n} with $n = \frac{1}{2}(j-2) + \frac{1}{2}\sum_{i=3}^{\infty}(i-2)\, n_i$. If we aim at all powers higher or equal to N^{-n}, we can cut off

$$i = 3, \dots 2n+4 \tag{2.56}$$
$$n_i = 0, \dots 2n+2 \tag{2.57}$$
$$j = 0, \dots 2n+2 \,. \tag{2.58}$$

We check that

$$\frac{(G_4)_{\text{NLO, non-}X_{1234}}}{\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle} = 0 + O\left(N^{-n}\right) \tag{2.59}$$
$$\frac{(G_4)_{\text{NLO, }X_{1234}}}{\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle} = \frac{\lambda (N+1)}{8\pi^2 N^2} \frac{r\left[N(r-1)+r+1\right]}{(1-r)^3} F^{(1)}(z,\bar{z}) + O\left(N^{-n}\right) \tag{2.60}$$

for any n, which is consistent with the absence of divergences and the exponential suppression of r^N. Using a crossing-transformation, it's easy to see that the suppressed terms $\sim r^N$ come from the subdominant saddle-pt.

2.6 Effective theory - simplest 4-pt - one loop (bis)

We can work with the original action without the Jacobian.

$$\frac{(G_4)_{\text{NLO}}}{\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle} = \frac{1}{\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle} \frac{1}{Z_\rho} \int d\rho \, \underbrace{r_{12} r_{13} \left(\left\langle \mathcal{O}_O^S \right\rangle_\chi + \left\langle \mathcal{O}_X^S \right\rangle_\chi + \left\langle \mathcal{O}_H^S \right\rangle_\chi \right) \exp\left(-N S_{\text{eff}}^{(\text{int})}\right)}_{\text{expand in series}} \exp\left(-N S_{\text{eff}}^{(\text{quad})}\right) \,. \tag{2.61}$$

The counting argument has again

$$r_{12} r_{34} \left(\left\langle \mathcal{O}_O^S \right\rangle_\chi + \left\langle \mathcal{O}_X^S \right\rangle_\chi + \left\langle \mathcal{O}_H^S \right\rangle_\chi \right) = N \sum_{j=0}^{\infty} \text{field}^j \tag{2.62}$$

and continues as (**).

2.7 Effective theory - simplest 4-pt - tree level for $r = 1$

The space of saddle-pt gets enlarged from 2d to 4d. It reads

$$\begin{cases} \rho_{12} &= R_{12} \exp(i\theta_{12}) \\ \rho_{34} &= R_{12} \exp(i\theta_{34}) \\ \rho_{23} &= -\sqrt{\frac{g^2}{2} - R_{12}^2} \exp\left(i(\theta_{14} - \theta_{12} - \theta_{34})\right) \\ \rho_{41} &= \sqrt{\frac{g^2}{2} - R_{12}^2} \exp(-i\theta_{14}) \end{cases} \tag{2.63}$$

with free parameters $R_{12} \in \left(0, \frac{g}{\sqrt{2}}\right)$ and $\theta_{12}, \theta_{34}, \theta_{14} \in (0, 2\pi)$. Choosing the new variables [6]

$$\begin{cases} \rho_{12} &= r \cos\frac{\alpha}{2} \cos\frac{\theta_1}{2} \exp\left(\frac{i}{4}(+2\varphi_1 + \chi + \xi)\right) \\ \rho_{34} &= r \cos\frac{\alpha}{2} \sin\frac{\theta_1}{2} \exp\left(\frac{i}{4}(-2\varphi_1 + \chi + \xi)\right) \\ \rho_{23} &= r \sin\frac{\alpha}{2} \cos\frac{\theta_2}{2} \exp\left(\frac{i}{4}(+2\varphi_2 - \chi + \xi)\right) \\ \rho_{41} &= r \sin\frac{\alpha}{2} \sin\frac{\theta_2}{2} \exp\left(\frac{i}{4}(-2\varphi_2 - \chi + \xi)\right) \end{cases} \tag{2.64}$$

with range $r \in (0, \infty)$, $\alpha, \theta_1, \theta_2 \in (0, \pi)$, $\varphi_1, \varphi_2 \in (0, 2\pi)$, $\chi \in (0, 4\pi)$ and $\xi \in (0, 8\pi)$, the saddle-pts are

$$r = g \qquad \alpha = 0 \qquad \theta_1 = \frac{\pi}{2} \qquad \forall \theta_2, \varphi_1, \varphi_2, \chi, \xi \tag{2.65}$$

and

$$r = g \qquad \alpha = \pi \qquad \theta_2 = \frac{\pi}{2} \qquad \forall \theta_1, \varphi_1, \varphi_2, \chi, \xi \tag{2.66}$$

and

$$r=g \qquad \theta_1=\theta_2=\frac{\pi}{2} \qquad \xi=(2n+1)\,\pi \qquad n=0,1,2,3 \qquad \forall\alpha,\varphi_1,\varphi_2,\chi\,, \tag{2.67}$$

while the path-integral measure becomes

$$\int d\rho=\int_0^\infty dr\int_0^\pi d\alpha\, d\theta_1\, d\theta_2\int_0^{2\pi} d\alpha\, d\varphi_1\, d\varphi_2\int_0^{4\pi} d\chi\int_0^{8\pi} d\xi\int_{-\infty}^{\infty}\prod_{(i,j)\in\{(1,3),(2,4)\}} d\mathrm{Re}\,(\rho_{ij})\; d\mathrm{Im}\,(\rho_{ij})\;. \tag{2.68}$$

We expand in non-moduli variables ρ_{ij}, r, θ_1, θ_2 and ξ around the saddle-pt and the counting argument is (**). [wrong result]

2.8 Effective theory - simplest 4-pt - one loop for $r=1$

We take into account the 1-loop interactions. The counting is (**) with $n\to n+1$. [wrong result]

2.9 Effective theory - 4-pt

We integrated the simplest correlator at finite N in Mathematica and reproduced the correct result. The same geometric sum of the PCGG method arises from the use of the multinomial theorem and Gaussian integration. We notice that the integrals in ρ_{13} and ρ_{24} are exactly Gaussian and can be done immediately.

We did the same for the general 4-pt. The saddle pt are now 3 [3]

$$|\rho_{12}|^2=|\rho_{34}|^2=\frac{g^2}{2} \qquad \text{or} \qquad |\rho_{13}|^2=|\rho_{24}|^2=\frac{g^2}{2} \qquad \text{or} \qquad |\rho_{14}|^2=|\rho_{23}|^2=\frac{g^2}{2}\,. \tag{2.69}$$

The sum looks similar to that in the PCGG method and so it is not solvable in closed form.

The punchline is that final sums look the same in the two methods.

At one loop, we check finiteness, calculate the finite parts

$$\frac{(G_4)_{\text{LO}}}{\langle\mathcal{O}_1\mathcal{O}_2\rangle\langle\mathcal{O}_3\mathcal{O}_4\rangle}=f\,(u_1,u_2)+f\left(\frac{1}{u_1},\frac{u_2}{u_1}\right)u_1^N+f\,(u_2,u_1)\left(\frac{u_1}{u_2}\right)^N \tag{2.70}$$

$$f\,(u_1,u_2)=\frac{u_2}{(1-u_1)\,(u_2-u_1)}-\frac{2u_1^2u_2}{(1-u_1)^2\,(u_2-u_1)^2\,N}+\frac{2u_1^2u_2\left(3u_1^2+u_1u_2+u_1+u_2\right)}{(1-u_1)^3\,(u_2-u_1)^3\,N^2}+O\left(N^{-3}\right) \tag{2.71}$$

$$\frac{(G_4)_{\text{NLO}}}{\langle\mathcal{O}_1\mathcal{O}_2\rangle\langle\mathcal{O}_3\mathcal{O}_4\rangle}=f\,(u_1,u_2)+f\left(\frac{1}{u_1},\frac{u_2}{u_1}\right)u_1^N+f\,(u_2,u_1)\left(\frac{u_1}{u_2}\right)^N \tag{2.72}$$

$$f\,(u_1,u_2)=-\frac{2g^2\alpha^2\bar\alpha^2z\bar z\,(1-z)\,(1-\bar z)\,(z-\alpha)\,(z-\bar\alpha)\,(\bar z-\alpha)\,(\bar z-\bar\alpha)}{(z\bar z-\alpha\bar\alpha)^2\left[z\bar z\,(\alpha+\bar\alpha-1)-\alpha\bar\alpha\,(z+\bar z-1)\right]^2}F^{(1)}\,(z,\bar z)+O\left(\frac{1}{N}\right) \tag{2.73}$$

with $u_1=\frac{z\bar z}{\alpha\bar\alpha}$ and $u_2=\frac{(1-z)(1-\bar z)}{(1-\alpha)(1-\bar\alpha)}$, and check their limit matches the simplest 4-pt. In particular, the $1/N$-corrections vanish.

Here in formula, we used the contribution of a single saddle-pt and added those of the other 2 using crossing symmetry

$$\frac{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle}{\langle\mathcal{O}_1\mathcal{O}_2\rangle\langle\mathcal{O}_3\mathcal{O}_4\rangle}=\left(\frac{z\bar z\,(1-\alpha)\,(1-\bar\alpha)}{\alpha\bar\alpha\,(1-z)\,(1-\bar z)}\right)^N\frac{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle}{\langle\mathcal{O}_1\mathcal{O}_4\rangle\langle\mathcal{O}_2\mathcal{O}_3\rangle}\overset{2\leftrightarrow4}{=}\left(\frac{z\bar z\,(1-\alpha)\,(1-\bar\alpha)}{\alpha\bar\alpha\,(1-z)\,(1-\bar z)}\right)^N\left(\frac{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle}{\langle\mathcal{O}_1\mathcal{O}_2\rangle\langle\mathcal{O}_3\mathcal{O}_4\rangle}\right)_{z\to1-z,\,\alpha\to1-\alpha} \tag{2.74}$$

$$\frac{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle}{\langle\mathcal{O}_1\mathcal{O}_2\rangle\langle\mathcal{O}_3\mathcal{O}_4\rangle}=\left(\frac{z\bar z}{\alpha\bar\alpha}\right)^N\frac{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle}{\langle\mathcal{O}_1\mathcal{O}_3\rangle\langle\mathcal{O}_2\mathcal{O}_4\rangle}\overset{2\leftrightarrow3}{=}\left(\frac{z\bar z}{\alpha\bar\alpha}\right)^N\left(\frac{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle}{\langle\mathcal{O}_1\mathcal{O}_2\rangle\langle\mathcal{O}_3\mathcal{O}_4\rangle}\right)_{z\to1/z,\,\alpha\to1/\alpha} \tag{2.75}$$

and $F^{(1)}\,(z,\bar z)=F^{(1)}\,(1-z,1-\bar z)$ and $F^{(1)}\left(\frac{1}{z},\frac{1}{\bar z}\right)=z\bar zF^{(1)}\,(z,\bar z)$.

2.10 PCGG - tree level

The 6 bridge lengths depend on 2

$$\ell_{12} \tag{2.76}$$

[3] Finding is very non-trivial and possible only in the variables $\rho_{ij}=r_{ij}e^{i\theta_{ij}}$.

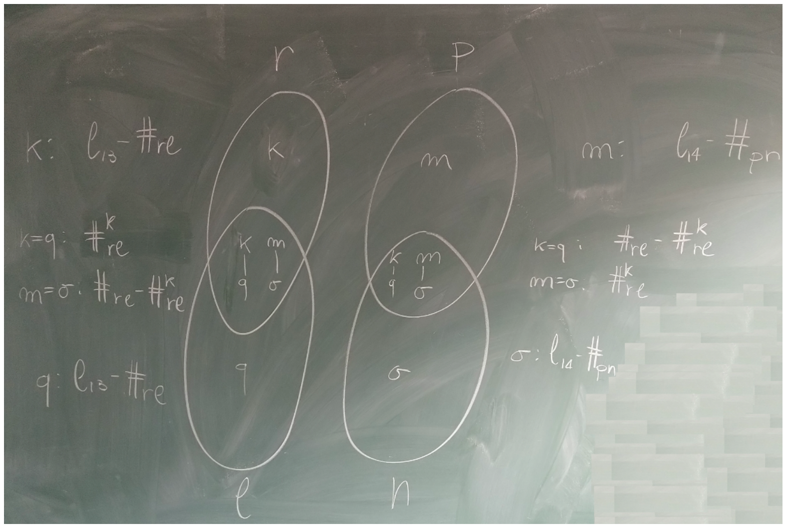
$$
\begin{aligned}
&\ell_{13} && (2.77)\\
&\ell_{14} = N - \ell_{12} - \ell_{13} && (2.78)\\
&\ell_{23} = N - \ell_{12} - \ell_{13} && (2.79)\\
&\ell_{24} = \ell_{13} && (2.80)\\
&\ell_{34} = \ell_{12} && (2.81)
\end{aligned}
$$

which take independent values within the bounds $0 \leq \ell_{12} + \ell_{13} \leq N$.

$$
\begin{aligned}
&\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle_{\mathrm{LO}} \qquad (2.82)\\
=&\frac{1}{(N!)^4} \left(\varepsilon_{i_1^{(1)} \ldots i_N^{(1)}} \varepsilon_{j_1^{(1)} \ldots j_N^{(1)}}\right) \ldots \left(\varepsilon_{i_1^{(4)} \ldots i_N^{(4)}} \varepsilon_{j_1^{(4)} \ldots j_N^{(4)}}\right) \left\langle (Y_1 \cdot \Phi)_{i_1^{(1)} j_1^{(1)}} \ldots (Y_1 \cdot \Phi)_{i_N^{(4)} j_N^{(4)}} \right\rangle_{\mathrm{LO}}\\
=&\sum_{\{\ell_{ij}\}} \frac{1}{(N!)^4} \binom{N}{\ell_{12}\ \ell_{13}\ \ell_{14}} \binom{N}{\ell_{12}\ \ell_{23}\ \ell_{24}} \binom{N}{\ell_{13}\ \ell_{23}\ \ell_{34}} \binom{N}{\ell_{14}\ \ell_{24}\ \ell_{34}}\\
&\ell_{12}! \ell_{13}! \ell_{14}! \ell_{23}! \ell_{24}! \ell_{34}!\\
&\left(\frac{1}{2} \frac{\lambda}{N} I_{12} Y_1 \cdot Y_2\right)^{\ell_{12}} \left(\frac{1}{2} \frac{\lambda}{N} I_{13} Y_1 \cdot Y_3\right)^{\ell_{13}} \left(\frac{1}{2} \frac{\lambda}{N} I_{14} Y_1 \cdot Y_4\right)^{\ell_{14}} \left(\frac{1}{2} \frac{\lambda}{N} I_{23} Y_2 \cdot Y_3\right)^{\ell_{23}} \left(\frac{1}{2} \frac{\lambda}{N} I_{24} Y_2 \cdot Y_4\right)^{\ell_{24}} \left(\frac{1}{2} \frac{\lambda}{N} I_{34} Y_3 \cdot Y_4\right)^{\ell_{34}}\\
&\left(\varepsilon_{i_1 \ldots i_{\ell_{12}} k_1 \ldots k_{\ell_{13}} m_1 \ldots m_{\ell_{14}}} \varepsilon^{j_1 \ldots j_{\ell_{12}} l_1 \ldots l_{\ell_{13}} n_1 \ldots n_{\ell_{14}}}\right) \left(\varepsilon_{j_1 \ldots j_{\ell_{12}} o_1 \ldots o_{\ell_{23}} q_1 \ldots q_{\ell_{24}}} \varepsilon^{i_1 \ldots i_{\ell_{12}} p_1 \ldots p_{\ell_{23}} r_1 \ldots r_{\ell_{24}}}\right)\\
&\left(\varepsilon_{l_1 \ldots l_{\ell_{13}} p_1 \ldots p_{\ell_{23}} s_1 \ldots s_{\ell_{34}}} \varepsilon^{k_1 \ldots k_{\ell_{13}} o_1 \ldots o_{\ell_{23}} t_1 \ldots t_{\ell_{34}}}\right) \left(\varepsilon_{n_1 \ldots n_{\ell_{14}} r_1 \ldots r_{\ell_{24}} t_1 \ldots t_{\ell_{34}}} \varepsilon^{m_1 \ldots m_{\ell_{14}} q_1 \ldots q_{\ell_{24}} s_1 \ldots s_{\ell_{34}}}\right)\\
=&\left(\frac{\lambda}{2N}\right)^{2N} \sum_{\ell_{12}=0}^{N} \sum_{\ell_{13}=0}^{N-\ell_{12}} \frac{(I_{12} I_{34} Y_1 \cdot Y_2 Y_3 \cdot Y_4)^{\ell_{12}} (I_{13} I_{24} Y_1 \cdot Y_3 Y_2 \cdot Y_4)^{\ell_{13}} (I_{14} I_{23} Y_1 \cdot Y_4 Y_2 \cdot Y_3)^{N-\ell_{12}-\ell_{13}}}{\left[\ell_{12}! \ell_{13}! (N - \ell_{12} - \ell_{13})!\right]^2}\\
&\left(\varepsilon_{i_1 \ldots i_{\ell_{12}} k_1 \ldots k_{\ell_{13}} m_1 \ldots m_{\ell_{14}}} \varepsilon^{j_1 \ldots j_{\ell_{12}} l_1 \ldots l_{\ell_{13}} n_1 \ldots n_{\ell_{14}}}\right) \left(\varepsilon_{j_1 \ldots j_{\ell_{12}} o_1 \ldots o_{\ell_{23}} q_1 \ldots q_{\ell_{24}}} \varepsilon^{i_1 \ldots i_{\ell_{12}} p_1 \ldots p_{\ell_{23}} r_1 \ldots r_{\ell_{24}}}\right)\\
&\left(\varepsilon_{l_1 \ldots l_{\ell_{13}} p_1 \ldots p_{\ell_{23}} s_1 \ldots s_{\ell_{34}}} \varepsilon^{k_1 \ldots k_{\ell_{13}} o_1 \ldots o_{\ell_{23}} t_1 \ldots t_{\ell_{34}}}\right) \left(\varepsilon_{n_1 \ldots n_{\ell_{14}} r_1 \ldots r_{\ell_{24}} t_1 \ldots t_{\ell_{34}}} \varepsilon^{m_1 \ldots m_{\ell_{14}} q_1 \ldots q_{\ell_{24}} s_1 \ldots s_{\ell_{34}}}\right)
\end{aligned}
$$

The epsilons give

$$
\begin{aligned}
&(\ell_{12}!)^4\, \delta^{r_1 \ldots r_{\ell_{13}} p_1 \ldots p_{\ell_{14}}}_{\overline{k_1 \ldots k_{\ell_{13}} m_1 \ldots m_{\ell_{14}}}}\, \delta^{\underline{\underline{l_1 \ldots l_{\ell_{13}}}}\, \underline{\underline{\underline{n_1 \ldots n_{\ell_{14}}}}}}_{\overline{q_1 \ldots q_{\ell_{13}}}\, \overline{\overline{o_1 \ldots o_{\ell_{14}}}}}\, \delta^{q_1 \ldots q_{\ell_{13}} m_1 \ldots m_{\ell_{14}}}_{\underline{\underline{l_1 \ldots l_{\ell_{13}}}}\, \underline{p_1 \ldots p_{\ell_{14}}}}\, \delta^{k_1 \ldots k_{\ell_{13}} o_1 \ldots o_{\ell_{14}}}_{\underline{r_1 \ldots r_{\ell_{13}}}\, \underline{\underline{\underline{n_1 \ldots n_{\ell_{14}}}}}} \qquad (2.83)\\
=&(\ell_{12}!)^4 \sum_{\#_{rl}=0}^{\ell_{13}} \sum_{\#_{pn}=0}^{\ell_{14}} \underbrace{\binom{N}{\#_{rl}}}_{\text{fill } \{r\} \cap \{l\}} \underbrace{\binom{N - \#_{rl}}{\#_{pn}}}_{\text{fill } \{p\} \cap \{n\}} \underbrace{\binom{N - \#_{rl} - \#_{pn}}{\ell_{13} - \#_{rl}}}_{\text{fill } \{r\} \backslash (\{r\} \cap \{l\})} \underbrace{\binom{N - \ell_{13} - \#_{pn}}{\ell_{13} - \#_{rl}}}_{\text{fill } \{l\} \backslash (\{r\} \cap \{l\})}\\
&\underbrace{\binom{N - 2\ell_{13} + \#_{rl} - \#_{pn}}{\ell_{14} - \#_{pn}}}_{\text{fill } \{p\} \backslash (\{p\} \cap \{n\})} \underbrace{\binom{N - 2\ell_{13} - \ell_{14} + \#_{rl}}{\ell_{14} - \#_{pn}}}_{\text{fill } \{n\} \backslash (\{p\} \cap \{n\})} \underbrace{(\ell_{13}!)^2 (\ell_{14}!)^2}_{\text{order } \{r\}\ ,\{l\}\ ,\{p\}\ ,\{n\}}\\
&\sum_{\#_{rl}^k = \max(0, \#_{rl} - \#_{pn})}^{\#_{rl}} \underbrace{\binom{\#_{rl}}{\#_{rl}^k}}_{\text{choose } k's \text{ in } \{r\} \cap \{l\}} \underbrace{\binom{\#_{pn}}{\#_{rl} - \#_{rl}^k}}_{\text{choose } k's \text{ in } \{p\} \cap \{n\}} \underbrace{(\ell_{13}!)^2 (\ell_{14}!)^2}_{\text{order } \{k\}, \{m\}, \{q\}, \{o\}}\\
=&N!\, (\ell_{12}!)^4 (\ell_{13}!)^4 (\ell_{14}!)^4 \sum_{\#_{rl}=0}^{\ell_{13}} \sum_{\#_{pn}=\max(0, -N + 2\ell_{13} + 2\ell_{14} - \#_{rl})}^{\ell_{14}} \sum_{\#_{rl}^k = \max(0, \#_{rl} - \#_{pn})}^{\#_{rl}}\\
&\frac{1}{\left[(\ell_{13} - \#_{rl})!\, (\ell_{14} - \#_{pn})!\, \left(\#_{rl} - \#_{rl}^k\right)!\right]^2 (N - 2\ell_{13} - 2\ell_{14} + \#_{rl} + \#_{pn})!\, \#_{rl}^k!\, \left(\#_{pn} - \#_{rl} + \#_{rl}^k\right)!}
\end{aligned}
$$



so

$$
\begin{aligned}
&\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle_{\mathrm{LO}} \\
=&N!\left(\frac{\lambda}{2N}\right)^{2N}\sum_{\ell_{12}=0}^{N}\sum_{\ell_{13}=0}^{N-\ell_{12}}\left(I_{12}I_{34}Y_1\cdot Y_2Y_3\cdot Y_4\right)^{\ell_{12}}\left(I_{13}I_{24}Y_1\cdot Y_3Y_2\cdot Y_4\right)^{\ell_{13}}\left(I_{14}I_{23}Y_1\cdot Y_4Y_2\cdot Y_3\right)^{N-\ell_{12}-\ell_{13}} \\
&\sum_{\#_{rl}=0}^{\ell_{13}}\sum_{\#_{pn}=\max(0,N-2\ell_{12}-\#_{rl})}^{N-\ell_{12}-\ell_{13}}\sum_{\#_{rl}^k=\max(0,\#_{rl}-\#_{pn})}^{\#_{rl}} \\
&\left[\frac{\ell_{12}!\ell_{13}!\left(N-\ell_{12}-\ell_{13}\right)!}{\left(\ell_{13}-\#_{rl}\right)!\left(N-\ell_{12}-\ell_{13}-\#_{pn}\right)!\left(\#_{rl}-\#_{rl}^k\right)!}\right]^2\frac{1}{\left(-N+2\ell_{12}+\#_{rl}+\#_{pn}\right)!\#_{rl}^k!\left(\#_{pn}-\#_{rl}+\#_{rl}^k\right)!} \\
=&N!\left(\frac{Y_1\cdot Y_2Y_3\cdot Y_4}{x_{12}^2x_{34}^2}\right)^N\left(\frac{\lambda}{8\pi^2N}\right)^{2N}\sum_{\ell_{12}=0}^{N}\sum_{\ell_{13}=0}^{N-\ell_{12}}\left(\frac{z\bar{z}}{\alpha\bar{\alpha}}\right)^{N-\ell_{12}}\left[\frac{\left(1-z\right)\left(1-\bar{z}\right)}{\left(1-\alpha\right)\left(1-\bar{\alpha}\right)}\right]^{\ell_{12}+\ell_{13}-N} \\
&\sum_{\#_{rl}=0}^{\ell_{13}}\sum_{\#_{pn}=\max(0,N-2\ell_{12}-\#_{rl})}^{N-\ell_{12}-\ell_{13}}\sum_{\#_{rl}^k=\max(0,\#_{rl}-\#_{pn})}^{\#_{rl}} \\
&\left[\frac{\ell_{12}!\ell_{13}!\left(N-\ell_{12}-\ell_{13}\right)!}{\left(\ell_{13}-\#_{rl}\right)!\left(N-\ell_{12}-\ell_{13}-\#_{pn}\right)!\left(\#_{rl}-\#_{rl}^k\right)!}\right]^2\frac{1}{\left(-N+2\ell_{12}+\#_{rl}+\#_{pn}\right)!\#_{rl}^k!\left(\#_{pn}-\#_{rl}+\#_{rl}^k\right)!}
\end{aligned}
\tag{2.84}
$$

$$
\begin{aligned}
&\frac{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle_{\mathrm{LO}}}{\langle \mathcal{O}_1\mathcal{O}_2\rangle\langle \mathcal{O}_3\mathcal{O}_4\rangle} \\
=&\frac{1}{N!}\sum_{\ell_{12}=0}^{N}\sum_{\ell_{13}=0}^{N-\ell_{12}}\left(\frac{z\bar{z}}{\alpha\bar{\alpha}}\right)^{N-\ell_{12}}\left[\frac{\left(1-z\right)\left(1-\bar{z}\right)}{\left(1-\alpha\right)\left(1-\bar{\alpha}\right)}\right]^{\ell_{12}+\ell_{13}-N}\sum_{\#_{rl}=0}^{\ell_{13}}\sum_{\#_{pn}=\max(0,N-2\ell_{12}-\#_{rl})}^{N-\ell_{12}-\ell_{13}}\sum_{\#_{rl}^k=\max(0,\#_{rl}-\#_{pn})}^{\#_{rl}} \\
&\left[\frac{\ell_{12}!\ell_{13}!\left(N-\ell_{12}-\ell_{13}\right)!}{\left(\ell_{13}-\#_{rl}\right)!\left(N-\ell_{12}-\ell_{13}-\#_{pn}\right)!\left(\#_{rl}-\#_{rl}^k\right)!}\right]^2\frac{1}{\left(-N+2\ell_{12}+\#_{rl}+\#_{pn}\right)!\#_{rl}^k!\left(\#_{pn}-\#_{rl}+\#_{rl}^k\right)!}.
\end{aligned}
\tag{2.85}
$$

It is hard to resum at fixed ℓ_{12} and ℓ_{13}. The simplest 4-pt is easy ($r\equiv\frac{z\bar{z}(1-\alpha)(1-\bar{\alpha})}{\alpha\bar{\alpha}(1-z)(1-\bar{z})}=\left(\frac{\langle\mathcal{O}_1\mathcal{O}_4\rangle\langle\mathcal{O}_2\mathcal{O}_3\rangle}{\langle\mathcal{O}_1\mathcal{O}_2\rangle\langle\mathcal{O}_3\mathcal{O}_4\rangle}\right)^{1/N}$ with $\alpha,\bar{\alpha}\to\infty$ and $\frac{(1-\alpha)(1-\bar{\alpha})}{\alpha\bar{\alpha}}$ finite).

$$
\begin{aligned}
&\frac{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle_{\mathrm{LO}}^{\ell_{13}=0}}{\langle \mathcal{O}_1\mathcal{O}_2\rangle\langle \mathcal{O}_3\mathcal{O}_4\rangle} \\
=&\frac{1}{N!}\sum_{\ell_{12}=0}^{N}\sum_{\#_{pn}=\max(0,N-2\ell_{12})}^{N-\ell_{12}}\frac{r^{N-\ell_{12}}}{\#_{pn}!\left(-N+2\ell_{12}+\#_{pn}\right)!}\left[\frac{\ell_{12}!\left(N-\ell_{12}\right)!}{\left(N-\ell_{12}-\#_{pn}\right)!}\right]^2 \\
=&\sum_{\ell_{12}=0}^{N}r^{N-\ell_{12}}
\end{aligned}
\tag{2.86}
$$

$$
\begin{aligned}
&= \frac{1-r^{N+1}}{1-r} \\
&\overset{\text{finite } N}{=} \begin{cases} 1 & r \to 0 \\ N+1 & r = 1 \\ r^N & r \to \infty \end{cases} \\
&\overset{N \gg 1}{=} \begin{cases} \frac{1}{1-r} & r < 1 \\ N & r = 1 \\ r^N & r > 1 \end{cases}
\end{aligned}
$$

The drastic resummation in the last steps indicates that there may be a clever way to obtain the result. Crossing symmetries implies

$$
\frac{\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle_{\mathrm{LO}}^{\ell_{13}=0}}{\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle} = r^N \frac{\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle_{\mathrm{LO}}^{\ell_{13}=0}}{\langle \mathcal{O}_1 \mathcal{O}_4 \rangle \langle \mathcal{O}_2 \mathcal{O}_3 \rangle} \overset{2 \leftrightarrow 4}{=} r^N \left(\frac{\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle_{\mathrm{LO}}^{\ell_{13}=0}}{\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle} \right)_{r \to 1/r} . \tag{2.87}
$$

At finite N, we can plot the $N+1$ modes labeled by ℓ_{12}:

- modes with $\ell_{12} \sim N$ dominate when $r \to 0$,
- modes with $\ell_{12} \sim 0$ dominate when $r \to \infty$,
- all modes equally contribute when $r = 1$.

The labeling is meaningful only at weak coupling because it relies on the concept of bridge length. At large N, the correlator stops converging at $r = 1$ because an arbitrarily large number of propagators ($0 \leq 2\ell_{12} \leq 2N$) of equal weight ($r^{N-\ell_{12}} \sim 1$) is exchanged in the s-channel. The critical value $r = 1$ should be the Hagedorn "temperature" of the 4-pt function in the free theory. On the contrary, it converges at $r < 1$ because the exchange of propagators is exponentially damped.

A "symmetric" normalization would be related to this above as

$$
\frac{1}{\sqrt{\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle \langle \mathcal{O}_1 \mathcal{O}_4 \rangle \langle \mathcal{O}_2 \mathcal{O}_3 \rangle}} = \frac{r^{-N/2}}{\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle} . \tag{2.88}
$$

Indeed the multiplication by $r^{-N/2}$ symmetrizes many results.

2.11 PCGG - simplest 4-pt - one loop

2.11.1 2-det interactions

$$
\begin{aligned}
&\frac{1}{(N!)^4} \left(\varepsilon_{i_1^{(1)} \dots i_N^{(1)}} \varepsilon_{j_1^{(1)} \dots j_N^{(1)}} \right) \dots \left(\varepsilon_{i_1^{(4)} \dots i_N^{(4)}} \varepsilon_{j_1^{(4)} \dots j_N^{(4)}} \right) \left\langle (Y_1 \cdot \Phi)_{i_1^{(1)} j_1^{(1)}} \dots (Y_1 \cdot \Phi)_{i_N^{(4)} j_N^{(4)}} V_O \right\rangle_{\mathrm{LO}} \qquad (2.89) \\
&= \sum_{\ell_{13}=1}^{N-1} \left\langle \mathcal{G}_{2\ell_{13}}^{\mathrm{NLO}} (x_1, x_2) \, \mathcal{G}_{2\ell_{13}}^{\mathrm{LO}} (x_3, x_4) \right\rangle_{\mathrm{LO}}^{\text{no contractions } 1-2,\, 3-4} + (\text{cyclic perm. of ops.}) \\
&= \sum_{\ell_{13}=1}^{N-1} 2 \left[\frac{(N-\ell_{13})!}{(\ell_{13}!)^2} \right]^2 \left\{ -\ell_{13} (N - \ell_{13} - 1) \, \delta_{j_1^{(2)} \dots j_{\ell_{13}}^{(2)}}^{i_1^{(1)} \dots i_{\ell_{13}}^{(1)}} \delta_{i_1^{(2)} \dots i_{\ell_{13}}^{(2)}}^{j_1^{(1)} \dots j_{\ell_{13}}^{(1)}} \delta_{j_1^{(4)} \dots j_{\ell_{13}}^{(4)}}^{i_1^{(3)} \dots i_{\ell_{13}}^{(3)}} \delta_{i_1^{(4)} \dots i_{\ell_{13}}^{(4)}}^{j_1^{(3)} \dots j_{\ell_{13}}^{(3)}} \right. \\
&\quad \left. + M_{j_1^{(2)} \dots j_{\ell_{13}}^{(2)} ; i_1^{(2)} \dots i_{\ell_{13}}^{(2)}}^{i_1^{(1)} \dots i_{\ell_{13}}^{(1)} ; j_1^{(1)} \dots j_{\ell_{13}}^{(1)}} \delta_{j_1^{(4)} \dots j_{\ell_{13}}^{(4)}}^{i_1^{(3)} \dots i_{\ell_{13}}^{(3)}} \delta_{i_1^{(4)} \dots i_{\ell_{13}}^{(4)}}^{j_1^{(3)} \dots j_{\ell_{13}}^{(3)}} \right\} \left(\frac{\lambda}{2N} \right)^{2N - 2\ell_{13} + 1} \left(I_{12} I_{34} Y_1 \cdot Y_2 Y_3 \cdot Y_4 \right)^{N - \ell_{13}} I_{12}^{-1} \left(Y_{112} + Y_{122} \right) \\
&\quad \left\langle (Y_1 \cdot \Phi)_{i_1^{(1)} j_1^{(1)}} \dots (Y_4 \cdot \Phi)_{i_{\ell_{13}}^{(4)} j_{\ell_{13}}^{(4)}} \right\rangle_{\mathrm{LO}}^{\text{no contractions } 1-2,\, 3-4} + (\text{cyclic perm. of ops.}) \\
&= \sum_{\ell_{13}=1}^{N-1} 2 \left[\frac{(N-\ell_{13})!}{\ell_{13}!} \right]^2 \left\{ -\ell_{13} (N - \ell_{13} - 1) \, \delta_{i_1^{(3)} \dots i_{\ell_{13}}^{(3)}}^{i_1^{(1)} \dots i_{\ell_{13}}^{(1)}} \delta_{i_1^{(2)} \dots i_{\ell_{13}}^{(2)}}^{j_1^{(1)} \dots j_{\ell_{13}}^{(1)}} \delta_{i_1^{(1)} \dots i_{\ell_{13}}^{(1)}}^{i_1^{(3)} \dots i_{\ell_{13}}^{(3)}} \delta_{j_1^{(1)} \dots j_{\ell_{13}}^{(1)}}^{i_1^{(2)} \dots i_{\ell_{13}}^{(2)}} \right. \\
&\quad \left. + M_{i_1^{(3)} \dots i_{\ell_{13}}^{(3)} ; i_1^{(2)} \dots i_{\ell_{13}}^{(2)}}^{i_1^{(1)} \dots i_{\ell_{13}}^{(1)} ; j_1^{(1)} \dots j_{\ell_{13}}^{(1)}} \delta_{i_1^{(1)} \dots i_{\ell_{13}}^{(1)}}^{i_1^{(3)} \dots i_{\ell_{13}}^{(3)}} \delta_{j_1^{(1)} \dots j_{\ell_{13}}^{(1)}}^{i_1^{(2)} \dots i_{\ell_{13}}^{(2)}} \right\} \left(\frac{\lambda}{2N} \right)^{2N+1} \\
&\quad \left(I_{12} I_{34} Y_1 \cdot Y_2 Y_3 \cdot Y_4 \right)^{N - \ell_{13}} \left(I_{14} I_{23} Y_1 \cdot Y_2 Y_2 \cdot Y_3 \right)^{\ell_{13}} I_{12}^{-1} \left(Y_{112} + Y_{122} \right) + (\text{cyclic perm. of ops.}) \\
&= - \sum_{\ell_{13}=1}^{N-1} \frac{2 (N!)^2 (N+1) (N - \ell_{13}) \ell_{13}}{N} \left(\frac{\lambda}{2N} \right)^{2N+1}
\end{aligned}
$$

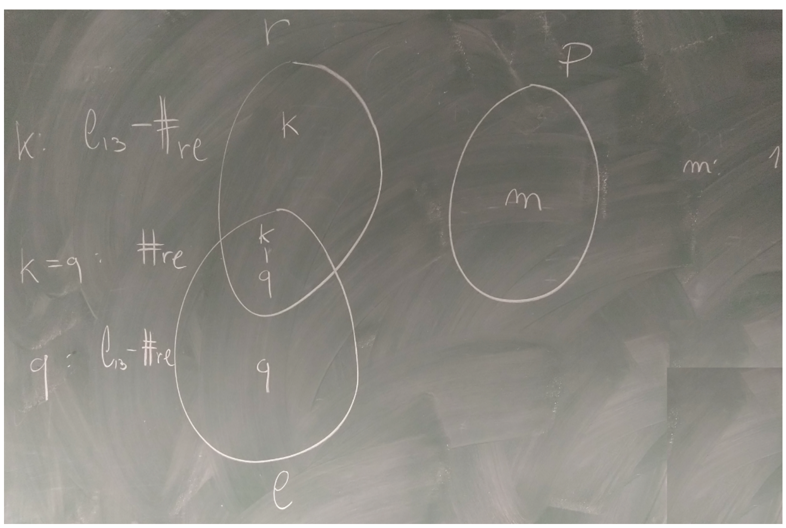
$$\left(I_{12}I_{34}Y_1\cdot Y_2Y_3\cdot Y_4\right)^{N-\ell_{13}}\left(I_{14}I_{23}Y_1\cdot Y_2Y_2\cdot Y_3\right)^{\ell_{13}} I_{12}^{-1}\left(Y_{112}+Y_{122}\right)+(\text{cyclic perm. of ops.})$$

The contraction $\delta\delta\delta\delta$ gives

$$\left[\frac{N!\ell_{13}!}{(N-\ell_{13})!}\right]^2 \tag{2.90}$$

and $M\delta\delta$

$$\begin{aligned}
&\left(\delta^{i^{(1)}_1\dots i^{(1)}_{\ell_{13}} i}_{i^{(3)}_1\dots i^{(3)}_{\ell_{13}} l}\delta^{j^{(1)}_1\dots j^{(1)}_{\ell_{13}} i}_{i^{(2)}_1\dots i^{(2)}_{\ell_{13}} l} - N\delta^{i^{(1)}_1\dots i^{(1)}_{\ell_{13}}}_{i^{(3)}_1\dots i^{(3)}_{\ell_{13}}}\delta^{j^{(1)}_1\dots j^{(1)}_{\ell_{13}}}_{i^{(2)}_1\dots i^{(2)}_{\ell_{13}}}\right)\delta^{i^{(3)}_1\dots i^{(3)}_{\ell_{13}}}_{i^{(1)}_1\dots i^{(1)}_{\ell_{13}}}\delta^{i^{(2)}_1\dots i^{(2)}_{\ell_{13}}}_{j^{(1)}_1\dots j^{(1)}_{\ell_{13}}} \qquad (2.91)\\
=&\delta^{\underline{i^{(1)}_1\dots i^{(1)}_{\ell_{13}} i}}_{i^{(3)}_1\dots i^{(3)}_{\ell_{13}} l}\delta^{\underline{\underline{j^{(1)}_1\dots j^{(1)}_{\ell_{13}}}}}_{i^{(2)}_1\dots i^{(2)}_{\ell_{13}}}\delta^{i^{(2)}_1\dots i^{(2)}_{\ell_{13}} l}_{\underline{\underline{j^{(1)}_1\dots j^{(1)}_{\ell_{13}}}}\,\underline{i}}\delta^{i^{(3)}_1\dots i^{(3)}_{\ell_{13}}}_{\underline{i^{(1)}_1\dots i^{(1)}_{\ell_{13}}}} - N\left[\frac{N!\ell_{13}!}{(N-\ell_{13})!}\right]^2\\
=&\delta^{\underline{r_1\dots r_{\ell_{13}}p}}_{k_1\dots k_{\ell_{13}}m}\delta^{\underline{\underline{l_1\dots l_{\ell_{13}}}}}_{q_1\dots q_{\ell_{13}}}\delta^{q_1\dots q_{\ell_{13}}m}_{\underline{\underline{l_1\dots l_{\ell_{13}}}}\,\underline{p}}\delta^{k_1\dots k_{\ell_{13}}}_{\underline{r_1\dots r_{\ell_{13}}}} - N\left[\frac{N!\ell_{13}!}{(N-\ell_{13})!}\right]^2\\
=&\sum_{\#_{rl}=0}^{\ell_{13}}\underbrace{\binom{N}{\#_{rl}}}_{\text{fill }\{r\}\cap\{l\}}\underbrace{\binom{N-\#_{rl}}{\ell_{13}-\#_{rl}}}_{\text{fill }\{r\}\setminus(\{r\}\cap\{l\})}\underbrace{\binom{N-\ell_{13}}{\ell_{13}-\#_{rl}}}_{\text{fill }\{l\}\setminus(\{r\}\cap\{l\})}\underbrace{\binom{N-2\ell_{13}+\#_{rl}}{1}}_{\text{fill }\{p\}}\underbrace{(\ell_{13}!)^2}_{\text{order }\{r\},\{l\}}\underbrace{(\ell_{13}!)^2}_{\text{order }\{k\},\{q\}} - N\left[\frac{N!\ell_{13}!}{(N-\ell_{13})!}\right]^2\\
=&\sum_{\#_{rl}=0}^{\ell_{13}}\frac{N!\,(\ell_{13}!)^4}{(\ell_{13}-\#_{rl})!^2\,(N-2\ell_{13}+\#_{rl}-1)!\#_{rl}!} - N\left[\frac{N!\ell_{13}!}{(N-\ell_{13})!}\right]^2\\
=&\frac{1}{N}\left(\frac{N!\ell_{13}!}{(N-\ell_{13}-1)!}\right)^2 - N\left[\frac{N!\ell_{13}!}{(N-\ell_{13})!}\right]^2\\
=&-\frac{\ell_{13}\left(2N-\ell_{13}\right)}{N}\left[\frac{N!\ell_{13}!}{(N-\ell_{13})!}\right]^2 .
\end{aligned}$$



2.11.2 3-det interactions

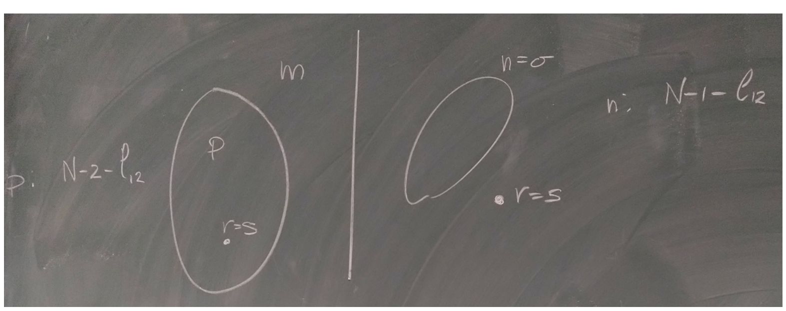
$$0 \le \ell_{12} \le N-2 \tag{2.92}$$
$$\ell_{14} = N-1-\ell_{12} \tag{2.93}$$
$$\ell_{23} = N-2-\ell_{12} \tag{2.94}$$
$$\ell_{34} = 1+\ell_{12} \tag{2.95}$$

$$\begin{aligned}
&\frac{1}{(N!)^4}\left(\varepsilon_{i^{(1)}_1\dots i^{(1)}_N}\varepsilon_{j^{(1)}_1\dots j^{(1)}_N}\right)\dots\left(\varepsilon_{i^{(4)}_1\dots i^{(4)}_N}\varepsilon_{j^{(4)}_1\dots j^{(4)}_N}\right)\left\langle (Y_1\cdot\Phi)_{i^{(1)}_1 j^{(1)}_1}\dots(Y_1\cdot\Phi)_{i^{(4)}_N j^{(4)}_N} V_{X+H}\right\rangle_{\text{LO}} \qquad (2.96)\\
=&\sum_{\{\ell_{ij}\}}\frac{1}{(N!)^4}\binom{N}{1}^2\binom{N}{2}
\end{aligned}$$

$$
\begin{aligned}
&\binom{N-1}{\ell_{12}\,\ell_{14}}\binom{N-2}{\ell_{12}\,\ell_{23}}\binom{N-1}{\ell_{23}\,\ell_{34}}\binom{N}{\ell_{14}\,\ell_{34}}\ell_{12}!\ell_{14}!\ell_{23}!\ell_{34}!\\
&\left(\frac{1}{2}\frac{\lambda}{N}I_{12}Y_1\cdot Y_2\right)^{\ell_{12}}\left(\frac{1}{2}\frac{\lambda}{N}I_{14}Y_1\cdot Y_4\right)^{\ell_{14}}\left(\frac{1}{2}\frac{\lambda}{N}I_{23}Y_2\cdot Y_3\right)^{\ell_{23}}\left(\frac{1}{2}\frac{\lambda}{N}I_{34}Y_3\cdot Y_4\right)^{\ell_{34}}\\
&\left(\varepsilon_{i^{(1)}i_1\ldots i_{\ell_{12}}m_1\ldots m_{\ell_{14}}}\varepsilon^{j^{(1)}j_1\ldots j_{\ell_{12}}n_1\ldots n_{\ell_{14}}}\right)\left(\varepsilon_{i^{(2)}i^{(3)}j_1\ldots j_{\ell_{12}}o_1\ldots o_{\ell_{23}}}\varepsilon^{j^{(2)}j^{(3)}i_1\ldots i_{\ell_{12}}p_1\ldots p_{\ell_{23}}}\right)\\
&\left(\varepsilon_{i^{(4)}p_1\ldots p_{\ell_{23}}s_1\ldots s_{\ell_{34}}}\varepsilon^{j^{(4)}o_1\ldots o_{\ell_{23}}t_1\ldots t_{\ell_{34}}}\right)\left(\varepsilon_{n_1\ldots n_{\ell_{14}}t_1\ldots t_{\ell_{34}}}\varepsilon^{m_1\ldots m_{\ell_{14}}s_1\ldots s_{\ell_{34}}}\right)\\
&\left\langle (Y_1\cdot\Phi)_{i^{(1)}j^{(1)}}\,(Y_2\cdot\Phi)_{i^{(2)}j^{(2)}}\,(Y_2\cdot\Phi)_{i^{(3)}j^{(3)}}\,(Y_3\cdot\Phi)_{i^{(4)}j^{(4)}}\,V_{X+H}\right\rangle_{\mathrm{LO}}+(\text{cyclic perm. of ops.})\\
=&\left(\frac{\lambda}{2N}\right)^{2N+1}\sum_{\ell_{12}=0}^{N-2}\left(I_{12}I_{34}Y_1\cdot Y_2Y_3\cdot Y_4\right)^{\ell_{12}+1}\left(I_{14}I_{23}Y_1\cdot Y_4Y_2\cdot Y_3\right)^{N-1-\ell_{12}}\\
&\frac{\ell_{12}!\,(1+\ell_{12})!}{(N-1-\ell_{12})!\,(N-2-\ell_{12})!}\left(-I_{12}^{-1}I_{23}^{-1}X_{1223}+F_{12,23}\right)\\
&\Big(-\delta^{i^{(1)}i^{(2)}p_1\ldots p_{N-2-\ell_{12}}}_{i^{(1)}m_1\ldots m_{N-1-\ell_{12}}}\delta^{m_1\ldots m_{N-1-\ell_{12}}}_{i^{(4)}p_1\ldots p_{N-2-\ell_{12}}}\delta^{i^{(4)}n_1\ldots n_{N-1-\ell_{12}}}_{i^{(2)}i^{(3)}o_1\ldots o_{N-2-\ell_{12}}}\delta^{i^{(3)}o_1\ldots o_{N-2-\ell_{12}}}_{n_1\ldots n_{N-1-\ell_{12}}}\\
&+\delta^{i^{(1)}i^{(4)}p_1\ldots p_{N-2-\ell_{12}}}_{i^{(1)}m_1\ldots m_{N-1-\ell_{12}}}\delta^{m_1\ldots m_{N-1-\ell_{12}}}_{i^{(4)}p_1\ldots p_{N-2-\ell_{12}}}\delta^{i^{(3)}n_1\ldots n_{N-1-\ell_{12}}}_{i^{(2)}i^{(3)}o_1\ldots o_{N-2-\ell_{12}}}\delta^{i^{(2)}o_1\ldots o_{N-2-\ell_{12}}}_{n_1\ldots n_{N-1-\ell_{12}}}\\
&-\delta^{i^{(3)}i^{(4)}p_1\ldots p_{N-2-\ell_{12}}}_{i^{(1)}m_1\ldots m_{N-1-\ell_{12}}}\delta^{m_1\ldots m_{N-1-\ell_{12}}}_{i^{(4)}p_1\ldots p_{N-2-\ell_{12}}}\delta^{i^{(2)}n_1\ldots n_{N-1-\ell_{12}}}_{i^{(2)}i^{(3)}o_1\ldots o_{N-2-\ell_{12}}}\delta^{i^{(1)}o_1\ldots o_{N-2-\ell_{12}}}_{n_1\ldots n_{N-1-\ell_{12}}}\\
&-\delta^{i^{(1)}i^{(4)}p_1\ldots p_{N-2-\ell_{12}}}_{i^{(1)}m_1\ldots m_{N-1-\ell_{12}}}\delta^{m_1\ldots m_{N-1-\ell_{12}}}_{i^{(4)}p_1\ldots p_{N-2-\ell_{12}}}\delta^{i^{(2)}n_1\ldots n_{N-1-\ell_{12}}}_{i^{(2)}i^{(3)}o_1\ldots o_{N-2-\ell_{12}}}\delta^{i^{(3)}o_1\ldots o_{N-2-\ell_{12}}}_{n_1\ldots n_{N-1-\ell_{12}}}\Big)\\
&+(\text{cyclic perm. of ops.})\\
=&\left(\frac{\lambda}{2N}\right)^{2N+1}\sum_{\ell_{12}=0}^{N-2}\left(I_{12}I_{34}Y_1\cdot Y_2Y_3\cdot Y_4\right)^{\ell_{12}+1}\left(I_{14}I_{23}Y_1\cdot Y_4Y_2\cdot Y_3\right)^{N-1-\ell_{12}}\\
&\frac{2\,(N!)^2\,(N+1)\,(\ell_{12}+1)\,(N-1-\ell_{12})}{N}\left(I_{12}^{-1}I_{23}^{-1}X_{1223}-F_{12,23}\right)+(\text{cyclic perm. of ops.})
\end{aligned}
$$

Contraction

$$
\begin{aligned}
&\delta^{i^{(1)}i^{(2)}p_1\ldots p_{N-2-\ell_{12}}}_{i^{(1)}m_1\ldots m_{N-1-\ell_{12}}}\delta^{m_1\ldots m_{N-1-\ell_{12}}}_{i^{(4)}p_1\ldots p_{N-2-\ell_{12}}}\delta^{i^{(4)}n_1\ldots n_{N-1-\ell_{12}}}_{i^{(2)}i^{(3)}o_1\ldots o_{N-2-\ell_{12}}}\delta^{i^{(3)}o_1\ldots o_{N-2-\ell_{12}}}_{n_1\ldots n_{N-1-\ell_{12}}}\\
=&(1+\ell_{12})\,\delta^{rp_1\ldots p_{N-2-\ell_{12}}}_{m_1\ldots m_{N-1-\ell_{12}}}\delta^{m_1\ldots m_{N-1-\ell_{12}}}_{sp_1\ldots p_{N-2-\ell_{12}}}\delta^{sn_1\ldots n_{N-1-\ell_{12}}}_{ro_1\ldots o_{N-1-\ell_{12}}}\delta^{o_1\ldots o_{N-1-\ell_{12}}}_{n_1\ldots n_{N-1-\ell_{12}}}\\
=&(1+\ell_{12})\underbrace{\binom{N}{1}}_{\text{choose } r=s}\underbrace{\binom{N-1}{N-2-\ell_{12}}}_{\text{fill }\{p\}}\underbrace{(N-2-\ell_{12})!}_{\text{order }\{p\}}\underbrace{(N-1-\ell_{12})!}_{\text{order }\{m\}}\\
&\underbrace{\binom{N-1}{N-1-\ell_{12}}}_{\text{fill }\{n\}}\underbrace{[(N-1-\ell_{12})!]^2}_{\text{order }\{n\}\ ,\{o\}}\\
=&\frac{1}{N}\left[\frac{N!\,(N-1-\ell_{12})!}{\ell_{12}!}\right]^2
\end{aligned}
\tag{2.97}
$$



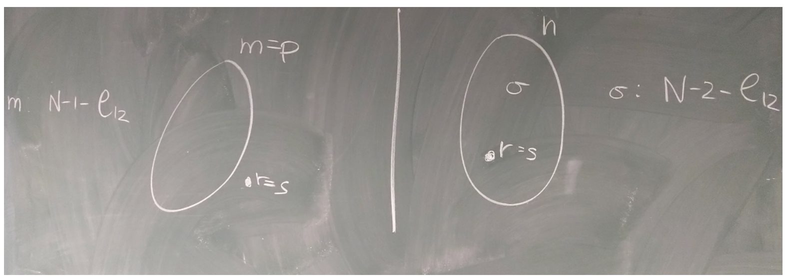
Contraction

$$
\delta^{i^{(1)}i^{(4)}p_1\ldots p_{N-2-\ell_{12}}}_{i^{(1)}m_1\ldots m_{N-1-\ell_{12}}}\delta^{m_1\ldots m_{N-1-\ell_{12}}}_{i^{(4)}p_1\ldots p_{N-2-\ell_{12}}}\delta^{i^{(3)}n_1\ldots n_{N-1-\ell_{12}}}_{i^{(2)}i^{(3)}o_1\ldots o_{N-2-\ell_{12}}}\delta^{i^{(2)}o_1\ldots o_{N-2-\ell_{12}}}_{n_1\ldots n_{N-1-\ell_{12}}}
\tag{2.98}
$$

$$= -\,(1+\ell_{12})^2\,\delta^{p_1\ldots p_{N-1-\ell_{12}}}_{m_1\ldots m_{N-1-\ell_{12}}}\,\delta^{m_1\ldots m_{N-1-\ell_{12}}}_{p_1\ldots p_{N-1-\ell_{12}}}\,\delta^{n_1\ldots n_{N-1-\ell_{12}}}_{o_1\ldots o_{N-1-\ell_{12}}}\,\delta^{o_1\ldots o_{N-1-\ell_{12}}}_{n_1\ldots n_{N-1-\ell_{12}}}$$
$$= -\left[\frac{N!\,(N-1-\ell_{12})!}{\ell_{12}!}\right]^2$$

Contraction (use that in $M\delta\delta$)

$$\begin{aligned}
&\delta^{i^{(3)}i^{(4)}p_1\ldots p_{N-2-\ell_{12}}}_{i^{(1)}m_1\ldots m_{N-1-\ell_{12}}}\,\delta^{m_1\ldots m_{N-1-\ell_{12}}}_{i^{(4)}p_1\ldots p_{N-2-\ell_{12}}}\,\delta^{i^{(2)}n_1\ldots n_{N-1-\ell_{12}}}_{i^{(2)}i^{(3)}o_1\ldots o_{N-2-\ell_{12}}}\,\delta^{i^{(1)}o_1\ldots o_{N-2-\ell_{12}}}_{n_1\ldots n_{N-1-\ell_{12}}} \\
&= (1+\ell_{12})\,\delta^{rp_1\ldots p_{N-1-\ell_{12}}}_{sm_1\ldots m_{N-1-\ell_{12}}}\,\delta^{m_1\ldots m_{N-1-\ell_{12}}}_{p_1\ldots p_{N-1-\ell_{12}}}\,\delta^{n_1\ldots n_{N-1-\ell_{12}}}_{ro_1\ldots o_{N-2-\ell_{12}}}\,\delta^{so_1\ldots o_{N-2-\ell_{12}}}_{n_1\ldots n_{N-1-\ell_{12}}} \\
&= (1+\ell_{12})\;\underbrace{\binom{N}{1}}_{\text{choose } r=s}\;\underbrace{\binom{N-1}{N-1-\ell_{12}}}_{\text{fill }\{p\}}\underbrace{[(N-1-\ell_{12})!]^2}_{\text{order }\{p\},\{m\}} \\
&\quad\underbrace{\binom{N-1}{N-2-\ell_{12}}}_{\text{fill }\{s\}}\underbrace{(N-1-\ell_{12})!}_{\text{order }\{n\}}\underbrace{(N-2-\ell_{12})!}_{\text{order }\{s\}} \\
&= \frac{1}{N}\left[\frac{N!\,(N-1-\ell_{12})!}{\ell_{12}!}\right]^2
\end{aligned} \tag{2.99}$$



Contraction

$$\begin{aligned}
&\delta^{i^{(1)}i^{(4)}p_1\ldots p_{N-2-\ell_{12}}}_{i^{(1)}m_1\ldots m_{N-1-\ell_{12}}}\,\delta^{m_1\ldots m_{N-1-\ell_{12}}}_{i^{(4)}p_1\ldots p_{N-2-\ell_{12}}}\,\delta^{i^{(2)}n_1\ldots n_{N-1-\ell_{12}}}_{i^{(2)}i^{(3)}o_1\ldots o_{N-2-\ell_{12}}}\,\delta^{i^{(3)}o_1\ldots o_{N-2-\ell_{12}}}_{n_1\ldots n_{N-1-\ell_{12}}} \\
&= \left[\frac{N!\,(N-1-\ell_{12})!}{\ell_{12}!}\right]^2
\end{aligned} \tag{2.100}$$

2.11.3 4-det interactions [changed sign by hand]

$$0 \le \ell_{12} \le N-1 \tag{2.101}$$
$$\ell_{14} = N-1-\ell_{12} \tag{2.102}$$
$$\ell_{23} = N-1-\ell_{12} \tag{2.103}$$
$$\ell_{34} = \ell_{12} \tag{2.104}$$

$$\begin{aligned}
&\frac{1}{(N!)^4}\left(\varepsilon_{i_1^{(1)}\ldots i_N^{(1)}}\varepsilon_{j_1^{(1)}\ldots j_N^{(1)}}\right)\ldots\left(\varepsilon_{i_1^{(4)}\ldots i_N^{(4)}}\varepsilon_{j_1^{(4)}\ldots j_N^{(4)}}\right)\left\langle (Y_1\cdot\Phi)_{i_1^{(1)}j_1^{(1)}}\ldots(Y_1\cdot\Phi)_{i_N^{(4)}j_N^{(4)}}\,V_{X+H}\right\rangle_{\mathrm{LO}} \\
&= \sum_{\{\ell_{ij}\}}\frac{1}{(N!)^4}\binom{N}{1}^4 \\
&\binom{N-1}{\ell_{12}\;\ell_{14}}\binom{N-1}{\ell_{12}\;\ell_{23}}\binom{N-1}{\ell_{23}\;\ell_{34}}\binom{N-1}{\ell_{14}\;\ell_{34}}\ell_{12}!\ell_{14}!\ell_{23}!\ell_{34}! \\
&\left(\frac{1}{2}\frac{\lambda}{N}I_{12}Y_1\cdot Y_2\right)^{\ell_{12}}\left(\frac{1}{2}\frac{\lambda}{N}I_{14}Y_1\cdot Y_4\right)^{\ell_{14}}\left(\frac{1}{2}\frac{\lambda}{N}I_{23}Y_2\cdot Y_3\right)^{\ell_{23}}\left(\frac{1}{2}\frac{\lambda}{N}I_{34}Y_3\cdot Y_4\right)^{\ell_{34}} \\
&\left(\varepsilon_{i^{(1)}i_1\ldots i_{\ell_{12}}m_1\ldots m_{\ell_{14}}}\varepsilon^{j^{(1)}j_1\ldots j_{\ell_{12}}n_1\ldots n_{\ell_{14}}}\right)\left(\varepsilon_{i^{(2)}j_1\ldots j_{\ell_{12}}o_1\ldots o_{\ell_{23}}}\varepsilon^{j^{(2)}i_1\ldots i_{\ell_{12}}p_1\ldots p_{\ell_{23}}}\right) \\
&\left(\varepsilon_{i^{(3)}p_1\ldots p_{\ell_{23}}s_1\ldots s_{\ell_{34}}}\varepsilon^{j^{(3)}o_1\ldots o_{\ell_{23}}t_1\ldots t_{\ell_{34}}}\right)\left(\varepsilon_{i^{(4)}n_1\ldots n_{\ell_{14}}t_1\ldots t_{\ell_{34}}}\varepsilon^{j^{(4)}m_1\ldots m_{\ell_{14}}s_1\ldots s_{\ell_{34}}}\right) \\
&\left\langle (Y_1\cdot\Phi)_{i^{(1)}j^{(1)}}\,(Y_2\cdot\Phi)_{i^{(2)}j^{(2)}}\,(Y_3\cdot\Phi)_{i^{(3)}j^{(3)}}\,(Y_4\cdot\Phi)_{i^{(4)}j^{(4)}}\,V_{X+H}\right\rangle_{\mathrm{LO}}
\end{aligned} \tag{2.105}$$

$$
\begin{aligned}
&= \left(\frac{\lambda}{2N}\right)^{2N+1} \sum_{\ell_{12}=0}^{N-1} \left(I_{12} I_{34} Y_1 \cdot Y_2 Y_3 \cdot Y_4\right)^{\ell_{12}} \left(I_{14} I_{23} Y_1 \cdot Y_4 Y_2 \cdot Y_3\right)^{N-1-\ell_{12}} 2 \left[\frac{\ell_{12}!}{(N-1-\ell_{12})!}\right]^2 \\
&\quad \Big\{ \Big(-\delta^{i^{(1)} p_1 \dots p_{N-1-\ell_{12}}}_{i^{(1)} m_1 \dots m_{N-1-\ell_{12}}} \delta^{i^{(3)} m_1 \dots m_{N-1-\ell_{12}}}_{i^{(3)} p_1 \dots p_{N-1-\ell_{12}}} \delta^{i^{(4)} n_1 \dots n_{N-1-\ell_{12}}}_{i^{(2)} o_1 \dots o_{N-1-\ell_{12}}} \delta^{i^{(2)} o_1 \dots o_{N-1-\ell_{12}}}_{i^{(4)} n_1 \dots n_{N-1-\ell_{12}}} \\
&\quad + \delta^{i^{(1)} p_1 \dots p_{N-1-\ell_{12}}}_{i^{(1)} m_1 \dots m_{N-1-\ell_{12}}} \delta^{i^{(2)} m_1 \dots m_{N-1-\ell_{12}}}_{i^{(3)} p_1 \dots p_{N-1-\ell_{12}}} \delta^{i^{(3)} n_1 \dots n_{N-1-\ell_{12}}}_{i^{(2)} o_1 \dots o_{N-1-\ell_{12}}} \delta^{i^{(4)} o_1 \dots o_{N-1-\ell_{12}}}_{i^{(4)} n_1 \dots n_{N-1-\ell_{12}}} \Big) \left[X_{1234} \left(Y_1 \cdot Y_4\right) \left(Y_2 \cdot Y_3\right) + I_{12} I_{34} F_{12,34} \left(Y_1 \cdot Y_2\right) \left(Y_3 \cdot Y_4\right)\right] \\
&\quad + \Big(-\delta^{i^{(3)} p_1 \dots p_{N-1-\ell_{12}}}_{i^{(1)} m_1 \dots m_{N-1-\ell_{12}}} \delta^{i^{(2)} m_1 \dots m_{N-1-\ell_{12}}}_{i^{(3)} p_1 \dots p_{N-1-\ell_{12}}} \delta^{i^{(4)} n_1 \dots n_{N-1-\ell_{12}}}_{i^{(2)} o_1 \dots o_{N-1-\ell_{12}}} \delta^{i^{(1)} o_1 \dots o_{N-1-\ell_{12}}}_{i^{(4)} n_1 \dots n_{N-1-\ell_{12}}} \\
&\quad + \delta^{i^{(4)} p_1 \dots p_{N-1-\ell_{12}}}_{i^{(1)} m_1 \dots m_{N-1-\ell_{12}}} \delta^{i^{(3)} m_1 \dots m_{N-1-\ell_{12}}}_{i^{(3)} p_1 \dots p_{N-1-\ell_{12}}} \delta^{i^{(2)} n_1 \dots n_{N-1-\ell_{12}}}_{i^{(2)} o_1 \dots o_{N-1-\ell_{12}}} \delta^{i^{(1)} o_1 \dots o_{N-1-\ell_{12}}}_{i^{(4)} n_1 \dots n_{N-1-\ell_{12}}} \Big) \left[-X_{1234} \left(Y_1 \cdot Y_2\right) \left(Y_3 \cdot Y_4\right) + X_{1234} \left(Y_1 \cdot Y_4\right) \left(Y_2 \cdot Y_3\right)\right] \\
&\quad - \Big(-\delta^{i^{(3)} p_1 \dots p_{N-1-\ell_{12}}}_{i^{(1)} m_1 \dots m_{N-1-\ell_{12}}} \delta^{i^{(1)} m_1 \dots m_{N-1-\ell_{12}}}_{i^{(3)} p_1 \dots p_{N-1-\ell_{12}}} \delta^{i^{(2)} n_1 \dots n_{N-1-\ell_{12}}}_{i^{(2)} o_1 \dots o_{N-1-\ell_{12}}} \delta^{i^{(4)} o_1 \dots o_{N-1-\ell_{12}}}_{i^{(4)} n_1 \dots n_{N-1-\ell_{12}}} \\
&\quad + \delta^{i^{(4)} p_1 \dots p_{N-1-\ell_{12}}}_{i^{(1)} m_1 \dots m_{N-1-\ell_{12}}} \delta^{i^{(1)} m_1 \dots m_{N-1-\ell_{12}}}_{i^{(3)} p_1 \dots p_{N-1-\ell_{12}}} \delta^{i^{(3)} n_1 \dots n_{N-1-\ell_{12}}}_{i^{(2)} o_1 \dots o_{N-1-\ell_{12}}} \delta^{i^{(2)} o_1 \dots o_{N-1-\ell_{12}}}_{i^{(4)} n_1 \dots n_{N-1-\ell_{12}}} \Big) \left[-X_{1234} \left(Y_1 \cdot Y_2\right) \left(Y_3 \cdot Y_4\right) + I_{14} I_{23} F_{14,23} \left(Y_1 \cdot Y_4\right) \left(Y_2 \cdot Y_3\right)\right] \Big\} \\
&= \left(\frac{\lambda}{2N}\right)^{2N+1} \sum_{\ell_{12}=0}^{N-1} \left(I_{12} I_{34} Y_1 \cdot Y_2 Y_3 \cdot Y_4\right)^{\ell_{12}} \left(I_{14} I_{23} Y_1 \cdot Y_4 Y_2 \cdot Y_3\right)^{N-1-\ell_{12}} \frac{2 \left(N!\right)^2 \left(N+1\right)}{N} \\
&\quad \left\{ \left(-1 - 2\ell_{12} + \ell_{12}^2 + N - \ell_{12} N\right) X_{1234} \left(Y_1 \cdot Y_2\right) \left(Y_3 \cdot Y_4\right) + \left(2 + 4\ell_{12} + \ell_{12}^2 - 2N - \ell_{12} N\right) X_{1234} \left(Y_1 \cdot Y_4\right) \left(Y_2 \cdot Y_3\right) \right. \\
&\quad \left. - \left(N - 1 - \ell_{12}\right) \left(\ell_{12} + 1\right) I_{12} I_{34} F_{12,34} \left(Y_1 \cdot Y_2\right) \left(Y_3 \cdot Y_4\right) + \ell_{12} \left(N - \ell_{12}\right) I_{14} I_{23} F_{14,23} \left(Y_1 \cdot Y_4\right) \left(Y_2 \cdot Y_3\right) \right\}
\end{aligned}
$$

Contraction IIX

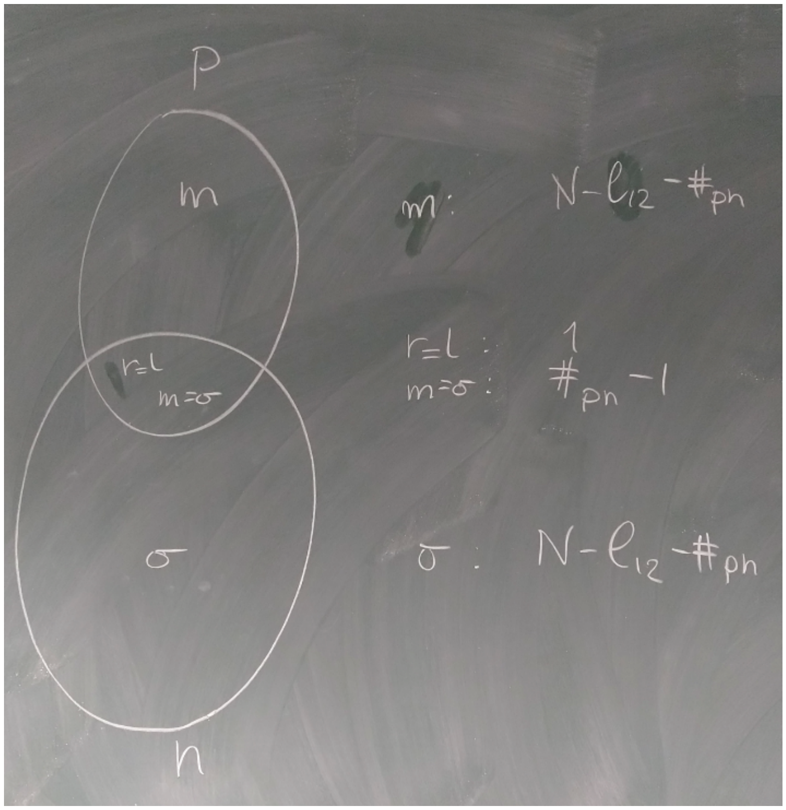
$$
\begin{aligned}
& \delta^{i^{(1)} p_1 \dots p_{N-1-\ell_{12}}}_{i^{(1)} m_1 \dots m_{N-1-\ell_{12}}} \delta^{i^{(3)} m_1 \dots m_{N-1-\ell_{12}}}_{i^{(3)} p_1 \dots p_{N-1-\ell_{12}}} \delta^{i^{(4)} n_1 \dots n_{N-1-\ell_{12}}}_{i^{(2)} o_1 \dots o_{N-1-\ell_{12}}} \delta^{i^{(2)} o_1 \dots o_{N-1-\ell_{12}}}_{i^{(4)} n_1 \dots n_{N-1-\ell_{12}}} \\
&= \left(1 + \ell_{12}\right)^2 \delta^{p_1 \dots p_{N-1-\ell_{12}}}_{m_1 \dots m_{N-1-\ell_{12}}} \delta^{m_1 \dots m_{N-1-\ell_{12}}}_{p_1 \dots p_{N-1-\ell_{12}}} \times \delta^{n_1 \dots n_{N-\ell_{12}}}_{o_1 \dots o_{N-\ell_{12}}} \delta^{o_1 \dots o_{N-\ell_{12}}}_{n_1 \dots n_{N-\ell_{12}}} \\
&= \left(1 + \ell_{12}\right)^2 \frac{N! \left(N - 1 - \ell_{12}\right)!}{\left(\ell_{12} + 1\right)!} \times \frac{N! \left(N - \ell_{12}\right)!}{\ell_{12}!} \\
&= \left(\frac{N!}{\ell_{12}!}\right)^2 \left(\ell_{12} + 1\right) \left(N - 1 - \ell_{12}\right)! \left(N - \ell_{12}\right)!
\end{aligned} \tag{2.106}
$$

Contraction IXI (use that in $M\delta\delta$)

$$
\begin{aligned}
& \delta^{i^{(1)} p_1 \dots p_{N-1-\ell_{12}}}_{i^{(1)} m_1 \dots m_{N-1-\ell_{12}}} \delta^{i^{(2)} m_1 \dots m_{N-1-\ell_{12}}}_{i^{(3)} p_1 \dots p_{N-1-\ell_{12}}} \delta^{i^{(3)} n_1 \dots n_{N-1-\ell_{12}}}_{i^{(2)} o_1 \dots o_{N-1-\ell_{12}}} \delta^{i^{(4)} o_1 \dots o_{N-1-\ell_{12}}}_{i^{(4)} n_1 \dots N-1-\ell_{12}} \\
&= \left(1 + \ell_{12}\right)^2 \delta^{p_1 \dots p_{N-1-\ell_{12}}}_{m_1 \dots m_{N-1-\ell_{12}}} \delta^{l m_1 \dots m_{N-1-\ell_{12}}}_{q p_1 \dots p_{N-1-\ell_{12}}} \delta^{q n_1 \dots n_{N-1-\ell_{12}}}_{l o_1 \dots o_{N-1-\ell_{12}}} \delta^{o_1 \dots o_{N-1-\ell_{12}}}_{n_1 \dots n_{N-1-\ell_{12}}} \\
&= \left(1 + \ell_{12}\right)^2 \frac{1}{N} \left[\frac{N! \left(N - 1 - \ell_{12}\right)!}{\ell_{12}!}\right]^2 \\
&= \frac{1}{N} \left[\frac{N! \left(N - 1 - \ell_{12}\right)! \left(1 + \ell_{12}\right)}{\ell_{12}!}\right]^2
\end{aligned} \tag{2.107}
$$

Contraction $\backslash\backslash\backslash\backslash$

$$
\begin{aligned}
& \delta^{i^{(3)} p_1 \dots p_{N-1-\ell_{12}}}_{i^{(1)} m_1 \dots m_{N-1-\ell_{12}}} \delta^{i^{(2)} m_1 \dots m_{N-1-\ell_{12}}}_{i^{(3)} p_1 \dots p_{N-1-\ell_{12}}} \delta^{i^{(4)} n_1 \dots n_{N-1-\ell_{12}}}_{i^{(2)} o_1 \dots o_{N-1-\ell_{12}}} \delta^{i^{(1)} o_1 \dots o_{N-1-\ell_{12}}}_{i^{(4)} n_1 \dots n_{N-1-\ell_{12}}} \\
&= \delta^{p_1 \dots p_{N-\ell_{12}}}_{r m_1 \dots m_{N-1-\ell_{12}}} \delta^{l m_1 \dots m_{N-1-\ell_{12}}}_{p_1 \dots p_{N-\ell_{12}}} \delta^{n_1 \dots n_{N-\ell_{12}}}_{l o_1 \dots o_{N-1-\ell_{12}}} \delta^{r o_1 \dots o_{N-1-\ell_{12}}}_{n_1 \dots n_{N-\ell_{12}}} \\
&= \sum_{\#_{pn} = \max(1, N - 2\ell_{12})}^{N-\ell_{12}} \underbrace{\binom{N}{\#_{pn}}}_{\text{fill } \{p\} \cap \{n\}} \underbrace{\binom{N - \#_{pn}}{N - \ell_{12} - \#_{pn}}}_{\text{fill } \{p\} \backslash (\{p\} \cap \{n\})} \underbrace{\binom{\ell_{12}}{N - \ell_{12} - \#_{pn}}}_{\text{fill } \{n\} \backslash (\{p\} \cap \{n\})} \underbrace{\left[\left(N - \ell_{12}\right)!\right]^2}_{\text{order } \{p\}, \{n\}} \\
&\quad \underbrace{\binom{\#_{pn}}{1}}_{\text{choose } r \text{ in } \{p\} \cap \{n\}} \underbrace{\left[\left(N - \ell_{12} - 1\right)!\right]^2}_{\text{order } \{m\}, \{o\}} \\
&= \frac{1}{N} \left[\frac{N! \left(N - \ell_{12}\right)!}{\ell_{12}!}\right]^2
\end{aligned} \tag{2.108}
$$



2.11.4 Sum

$$
\begin{aligned}
&\frac{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle^{\ell_{13}=0}_{\text{NLO}}}{\langle \mathcal{O}_1\mathcal{O}_2\rangle\langle \mathcal{O}_3\mathcal{O}_4\rangle} \\
&= \frac{\lambda\,(N+1)}{8\pi^2N^2}\frac{r\left\{r^N\left[N\,(r-1)-r-1\right]+N\,(r-1)+r+1\right\}}{(1-r)^3}F^{(1)}\,(z,\bar{z}) \\
\overset{N\gg 1}{=} &-\frac{\lambda}{8\pi^2}F^{(1)}\,(z,\bar{z})\times\begin{cases}\frac{r}{(1-r)^2} & r<1\\ \frac{N^2}{6} & r=1\\ \frac{r^{N+1}}{(1-r)^2} & r>1\end{cases}
\end{aligned}
\tag{2.109}
$$

The result is crossing-symmetric, taking note of the property $F^{(1)}\,(z,\bar{z})=F^{(1)}\,(1-z,1-\bar{z})$.

3 Miscellanea

Wick contractions

$$
\begin{aligned}
&\left\langle (Y_1\cdot\Phi)_{i^{(1)}j^{(1)}}\,(Y_2\cdot\Phi)_{i^{(2)}j^{(2)}}\,(Y_2\cdot\Phi)_{i^{(3)}j^{(3)}}\,(Y_3\cdot\Phi)_{i^{(4)}j^{(4)}}\,V_{X+H}\right\rangle_{\text{LO}} \\
=&\frac{\lambda^3}{N^3}T^{a_1}_{i^{(1)}j^{(1)}}T^{a_2}_{i^{(2)}j^{(2)}}T^{a_3}_{i^{(3)}j^{(3)}}T^{a_4}_{i^{(4)}j^{(4)}}\,(Y_1\cdot Y_2)\,(Y_2\cdot Y_3) \\
&\left(f^{aa_1a_2}f^{aa_3a_4}+f^{aa_1a_3}f^{aa_2a_4}\right)(-X_{1223}+I_{12}I_{23}F_{12,23})
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
&\left\langle (Y_1\cdot\Phi)_{i^{(1)}j^{(1)}}\,(Y_2\cdot\Phi)_{i^{(2)}j^{(2)}}\,(Y_3\cdot\Phi)_{i^{(3)}j^{(3)}}\,(Y_4\cdot\Phi)_{i^{(4)}j^{(4)}}\,V_{X+H}\right\rangle_{\text{LO}} \\
=&\frac{\lambda^3}{N^3}T^{a_1}_{i^{(1)}j^{(1)}}T^{a_2}_{i^{(2)}j^{(2)}}T^{a_3}_{i^{(3)}j^{(3)}}T^{a_4}_{i^{(4)}j^{(4)}} \\
&[(-f^{aa_1a_3}f^{aa_2a_4}X_{1234}-f^{aa_1a_4}f^{aa_2a_3}X_{1234}+f^{aa_1a_2}f^{aa_3a_4}I_{12}I_{34}F_{12,34})\,(Y_1\cdot Y_2)\,(Y_3\cdot Y_4) \\
&+(f^{aa_1a_2}f^{aa_3a_4}X_{1234}+f^{aa_1a_3}f^{aa_2a_4}X_{1234}+f^{aa_1a_4}f^{aa_2a_3}I_{14}I_{23}F_{14,23})\,(Y_1\cdot Y_4)\,(Y_2\cdot Y_3)]
\end{aligned}
\tag{3.2}
$$

Color contractions

$$
f^{aa_1a_2}f^{aa_3a_4}T^{a_1}_{i^{(1)}j^{(1)}}T^{a_2}_{i^{(2)}j^{(2)}}T^{a_3}_{i^{(3)}j^{(3)}}T^{a_4}_{i^{(4)}j^{(4)}}=\frac{1}{8}\left(-\delta_{i^{(1)}j^{(2)}}\delta_{i^{(2)}j^{(3)}}\delta_{i^{(3)}j^{(4)}}\delta_{i^{(4)}j^{(1)}}+\delta_{i^{(1)}j^{(2)}}\delta_{i^{(2)}j^{(4)}}\delta_{i^{(3)}j^{(1)}}\delta_{i^{(4)}j^{(3)}}\right.
\tag{3.3}
$$

$$+\delta_{i^{(1)}j^{(3)}}\delta_{i^{(2)}j^{(1)}}\delta_{i^{(3)}j^{(4)}}\delta_{i^{(4)}j^{(2)}} - \delta_{i^{(1)}j^{(4)}}\delta_{i^{(2)}j^{(1)}}\delta_{i^{(3)}j^{(2)}}\delta_{i^{(4)}j^{(3)}}\big)$$

$$\begin{aligned} f^{aa_1a_3}f^{aa_2a_4}T^{a_1}_{i^{(1)}j^{(1)}}T^{a_2}_{i^{(2)}j^{(2)}}T^{a_3}_{i^{(3)}j^{(3)}}T^{a_4}_{i^{(4)}j^{(4)}} = \frac{1}{8}\big(&-\delta_{i^{(1)}j^{(3)}}\delta_{i^{(3)}j^{(2)}}\delta_{i^{(2)}j^{(4)}}\delta_{i^{(4)}j^{(1)}} + \delta_{i^{(1)}j^{(3)}}\delta_{i^{(3)}j^{(4)}}\delta_{i^{(2)}j^{(1)}}\delta_{i^{(4)}j^{(2)}} \\ &+\delta_{i^{(1)}j^{(2)}}\delta_{i^{(3)}j^{(1)}}\delta_{i^{(2)}j^{(4)}}\delta_{i^{(4)}j^{(3)}} - \delta_{i^{(1)}j^{(4)}}\delta_{i^{(3)}j^{(1)}}\delta_{i^{(2)}j^{(3)}}\delta_{i^{(4)}j^{(2)}}\big) \end{aligned} \tag{3.4}$$

$$\begin{aligned} f^{aa_1a_4}f^{aa_3a_2}T^{a_1}_{i^{(1)}j^{(1)}}T^{a_2}_{i^{(2)}j^{(2)}}T^{a_3}_{i^{(3)}j^{(3)}}T^{a_4}_{i^{(4)}j^{(4)}} = \frac{1}{8}\big(&-\delta_{i^{(1)}j^{(4)}}\delta_{i^{(4)}j^{(3)}}\delta_{i^{(3)}j^{(2)}}\delta_{i^{(2)}j^{(1)}} + \delta_{i^{(1)}j^{(4)}}\delta_{i^{(4)}j^{(2)}}\delta_{i^{(3)}j^{(1)}}\delta_{i^{(2)}j^{(3)}} \\ &+\delta_{i^{(1)}j^{(3)}}\delta_{i^{(4)}j^{(1)}}\delta_{i^{(3)}j^{(2)}}\delta_{i^{(2)}j^{(4)}} - \delta_{i^{(1)}j^{(2)}}\delta_{i^{(4)}j^{(1)}}\delta_{i^{(3)}j^{(4)}}\delta_{i^{(2)}j^{(3)}}\big) \end{aligned} \tag{3.5}$$

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