Deformations of the circular Wilson loop and spectral-parameter independence

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Spectral deformations of Wilson loops

We are concerned with Wilson loop operators on curves $(x_1(\theta), x_2(\theta))$ that lie in $\mathbb{R}^2$ in planar $\mathcal{N}=4$ SYM theory:

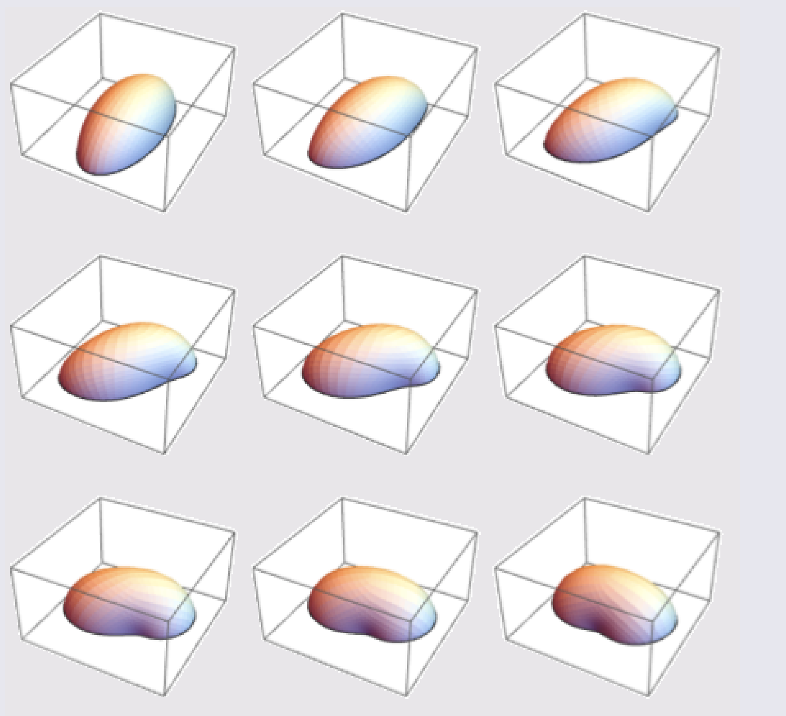
$$\mathcal{W} = \frac{1}{N}\operatorname{tr}\mathcal{P}\exp\left[\oint d\theta\,\left(i\dot{x}^\mu A_\mu + |\dot{x}|\phi^1\right)\right].$$

Using holography, finding their expectation values at large coupling λ amounts to solve a minimal-area problem in hyperbolic space $H_3 \subset AdS_5$.

Kruczenski et al. developed a method to construct minimal surfaces through Pohlmeyer reduction of the string sigma-model [1]. They also discovered an exact symmetry that changes the shape of the surface while preserving the area [1]. The associated degree of freedom is the spectral parameter $e^{i\varphi}$ of the string integrability framework and a connection to Yangian symmetry was established [2].

The symmetry implies the existence of one-parameter families of Wilson loops on different curves but with the same expectation value at $\lambda \gg 1$.

Spectral deformations are non-trivial to generate because they act non-locally on the entire surface and not many minimal surfaces are known analytically.



Spectral deformations of a minimal surface and the elliptical boundary curve for $\varphi = 0, \frac{\pi}{8}, \frac{2\pi}{8}, \ldots \pi$ [2].

Wavy deformations of the circular Wilson loop

We focus exclusively on spectral deformations of curves that deviate from the unit circle by a small amount ϵ

$$X(\theta) \equiv x_1(\theta) + i x_2(\theta) = \exp\left[i\theta + \sum_{n=1}^{\infty}\epsilon^n g_n(\theta)\right], \qquad g_n(\theta) = \sum_{m=-\infty}^{\infty} g_{n,m}e^{im\theta}, \qquad \theta\in[0,2\pi],$$

for which the Wilson loop has the expansion $\langle\mathcal{W}\rangle \equiv \langle\mathcal{W}\rangle_{\epsilon^0} + \epsilon^2\langle\mathcal{W}\rangle_{\epsilon^2} + \epsilon^4\langle\mathcal{W}\rangle_{\epsilon^4} + \ldots$ with

$$\langle\mathcal{W}\rangle_{\epsilon^0} = \frac{2}{\sqrt{\lambda}}I_1(\sqrt{\lambda}), \qquad \frac{\langle\mathcal{W}\rangle_{\epsilon^2}}{\langle\mathcal{W}\rangle_{\epsilon^0}} = \frac{\pi^2}{2}B(\lambda)\sum_{m=-\infty}^{\infty}|m|\left(m^2-1\right)|g_{1,m}|^2, \qquad B(\lambda) = \frac{\lambda}{2\pi^2}\frac{\partial}{\partial\lambda}\log\langle\mathcal{W}\rangle_{\epsilon^0}.$$

The 1/2-BPS circular Wilson loop $\langle\mathcal{W}\rangle_{\epsilon^0}$ is given by a modified Bessel function via supersymmetric localization. The "wavy approximation" $\langle\mathcal{W}\rangle_{\epsilon^2}$ is a functional of the geometry of the curve by conformal symmetry [3] and the overall factor is the Bremsstrahlung function $B(\lambda)$ computed exactly using localization-based arguments.

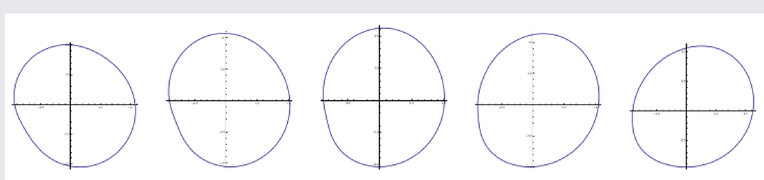
Let us make a spectral (φ-)deformation of the wavy circle. Is $\langle\mathcal{W}\rangle$ invariant?

Yes at $\lambda \gg 1$ because spectral deformations are an exact symmetry of the worldsheet area [1].

No at $\lambda \ll 1$, but the dependence on φ is mild because φ shows up at an unexpected high power in ϵ [4].

Spectral deformation simply rotates the circle $\longrightarrow$ $\langle\mathcal{W}\rangle_{\epsilon^0}$ is φ-independent
All-loop structure of the wavy approximation [3] $\longrightarrow$ $\langle\mathcal{W}\rangle_{\epsilon^2}$ is φ-independent
No comprehensive survey so far [4] $\longrightarrow$ $\langle\mathcal{W}\rangle_{\epsilon^4}$ and $\langle\mathcal{W}\rangle_{\epsilon^6}$ are φ-independent?
Several curves [4] like the example below $\longrightarrow$ $\langle\mathcal{W}\rangle_{\epsilon^8}$ is φ-dependent

For example, the spectral deformation of a particular wavy circle [4], here depicted for $\varphi = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$ and fixed ϵ, introduces φ in the expectation value at $\lambda \ll 1$:



$$\langle\mathcal{W}\rangle = 1 + \lambda\left[\frac{1}{8} + \frac{3}{8}\epsilon^2 + \frac{773}{64}\epsilon^4 + \frac{57\,359}{1256}\epsilon^6 + \frac{1\,182\,155\,647 + 62\,208\cos(2\varphi)}{286\,720}\epsilon^8 + O(\epsilon^{10})\right] + O(\lambda^2).$$

Statement of the problem

Is $\langle\mathcal{W}\rangle_{\epsilon^4}$ invariant under spectral deformations?

Our work answers positively at 1 and 2 loops at $\lambda \ll 1$:

- construct the most general wavy circle (with shape controlled by $\epsilon \ll 1$),
- deform the curve with a spectral deformation (controlled by $\varphi \in [0, 2\pi]$),
- compute the Wilson loop expectation value at 1 and 2 loops,
- extract the parts proportional to $\lambda\epsilon^4$ and $\lambda^2\epsilon^4$,
- test their independence from φ.

We also derive a Fourier representation of the 1,2-loop expectation value and an efficient algorithm to compute it.

Is $\langle\mathcal{W}\rangle_{\epsilon^6}$ invariant under spectral deformations?

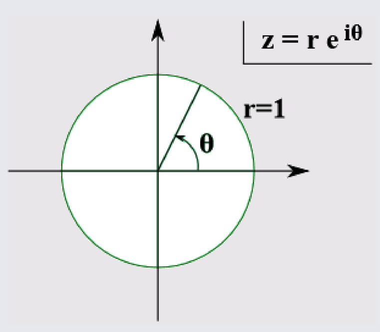
No: we find one curve at least for which $\langle\mathcal{W}\rangle^{\text{1-loop}}_{\epsilon^6}$ depends on φ (not reported here).

Holomorphic function

The worldsheet disk $|z| \le 1$ maps to the minimal surface and the boundary curve $z = 1$ to the Wilson loop path.

In Kruczenski's approach [1] a minimal surface is determined by the holomorphic function $f(z)$ and the real function $\alpha(z,\bar{z})$. The circular solution has $f(z) = 0$, so our physical input is the set of "small" Taylor coefficients of

$$f(z) = \epsilon\sum_{p=0}^{\infty} a_p z^p.$$



Spectral information is carried in a conformally-invariant way by rescaling $f(z) \to e^{i\varphi}f(z)$.

Construction of the curve

The function $\alpha(z,\bar{z})$ solves the generalized cosh-Gordon equation $\partial\bar{\partial}\alpha(z,\bar{z}) = e^{2\alpha(z,\bar{z})} + |f(z)|^2 e^{-2\alpha(z,\bar{z})}$:

$$\left[\partial_r^2 + \frac{\partial_r}{r} + \frac{\partial_\theta}{r^2} - \frac{8}{(1-r^2)^2}\right]\alpha^{(2)}(z,\bar{z}) = 4(1-r^2)^2|f(z)/\epsilon|^2, \qquad \alpha(z,\bar{z}) \equiv -\log(1-r^2) + \epsilon^2\alpha^{(2)}(z,\bar{z}) + O(\epsilon^4).$$

The limit $\xi = 1 - r^2 \to 0$ determines the function $\beta_2(\theta)$:

$$\alpha(z,\bar{z}) = -\ln\xi + \xi^2(1+\xi)\beta_2(\theta) + O(\xi^4), \qquad \beta_2(\theta) \equiv 0 + \epsilon^2\beta_2^{(2)}(\theta) + O(\epsilon^4).$$

The boundary curve $X(\theta) \equiv \exp\left[i\theta + \sum_{n=1}^{\infty}\epsilon^n g_n(\theta)\right]$ solves the Schwarzian-derivative differential equation

$$\frac{X'''(\theta)}{X'(\theta)} - \frac{3}{2}\left(\frac{X''(\theta)}{X'(\theta)}\right)^2 = \frac{1}{2} - 12\beta_2(\theta) - 4i\,\mathrm{Im}\left(e^{2i\theta}e^{i\varphi}f(e^{i\theta})\right),$$

which splits into the lowest orders

$$L_2 g_1 = 4\mathrm{Im}\left(e^{2i\theta}e^{i\varphi}f(e^{i\theta})/\epsilon\right), \qquad L_2 g_2 = \frac{i}{2}g_1'^2 - \frac{3i}{2}g_1''^2 - ig_1'g_1''' - 12i\beta_2^{(2)},$$
$$L_2 g_3 = g_1'''(g_1'^2 - ig_2') - 3ig_1''g_2'' + g_1'(3g_1''^2 + ig_2' - ig_2''').$$

The solutions return the Wilson loop path $X(\theta)$ in terms of the input coefficients a_p of the holomorphic function, the "wavy" parameter ϵ and the spectral parameter φ.

1- and 2-loop expectation value

$$\langle\mathcal{W}\rangle^{\text{1-loop}} = \frac{\lambda}{16\pi^2}\oint d\theta_1\,d\theta_2\,I(\theta_1,\theta_2), \qquad I(\theta_1,\theta_2) \equiv -\frac{\mathrm{Re}(\dot{X}(\theta_1)\dot{\bar{X}}(\theta_2)) - |\dot{X}(\theta_1)||\dot{X}(\theta_2)|}{|X(\theta_1)-X(\theta_2)|^2},$$

$$\langle\mathcal{W}\rangle^{\text{2-loop}} = -\frac{\lambda^2}{128\pi^4}\oint d\theta_1\,d\theta_2\,d\theta_3\left[\varepsilon(\theta_1,\theta_2,\theta_3)I(\theta_1,\theta_3)\frac{\mathrm{Re}((X(\theta_3)-X(\theta_2))\dot{\bar{X}}(\theta_2))}{|X(\theta_3)-X(\theta_2)|^2}\log\frac{|X(\theta_1)-X(\theta_2)|^2}{|X(\theta_3)-X(\theta_1)|^2}\right]$$
$$+ \frac{\lambda^2}{2}\left(\frac{1}{16\pi^2}\oint d\theta_1\,d\theta_2\,I(\theta_1,\theta_2)\right)^2 - \frac{\lambda^2}{64\pi^4}\int_{\theta_1>\theta_2>\theta_3>\theta_4} d\theta_1\,d\theta_2\,d\theta_3\,d\theta_4\,I(\theta_1,\theta_3)I(\theta_2,\theta_4).$$

The integrands proportional to $\lambda\epsilon^4$ and $\lambda^2\epsilon^4$ are made of few types of building blocks. Some basic properties allow to "regularize" these integrals by means of recurrence relations, which are solvable in closed form in the simplest cases, e.g.

$$A^p_{n_1,n_2} \equiv \oint d\theta_1 d\theta_2\frac{e^{in_1\theta_1+in_2\theta_2}}{\left(e^{i\theta_1}-e^{i\theta_2}\right)^p}, \qquad A^p_{n_1,n_2} = (-)^p A^p_{n_2,n_1},$$
$$A^0_{n_1,n_2} = 4\pi^2\delta_{0,n_1}\delta_{0,n_2}, \qquad \textstyle\sum_{k=0}^{p}\binom{p}{k}(-)^k A^p_{n_1+p-k,n_2+k} = A^0_{n_1,n_2}$$
$$\longrightarrow \quad A^2_{n_1,n_2} = \pi^2|n_1-n_2|\,\delta_{2,n_1+n_2} + C^{(1)}_{n_1+n_2},$$
$$A^4_{n_1,n_2} = \frac{\pi^2}{24}|n_1-n_2|\left[(n_1-n_2)^2-4\right]\delta_{4,n_1+n_2} + C^{(2)}_{n_1+n_2}(n_1-n_2)^2 + C^{(3)}_{n_1+n_2}.$$

Result

The order $\lambda\epsilon^4$ is quite constrained: monomials have the form $a_{p_1}a_{p_2}\bar{a}_{p_3}\bar{a}_{p_4}$ with $p_1+p_2-p_3-p_4 = 0$ and the spectral parameter cancels completely.

$$\langle\mathcal{W}\rangle^{\text{1-loop}}_{\epsilon^4} = \lambda\epsilon^4\sum_{p=0}^{\infty}\frac{2\left(17p^4+136p^3+412p^2+560p+291\right)|a_p|^4}{3(p+1)^3(p+2)^3(p+3)^3(2p+3)(2p+5)} + \lambda\epsilon^4\sum_{p=0}^{\infty}\sum_{q=0}^{p-1}\left(\frac{8|a_p|^2|a_q|^2}{3(p+1)^2(p+2)^2(p+3)^2(q+1)(q+2)(q+3)}\right.$$
$$\times\frac{1}{(p+q+3)(p+q+4)(p+q+5)}(p^5+9p^4q+25p^3q^2+7p^2q^3-6pq^4-2q^5+28p^4+172p^3q+192p^2q^2-20pq^3-32q^4+280p^3$$
$$+908p^2q+329pq^2-149q^3+1160p^2+1684pq-76q^2+2023p+799q+1164) + \frac{4\bar{a}_{2p-q}\bar{a}_q a_p^2 + a_{2p-q}a_q\bar{a}_p^2}{3(p+1)^2(p+2)^2(p+3)^2(2p+3)(2p+5)}$$
$$\left.\times\frac{72p^4-71p^3q+16p^2q^2+434p^3-362p^2q+64pq^2+937p^2-585pq+60q^2+866p-306q+291}{(2p-q+1)(2p-q+2)(2p-q+3)}\right) + \lambda\epsilon^4\sum_{p=0}^{\infty}\sum_{q=0}^{p-1}\sum_{r=0}^{q-1}\frac{1}{3(p+1)(p+2)}$$
$$\times\frac{8a_{p+q-r}a_r\bar{a}_p\bar{a}_q + a_p a_q\bar{a}_{p+q-r}\bar{a}_r}{(p+3)(q+1)(q+2)(q+3)(p+q+3)(p+q+4)(p+q+5)(p+q-r+1)(p+q-r+2)(p+q-r+3)}(p^5+16p^4q-7p^4r$$
$$+55p^3q^2-34p^3qr+4p^3r^2+55p^2q^3-60p^2q^2r+12p^2qr^2+16pq^4-34pq^3r+12pq^2r^2+q^5+4q^3r^2-7q^4r+28p^4+280p^3q-108p^3r$$
$$+540p^2q^2-396p^2qr+48p^2r^2+280pq^3-396pq^2r+96pqr^2+28q^4+48q^2r^2-108q^3r+280p^3+1525p^2q-617p^2r+1525pq^2-1384pqr$$
$$+188pr^2+280q^3-617q^2r+188qr^2+1160p^2+3160pq-1476pr+1160q^2-1476qr+240r^2+2023p+2023q-1224r+1164)$$

An analogue expression for $\langle\mathcal{W}\rangle^{\text{2-loop}}_{\epsilon^4}$ is not available in closed form, but recurrence relations calculate it once p_1, p_2, p_3, p_4 are given. So far we have scanned the intervals

$$0 \le p_4 \le 5, \quad p_4+1 \le p_3 \le p_4+6, \quad p_3+1 \le p_2 \le p_3+6, \quad p_2+1 \le p_1 \le p_2+6$$

and found that only monomials of the form $a_{p_1}a_{p_2}\bar{a}_{p_3}\bar{a}_{p_4}$ can be non-zero, so the spectral parameter cancels.

Conclusion

We study small deformations of the 1/2-BPS circular Wilson loop: the ϵ^4- and ϵ^6-correction to the expectation value (first orders not predicted by localization) and their behavior under spectral deformations.

Spectral invariance of the ϵ^4-correction $\langle\mathcal{W}\rangle_{\epsilon^4}$ is proved analytically at 1 loop and it is supported by numerical evidence at 2 loops. Higher powers break the invariance. The recursive algorithm that computes the 1- and 2-loop expectation value, given the parametrization of the wavy circle in Fourier series, comes as a useful by-product.

Can one make a connection to operator insertions in the CFT_1 supported by the circular Wilson loop [5]?

Since $\langle\mathcal{W}\rangle_{\epsilon^4}$ is invariant at strong coupling [1], the invariance may hold at finite coupling. Does it come from integrability?

$\langle\mathcal{W}\rangle_{\epsilon^2}$ defines the Bremsstrahlung function. What is the interpretation of $\langle\mathcal{W}\rangle_{\epsilon^4}$ in the language of accelerated quarks?

References

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