TMA947 Assignment Exercises 4

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Contents

1 HW 4.1

Problem Statement

Consider the linear programming problem:

$$\text{minimize } -x_2 \tag{4.1}$$

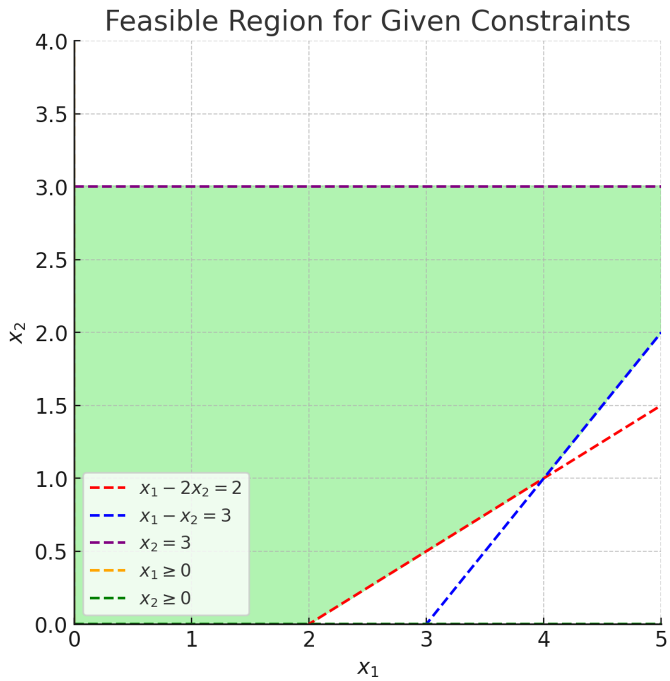
subject to

$$x_1 - 2x_2 \leq 2, \tag{4.2}$$

$$x_1 - x_2 \leq 3, \tag{4.3}$$

$$x_2 \leq 3, \tag{4.4}$$

$$x_1, x_2 \geq 0. \tag{4.5}$$



Interpretation

Minimizing $-x_2$ is equivalent to **maximizing** x_2. Therefore, the objective is to determine the maximum possible value of x_2 within the feasible region defined by the constraints.

Graphical Solution

1. Plot the Constraints

- **Constraint (4.2):** $x_1 - 2x_2 \leq 2$
  - When $x_2 = 0$: $x_1 \leq 2$
  - When $x_1 = 0$: $x_2 \geq -1$ (but $x_2 \geq 0$)
- **Constraint (4.3):** $x_1 - x_2 \leq 3$
  - When $x_2 = 0$: $x_1 \leq 3$
  - When $x_1 = 0$: $x_2 \geq -3$ (but $x_2 \geq 0$)
- **Constraint (4.4):** $x_2 \leq 3$
- **Non-negativity Constraints:** $x_1, x_2 \geq 0$

2. Identify Intersection Points (Vertices)

The feasible region is a polygon in the first quadrant bounded by the above constraints. The vertices (corner points) of this region are:

1. **Vertex A:** $(0, 0)$ — Intersection of $x_1 = 0$ and $x_2 = 0$
2. **Vertex B:** $(2, 0)$ — Intersection of $x_1 - 2x_2 = 2$ and $x_2 = 0$
3. **Vertex C:** $(4, 1)$ — Intersection of $x_1 - 2x_2 = 2$ and $x_1 - x_2 = 3$
4. **Vertex D:** $(6, 3)$ — Intersection of $x_1 - x_2 = 3$ and $x_2 = 3$
5. **Vertex E:** $(0, 3)$ — Intersection of $x_1 = 0$ and $x_2 = 3$

3. Evaluate the Objective Function at Each Vertex

Since the goal is to maximize x_2, the highest x_2 values within the feasible region are considered.

$$
\begin{aligned}
\text{Vertex A:} \quad & x_2 = 0 \\
\text{Vertex B:} \quad & x_2 = 0 \\
\text{Vertex C:} \quad & x_2 = 1 \\
\text{Vertex D:} \quad & x_2 = 3 \\
\text{Vertex E:} \quad & x_2 = 3
\end{aligned}
$$

The maximum x_2 value is **3**, achieved at both **Vertex D** $(6, 3)$ and **Vertex E** $(0, 3)$.

Optimal Solutions

1. Non-Uniqueness of the Optimal Solution

The optimal value $x_2 = 3$ is achieved along the entire line segment connecting **Vertex D** $(6, 3)$ and **Vertex E** $(0, 3)$. Therefore, **the optimal solution is not unique**. All points $(x_1, 3)$ where $0 \leq x_1 \leq 6$ are optimal.

2. Basic Feasible Solutions

Basic feasible solutions correspond to the vertices of the feasible region. Among these, only **Vertex D** $(6, 3)$ and **Vertex E** $(0, 3)$ lie on the optimal face $x_2 = 3$. Therefore, **there are two basic feasible solutions that are optimal**.

Conclusion

- **Optimal Solution:** All points $(x_1, 3)$ where $0 \leq x_1 \leq 6$.
- **Uniqueness:** The optimal solution is **not unique**.
- **Basic Feasible Solutions Optimal: Two** — specifically, $(0, 3)$ and $(6, 3)$.

2 E 4.3

Consider the polyhedron defined by the following inequalities:

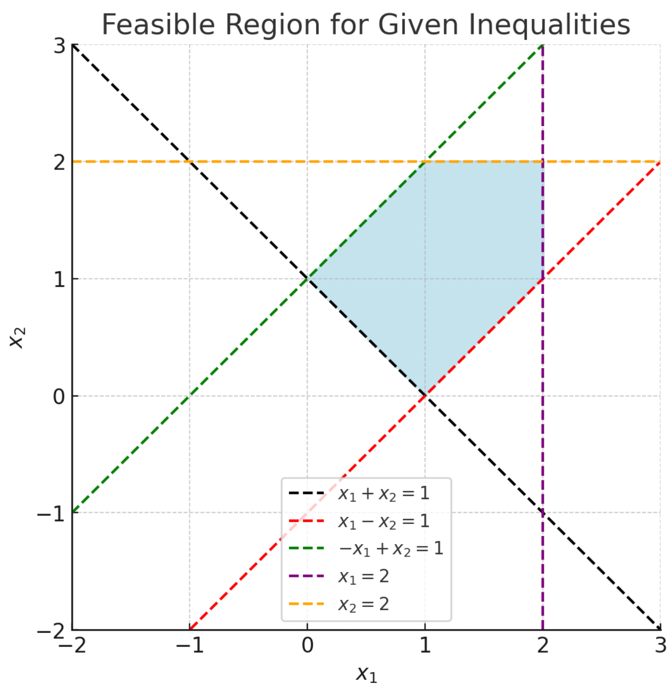
$$
\begin{aligned}
x_1 + x_2 &\leq 1, \\
x_1 - x_2 &\leq 1, \\
-x_1 + x_2 &\leq 1, \\
x_1 &\leq 2, \\
x_2 &\leq 2.
\end{aligned}
$$

Objective

Find the Basic Feasible Solution (BFS) that corresponds to the extreme point $(2, 2)^T$. Construct new basic solutions by using four out of the five columns included in the BFS corresponding to $(2, 2)^T$ and one column that was previously not included.

Determine whether any BFS can be obtained through this method, specify which ones, and explain the theoretical reasoning behind the findings.

Solution



Identifying Extreme Points (Vertices) To identify the extreme points (vertices) of the feasible region, solve the intersections of the constraints:

- $(1, 0)$
- $(0, 1)$
- $(2, 1)$
- $(1, 2)$
- $(2, 2)$

At the extreme point $(2, 2)$, the active (tight) constraints are:

$$x_1 \leq 2, \tag{1}$$
$$x_2 \leq 2. \tag{2}$$

Introducing slack variables to convert inequalities to equalities, the system becomes:

$$x_1 + x_2 - s_1 = 1, \tag{3}$$
$$x_1 - x_2 + s_2 = 1, \tag{4}$$
$$-x_1 + x_2 + s_3 = 1, \tag{5}$$
$$x_1 + s_4 = 2, \tag{6}$$
$$x_2 + s_5 = 2. \tag{7}$$

At $(2, 2)$, the slack variables are:

$$\begin{aligned} s_1 &= 3, \\ s_2 &= 1, \\ s_3 &= 1, \\ s_4 &= 0, \\ s_5 &= 0. \end{aligned}$$

Basis Variables: x_1, x_2, s_1, s_2, s_3.

Non-Basis Variables: s_4, s_5 (set to zero).

Constructing New Basic Solutions To construct new BFS, replace one of the current basis variables $(x_1, x_2, s_1, s_2, s_3)$ with a non-basis variable (s_4 or s_5). The feasible replacements are as follows:

Case 1: Replace s_3 with s_4 **New Basis:** x_1, x_2, s_1, s_2, s_4.

Setting: $s_3 = 0$ and $s_5 = 0$.

Solving the System:

$$x_1 + x_2 - s_1 = 1, \tag{8}$$
$$x_1 - x_2 + s_2 = 1, \tag{9}$$
$$-x_1 + x_2 = 1, \tag{10}$$
$$x_1 + s_4 = 2, \tag{11}$$
$$x_2 = 2. \tag{12}$$

Solution:

$$x_2 = 2,$$
$$x_1 = 1,$$
$$s_4 = 1,$$
$$s_2 = 2,$$
$$s_1 = 2.$$

Feasibility: All variables are non-negative.

Resulting BFS: $(1, 2)^T$.

Case 2: Replace s_2 with s_5 **New Basis:** x_1, x_2, s_1, s_5, s_3.

Setting: $s_2 = 0$ and $s_4 = 0$.

Solving the System:

$$x_1 + x_2 - s_1 = 1, \tag{13}$$
$$x_1 - x_2 = 1, \tag{14}$$
$$-x_1 + x_2 + s_3 = 1, \tag{15}$$
$$x_1 = 2, \tag{16}$$
$$x_2 + s_5 = 1. \tag{17}$$

Solution:

$$x_1 = 2,$$
$$x_2 = 1,$$
$$s_3 = 2,$$
$$s_5 = 1,$$
$$s_1 = 2.$$

Feasibility: All variables are non-negative.

Resulting BFS: $(2, 1)^T$.

Other Replacement Attempts

- Replacing s_1 with s_4 or s_5 leads to infeasibility.
- Replacing x_1 or x_2 with s_4 or s_5 also leads to infeasibility.

Conclusion on Obtained BFS From the BFS at $(2,2)^T$, by replacing one basic variable with a non-basic variable, the following feasible solutions are obtained:

- $(2,1)^T$
- $(1,2)^T$

Attempts to obtain other BFS, such as $(1,0)^T$ and $(0,1)^T$, directly from $(2,2)^T$ through single replacements were unsuccessful due to infeasibility.

Adjacency in BFS: The theory of linear programming states that two BFS are adjacent if they differ by exactly one basic variable. In this context, $(2,2)^T$ is adjacent to $(1,2)^T$ and $(2,1)^T$ but not directly to $(1,0)^T$ or $(0,1)^T$.

Conclusion: Only BFS adjacent to $(2,2)^T$, namely $(2,1)^T$ and $(1,2)^T$, can be obtained by replacing one basic variable at a time. The BFS $(1,0)^T$ and $(0,1)^T$ cannot be directly reached from $(2,2)^T$ through a single replacement.

Summary Answer

Starting at the basic feasible solution $(2,2)$, replacing one of its five basic columns with a previously excluded column yields only the solutions $(2,1)$ and $(1,2)$. No other BFS can be obtained in this manner. According to theory, only adjacent extreme points are reachable by exchanging a single basic variable.

3 E 4.7

Introduction

In the simplex method for solving linear programming (LP) problems, the selection of entering and exiting variables is crucial. These choices determine the path the algorithm takes toward the optimal solution while maintaining feasibility. This analysis examines two specific choices made during simplex iterations:

1. Choosing the entering variable to be a non-basic variable with a negative reduced cost, but not the most negative one.
2. Choosing the exiting variable as a basic variable with a positive $(B^{-1}N_j)_i$ component, but not fulfilling the minimum ratio test.

Choice 1: Selecting the Entering Variable

Scenario

The entering variable is chosen as a non-basic variable with a negative reduced cost, but it is not necessarily the one with the most negative reduced cost.

Implications

- The simplex method requires that any non-basic variable with a negative reduced cost can enter the basis to potentially improve the objective function.
- Selecting the most negative reduced cost variable (steepest descent rule) can lead to faster convergence.
- Choosing a different variable with a negative reduced cost still maintains the feasibility and correctness of the algorithm.

Conclusion

This choice is *not* a critical mistake. While it may affect the efficiency by potentially increasing the number of iterations needed to reach the optimal solution, it does not compromise the correctness or feasibility of the final outcome.

Choice 2: Selecting the Exiting Variable

Scenario

The exiting variable is chosen as a basic variable with a positive $(B^{-1}N_j)_i$ component, but the selection does not adhere to the minimum ratio test.

Implications

Minimum Ratio Test

- The minimum ratio test ensures that the new basic solution remains feasible by preventing any basic variable from becoming negative.
- Mathematically, for each candidate exiting variable x_B, the ratio $\frac{(B^{-1}b)_i}{(B^{-1}N_j)_i}$ is computed.
- The smallest non-negative ratio determines which variable exits the basis.

Consequences of Skipping the Test

- **Infeasibility:** Without the minimum ratio test, the entering variable may increase beyond feasible limits, causing some basic variables to become negative and violating the $x \geq 0$ constraints.
- **Cycling or Failure to Converge:** The algorithm might cycle indefinitely or converge to a suboptimal solution if the exiting variable is not chosen correctly.

Conclusion

This choice is a *critical mistake.* Ignoring the minimum ratio test can lead to infeasibility, prevent the algorithm from reaching the optimal solution, or cause it to cycle without termination.

Mathematical Explanation

Consider a linear program in standard form:

$$\begin{aligned} \text{Maximize} \quad & c^\top x \\ \text{Subject to} \quad & Ax = b, \\ & x \geq 0. \end{aligned}$$

During each iteration of the simplex method:

Entering Variable Selection

1. Identify non-basic variables with negative reduced costs $c_j - z_j < 0$.
2. Any such variable can enter the basis to improve the objective function.

Exiting Variable Selection (Minimum Ratio Test)

1. Compute $B^{-1}N_j$, where N_j is the column of the entering variable.
2. For each basic variable x_B, calculate the ratio $\frac{(B^{-1}b)_i}{(B^{-1}N_j)_i}$ for $(B^{-1}N_j)_i > 0$.
3. The smallest non-negative ratio determines which variable exits the basis to maintain feasibility.

Consequences of Skipping the Minimum Ratio Test

- Suppose an exiting variable x_{B_k} is selected without ensuring it has the smallest ratio.
- This could allow the entering variable x_j to increase beyond feasible limits, causing some x_i to become negative.
- The feasibility of the current basic solution is compromised, potentially leading the algorithm away from the optimal path or causing cycling.

Summary

- **Choice 1:** Selecting an entering variable with a negative reduced cost, even if not the most negative, is acceptable. It does not hinder the algorithm's ability to find the optimal solution but may affect the number of iterations required.
- **Choice 2:** Selecting an exiting variable without performing the minimum ratio test is a critical mistake. It undermines the feasibility and correctness of the simplex method, potentially preventing the algorithm from finding the optimal solution.

Therefore, choosing the exiting variable without performing the minimum ratio test (Option 2) is a critical mistake.

4 E 4.8

Optimization Problem

Solve the following linear program using Phase I and II of the simplex method:

$$\text{minimize } z = 2x_1 - x_2 + x_3$$

subject to

$$x_1 + 2x_2 - x_3 \leq 7,$$

$$-2x_1 + x_2 - 3x_3 \leq -3,$$

$$x_1, x_2, x_3 \geq 0.$$

Solution

Step 1: Convert to Standard Form

First, convert the inequalities to equalities by introducing slack and artificial variables.

First Constraint:

$$x_1 + 2x_2 - x_3 + s_1 = 7 \quad (s_1 \geq 0)$$

Second Constraint: Since the right-hand side is negative and the inequality is $\leq$, multiply by -1 to obtain:

$$2x_1 - x_2 + 3x_3 - s_2 + a_2 = 3 \quad (s_2, a_2 \geq 0)$$

To handle this $\geq$ constraint, subtract a surplus variable s_2 and add an artificial variable a_2.

Variables:

$$x_1, x_2, x_3, s_1, s_2, a_2 \geq 0$$

Phase I: Finding an Initial Feasible Solution

Objective for Phase I:

$$\text{Minimize } w = a_2$$

Initial Basic Variables:

$$B = \{s_1, a_2\}$$

Non-Basic Variables:

$$N = \{x_1, x_2, x_3, s_2\}$$

Initial Setup

Matrix A:

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & -1 & 1 \end{bmatrix}$$

Partitioned A Matrix:

$$A = [B \mid N] = \left[\begin{array}{cc|cccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 3 & -1 \end{array} \right]$$

Cost Vectors:

$$c_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c_N = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

Current Solution:

$$x_1 = x_2 = x_3 = s_2 = 0, \quad s_1 = 7, \quad a_2 = 3$$

Iteration 1: Pivoting to Remove Artificial Variable

1. **Compute $y = c_B B^{-1}$:**

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow B^{-1} = B$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \times B^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

2. **Compute Reduced Costs $\bar{c} = c_N - yN$:**

$$N = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & -1 & 3 & -1 \end{bmatrix}$$

$$\bar{c} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & -1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -3 & 1 \end{bmatrix}$$

3. **Determine Entering Variable:** Choose the variable with the most negative $\bar{c}$, which is x_3 with $\bar{c}_{x_3} = -3$.

4. **Minimum Ratio Test:** Only consider rows where the coefficient of x_3 is positive.

$$\begin{cases} -1 \text{ (Row 1)} \\ 3 \text{ (Row 2)} \end{cases}$$

Only Row 2 is eligible:

$$\text{Ratio} = \frac{3}{3} = 1$$

Leaving Variable: a_2

5. **Pivot Operation:**

$$\text{New Basis: } B = \{s_1, x_3\}$$

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \quad N = \{x_1, x_2, s_2, a_2\}$$

$$c_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad c_N = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{c} = c_N - yN = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & -1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

Since all $\bar{c} \geq 0$, Phase I is complete with $w = 0$ (feasibility achieved).

Phase II: Optimizing the Original Objective Function

Objective for Phase II:

$$\text{Minimize } z = 2x_1 - x_2 + x_3$$

Current Basis:

$$B = \{s_1, x_3\}$$

Non-Basic Variables:

$$N = \{x_1, x_2, s_2\}$$

Setup for Phase II

1. **Partitioned A Matrix:**

$$A = [B \mid N] = \left[\begin{array}{cc|ccc} 1 & -1 & 1 & 2 & 0 \\ 0 & 3 & 2 & -1 & -1 \end{array}\right]$$

2. **Cost Vectors:**

$$c_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad c_N = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}$$

3. **Compute $y = c_B B^{-1}$:**

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \Rightarrow B^{-1} = \frac{1}{3} \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 \end{bmatrix} \times B^{-1} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

4. **Compute Reduced Costs $\bar{c} = c_N - yN$:**

$$N = \{x_1, x_2, s_2\} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & -1 \end{bmatrix}$$

$$\bar{c} = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}$$

5. **Determine Entering Variable:** Choose the variable with the most negative $\bar{c}$, which is x_2 with $\bar{c}_{x_2} = -1$.

6. **Minimum Ratio Test:** Compute $B^{-1}A_{x_2}$:

$$A_{x_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$B^{-1}A_{x_2} = \frac{1}{3}\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6-1 \\ 0-1 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ -\frac{1}{3} \end{bmatrix}$$

Only positive entries are considered for the ratio:

$$\text{Row 1: } \frac{7}{\frac{5}{3}} = \frac{21}{5} = 4.2$$

$$\text{Row 2: Ignored since coefficient is negative}$$

Leaving Variable: s_1

7. **Pivot Operation:**

$$\text{New Basis: } B = \{x_2, x_3\}$$

$$B = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}, \quad N = \{x_1, s_2\}$$

$$c_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad c_N = \begin{bmatrix} 2 & 0 \end{bmatrix}$$

$$B^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$$

$$y = \begin{bmatrix} -1 & 0 \end{bmatrix} \times B^{-1} = \begin{bmatrix} -\frac{3}{7} & 0 \end{bmatrix}$$

8. **Compute Reduced Costs $\bar{c} = c_N - yN$:**

$$N = \{x_1, s_2\} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$\bar{c} = \begin{bmatrix} 2 & 0 \end{bmatrix} - \begin{bmatrix} -\frac{3}{7} & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2+\frac{3}{7} & 0+0 \end{bmatrix} = \begin{bmatrix} \frac{17}{7} & 0 \end{bmatrix}$$

9. **Optimality Check:** All $\bar{c} \geq 0$ ($\frac{17}{7} > 0$, $0 \geq 0$), thus the current solution is optimal.

Final Optimal Solution

Non-Basic Variables:

$$x_1 = 0, \quad s_2 = 0$$

Basic Variables:

$$x_2 = 4.8, \quad x_3 = 2.6$$

Objective Function Value:

$$z = 2(0) - 4.8 + 2.6 = -2.2$$

Summary of Optimal Solution

- **Variables:**
$$x_1 = 0, \quad x_2 = 4.8, \quad x_3 = 2.6$$
- **Minimum Objective Value:**
$$z = -2.2$$
- **Slack Variables:**
$$s_1 = 0, \quad s_2 = 0$$

This solution satisfies all the constraints and minimizes the objective function as required.