TMA373 Notes (Expanded Version with Dark Green Additions)

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1 Terminology

Goal: Present/recall notions and problems we shall consider in the lecture.

- A **differential equation (DE)** is an equation that relates an unknown function (or more) and its derivative(s).
- An **ordinary differential equation (ODE)** is a DE, where the unknown function depends only on one variable (e.g., $y(x)$ or $x(t)$).
  - **Linearity:** An ODE is linear if it can be written in the form

    $$a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t),$$

    where $a_i(t)$ are given functions and $f(t)$ is the source/forcing term. Otherwise, the ODE is *nonlinear*.
- To determine a unique solution to an ODE, one needs additional conditions:
  - An **initial value problem (IVP)** consists of an ODE with an initial condition. For example:

    $$\frac{d}{dt}P(t) = \lambda P(t),$$
    $$P(t_0) = P_0.$$

    Here, P_0 is the initial population size, and $P(t)$ describes the population at time t. The existence and uniqueness theorem (e.g., Picard–Lindelöf) provides criteria under which there is a unique solution.
  - A **boundary value problem (BVP)** consists of an ODE with boundary conditions. For example:

    $$-u''(x) + 4u(x) = \cos(x), \quad x \in (0,1),$$
    $$u(0) = 0, \quad u(1) = 5.$$

Here, the solution $u(x)$ is specified at the boundaries $x = 0$ and $x = 1$. BVPs often arise in steady-state heat conduction, static deflection of beams, and other equilibrium problems.

- A **partial differential equation (PDE)** is a DE where the unknown function depends on two or more variables (e.g., $u(x, y)$ or $u(t, x, y, z)$).

Examples of PDEs:

- **Laplace's equation:**
$$\Delta u = 0,$$
where Δ is the Laplace operator, defined as:
$$\Delta u(x) = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}.$$
In 2D, this reads $u_{xx} + u_{yy} = 0$. This equation is used to describe steady-state heat distribution, gravitational/electrostatic potential fields, etc.

- **Heat equation:**
$$u_t - \Delta u = f.$$
In 1D, this becomes $u_t - u_{xx} = f(x, t)$, where $u(x, t)$ describes temperature over time and space. It models diffusion processes and is parabolic in nature.

- **Wave equation:**
$$u_{tt} - \Delta u = g.$$
In 1D, this becomes $u_{tt} - u_{xx} = g(x, t)$, where $u(x, t)$ describes the motion of a string. It is hyperbolic in nature and preserves energy.

Conditions for PDEs: For the 1D heat equation on $[0, 1]$:

$$\begin{aligned} u_t(x,t) - u_{xx}(x,t) &= f(x,t), \quad x \in (0,1),\ t \in (0,T], \\ u(0,t) = 0, \quad u(1,t) &= \sin(t), \quad t \in (0,T], \\ u(x,0) &= 3x, \quad x \in (0,1). \end{aligned}$$

Here, $u(x, 0)$ specifies the initial temperature distribution, while $u(0, t)$ and $u(1, t)$ are boundary conditions.

Classification of Linear Second-Order PDEs:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

where $A, B, C, D, E, F, G \in \mathbb{R}$.

The classification depends on the discriminant $d = B^2 - 4AC$:

- Elliptic if $d < 0$ (e.g., Laplace equation).
- Parabolic if $d = 0$ (e.g., heat equation).

- Hyperbolic if $d > 0$ (e.g., wave equation).

This classification influences the type of boundary/initial data needed and the nature of solution propagation.

Applications:

- ODEs appear in thermodynamics, classical mechanics, medicine (e.g., pharmacokinetics), population dynamics, electrical engineering (e.g., circuit analysis), finance (e.g., interest rate models), etc.
- PDEs appear in fluid dynamics (Navier–Stokes), quantum mechanics (Schrödinger equation), electrodynamics (Maxwell's equations), elasticity (stress-strain relationships), and more.

Further Definitions and Explanations

- **General solution vs. particular solution:** For an ODE, the *general solution* includes constants of integration, while a *particular solution* is any one solution that satisfies the non-homogeneous part. When boundary or initial conditions are applied, one obtains a unique *particular* solution within the family of general solutions.
- **Well-posedness (Hadamard):** A problem is well-posed if (1) a solution exists, (2) the solution is unique, and (3) the solution depends continuously on the input data (the problem is stable). PDEs and ODEs are typically analyzed with respect to well-posedness before numerical methods are considered.

2 Mathematical Tools

Goal: Introduce some spaces of functions and several mathematical tools and results. This will help us study and solve (numerically) differential equations in the next chapters.

- A set V is called a **vector space (VS)** if, for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$, the following properties hold:
  1. Associativity: $(u + v) + w = u + (v + w)$.
  2. Commutativity: $u + v = v + u$.
  3. Existence of zero: $\exists\, 0 \in V$ such that $u + 0 = u$.
  4. Existence of inverse: $\exists\, (-u) \in V$ such that $u + (-u) = 0$.
  5. Scalar multiplication: $\alpha(\beta u) = (\alpha\beta)u$.
  6. Identity: $1u = u$ for all $u \in V$.

7. Distributivity: $\alpha(u+v) = \alpha u + \alpha v$ and $(\alpha+\beta)u = \alpha u + \beta u$.

- The space of all polynomials of degree $\leq n$, defined on $\mathbb{R}$, is denoted as:

$$P^{(n)}(\mathbb{R}) = \{a_0 + a_1 x + \cdots + a_n x^n : a_0, a_1, \ldots, a_n \in \mathbb{R}\}.$$

  This is a finite-dimensional vector space of dimension $n+1$.

- A **subspace** $U \subset V$ is defined as a subset where $\alpha u + \beta v \in U$ for all $u, v \in U$ and $\alpha, \beta \in \mathbb{R}$. A subspace itself must satisfy the vector space axioms (closed under addition and scalar multiplication).

- The **span** of vectors $v_1, v_2, \ldots, v_n \in V$ is:

$$\text{span}(v_1, \ldots, v_n) = \{a_1 v_1 + \cdots + a_n v_n : a_1, \ldots, a_n \in \mathbb{R}\}.$$

- A set $\{v_1, \ldots, v_n\}$ is **linearly independent** if $a_1 v_1 + \cdots + a_n v_n = 0$ implies $a_1 = \cdots = a_n = 0$.

- A **basis** of V is a linearly independent set that spans V. The **dimension** of V is the number of basis elements (finite or infinite).

- An **inner product** $(\cdot, \cdot) : V \times V \to \mathbb{R}$ satisfies:

  1. Symmetry: $(u, v) = (v, u)$ (or Hermitian symmetry in complex spaces).
  2. Linearity: $(u + \alpha v, w) = (u, w) + \alpha(v, w)$.
  3. Positivity: $(u, u) \geq 0$.
  4. $(u, u) = 0$ if and only if $u = 0$.

  Such a structure allows us to define norms and measure angles/distances in V.

- Examples of spaces:

  – **Square integrable functions:** $L^2([a, b])$ with inner product:

$$(f, g)_{L^2} = \int_a^b f(x)\, g(x)\, dx.$$

    This is one of the Lebesgue function spaces L^p; L^2 spaces are particularly important for PDE theory.

  – **Continuous functions:** $C^k(\Omega)$ consists of k-times continuously differentiable functions on Ω.

  – **Sobolev spaces:** $H^k(\Omega) = \{f \in L^2(\Omega) : D^\alpha f \in L^2(\Omega) \text{ for all } |\alpha| \leq k\}$. These spaces are crucial in modern PDE theory, providing a framework to handle weak derivatives and weaker continuity requirements.

- **Key inequalities:**

- Cauchy–Schwarz: $|(u, v)| \leq \|u\|\|v\|$.
- Triangle: $\|u + v\| \leq \|u\| + \|v\|$.
- Minkowski: $\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$.
- Hölder: $\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}$, where $\frac{1}{p} + \frac{1}{q} = 1$.
- Poincaré/Friedrichs inequalities: Provide bounds for functions in terms of their derivatives if certain boundary conditions are satisfied. Essential in PDE analysis.

- **Lax–Milgram Theorem:** For a Hilbert space H, and a bounded, coercive bilinear form $a(\cdot, \cdot)$, and linear functional ℓ, there exists a unique $u \in H$ such that:
$$a(u, v) = \ell(v) \quad \forall v \in H.$$
This is the cornerstone of the variational approach to solving PDEs, guaranteeing unique solutions to certain classes of PDEs.

Applications: Sobolev spaces are essential in theoretical analysis of PDEs, variational problems, and numerical methods (finite element methods, for instance). They also play a role in optimization, control problems, and in establishing regularity of solutions to PDEs.

Further Explanations

- **Hilbert spaces:** Complete inner product spaces. In these spaces, every Cauchy sequence converges, which is critical for many proofs of existence and stability (e.g., using Lax–Milgram).
- **Reflexive Banach spaces:** Sobolev spaces $H^k(\Omega)$ are Hilbert spaces and thus reflexive, meaning bounded linear functionals can be represented in an associated dual space. This underpins many PDE existence theorems (e.g., via Galerkin or monotone operator methods).

3 Interpolation and Numerical Integration

Goal: Interpolation aims to pass a simple function (e.g., a polynomial) through a given set of data points. Numerical integration provides numerical approximations for integrals of the form:
$$\int_a^b f(x)\, dx.$$

- Let q be a positive integer. Consider an interval $[a, b]$ and a grid of $q + 1$ distinct points $x_0 = a < x_1 < \cdots < x_q = b$. The **Lagrange polynomials** are defined as:

$$\lambda_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{q} \frac{x - x_j}{x_i - x_j}, \quad \text{for } i = 0, 1, \ldots, q.$$

The space of polynomials of degree at most q, denoted $P^{(q)}([a, b])$, is spanned by these polynomials:

$$P^{(q)}([a, b]) = \text{span}(\lambda_0(x), \lambda_1(x), \ldots, \lambda_q(x)).$$

- For a continuous function $f : [a, b] \to \mathbb{R}$ and $q + 1$ interpolation points $(x_j, f(x_j))_{j=0}^{q}$, the **Lagrange interpolant** $\pi_q f \in P^{(q)}([a, b])$ satisfies:

$$\pi_q f(x_j) = f(x_j), \quad j = 0, 1, \ldots, q.$$

Interpolation polynomials are used in numerical methods, such as finite elements, or simply for approximating data.

- **Piecewise Linear Interpolation:** Consider a uniform partition of $[a, b]$, $\tau_h : x_0 = a < x_1 < \cdots < x_{m+1} = b$, with mesh size $h_j = x_j - x_{j-1}$. Define **hat functions** $\{\phi_j\}_{j=0}^{m+1}$ as:

$$\phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h_j}, & x_{j-1} \leq x \leq x_j, \\ \frac{x_{j+1} - x}{h_{j+1}}, & x_j \leq x \leq x_{j+1}, \\ 0, & \text{otherwise.} \end{cases}$$

The space of continuous piecewise linear functions, V_h, is spanned by these hat functions:

$$V_h = \text{span}(\phi_0, \phi_1, \ldots, \phi_{m+1}).$$

The piecewise linear interpolant of f is given by:

$$\pi_h f(x) = \sum_{j=0}^{m+1} f(x_j)\phi_j(x), \quad x \in [a, b].$$

- **Error Bounds:** For $f \in C^2([a, b])$, and a uniform partition with constant mesh h:

$$\begin{aligned} \|\pi_h f - f\|_{L^p} &\leq Ch^2 \|f''\|_{L^p}, \\ \|\pi_h f - f\|_{L^p} &\leq Ch \|f'\|_{L^p}, \\ \|(\pi_h f)' - f'\|_{L^p} &\leq Ch \|f''\|_{L^p}, \end{aligned}$$

for $p = 1, 2, \infty$. For non-uniform partitions, similar bounds hold using the mesh function $h(x)$, though constants may depend on the partition's shape regularity.

- **Quadrature Rules:** Classical methods for approximating $\int_a^b f(x)\,dx$ include:
  - **Midpoint Rule:**
  $$\int_a^b f(x)\,dx \approx (b-a) f\left(\frac{a+b}{2}\right).$$
  - **Trapezoidal Rule:**
  $$\int_a^b f(x)\,dx \approx \frac{b-a}{2}[f(a)+f(b)].$$
  - **Simpson's Rule:**
  $$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\Big[f(a)+4f\Big(\tfrac{a+b}{2}\Big)+f(b)\Big].$$

  For a uniform partition $a = x_0 < x_1 < \cdots < x_N = b$, we apply these rules to each subinterval and sum:
  $$\int_a^b f(x)\,dx \approx \sum_{j=0}^{N-1} QF(x_j, x_{j+1}, f),$$
  where QF denotes the quadrature formula applied to each subinterval.

Applications:

- Interpolation is used to approximate function values at unsampled points (e.g., in data fitting, curve fitting, or computer graphics).
- Quadrature formulas are essential for solving integral equations, numerical PDE methods (e.g., assembling finite element matrices), and in any scenario where exact integration is cumbersome.

Further Explanations

- **Gaussian Quadrature:** Uses optimally chosen integration points (Gauss points) for polynomials up to degree $2n-1$ exactly with n points in $[a, b]$. Leads to very efficient integration if f is smooth.
- **Adaptive Quadrature:** Refinement of subintervals where f varies rapidly. Often based on estimating local errors.

4 Numerical Methods for IVP

Goal: Present basic numerical methods for the initial value problem (IVP):

$$\begin{cases} \dot{y}(t) = f(y(t)), & t \in (0, T], \\ y(0) = y_0, \end{cases}$$

where $T > 0$, $y_0 \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ are given. The derivative $\dot{y}(t)$ denotes $\frac{d}{dt}y(t)$. The methods described can be adapted for $f(t, y)$, $y(t_0)$, or vector-valued problems.

- **Variation of Constants (VOC) Formula:**

$$\dot{u}(t) + a(t)u(t) = f(t), \quad t \in (0, T],$$
$$u(0) = u_0,$$

  where f, a, and u_0 are given, $a(t) \geq 0$ is bounded, and f, a are continuous. The solution is given by:

$$u(t) = u_0 e^{-A(t)} + \int_0^t e^{-(A(t)-A(s))} f(s)\, ds,$$

  where $A(t) = \int_0^t a(s)\, ds$. This formula generalizes to second-order problems, systems of IVPs, PDEs, and nonlinear problems.

- **Numerical Time Integrators:** Consider the IVP above with a time step $k = \frac{T}{N}$ and a time grid $0 = t_0 < t_1 < \cdots < t_N = T$. Define numerical methods starting with $y_0 = y(0)$:

  - **Forward (Explicit) Euler Scheme:**

$$y_{n+1} = y_n + k\, f(y_n).$$

    Simple to implement but conditionally stable. Small time steps may be required (CFL condition).

  - **Backward (Implicit) Euler Scheme:**

$$y_{n+1} = y_n + k\, f(y_{n+1}).$$

    Requires solving for y_{n+1} implicitly (e.g., via Newton's method). This scheme is A-stable, good for stiff problems.

  - **Crank–Nicolson Scheme:**

$$y_{n+1} = y_n + \frac{k}{2}\big[f(y_n) + f(y_{n+1})\big].$$

    Second-order accurate, implicit, and often used for diffusion/heat equations in PDE contexts.

These methods provide approximations $y_n \approx y(t_n)$.

Applications: IVPs are ubiquitous in science and engineering (e.g., motion of particles under forces, chemical kinetics). Since many IVPs cannot be solved exactly, numerical methods like Euler's schemes, Runge–Kutta, and multistep methods are widely used.

Further Explanations

- **Local truncation error (LTE)**: Measures the error introduced in a single step of the method.
- **Stability regions**: For linear test equations, one checks whether the numerical solution remains bounded. Explicit methods have limited stability regions, while implicit methods often have large (or infinite) stability regions.
- **Higher-order methods**: Runge–Kutta and linear multistep methods can achieve higher accuracy per step at the expense of more function evaluations or more complex formulations.

5 FEM for Two-Point BVP

Goal: Present and analyze the Finite Element Method (FEM) for classical two-point boundary value problems (BVPs).

- **Model Problem:** Consider the BVP:

$$\begin{cases} -(a(x)u'(x))' = f(x), \quad x \in (0,1), \\ u(0) = 0, \quad u(1) = 0, \end{cases}$$

  where f and a are given (e.g., $f \in L^2(0,1)$, and $a(x) \geq \alpha_0 > 0$ is continuous or piecewise continuous). This could model steady-state heat conduction with variable conductivity $a(x)$.

- **Variational Formulation (VF):**

  Find $u \in H_0^1(0,1)$ such that $\displaystyle\int_0^1 a(x)\, u'(x)\, v'(x)\, dx = \int_0^1 f(x)\, v(x)\, dx \quad \forall v \in H_0^1(0,1).$

  We use integration by parts and the fact that $v(0) = v(1) = 0$ to move derivatives off of u onto the test function v, forming this weak/variational formulation.

- **Finite Element Problem (FE):** Define $V_h^0 \subset H_0^1(0,1)$. Solve:

  Find $u_h \in V_h^0$ such that $\displaystyle\int_0^1 a(x)u_h'(x)v_h'(x)\, dx = \int_0^1 f(x)v_h(x)\, dx \quad \forall v_h \in V_h^0.$

This leads to the linear system $S\zeta = b$, where S is the **stiffness matrix** and b is the **load vector**:

$$s_{ij} = \int_0^1 a(x)\, \phi_i'(x)\, \phi_j'(x)\, dx, \quad b_i = \int_0^1 f(x)\, \phi_i(x)\, dx.$$

The unknown vector ζ contains the nodal values $u_h(x_j)$ for interior nodes.

- **Error Estimates:**
  - *A priori:*
    $$\|u - u_h\|_{H^1(0,1)} \le Ch\|u\|_{H^2(0,1)},$$
    under suitable regularity ($u \in H^2(0,1)$).
  - *A posteriori:*
    $$\|u - u_h\|_{H^1(0,1)} \le C \left(\int_0^1 \frac{h^2(x)}{a(x)}\, R^2(u_h(x))\, dx \right)^{1/2},$$
    where $R(u_h)$ is the residual. Used in adaptive methods.
- **Adaptivity:** Use a posteriori error estimates to locally refine the mesh and improve numerical approximations. The goal is to place more degrees of freedom in regions with steep solution gradients or near singularities.
- **Example BVPs:**
  - Homogeneous Dirichlet BC: $u(0) = 0,\ u(1) = 0$.
  - Non-homogeneous Dirichlet BC: $u(0) = \alpha,\ u(1) = \beta$. (Homogenize BCs via transformations.)
  - Mixed BC: $u(0) = 0,\ u'(1) = \beta$. (In the VF, boundary terms adjust accordingly.)

Applications: FEM is widely used in engineering and physics for solving complex boundary value problems, such as mechanical design, structural analysis (beams, rods), and 1D flow or diffusion problems.

Further Explanations

- **Energy norm:** $\|v\|_E^2 = \int_0^1 a(x)\, |v'(x)|^2\, dx$. Minimizing the energy norm error is equivalent to solving the variational problem.
- **Condition number of stiffness matrix:** The matrix S can become ill-conditioned if h is very small or if $a(x)$ varies significantly. Preconditioning is often necessary in large-scale problems.

6 The Heat Equation in 1D

Goal: Briefly study the exact solution to some heat equations and present a numerical discretization.

- **Model Problem:** Consider the heat equation:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t), & 0 < x < 1,\ 0 < t \le T, \\ u(0,t) = 0, \quad u_x(1,t) = 0, & 0 < t \le T, \\ u(x,0) = u_0(x), & 0 < x < 1, \end{cases}$$

  The boundary conditions here are a mix of Dirichlet at $x = 0$ and Neumann at $x = 1$.

- **Stability Estimates:**

$$\|u(\cdot,t)\|_{L^2(0,1)} \le \|u_0\|_{L^2(0,1)} + \int_0^t \|f(\cdot,s)\|_{L^2(0,1)}\,ds,$$
$$\|u_x(\cdot,t)\|^2_{L^2(0,1)} \le \|u_0'\|^2_{L^2(0,1)} + \int_0^t \|f(\cdot,s)\|^2_{L^2(0,1)}\,ds.$$

  These estimates show that the solution does not blow up in finite time if data is bounded. For $f = 0$, solutions often exhibit exponential decay of higher modes, reflecting diffusive damping.

- **Variational Formulation (VF):**

$$(u_t(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v_x)_{L^2} = (f(\cdot,t), v)_{L^2} \quad \forall v \in H^1(0,1),$$

  with $u(\cdot,0) = u_0(\cdot)$. The boundary condition $u_x(1,t) = 0$ modifies the boundary integral terms accordingly (Neumann boundary term).

- **Finite Element Problem (FE):**

$$(u_{h,t}(\cdot,t), v_h)_{L^2} + (u_{h,x}(\cdot,t), v_{h,x})_{L^2} = (f(\cdot,t), v_h)_{L^2} \quad \forall v_h \in V_h^0,$$

  with $u_h(x,0) = \pi_h u_0(x)$. This leads to a system of ODEs in time for the nodal values.

- **System of ODEs:** Let $u_h(x,t) = \sum_{j=1}^m \zeta_j(t)\phi_j(x)$. Then

$$M\dot{\zeta}(t) + S\zeta(t) = F(t),$$

  where M is the mass matrix, S the stiffness matrix, and $F(t)$ is the load vector. We then discretize in time (e.g., backward Euler or Crank–Nicolson) to get a fully discrete method.

- **Error Analysis:**

$$\|u(\cdot,t) - u_h(\cdot,t)\|_{L^2} \le Ch^2,$$
$$\|u(\cdot,t_n) - u_h^n\|_{L^2} \le C_1h^2 + C_2k.$$

The heat equation's smoothing properties often allow higher spatial accuracy once $t > 0$.

Applications: The heat equation models heat transfer, diffusion processes in chemistry, population dynamics (diffusion of species), and more. Its numerical solutions are also central in large-scale PDE simulations (e.g., industrial processes).

Further Explanations

- **Stiffness in time**: Parabolic problems often require small time steps if treated explicitly (due to diffusive CFL constraints). Implicit schemes allow larger time steps but require solving linear systems.
- **Thermal conductivity**: More complex models might have $-(k(x)u_x)_x$ in place of $-u_{xx}$, similar to $a(x)$ in the elliptic case.

7 The Wave Equation in 1D

Goal: Briefly study the exact solution of the wave equation and present a numerical discretization of this PDE.

- **Model Problem:** Consider the inhomogeneous wave equation with homogeneous Dirichlet boundary conditions:

  $$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = f(x,t), & 0 < x < 1,\ 0 < t \leq T, \\ u(0,t) = 0, \quad u(1,t) = 0, & 0 < t \leq T, \\ u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), & 0 < x < 1, \end{cases}$$

  where u_0, v_0, and f are given functions.

- **Energy Conservation (Homogeneous Case):** For $f \equiv 0$, the wave equation satisfies:

  $$\frac{1}{2}\|u_t(\cdot,t)\|_{L^2}^2 + \frac{1}{2}\|u_x(\cdot,t)\|_{L^2}^2 = \frac{1}{2}\|v_0\|_{L^2}^2 + \frac{1}{2}\|u_0'\|_{L^2}^2, \quad \forall t \in [0,T].$$

  This implies the total mechanical energy (kinetic + potential) is conserved over time.

- **Variational Formulation (VF):**

  $$(u_{tt}(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v_x)_{L^2} = (f(\cdot,t), v)_{L^2} \quad \forall v \in H_0^1(0,1).$$

  with initial conditions $u(\cdot,0) = u_0$, $u_t(\cdot,0) = v_0$.

- **Finite Element Problem (FE):** Define $V_h^0 \subset H_0^1(0,1)$. The FE problem reads:

$$(u_{h,tt}(\cdot,t), v_h)_{L^2} + (u_{h,x}(\cdot,t), v_{h,x})_{L^2} = (f(\cdot,t), v_h)_{L^2} \quad \forall v_h \in V_h^0,$$

  with initial conditions $u_h(\cdot,0) = \pi_h u_0(x)$ and $u_{h,t}(\cdot,0) = \pi_h v_0(x)$.

- **System of ODEs:**

$$M\ddot{\zeta}(t) + S\zeta(t) = F(t),$$

  where M (mass matrix), S (stiffness matrix), and $F(t)$ (load vector) are given by integrals of basis functions.

- **Time Discretization:** One may use Crank–Nicolson or other energy-conserving methods (like the leapfrog scheme in FD methods). The discrete solutions approximate $u_h^n \approx u(\cdot, t_n)$.

Applications: The wave equation models vibrations in strings, membranes, structural components, and acoustic pressure waves, among others.

Further Explanations

- **Stability in wave equations**: The solution is typically stable if boundary conditions are well-posed (e.g., Dirichlet or Neumann). Numerical methods must preserve energy or dissipate it appropriately when damping is present.
- **Courant condition** in FD: For explicit FD methods of the wave equation, a strict k/h constraint is needed. In FEM with certain time integrators, one gets analogous stability conditions.

8 On Our Way to FEM in 2D

Goal: Extend integration by parts, piecewise linear functions, and linear interpolation to 2D. Prepare for finite element (FE) discretization of PDEs in higher dimensions.

- **Green's Formula:** A generalization of integration by parts in 2D (or higher):

$$\int_\Omega \Delta u\, v\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n}\, v\, ds - \int_\Omega \nabla u \cdot \nabla v\, dx,$$

  where $\frac{\partial u}{\partial n} = \mathbf{n} \cdot \nabla u$ is the outward normal derivative on $\partial\Omega$. This is crucial for deriving weak formulations of 2D PDEs.

- **Poisson's Equation:**

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

The variational formulation reads:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f \, v \, dx \quad \forall v \in H_0^1(\Omega).$$

- **Triangulation of Ω:** A mesh $\mathcal{T}_h$ of Ω consists of triangles K such that $\Omega = \bigcup_{K \in \mathcal{T}_h} K$. Elements must satisfy shape-regularity conditions (to avoid extremely skinny or degenerate triangles).
- **Piecewise Linear Functions:** On each triangle K, consider $P_1(K)$, the space of linear polynomials in x and y. Nodal basis functions are constructed by mapping from a reference element.
- **FE Spaces:** Define $V_h = \{v \in C^0(\Omega) \mid v|_K \in P_1(K) \ \forall K \in \mathcal{T}_h\}$. For homogeneous Dirichlet, impose $v|_{\partial\Omega} = 0$.
- **Interpolation in 2D:**

  $$\pi_h f = \sum_{j=1}^{n_p} f(N_j)\phi_j,$$

  where N_j are the mesh nodes and ϕ_j the corresponding 2D hat functions. Error estimates in H^1 and L^2 norms are extensions of the 1D case.
- **L^2-Projection:** For $f \in L^2(\Omega)$, the L^2-projection $P_h f \in V_h$ satisfies:

  $$\int_\Omega (f - P_h f) v \, dx = 0 \quad \forall v \in V_h.$$

  One can prove $\|P_h f - f\|_{L^2(\Omega)} \le C h^r \|f\|_{H^r(\Omega)}$.

Applications: Poisson's equation in 2D or 3D arises in electrostatics, gravitation, fluid pressure calculations, and potential flows.

Further Explanations

- **Reference triangle**: Typically with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Mappings from a generic physical element to the reference element unify the definition of basis functions.
- **Affine transformations**: Used to map any triangle K to the reference triangle. Jacobian of this map scales integrals over K.

9 FEM for Poisson's Equation in 2D

Goal: Derive FEM for Poisson's equation and provide error estimates.

- **Model Problem:**
$$\begin{cases} -\Delta u = f, & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$
with $f : \Omega \to \mathbb{R}$ given.
- **Variational Formulation (VF):**
$$(\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega).$$
- **Finite Element Space:**
$$V_h^0(\Omega) = \{v \in V_h \mid v|_{\partial\Omega} = 0\}.$$
- **Finite Element Problem (FE):**
$$(\nabla u_h, \nabla v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h^0(\Omega).$$
Becomes $S\zeta = b$ upon basis choice.
- **Error Estimates:**
$$\|u - u_h\|_{H^1(\Omega)} \le C\, h\, \|u\|_{H^2(\Omega)}, \quad \|u - u_h\|_{L^2(\Omega)} \le C\, h^2\, \|u\|_{H^2(\Omega)}.$$

Applications: Poisson's equation arises in electrostatic potential, gravitational potential, fluid flow under incompressibility constraints, and more.

Further Explanations

- **Assembly process**: Summation of element-level stiffness matrices and load vectors into global matrices. Efficient algorithms (e.g., using local numbering and global indexing) are key to large-scale FEM codes.
- **Boundary layers**: If f or boundary data changes rapidly near corners or boundaries, solutions can have boundary layers. Mesh refinement is typically needed there.

10 FEM for Heat Equations in Higher Dimensions

Goal: Study the exact solution of the heat equation (stability), derive a finite element (FE) discretization, and provide a semi-discrete error estimate for the FEM.

- **Model Problem:**

$$\begin{cases} u_t - \Delta u = f, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(\cdot, 0) = u_0, & \text{in } \Omega, \end{cases}$$

  $\Omega \subset \mathbb{R}^d$ with $d \geq 1$.

- **Stability Estimates:**

$$\|u(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} + \int_0^t \|f(s)\|_{L^2(\Omega)}\, ds,$$

$$\|\nabla u(\cdot, t)\|_{L^2(\Omega)}^2 \leq \|\nabla u_0\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla f(\cdot, s)\|_{L^2(\Omega)}^2\, ds.$$

- **Variational Formulation (VF):**

$$(u_t, v) + (\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega),$$

  with $u(\cdot, 0) = u_0$.

- **Finite Element Problem (FE):**

$$(u_{h,t}, v_h) + (\nabla u_h, \nabla v_h) = (f, v_h), \quad \forall v_h \in V_h^0(\Omega),$$

  leading to a system of ODEs $M\,\dot{\zeta}(t) + S\,\zeta(t) = F(t)$ with $\zeta(0)$ determined by u_0.

- **Error Estimates (Semi-discrete):**

$$\|u(\cdot, t) - u_h(\cdot, t)\|_{L^2(\Omega)} \leq C\, h^2 \Big(\|u_0\|_{H^2(\Omega)} + \int_0^t \|u_t(\cdot, s)\|_{H^2(\Omega)} ds \Big).$$

Applications: The heat equation in multiple dimensions is central to many fields, from geology (heat in the Earth's crust) to industrial engineering (cooling/heating processes).

Further Explanations

- **Parabolic regularization**: Solutions to the heat equation become smoother over time, assisting numerical methods in obtaining accurate solutions after an initial transient.
- **Non-homogeneous boundary conditions**: One must handle these carefully in the variational form, adding boundary integrals or adjusting solution spaces.

11 FEM for Wave Equations in Higher Dimensions

Goal: Study the exact solution of the wave equation (conservation of energy), derive a finite element (FE) discretization, and provide a semi-discrete error estimate for the FEM.

- **Model Problem:**
$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \partial\Omega, \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = v_0, \end{cases}$$
for $\Omega \subset \mathbb{R}^d$.

- **Energy Conservation (Homogeneous case):**
$$\frac{1}{2}\|\nabla u(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_t(\cdot,t)\|^2_{L^2(\Omega)} = \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_0\|^2_{L^2(\Omega)}.$$

- **Variational Formulation (VF):**
$$(u_{tt}, v) + (\nabla u, \nabla v) = (f, v), \quad \forall v \in H^1_0(\Omega).$$

- **Finite Element Problem (FE):**
$$(u_{h,tt}, v_h) + (\nabla u_h, \nabla v_h) = (f, v_h), \quad \forall v_h \in V_h^0(\Omega).$$
with $u_h(\cdot,0) = \pi_h u_0$, $u_{h,t}(\cdot,0) = \pi_h v_0$.

- **System of ODEs:**
$$M\,\ddot{\zeta}(t) + S\,\zeta(t) = F(t).$$
We then discretize in time.

- **Error Estimates (Semi-discrete):**
$$\|u_h(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} \le Ch^2\big(\|u_0\|_{H^2(\Omega)} + \|v_0\|_{H^1(\Omega)}\big),$$
plus time discretization terms if fully discretized.

Applications: Vibrations of membranes/plates in 2D, acoustic wave propagation in 3D, seismic wave modeling, etc.

Further Explanations

- **Explicit or implicit time stepping**: For wave equations, explicit methods (like leapfrog or central differences) are common but require a strict CFL condition. Implicit methods are unconditionally stable but require more computation per step.
- **Energy-preserving schemes**: Certain numerical schemes preserve a discrete energy, preventing non-physical growth or decay of numerical solutions.

12 The Finite Element Concept

Goal: Study the concept of the finite element.

- **Definition:** A finite element consists of the triplet (K, P, Σ):
  - $K \subset \mathbb{R}^d$ is a polygon or polyhedron (the element domain).
  - P is a finite-dimensional polynomial function space on K (e.g., $P_1(K)$ for linear, $P_2(K)$ for quadratic).
  - Σ is a P-unisolvent set of linear functionals on P.

  Being P-unisolvent means the only polynomial p for which $L_j(p) = 0$ for all j is $p = 0$. Typically, Σ are point evaluations at vertices, midpoints, etc.

- **Examples of Finite Elements:**
  - **1D Lagrange $P^{(k)}$ elements:**

    $$K = [x_i, x_{i+1}], \quad P = P^{(k)}(K), \quad \Sigma = \{\delta_{x_0}, \ldots, \delta_{x_k}\},$$

    where $\delta_{x_j}(f) = f(x_j)$.
  - **2D Linear Lagrange Element (Triangle)**:

    $$K = \triangle, \quad P = P_1(K), \quad \Sigma = \{\text{evaluation at vertices}\}.$$

    The shape functions are constructed so that $\phi_i(N_j) = \delta_{ij}$, where N_j are the triangle's vertices.

- **Higher Order Elements:** For Poisson's equation in 2D or 3D, using $P^{(p)}(K)$ polynomials and C^0 continuity can provide higher convergence rates. For instance,

  $$\|u - u_h\|_{H^1(\Omega)} \leq Ch^p |u|_{H^{p+1}(\Omega)}.$$

  This typically increases the complexity of basis functions but reduces the number of elements needed for a given accuracy.

- **Variational Crimes:** Errors introduced by numerical integration (approximating element integrals), approximate geometry (curved boundaries approximated by flat edges), or reduced integration schemes. These can alter the bilinear form $a(\cdot,\cdot)$ and the functional $\ell(\cdot)$, thus affecting convergence rates.

Applications: See previous examples. Finite elements are used throughout computational engineering, from structural mechanics (stress analysis) to electromagnetic waves.

Further Explanations

- **Isoparametric concept**: The same shape functions are used to approximate geometry and solution fields, allowing accurate representation of curved domains.
- **Mixed/hybrid elements**: For certain PDEs (e.g., Stokes flow), one needs vector-valued or mixed formulations with constraints. This leads to specialized finite element families (e.g., Raviart–Thomas, Taylor–Hood).

13 Further Topics

Goal: Analyze stability of time integrators and present finite difference (FD) methods for Poisson's equation in 2D.

- **Stability Analysis:**

$$\dot{y}(t) = \lambda\, y(t), \quad y(0) = y_0, \quad \lambda \in \mathbb{C}.$$

  A time integrator yields $y_{n+1} = R(\lambda k)\, y_n$. The *stability region* is

$$S = \{z \in \mathbb{C} \mid |R(z)| \le 1\}.$$

  For parabolic PDEs (heat equation), $\lambda < 0$ can be large in magnitude, requiring $k|\lambda|$ to lie within S for stability. Explicit Euler has a small region, forcing $k \sim O(h^2)$, while implicit methods have larger regions (A-stable for backward Euler).

- **Finite Difference for Derivatives:**

$$y'(x) \approx \frac{y(x+h) - y(x)}{h}, \quad y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}.$$

- **Finite Difference for Poisson's Equation in 2D:**

$$\Delta u = u_{xx} + u_{yy} = f(x,y), \quad (x,y) \in [0,1]^2.$$

With uniform grid $x_i = ih$, $y_j = jh$ ($h = 1/(n+1)$), the standard 5-point stencil:

$$\Delta_h u_{ij} = \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2}.$$

Setting $\Delta_h u_{ij} = f_{ij}$ for (i, j) in the interior yields a sparse linear system.

- **Convergence:**

$$\|u - u_h\|_\infty \le C\, h^2,$$

assuming enough smoothness for u.

Applications:

- **Stability analysis** ensures that numerical solutions do not blow up.
- **Finite differences** are simpler than FEM but less flexible for complex geometries.

Further Explanations

- **Spectral radius** of iteration matrices: Important in analyzing iterative solvers for $A\mathbf{u} = \mathbf{f}$.
- **Multigrid methods**: Extremely efficient for FD or FEM discretizations of Poisson-like equations.
- **High-dimensional PDEs**: FD quickly becomes intractable on large grids, driving interest in advanced methods or HPC techniques.