Real Space Renormalization Group

Abstract

Study on the spectrum of the quantum transverse Ising Model.

1 Theory

The aim of this assignment was to apply the Real Space Renormalization Algorithm to the transverse-field Ising Model Hamiltonian.

1.1 Transverse Field Ising Model

The Hamiltonian of the Transverse Field Ising Model is:

$$\hat{H} = \lambda \sum_{i=1}^{N} \sigma_z^i + \sum_{i=1}^{N-1} \sigma_x^{i+1} \sigma_x^i \tag{1}$$

Where λ is a parameter linked to the intensity of an external field, and $\sigma_{x/z}^i$ are the operators obtained by

$$\sigma_{x/z}^i = \mathbb{1}_{2^{i-1}} \otimes \sigma_{x/z} \otimes \mathbb{1}_{2^{N-i}} \tag{2}$$

with $\sigma_{x/z}$ the Pauli matrices on the x or z axis.

1.2 Real Space Renormalization Group

The algorithm used to obtain an approximated solution of our problem is the Real Space Renormalization Group algorithm. Starting from H_i the Hamiltonian of a set of interacting particles it is possible to duplicate the number of particles of our system.

$$H_i^{'} = H_i \otimes \mathbb{1} + \mathbb{1} \otimes H_i + H_i^* \tag{3}$$

Where the tensor products represent the non interacting part of the left and right side subsets, while H_i^* is the interaction term between the two sides. Then project the duplicated system on to the half eigen-states with lowest energy.

$$H_{i+1} = P_i^{-1} H_i^{'} Pi \tag{4}$$

where P is a projector obtained from rejecting half of the columns from the eigenvector matrix of $H_i^{'}$ while P^{-1} is obtained by rejecting half of the rows from its inverse. Now it is possible to reiterate the algorithm from [Eq.3]. In this case the H_i^* can be computed as follows, evolving its form in the iterations of the algorithm.

$$H_i^* = \sigma_i^L \cdot \sigma_i^R \tag{5}$$
$$\sigma_{i+1}^L = \mathbb{1} \otimes P_i^{-1} \sigma_i^L P_i \tag{6}$$
$$\sigma_{i+1}^R = P_i^{-1} \sigma_i^R P_i \otimes \mathbb{1} \tag{7}$$

Where $\sigma_0^L = \sigma_x^l$, $\sigma_0^R = \sigma_x^r$ the matrices appearing in the interaction term in the Hamiltonian [Eq.1], for the last particle of the left side (l) and the first particle of the right side (r). After a suitable number of iteration it is possible to evaluate the eigenvalues of the last Hamiltonian and obtain information about the energies.

2 Code Development

The core of the code can be appreciated in the following listing. In the program it is found inside a *do loop* used to iterate the RSRG algorithm. As first H_i' is diagonalized to obtain its eigenvectors which then are used to compute the projectors. Then H_i' is reduced to H_{i+1}. After these procedures the matrices σ_{i+1}^R and σ_{i+1}^L which are necessary to evaluate the new interaction term H_{i+1}^* are computed. As last the new Hamiltonian H_{i+1} is obtained.

```
!1)Diagonalize H_{i}
call eigz(test_system, size(test_system, dim = 1), energies)
!2) Obtain Projector P_{i} and P^{-1}_{i}
Prf = test_system(:,1:2**N)
call invz(test_system, size(test_system, dim = 1) )
Prb = test_system(1:2**N,:)
!3) Get H*_{i}
test_system = cmplx( dble(0.0),dble(0.0) )
do ii = 1, 2**N
    rdHam(ii,ii) = energies(ii)
end do
!4) Get sigma^{L/R}_{i+1}
opn= matmul(Prb, matmul(opn,Prf))
opn1= matmul(Prb, matmul(opn1,Prf))
opn= getbigmat(opn,2,2)
opn1= getbigmat(opn1,1,2)
!5) Get H_{i+1} Hamiltonian
test_system = getbigmat(rdHam,1,2) + getbigmat(rdHam, 2,2) +
    matmul(opn,opn1)
```

All the functions used derives from older exercise, and in particular:

- **eigz** : Computes eigenvalues and eigenvectors of an hermitian matrix.

- **invz** : Computes inverse of an hermitian matrix.
- **getbigmat** : Computes $(\bigotimes_1^{N-m-1} \mathbb{1}_C) \otimes A_C \otimes (\bigotimes_1^{m} \mathbb{1}_C)$ given a square matrix A of size $C \times C$ and the two parameters N and m

The free parameters of the code are λ the field term of the Hamiltonian, the dimension used for the system, and the number of iteration of the algorithm.

3 Results

In the following figure it is possible to see a comparison between the mean field solution and the result obtained for 2048 particles obtained by starting from a subset of 4 and performing 9 iterations.

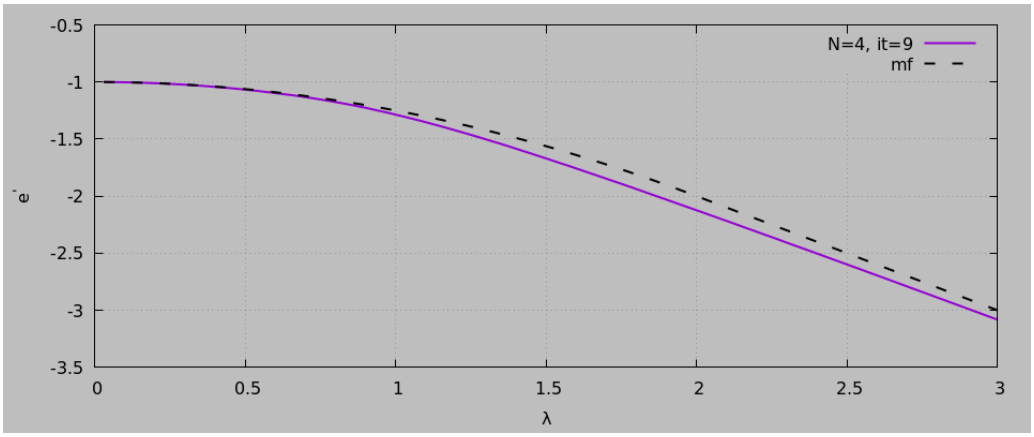

Figure 1: On the x axis is represented the field parameter $lambda$, while in the y axis, the per particle energy of the ground state.

4 Self Development

I found this last exercise really interesting , both RSRG and RDRG are powerful tools to investigate many bodies quantum systems. Unfortunately my attempts to implement RDRG have been unsuccessful. About the RSRG, various test have been done, varying the number of iterations and the dimension of the Hamiltonian. The results clearly differed among each other, but the deviations were not enough great to be seen in a graph with the scale like the graph in [Fig.1], so for the future could be also interesting to perform a more accurate statistical analysis on the results of the RSRG.