TMA947 Assignment Exercises 5

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October 2024

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1 HW 5.1

Problem Statement

Consider the optimization problem:

$$\text{minimize } f(x) + \nu\chi(x) \quad \text{subject to } x \in \mathbb{R}^n \quad \text{(5.3)-(5.4)}$$

The objective is to demonstrate that this is a **convex optimization problem**.

Given Conditions

- $f : \mathbb{R}^n \to \mathbb{R}$ is convex.
- $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, m$ are convex functions.
- $\psi(s) = s^2$ is a convex function on $\mathbb{R}$.

The penalty term $\chi(x)$ is defined as:

$$\chi(x) = \sum_{i=1}^{m} \psi\left(\max\{0, g_i(x)\}\right) = \sum_{i=1}^{m} \left(\max\{0, g_i(x)\}\right)^2$$

Proof of Convexity

1. **Convexity of** $\max\{0, g_i(x)\}$**:**

- The function $\max\{0, g_i(x)\}$ is the **positive part** of $g_i(x)$.
- Since $g_i(x)$ is convex, and the maximum of convex functions is convex, $\max\{0, g_i(x)\}$ is convex.

2. **Convexity of $\psi(\max\{0, g_i(x)\})$:**
   - The function $\psi(s) = s^2$ is convex and **non-decreasing** on $\mathbb{R}_+$.
   - The composition of a convex, non-decreasing function with a convex function preserves convexity.
   - Therefore, $\psi\left(\max\{0, g_i(x)\}\right) = \left(\max\{0, g_i(x)\}\right)^2$ is convex.
3. **Convexity of the Sum $\chi(x)$:**
   - $\chi(x)$ is the sum of convex functions $\left(\max\{0, g_i(x)\right)^2$ for $i = 1, \ldots, m$.
   - The sum of convex functions is convex.
   - Hence, $\chi(x)$ is convex.
4. **Convexity of the Objective Function $f(x) + \nu\chi(x)$:**
   - Both $f(x)$ and $\chi(x)$ are convex.
   - A positive linear combination of convex functions (with $\nu > 0$) is convex.
   - Therefore, $f(x) + \nu\chi(x)$ is convex.

Conclusion

Since the objective function $f(x) + \nu\chi(x)$ is convex and the feasible region $\mathbb{R}^n$ is a convex set, the problem

$$\text{minimize } f(x) + \nu\chi(x) \quad \text{subject to } x \in \mathbb{R}^n$$

is indeed a **convex optimization problem**.

Final Answer

Because f is convex, each $\max\{0, g_i(x)\}$ is convex, and squaring preserves convexity, the penalty $\chi(x)$ is convex. Thus, $f(x)+\nu\chi(x)$ is a convex function on $\mathbb{R}^n$, making problem (5.3)-(5.4) a convex optimization problem.

2 E 5.3

Problem Statement

Consider the following Linear Programming (LP) problem:

$$\text{Minimize} \quad 9x_1 + 3x_2 + 2x_3 + 2x_4$$
$$\text{Subject to} \quad \begin{cases} x_1 + x_2 + x_3 + x_4 \geq 1, \\ 3x_1 - x_2 + 2x_4 \geq 1, \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \leq 0. \end{cases}$$

1. Use LP duality and a graphical solution to obtain the optimal objective value z^*.
2. Use complementary slackness to obtain the optimal solution $\mathbf{x}^*$.

Solution

Part (a): Optimal Objective Value Using LP Duality and Graphical Solution

1. Primal Problem Reformulation The primal LP problem is given by:

$$\text{Minimize} \quad 9x_1 + 3x_2 + 2x_3 + 2x_4$$
$$\text{Subject to} \quad \begin{cases} x_1 + x_2 + x_3 + x_4 \geq 1, & (1) \\ 3x_1 - x_2 + 2x_4 \geq 1, & (2) \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \leq 0. & (3) \end{cases}$$

Variable Transformation To standardize the constraints, the inequality $x_4 \leq 0$ is addressed by introducing a substitution:

$$x_4' = -x_4 \quad \Rightarrow \quad x_4' \geq 0.$$

Substituting $x_4 = -x_4'$ into the primal problem, the reformulated problem becomes:

$$\text{Minimize} \quad 9x_1 + 3x_2 + 2x_3 - 2x_4'$$
$$\text{Subject to} \quad \begin{cases} x_1 + x_2 + x_3 - x_4' \geq 1, & (1') \\ 3x_1 - x_2 - 2x_4' \geq 1, & (2') \\ x_1,\ x_2,\ x_3,\ x_4' \geq 0. & (3') \end{cases}$$

2. Dual Problem Formulation The dual of a minimization problem with $\geq$ constraints is a maximization problem. Let y_1 and y_2 be the dual variables corresponding to constraints $(1')$ and $(2')$, respectively.

$$\text{Maximize} \quad y_1 + y_2$$
$$\text{Subject to} \quad \begin{cases} y_1 + 3y_2 \leq 9, & (a) \\ y_1 - y_2 \leq 3, & (b) \\ y_1 \leq 2, & (c) \\ y_1 + 2y_2 \geq 2, & (d) \\ y_1,\ y_2 \geq 0. & (e) \end{cases}$$

3. Graphical Solution of the Dual Problem To determine the optimal objective value z^*, the feasible region defined by the dual constraints is analyzed, and the point that maximizes $y_1 + y_2$ is identified.

Plotting the Constraints

1. $y_1 + 3y_2 \leq 9$
2. $y_1 - y_2 \leq 3$
3. $y_1 \leq 2$
4. $y_1 + 2y_2 \geq 2$
5. $y_1,\ y_2 \geq 0$

Identifying Intersection Points

- **Intersection of $y_1 = 2$ and $y_1 + 2y_2 = 2$:**

$$\begin{cases} y_1 = 2, \\ 2 + 2y_2 = 2 \quad \Rightarrow \quad y_2 = 0. \end{cases}$$

  Point: $(2, 0)$

- **Intersection of $y_1 = 2$ and $y_1 + 3y_2 = 9$:**

$$\begin{cases} y_1 = 2, \\ 2 + 3y_2 = 9 \quad \Rightarrow \quad y_2 = \frac{7}{3}. \end{cases}$$

  Point: $\left(2, \frac{7}{3}\right)$

- **Intersection of $y_1 = 0$ and $y_1 + 2y_2 = 2$:**

$$\begin{cases} y_1 = 0, \\ 0 + 2y_2 = 2 \quad \Rightarrow \quad y_2 = 1. \end{cases}$$

  Point: $(0, 1)$

Feasible Corner Points

The feasible region is bounded by the following corner points:

$$(0, 1),$$
$$(2, \frac{7}{3}),$$
$$(2, 0).$$

Evaluating the Objective Function at Each Corner Point

$$\begin{aligned}
\text{At } (0,1): &\quad y_1 + y_2 = 0 + 1 = 1, \\
\text{At } \left(2, \frac{7}{3}\right): &\quad y_1 + y_2 = 2 + \frac{7}{3} = \frac{13}{3} \approx 4.\overline{3}, \\
\text{At } (2,0): &\quad y_1 + y_2 = 2 + 0 = 2.
\end{aligned}$$

Optimal Dual Solution

The maximum value of $y_1 + y_2$ is $\frac{13}{3}$ at the point $\left(2, \frac{7}{3}\right)$.

Conclusion for Part (a)

By **LP duality**, the optimal objective value of the dual problem equals the optimal objective value of the primal problem. Therefore, the optimal objective value is:

$$\boxed{z^* = \frac{13}{3} \approx 4.\overline{3}}$$

Part (b): Optimal Primal Solution Using Complementary Slackness

1. Recap of Optimal Dual Variables

From Part (a), the optimal dual solution is:

$$\boxed{y^* = (y_1^*, y_2^*) = \left(2, \frac{7}{3}\right)}$$

2. Complementary Slackness Conditions

Complementary slackness provides conditions that relate the optimal solutions of the primal and dual problems. Specifically:

- For each primal constraint i, y_i^* multiplied by the slack of constraint i equals zero.
- For each dual constraint j, x_j^* multiplied by the slack of constraint j equals zero.

Mathematically, the conditions are:

$$\begin{aligned}
&y_1^* \cdot (x_1^* + x_2^* + x_3^* - x_4'^* - 1) = 0, && (1) \\
&y_2^* \cdot (3x_1^* - x_2^* - 2x_4'^* - 1) = 0, && (2) \\
&x_1^* \cdot (9 - (y_1^* + 3y_2^*)) = 0, && (a') \\
&x_2^* \cdot (3 - (y_1^* - y_2^*)) = 0, && (b') \\
&x_3^* \cdot (2 - y_1^*) = 0, && (c') \\
&x_4'^* \cdot (0 - (-2y_1^* - 2y_2^*)) = 0. && (d')
\end{aligned}$$

3. Applying Complementary Slackness

Dual Constraints Evaluation

Given $y^* = \left(2, \frac{7}{3}\right)$, evaluate each dual complementary slackness condition:

$$\begin{aligned}
&\text{Condition } (a'): \quad 9 - \left(2 + 3 \cdot \frac{7}{3}\right) = 9 - 9 = 0,\\
&\text{Condition } (b'): \quad 3 - \left(2 - \frac{7}{3}\right) = 3 - \left(\frac{6}{3} - \frac{7}{3}\right) = 3 + \frac{1}{3} = \frac{10}{3},\\
&\text{Condition } (c'): \quad 2 - 2 = 0,\\
&\text{Condition } (d'): \quad 0 - \left(-2 \cdot 2 - 2 \cdot \frac{7}{3}\right) = 0 - \left(-4 - \frac{14}{3}\right) = 4 + \frac{14}{3} = \frac{26}{3}.
\end{aligned}$$

Determining Primal Variables

Applying the complementary slackness conditions:

- **Condition (a'):**
$$x_1^* \cdot 0 = 0 \quad \Rightarrow \quad \text{No restriction on } x_1^*.$$
- **Condition (b'):**
$$x_2^* \cdot \frac{10}{3} = 0 \quad \Rightarrow \quad x_2^* = 0.$$
- **Condition (c'):**
$$x_3^* \cdot 0 = 0 \quad \Rightarrow \quad \text{No restriction on } x_3^*.$$
- **Condition (d'):**
$$x_4'^* \cdot \frac{26}{3} = 0 \quad \Rightarrow \quad x_4'^* = 0.$$

Primal Constraints from Complementary Slackness

Using conditions (1) and (2):

$$\begin{aligned}
&y_1^* \cdot (x_1^* + x_2^* + x_3^* - x_4'^* - 1) = 0 \quad \Rightarrow \quad 2(x_1^* + 0 + x_3^* - 0 - 1) = 0 \quad \Rightarrow \quad x_1^* + x_3^* = 1,\\
&y_2^* \cdot (3x_1^* - x_2^* - 2x_4'^* - 1) = 0 \quad \Rightarrow \quad \frac{7}{3}(3x_1^* - 0 - 0 - 1) = 0 \quad \Rightarrow \quad 3x_1^* - 1 = 0 \quad \Rightarrow \quad x_1^* = \frac{1}{3}.
\end{aligned}$$

Substituting $x_1^* = \frac{1}{3}$ into $x_1^* + x_3^* = 1$:

$$\frac{1}{3} + x_3^* = 1 \quad \Rightarrow \quad x_3^* = \frac{2}{3}.$$

4. Optimal Primal Solution

Combining the results:

$$\begin{aligned}
x_1^* &= \frac{1}{3}, \\
x_2^* &= 0, \\
x_3^* &= \frac{2}{3}, \\
x_4'^* &= 0 \quad \Rightarrow \quad x_4^* = -x_4'^* = 0.
\end{aligned}$$

Thus, the optimal primal solution is:

$$\boxed{\mathbf{x}^* = \left(\frac{1}{3},\ 0,\ \frac{2}{3},\ 0\right)}$$

5. Verification of Optimality

Primal Constraints Satisfaction

$$\begin{aligned}
&x_1^* + x_2^* + x_3^* + x_4 = \frac{1}{3} + 0 + \frac{2}{3} + 0 = 1 \geq 1 \quad \text{(Satisfied)}, \\
&3x_1^* - x_2^* + 2x_4 = 3 \cdot \frac{1}{3} - 0 + 0 = 1 \geq 1 \quad \text{(Satisfied)}, \\
&x_1^* \geq 0,\ x_2^* \geq 0,\ x_3^* \geq 0,\ x_4^* \leq 0 \quad \text{(All satisfied)}.
\end{aligned}$$

Primal Objective Value

$$\begin{aligned}
9x_1^* + 3x_2^* + 2x_3^* + 2x_4^* &= 9 \cdot \frac{1}{3} + 3 \cdot 0 + 2 \cdot \frac{2}{3} + 2 \cdot 0 \\
&= 3 + 0 + \frac{4}{3} + 0 \\
&= \frac{13}{3},
\end{aligned}$$

which matches the dual optimal value $z^* = \frac{13}{3}$, confirming strong duality and optimality.

Conclusion

1. **Optimal Objective Value:**

   Using LP duality and a graphical solution, the optimal objective value is:

   $$\boxed{z^* = \frac{13}{3} \approx 4.\overline{3}}$$

2. **Optimal Primal Solution:**

   Using complementary slackness, the optimal primal solution is:

   $$\boxed{\mathbf{x}^* = \left(\frac{1}{3},\ 0,\ \frac{2}{3},\ 0\right)}$$

3 E 5.4

Primal Problem

The primal relaxation of the standard linear programming (LP) problem is defined as follows:

$$\begin{aligned} \text{Minimize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{Subject to} \quad & \mathbf{a}_i^\top \mathbf{x} \geq b_i - v_i, \quad \forall i = 1, 2, \ldots, n \\ & \sum_{i=1}^{n} v_i \leq \epsilon \\ & \mathbf{x} \geq \mathbf{0}, \quad \mathbf{v} \geq \mathbf{0} \end{aligned}$$

where:

- $\mathbf{x}$ are the decision variables.
- v_i represent the allowed violations of the constraints.
- ϵ bounds the total sum of violations.

Dual Problem Formulation

Dual variables corresponding to each constraint in the primal are introduced.

Dual Variables

- Let $y_i \geq 0$ be the dual variable associated with the i-th constraint $\mathbf{a}_i^\top \mathbf{x} \geq b_i - v_i$.
- Let $z \geq 0$ be the dual variable associated with the constraint $\sum_{i=1}^{n} v_i \leq \epsilon$.

Dual Objective

The dual objective aims to maximize the weighted sum of the right-hand sides of the primal constraints minus the penalty for violations:

$$\text{Maximize} \quad \mathbf{b}^\top \mathbf{y} - \epsilon z$$

Dual Constraints

For each primal variable x_j, the dual constraint ensures that the weighted sum of coefficients does not exceed the corresponding cost in $\mathbf{c}$:

$$\sum_{i=1}^{n} a_{ij} y_i \leq c_j, \quad \forall j = 1, 2, \ldots, m$$

For each violation variable v_i, the dual constraint links y_i and z to ensure that the shadow price y_i does not exceed the penalty z:

$$y_i \leq z, \quad \forall i = 1, 2, \ldots, n$$

Complete Dual Formulation

Combining the above elements, the dual can be expressed as:

$$\begin{aligned} \text{Maximize} \quad & \mathbf{b}^\top \mathbf{y} - \epsilon z \\ \text{Subject to} \quad & A^\top \mathbf{y} \leq \mathbf{c} \\ & \mathbf{y} \leq z\mathbf{e} \\ & \mathbf{y} \geq \mathbf{0}, \quad z \geq 0 \end{aligned}$$

where:

- $A^\top$ is the transpose of the constraint matrix A.
- $\mathbf{e}$ is a vector of ones.

Interpretation of the Dual

The dual problem provides insights into the trade-offs between the objective and the constraints of the primal problem:

- **Dual Variables y_i:** Represent the *shadow prices* or *marginal values* of the original constraints. Specifically, y_i indicates how much the objective function would improve if the right-hand side b_i of the i-th constraint were relaxed by one unit.
- **Dual Variable z:** Represents the *cost* associated with allowing constraint violations. It quantifies the trade-off between strictly adhering to the original constraints and permitting some flexibility (violations) up to the total allowed violation ϵ.
- **Objective Function $\mathbf{b}^\top \mathbf{y} - \epsilon z$:** The dual seeks to maximize the total benefit derived from the constraints (weighted by y_i) while minimizing the penalty associated with allowing violations (weighted by z).
- **Constraints $A^\top \mathbf{y} \leq \mathbf{c}$ and $\mathbf{y} \leq z\mathbf{e}$:** Ensure that the allocation of dual resources (represented by y_i and z) does not exceed the costs specified by the objective coefficients $\mathbf{c}$ and that the shadow prices y_i do not surpass the penalty z.

Final Dual Formulation

$$\begin{aligned} \text{Maximize} \quad & \mathbf{b}^\top \mathbf{y} - \epsilon z \\ \text{Subject to} \quad & A^\top \mathbf{y} \leq \mathbf{c} \\ & y_i \leq z, \quad \forall i = 1, 2, \ldots, n \\ & \mathbf{y} \geq \mathbf{0}, \quad z \geq 0 \end{aligned}$$

Summary

- **Dual Variables:** y_i (shadow prices) and z (violation penalty).
- **Dual Objective:** Maximize the benefit from constraints minus the cost of violations.

- **Dual Constraints:** Ensure that the allocation of dual resources does not exceed costs and that shadow prices are bounded by the violation penalty.

This dual formulation captures the optimal balance between minimizing costs and allowing a controlled amount of constraint violations, providing valuable information on how sensitive the primal solution is to changes in the constraints and the permissible violations.

4 E 5.5

Problem Statement

Determine the marginal change in the optimal objective z with respect to the parameter a for the following linear program:

$$\begin{aligned} \text{Minimize} \quad & z = -2x_1 + x_2 \\ \text{Subject to} \quad & x_1 - 3x_2 \geq a, \\ & x_1 \geq 0, \\ & 0 \leq x_2 \leq 2. \end{aligned}$$

Solution

1. Standard Form Conversion

Assuming $a < 0$, the problem is converted to standard form by introducing slack variables s_1 and s_2:

$$\begin{aligned} \text{Minimize} \quad & z = -2x_1 + x_2 \\ \text{Subject to} \quad & -x_1 + 3x_2 - s_1 = -a, \\ & x_2 + s_2 = 2, \\ & x_1, x_2, s_1, s_2 \geq 0. \end{aligned}$$

2. Optimal Solution at $a = -3$

Given that x_1 and x_2 are basic variables at $a = -3$, the slack variables s_1 and s_2 are non-basic and set to zero:

$$-x_1 + 3x_2 = -a, \quad \text{since } s_1 = 0,$$
$$x_2 = 2, \quad \text{since } s_2 = 0.$$

Substituting $x_2 = 2$ into the first equation:

$$\begin{aligned} -x_1 + 3(2) &= -a \\ -x_1 + 6 &= -a \\ x_1 &= a + 6. \end{aligned}$$

At $a = -3$:

$$\begin{aligned} x_1 &= -3 + 6 = 3, \\ x_2 &= 2. \end{aligned}$$

The objective value is:

$$z = -2(3) + 2 = -6 + 2 = -4.$$

3. Calculating the Marginal Change $\frac{dz}{da}$

Express z in terms of a:

$$\begin{aligned} x_1 &= a + 6, \\ z &= -2x_1 + x_2 = -2(a + 6) + 2 = -2a - 12 + 2 = -2a - 10. \end{aligned}$$

Taking the derivative of z with respect to a:

$$\frac{dz}{da} = -2.$$

Conclusion

The marginal change in the optimal objective z with respect to a when a is varied from its current value of -3 is $\mathbf{-2}$.

Answer: The marginal change in the optimal objective is -2.