Computer Exercise 2

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Contents

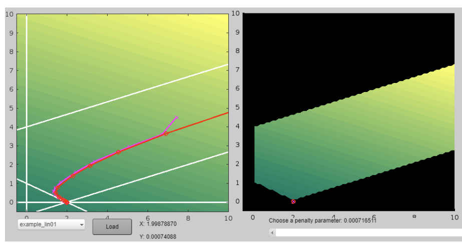
1 Part 1: Constrained Optimization: Penalty Methods

There are four nonlinear problems, two convex (example_nl{01,02}.m), and two non-convex (example_nl{03,04}.m), as well as three linear problems (example_lin[01-03].m); you can find the problem formulations in the Appendix.

1.1 Interior Point Method

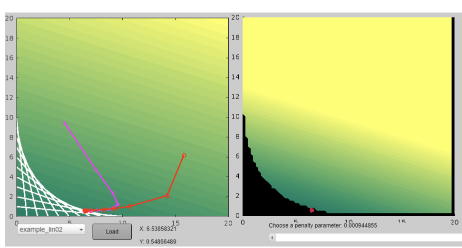
Using the interior point method, solve the LPs 01–03. Do we always find an optimal extreme point (problem 03)? Notice how the algorithm follows closely the central path and goes "directly" to the global minimum point (i.e., it skips visiting the extreme points), if you change the penalty parameter smoothly. Compare with the Simplex method.

1.1.1 Interior Point Method - example_lin01



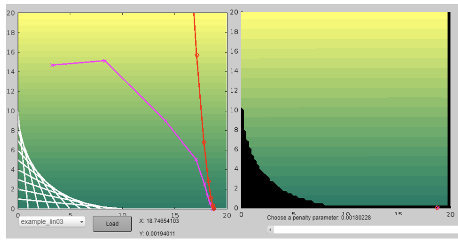
The algorithm reached an optimal point in $x^* = (2, 0)$. The point is also an extreme point.

1.1.2 Interior Point Method - example_lin02



The algorithm reached a optimal point in $x^* = (6.55, 0.55)$. The point is also an extreme point.

1.1.3 Interior Point Method - example_lin03



In this case the optimal points are found in $x^* = (10 \leq x_1 \leq 20, x_2 = 0)$.

The algorithm reached an optimal point in $x^* = (18.74, 0)$, however this is not an extreme point. The point reached is dependent on the initial conditions.

In this last example it can be seen how the Interior Point Method and the Simplex Method differs in the way they approach an optimal point. The latter, visits one extreme point at a time and evaluates the optimality, while the first approach the optimal point directly.

If the Simplex Method was used in the last case, we would have found an optimal point that would have been also an extreme point, namely $x^* = (10, 0)$.

1.2 KKT Points

Change the directory to Files computerexercise2/kkt and type go at the Matlab prompt. Find the KKT points of the nonlinear problems (example_nl[01-04].m). Are the KKT conditions sufficient for the global optimality (problem 03)? Are they necessary (problem 04)?

1.2.1 KKT Points - example_nl01

The only KKT point in this problem is found in (2, 1), where the gradient can be expressed as a non-negative linear combination of the active constraints.

The problem is convex, the KKT conditions are sufficient.

The constraints are affine, therefore Abadie's CQ holds and the KKT conditions are necessary.

1.2.2 KKT Points - example_nl02

The KKT points are found all in $x_1 = 2$, with $1 <= x <= 4$ and $1 <= x_2 <= 4$ with $2 <= x_2 <= 4$.

The problem is convex, the KKT conditions are sufficient.

The constraints are affine, therefore Abadie's CQ holds and the KKT conditions are necessary.

1.2.3 KKT Points - example_nl03

The KKT points are found in

- (0.71, 0.77)
- (3.18, 0.76)
- (2.58, 2.58)
- (0.44, 3.15)
- (2.03, 0.75)
- (0.99, 1.87)

The problem is not convex so the KKT conditions are not sufficient.

One constraint is not convex, so Slater CQ does not hold, however LICQ holds at the solutions, hence Abadie's holds at these points so the KKT conditions are necessary.

1.2.4 KKT Points - example_nl04

There are no KKT points in this problem. The only KKT point is (0.371, 0.00396)

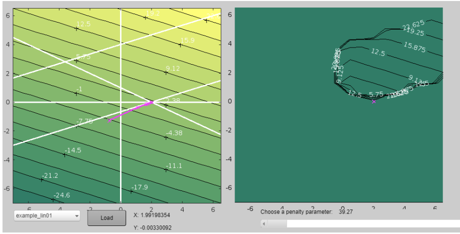
The problem is not convex so the KKT conditions are not sufficient.

One constraint is not convex so Slater CQ does not hold. LICQ does not hold. Abadie's does not hold, so the KKT conditions are not necessary.

1.3 Exterior Point Method

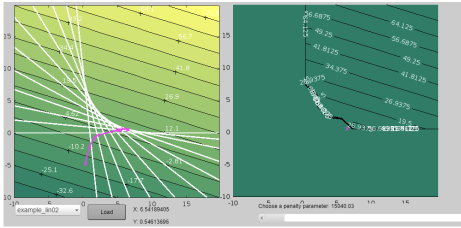
Using the exterior penalty algorithm, solve the nonlinear and linear problems. Can you get different "optimal" solutions by changing the penalty parameter in a different manner or by starting from different points (problem 03)? Tricky: Can you think of a reason for the slow convergence in problem 04 (hint: KKT)?

1.3.1 Exterior Point Method - example_lin01



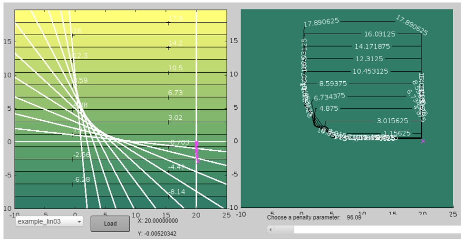
The algorithm reached an optimal point in $x^* = (2, 0)$. Ran progressively until penalty parameter $\nu = 0.39$.

1.3.2 Exterior Point Method - example_lin02



The algorithm reached an optimal point in $x^* = (6.54, 0.55)$. Ran progressively until penalty parameter $\nu = 15000$.

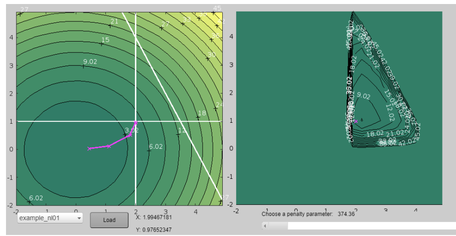
1.3.3 Exterior Point Method - example_lin03



The algorithm reached an optimal point in $x^* = (20, 0)$. Ran progressively until penalty parameter $\nu = 96$, starting from $\nu = 0.10$.

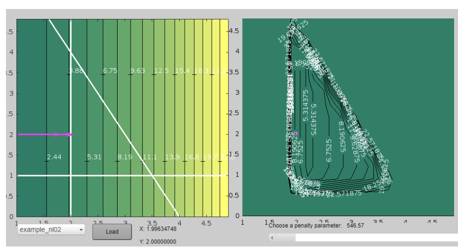
Starting from $\nu = 32$, the algorithm reached an optimal point in $x^* = (10.15, 0)$.

1.3.4 Exterior Point Method - example_nl01



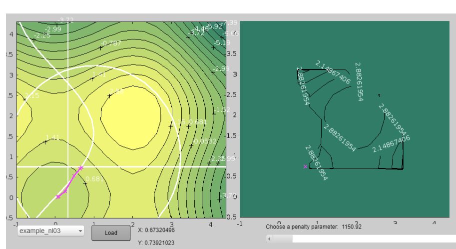
The algorithm reached an optimal point in $x^* = (2, 1)$. Ran progressively until penalty parameter $\nu = 374$, starting from $\nu = 0.10$.

1.3.5 Exterior Point Method - example_nl02



The algorithm reached an optimal point in $x^* = (2, 2)$. Ran progressively until penalty parameter $\nu = 374$, starting from $\nu = 0.10$.

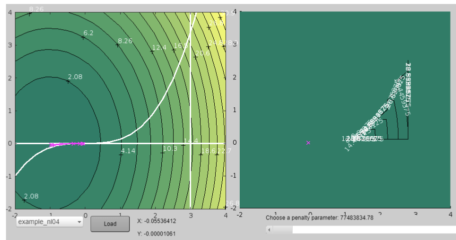
1.3.6 Exterior Point Method - example_nl03



The algorithm reached an optimal point in $x^* = (0.67, 0.74)$. Ran progressively until penalty parameter $\nu = 1150$, starting from $\nu = 0.10$.

In this case, we have multiple KKT points to which the algorithm can converge to.

1.3.7 Exterior Point Method - example_nl04



The algorithm reached an optimal point in $x^* = (-0.05, 0)$. Ran progressively until penalty parameter $\nu = 10^8$, starting from $\nu = 0.10$.

The convergence to an optimal point required a lot of iterations in this case. The reason for this can be attributed to the fact that there are no KKT points.

2 Part 2: Constrained Optimization: MATLAB's and Optimization Toolbox

2.1 Problem 1

Given is the following optimization problem:

$$\max f(x) = x_1 - 2x_1^2 + 2x_2 - x_2^2 + x_1 x_2,$$

subject to the constraints:

$$x_1^2 - x_2 \le 0,$$

$$2x_1 - x_2 \ge 0.$$

(a) Solve the problem using `fmincon`.

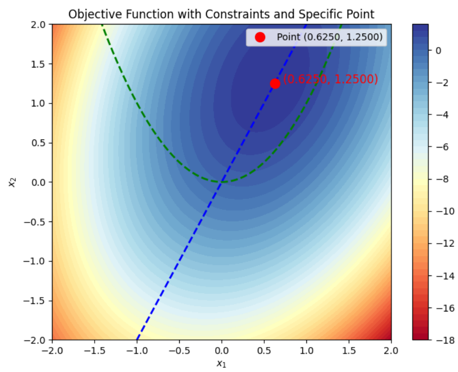
(b) State the KKT conditions, examine the convexity of the problem, and verify that the obtained solution is a global maximum.

```
Optimal solution:
    0.6250    1.2500

Optimal value:    -1.5625
Lagrange multipliers:
Upper bounds    :
     0     0

Lower bounds    :
     0     0

Lin. ineq.      :
```




```
    0.1250

Lin. eq.         :
Nonlin. ineq.    :
   4.5309e-07

Nonlin. eq.      :

f =

   -1.5625

Gradient of the objective function (df):
    0.2500    -0.1250

Gradient of Constraint 1 (Nonlinear):
    1.2500    -1.0000

Gradient of Constraint 2 (Linear):
    -2
     1

Nonlinear constraint (C) at the optimal solution:
   -0.8594

Linear constraint (A*x) at the optimal solution:
  -4.0905e-06

Hessian of the objective function (hf):
     4    -1
    -1     2
```

The `fmincon` algorithm returns an optimal value of 1.5625 at $(0.6250, 1.2500)$.

This is posed as a maximization problem, we can translate the problem into a minimization one by taking the negative of the objective function. This will be the function we'll consider.

The Hessian of the objective function is positive definite, therefore the function is convex. The two constraint are convex as well (the linear one is banal and the non linear one gives a positive semi-definite matrix).

The problem is convex then and KKT is sufficient.

For this reason we can say that the maximum solution found is a global maximum.

2.2 Problem 2

Given is the following optimization problem:

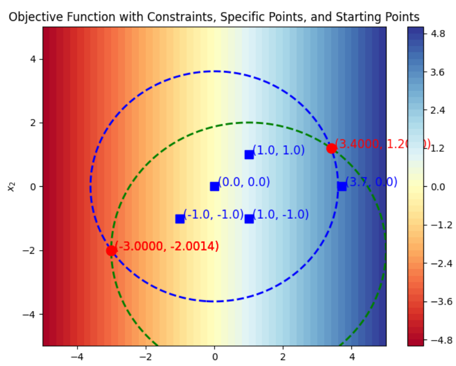
$$\min f(x) = x_1,$$

subject to the constraints:

$$(x_1 - 1)^2 + (x_2 + 2)^2 \leq 16,$$

$$x_1^2 + x_2^2 \geq 13.$$

Solve the problem from at least five starting points. Describe what happens. Which point is the best one? Can you guarantee that this is a global minimum? Fun points to try are $(1, 1)^T$, $(0, 0)^T$, $(3.7, 0)^T$, and $(-1, -1)^T$.



- Starting Point: (1, 1), Optimal Solution: (3.4, 1.2), Optimal Value: 3.4
- Starting Point: (0, 0), Optimal Solution: (-3, -2), Optimal Value: -3
- Starting Point: (3.7, 0), Optimal Solution: (3.4, 1.2), Optimal Value: 3.4
- Starting Point: (-1, -1), Optimal Solution: (-3, -2), Optimal Value: -3
- Starting Point: (1, -1), Optimal Solution: (-3, -2), Optimal Value: -3

The best points are (0, 0), (-1, -1), (1, -1) because the algorithm converges to the global minimum, while by choosing the other points it doesn't.

The objective function and the first constraint are convex, but the second constraint isn't. We don't have a convex problem.

KKT is therefore not sufficient and no solution found can be guaranteed to be global minimum.

For the point (-3, -2), the negative gradient can be expressed as a non-negative linear combination of the gradients of the active constraints, this is a KKT point.

For the point (3.4, 1.2), this is not the case, it's not a KKT point.