CLT FOR RANGE OF RANDOM WALKS ON HYPERBOLIC GROUPS

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Abstract. We prove a central limit theorem for random walks with a finitely supported step distribution on wreath products of the form $A \wr H = \bigoplus_H A \wr H$, where A is a non-trivial finite group and H is a non-elementary hyperbolic group.

1. Introduction

In this paper we consider wreath products where the group of lamps is a finite group, and the base group is a hyperbolic group. We will consider a word length in these wreath products given by the switch-walk-switch generating set with respect to some fixed generating set of H (see Subsection 2.3.1).

Theorem 1.1. *Let A be a non-trivial finite group and let H be a non-elementary hyperbolic group. Endow $A \wr H$ with the switch-walk-switch word metric $|\cdot|_{\mathrm{sws}}$ associated with some arbitrary finite generating set of H. Consider a finitely supported probability measure μ on $A \wr H$ such that $\mathrm{supp}(\mu_H)$ is non-elementary. Denote by $\{w_n\}_{n\geq 0}$ the μ-random walk on $A \wr H$, and let $C = \lim_{n\to\infty} \frac{\mathbb{E}(|w_n|_{\mathrm{sws}})}{n}$ be the drift of the μ-random walk on $A \wr H$. Then the sequence of normalized random variables $\frac{|w_n|_{\mathrm{sws}} - Cn}{\sqrt{n}}$, $n \geq 1$, converges in law to a non-degenerate gaussian random variable.*

This result is new even for simple random walks on $\mathbb{Z}/2\mathbb{Z} \wr F_2$.

1.1. Background.

- The CLT for non-abelian free groups is due to [SS87] and [Led01]. Then for non-elementary hyperbolic groups with a finite exponential moment is due to [Bjö10]. This was generalized for any finite second moment measure in [BQ16a]. The last two results hold more generally for group acting on a Gromov hyperbolic space by isometries. [BQ16b] show a CLT for random walks on $\mathrm{GL}_d(\mathbb{R})$ with a finite second moment. See also [Gou17].
- [EZ22] prove a law of large numbers for random walks on $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^2$ with a finite $(2+\varepsilon)$-moment, for some $\varepsilon > 0$. They also discuss limit laws in other wreath products.
- [BFGK24] prove a central limit theorem for random walks on the group of affine transformations of a horospherical product of Gromov hyperbolic spaces.
- [Cho23] proves a central limit theorem for groups acting with contracting elements.
- [GTT22] prove a CLT with respect to the counting measure on the Cayley graph of a group acting on a hyperbolic space.
- [Hor18] proves a CLT for random walks on mapping class groups and $\mathrm{Out}(F_n)$.
- [Bar22] proves a CLT for groups acting on a CAT(0) space.
- [Gil08] proves that the drift of $\mathbb{Z}/2\mathbb{Z} \wr G$ is strictly larger than that of its projection to G.
- [MSŠ23] prove a LLN and CLT for the capacity of the range of a random walk on a group.
- [Sal01] proves a LLN and CLT for a simple random walk on a free group, conditioned on the boundary point.

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2. Preliminaries

2.1. Notation for graphs and paths.

- We will work with undirected, unlabeled graphs (V, E).
- A path $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n)$ is an ordered sequence of vertices in the graph.

2.2. Hyperbolic groups.

Say basic things about hyperbolicity; explain pivots

2.3. Wreath products.

We consider the wreath products $A \wr H$, where A is a finite non-trivial group and H is a finitely generated group. Let S_H be a finite and symmetric generating set of H.

2.3.1. *The switch-walk-switch word metric.* We consider the *switch-walk-switch* S_{sws} generating set of $A \wr H$, given by

$$S_{\text{sws}} := \Big\{ (\delta_a, 0)(\mathbf{0}, s)(\delta_{a'}, 0) \Big| a, a' \in A \text{ and } s \in S_H \Big\}.$$

The following goes back to [Par92, Theorem 1.2].

Proposition 2.1. *For any $g = (f, x) \in A \wr H$, the word length of g with respect to the standard generating set is*

$$|g|_{\text{sws}} = \mathrm{TSP}(e_H, x, \mathrm{supp}(f)).$$

2.4. Random walks on groups.

Let G be a countable group and consider a probability measure μ on G. Consider the product space $\Omega := G^{\mathbb{Z}_+}$ endowed with the product σ-field. For each $n \geq 1$ we denote by

$$X_n : \Omega \to G$$
$$w := (w_1, w_2, \cdots) \mapsto X_n(w) := w_n$$

the n-th coordinate map. We endow Ω with the product probability measure $\mu^{\mathbb{Z}_+}$.

We denote by

$$\theta : \Omega \to \Omega$$
$$w := (w_1, w_2, \cdots) \mapsto \theta(w) := (w_2, w_3, \dots)$$

the shift map in the space of increments.

Now we define the μ-random walk $\{Z_n\}_{n \geq 0}$ on G as follows. We define $Z_0(w) = e_G$ for each $w \in \Omega$, and for each $n \geq 1$ we define

$$Z_n(w) := Z_{n-1}(w) \cdot X_n(w).$$

We remark that $Z_n(w)(Z_m \circ \theta^n)(w) = Z_{n+m}(w)$, for each $w \in \Omega$ and $n, m \geq 1$.

3. Mathieu-Sisto's deviation inequalities and consequences

3.1. Defective adapted cocycles and the central limit theorem.

A sequence $\mathcal{Q} = \{Q_n\}_{n \geq 1}$ of maps $Q_n : \Omega \to \mathbb{R}$ such that Q_n is measurable with respect to $\sigma(X_1, \dots, X_n)$, for each $n \geq 1$, is called a *defective adapted cocycle*. We will use the convention $Q_0 \equiv 0$. The *defect of* $\mathcal{Q}$ is the collection of maps $\Psi = \{\Psi_{n,m}\}_{n,m \geq 0}$ defined by

$$\Psi_{n,m}(w) = Q_{n+m}(w) - Q_n(w) - (Q_m \circ \theta^n)(w), \text{ for each } w \in \Omega \text{ and } n, m \geq 0.$$

The following result states that the central limit theorem holds for defective adapted cocycles that satisfy a second-moment deviation inequality.

Theorem 3.1 ([MS20, Theorem 4.2])**.** *Let G be a countable group endowed with a probability measure μ. Consider $\mathcal{Q}$ a defective adapted cocycle on $\Omega = G^{\mathbb{Z}_+}$, and denote by $\{\Psi_{n,m}\}_{n,m \geq 0}$ its defect. Suppose that*

1. $\mathbb{E}[|Q_1|^2] < \infty$, *and*
2. $\sup_{m,n \geq 0} \left\{ \mathbb{E}\left[|\Psi_{n,m}|^2 \right] \right\} < \infty$.

Then, there exist constants $\ell, \sigma \in \mathbb{R}$ such that the random variables $\frac{1}{\sqrt{n}}(Q_n - \ell n)$ converge in law to a Gaussian random variable with zero mean and variance σ^2.

Furthermore, it is proved in [MS20, Theorem 3.3] that the constant ℓ that appears in the statement of Theorem 3.1 satisfies that $\frac{1}{n}Q_n$ converges to ℓ in L_1 as $n \to \infty$.

We will use this result for the defective adapted cocycle obtained from the word length of the μ-random walk on G at time n. That is, we will consider some word metric d on G, and define $Q_n \coloneqq d(e_G, Z_n)$, for each $n \geq 1$. Since we will be working with finitely supported probability measures, it holds immediately that $\mathbb{E}[|Q_1|^2] < \infty$.

The objective of the following sections of this paper is to prove that there exists a constant $C > 0$ such that

$$\mathbb{E}[d(e_G, Z_{n+m}) - d(e_G, Z_n) - d(e_G, Z_m \circ \theta^n)] = \mathbb{E}[|\Psi_{n,m}|^2] \leq C, \text{ for each } n, m \geq 0,$$

where $G = A \wr H$, d will be the switch-walk-switch word metric and μ is a finitely supported probability measure, as in the hypotheses of Theorem 1.1.

3.2. **Continuity results.** The following results are obtained in Sections 5 and 6 of Mathieu-Sisto paper.

(1) Continuity and differentiability of drift .
(2) Continuity and differentiability of asymptotic entropy.

We will apply them to lamplighters over hyperbolic groups.

3.3. **Polylog deviation inequalities.** In this section we explain a generalization of [MS20, Theorems 4.1 & 4.2] (see Theorem 3.1) remember to do the modification of n to m+n.

Theorem 3.2. *Suppose that Q_n is a defective adapted cocycle with defect*

$$\Psi_{n,m}(w) = Q_{n+m}(w) - Q_n(w) - (Q_m \circ \theta^n)(w), \text{ for each } w \in \Omega \text{ and } n, m \geq 0.$$

Suppose that for some fixed polynomial p and $N_0 \in \mathbb{N}$ we have that

$$\mathbb{E}[|\Psi_{n,m}|^2] \leq p(\log(n+m))$$

whenever $n, m \geq N_0$. Then a CLT holds for Q_n.

Here are the places in the proof of Mathieu-Sisto's CLT where the deviation inequality is used:

(1) The deviation inequality is used in multiple occasions during the proof of Lemma 4.6. One should do the appropriate modifications.

The proof follows very similarly to that of [MS20, Theorems 4.1& 4.2]. We now explain the parts of the proof where minor modifications are needed.

The first step in the proof is [MS20, Theorem 4.4] and its application to obtain [MS20, Theorem 4.2]. With an analogous proof, one obtains

$$V(Q_n) \leq n\left(p\mathbb{E}(Q_1^2) + 16\log(n)\right),$$

where p is the polynomial from the second moment deviation inequality.

The second part of the proof is in [MS20, Lemma 4.5]. Here, we can use [Ham62, Theorem 2] to obtain the following statement.

Lemma 3.3. *Let $(a_n)_{n\geq 0}$ be a sequence of real numbers and let p be a polynomial. Suppose that there is $b \geq 0$ such that*

$$a_{n+m} \leq a_n + a_m + bp(\log(n+m)), \text{ for each } n, m \geq 1.$$

Then $\lim_{n\to\infty} \frac{a_n}{n}$ exists.

The most important part of the proof is the following lemma.

Lemma 3.4. *Let $(Q_n)_n$ be a DAC with a finite second moment and that satisfies a second-moment deviation inequality. Then*

$$\lim_{M\to\infty}\limsup_{n\to\infty}\frac{1}{n}V\Big(Q_n-\sum_{j=0}^{\lceil n/2^M\rceil-1}Q_{2^M}\circ\theta_{j2^M}\Big)=0.$$

Proof. This should be done more carefully. □

4. General bounds for the defect of the TSP

In this section we explain the key inequality for the TSP along the trajectory of the random walk. We will use this inequality to bound the norm of defective adapted cocycles in the next sections.

Let (X,d) be a proper geodesic metric space. Recall that for the points $P,Q\in X$ and a finite set of points L, we denote by the $TSP(P,L,Q)$ the length of the shortest path in X which starts at P, ends at Q, and visits every point in L. Let α be some solution to the $TSP(P,L,Q)$ and let us list the points of $L=\{l_1,\dots,l_k\}$ in the order of their fist appearances along α $L=(l_{\pi(1)},\dots,l_{\pi(k)})$. Then we have the following equality

$$TSP(P,L,Q)=d(P,l_{\pi(1)})+\sum_{i=1}^{k-1}d(l_{\pi(i)},l_{\pi(i+1)})+d(l_{\pi(k)},Q)$$

and the permutation $\pi\in\mathrm{Sym}(k)$ determines α uniquely up to the choice of the geodesic segments connecting P with $l_{\pi(1)}$, Q with $l_{\pi(k)}$ and $l_{\pi(i)}$ with $l_{\pi(i+1)}$ for $i=1,\dots,k-1$. For any solution α of the $TSP(P,L,Q)$, we will refer to the points in $\{P,Q\}\cup L$ as the *nodes* of α. In the proof of the next lemma we will view α as a homeomorphism from some segment into X, and whenever we talk about a subpath of α we mean the restriction of this homeomorhpism to a closed subsegment. Here we should change the symmetric difference by a union

Lemma 4.1. *Let P,Q,R be three distinct points in the metric space X. Pick two finite sets $L_1,L_2\subset X$. We will call the points in $L_1\cup L_2\cup\{P,Q,R\}$ the marked points. Let $I,C,T\subseteq X$ be three bounded sets such that the following conditions hold.*

1. *$P\in I$, $Q\in C$ and $R\in T$.*
2. *$I\cap T=\emptyset$.*
3. *$L_1\subseteq I\cup C$ and $L_2\subseteq C\cup T$.*

Moreover, assume that there are two compact sets $B_1,B_2\subseteq X$ such that the following conditions hold.

1. *$B_1\cap T=B_2\cap I=\emptyset$.*
2. *Any geodesic segment joining a marked point in I with a marked point in $C\cup T$ intersects B_1, and any geodesic segment joining a marked point in T with a marked point in $I\cup C$ intersects B_2.*
3. *If x is any marked point in I and y is any marked point in C, then for any geodesic segment γ connecting x and y the closest to x point in $\gamma\cap(B_1\cup B_2)$ is in B_1 and the closest to y point in $\gamma\cap(B_1\cup B_2)$ is in B_2.*
4. *If D_1 is the maximum of diameters of B_1 and B_2, then $d(B_1,B_2)>2D_1$.*

Let D_2 denote the diameter of $B_1\cup C\cup B_2$ and let N be the number of marked points in C. Then the following inequality holds.

$$0\le TSP(P,L_1,Q)+TSP(Q,L_2,R)-TSP(P,L_1\Delta L_2,R)\le N(12D_1+2D_2)$$

Proof. Since the concatenation of any solution of $TSP(P,L_1,Q)$ with any solution of $TSP(Q,L_2,R)$ at Q produces a path that starts in P, ends at R, and visits every point in $L_1\Delta L_2$, the first inequality follows.

In order to prove the second part of the inequality we will show that any solution α of the $TSP(P, L_1\Delta L_2, R)$ contains two non-overlapping parts α_I and α_T, such that α_I is close to the solution of $TSP(P, L_1, Q)$ and α_T is close to the solution of $TSP(Q, L_2, R)$.

Let α be any path from P to R realizing the solution to $TSP(P, L_1\Delta L_2, R)$. We define *the trace* of α in $I \cup B_1$ denoted by α_I as follows. Let π be the permutation of $L_1\Delta L_2 = \{x_1, \dots, x_k\}$ that defines α and let s_i be a geodesic segment of α joining $x_{\pi(i)}$ and $x_{\pi(i+1)}$. Then, if both endpoints of s_i belong to I we include s_i into the trace α_I. If exactly one of the endpoints of s_i, let us denote it by x, is in I, then we can find the closest to x intersection of s_i with B_1, denoted by y, and add the part of s_i between x and y to the trace α_I. Otherwise, no new points from s_i are added to the trace.

It is easy to see that α_I is a union of maximal subpaths $p_1, \dots p_t$ of α, such that for any $i = 1, \dots, t$ the subpath p_t has both of its endpoints in $B_1 \cup \{P\}$. Now, we are going to show that one can add a collection of geodesic segments of total length at most $N(6D_1 + D_2)$ to α_I to get a path that starts at P, ends at Q, and visits every point in L_1.

Since the sets I, C, T satisfy the conditions (1)-(3) from the statement of the lemma, we have $L_1 \cap (I \setminus C) = (L_1\Delta L_2) \cap (I \setminus C)$, so α_I already contains every point in $L_1 \cap (I \setminus C)$. Therefore, if β is any path that starts at P, contains α_I, and ends at a point in B_1, then one can extend β to a path that visits every point of L_1 and ends at Q as follows. First, we connect the points in $L_1 \cap C$ by geodesic segments in arbitrary order, and then one of the endpoints of the resulting path is connected with the endpoint of β that lies in B_1, while the other end of this path is connected to Q. Notice that the total length of the geodesic segments that we add to β in this procedure will not exceed ND_2.

Next we will construct a suitable path β by joining the subpaths $p_1, \dots, p_t$ of α_I by at most $6N$ geodesic segments, with each segment having the length at most D_1. We need the following fact.

Claim 1. Let $p_1, \dots, p_t$ be the list of the subpaths of α_I defined above. Then $t \leq 6N$.

Proof. Let π be the permutation defining α and let $s_1, ...s_{k+1}$ be the corresponding geodesic segments. Notice that by definition each of these segments corresponds to the first visit of α to a new point from $L_1\Delta L_2$.

We will trace the paths $p_1, \dots, p_t$ as we travel along α. By definition, each path p_i, except for p_1, starts and ends with subsegments of uniquely determined geodesic segments s_i' and s_i of α which connect marked points with points in B_1. The other endpoint of s_i must be a new marked point that belongs either to $C \setminus T$ or to T. If this marked endpoint is in T we will call s_i a *leap*, and if it is in $C \setminus T$, we will call s_i a *step*. It is easy to see that the number of *steps* can not exceed the number of marked points in $C \setminus T$ and this number is less or equal to N. Therefore, it suffices to show that the number of leaps can not exceed $5N$.

We will prove even stronger statement, namely, that the number of the geodesic segments of α joining marked vertices in I with marked vertices in T can not exceed $5N$. For the following combinatorial argument, it will be convenient to introduce the coding of the nodes of α by symbols $\mathcal{I}, \mathcal{C}, \mathcal{T}$. Naturally, a point is coded by $\mathcal{I}$ if it belongs to I, by $\mathcal{C}$ if it belongs to $C \setminus (I \cup T)$ and by $\mathcal{T}$, if the point is in T. Thus, if α has $k+2$ nodes including P and R, the coding will produce a word ω of length $k+2$ starting with letter $\mathcal{I}$ and ending with $\mathcal{T}$ which contains no more than N letters $\mathcal{C}$. Moreover, every geodesic segment of α joining a node in I with a node in T corresponds to a unique subword $\mathcal{IT}$ or $\mathcal{TI}$ in ω. We claim that if the number of subwords $\mathcal{IT}$ or $\mathcal{TI}$ in ω is at least $5N$, then α could be replaced by a strictly shorter path contradicting the definition of the solution to the TSP. Indeed, assume that ω contains at least $5N$ subwords of the form $\mathcal{IT}$ or $\mathcal{TI}$, then since it contains only N letters $\mathcal{C}$, there exists a subword of ω which contains 4 consecutive subwords of the form $\mathcal{IT}$ or $\mathcal{TI}$ and does not contain letter $\mathcal{C}$. Hence this word contains a subword of the form $\mathcal{I}\mathcal{T}^a\mathcal{I}^b\mathcal{T}$ where $a, b \geq 1$. In geometric terms this means that α contains a path γ of the following form:

(1) γ contains three geodesic segments of α P_1Q_1, Q_2P_2, and P_3Q_3 such that $P_1, P_2, P_3 \in I$ and $Q_1, Q_2, Q_3 \in T$, and γ starts with the segment P_1Q_1 and ends with the segment P_3Q_3.
(2) all nodes of γ between Q_1 and Q_2 are contained in T, while all of its nodes visited between P_2 and P_3 belong to I.

Notice that conditions (1)-(3) on B_1 and B_2 imply that each of the segments P_1Q_1 and P_2Q_2 intersects both B_1 and B_2, and the length of each of these segments is at least $2D_1$.

Now we are going to run a "surgery" procedure that produces a suitable shortening γ' of γ. Let x_1 and x_2 be closest to P_1 and to P_2, respectively, intersections of P_1Q_1 and of Q_2P_2 with B_1, and for $i = 1, 2, 3$ we define y_i as the closest to Q_i intersection of P_iQ_i with B_2. Then γ' is constructed as follows. It starts at P_1 and tracks P_1Q_1 until it reaches x_1, then γ' moves from x_1 to x_2 and follows γ until it reaches y_3 visiting all of the nodes between P_2 and P_3 in the process. After γ' reaches y_3 it moves to y_1 and follows γ until it reaches y_2 visiting all of the nodes between Q_1 and Q_2 in the process. Finally, from y_2 γ' moves to y_3 and follows γ to Q_3.

It is easy to see that γ' starts at P_1, ends at Q_3, and visits all of the nodes of γ. Moreover, γ' is obtained by removing segments x_1y_1 and x_2y_2, each of length greater than $2D_1$, from γ, and adding the segments x_1x_2, y_3y_1 and y_2y_3. Since D_1 is the maximum of the diameters of B_1 and B_2, the total length of the segments added does not exceed $3D_1$, so γ' is indeed strictly shorter than γ.

Finally, if one replaces γ in α by γ', the resulting path still starts at P, ends at R, and visits every point in $L_1\Delta L_2$, but is shorter than α, and this contradicts the assumption that α realizes the solution to the $TSP(P, L_1\Delta L_2, R)$. Therefore, the number of leaps is also bounded by $5N$, and $t \leq 6N$. □

Now the construction of β is completed as follows. For $i = 1, \ldots, t-1$ connect the ending point of p_i in B_1 with the starting point of p_{i+1} in B_1 by a geodesic segment, by definition of D_1 such a segment of would have length at most D_1. Since $t \leq 6N$, we have added the segments of total length at most $6ND$ to α_I, and it is easy to see that the resulting path starts at P, contains α_I, and ends at a point in B_1.

Therefore, we can conclude that the length $l(\alpha_I)$ satisfies the inequality

$$l(\alpha_I) \geq TSP(P, L_1, Q) - N(6D_1 + D_2).$$

The trace α_T is defined similarly, and a similar argument shows that

$$l(\alpha_T) \geq TSP(P, L_2, Q) - N(6D_1 + D_2).$$

It is easy to see that no node of α appears in both α_I and α_T, so they have no subpath of α in common, and therefore, we have

$$TSP(P, L_1, Q) + TSP(Q, L_2, R) - N(12D_1 + 2D_2) \leq l(\alpha_I) + l(\alpha_T) \leq l(\alpha)$$

This completes the proof of the second inequality.

□

5. Acylindrically hyperbolic groups

In this section we will use lemma 4.1 to obtain the upper bounds on the moments of the defective adapted cocycles when the base group H is acylindrically hyperbolic.

We remind that our goal is to prove an inequality of the following kind: there is a polynomial with positive coefficients p (probably power 5 or 6) such that for any $n.m \geq 1$ we have

$$\mathbb{E}(|\|\Psi_{n,m}|^2) \leq p(\log(n) + \log(m)).$$

With this in mind, for finitely supported random walk, then we can restrict to supposing $n > C\log(n+m)$ for some big constant C. Indeed, if $n \leq C\log(n+m)$, then $\Psi_{n,m} \leq Cn$ (it is after all a difference of the word lengths).

The following proposition is a straightforward corollary of Theorem 9.1 and Theorem 10.7 in [MS20]

Proposition 5.1. *Let H be a finitely generated acylindrically hyperbolic group and let μ_H be a symmetric non-elementary probability measure on H with finite exponential moment. Choose arbitrary finite symmetric generating set of H and let d_H be the corresponding word metric on H. Finally, we denote by Z_n the Then the following statements hold.*

(1) *There exists a constant K such that for any $n \geq 1$*

$$\mathbb{P}^{\mu_H}\left(d_H(Z_n, e_H) \leq n/K\right) \leq Ke^{-n/K}$$

niform geodesic tracking *There is a constant C such that for any $n \geq 1$, for each pair (i,j) with $1 \leq i < j \leq n$ and each geodesic segment α joining Z_i and Z_j we have*

$$\mathbb{P}^{\mu_H}\left(\max_{i\leq k\leq j} d_H(Z_k, \alpha) \geq C\log n\right) \leq C/n^4$$

Proof. The first statement immediately follows from Theorem 9.1, Remark 10.2 and statement 1 in Proposition 10.3 in [MS20]

The second statement follows from Theorem 10.7 and Remark 10.2 in [MS20] combined with the union bound. □

We should decide whether to replace the first item with union bound.

Lemma 5.2. *Let K_0 be the constant of Proposition 5.1. Then there is a constant K_1 such that with probability $1 - \frac{1}{(n+m)^2}$ we have the following. As soon as $|i-j| \geq K_1\log(n+m)$, for $i, j \in \{0, \ldots n+m\}$, we have $d_H(Z_i, Z_j) \geq \frac{|i-j|}{K_0}$.*

Proof. Union bound using Proposition 5.1. □

Our aim is to reduce the situation to a deterministic setting that occurs with high probability. Then, we apply the combinatorial argument, and afterwards we estimate the moments associated with the constants that appear in the combinatorial lemma, using that the deterministic setting occurs with high probability.

From now on consider $n, m \geq 1$ such that $n, m \geq C_0\log(n+m)$.

From now on, using the uniform geodesic tracking from Proposition 5.1, we will assume that there is a constant C_1 for each pair (i,j) with $1 \leq i < j \leq n+m$ and each geodesic segment α joining Z_i and Z_j we have

$$\max_{i\leq k\leq j} d_H(Z_k, \alpha) \leq C_1\log(n+m),$$

with probability at least $1 - C_1\frac{1}{(n+m)^4}$.

From now on, let us denote

$$W = C_1\log(n+m).$$

5.1. **Definitions of $\mathfrak{I}$, $\mathfrak{M}$ and $\mathfrak{T}$.** We will first define the sets $\mathfrak{I}$, $\mathfrak{M}$ and $\mathfrak{T}$, and then explain why they have the properties that we need.

Now we fix a constant C_2 much larger than C_1.

We take $R = d_H(e_H, Z_m)$. Let us fix a geodesic α from e_H to Z_{n+m}. Consider the neighborhood of α of radius W:

$$N_W(\alpha) = \{g \in H \mid d_H(\alpha, g) \leq W\}.$$

We define

$$\mathfrak{I} := \{g \in H \mid d_H(g, e_H) \leq R - C_2 \log(n+m)\} \cap N_W(\alpha).$$

$$\mathfrak{T} := \{g \in H \mid d_H(g, e_H) \geq R + C_2 \log(n+m)\} \cap N_W(\alpha).$$

$$\mathfrak{M} := \{g \in H \mid R - C_2 \log(n+m) \leq d_H(g, e_H) \leq R + C_2 \log(n+m)\} \cap N_W(\alpha).$$

Matthieu-Sisto, with our choice of n and m, will guarantee that $\mathfrak{I}$ and $\mathfrak{T}$ will be non-empty.

By definition we have $Z_0 = e_H \in \mathfrak{I}$ and $Z_m \in \mathfrak{M}$. We will prove that with high probability, the trajectory of the random walk between times 1 and m does not enter $\mathfrak{T}$, trajectory between m to $m+n$ does not enter $\mathfrak{I}$. In particular, $Z_{m+n} \in \mathfrak{T}$ with high probability.

From the definition, we have $\mathfrak{I} \cap \mathfrak{T} = \varnothing$. For us, $L_1 = \{Z_0, Z_1, \dots, Z_m\}$ and $L_2 = \{Z_{m+1}, \dots, Z_{m+n}\}$.

Let us now define B_1 and B_2.

$$B_1 := \{g \in H \mid d_H(g, e_H) = R - C_2 \log(n+m)\} \cap N_{4W}(\alpha).$$

$$B_2 := \{g \in H \mid d_H(g, e_H) = R + C_2 \log(n+m)\} \cap N_{4W}(\alpha).$$

By definition we have $B_1 \cap \mathfrak{I} = B_2 \cap \mathfrak{T} = \varnothing$. This verifies the first condition. The third condition follows from the definition. The fourth condition will follow from the fact that we chose $C_2 >> C_1$. The second condition will be the most technically challenging to verify. It will follow from our assumption of uniform geodesic tracking.

Now we will prove that the sets we have defined together with the points Z_0, Z_m and Z_{n+m} verify the conditions of the combinatorial lemma.

Lemma 5.3. *If we choose $C_2 >> C_1$ large enough, under the assumptions of uniform geodesic tracking and Lemma 5.2, we can guarantee that $\{Z_0, Z_1, \dots, Z_m\} \cap \mathfrak{T} = \{Z_m, Z_{m+1}, \dots, Z_{m+n}\} \cap \mathfrak{I} = \varnothing$.*

Proof. Otherwise, you can find points Z_i, Z_j such that $d_H(Z_i, Z_j) \leq \frac{|i-j|}{K_0}$ with $|i-j| \geq 3C_1 K_1 \log(n+m)$.

Now, if we choose $C_2 > 3C_1K_1 \cdot$ (*step length*) the random walk cannot possibly enter $\mathfrak{I}$ after moment m. This proves that $\{Z_m, Z_{m+1}, \dots, Z_{m+n}\} \cap \mathfrak{I} = \varnothing$. Then a similar reasoning shows $\{Z_0, Z_1, \dots, Z_m\} \cap \mathfrak{T} = \varnothing$.

$\square$

Lemma 5.4. *Any geodesic segment joining a point of the trajectory of the random walk $(Z_k)_k$ inside $\mathfrak{I}$ with a marked point in $\mathfrak{M} \cup \mathfrak{T}$ intersects B_1, and any geodesic segment joining a point of the trajectory of the random walk $(Z_k)_k$ inside $\mathfrak{T}$ with a marked point in $\mathfrak{I} \cup \mathfrak{M}$ intersects B_2.*

Proof. We will provide the proof for B_1; for B_2 it is analogous.

Suppose that we have $Z_i \in \mathfrak{I}$ and $Z_j \in \mathfrak{M} \cup \mathfrak{T}$. Then we can track trajectory of the random walk between i and j and join them with geodesic segments. This gives a path that we call $P_{i,j}$.

Let us denote by C the maximal jump of the random walk. Denote by γ a geodesic joining Z_i and Z_j. Then for any $y \in P_{i,j}$ is within distance at most $W + C$ from both α and γ (this follows from uniform geodesic tracking).

To prove our claim, it suffices to show that any point in γ is within $4W$ distance of α. Then the claim follows from continuity of distance (i.e. at some point the path crosses the appropriate sphere).

Take any $y \in \gamma$. We will prove that there is a point in α at distance at most $4W$ from y. Let us split γ into two closed segments $[Z_i, y]$ and $[y, Z_j]$. Then any point in $P_{i,j}$ is at distance at most $W + C$ from at least one of these subsegments. By connectivity, we can find a point $z \in P_{i,j}$ such that z is at distance at most $W + C$ from both segments. Let us denote $\pi_1(z) \in [Z_i, y]$ and $\pi_2(z) \in [y, Z_j]$ such that $d_H(z, \pi_1(z)) \leq W + C$ and $d_H(z, \pi_2(z)) \leq W + C$. The subpath of γ that connects $\pi_1(z)$ and $\pi_2(z)$ will be geodesic and contain y. Then, by triangular inequality, the length of this geodesic subpath is at most $2W + 2C$. suppose without losing generality that $d_H(y, \pi_1(z)) \leq W + C$. Then the path that connects y to $\pi_1(z)$ to z and then to the projection of z on α has length at most $3W + 3C$, and we can assume that W is larger than $3C$. Hence we conclude the upper bound of $4W$. □

5.2. **Estimates for the moments of N, D_1 and D_2.** From what we have done so far we have $D_1 \leq 16W$ and $D_2 \leq 16W + C_2 \log(n + m)$. Now we will bound N.

For N we have that it is bounded by the diameter of $\mathfrak{M}$ times K_0. This follows from uniform progress of the random walk. This gives the upper bound $N \leq K_0(16W + 2C + 2\log(n + m)) + 1$.

6. Hyperbolic groups

We do as above, following similarly to the combinatorial proof I wrote with Kunal using pivots. We need to phrase it, as above, in terms of the combinatorial lemma.

7. CLT for the lamplighter over a hyperbolic group (using pivots)

Let us consider a non-trivial finite group A and a non-elementary hyperbolic group H. Choose an arbitrary symmetric finite generating set S_H of H. Let us denote by d_H (resp. $|\cdot|_H$) the associated word metric (resp. word length) on H, and by d_{sws} (resp. $|\cdot|_{\text{sws}}$) the switch-walk-switch word metric (resp. word length) on $A \wr H$. Additionally, let us consider $\delta \geq 0$ to be a hyperbolicity constant of $\mathrm{Cay}(H, S_H)$.

7.1. Pivots.

Definition 1. Given a path $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ on $\mathrm{Cay}(H, S_H)$ and $g \in H$, let $\pi_\gamma(g)$ be the set of elements visited by γ that minimize the word metric to g. That is, if we denote by $d_H(g, \gamma) := \min\{d_H(g, \gamma_i) \mid i = 1, 2 \ldots, k\}$ the minimal distance between g and any element of the path γ, then

$$\pi_\gamma(g) := \{h \in G \mid h = \gamma_i \text{ for some } i \in \{1, \ldots, k\} \text{ and } d_S(g, h) = d_H(g, \gamma)\}. \tag{1}$$

We now introduce the definition of pivots that we will use in the proof of Theorem 1.1. We refer to [Gou22, Section 4A] for details.

Definition 2. Let $C, D > 0$, $L \geq 20C + 100\delta + 1$, and $N \in \mathbb{Z}_{\geq 1}$. Let $\mathbf{w} = \{w_n\}_{n \geq 1} \in \Omega = (A \wr H)^{\mathbb{Z}_+}$, and consider the associated trajectory of the random walk $\{Z_n(\mathbf{w})\}_{n \geq 0}$. To avoid having too much notation we will omit the dependence of Z_n on $\mathbf{w}$. Denote by Z_n^H the projection of Z_n to H, for each $n \geq 0$. A time instant $m \geq 1$ is called a (C, D, L, N)-*pivot* for $\mathbf{w}$ if the following conditions hold.

(1) $d_H\left(Z_m^H, Z_{m+N}^H\right) \geq L$.

Let γ be an arbitrary geodesic path in $\mathrm{Cay}(H, S_H)$ that connects Z_m^H to Z_{m+N}^H. Then

(2) for all $0 \leq k \leq m$ we have $d_H\left(\pi_\gamma\left(Z_k^H\right), Z_m^H\right) \leq C$,

(3) for all $m \leq k \leq m + N$ we have $d_H\left(Z_k^H, \gamma\right) \leq D$, and

(4) for all $k \geq m + N$, we have $d_H\left(\pi_\gamma\left(Z_k^H\right), Z_{m+N}^H\right) \leq C$.

The following lemma will be our main tool.

Lemma 7.1. *For any $C, D, \delta > 0$, and any $L \geq 20C + 100\delta + 1$, there exists $N, R > 0$ such that for all $k \geq 1$ we have*

$$\sup_{i \geq 1} \mathbb{P}\,(\text{there is no } (C,D,L,N)\text{-pivot between instants } i \text{ and } i+k) \leq Re^{-k/R}.$$

7.2. **The structure of the TSP along pivot times.** Suppose that we are looking at a sample path $\{Z_n\}_{n\geq 0}$ of the μ-random walk on $A \wr H$, and that we have a (C, D, L, N)-pivot time m. Denote by $Z_n = (\varphi_n, Z_n^H)$ the lamp configuration and projection to H of Z_n, respectively. Let γ be an arbitrary geodesic path from Z_m^H to Z_{m+N}^H on $\mathrm{Cay}(H, S_H)$. Then, using that the support of μ is finite, we can find constants $C, D \geq 1$ such that the support of φ_n can be decomposed as a disjoint union

$$\mathrm{supp}(\varphi_n) = P_1' \cup P_2' \cup P_3',$$

that satisfies the following properties. Let us denote $P_1 = P_1' \cup \{Z_{m+u}^H\}$, $P_2 = P_2' \cup \{Z_m^H, Z_{m+N}^H\}$ and $P_3 = P_3' \cup \{Z_{m+N}^H\}$. Then we have $d_H(Z_m^H, Z_{m+N}^H) \geq L$, together with

(1) for all $g \in P_1$, we have $d_H\left(\pi_\gamma(g), Z_m^H\right) \leq C$,
(2) for all $g \in P_2$ we have $d_H(g, \gamma) \leq D$, and
(3) for all $g \in P_3$, we have $d_H\left(\pi_\gamma(g), Z_{m+N}^H\right) \leq C$.

Definition 3. Let $g = (f, x) \in A \wr H$ and let η be a path on $\mathrm{Cay}(A \wr H, S_{\mathrm{sws}})$ that realizes $|g|_{S_{\mathrm{sws}}}$. We will identify η with a path of minimal length on $\mathrm{Cay}(H, S_H)$ that starts at e_H, visits all elements in $\mathrm{supp}(f)$, and finishes at x. Consider the finite subsets $P_1, P_2, P_3 \subseteq H$ as above. We define the associated *coding* of η as the word u in the alphabet $\{P_1, P_2, P_3\}$, such that $u_i = P_j$ if and only if at the i-th step of η, there is a lamp at a position in P_j which was modified for the first time.

Lemma 7.2. *Let η be a solution to the TSP problem associated with Z_n (i.e. it visits the support of φ_n, begins at $Z_0^H = e_H$ and finishes at Z_n^H). Then the coding of η does not have a subword of the form $P_1 P_3^{\varepsilon} P_1^{\varepsilon'} P_3$, for $\varepsilon, \varepsilon' \geq 1$.*

Proof. Suppose that the coding of η contains a subword of the form $P_1 P_3^{\varepsilon} P_1^{\varepsilon'} P_3$, for $\varepsilon, \varepsilon' \geq 1$. In particular, during this coding the path does not modify any lamps in the set P_2, and for each transition in the coding between P_1 and P_3, or vice-versa, it must cross P_2. Each of these crossings contributes at least $L - 2C$ units to the length of the path.

Let us decompose the path η as a concatenation of paths $\eta = \eta_1\eta_2\eta_3\eta_4\eta_5\eta_6$ such that

- the path η_1 is the path η until the last time it visits an element of P_1, before the first occurrence of P_3 in the subword of the coding that we are considering,
- the coding of the path η_2 is empty,
- the coding of the path η_3 starts with P_3, and η_3 follows η until the last time it visits an element of P_3 before the next occurrence of P_1 in the subword of the coding that we consider,
- the coding of the path η_4 is empty,
- the coding of the path η_5 starts with P_1 and it follows η until the first time it visits an element of P_3, and
- the path η_6 follows η starting from the last point visited by η_5.

Now we consider the following modification of the path η, which we call $\widetilde{\eta} = \eta_1\xi_1\eta_4\eta_5\xi_2\eta_3\xi_3\eta_6$ where

- the path ξ_1 connects geodesically the last element of η_1 with the first element of η_4,
- the path ξ_2 connects geodesically the last element of η_5 with the first element of η_3, and
- the path ξ_3 connects geodesically the last element of η_3 with the first element of η_6.

From this, we removed the paths η_2 and η_4, and added the paths ξ_1, ξ_2, ξ_3. Then, we have

$$\begin{aligned}\text{length}(\widetilde{\eta}) &= \text{length}(\eta) - \text{length}(\eta_2) - \text{length}(\eta_4) + \text{length}(\xi_1) + \text{length}(\xi_2) + \text{length}(\xi_3) \\ &\leq \text{length}(\eta) - 2(L - 2C) + 2(C + 2) + C + 3 \\ &= \text{length}(\eta) - 2L + 7C + 7 < \text{length}(\eta),\end{aligned}$$

since $L > 20C + 100\delta + 1$. This is a contradiction, since the path $\widetilde{\eta}$ is strictly shorter than η and it is also a solution to the TSP associated with Z_n. □

Definition 4. Given a path η in $\text{Cay}(A \wr H, S_{\text{sws}})$, let us call a *backtracking* a subpath of η that has an associated coding of the form $P_1 P_2^* P_3^{\varepsilon} P_2^* P_1^{\varepsilon'} P_2^* P_3$ for $\varepsilon, \varepsilon' \geq 1$. Here the $*$ symbolizes 0 or more occurrences of the symbol.

Lemma 7.3. *Let η be a solution of the TSP associated with Z_n. Then the number of backtrackings of η is at most $2|P_2|$.*

Proof. Indeed, thanks to Lemma 7.2, if η has a backtracking, then during this subpath there must be at least one lamp in a position on P_2 that is modified. Since an optimal path needs to modify a lamp only once, and there are $|P_2|$ lamp positions in P_2, one obtains the result (the extra factor of 2 comes from the fact that consecutive backtrackings may share a subpath).

□

Lemma 7.4. *Consider the trajectory $\{Z_k\}_{k=0}^n$ of the μ-random walk up to time n, together with the sets P_1, P_2 and P_3 as above. Let T be the length of a solution to the TSP problem associated with Z_n. Then there exists a path η that starts at Z_0^H, finishes at Z_n^H and visits all points in $\text{supp}(\varphi_n)$ such that $\text{length}(\eta) \leq T + 100N(L + 2D)$, and such that, in the coding of η, all instances of P_1 appear before any instance of P_3.*

Proof. Let us denote by η_0 the solution to the TSP associated with Z_n. Note that thanks to Lemma 7.3, the number of backtrackings in η_0 is at most $2|P_2|$.

We modify the path η_0 as follows: first do all excursions of η_0 on P_1 concatenating them in the boundary between P_1 and P_2. This adds at most $\#|\text{excursions in } P_1| \cdot 2CD$ units of length. Then visit all elements of P_2, and finish at Z_{m+N}^H. This adds $K(C, D, \ell(\gamma), N)$ units of length. Then do all excursion in P_3. This adds again at most $\#|\text{excursions in } P_3| \cdot 2CD$ units of length. Note that $\#|\text{excursions in } P_1| + \#|\text{excursions in } P_3| \leq 4\#|\text{backtrackings}| \leq 8\#|P_2|$. Hence, the new path has a larger length only by a constant written in terms of the parameters of the pivot time. □

In other words, we are saying that trying to solve the problem by first visiting all elements of P_1, and then visiting all elements of P_3, and crossing the middle section only once, is at a bounded length of being optimal.

Lemma 7.5. *For any $N \in \mathbb{N}$ there exists some $C \geq 1$ such that the following holds. Let $m \in \mathbb{N}$ be an integer and let U be the waiting time until the first pivot after time m. Then we have*

$$\sup_{m \geq 1} \mathbb{E}\left[\left(\sum_{i=m}^{m+U+N} |g_i|\right)^2\right] \leq C.$$

Proof. Recall that the way Gouëzel constructs pivots is as follows: if we let $S \subset H$ be our finite Schottky set, then we can decompose some convolution power μ_H^{*N} as

$$\mu_H^{*N} = \alpha \mu_S + (1 - \alpha)\nu$$

for some $\alpha > 0$. Then we draw our increments as follows: let $\{\varepsilon_i\}_i$ be i.i.d. Bernoulli(α) random variables. If $\varepsilon_i = 1$, we draw $g_i' = s_i$ according to μ_S. Else we draw $g_i' = w_i$

according to ν. We observe that the sequence $\{g_1'...g_k'\}_k$ has the same distribution as $\{g_1...g_k\}_k$ for $g_i \sim \mu_H^N$.

Now we denote the resampled random walk by $g_1'...g_n' = w_1...w_{k_1}s_1w_{k_1+1}...w_{k_2}s_2...$, where the strings between $s_i's$ may be empty. Now each string $w_{k_{i-1}+1}...w_{k_i}s_i$ is distributed according to $\nu^Z * \mu_S$, where Z is a geometric random variable with parameter α.

Now Gouezel tells us that, conditional on any realization of the increments drawn from ν, the number of μ_S increments ℓ until we see a pivot has an exponential tail [Gou22, Lemma 4.9]. This implies that

$$\mathbb{E}\left[\left(\sum_{i=m}^{m+U+N}|g_i|\right)^2 \middle| \{w_i\}_i\right] \leq \mathbb{E}\left[\left(L\ell + \sum_{i=0}^{\ell-1}\sum_{k=k_i}^{k_{i+1}-1}|w_i|\right)^2 \middle| \{w_i\}_i\right].$$

Now we can integrate over the possible values of w_i and use independence in order to conclude that

$$\mathbb{E}\left[\left(\sum_{i=m}^{m+U+N}|g_i|\right)^2\right]$$

is bounded uniformly over m. □

7.3. **Proof of the CLT.** Let us define $\Psi_{n,m} = |Z_n| - |Z_m^{-1}Z_n| - |Z_m|$. Thanks to Theorem 3.1, it suffices to show that $\sup_{m,n\geq 1}\mathbb{E}(|\Psi_{n,m}|^2)$ is bounded.

We will do the proof for finitely supported μ.

We fix m and n. Let $m+u$ be the first instant after m that you see a pivot.

If $m+u+N > n$, then we use Lemma 7.5

$$\mathbb{E}|\Psi_{n,m}|^2 \leq \sup_{m\geq 1}\mathbb{E}\left[\left(\sum_{i=m}^{m+u}|g_i|\right)^2\right] \leq C.$$

Otherwise, $m+u+N \leq n$ and we do the following.

(1) The three conditions at the beginning of this subsection are satisfies.
(2) Our objective is to get a good upper bound for $\Phi_{n,m}$ in the inequality
$$|Z_n| \geq |Z_m| + |Z_m^{-1}Z_n| - \Phi_{n,m}.$$
(3) We first note that $||Z_m| - |Z_{m+u}||$ has a finite second moment. Indeed, this amount is controlled by the increments done during u steps, and we know the distribution of how large u can be. That is, we use Lemma 7.5 to justify this. The same is true for $\left||Z_m^{-1}Z_n| - |Z_{m+u}^{-1}Z_n|\right|$. Again, this follows from a triangular inequality and Lemma 7.5.
(4) From this, we just need a good upper bound for $\Phi_{n,m}$ in the inequality
$$|Z_n| \geq |Z_{m+u}| + |Z_{m+u}^{-1}Z_n| - \Phi_{n,m}.$$
(5) We note that $\left||Z_{m+u+N}^{-1}Z_n| - |Z_{m+u}^{-1}Z_n|\right|$ is a bounded constant (since it only depends on N), and in particular has a finite second moment.
(6) From this, we just need a good upper bound for $\Phi_{n,m}$ in the inequality
$$|Z_n| \geq |Z_{m+u}| + |Z_{m+u+N}^{-1}Z_n| - \Phi_{n,m}.$$
(7) We look at the TSP between time 0 and n, we use the path η from the previous lemma to get a path which is near optimal and crosses only once the neighborhood of γ.

(8) From this path we obtain near-optimal paths from $|Z_{m+u}|$ and for $|Z_{m+u+N}^{-1}Z_n|$, by doing surgery near the endpoints of γ and possibly adding a constant bounded amount of length.

Indeed, we first take the path from the starting point to the last visit to P_1, and we connect it to Z_{m+u}. This is at most $Optimal + L + 2D$. Similarly we look at the first time we enter P_3, and connect that to a path to Z_{m+u+N}. This again adds at most $Optimal + L + 2D$.

(9) From this, we directly apply Theorem 3.1.

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