# CLT FOR RANGE OF RANDOM WALKS ON HYPERBOLIC GROUPS

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ABSTRACT. We prove a central limit theorem for random walks with a finitely supported step distribution on wreath products of the form $F \wr H = \bigoplus_H F \wr H$, where $F$ is a non-trivial finite group and $H$ is a non-elementary hyperbolic group.

## 1. Introduction

(1) Talk about random walks on groups; trying to prove limit laws in general
(2) Make a list of things known about lamplighters and why they are relevant
(3) Say that in this paper we concentrate on the CLT

Here are some things that we should cite... I am missing many more but this is a good start. [Sal01].

(1) The CLT for non-abelian free groups is due to [SS87] and [Led01]. Then for non-elementary hyperbolic groups with a finite exponential moment is due to [Bjö10]. This was generalized for any finite second moment measure in [BQ16a]. The last two results hold more generally for group acting on a Gromov hyperbolic space by isometries. [BQ16b] show a CLT for random walks on $\mathrm{GL}_d(\mathbb{R})$ with a finite second moment. See also [Gou17].
(2) [EZ22] prove a law of large numbers for random walks on $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^2$ with a finite $(2+\varepsilon)$-moment, for some $\varepsilon > 0$. They also discuss limit laws in other wreath products.
(3) [BFGK24] prove a central limit theorem for random walks on the group of affine transformations of a horospherical product of Gromov hyperbolic spaces.
(4) [Cho23a] proves a central limit theorem for groups acting with contracting elements.
(5) [GTT22] prove a CLT with respect to the counting measure on the Cayley graph of a group acting on a hyperbolic space.
(6) [Hor18] proves a CLT for random walks on mapping class groups and $\mathrm{Out}(F_n)$.
(7) [Bar22] proves a CLT for groups acting on a CAT(0) space.
(8) [Gil08] proves that the drift of $\mathbb{Z}/2\mathbb{Z} \wr G$ is strictly larger than that of its projection to $G$.
(9) [MSŠ23] prove a LLN and CLT for the capacity of the range of a random walk on a group.
(10) [Sal01] proves a LLN and CLT for a simple random walk on a free group, conditioned on the boundary point.

### 1.1. Main results.

Consider the switch-walk-switch word length $|\cdot|$ on $A \wr H$.

**Theorem 1.1.** *Let $A$ be a non-trivial group and $H$ a non-elementary hyperbolic group. Consider a probability measure $\mu$ on $A \wr H$ such that $\mu_H$ is non-elementary and has a finite second moment, and such that $\mu$ has bounded lamp range. Denote by $\{w_n\}_{n\geq 0}$ the $\mu$-random walk on $A \wr H$, and let $C = \lim_{n\to\infty} \frac{\mathbb{E}(|w_n|)}{n}$ be the drift of the $\mu$-random walk on $A \wr B$. Then the sequence of normalized random variables $\frac{|w_n|-Cn}{\sqrt{n}}$, $n \geq 1$, converges in law to a non-degenerate gaussian law.*

What are the moment hypotheses for acylindrically hyperbolic base groups? Finite support? exponential tails?

We prove a central limit theorem for the range of random walks with a finite second moment on hyperbolic groups.

**Theorem 1.2.** *Let $H$ be a non-elementary hyperbolic group and let $\mu$ be a non-elementary probability measure on $H$ with a finite second moment. Let $C$ be the probability that the $\mu$-random walk on $H$ starting at $e_H$ never returns to $e_H$. Then the sequence of normalized random variables $\frac{|R_n|-Cn}{\sqrt{n}}$, $n \geq 1$, converges in law to a non-degenerate gaussian law.*

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## 2. Preliminaries

**2.1. Hyperbolic groups.** Say basic things about hyperbolicity; explain pivots

**2.2. Lamplighter groups.** We consider the wreath products $A \wr H$, where $A$ is a finite non-trivial group and $H$ is a finitely generated group. Let $S_H$ be a finite and symmetric generating set of $H$. Then we consider the *switch-walk-switch* $S_{\mathrm{sws}}$ generating set of $A \wr H$, given by

$$S_{\mathrm{sws}} := \Big\{(\delta_a, 0)(\mathbf{0}, s)(\delta_{a'}, 0) \Big| a, a' \in A \text{ and } s \in S_H\Big\}.$$

**Theorem 2.1** ([Par92, Theorem 1.2])**.** *For any $g = (f, x) \in A \wr H$, the word length of $g$ with respect to the standard generating set is*

$$|g| = \mathrm{TSP}(e_H, x, \mathrm{supp}(f)).$$

**2.3. Random walks on groups.**

(1) Recall basic concepts of random walks on groups.

**2.4. Defective adapted cocycles and the central limit theorem.**

(1) Introduce all necessary results and definitions of Mathieu-Sisto.
(2) Introduce pivots and the results of Gouëzel that we will use.

We will use the following general criterion of Mathieu-Sisto.

**Definition 1** (Defective adapted cocycle)**.**

**Theorem 2.2** ([MS20])**.** *Suppose that $Q_n$ is a defective adapted cocycle with defect*

$$\Phi_{m,n} := Q_n - (Q_m + w_m Q_{m-n}).$$

*Suppose that there exists $C > 0$ such that*

$$\sup_{m,n \geq 1} \mathbb{E}[|\Phi_{m,n}|^2] \leq C.$$

*Then the CLT holds for $Q_n$.*

In this section we explain a generalization of [MS20, Theorems 4.1 & 4.2] (see Theorem 2.2)

**Theorem 2.3.** *Suppose that $Q_n$ is a defective adapted cocycle with defect*

$$\Phi_{m,n} := Q_n - (Q_m + Z_m Q_{m-n})$$

*and suppose that for some fixed polynomial $p$ and $N_0 \in \mathbb{N}$ we have that*

$$\mathbb{E}[|\Phi_{m,n}|^2] \leq p(\log(n))$$

*whenever $m, n - m \geq N_0$. Then a CLT holds for $Q_n$.*

Is this the formulation we want?

Here are the places in the proof of Mathieu-Sisto's CLT where the deviation inequality is used:

(1) In Theorem 4.4 one obtains $V(Q_n) \leq n\Big(p(4\mathbb{E}(Q_1^2) + 16\log(n))\Big)$
(2) Then in Lemma 4.5 it is cited a result of Hammersley [Ham62, Theorem 2]. This should be replaced by an inequality of the form $a_{n+m} \leq a_n + a_m + bp(log(n+m))$.

**Lemma 2.4.** *Let $\{a_n\}_{n\geq 0}$ be a sequence of non-negative real numbers. Suppose that there exists $b \geq 0$ and a polynomial $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$a_{n+m} \leq a_n + a_m + b\sqrt{p(log(a_n + a_m))}, \text{ for each } m, n \geq 0.$$

*Then the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists.*

(3) The deviation inequality is used in multiple occasions during the proof of Lemma 4.6. One should do the appropriate modifications.

## 3. CLT for the lamplighter over a hyperbolic group (using pivots)

3.1. **Pivots.** Let us consider a non-elementary hyperbolic group $H$, and let us fix a finite generating set $S_H$. Let $\delta \geq 0$ be the hyperbolicity constant of $\mathrm{Cay}(H, S_H)$, and let us denote by $d_H : H \times H \to \mathbb{Z}_{\geq 0}$ the word metric on $H$ with respect to $S_H$.

**Definition 2.** Given a path $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ on $\mathrm{Cay}(H, S_H)$ and $g \in H$, let $\pi_\gamma(g)$ be the set of elements of $\gamma$ that minimize the word metric to $g$. That is, we define

$$\pi_\gamma(g) \coloneqq \{\gamma_i \in \gamma \mid d_S(\gamma_i, g) \leq d_S(\gamma_j, g) \text{ for all } j = 1, \ldots, k\}\,. \tag{1}$$

We now introduce the definition of pivots that we will use in the proof of Theorem **??**. We refer to [Gou22, Section 4A] for details.

**Definition 3.** Let $C, D > 0$, $L \geq 20C + 100\delta + 1$, and $N \in \mathbb{Z}_{\geq 1}$. Let $\mathbf{w} = \{w_n\}_{n\geq 0} \in (A \wr H)^{\mathbb{N}}$ be a sample path of the $\mu$-random walk on $A \wr H$. Denote by $w_n^H$ the projection to $H$ of $w_n$, for each $n \geq 0$. A time instant $m \geq 1$ is a $(C, D, L, N)$-*pivot* for the sample path $\mathbf{w}$ if the following three conditions hold.

(1) $d_H\left(w_m^H, w_{m+N}^H\right) \geq L$.

Let $\gamma$ be an arbitrary geodesic path in $\mathrm{Cay}(H, S_H)$ that connects $w_m^H$ to $w_{m+N}^H$. Then

(2)
$$d_H\left(\pi_\gamma\left(w_k^H\right), w_m^H\right) \leq C, \text{ for all } k \in \{0, 1, \ldots, m\},$$

(3) for all $m \leq k \leq m + N$ we have $d_H\left(w_k^H, \gamma\right) \leq D$, and

(4) for all $k \geq m + N$, we have $d_H\left(\pi_\gamma\left(w_k^H\right), w_{m+N}^H\right) \leq C$.

The following lemma will be our main tool.

**Lemma 3.1.** *For any $C, D, \delta > 0$, and any $L \geq 20C + 100\delta + 1$, there exists $N, R > 0$ large such that*

$$\sup_{i\geq 1} \mathbb{P}\left(\exists m \in [i, i + k] \text{ such that } m \text{ is an } (C, D, L, N)\text{-pivot}\right) \geq 1 - Re^{-k/R}.$$

*Sketch of proof.* This is proven in proposition 4.11 in Gouezel's paper, where for Gouezel's definition of pivots, conditions 1,2, and 4 are met. To see why Gouezel's proof implies the lemma we state, we observe that for Gouezel's definition of pivots, the increments $w_n^{-1} w_{n+N}$ are drawn from some explicit finite set of isometries $S \subset H$ that receive positive support from $\mu_H^N$. For this finite set of isometries, we can pick some $D > 0$ large enough so that $\mu^N$ gives positive mass to each of the sets $\pi_H^{-1}(s) = \{(\varphi, s), \mathrm{supp}\varphi \subset B_D([e, s])\}$. Then tracing through the rest of Gouezel's proof we have the estimate required. ☐

3.2. **TSP structure along pivots.**

**Proposition 3.2.** *Suppose that we are looking at a sample path $\{w_n\}_{n\geq 0}$ and that we have a pivoting time $m$. Then the group element $w_n = (f_n, x_n)$ satisfies the following. The support of $f_n$ can be decomposed as a disjoint union*

$$\mathrm{supp}(f_n) = P_1' \cup P_2' \cup P_3',$$

*that satisfies the following properties. Let us denote $P_1 = P_1' \cup \{w_{m+u}^H\}$, $P_2 = P_2' \cup \{w_{m+u}^H, w_{m+u+N}^H\}$ and $P_3 = P_3' \cup \{w_{m+u+N}^H\}$. Let $\gamma$ be an arbitrary geodesic path from $w_m^H$ to $w_{m+N}^H$ on $\mathrm{Cay}(H, S_H)$. Then we have*

(1) *for all $g \in P_1$, we have $d_H\left(\pi_\gamma(g), w_m^H\right) \leq C$,*

(2) *for all $g \in P_2$ we have $d_H(g, \gamma) \leq D$, and*

(3) *for all $g \in P_3$, we have $d_D\left(\pi_\gamma(g), w_{m+N}^H\right) \leq C$.*

**Definition 4.** Let $g = (f, x) \in A \wr H$ and suppose that $\mathrm{supp}(f) = P_1 \cup P_2 \cup P_3$. Let $\eta$ be a path on $\mathrm{Cay}(A \wr H, S_{\mathrm{sws}})$ that realizes $|g|_{S_{\mathrm{sws}}}$. We define the associated *coding* of $\eta$ as the word $u$ in the alphabet $\{P_1, P_2, P_3\}$, such that $u_i = P_j$ if and only if at the $i$-th step of $\eta$, there is a lamp at a position in $P_j$ which was modified for the first time.

We are going to abuse notation (in this draft) and not make a distinction between the elements visited by a path, and the coding in the alphabet $\{P_1, P_2, P_3\}$ associated with it.

**Definition 5.** Given a path $\eta$, let us call a *backtracking* a subpath of $\eta$ that is of the form $P_1 P_2^* P_3^\varepsilon P_2^* P_1^{\varepsilon'} P_2^* P_3$ for $\varepsilon, \varepsilon' \geq 1$. Here the $*$ symbolizes 0 or more occurrences.

**Lemma 3.3.** *Let $\eta$ be a solution to the TSP for $|w_n|$. Then the coding of $\eta$ does not have a subword of the form $P_1 P_3^\varepsilon P_1^{\varepsilon'} P_3$, for $\varepsilon, \varepsilon' \geq 1$.*

*Proof.* Surgery, meaning that you glue together the excursions to $P_1$, and you glue together the excursions to $P_3$, and connect them with any path through $P_2$. This gives something even shorter than optimal since each gluing strictly reduces the length of the path. □

**Corollary 3.4.** *Let $\eta$ be a solution to the TSP for $|w_n|$. Then the number of backtrackings of $\eta$ is at most $|P_2|$.*

*Proof.* Every backtracking must contain at least one element of $P_2$. (Recall that the path only has an element in its coding if it has not been visited before). □

**Lemma 3.5.** *Consider a sequence of points $\{w_n\}_n$ of $H$, that satisfies the decomposition of* $\operatorname{supp}(f_n)$ *given by the three conditions of Proposition 3.2.*

*Let $T$ be the length of a solution to* $\mathrm{TSP}(w_0, w_n, \operatorname{supp}(f_n)) = \mathrm{TSP}\left(w_0^H, w_n^H, P_1 \cup P_2 \cup P_3\right) = |w_n|_{A\wr H}$. *Then there exists a path $\eta$ that starts at $w_0^H$, finishes at $w_n^H$ and visits all points in* $\operatorname{supp}(f_n)$ *such that* $\mathrm{length}(\eta) \leq T + 100N(L+2D)$*, and such that, in the coding of $\eta$, all the elements of $P_1$ appear before any of the elements of $P_3$.*

*Proof.* Let us first consider $\eta_0$ the optimal solution to the TSP.

Lemma 3.3 implies Corollary 3.4 that the total number of backtrackings is the size of $P_2$.

Finally, the argument goes as follows: first do all excursions of $\eta_0$ on $P_1$, then visit all elements in $P_2$, and then do all excursion in $P_3$. In total we added at most $2D\times$(number of backtrackings)+(length of solution of TSP in P2 that visits all elements in P2). And the number of backtrackings is at most $|P_2|$ by the previous claim.

□

In other words, we are saying that trying to solve the problem by first visiting all elements of $P_1$, and then visiting all elements of $P_3$, and crossing the middle section only once, is at a bounded length of being optimal.

**Lemma 3.6.** *For any $N \in \mathbb{N}$ there exists some $C > 0$ such that the following holds. set $m \in \mathbb{N}$ be an integer and let $U$ be the waiting time until the first pivot after time $m$. Then we have*

$$\sup_{m \geq 1} \mathbb{E}\left[\left(\sum_{i=m}^{m+U+N} |g_i|\right)^2\right] \leq C.$$

*Proof.* I'll go into the construction of pivots and explain why this is true. Maybe there's a simpler reasoning using only the exponential estimates on $U$.

Recall that the way Gouezel constructs pivots is as follows: if we let $S \subset H$ be our finite Schottky set, then we can decompose some convolution power $\mu_H^N$ as

$$\mu_H^N = \alpha\mu_S + (1-\alpha)\nu$$

for some positive $\alpha > 0$. Then we draw our increments as follows: let $\{\varepsilon_i\}_i$ be i.i.d. Bernoulli($\alpha$) random variables. If $\varepsilon_i = 1$, we draw $g_i' = s_i$ according to $\mu_S$. Else we draw $g_i' = w_i$ according to $\nu$. We observe that the sequence $\{g_1' ... g_k'\}_k$ has the same distribution as $\{g_1 ... g_k\}_k$ for $g_i \sim \mu_H^N$.

Now we denote the resampled random walk by $g_1' ... g_n' = w_1 ... w_{k_1} s_1 w_{k_1+1} ... w_{k_2} s_2 ...$, where the strings between $s_i's$ may be empty. Now each string $w_{k_{i-1}+1} ... w_{k_i} s_i$ is distributed according to $\nu^Z * \mu_S$, where $Z$ is a geometric random variable with parameter $\alpha$.

Now Gouezel tells us that, conditional on any realization of the increments drawn from $\nu$, the number of $\mu_S$ increments $\ell$ until we see a pivot has an exponential tail. This implies that

$$\mathbb{E}\left[\left(\sum_{i=m}^{m+U+N} |g_i|\right)^2 |\{w_i\}_i\right] \leq \mathbb{E}\left[\left(L\ell + \sum_{i=0}^{\ell-1}\sum_{k=K_i}^{K_{i+1}-1} |w_i|\right)^2 |\{w_i\}_i\right].$$

Now we can integrate over the possible values of $w_i$ and use independence in order to conclude that

$$\mathbb{E}\left[\left(\sum_{i=m}^{m+U+N}|g_i|\right)^2\right]$$

is bounded uniformly over $m$. □

3.3. **Proof of the CLT.** Let us define $\Phi_{n,m} = |w_n| - |w_m^{-1}w_n| - |w_m|$. Thanks to Theorem 2.2, it suffices to show that $\sup_{m,n\geq 1}\mathbb{E}(|\Phi_{n,m}|^2)$ is bounded.

We will do the proof for finitely supported $\mu$.

We fix $m$ and $n$. Let $m+u$ be the first instant after $m$ that you see a pivot.

If $m+u+N > n$, then we use Lemma 3.6

$$\mathbb{E}|\Phi_{n,m}|^2 \leq \sup_{m\geq 1}\mathbb{E}\left[\left(\sum_{i=m}^{m+u}|g_i|\right)^2\right] \leq C.$$

Otherwise, $m+u+N \leq n$ and we do the following.

(1) The three conditions at the beginning of this subsection are satisfies.
(2) Our objective is to get a good upper bound for $\Phi_{n,m}$ in the inequality
$$|w_n| \geq |w_m| + |w_m^{-1}w_n| - \Phi_{n,m}.$$
(3) We first note that $||w_m| - |w_{m+u}||$ has a finite second moment. Indeed, this amount is controlled by the increments done during $u$ steps, and we know the distribution of how large $u$ can be. That is, we use Lemma 3.6 to justify this. The same is true for $\big||w_m^{-1}w_n| - |w_{m+u}^{-1}w_n|\big|$. Again, this follows from a triangular inequality and Lemma 3.6.
(4) From this, we just need a good upper bound for $\Phi_{n,m}$ in the inequality
$$|w_n| \geq |w_{m+u}| + |w_{m+u}^{-1}w_n| - \Phi_{n,m}.$$
(5) We note that $\big||w_{m+u+N}^{-1}w_n| - |w_{m+u}^{-1}w_n|\big|$ is a bounded constant (since it only depends on $N$), and in particular has a finite second moment.
(6) From this, we just need a good upper bound for $\Phi_{n,m}$ in the inequality
$$|w_n| \geq |w_{m+u}| + |w_{m+u+N}^{-1}w_n| - \Phi_{n,m}.$$
(7) We look at the TSP between time 0 and $n$, we use the path $\eta$ from the previous lemma to get a path which is near optimal and crosses only once the neighborhood of $\gamma$.
(8) From this path we obtain near-optimal paths from $|w_{m+u}|$ and for $|w_{m+u+N}^{-1}w_n|$, by doing surgery near the endpoints of $\gamma$ and possibly adding a constant bounded amount of length.

    Indeed, we first take the path from the starting point to the last visit to $P_1$, and we connect it to $w_{m+u}$. This is at most *Optimal* $+ L + 2D$. Similarly we look at the first time we enter $P_3$, and connect that to a path to $w_{m+u+N}$. This again adds at most *Optimal* $+ L + 2D$.
(9) From this, we directly apply 2.2.

## 4. The CLT for the range of random walks on hyperbolic groups

We borrow the framework from [MS20] for proving a CLT - We observe the following trivial fact that whenever $1 \leq m \leq n$ and denoting $R_{m,n}$ for the range between times $m$ and $n$ we have

$$|R_n| = |R_m| + |R_{m,n}| - |R_m \cap R_{m,n}|.$$

In the language of [MS20], we say that $\{|R_n|\}_{n\geq 1}$ is a *defective adapted cocycle* with defect $\Phi_{m,n} := |R_m \cap R_{m,n}|$.

By theorem 4.2 in [MS20], to prove a CLT for the sequence $|R_n|$ it is enough to show a second-moment deviation inequality: that

$$\mathbb{E}[\Phi_{m,n}^2] \leq C.$$

For some $C$ not depending on $m, n$. We instead prove a stronger version of the deviation inequality:

**Proposition 4.1.** *There exists $C > 0$ such that for any $1 \leq m \leq n$.*
$$\mathbb{P}(\Phi_{m,n} \geq k) \leq Ce^{-k/C},$$

*Proof.* Let $\hat{R}_n$ denote the range of the reversed random walk - that is, the random walk driving by $\hat{\mu}$.' If is enough to show that for any $n, n' \in \mathbb{N}$ we have
$$\mathbb{P}(\sup_{n,n'} |\hat{R}_n \cap R_n| \geq k) \leq Ce^{-k/C}.$$

This is an immediate consequence of lemma 5.3 of [Cho23b]. (maybe this is actually Lemma 4.9 of the arxiv version of [Cho23b]?) □

This concludes the proof of Theorem 1.2.

I have a question: do we know of ANY random walk that is transient but does not satisfy a CLT for range? I think this may be unknown

Whenever you have a positive density of cut times, I think you should be able to make a regeneration-type argument to prove a CLT for the range, so you need to look for transient random walks which travel sublinearly, maybe $\mathbb{Z}^3$ is a good candidate?

Yes indeed, on $Z^3$ the variance of the range is not like $\sqrt{n}$. However, with the appropriate normalization there is a CLT; this is proved here [JP70, JP71]. Maybe one can construct groups that interpolate between $\mathbb{Z}^3$ and something else to find a group that has different limit laws along different subsequences?

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