String perturbation theory for Maldacena-Wilson loops with heat kernel method

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Contents

1 Introduction

The 1/4-BPS latitude Wilson loops in $\mathcal{N}=4$ SYM [1, 2] were holographically computed as a string partition functions in [3] at one loop in the strong-coupling regime. This family of operators is defined along a unit circle and it is spanned by the angle $\theta_0 \in \left[0, \frac{\pi}{2}\right]$, parametrizing the coupling to the scalars in the gauge theory.

The semiclassical expansion of the string effective action $\Gamma\left(\lambda, \theta_0\right)$

$$\Gamma\left(\lambda, \theta_0\right) \equiv -\log Z\left(\lambda, \theta_0\right) = \Gamma_0\left(\lambda, \theta_0\right) + \Gamma_1\left(\theta_0\right) + \ldots \tag{1.1}$$

starts with (minus) the *regularized* classical area of the worldsheet

$$\Gamma_0\left(\lambda, \theta_0\right) = \sqrt{\lambda}\cos\theta_0 \tag{1.2}$$

and it receives a perturbative correction around the (dominant) saddle-point of the string partition function. This next-to-leading term comes from the one-loop determinants of the operators describing the semiclassical fluctuations around the background

$$\Gamma_1\left(\theta_0\right) = -\log Z_1\left(\theta_0\right) = -\log \frac{\prod_{p_{12}, p_{56}, p_{89}=\pm 1} \mathrm{Det}^{1/4} \mathcal{O}^2_{p_{12}, p_{56}, p_{89}}\left(\theta_0\right)}{\mathrm{Det}^{3/2}\mathcal{O}_1\left(\theta_0\right) \mathrm{Det}^{3/2}\mathcal{O}_2\left(\theta_0\right) \mathrm{Det}^{1/2}\mathcal{O}_{3+}\left(\theta_0\right) \mathrm{Det}^{1/2}\mathcal{O}_{3-}\left(\theta_0\right)} \,. \tag{1.3}$$

Higher-loop corrections would arise from multiple functional derivatives of the classical action and have not been computed.

The one-loop effective action for the string dual to the latitude Wilson loops, *normalized* to the 1/2-BPS circular case $\theta_0 = 0$, is predicted by gauge theory to be [1, 4]

$$\Gamma_1\left(\theta_0\right) - \Gamma_1\left(0\right) = \frac{3}{2}\log\cos\theta_0 \,. \tag{1.4}$$

This difference was computed for any value of the R-symmetry angle θ_0 using the Gel'fand-Yaglom method in [3], but it was found to be in disagreement with the gauge-theory prediction $\frac{3}{2}\log\cos\theta_0$ in the formula above.

The aim of the project is to repeat the computation of the effective action of the *normalized* latitude using heat kernel techniques. Since the heat kernel propagators of the Laplace and Dirac operators are known on the hyperbolic plane H^2 (the geometry of the worldsheet dual to the circular Wilson loop $\theta_0 = 0$), it is likely that we can reach our goal only for small values of θ_0. In other words, we aim for the first θ_0^2 correction to the one-loop effective action

$$\Gamma_1\left(\theta_0\right) - \Gamma_1\left(0\right) = -\frac{3}{4}\theta_0^2 - \frac{1}{8}\theta_0^2 + O\left(\theta_0^4\right) \,. \tag{1.5}$$

A possible extension of the project may concern the inclusion of a multiple string wrapping with winding number $k \neq 1$ and the study of other SUSY Wilson loops with near-H^2 or near-S^2 string worldsheets.

2 String action for the latitude Wilson loops

The holographic description of 1/4-BPS latitude Wilson loops is reviewed in [3]. The relation between the R-symmetry angle θ_0 and the convenient parameter σ_0 is

$$\tanh \sigma_0 = \cos \theta_0 \, . \tag{2.1}$$

The classical worldsheet is a conformally-flat Euclidean space

$$ds^2 = \Omega^2(\sigma)\left(d\tau^2 + d\sigma^2\right) \qquad \sigma > 0 \qquad \tau \in [0, 2\pi) \tag{2.2}$$

which is a "small" deformation of H^2 for small values of the angle θ_0

$$\Omega^2(\sigma) \equiv \frac{1}{\sinh^2 \sigma} + \frac{1}{\cosh^2(\sigma + \sigma_0)} = \frac{1}{\sinh^2 \sigma} + O\left(\theta_0^2\right) \, . \tag{2.3}$$

In the case of the circular Wilson loop $\theta_0 = 0$, the dual geometry becomes exactly H^2:

$$ds^2\big|_{\theta_0 = 0} = \frac{d\tau^2 + d\sigma^2}{\sinh^2 \sigma} = d\rho^2 + \sinh^2 \rho d\tau^2 \qquad \sinh \rho = \frac{1}{\sinh \sigma} \, . \tag{2.4}$$

We expanded the type IIB Green-Schwarz action around the stable string solution ending on a latitude Wilson loop. The resulting bosonic and fermionic action, quadratic in the fluctuation fields, are reported in what follows.

2.1 Operators for finite $\theta_0 \in \left[0, \frac{\pi}{2}\right]$

The bosonic operators read

$$\mathcal{O}_1(\theta_0) \equiv \frac{1}{\Omega^2}\left(-\partial_\tau^2 - \partial_\sigma^2 + \frac{2}{\sinh^2 \sigma}\right) \tag{2.5}$$

$$\mathcal{O}_2(\sigma_0) \equiv \frac{1}{\Omega^2}\left(-\partial_\tau^2 - \partial_\sigma^2 - \frac{2}{\cosh^2(\sigma + \sigma_0)}\right) \tag{2.6}$$

$$\mathcal{O}_{3\pm}(\sigma_0) \equiv \frac{1}{\Omega^2}\left(-\partial_\tau^2 - \partial_\sigma^2 \pm 2i\left(\tanh(2\sigma + \sigma_0) - 1\right)\partial_\tau - 1 - 2\tanh(2\sigma + \sigma_0) + 3\tanh^2(2\sigma + \sigma_0)\right) \tag{2.7}$$

while the eight fermionic operators are labeled by the triplet (p_{12}, p_{56}, p_{89}), each label being either $+1$ or -1, as

$$\mathcal{O}_{p_{12},p_{56},p_{89}}(\sigma_0) \equiv \frac{1}{\Omega}\left(-i\partial_\tau + p_{56}\frac{1 - \tanh(2\sigma + \sigma_0)}{2}\right)\sigma_2 + \frac{i}{\Omega}\left(\partial_\sigma + \frac{\Omega'}{2\Omega}\right)\sigma_1 - \frac{p_{12}p_{56}}{\Omega^2\cosh^2(\sigma + \sigma_0)}\mathbb{I}_2 + \frac{p_{12}}{\Omega^2\sinh^2\sigma}\sigma_3. \tag{2.8}$$

We keep all the three labels for clarity, although p_{89} does not effectively play any role. The one-loop effective action is formally given by

$$\Gamma_1(\theta_0) = \frac{3}{2}\log \mathrm{Det}\mathcal{O}_1 + \frac{3}{2}\log \mathrm{Det}\mathcal{O}_2 + \frac{1}{2}\log \mathrm{Det}\mathcal{O}_{3+} + \frac{1}{2}\log \mathrm{Det}\mathcal{O}_{3-} - \frac{1}{4}\sum_{p_{12},p_{56},p_{89}=\pm 1}\log \mathrm{Det}\mathcal{O}^2_{p_{12},p_{56},p_{89}}. \tag{2.9}$$

2.2 Operators for $\theta_0 \sim 0$

For the present project we expand the relevant differential operators for small θ_0 around the circular Wilson loop $\theta_0 = 0$:

$$\mathcal{O}_i = \bar{\mathcal{O}}_i + \theta_0^2\tilde{\mathcal{O}}_i + \ldots \qquad i = 1,\ 2,\ 3\pm \tag{2.10}$$

$$\mathcal{O}_{p_{12},p_{56},p_{89}} = \bar{\mathcal{O}}_{p_{12},p_{56},p_{89}} + \theta_0^2\tilde{\mathcal{O}}_{p_{12},p_{56},p_{89}} + \ldots \, . \tag{2.11}$$

We can also consider the fermionic second-order differential operators and its expansion

$$\mathcal{O}^2_{p_{12},p_{56},p_{89}} = \bar{\mathcal{O}}^2_{p_{12},p_{56},p_{89}} + \theta_0^2\left\{\bar{\mathcal{O}}_{p_{12},p_{56},p_{89}}, \tilde{\mathcal{O}}_{p_{12},p_{56},p_{89}}\right\} + \ldots \, . \tag{2.12}$$

The operators for the circular case are the (self-adjoint) massive Laplace and Dirac ones on the hyperbolic plane H^2.

$$\bar{\mathcal{O}}_1 = -\Delta_{\rho,\tau} + 2 \tag{2.13}$$

$$\bar{\mathcal{O}}_2 = \bar{\mathcal{O}}_{3\pm} = -\Delta_{\rho,\tau} \tag{2.14}$$

$$\bar{\mathcal{O}}_{p_{12},p_{56},p_{89}} = -i\nabla\!\!\!\!/\,_{\rho,\tau} + p_{12}\sigma_3 \tag{2.15}$$

The spectrum of physical excitations is composed of 3 massive scalars $\left(m^2=2\right)$, 5 massless scalars and 8 massive Majorana fermions $(m^2=1)$ propagating in the hyperbolic background. The first corrections to these operators are [1]

$$\tilde{\mathcal{O}}_1=\tilde{\mathcal{O}}_2=\frac{1}{(1+\cosh\rho)^2}\left(\Delta_{\rho,\tau}-2\right) \tag{2.16}$$

$$\tilde{\mathcal{O}}_{3\pm}=\frac{1}{(1+\cosh\rho)^2}\left[\Delta_{\rho,\tau}-\left(\frac{1-\cosh\rho}{\sinh\rho}\right)^2\left(2\pm i\partial_\tau\right)\right] \tag{2.17}$$

$$\tilde{\mathcal{O}}_{p_{12},p_{56},p_{89}}=\frac{i}{2\,(1+\cosh\rho)^2}\nabla\!\!\!\!/_{\rho,\tau}-\frac{i\,(1-\cosh\rho)}{2\sinh\rho\,(1+\cosh\rho)^2}\sigma_1+\frac{p_{56}\sinh^3\rho}{4\,(1+\cosh\rho)^4}\sigma_2-p_{12}\frac{\sigma_3+p_{56}\mathbb{I}_2}{(1+\cosh\rho)^2}\,. \tag{2.18}$$

2.3 Effective action at $\theta_0=0$

The one-loop effective action for the circular Wilson loop [2]

$$\bar{\Gamma}_1=\frac{3}{2}\log\mathrm{Det}\bar{\mathcal{O}}_1+\frac{3}{2}\log\mathrm{Det}\bar{\mathcal{O}}_2+\frac{1}{2}\log\mathrm{Det}\bar{\mathcal{O}}_{3+}+\frac{1}{2}\log\mathrm{Det}\bar{\mathcal{O}}_{3-}-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}=\pm1}\log\mathrm{Det}\bar{\mathcal{O}}^2_{p_{12},p_{56},p_{89}} \tag{2.19}$$

was computed with the Gel'fand-Yaglom method in [5]. We show how to recover the same result using the heat kernel method [6, 7].

The worldsheet bounded by the unit circle of the 1/2-BPS Wilson loop has the geometry of the (global) hyperbolic plane H^2

$$ds^2\Big|_{\theta_0=0}=d\rho^2+\sinh^2\rho d\tau^2 \tag{2.20}$$

with IR-regularized volume

$$V_{H^2}\equiv\int_0^{2\pi}d\tau\int_0^{\mathrm{arccosh}\frac{1}{\epsilon}}d\rho\,\sinh\rho=\frac{2\pi}{\epsilon}-2\pi\to-2\pi\,. \tag{2.21}$$

In the formula above, we used the relation between the Poincare' radial coordinate z, the geodesic radial coordinate ρ and the worldsheet radial coordinate σ (for this classical solution)

$$z=\tanh\sigma\qquad\sinh\rho=\frac{1}{\sinh\sigma} \tag{2.22}$$

to put a lower cutoff ϵ on z as $\epsilon<z<1$, hence $0<\rho<\mathrm{arccosh}\frac{1}{\epsilon}$.

Let us consider the propagation of the worldsheet excitations on the hyperbolic background geometry with Dirichlet conditions on the S^1 boundary at $\rho=\infty$. The scalars enter the Lagrangian with the scalar Laplace operator $-\Delta+m^2$ and the spinors with the Dirac operator $-i\nabla\!\!\!\!/+m$, where m is the mass of the relevant field [8, 9, 10]. The heat kernel propagators of a scalar and a Dirac spinor field [3] read

$$K_{-\Delta+m^2}\left(x,x';t\right)=\frac{1}{2\pi}\int_0^\infty dv\,v\tanh\left(\pi v\right)P_{-\frac{1}{2}+iv}\left(\cosh d\left(x,x'\right)\right)e^{-t\left(v^2+\frac{1}{4}+m^2\right)} \tag{2.23}$$

$$K_{-\nabla\!\!\!\!/^2+m^2}\left(x,x';t\right)=U\left(x,x'\right)\frac{1}{2\pi}\int_0^\infty dv\,v\coth\pi v\,\cosh\frac{d\left(x,x'\right)}{2}\,{}_2F_1\left(1+iv,\,1-iv,\,1,\,\frac{1-\cosh d\left(x,x'\right)}{2}\right)e^{-t\left(v^2+m^2\right)} \tag{2.24}$$

where $U\left(x,x'\right)$ is the parallel spinor propagator (in a convenient matrix representation) [11]

$$U\left(x,x'\right)=\mathbb{I}_2\cos\left(\mathrm{atan}\frac{\cosh\frac{\rho+\rho'}{2}\tan\frac{\tau+\tau'}{2}}{\cosh\frac{\rho-\rho'}{2}}\right)+i\sigma_3\sin\left(\mathrm{atan}\frac{\cosh\frac{\rho+\rho'}{2}\tan\frac{\tau+\tau'}{2}}{\cosh\frac{\rho-\rho'}{2}}\right) \tag{2.25}$$

[1] The unperturbed and perturbed operators do not commute $\left[\bar{\mathcal{O}},\tilde{\mathcal{O}}\right]\neq0$, so the eigenspectrum changes non-trivially when the perturbation parameter θ_0 is switched on.

[2] The fermionic contribution is scaled down by a factor of 4 because of the Majorana condition and the square of the Dirac operator.

[3] The $\frac{1}{2}$ of the Majorana condition was not incorporated in the definition of the *Dirac* heat kernel propagator, at variance with [7, 11].

and it is a function of the geodesic distance between the manifold points x, x'

$$\cosh d\left(x, x'\right) = \cosh\rho \cosh\rho^{'} - \sinh\rho \sinh\rho^{'} \cos\left(\tau - \tau'\right) . \tag{2.26}$$

Besides the volume of H^2 (an IR-divergent quantity which required a regularization), we also have the usual UV divergences coming from the loop integrals. They arise from the small-t integral region, which corresponds to the short-distance heat propagation. This UV divergence can be regularized by a Mellin transform, namely the zeta function for Res large enough.

For each scalar we have

$$\begin{aligned}
\zeta_{-\Delta+m^2}(s) &= \frac{V_{H^2}}{2\pi}\int_0^\infty dv \frac{v\tanh\pi v}{\left(v^2+m^2+\frac{1}{4}\right)^s} \qquad (2.27)\\
&= \frac{V_{H^2}}{2\pi}\left[\frac{\left(m^2+\frac{1}{4}\right)^{1-s}}{2(s-1)} - 2\int_0^\infty dv \frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+m^2+\frac{1}{4}\right)^s}\right]\\
\zeta^{'}_{-\Delta+m^2}(0) &= \frac{V_{H^2}}{2\pi}\left\{\frac{\left(m^2+\frac{1}{4}\right)}{2}\left[\log\left(m^2+\frac{1}{4}\right)-1\right] + 2\int_0^\infty dv \frac{v\log\left(v^2+m^2+\frac{1}{4}\right)}{e^{2\pi v}+1}\right\} \qquad (2.28)\\
&= -\frac{V_{H^2}}{2\pi}\left[\frac{1+\log 2}{12} - \log A + \int_0^{m^2+\frac{1}{4}} dx\, \psi\left(\sqrt{x}+\frac{1}{2}\right)\right]
\end{aligned}$$

while for each Dirac spinor

$$\begin{aligned}
\zeta_{-\nabla\!\!\!/^2+m^2}(s) &= \frac{V_{H^2}}{\pi}\int_0^\infty dv \frac{v\coth\pi v}{\left(v^2+m^2\right)^s} \qquad (2.29)\\
&= \frac{V_{H^2}}{\pi}\left[\frac{\left(m^2\right)^{1-s}}{2(s-1)} + 2\int_0^\infty dv \frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+m^2\right)^s}\right]\\
\zeta^{'}_{-\nabla\!\!\!/^2+m^2}(0) &= \frac{V_{H^2}}{\pi}\left[\frac{m^2}{2}\left(\log m^2 - 1\right) - 2\int_0^\infty dv \frac{v\log\left(v^2+m^2\right)}{e^{2\pi v}-1}\right] \qquad (2.30)\\
&= \frac{V_{H^2}}{\pi}\left[-\frac{1}{6} + 2\log A + \sqrt{m^2} + \int_0^{m^2} dx\, \psi\left(\sqrt{x}\right)\right] .
\end{aligned}$$

The total zeta function is additive [4]

$$\begin{aligned}
\bar{\Gamma}_1 &= \frac{d}{ds}\left(-\frac{3}{2}\bar{\zeta}_{\mathcal{O}_1}(s) - \frac{3}{2}\bar{\zeta}_{\mathcal{O}_2}(s) - \frac{1}{2}\bar{\zeta}_{\mathcal{O}_{3+}}(s) - \frac{1}{2}\bar{\zeta}_{\mathcal{O}_{3-}}(s) + \frac{1}{4}\sum_{p_{12},p_{56},p_{89}} \bar{\zeta}_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(s)\right)\Bigg|_{s=0}\\
&= -\frac{3}{2}\left(-\frac{25}{12} + \frac{3}{2}\log 2\pi - 2\log A\right) - \frac{5}{2}\left(-\frac{1}{12} + \frac{1}{2}\log 2\pi - 2\log A\right) + \frac{8}{4}\left(-\frac{5}{3} + 2\log 2\pi - 4\log A\right)\\
&= \frac{1}{2}\log 2\pi. \qquad (2.31)
\end{aligned}$$

$A \approx 1.28243$ is the Glaisher constant. This agrees with the one-loop partition function evaluated in [5] with Gel'fand-Yaglom and the "SUSY-preserving" regularization of bosonic and fermionic Fourier frequencies. We remind that the it differs from the localization prediction $\bar{\Gamma}_1 = \frac{1}{2}\log\frac{\pi}{2}$, possibly due to the string path-integral ambiguities.

2.4 Effective action at order θ_0^2

What is physically more interesting to compute is the first deviation of a 1/4-BPS Wilson loop $\tilde{\Gamma}_1$ from the 1/2-BPS case $\bar{\Gamma}_1$:

$$\Gamma_1(\theta_0) = \bar{\Gamma}_1 + \theta_0^2\tilde{\Gamma}_1 + \ldots . \tag{2.32}$$

[4] The factor of $\frac{1}{2}$ for each fermion accounts for the fact that we used the heat kernel of the Dirac operator squared.

In principle $\Gamma_1(\theta_0)$ is obtained from the path-integral [5]

$$Z_1(\theta_0)=\int\prod_{i=1}^{8}\mathcal{D}\xi_i\exp\left[-\int_{\tau,\sigma}\Omega^2\left(\sum_{i=1,2,3}\xi_i\mathcal{O}_1\xi_i+\sum_{i=4,5,6}\xi_i\mathcal{O}_2\xi_i+\xi_7\mathcal{O}_{3+}\xi_7+\xi_8\mathcal{O}_{3-}\xi_8\right)\right]$$
$$\times\int\prod_{p_{12},p_{56},p_{89}=\pm1}\mathcal{D}\Psi_{p_{12},p_{56},p_{89}}\left[\exp-\int_{\tau,\sigma}\Omega^2\left(\sum_{p_{12},p_{56},p_{89}=\pm1}\bar{\Psi}_{p_{12},p_{56},p_{89}}\mathcal{O}_{p_{12},p_{56},p_{89}}\Psi_{p_{12},p_{56},p_{89}}\right)\right]. \quad (2.33)$$

For small angle, we expand the operators and perform a sort of perturbation theory for small "coupling constant" θ_0 via diagrammatical methods. One can then proceed with splitting the Lagrangian into the "free" and "interacting" part.

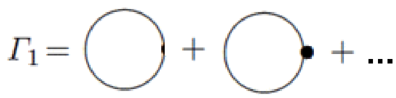

Figure 1: The small-angle effective action $\Gamma_1(\theta_0)$ is an infinite sum of one-loop vacuum diagrams with insertions, each carrying a power θ_0^2.

On the other hand, we found more convenient and safe to develop a perturbation theory for the functional determinants, by constructing the heat kernel propagators order-by-order. The starting point is the heat equation [6]

$$(\partial_t+\mathcal{O}_x)K_{\mathcal{O}}\left(x,x';t\right)=0 \qquad K_{\mathcal{O}}\left(x,x';0\right)=\frac{1}{\sqrt{g}}\delta\left(x-x'\right)\mathbb{I} \quad (2.34)$$

which decomposes into an infinite set of differential equations

$$\left(\partial_t+\bar{\mathcal{O}}_x\right)\bar{K}_{\mathcal{O}}\left(x,x';t\right)=0 \qquad \bar{K}_{\mathcal{O}}\left(x,x';0\right)=\frac{1}{\sqrt{\bar{g}}}\delta\left(x-x'\right)\mathbb{I} \quad (2.35)$$
$$\left(\partial_t+\bar{\mathcal{O}}_x\right)\tilde{K}_{\mathcal{O}}\left(x,x';t\right)+\tilde{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x';t\right)=0 \qquad \tilde{K}_{\mathcal{O}}\left(x,x';0\right)=-\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta\left(x-x'\right)\mathbb{I} \quad (2.36)$$
$$\cdots$$

where the point manifold x is the collection of coordinates (ρ,τ), $\delta(x)\equiv\delta(\rho)\delta(\tau)$ is the "flat" Dirac delta function and we posed

$$\begin{aligned}\mathcal{O}&=\bar{\mathcal{O}}+\theta_0^2\tilde{\mathcal{O}}+\ldots\\ g&=\bar{g}+\theta_0^2\tilde{g}+\ldots\\ K_{\mathcal{O}}\left(x,x';t\right)&=\bar{K}_{\mathcal{O}}\left(x,x';t\right)+\theta_0^2\tilde{K}_{\mathcal{O}}\left(x,x';t\right)+\ldots\\ K_{\mathcal{O}}(t)&=\int_x\sqrt{g}K_{\mathcal{O}}\left(x,x';t\right)\\ \zeta_{\mathcal{O}}(s)&=\bar{\zeta}_{\mathcal{O}}(s)+\theta_0^2\tilde{\zeta}_{\mathcal{O}}(s)+\ldots\\ \frac{1}{\sqrt{g}}\delta(x)&=\left(\frac{1}{\sqrt{\bar{g}}}-\theta_0^2\frac{\tilde{g}}{2\bar{g}^{3/2}}+\ldots\right)\delta(x)\,.\end{aligned} \quad (2.37)$$

[5] We do not integrate over the adjoints of $\Psi_{p_{12},p_{56},p_{89}}$ since they are Majorana fields.
[6] $\mathbb{I}$ is the unit matrix of the internal space (1×1 for scalars and 2×2 for fermions).

One can solve the heat equation at order θ_0^2

$$\tilde{K}_{\mathcal{O}}\left(x,x^{'};t\right) = -\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta\left(x-x^{'}\right)\mathbb{I} + \int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\bar{K}_{\mathcal{O}}\left(x,x^{''};t-t^{'}\right)\bar{\mathcal{O}}_{x^{''}}\left(\frac{\tilde{g}}{2\bar{g}^{3/2}}\delta\left(x^{''}-x^{'}\right)\right) \tag{2.38}$$

$$-\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\bar{K}_{\mathcal{O}}\left(x,x^{''};t-t^{'}\right)\tilde{\mathcal{O}}_{x^{''}}\bar{K}_{\mathcal{O}}\left(x^{''},x^{'};t^{'}\right)$$

$$\tilde{K}_{\mathcal{O}}(t) = \int_x\frac{\tilde{g}}{2\sqrt{\bar{g}}}\bar{K}_{\mathcal{O}}(x,x;t) + \int_x\sqrt{\bar{g}}\tilde{K}_{\mathcal{O}}(x,x;t) \tag{2.39}$$

$$= -t\int_x\sqrt{\bar{g}}\mathrm{tr}\left[\tilde{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x,x^{'};t\right)\right]_{x=x^{'}}$$

$$\tilde{\zeta}_{\mathcal{O}}(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\tilde{K}_{\mathcal{O}}(t)\ . \tag{2.40}$$

For each field of the theory, we compute the expressions

$$\frac{3}{2}\tilde{K}_{\mathcal{O}_1}(t) = -\frac{3t}{2}\left(\int_x\frac{\sinh\rho}{(1+\cosh\rho)^2}\right)\left[(\Delta_x-2)\bar{K}_{-\Delta+2}\left(x,x^{'};t\right)\right]_{x=x^{'}} \tag{2.41}$$

$$= \frac{3t}{4}\int_0^\infty dv\, v\tanh(\pi v)\left(v^2+\frac{9}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}$$

$$\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_1}(s) = \frac{3s}{4}\int_0^\infty dv\frac{v\tanh(\pi v)}{\left(v^2+\frac{9}{4}\right)^s} \tag{2.42}$$

$$= \frac{3s}{4}\int_0^\infty dv\frac{1}{\left(v^2+\frac{9}{4}\right)^s}\left(v-\frac{2v}{e^{2\pi v}+1}\right)$$

$$= \frac{3s}{8(s-1)}\left(\frac{9}{4}\right)^{1-s} - \frac{3s}{2}\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^s}$$

$$\frac{3}{2}\tilde{\zeta}^{'}_{\mathcal{O}_1}(0) = \frac{d}{ds}\left[\frac{3s}{8(s-1)}\left(\frac{9}{4}\right)^{1-s}\right]_{s=0} - \frac{3}{2}\frac{1}{48} \tag{2.43}$$

$$= -\frac{7}{8}$$

$$\frac{3}{2}\tilde{K}_{\mathcal{O}_2}(t) = -\frac{3t}{2}\left(\int_x\frac{\sinh\rho}{(1+\cosh\rho)^2}\right)\left[(\Delta_x-2)\bar{K}_{-\Delta}\left(x,x^{'};t\right)\right]_{x=x^{'}} \tag{2.44}$$

$$= \frac{3t}{4}\int_0^\infty dv\, v\tanh(\pi v)\left(v^2+\frac{9}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}$$

$$\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_2}(s) = \frac{3s}{4}\int_0^\infty dv\frac{v^2+\frac{9}{4}}{\left(v^2+\frac{1}{4}\right)^{s+1}}v\tanh(\pi v) \tag{2.45}$$

$$= \int_0^\infty dv\frac{1}{\left(v^2+\frac{1}{4}\right)^s}\left[\frac{3s}{4}+\frac{3s}{2\left(v^2+\frac{1}{4}\right)}\right]\left(v-\frac{2v}{e^{2\pi v}+1}\right)$$

$$= \frac{3s}{8(s-1)}\left(\frac{1}{4}\right)^{1-s} + \frac{3}{4}\left(\frac{1}{4}\right)^{-s}$$

$$-\frac{3s}{2}\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^s} - 3s\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^{s+1}}$$

$$\frac{3}{2}\tilde{\zeta}^{'}_{\mathcal{O}_2}(0) = \frac{d}{ds}\left[\frac{3s}{8(s-1)}\left(\frac{1}{4}\right)^{1-s}+\frac{3}{4}\left(\frac{1}{4}\right)^{-s}\right]_{s=0} - \frac{3}{2}\frac{1}{48} - 3\left(\frac{\log 2}{2}-\frac{\gamma}{2}\right) \tag{2.46}$$

$$= -\frac{1}{8}+\frac{3\gamma}{2}$$

$$\frac{1}{2}\tilde{K}_{\mathcal{O}_{3+}}(t)+\frac{1}{2}\tilde{K}_{\mathcal{O}_{3-}}(t)=-t\int_x\frac{\sinh\rho}{(1+\cosh\rho)^2}\left[\left(\Delta_x-2\left(\frac{1-\cosh\rho}{\sinh\rho}\right)^2\right)\bar{K}_{-\Delta}\left(x,x^{'};t\right)\right]_{x=x'} \tag{2.47}$$
$$=\frac{t}{2}\int_0^\infty dv\,v\tanh(\pi v)\left(v^2+\frac{5}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}$$

$$\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3+}}(s)+\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3-}}(s)=\frac{s}{2}\int_0^\infty dv\frac{v^2+\frac{5}{4}}{\left(v^2+\frac{1}{4}\right)^{s+1}}v\tanh(\pi v) \tag{2.48}$$
$$=\int_0^\infty dv\frac{1}{\left(v^2+\frac{1}{4}\right)^s}\left[\frac{s}{2}+\frac{s}{2\left(v^2+\frac{1}{4}\right)}\right]\left(v-\frac{2v}{e^{2\pi v}+1}\right)$$
$$=\frac{s}{4(s-1)}\left(\frac{1}{4}\right)^{1-s}+\frac{1}{4}\left(\frac{1}{4}\right)^{-s}$$
$$-s\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^s}-s\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^{s+1}}$$

$$\frac{1}{2}\tilde{\zeta}'_{\mathcal{O}_{3+}}(0)+\frac{1}{2}\tilde{\zeta}'_{\mathcal{O}_{3-}}(0)=\frac{d}{ds}\left[\frac{s}{4(s-1)}\left(\frac{1}{4}\right)^{1-s}+\frac{1}{4}\left(\frac{1}{4}\right)^{-s}\right]_{s=0}-\frac{1}{48}-\left(\frac{\log 2}{2}-\frac{\gamma}{2}\right) \tag{2.49}$$
$$=-\frac{1}{12}+\frac{\gamma}{2}$$

$$-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\tilde{K}_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(t)=\frac{t}{4}\sum_{p_{12},p_{56},p_{89}}\int_x\sinh\rho\,\mathrm{tr}\left[\left\{\bar{\mathcal{O}}^x_{p_{12},p_{56},p_{89}},\tilde{\mathcal{O}}^x_{p_{12},p_{56},p_{89}}\right\}\bar{K}_{-\nabla\!\!\!\!/^2+1}\left(x,x^{'};t\right)\right]_{x=x'} \tag{2.50}$$
$$=-2t\int_0^\infty dv\,v\coth(\pi v)\left(v^2+2\right)e^{-t\left(v^2+1\right)}$$

$$-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\tilde{\zeta}_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(s)=-2s\int_0^\infty dv\frac{v^2+2}{v^2+1}\frac{v\coth(\pi v)}{\left(v^2+1\right)^s} \tag{2.51}$$
$$=\int_0^\infty dv\frac{1}{\left(v^2+1\right)^s}\left(-2s-\frac{2s}{v^2+1}\right)\left(v+\frac{2v}{e^{2\pi v}-1}\right)$$
$$=-\frac{s}{s-1}-1-4s\int_0^\infty dv\frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)^s}-4s\int_0^\infty dv\frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)^{s+1}}$$

$$-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\tilde{\zeta}'_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(0)=\frac{d}{ds}\left(-\frac{s}{s-1}\right)_{s=0}-4\frac{1}{24}-4\left(-\frac{1}{4}+\frac{\gamma}{2}\right) \tag{2.52}$$
$$=\frac{11}{6}-2\gamma$$

and, summing over the field content of the theory, we obtain

$$\tilde{\Gamma}=-\frac{d}{ds}\left(\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_1}(s)+\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_2}(s)+\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3+}}(s)+\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3-}}(s)-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\tilde{\zeta}_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(s)\right)\Bigg|_{s=0}=-\frac{3}{4}. \tag{2.53}$$

2.5 Analytic continuations in $\tilde{\zeta}(s)$

We consider the sum of zeta-functions

$$\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_1}(s)+\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_2}(s)+\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3+}}(s)+\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_{3-}}(s)-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\tilde{\zeta}_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(s) \tag{2.54}$$
$$=\int_0^\infty dv\left[\frac{3s}{4}\frac{1}{\left(v^2+\frac{9}{4}\right)^s}v\right]+\int_0^\infty dv\left[\frac{v}{\left(v^2+\frac{1}{4}\right)^s}\left(\frac{3s}{4}+\frac{3s}{2\left(v^2+\frac{1}{4}\right)}\right)\right]$$
$$+\int_0^\infty dv\left[\frac{v}{\left(v^2+\frac{1}{4}\right)^s}\left(\frac{s}{2}+\frac{s}{2\left(v^2+\frac{1}{4}\right)}\right)\right]+\int_0^\infty dv\left[\frac{v}{\left(v^2+1\right)^s}\left(-2s-\frac{2s}{v^2+1}\right)\right]$$
$$+\int_0^\infty dv\left(\text{exponentially suppressed integrands }\sim e^{-2\pi v}\text{ for large }v\right). \tag{2.55}$$

In the previous section, we treated each of the four summands in square brackets above individually:

- we integrated $\int_0^\infty dv\,[\dots]$ for s large enough,
- analytically continued to a neighborhood of $s=0$ by simply neglecting the restriction on s,
- derived $\frac{d}{ds}[\dots]_{s=0}$.

Summing up these terms produced a contribution

$$\begin{aligned}&\frac{d}{ds}\left[\frac{3s}{8(s-1)}\left(\frac{9}{4}\right)^{1-s}\right]_{s=0}+\frac{d}{ds}\left[\frac{3s}{8(s-1)}\left(\frac{1}{4}\right)^{1-s}+\frac{3}{4}\left(\frac{1}{4}\right)^{-s}\right]_{s=0}\\&+\frac{d}{ds}\left[\frac{s}{4(s-1)}\left(\frac{1}{4}\right)^{1-s}+\frac{1}{4}\left(\frac{1}{4}\right)^{-s}\right]_{s=0}+\frac{d}{ds}\left(-\frac{s}{s-1}\right)_{s=0}\\=&2\log 2\,.\end{aligned}\tag{2.56}$$

Alternatively, we can directly start with <u>the sum of the four terms</u> and

- derive $\frac{d}{ds}[\dots]_{s=0}$ and skip the analytic continuation because the sum is well-defined around $s=0$,
- integrate $\int_0^\infty dv\,[\dots]$.

The new algorithm yields the same result

$$\begin{aligned}&\int_0^\infty dv\frac{d}{ds}\left\{\left[\frac{3s}{4}\frac{1}{\left(v^2+\frac{9}{4}\right)^s}v\right]+\left[\frac{v}{\left(v^2+\frac{1}{4}\right)^s}\left(\frac{3s}{4}+\frac{3s}{2\left(v^2+\frac{1}{4}\right)}\right)\right]\right.\\&\left.+\left[\frac{v}{\left(v^2+\frac{1}{4}\right)^s}\left(\frac{s}{2}+\frac{s}{2\left(v^2+\frac{1}{4}\right)}\right)\right]+\left[\frac{v}{\left(v^2+1\right)^s}\left(-2s-\frac{2s}{v^2+1}\right)\right]\right\}_{s=0}\\=&2\log 2\,.\end{aligned}\tag{2.57}$$

2.6 Small-t asymptotics of $\tilde{K}(t)$

$$\begin{aligned}\frac{3}{2}\tilde{K}_{\mathcal{O}_1}(t)&=\frac{3t}{4}\int_0^\infty dv\,v\tanh(\pi v)\left(v^2+\frac{9}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}\\&=\frac{3t}{4}\int_0^\infty dv\left(v-\frac{2v}{e^{2\pi v}+1}\right)\left(v^2+\frac{9}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}\\&=\frac{3t}{4}\int_0^\infty dv\,v\left(v^2+\frac{9}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}-\frac{3t}{4}\int_0^\infty dv\frac{2v}{e^{2\pi v}+1}\left(v^2+\frac{9}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}\\&=\frac{3(4+9t)}{32t}e^{-9t/4}-\frac{3t}{4}\int_0^\infty dv\frac{2v}{e^{2\pi v}+1}\left(v^2+\frac{9}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}\\&=\frac{3}{8t}+O(t)\end{aligned}\tag{2.58}$$

$$\begin{aligned}\frac{3}{2}\tilde{K}_{\mathcal{O}_2}(t)&=\frac{3t}{4}\int_0^\infty dv\,v\tanh(\pi v)\left(v^2+\frac{9}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}\\&=\frac{3t}{4}\int_0^\infty dv\left(v-\frac{2v}{e^{2\pi v}+1}\right)\left(v^2+\frac{9}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}\\&=\frac{3t}{4}\int_0^\infty dv\,v\left(v^2+\frac{9}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}-\frac{3t}{4}\int_0^\infty dv\frac{2v}{e^{2\pi v}+1}\left(v^2+\frac{9}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}\\&=\frac{3(4+9t)}{32t}e^{-t/4}-\frac{3t}{4}\int_0^\infty dv\frac{2v}{e^{2\pi v}+1}\left(v^2+\frac{9}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}\\&=\frac{3}{8t}+\frac{3}{4}+O(t)\end{aligned}\tag{2.59}$$

$$\begin{aligned}
\frac{1}{2}\tilde{K}_{\mathcal{O}_{3+}}(t)+\frac{1}{2}\tilde{K}_{\mathcal{O}_{3-}}(t) &= \frac{t}{2}\int_0^\infty dv\, v\tanh(\pi v)\left(v^2+\frac{5}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)} \\
&= \frac{t}{2}\int_0^\infty dv\left(v-\frac{2v}{e^{2\pi v}+1}\right)\left(v^2+\frac{5}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)} \\
&= \frac{t}{2}\int_0^\infty dv\, v\left(v^2+\frac{5}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}-t\int_0^\infty dv\frac{v}{e^{2\pi v}+1}\left(v^2+\frac{5}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)} \\
&= \frac{4+5t}{16t}e^{-t/4}-t\int_0^\infty dv\frac{v}{e^{2\pi v}+1}\left(v^2+\frac{5}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)} \\
&\frac{1}{4t}+\frac{1}{4}+O(t)
\end{aligned} \tag{2.60}$$

$$\begin{aligned}
-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\tilde{K}_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(t) &= -2t\int_0^\infty dv\, v\coth(\pi v)\left(v^2+2\right)e^{-t\left(v^2+1\right)} \\
&= -2t\int_0^\infty dv\left(v+\frac{2v}{e^{2\pi v}-1}\right)\left(v^2+2\right)e^{-t\left(v^2+1\right)} \\
&= -2t\int_0^\infty dv\, v\left(v^2+2\right)e^{-t\left(v^2+1\right)}-t\int_0^\infty dv\frac{v}{e^{2\pi v}-1}\left(v^2+2\right)e^{-t\left(v^2+1\right)} \\
&= \frac{-1-2t}{t}e^{-t}-t\int_0^\infty dv\frac{v}{e^{2\pi v}-1}\left(v^2+2\right)e^{-t\left(v^2+1\right)} \\
&= -\frac{1}{t}-1+O(t)
\end{aligned} \tag{2.61}$$

$$\frac{3}{2}\tilde{K}_{\mathcal{O}_1}(t)+\frac{3}{2}\tilde{K}_{\mathcal{O}_2}(t)+\frac{1}{2}\tilde{K}_{\mathcal{O}_{3+}}(t)+\frac{1}{2}\tilde{K}_{\mathcal{O}_{3-}}(t)-\frac{1}{4}\sum_{p_{12},p_{56},p_{89}}\tilde{K}_{\mathcal{O}^2_{p_{12},p_{56},p_{89}}}(t)=O(t) \tag{2.62}$$

3 String action for the k-wound circular Wilson loop

We review the string action in [5]. The induced metric on this solution is

$$ds^2 = \frac{k^2}{\sinh^2 k\sigma}\left(d\sigma^2 + d\tau^2\right) \qquad \sigma > 0 \qquad \tau \in [0, 2\pi) \tag{3.1}$$

$$ds^2\big|_{k=1} = \frac{d\tau^2 + d\sigma^2}{\sinh^2 \sigma} = d\rho^2 + \sinh^2 \rho d\tau^2 \qquad \sinh\rho = \frac{1}{\sinh k\sigma}\,. \tag{3.2}$$

We use the coordinates ρ and τ (note k in the definition). We made the change of coordinate (1.5) [11] [7] so that the regularized range of the new variable is k-independent:

$$z \in [\epsilon, \infty) \quad \rightarrow \quad \sigma \in \left[\frac{1}{k}\text{arctanh}\epsilon, \infty\right) \quad \rightarrow \quad \rho \in \left[0, \text{arccosh}\frac{1}{\epsilon}\right] \tag{3.3}$$

The winding number k is analytically continued to real values for perturbation theory to apply.

3.1 Operators for finite $k \in \mathbb{R}$

Bosonic and fermionic operators read

$$\mathcal{O}_0\left(k\right) \equiv \frac{\sinh^2 k\sigma}{k^2}\left(-\partial_\tau^2 - \partial_\sigma^2\right) \tag{3.4}$$

$$\mathcal{O}_1\left(k\right) \equiv \mathcal{O}_0\left(k\right) + 2 \tag{3.5}$$

$$\mathcal{O}_F\left(k\right) \equiv i\frac{\sinh k\sigma}{k}\sigma_1\partial_\sigma - i\frac{\sinh k\sigma}{k}\sigma_2\partial_\tau - \frac{i}{2}\cosh k\sigma\sigma_1 + \sigma_3 \tag{3.6}$$

with the one-loop effective action being

$$\Gamma_1\left(k\right) = \frac{5}{2}\log \text{Det}\mathcal{O}_0 + \frac{3}{2}\log \text{Det}\mathcal{O}_1 - \frac{8}{4}\log \text{Det}\mathcal{O}_F^2\,. \tag{3.7}$$

3.2 Operators for $k \sim 1$

Around the circular loop $k = 1$, the operators get corrected as

$$\mathcal{O}_i = \bar{\mathcal{O}}_i + (k-1)\,\tilde{\mathcal{O}}_i + \ldots \qquad i = 0, 1 \tag{3.8}$$

$$\mathcal{O}_F = \bar{\mathcal{O}}_F + (k-1)\,\tilde{\mathcal{O}}_F + \ldots \tag{3.9}$$

$$\mathcal{O}_F^2 = \bar{\mathcal{O}}_F^2 + (k-1)\left\{\bar{\mathcal{O}}_F, \tilde{\mathcal{O}}_F\right\} + \ldots \tag{3.10}$$

with

$$\bar{\mathcal{O}}_0 = -\Delta_{\rho,\tau} \tag{3.11}$$

$$\bar{\mathcal{O}}_1 = -\Delta_{\rho,\tau} + 2 \tag{3.12}$$

$$\bar{\mathcal{O}}_F = -i\nabla\!\!\!\!/\,_{\rho,\tau} + \sigma_3 \tag{3.13}$$

$$\tilde{\mathcal{O}}_0 = \tilde{\mathcal{O}}_1 = \frac{2}{\sinh^2 \rho}\partial_\tau^2 \tag{3.14}$$

$$\tilde{\mathcal{O}}_F = \frac{i\sigma_2}{\sinh\rho}\partial_\tau \tag{3.15}$$

[7] There is a minus sign missing.

3.3 Effective action at order $k-1$

We use the same formulas of the latitude loop with $x=(\rho,\tau)$, $\delta(x)\equiv\delta(\rho)\,\delta(\tau)$ and $k-1$ replacing the parameter θ_0^2.

$$\begin{aligned}\frac{5}{2}\tilde{K}_{\mathcal{O}_0}(t) &= -\frac{5t}{2}\int_x \frac{2}{\sinh\rho}\left[\partial_\tau^2\bar{K}_{-\Delta}\left(x,x^{'};t\right)\right]_{x=x'} \\ &= -\frac{5\left(-\frac{1}{\epsilon}+1\right)t}{2}\int_0^\infty dv\, v\tanh(\pi v)\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}\end{aligned} \tag{3.16}$$

$$\begin{aligned}\frac{5}{2}\tilde{\zeta}_{\mathcal{O}_0}(s) &= -\frac{5\left(-\frac{1}{\epsilon}+1\right)s}{2}\int_0^\infty dv\frac{v\tanh(\pi v)}{\left(v^2+\frac{1}{4}\right)^s} \\ &= -\frac{5\left(-\frac{1}{\epsilon}+1\right)s}{2}\int_0^\infty dv\frac{1}{\left(v^2+\frac{1}{4}\right)^s}\left(v-\frac{2v}{e^{2\pi v}+1}\right) \\ &= -\frac{5\left(-\frac{1}{\epsilon}+1\right)s}{4(s-1)}\left(\frac{1}{4}\right)^{1-s}+5\left(-\frac{1}{\epsilon}+1\right)s\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^s}\end{aligned} \tag{3.17}$$

$$\begin{aligned}\frac{5}{2}\tilde{\zeta}'_{\mathcal{O}_0}(s) &= \left(-\frac{1}{\epsilon}+1\right)\frac{d}{ds}\left[-\frac{5s}{4(s-1)}\left(\frac{1}{4}\right)^{1-s}\right]_{s=0}+5\left(-\frac{1}{\epsilon}+1\right)\frac{1}{48} \\ &= \frac{5}{12}\left(-\frac{1}{\epsilon}+1\right)\end{aligned} \tag{3.18}$$

$$\begin{aligned}\frac{3}{2}\tilde{K}_{\mathcal{O}_1}(t) &- \frac{3t}{2}\int_x \frac{2}{\sinh\rho}\left[\partial_\tau^2\bar{K}_{-\Delta+2}\left(x,x^{'};t\right)\right]_{x=x'} \\ &= -\frac{3\left(-\frac{1}{\epsilon}+1\right)t}{2}\int_0^\infty dv\, v\tanh(\pi v)\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}\end{aligned} \tag{3.19}$$

$$\begin{aligned}\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_1}(s) &= -\frac{3\left(-\frac{1}{\epsilon}+1\right)s}{2}\int_0^\infty dv\, v\tanh(\pi v)\,\frac{v^2+\frac{1}{4}}{\left(v^2+\frac{9}{4}\right)^s} \\ &= -\frac{3\left(-\frac{1}{\epsilon}+1\right)s}{2}\int_0^\infty dv\frac{1}{\left(v^2+\frac{9}{4}\right)^s}\left(1-\frac{2}{v^2+\frac{9}{4}}\right)\left(v-\frac{2v}{e^{2\pi v}+1}\right) \\ &= -\frac{3\left(-\frac{1}{\epsilon}+1\right)s}{4(s-1)}\left(\frac{9}{4}\right)^{1-s}+\frac{3\left(-\frac{1}{\epsilon}+1\right)}{2}\left(\frac{9}{4}\right)^{-s} \\ &+3\left(-\frac{1}{\epsilon}+1\right)s\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^s}-6\left(-\frac{1}{\epsilon}+1\right)s\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^{s+1}}\end{aligned} \tag{3.20}$$

$$\begin{aligned}\frac{3}{2}\tilde{\zeta}'_{\mathcal{O}_1}(s) &= \left(-\frac{1}{\epsilon}+1\right)\frac{d}{ds}\left[-\frac{3s}{4(s-1)}\left(\frac{9}{4}\right)^{1-s}+\frac{3}{2}\left(\frac{9}{4}\right)^{-s}\right]_{s=0}+3\left(-\frac{1}{\epsilon}+1\right)\frac{1}{48}-6\left(-\frac{1}{\epsilon}+1\right)\left(-\frac{\log\frac{3}{2}}{2}+\frac{1}{2}-\frac{\gamma}{2}\right) \\ &= \left(-\frac{5}{4}+3\gamma\right)\left(-\frac{1}{\epsilon}+1\right)\end{aligned} \tag{3.21}$$

$$-\frac{8}{4}\tilde{K}_{\mathcal{O}_F^2}(t)=\frac{8t}{4}\int_x \sinh\rho \mathrm{tr}\left[\left\{\bar{\mathcal{O}}_F^x,\tilde{\mathcal{O}}_F^x\right\}\bar{K}_{-\nabla\!\!\!/^2+1}\left(x,x^{'};t\right)\right]_{x=x'} \tag{3.22}$$

$$=2\left(-\frac{1}{\epsilon}+1\right)t\int_0^\infty dv\, v\coth(\pi v)\left(2v^2+1\right)e^{-t\left(v^2+1\right)}$$

$$-\frac{8}{4}\tilde{\zeta}_{\mathcal{O}_F^2}(s)=2\left(-\frac{1}{\epsilon}+1\right)s\int_0^\infty dv\,\frac{2v^2+1}{\left(v^2+1\right)^{s+1}}v\coth(\pi v) \tag{3.23}$$

$$=2\left(-\frac{1}{\epsilon}+1\right)s\int_0^\infty dv\,\frac{1}{\left(v^2+1\right)^s}\left(2-\frac{1}{v^2+1}\right)\left(v+\frac{2v}{e^{2\pi v}-1}\right)$$

$$=\frac{2\left(-\frac{1}{\epsilon}+1\right)s}{s-1}-\left(-\frac{1}{\epsilon}+1\right)+8\left(-\frac{1}{\epsilon}+1\right)s\int_0^\infty dv\,\frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)^s}-4\left(-\frac{1}{\epsilon}+1\right)s\int_0^\infty dv\,\frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)^{s+1}}$$

$$-\frac{8}{4}\tilde{\zeta}_{\mathcal{O}_F^2}(s)=\left(-\frac{1}{\epsilon}+1\right)\frac{d}{ds}\left(\frac{2s}{s-1}-1\right)_{s=0}+8\left(-\frac{1}{\epsilon}+1\right)\frac{1}{24}-4\left(\frac{1}{\epsilon}-1\right)\left(-\frac{1}{4}+\frac{\gamma}{2}\right) \tag{3.24}$$

$$=\left(-\frac{2}{3}-2\gamma\right)\left(-\frac{1}{\epsilon}+1\right)$$

We drop the divergences. If we write the effective action as usual

$$\Gamma_1(k)=\bar{\Gamma}_1+(k-1)\tilde{\Gamma}_1+\ldots \tag{3.25}$$

then we get

$$\tilde{\Gamma}=-\frac{d}{ds}\left(\frac{5}{2}\tilde{\zeta}_{\mathcal{O}_0}(s)+\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_1}(s)-\frac{8}{4}\tilde{\zeta}_{\mathcal{O}_F^2}(s)\right)\Bigg|_{s=0}=\frac{3}{2}-\gamma\approx 0.922\,. \tag{3.26}$$

The first correction does not coincide with any of the three results in literature:

- localization $\tilde{\Gamma}_1=\frac{3}{2}$,
- Kruczenski Tirziu $\tilde{\Gamma}_1=\frac{3}{2}+\gamma\approx 2.077$,
- Bergamin Tseytlin $\tilde{\Gamma}_1=\frac{1}{2}\log 2\pi-\frac{1}{4}\int_0^\infty\frac{dy}{y\sinh y}\left[\left(5e^{-y}+3e^{-3y}\right)\left(\frac{y}{\sinh^2 y}-\coth y\right)+16e^{-2y}\left(\frac{y\cosh y}{\sinh^2 y}-\frac{1}{\sinh y}\right)\right]\approx$ 1.235.

3.4 Analytic continuations in $\tilde{\zeta}(s)$

We repeat the analysis done for the latitude. Unlike this case, we will conclude that we must always perform analytic continuation in s to arrive to a finite result. *Differences with the latitude are emphasized in italics.*

We consider the sum of zeta-functions

$$\frac{5}{2}\tilde{\zeta}_{\mathcal{O}_0}(s)+\frac{3}{2}\tilde{\zeta}_{\mathcal{O}_1}(s)-\frac{8}{4}\tilde{\zeta}_{\mathcal{O}_F^2}(s) \tag{3.27}$$

$$=\int_0^\infty dv\left[-\frac{5s}{2}\frac{v}{\left(v^2+\frac{1}{4}\right)^s}\right]+\int_0^\infty dv\left[-\frac{3s}{2}\frac{v}{\left(v^2+\frac{9}{4}\right)^s}\left(1-\frac{2}{v^2+\frac{9}{4}}\right)\right]+\int_0^\infty dv\left[2s\frac{v}{\left(v^2+1\right)^s}\left(2-\frac{1}{v^2+1}\right)\right]$$

$$+\int_0^\infty dv\left(\text{exponentially suppressed integrands }\sim e^{-2\pi v}\text{ for large }v\right)\,.$$

In the previous section, we treated each of the three summands in square brackets above <u>individually</u>:

- we integrated $\int_0^\infty dv\,[\ldots]$ for s large enough,
- analytically continued to a neighborhood of $s=0$ by simply neglecting the restriction on s,
- derived $\frac{d}{ds}[\ldots]_{s=0}$.

Summing up these terms produced a contribution

$$\frac{d}{ds}\left[-\frac{5s}{4(s-1)}\left(\frac{1}{4}\right)^{1-s}\right]_{s=0}+\frac{d}{ds}\left[-\frac{3s}{4(s-1)}\left(\frac{9}{4}\right)^{1-s}+\frac{3}{2}\left(\frac{9}{4}\right)^{-s}\right]_{s=0}+\frac{d}{ds}\left(\frac{2s}{s-1}\right)_{s=0} \tag{3.28}$$

$$=-3\log\left(\frac{3}{2}\right)\,.$$

Alternatively, we can directly start with <u>the sum of the three terms</u> and

- derive $\frac{d}{ds}\left[\ldots\right]_{s=0}$ *and intentionally decide to skip the continuation in* s,
- integrate $\int_0^\infty dv\left[\ldots\right]$.

The new algorithm fails to return a finite answer

$$\int_0^\infty dv\frac{d}{ds}\left\{\left[-\frac{5s}{2}\frac{v}{\left(v^2+\frac{1}{4}\right)^s}\right]+\left[-\frac{3s}{2}\frac{v}{\left(v^2+\frac{9}{4}\right)^s}\left(1-\frac{2}{v^2+\frac{9}{4}}\right)\right]+\left[2s\frac{v}{\left(v^2+1\right)^s}\left(2-\frac{1}{v^2+1}\right)\right]\right\}_{s=0}=\infty \tag{3.29}$$

because the sum (second method), as well as the individual terms (first method), would need to be continued to $s=0$.

3.5 Small-t asymptotics of $\tilde{K}\left(t\right)$

From our calculations above.

$$\begin{aligned}\frac{5}{2}\tilde{K}_{\mathcal{O}_0}\left(t\right)&=-\frac{5t}{2}\int_0^\infty dv\,v\tanh\left(\pi v\right)\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}\\&=-\frac{5t}{2}\int_0^\infty dv\left(v-\frac{2v}{e^{2\pi v}+1}\right)\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}\\&=-\frac{5\left(t+4\right)}{16t}e^{-t/4}+\frac{5t}{2}\int_0^\infty dv\frac{2v}{e^{2\pi v}+1}\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}\\&=-\frac{5}{4t}+O\left(t\right)\end{aligned} \tag{3.30}$$

$$\begin{aligned}\frac{3}{2}\tilde{K}_{\mathcal{O}_1}\left(t\right)&=-\frac{3t}{2}\int_0^\infty dv\,v\tanh\left(\pi v\right)\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}\\&=-\frac{3t}{2}\int_0^\infty dv\left(v-\frac{2v}{e^{2\pi v}+1}\right)\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}\\&=-\frac{3t}{2}\int_0^\infty dvv\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}+\frac{3t}{2}\int_0^\infty dv\frac{2v}{e^{2\pi v}+1}\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}\\&=-\frac{3\left(t+4\right)}{16t}e^{-9t/4}+O\left(t\right)\\&=-\frac{3}{4t}+\frac{3}{2}+O\left(t\right)\end{aligned} \tag{3.31}$$

$$\begin{aligned}-\frac{8}{4}\tilde{K}_{\mathcal{O}_F^2}\left(t\right)&=2t\int_0^\infty dv\,v\coth\left(\pi v\right)\left(2v^2+1\right)e^{-t\left(v^2+1\right)}\\&=2t\int_0^\infty dv\left(v+\frac{2v}{e^{2\pi v}-1}\right)\left(2v^2+1\right)e^{-t\left(v^2+1\right)}\\&=\frac{2+t}{t}e^{-t}+2t\int_0^\infty dv\frac{2v}{e^{2\pi v}-1}\left(2v^2+1\right)e^{-t\left(v^2+1\right)}\\&=\frac{2}{t}-1+O\left(t\right)\end{aligned} \tag{3.32}$$

$$\frac{5}{2}\tilde{K}_{\mathcal{O}_0}\left(t\right)+\frac{3}{2}\tilde{K}_{\mathcal{O}_1}\left(t\right)-\frac{8}{4}\tilde{K}_{\mathcal{O}_F^2}\left(t\right)=\frac{1}{2}+O\left(t\right) \tag{3.33}$$

From (2.19) and (3.17) [11] at $k\to 1$. Remember that our spinors are of Dirac type, not Majorana as in [11].

$$
\begin{aligned}
\tilde{K}_{-\Delta+m^2}(t) &= \bar{K}_{-\Delta+m^2}(t) + \frac{e^{-\left(m^2+\frac{1}{4}\right)t}}{\sqrt{4\pi t}} \int_0^\infty dy\, e^{-y^2/t} \frac{y - \sinh y \cosh y}{\sinh^3 y} \\
&= \bar{K}_{-\Delta+m^2}(t) + \frac{e^{-\left(m^2+\frac{1}{4}\right)t}}{\sqrt{4\pi}} \int_0^\infty du\, e^{-u^2} \frac{\sqrt{t}u - \sinh\left(\sqrt{t}u\right)\cosh\left(\sqrt{t}u\right)}{\sinh^3\left(\sqrt{t}u\right)} \\
&= -\frac{1}{2}\left[\frac{1}{t} + \left(-\frac{1}{3} - m^2\right) + O(t)\right] + \left(-\frac{1}{6} + O(t)\right) \\
&= -\frac{1}{2t} + \frac{m^2}{2} + O(t)
\end{aligned} \tag{3.34}
$$

$$
\begin{aligned}
\frac{5}{2}\tilde{K}_{\mathcal{O}_0}(t) + \frac{3}{2}\tilde{K}_{\mathcal{O}_1}(t) &= \left(-\frac{5}{4t} + O(t)\right) + \left(-\frac{3}{4t} + \frac{3}{2} + O(t)\right) \\
&= -\frac{2}{t} + \frac{3}{2} + O(t)
\end{aligned} \tag{3.35}
$$

$$
\begin{aligned}
\tilde{K}_{-\nabla\!\!\!/^2+m^2}(t) &= \bar{K}_{-\nabla\!\!\!/^2+m^2}(t) - 2 \times \frac{2e^{-m^2 t}}{\sqrt{4\pi t}} \int_0^\infty dy\, e^{-y^2/t} \frac{y\cosh y - \sinh y}{\sinh^3 y} \\
&= \bar{K}_{-\nabla\!\!\!/^2+m^2}(t) - \frac{4e^{-m^2 t}}{\sqrt{4\pi}} \int_0^\infty dy\, e^{-u^2} \frac{\left(\sqrt{t}u\right)\cosh\left(\sqrt{t}u\right) - \sinh\left(\sqrt{t}u\right)}{\sinh^3\left(\sqrt{t}u\right)} \\
&= \left[-\frac{1}{t} + \left(-\frac{1}{6} + m^2\right) + O(t)\right] + \left(-\frac{1}{3} + O(t)\right) \\
&= -\frac{1}{t} + \left(-\frac{1}{2} + m^2\right) + O(t)
\end{aligned} \tag{3.36}
$$

$$
-\frac{8}{4}\tilde{K}_{\mathcal{O}_F^2}(t) = \frac{2}{t} - 1 + O(t) \tag{3.37}
$$

$$
\frac{5}{2}\tilde{K}_{\mathcal{O}_0}(t) + \frac{3}{2}\tilde{K}_{\mathcal{O}_1}(t) - \frac{8}{4}\tilde{K}_{\mathcal{O}_F^2}(t) = \frac{1}{2} + O(t) \tag{3.38}
$$

4 String action for the Wilson cusp

This section summarizes the strong-coupling results of [12] for the Wilson cusp with vanishing R-symmetry angle $\theta = 0$. In the near Wilson line limit $\phi \sim 0$, the cusp disappears and we are left with a pair of antipodal lines on $\mathbb{S}^3 \times \mathbb{R}$. We refer to this paper (formulas, appendices etc.) in what follows.

We assume that the Wilson cusp couples to the scalar ϕ^1 (hence $\theta = 0$ and the constant coordinate $\vartheta = 0$ vanishes in the $\mathbb{S}^5$ sector). We begin by reviewing the classical string solution at given ϕ in Appendix B.

$$0 \le k < \frac{1}{\sqrt{2}} \qquad b = \frac{\sqrt{1-2k^2}}{k} \qquad p = \frac{b^2}{\sqrt{1+b^2}} \qquad q = 0\,. \tag{4.1}$$

The angular opening of the cusp $\pi - \phi$ is in one-to-one correspondence to the parameter k (B.10)

$$\pi - \phi = 2\frac{p^2}{b\sqrt{b^4+p^2}}\left[\Pi\left(\frac{b^4}{b^4+p^2}|k^2\right) - \mathbb{K}\left(k^2\right)\right]. \tag{4.2}$$

The one-loop computation was done in the set of coordinates for which the (Wick-rotated) induced metric (B.29) is

$$ds^2 = \frac{1-k^2}{\mathrm{cn}^2\left(\sigma|k^2\right)}\left(d\sigma^2 + d\tau^2\right) \qquad |\sigma| < \mathbb{K}\left(k^2\right) \qquad \tau \in \mathbb{R}\,. \tag{4.3}$$

This coordinatization would introduce an extra problem: as k varies, the metric and the operators change too as before, but now also the domain of σ. On the regularized worldsheet, we would have the same problem for both τ, σ because their cutoffs are k-dependent:

$$|\sigma| < \mathrm{cn}^{-1}\left(\frac{\sqrt{1+b^2}}{b\cosh\rho_0}|k^2\right), \qquad 0 < \tau < \mathcal{T} \equiv \frac{\sqrt{b^4+p^2}}{bp}T\,. \tag{4.4}$$

where cn^{-1} is the inverse Jacobi cosine, while ρ_0 and T are the (k-independent) cutoffs on the AdS_5 radius ρ and time t.

If we want to keep our heat kernel formulas simple, we should look for a new set of coordinates at the price of a more complicated metric. We make the replacement (B.23) for the time direction

$$\tau = \frac{\sqrt{b^4+p^2}}{bp}w \tag{4.5}$$

while we can define for the spatial direction

$$\sigma = \frac{2}{\pi}\mathbb{K}\left(k^2\right) r \tag{4.6}$$

so that the metric we will work with becomes

$$ds^2 = \frac{1-k^2}{\mathrm{cn}^2\left(\frac{2}{\pi}\mathbb{K}\left(k^2\right) r|k^2\right)}\left(\frac{4}{\pi^2}\mathbb{K}^2\left(k^2\right)dr^2 + \frac{b^4+p^2}{b^2p^2}dw^2\right) \qquad |r| < \frac{\pi}{2} \qquad w \in \mathbb{R} \tag{4.7}$$

and with

$$|r| < \frac{\pi}{2\mathbb{K}\left(k^2\right)}\mathrm{cn}^{-1}\left(\frac{\sqrt{1+b^2}}{b\cosh\rho_0}|k^2\right), \qquad 0 < w < T\,. \tag{4.8}$$

on the regularized worldsheet. Notice that the range of r (like the one of σ) is k-dependent. We could make it independent (as we did for w) with a more complicated definition of r that would include ρ_0. We didn't pursue this direction.

This is a deformation of H^2 for finite ϕ. The small-ϕ limit is reached for

$$k \to 0 \qquad b,\, p \to \infty \tag{4.9}$$

when the worldsheet approaches the infinite-strip geometry of H^2

$$ds^2\Big|_{k=0} = \frac{dr^2+dw^2}{\cos^2 r} = \frac{d\sigma^2+d\tau^2}{\cos^2\sigma}. \tag{4.10}$$

4.1 Operators for finite $k \in \left[0, \frac{1}{\sqrt{2}}\right)$

The bosonic operators read (D.6)-(D.8) [8]

$$
\begin{aligned}
\mathcal{O}_0(k) &\equiv \frac{\mathrm{cn}^2\left(\frac{2}{\pi}\mathbb{K}\left(k^2\right)r|k^2\right)}{1-k^2}\left(-\frac{\pi^2}{4\mathbb{K}^2\left(k^2\right)}\partial_r^2 - \frac{b^2p^2}{b^4+p^2}\partial_w^2\right) \\
\mathcal{O}_1(k) &\equiv \mathcal{O}_0(k) + 2 \\
\mathcal{O}_2(k) &\equiv \mathcal{O}_0(k) + 2 - 2\frac{k^2\mathrm{cn}^4\left(\frac{2}{\pi}\mathbb{K}\left(k^2\right)r|k^2\right)}{1-k^2}
\end{aligned}
\tag{4.11}
$$

while the fermionic operator (D.13) is

$$
\mathcal{O}_F(k) \equiv -i\frac{\mathrm{cn}\left(\frac{2}{\pi}\mathbb{K}\left(k^2\right)r|k^2\right)}{\sqrt{1-k^2}}\left(\frac{\pi}{2\mathbb{K}\left(k^2\right)}\partial_r + \frac{\mathrm{sn}\left(\frac{2}{\pi}\mathbb{K}\left(k^2\right)r|k^2\right)\mathrm{dn}\left(\frac{2}{\pi}\mathbb{K}\left(k^2\right)r|k^2\right)}{2\mathrm{cn}\left(\frac{2}{\pi}\mathbb{K}\left(k^2\right)r|k^2\right)}\right)\sigma_1 - i\frac{bp}{\sqrt{b^4+p^2}}\frac{\mathrm{cn}\left(\frac{2}{\pi}\mathbb{K}\left(k^2\right)r|k^2\right)}{\sqrt{1-k^2}}\sigma_2\partial_w + \sigma_3.
\tag{4.12}
$$

Taking into account the multiplicities, the one-loop effective action is given by

$$
\Gamma_1(k) = \frac{5}{2}\log\mathrm{Det}\mathcal{O}_0 + \frac{2}{2}\log\mathrm{Det}\mathcal{O}_1 + \frac{1}{2}\log\mathrm{Det}\mathcal{O}_2 - \frac{8}{4}\log\mathrm{Det}\mathcal{O}_F^2\,. \tag{4.13}
$$

4.2 Operators for $k \sim 0$

Around the pair of straight Wilson lines $\phi = k = 0$, the operators get corrected as

$$
\mathcal{O}_i = \bar{\mathcal{O}}_i + k^2\tilde{\mathcal{O}}_i + \dots \qquad i = 0,1,2 \tag{4.14}
$$

$$
\mathcal{O}_F = \bar{\mathcal{O}}_F + k^2\tilde{\mathcal{O}}_F + \dots \tag{4.15}
$$

$$
\mathcal{O}_F^2 = \bar{\mathcal{O}}_F^2 + k^2\left\{\bar{\mathcal{O}}_F, \tilde{\mathcal{O}}_F\right\} + \dots \tag{4.16}
$$

where the barred operators are the massive Laplacian and Dirac operators on H^2

$$
\bar{\mathcal{O}}_0 = -\Delta_{r,w} \tag{4.17}
$$

$$
\bar{\mathcal{O}}_1 = \bar{\mathcal{O}}_2 = -\Delta_{r,w} + 2 \tag{4.18}
$$

$$
\bar{\mathcal{O}}_F = -i\nabla\!\!\!/_{r,w} + \sigma_3 \tag{4.19}
$$

and the first corrections evaluate to

$$
\tilde{\mathcal{O}}_0 = \tilde{\mathcal{O}}_1 = \frac{\cos^2 r}{2}\left(-\cos^2 r\partial_r^2 + \left(2+\sin^2 r\right)\partial_w^2\right) \tag{4.20}
$$

$$
\tilde{\mathcal{O}}_2 = \frac{\cos^2 r}{2}\left(-\cos^2 r\partial_r^2 + \left(2+\sin^2 r\right)\partial_w^2 - 4\cos^2 r\right) \tag{4.21}
$$

$$
\tilde{\mathcal{O}}_F = -\frac{i\cos^3 r}{4}\sigma_1\partial_r - \frac{i\left(\cos 3r - 9\cos r\right)}{16}\sigma_2\partial_w - \frac{3i\sin r\cos^2 r}{8}\sigma_1\,. \tag{4.22}
$$

Unlike the bosonic operators of the latitude an the k-wound circle, the bosonic and fermionic $\tilde{\mathcal{O}}$'s of the cusp are not proportional to the Laplacian $\Delta_{r,w}$ and the Dirac operator $\nabla\!\!\!/_{r,w}$ respectively. This is due to the fact that we rescaled the initial coordinates σ, τ differently.

4.3 Effective action at $k = 0$

The one-loop effective action for the string dual to the antipodal lines on $\mathbb{S}^3 \times \mathbb{R}$

$$
\begin{aligned}
\bar{\Gamma}_1 &= \frac{5}{2}\log\mathrm{Det}\bar{\mathcal{O}}_0 + \frac{2}{2}\log\mathrm{Det}\bar{\mathcal{O}}_1 + \frac{1}{2}\log\mathrm{Det}\bar{\mathcal{O}}_2 - \frac{8}{4}\log\mathrm{Det}\bar{\mathcal{O}}_F^2 \\
&= \frac{d}{ds}\left(-\frac{5}{2}\bar{\zeta}_{\mathcal{O}_0}(s) - \frac{2}{2}\bar{\zeta}_{\mathcal{O}_1}(s) - \frac{1}{2}\bar{\zeta}_{\mathcal{O}_2}(s) + \frac{8}{4}\bar{\zeta}_{\mathcal{O}_F^2}(s)\right)\Bigg|_{s=0} \\
&= 0
\end{aligned}
$$

[8] We undo the rescaling present in the reference.

vanishes because the traced heat kernels (so the zeta-functions) are multiplied by the volume of the regularized H^2 of the infinite-strip geometry. Indeed, the relation between the AdS radial coordinate ρ and the worldsheet coordinate is (B.25) [12], i.e. for $k=0$

$$\sinh\rho = \tan r\,. \tag{4.23}$$

Thus the worldsheet has IR-regularized volume

$$V_{H^2} \equiv \int_0^T dw \int_{-\arctan(\sinh\rho_0)}^{\arctan(\sinh\rho_0)} \frac{dr}{\cos^2 r} = 2T\sinh\rho_0 \to 0\,. \tag{4.24}$$

that is double the divergence for the half-plane sitting on a straight line. We singled out a "cylinder" of height T and almost touching the boundary $\mathbb{S}^3\times\mathbb{R}$ $(0<\rho<\rho_0)$. The target-space w time has cutoff T.

4.4 Effective action at order k^2

We use the same formulas of the latitude loop, now with $x=(r,w)$, $\delta(x)\equiv\delta(r)\,\delta(w)$ and the small k^2 replacing the parameter θ_0^2. Notice that will approximate the cutoff on r (hidden in $\int_x$ in what follows) to the lowest order in k

$$\frac{\pi}{2\mathbb{K}\left(k^2\right)}\mathrm{cn}^{-1}\left(\frac{\sqrt{1+b^2}}{b\cosh\rho_0}|k^2\right) = \arctan\left(\sinh\rho_0\right) + O\left(k^2\right)$$

because further corrections do not affect the effective action at order k^2 (but they would at order k^4).

For each field of the theory, we compute the expressions

$$\begin{aligned}
\frac{5}{2}\tilde{K}_{\mathcal{O}_0}(t) &= -\frac{5t}{2}\int_x \frac{1}{\cos^2 r}\left[\tilde{\mathcal{O}}_0\bar{K}_{-\Delta}\left(x,x';t\right)\right]_{x=x'} \qquad (4.25)\\
&= \frac{5t}{16\pi}\int_x\left(\frac{3}{\cos^2 r}-2\right)\int_0^\infty dv\, v\tanh\pi v\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}\\
&= \frac{5\left(3e^{\rho_0}-2\pi\right)tT}{16\pi}\int_0^\infty dv\, v\tanh\pi v\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}\\
\frac{5}{2}\tilde{\zeta}_{\mathcal{O}_0}(s) &= \frac{5\left(3e^{\rho_0}-2\pi\right)sT}{16\pi}\int_0^\infty dv\frac{v\tanh\pi v}{\left(v^2+\frac{1}{4}\right)^s} \qquad (4.26)\\
&= \frac{5\left(3e^{\rho_0}-2\pi\right)sT}{16\pi}\int_0^\infty dv\frac{1}{\left(v^2+\frac{1}{4}\right)^s}\left(v-\frac{2v}{e^{2\pi v}+1}\right)\\
&= \frac{5\left(3e^{\rho_0}-2\pi\right)sT}{32\pi\,(s-1)}\left(\frac{1}{4}\right)^{1-s} - \frac{5\left(3e^{\rho_0}-2\pi\right)sT}{8\pi}\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^s}\\
\frac{5}{2}\tilde{\zeta}'_{\mathcal{O}_0}(0) &= \frac{d}{ds}\left[\frac{5\left(3e^{\rho_0}-2\pi\right)sT}{32\pi\,(s-1)}\left(\frac{1}{4}\right)^{1-s}\right]_{s=0} - \frac{5\left(3e^{\rho_0}-2\pi\right)T}{8\pi}\frac{1}{48} \qquad (4.27)\\
&= \frac{5T}{48} - \frac{5e^{\rho_0}T}{32\pi}
\end{aligned}$$

$$
\begin{aligned}
\frac{2}{2}\tilde{K}_{\mathcal{O}_1}(t) &= -\frac{2t}{2}\int_x \frac{1}{\cos^2 r}\left[\tilde{\mathcal{O}}_1\bar{K}_{-\Delta+2}\left(x,x^{'};t\right)\right]_{x=x'} && (4.28)\\
&= \frac{t}{8\pi}\int_x\left(\frac{3}{\cos^2 r}-2\right)\int_0^\infty dv\, v\tanh\pi v\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}\\
&= \frac{\left(3e^{\rho_0}-2\pi\right)tT}{8\pi}\int_0^\infty dv\, v\tanh\pi v\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}\\
\frac{2}{2}\tilde{\zeta}_{\mathcal{O}_1}(s) &= \frac{\left(3e^{\rho_0}-2\pi\right)sT}{8\pi}\int_0^\infty dv\frac{v^2+\frac{1}{4}}{\left(v^2+\frac{9}{4}\right)^{s+1}}v\tanh\pi v && (4.29)\\
&= \frac{\left(3e^{\rho_0}-2\pi\right)sT}{8\pi}\int_0^\infty dv\frac{1}{\left(v^2+\frac{9}{4}\right)^{s}}\left(1-\frac{2}{v^2+\frac{9}{4}}\right)\left(v-\frac{2v}{e^{2\pi v}+1}\right)\\
&= \frac{\left(3e^{\rho_0}-2\pi\right)sT}{16\pi\left(s-1\right)}\left(\frac{9}{4}\right)^{1-s}-\frac{\left(3e^{\rho_0}-2\pi\right)T}{8\pi}\left(\frac{9}{4}\right)^{-s}\\
&\quad-\frac{\left(3e^{\rho_0}-2\pi\right)sT}{4\pi}\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^{s}}+\frac{\left(3e^{\rho_0}-2\pi\right)sT}{2\pi}\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^{s+1}}\\
\frac{2}{2}\tilde{\zeta}^{'}_{\mathcal{O}_1}(0) &= \frac{d}{ds}\left[\frac{\left(3e^{\rho_0}-2\pi\right)sT}{16\pi\left(s-1\right)}\left(\frac{9}{4}\right)^{1-s}-\frac{\left(3e^{\rho_0}-2\pi\right)T}{8\pi}\left(\frac{9}{4}\right)^{-s}\right]_{s=0}-\frac{\left(3e^{\rho_0}-2\pi\right)T}{4\pi}\frac{1}{48}+\frac{\left(3e^{\rho_0}-2\pi\right)T}{2\pi}\left(-\frac{\log\frac{3}{2}}{2}+\frac{1}{2}-\frac{\gamma}{2}\right) && (4.30)\\
&= \frac{\left(-5+12\gamma\right)T}{24}+\frac{e^{\rho_0}\left(5-12\gamma\right)T}{16\pi}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\tilde{K}_{\mathcal{O}_2}(t) &= -\frac{t}{2}\int_x \frac{1}{\cos^2 r}\left[\tilde{\mathcal{O}}_2\bar{K}_{-\Delta+2}\left(x,x^{'};t\right)\right]_{x=x'} && (4.31)\\
&= \frac{t}{16\pi}\int_x\int_0^\infty dv\, v\tanh\pi v\left[\left(\frac{3}{\cos^2 r}-2\right)\left(v^2+\frac{1}{4}\right)+8\cos^2 r\right]e^{-t\left(v^2+\frac{9}{4}\right)}\\
&= \frac{tT}{16\pi}\int_0^\infty dv\, v\tanh\pi v\left[\left(3e^{\rho_0}-2\pi\right)\left(v^2+\frac{1}{4}\right)+4\pi\right]e^{-t\left(v^2+\frac{9}{4}\right)}\\
\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_2}(s) &= \frac{sT}{16\pi}\int_0^\infty dv\frac{\left(3e^{\rho_0}-2\pi\right)\left(v^2+\frac{1}{4}\right)+4\pi}{\left(v^2+\frac{9}{4}\right)^{s+1}}v\tanh\pi v && (4.32)\\
&= \frac{sT}{16\pi}\int_0^\infty dv\frac{1}{\left(v^2+\frac{9}{4}\right)^{s}}\left(3e^{\rho_0}-2\pi+\frac{8\pi-6e^{\rho_0}}{v^2+\frac{9}{4}}\right)\left(v-\frac{2v}{e^{2\pi v}+1}\right)\\
&= \frac{\left(3e^{\rho_0}-2\pi\right)sT}{32\pi\left(s-1\right)}\left(\frac{9}{4}\right)^{1-s}+\frac{\left(4\pi-3e^{\rho_0}\right)T}{16\pi}\left(\frac{9}{4}\right)^{-s}\\
&\quad-\frac{\left(3e^{\rho_0}-2\pi\right)sT}{8\pi}\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^{s}}-\frac{\left(4\pi-3e^{\rho_0}\right)sT}{4\pi}\int_0^\infty dv\frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^{s+1}}\\
\frac{1}{2}\tilde{\zeta}^{'}_{\mathcal{O}_2}(0) &= \frac{d}{ds}\left[\frac{\left(3e^{\rho_0}-2\pi\right)sT}{32\pi\left(s-1\right)}\left(\frac{9}{4}\right)^{1-s}+\frac{\left(4\pi-3e^{\rho_0}\right)T}{16\pi}\left(\frac{9}{4}\right)^{-s}\right]_{s=0} && (4.33)\\
&\quad-\frac{\left(3e^{\rho_0}-2\pi\right)T}{8\pi}\frac{1}{48}-\frac{\left(4\pi-3e^{\rho_0}\right)T}{4\pi}\left(-\frac{\log\frac{3}{2}}{2}+\frac{1}{2}-\frac{\gamma}{2}\right)\\
&= \frac{\left(-17+24\gamma\right)T}{48}+\frac{e^{\rho_0}\left(5-12\gamma\right)T}{32\pi}
\end{aligned}
$$

$$-\frac{8}{4}\tilde{K}_{\mathcal{O}_F}(t) = 2t\int_x \frac{1}{\cos^2 r}\mathrm{tr}\left[\left\{\bar{\mathcal{O}}_F^x, \tilde{\mathcal{O}}_F^x\right\}\bar{K}_{-\bar{\nabla}^2+1}\left(x, x^{'}; t\right)\right]_{x=x'} \tag{4.34}$$
$$= \frac{t}{\pi}\int_x\int_0^\infty dv\, v\coth\pi v\left[\left(1-\frac{3}{2\cos^2 r}\right)\left(v^2+1\right)-1-\frac{\cos 2r}{2}+\frac{3}{4\cos^2 r}\right]e^{-t\left(v^2+1\right)}$$
$$= \frac{tT}{\pi}\int_0^\infty dv\, v\coth\pi v\left[\left(\pi-\frac{3e^{\rho_0}}{2}\right)\left(v^2+1\right)+\frac{3e^{\rho_0}}{4}-\pi\right]e^{-t\left(v^2+1\right)}$$

$$-\frac{8}{4}\tilde{\zeta}_{\mathcal{O}_F^2}(s) = \frac{sT}{\pi}\int_0^\infty dv\frac{1}{\left(v^2+1\right)^s}\left(\pi-\frac{3e^{\rho_0}}{2}+\frac{\frac{3e^{\rho_0}}{4}-\pi}{v^2+1}\right)v\coth\pi v \tag{4.35}$$
$$= \frac{sT}{\pi}\int_0^\infty dv\frac{1}{\left(v^2+1\right)^s}\left(\pi-\frac{3}{2}e^{\rho_0}+\frac{\frac{3}{4}e^{\rho_0}-\pi}{v^2+1}\right)\left(v+\frac{2v}{e^{2\pi v}-1}\right)$$
$$= \frac{sT}{2\pi(s-1)}\left(\pi-\frac{3}{2}e^{\rho_0}\right)+\frac{T}{2\pi}\left(\frac{3}{4}e^{\rho_0}-\pi\right)$$
$$+\frac{2sT}{\pi}\left(\pi-\frac{3}{2}e^{\rho_0}\right)\int_0^\infty dv\frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)^s}+\frac{2sT}{\pi}\left(\frac{3}{4}e^{\rho_0}-\pi\right)\int_0^\infty dv\frac{v}{\left(e^{2\pi v}-1\right)\left(v^2+1\right)^{s+1}}$$

$$-\frac{8}{4}\tilde{\zeta}'_{\mathcal{O}_F^2}(0) = \frac{d}{ds}\left[\frac{sT}{2\pi(s-1)}\left(\pi-\frac{3}{2}e^{\rho_0}\right)\right]_{s=0} \tag{4.36}$$
$$+\frac{2T}{\pi}\left(\pi-\frac{3}{2}e^{\rho_0}\right)\frac{1}{24}+\frac{2T}{\pi}\left(\frac{3}{4}e^{\rho_0}-\pi\right)\left(-\frac{1}{4}+\frac{\gamma}{2}\right)$$
$$= \left(\frac{1}{12}-\gamma\right)T+\frac{e^{\rho_0}(3+\gamma)T}{4\pi}$$

and, summing over the field content of the theory and dropping the e^{ρ_0} divergencies

$$\tilde{\Gamma} = -\frac{d}{ds}\left(\frac{5}{2}\tilde{\zeta}_{\mathcal{O}_0}(s)+\frac{2}{2}\tilde{\zeta}_{\mathcal{O}_1}(s)+\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_2}(s)-\frac{8}{4}\tilde{\zeta}_{\mathcal{O}_F^2}(s)\right)\Big|_{s=0} = \frac{3T}{8}\,, \tag{4.37}$$

we obtain the Drukker-Forini result [9]

$$\Gamma_1(k) = \frac{3T}{8}k^2+O\left(k^2\right)\,. \tag{4.38}$$

4.5 Analytic continuations in $\tilde{\zeta}(s)$

We repeat the analysis done for the latitude to prove that analytic continuation in s is again superfluous. We drop the e^{ρ_0} divergencies in this section.

We consider the sum of zeta-functions

$$\frac{5}{2}\tilde{\zeta}_{\mathcal{O}_0}(s)+\frac{2}{2}\tilde{\zeta}_{\mathcal{O}_1}(s)+\frac{1}{2}\tilde{\zeta}_{\mathcal{O}_2}(s)-\frac{8}{4}\tilde{\zeta}_{\mathcal{O}_F}(s) \tag{4.39}$$
$$= \int_0^\infty dv\left[\frac{5(-2\pi)sT}{16\pi}\frac{v}{\left(v^2+\frac{1}{4}\right)^s}\right]+\int_0^\infty dv\left[\frac{(-2\pi)sT}{8\pi}\frac{v}{\left(v^2+\frac{9}{4}\right)^s}\left(1-\frac{2}{v^2+\frac{9}{4}}\right)\right]$$
$$+\int_0^\infty dv\left[\frac{sT}{16\pi}\frac{v}{\left(v^2+\frac{9}{4}\right)^s}\left(-2\pi+\frac{8\pi}{v^2+\frac{9}{4}}\right)\right]+\int_0^\infty dv\left[\frac{sT}{\pi}\frac{v}{\left(v^2+1\right)^s}\left(\pi+\frac{-\pi}{v^2+1}\right)\right]$$
$$+\int_0^\infty dv\left(\text{exponentially suppressed integrands } \sim e^{-2\pi v}\text{ for large } v\right)\,.$$

In the previous section, we treated each of the four summands in square brackets above <u>individually</u>:

- we integrated $\int_0^\infty dv\,[\ldots]$ for s large enough,
- analytically continued to a neighborhood of $s=0$ by simply neglecting the restriction on s,
- derived $\frac{d}{ds}[\ldots]_{s=0}$.

[9] See (D.47)-(D.48) [12]. We use $\mathcal{T} = T+O\left(k^2\right)$.

Summing up these terms produced a contribution

$$\begin{aligned}
&\frac{d}{ds}\left[\frac{5\,(-2\pi)\,sT}{32\pi\,(s-1)}\left(\frac{1}{4}\right)^{1-s}\right]_{s=0}+\frac{d}{ds}\left[\frac{(-2\pi)\,sT}{16\pi\,(s-1)}\left(\frac{9}{4}\right)^{1-s}-\frac{(-2\pi)\,T}{8\pi}\left(\frac{9}{4}\right)^{-s}\right]_{s=0} \\
&+\frac{d}{ds}\left[\frac{(-2\pi)\,sT}{32\pi\,(s-1)}\left(\frac{9}{4}\right)^{1-s}+\frac{(4\pi)\,T}{16\pi}\left(\frac{9}{4}\right)^{-s}\right]_{s=0}+\frac{d}{ds}\left[\frac{sT}{2\pi\,(s-1)}\,(\pi)\right]_{s=0} \\
=&-\log\left(\frac{3}{2}\right)T\,.
\end{aligned}\tag{4.40}$$

Alternatively, we can directly start with the sum of the four terms and

- derive $\frac{d}{ds}\,[\ldots]_{s=0}$ and skip the analytic continuation because the sum is well-defined around $s=0$,
- integrate $\int_0^\infty dv\,[\ldots]$.

The new algorithm yields the same result

$$\begin{aligned}
&\int_0^\infty dv\frac{d}{ds}\left\{\left[\frac{5\,(-2\pi)\,sT}{16\pi}\frac{v}{\left(v^2+\frac{1}{4}\right)^s}\right]+\left[\frac{(-2\pi)\,sT}{8\pi}\frac{v}{\left(v^2+\frac{9}{4}\right)^s}\left(1-\frac{2}{v^2+\frac{9}{4}}\right)\right]\right. \\
&\qquad\left.+\left[\frac{sT}{16\pi}\frac{v}{\left(v^2+\frac{9}{4}\right)^s}\left(-2\pi+\frac{8\pi}{v^2+\frac{9}{4}}\right)\right]+\left[\frac{sT}{\pi}\frac{v}{\left(v^2+1\right)^s}\left(\pi+\frac{-\pi}{v^2+1}\right)\right]\right\}_{s=0} \\
=&-\log\left(\frac{3}{2}\right)T\,.
\end{aligned}\tag{4.41}$$

4.6 Small-t asymptotics of $\tilde{K}\,(t)$

$$\begin{aligned}
\frac{5}{2}\tilde{K}_{\mathcal{O}_0}(t)&=\frac{5\,(3e^{\rho_0}-2\pi)\,tT}{16\pi}\int_0^\infty dv\,v\tanh\pi v\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)} \\
&=\frac{5\,(3e^{\rho_0}-2\pi)\,tT}{16\pi}\int_0^\infty dv\left(v-\frac{2v}{e^{2\pi v}+1}\right)\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)} \\
&=\frac{5\,(3e^{\rho_0}-2\pi)\,tT}{16\pi}\int_0^\infty dv\,v\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)}-\frac{5\,(3e^{\rho_0}-2\pi)\,tT}{8\pi}\int_0^\infty dv\frac{v}{e^{2\pi v}+1}\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)} \\
&=\frac{5\,(3e^{\rho_0}-2\pi)\,(4+t)\,tT}{128\pi t}e^{-t/4}-\frac{5\,(3e^{\rho_0}-2\pi)\,tT}{8\pi}\int_0^\infty dv\frac{v}{e^{2\pi v}+1}\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{1}{4}\right)} \\
&=\frac{5\,(3e^{\rho_0}-2\pi)\,T}{32\pi t}+O\,(t)
\end{aligned}\tag{4.42}$$

$$\begin{aligned}
\frac{2}{2}\tilde{K}_{\mathcal{O}_1}(t)&=\frac{(3e^{\rho_0}-2\pi)\,tT}{8\pi}\int_0^\infty dv\,v\tanh\pi v\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)} \\
&=\frac{(3e^{\rho_0}-2\pi)\,tT}{8\pi}\int_0^\infty dv\left(v-\frac{2v}{e^{2\pi v}+1}\right)\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)} \\
&=\frac{(3e^{\rho_0}-2\pi)\,tT}{8\pi}\int_0^\infty dv\,v\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)}-\frac{(3e^{\rho_0}-2\pi)\,tT}{4\pi}\int_0^\infty dv\frac{v}{e^{2\pi v}+1}\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)} \\
&=\frac{(3e^{\rho_0}-2\pi)\,(4+t)\,T}{64\pi t}e^{-9t/4}-\frac{(3e^{\rho_0}-2\pi)\,tT}{4\pi}\int_0^\infty dv\frac{v}{e^{2\pi v}+1}\left(v^2+\frac{1}{4}\right)e^{-t\left(v^2+\frac{9}{4}\right)} \\
&=\frac{(3e^{\rho_0}-2\pi)\,T}{16\pi t}+\frac{(3e^{\rho_0}-2\pi)\,T}{8\pi}+O\,(t)
\end{aligned}\tag{4.43}$$

$$\begin{aligned}\frac{1}{2}\tilde{K}_{\mathcal{O}_2}(t) &= \frac{tT}{16\pi}\int_0^\infty dv\, v\tanh\pi v\left[\left(3e^{\rho_0}-2\pi\right)\left(v^2+\frac{1}{4}\right)+4\pi\right]e^{-t\left(v^2+\frac{9}{4}\right)} \\ &= \frac{tT}{16\pi}\int_0^\infty dv\left(v-\frac{2v}{e^{2\pi v}+1}\right)\left[\left(3e^{\rho_0}-2\pi\right)\left(v^2+\frac{1}{4}\right)+4\pi\right]e^{-t\left(v^2+\frac{9}{4}\right)} \\ &= \frac{tT}{16\pi}\int_0^\infty dv\, v\left[\left(3e^{\rho_0}-2\pi\right)\left(v^2+\frac{1}{4}\right)+4\pi\right]e^{-t\left(v^2+\frac{9}{4}\right)}-\frac{tT}{8\pi}\int_0^\infty dv\frac{v}{e^{2\pi v}+1}\left[\left(3e^{\rho_0}-2\pi\right)\left(v^2+\frac{1}{4}\right)+4\pi\right]e^{-t\left(v^2+\frac{9}{4}\right)} \\ &= \frac{T\left[3e^{\rho_0}\left(4+t\right)+2\pi\left(-4+7t\right)\right]}{128\pi t}e^{-9t/4}-\frac{tT}{8\pi}\int_0^\infty dv\frac{v}{e^{2\pi v}+1}\left[\left(3e^{\rho_0}-2\pi\right)\left(v^2+\frac{1}{4}\right)+4\pi\right]e^{-t\left(v^2+\frac{9}{4}\right)} \\ &= \frac{\left(3e^{\rho_0}-2\pi\right)T}{32\pi t}-\frac{\left(3e^{\rho_0}-4\pi\right)T}{16\pi}+O\left(t\right)\end{aligned} \tag{4.44}$$

$$\begin{aligned}-\frac{8}{4}\tilde{K}_{\mathcal{O}_F}(t) &= \frac{tT}{\pi}\int_0^\infty dv\, v\coth\pi v\left[\left(\pi-\frac{3e^{\rho_0}}{2}\right)\left(v^2+1\right)+\frac{3e^{\rho_0}}{4}-\pi\right]e^{-t\left(v^2+1\right)} \\ &= \frac{tT}{\pi}\int_0^\infty dv\left(v+\frac{2v}{e^{2\pi v}-1}\right)\left[\left(\pi-\frac{3e^{\rho_0}}{2}\right)\left(v^2+1\right)+\frac{3e^{\rho_0}}{4}-\pi\right]e^{-t\left(v^2+1\right)} \\ &= \frac{tT}{\pi}\int_0^\infty dv\, v\left[\left(\pi-\frac{3e^{\rho_0}}{2}\right)\left(v^2+1\right)+\frac{3e^{\rho_0}}{4}-\pi\right]e^{-t\left(v^2+1\right)} \\ &+\frac{2tT}{\pi}\int_0^\infty dv\frac{v}{e^{2\pi v}-1}\left[\left(\pi-\frac{3e^{\rho_0}}{2}\right)\left(v^2+1\right)+\frac{3e^{\rho_0}}{4}-\pi\right]e^{-t\left(v^2+1\right)} \\ &= \frac{\left[4\pi-3e^{\rho_0}\left(2+t\right)\right]T}{8\pi t}e^{-t}+\frac{2tT}{\pi}\int_0^\infty dv\frac{v}{e^{2\pi v}-1}\left[\left(\pi-\frac{3e^{\rho_0}}{2}\right)\left(v^2+1\right)+\frac{3e^{\rho_0}}{4}-\pi\right]e^{-t\left(v^2+1\right)} \\ &= \frac{\left(-3e^{\rho_0}+2\pi\right)T}{4\pi t}+\frac{\left(3e^{\rho_0}-4\pi\right)T}{8\pi}+O\left(t\right)\end{aligned} \tag{4.45}$$

$$\frac{5}{2}\tilde{K}_{\mathcal{O}_0}(t)+\frac{2}{2}\tilde{K}_{\mathcal{O}_1}(t)+\frac{1}{2}\tilde{K}_{\mathcal{O}_2}(t)-\frac{8}{4}\tilde{K}_{\mathcal{O}_F}(t)=-\frac{3e^{\rho_0}T}{16\pi}+O\left(t\right) \tag{4.46}$$

5 The dependence of determinants on conformal factors

5.1 General definitions

Let us consider two d-dimensional spaces $\bar{\mathcal{M}}$ and $\mathcal{M}$ with conformally equivalent metrics $\bar{g}_{ij}$ and $g_{ij} = e^{2\alpha\Omega(x)}\bar{g}_{ij}$ and without boundary. The determinant of the metric g is an expansion in small α

$$\begin{aligned} g &= e^{2\alpha d\Omega}\bar{g} && (5.1)\\ &= \bar{g} + \alpha\tilde{g} + O\left(\alpha^2\right) \\ &= \bar{g}\left(1 + 2\alpha d\Omega + O\left(\alpha^2\right)\right) \\ \sqrt{g} &= \sqrt{\bar{g}}\left(1 + \alpha d\Omega + O\left(\alpha^2\right)\right) \, . && (5.2) \end{aligned}$$

We compute the difference of Ricci scalars

$$\begin{aligned} R &= e^{-2\alpha\Omega}\left[\bar{R} - \frac{2\alpha\left(d-1\right)}{\sqrt{\bar{g}}}\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\Omega\right) - \alpha^2\left(d-1\right)\left(d-2\right)\partial_i\Omega\partial_j\Omega\right] && (5.3)\\ &= \bar{R} + \alpha\tilde{R} + O\left(\alpha^2\right) \\ &= \bar{R} - 2\left[\Omega\bar{R} + \frac{d-1}{\sqrt{\bar{g}}}\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\Omega\right)\right]\alpha + O\left(\alpha^2\right) \, . \end{aligned}$$

In the next two sections we focus on the Seeley-de Witt coefficients a_0, a_2 for the d-dimensional scalar Laplacian with a spacetime-dependent mass on $\mathcal{M}$. Let us refer to it as $\mathcal{O}$. We will study the corrections $\tilde{a}_i$

$$a_i = \bar{a}_i + \alpha\tilde{a}_i + O\left(\alpha^2\right) \qquad\qquad i = 0, 2 \tag{5.4}$$

in two independent ways:

- we write the a_i [13] and expand in α,
- we take our expression for the heat kernel of $\mathcal{O}$ (in terms of the heat kernel for $\bar{\mathcal{O}}$)

$$\begin{aligned} K_{\mathcal{O}}\left(x, x^{'}; t\right) &= \bar{K}_{\mathcal{O}}\left(x, x^{'}; t\right) + \alpha\tilde{K}_{\mathcal{O}}\left(x, x^{'}; t\right) + O\left(\alpha^2\right) && (5.5)\\ &= \left(1 - \alpha\frac{\tilde{g}}{2\bar{g}}\right)\bar{K}_{\mathcal{O}}\left(x, x^{'}; t\right) + \alpha\int_0^t dt^{'}\int_{x^{''}}\sqrt{\bar{g}}\bar{K}_{\mathcal{O}}\left(x, x^{''}; t - t^{'}\right)\tilde{\mathcal{O}}_{x^{''}}\bar{K}_{\mathcal{O}}\left(x^{''}, x^{'}; t^{'}\right) + O\left(\alpha^2\right) \\ K_{\mathcal{O}}\left(t\right) &= \bar{K}_{\mathcal{O}}\left(t\right) + \alpha\tilde{K}_{\mathcal{O}}\left(t\right) + O\left(\alpha^2\right) && (5.6)\\ &= \int_x\sqrt{\bar{g}}\left[\bar{K}_{\mathcal{O}}\left(x, x; t\right) - \alpha t\left(\tilde{\mathcal{O}}_x\bar{K}_{\mathcal{O}}\left(x, x^{'}; t\right)\right)_{x=x^{'}}\right] + O\left(\alpha^2\right) \end{aligned}$$

  and, given the $\bar{a}_i$ of $\bar{\mathcal{O}}$ in [13], we recover the same corrections $\tilde{a}_i$.

The former is a mere manipulation of literature results, whereas the latter shows the ability of our method to capture the correction to the heat kernel (so to the Seeley coefficients) using *only* on the unperturbed one.

The finite part of the determinant is not captured by the small-t expansion, as we would need the heat kernel for any finite $t > 0$. The information on the divergence of the scalar Laplacian in d dimensions is carried by the $d+1$ coefficients a_i with $i = 0, \ldots d$. Although the check above should be extended to all the infinitely-many a_i ($i = 0, 1, \ldots$), we chose to restrict to $i = 0, 2$ because

- their expression is particularly compact to handle,
- only a_0, a_2 are relevant for divergences on two-dimensional worldsheets without boundary.

5.2 Massive Laplacian: literature in Gilkey

Let us focus on the Laplacian with an arbitrary non-constant mass term

$$\begin{aligned} \mathcal{O} &= -\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right) + X && (5.7)\\ &= -\frac{e^{-\alpha d\Omega}}{\sqrt{\bar{g}}}\partial_i\left(e^{\alpha(d-2)\Omega}\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\right) + X \, . \end{aligned}$$

The small-t expansion of the heat kernel of an operator $\mathcal{O}$ on $\mathcal{M}$

$$K_{\mathcal{O}}(x,x;t) = \frac{1}{(4\pi)^{d/2}} \sum_{k=0,2,\ldots} t^{(k-d)/2} b_{k/2} \tag{5.8}$$

$$K_{\mathcal{O}}(t) = \int_x \sqrt{g} K_{\mathcal{O}}(x,x;t) = \sum_{k=0,2,\ldots} t^{(k-d)/2} a_k \tag{5.9}$$

is given by the coefficients

$$b_0 = 1 \tag{5.10}$$

$$b_1 = \frac{1}{6}R - X \tag{5.11}$$

$$\ldots$$

$$a_0 = \frac{1}{(4\pi)^{d/2}} \int_x \sqrt{g} \tag{5.12}$$

$$a_2 = \frac{1}{(4\pi)^{d/2}} \int_x \sqrt{g} \left(\frac{1}{6}R - X\right) \tag{5.13}$$

$$\ldots$$

The determinant inherits the divergencies from the first $d+1$ Seeley coefficients [10]

$$\begin{aligned}(\log \mathrm{Det}\mathcal{O})_\infty &= -\int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} \left[K_{\mathcal{O}}(t)\right]_{\text{non-positive powers of } t} \\ &= \sum_{k=0,2,\ldots d-2} \frac{2a_k}{k-d} \Lambda^{d-k} - a_d \log \Lambda^2 \,.\end{aligned} \tag{5.14}$$

We think of the expressions above as series in small α, for the mass

$$X = \bar{X} + \alpha \tilde{X} + O\left(\alpha^2\right) , \tag{5.15}$$

the operator

$$\begin{aligned}\mathcal{O} &= \bar{\mathcal{O}} + \alpha \tilde{\mathcal{O}} + O\left(\alpha^2\right) \\ &= \left(-\frac{1}{\sqrt{\bar{g}}} \partial_i \left(\sqrt{\bar{g}} \bar{g}^{ij} \partial_j\right) + \bar{X}\right) + \left[-(d-2)\, \bar{g}^{ij} \partial_i \Omega \partial_j + \frac{2\Omega}{\sqrt{\bar{g}}} \partial_i \left(\sqrt{\bar{g}} \bar{g}^{ij} \partial_j\right) + \tilde{X}\right] \alpha + O\left(\alpha^2\right)\end{aligned} \tag{5.16}$$

and for the Seeley-de Witt coefficients

$$\bar{b}_0 = 1 \tag{5.17}$$

$$\bar{b}_1 = \frac{1}{6}\bar{R} - \bar{X} \tag{5.18}$$

$$\ldots$$

$$\tilde{b}_0 = 0 \tag{5.19}$$

$$\tilde{b}_1 = \frac{1}{6}\tilde{R} - \tilde{X} = -\frac{1}{3}\Omega \bar{R} - \frac{d-1}{3\sqrt{\bar{g}}} \partial_i \left(\sqrt{\bar{g}} \bar{g}^{ij} \partial_j \Omega\right) - \tilde{X} \tag{5.20}$$

$$\ldots$$

[10] To single out the divergencies we cannot employ the usual analytic continuation of the zeta-function in s, but the definition in Section 5.8 [14].

$$\bar{a}_0 = \frac{1}{(4\pi)^{d/2}} \int_x \sqrt{\bar{g}} \tag{5.21}$$

$$\bar{a}_2 = \frac{1}{(4\pi)^{d/2}} \int_x \sqrt{\bar{g}} \left(\frac{1}{6}\bar{R} - \bar{X}\right) \tag{5.22}$$

$$\dots$$

$$\tilde{a}_0 = \frac{d}{(4\pi)^{d/2}} \int_x \sqrt{\bar{g}}\Omega \tag{5.23}$$

$$\tilde{a}_2 = \frac{1}{(4\pi)^{d/2}} \int_x \sqrt{\bar{g}} \left(\frac{d-2}{6}\Omega\bar{R} - d\Omega\bar{X} + \tilde{X}\right) \tag{5.24}$$

$$\dots$$

5.3 Massive Laplacian: our formula

The perturbative heat kernel yields for any operator yields

$$\begin{aligned} K_{\mathcal{O}}(t) &= \int_x \sqrt{\bar{g}}\bar{K}_{\mathcal{O}}(x,x;t) - \alpha t \int_x \sqrt{\bar{g}} \left[\tilde{\mathcal{O}}_x \bar{K}_{\mathcal{O}}\left(x,x^{'};t\right)\right]_{x=x^{'}} + O\left(\alpha^2\right) \\ &= \bar{K}_{\mathcal{O}}(t) + \alpha t \int_x \left\{\left[(d-2)\sqrt{\bar{g}}\bar{g}^{ij}\partial_i\Omega\partial_j - 2\Omega\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\right) - \sqrt{\bar{g}}\tilde{X}\right]\bar{K}_{\mathcal{O}}\left(x,x^{'};t\right)\right\}_{x=x^{'}} + O\left(\alpha^2\right) \end{aligned} \tag{5.25}$$

integrate first term by parts

$$\begin{aligned} &= \bar{K}_{\mathcal{O}}(t) + \alpha t \int_x \Big[(d-2)\partial_j\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_i\Omega\bar{K}_{\mathcal{O}}\left(x,x^{'};t\right)\right) - (d-2)\partial_j\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_i\Omega\right)\bar{K}_{\mathcal{O}}\left(x,x^{'};t\right) \\ &\qquad -2\Omega\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\bar{K}_{\mathcal{O}}\left(x,x^{'};t\right)\right) - \sqrt{\bar{g}}\tilde{X}\bar{K}_{\mathcal{O}}\left(x,x^{'};t\right)\Big]_{x=x^{'}} + O\left(\alpha^2\right) \end{aligned}$$

use heat equation for $\bar{\mathcal{O}}$ and divergence theorem

$$\begin{aligned} &= \bar{K}_{\mathcal{O}}(t) + \alpha t\left[-(d-2)\int_x \partial_j\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_i\Omega\right)\bar{K}_{\mathcal{O}}(x,x;t) - \int_x\sqrt{\bar{g}}\left(2\Omega\bar{X}+\tilde{X}\right)\bar{K}_{\mathcal{O}}(x,x;t) - 2\int_x\sqrt{\bar{g}}\Omega\partial_t\bar{K}_{\mathcal{O}}(x,x;t)\right] \\ &\qquad + O\left(\alpha^2\right) \end{aligned}$$

plug time-expansion for $\bar{K}_{\mathcal{O}}(x,x;t)$ and use divergence theorem

$$= \bar{K}_{\mathcal{O}}(t) + \frac{\alpha}{(4\pi)^{d/2}}\left[dt^{-d/2}\int_x\sqrt{\bar{g}}\Omega + t^{(2-d)/2}\int_x\sqrt{\bar{g}}\left(\frac{d-2}{6}\Omega\bar{R} - d\Omega\bar{X} - \tilde{X}\right) + O\left(t^{(3-d)/2}\right)\right] + O\left(\alpha^2\right)\ .$$

From here we extract the coefficients

$$\tilde{a}_0 = \frac{d}{(4\pi)^{d/2}} \int_{\mathcal{M}} \sqrt{\bar{g}}\Omega \tag{5.26}$$

$$\tilde{a}_2 = \frac{1}{(4\pi)^{d/2}} \int_{\mathcal{M}} \sqrt{\bar{g}} \left(\frac{d-2}{6}\Omega\bar{R} - d\Omega\bar{X} + \tilde{X}\right) \tag{5.27}$$

that match the ones in literature at the end of the previous section.

References

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