CLT FOR RANGE OF RANDOM WALKS ON HYPERBOLIC GROUPS

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Abstract. We prove a central limit theorem for random walks with a finitely supported step distribution on wreath products of the form $A \wr H = \bigoplus_H A \wr H$, where A is a non-trivial finite group and H is a non-elementary hyperbolic group.

1. Introduction

In this paper we consider wreath products where the group of lamps is a finite group, and the base group is a hyperbolic group. We will consider a word length in these wreath products given by the switch-walk-switch generating set with respect to some fixed generating set of H (see Subsection 2.3.1).

Theorem 1.1. *Let A be a non-trivial finite group and H a non-elementary hyperbolic group, endowed with the switch-walk-switch word metric $|\cdot|_{\mathrm{sws}}$ associated with some arbitrary finite generating set of H. Consider a finitely supported probability measure μ on $A \wr H$ such that $\mathrm{supp}(\mu_H)$ is non-elementary. Denote by $\{w_n\}_{n\geq 0}$ the μ-random walk on $A \wr H$, and let $C = \lim_{n\to\infty} \frac{\mathbb{E}(|w_n|_{\mathrm{sws}})}{n}$ be the drift of the μ-random walk on $A \wr H$. Then the sequence of normalized random variables $\frac{|w_n|_{\mathrm{sws}} - Cn}{\sqrt{n}}$, $n \geq 1$, converges in law to a non-degenerate gaussian random variable.*

This result is new even for simple random walks on $\mathbb{Z}/2\mathbb{Z} \wr F_2$.

1.1. Background.

- The CLT for non-abelian free groups is due to [SS87] and [Led01]. Then for non-elementary hyperbolic groups with a finite exponential moment is due to [Bjö10]. This was generalized for any finite second moment measure in [BQ16a]. The last two results hold more generally for group acting on a Gromov hyperbolic space by isometries. [BQ16b] show a CLT for random walks on $\mathrm{GL}_d(\mathbb{R})$ with a finite second moment. See also [Gou17].
- [EZ22] prove a law of large numbers for random walks on $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^2$ with a finite $(2+\varepsilon)$-moment, for some $\varepsilon > 0$. They also discuss limit laws in other wreath products.
- [BFGK24] prove a central limit theorem for random walks on the group of affine transformations of a horospherical product of Gromov hyperbolic spaces.
- [Cho23] proves a central limit theorem for groups acting with contracting elements.
- [GTT22] prove a CLT with respect to the counting measure on the Cayley graph of a group acting on a hyperbolic space.
- [Hor18] proves a CLT for random walks on mapping class groups and $\mathrm{Out}(F_n)$.
- [Bar22] proves a CLT for groups acting on a CAT(0) space.
- [Gil08] proves that the drift of $\mathbb{Z}/2\mathbb{Z} \wr G$ is strictly larger than that of its projection to G.
- [MSŠ23] prove a LLN and CLT for the capacity of the range of a random walk on a group.
- [Sal01] proves a LLN and CLT for a simple random walk on a free group, conditioned on the boundary point.

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2. Preliminaries

2.1. **Notation for graphs and paths.**

- We will work with undirected, unlabeled graphs (V, E).
- A path $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ is an ordered sequence of vertices in the graph.

2.2. **Hyperbolic groups.** Say basic things about hyperbolicity; explain pivots

2.3. **Wreath products.** We consider the wreath products $A \wr H$, where A is a finite non-trivial group and H is a finitely generated group. Let S_H be a finite and symmetric generating set of H.

2.3.1. *The switch-walk-switch word metric.* We consider the *switch-walk-switch* S_{sws} generating set of $A \wr H$, given by

$$S_{\mathrm{sws}} \coloneqq \Big\{(\delta_a, 0)(\mathbf{0}, s)(\delta_{a'}, 0)\Big| a, a' \in A \text{ and } s \in S_H\Big\}.$$

Theorem 2.1 ([Par92, Theorem 1.2])**.** *For any $g = (f, x) \in A \wr H$, the word length of g with respect to the standard generating set is*

$$|g|_{\mathrm{sws}} = \mathrm{TSP}(e_H, x, \mathrm{supp}(f)).$$

2.4. **Random walks on groups.** Let G be a countable group and consider a probability measure μ on G. Consider the product space $\Omega \coloneqq G^{\mathbb{Z}_+}$ endowed with the product σ-field. For each $n \geq 1$ we denote by

$$X_n : \Omega \to G$$
$$w \coloneqq (w_1, w_2, \cdots) \mapsto X_n(w) \coloneqq w_n$$

the n-th coordinate map. We endow Ω with the product probability measure $\mu^{\mathbb{Z}_+}$.

We denote by

$$\theta : \Omega \to \Omega$$
$$w \coloneqq (w_1, w_2, \cdots) \mapsto \theta(w) \coloneqq (w_2, w_3, \ldots)$$

the shift map in the space of increments.

Now we define the μ-random walk $\{Z_n\}_{n\geq 0}$ on G as follows. We define $Z_0(w) = e_G$ for each $w \in \Omega$, and for each $n \geq 1$ we define

$$Z_n(w) \coloneqq Z_{n-1}(w) \cdot X_n(w).$$

We remark that $Z_n(w)(Z_m \circ \theta^n)(w) = Z_{n+m}(w)$, for each $w \in \Omega$ and $n, m \geq 1$.

2.5. **Defective adapted cocycles and the central limit theorem.** A sequence $\mathcal{Q} = \{Q_n\}_{n\geq 1}$ of maps $Q_n : \Omega \to \mathbb{R}$ such that Q_n is measurable with respect to $\sigma(X_1, \ldots, X_n)$, for each $n \geq 1$, is called a *defective adapted cocycle.* We will use the convention $Q_0 \equiv 0$. The *defect of* $\mathcal{Q}$ is the collection of maps $\Psi = \{\Psi_{n,m}\}_{n,m\geq 0}$ defined by

$$\Psi_{n,m}(w) = Q_{n+m}(w) - Q_n(w) - (Q_m \circ \theta^n)(w), \text{ for each } w \in \Omega \text{ and } n, m \geq 0.$$

The following result states that the central limit theorem holds for defective adapted cocycles that satisfy a second-moment deviation inequality.

Theorem 2.2 ([MS20, Theorem 4.2])**.** *Let G be a countable group endowed with a probability measure μ. Consider $\mathcal{Q}$ a defective adapted cocycle on $\Omega = G^{\mathbb{Z}_+}$, and denote by $\{\Psi_{n,m}\}_{n,m\geq 0}$ its defect. Suppose that*

1. $\mathbb{E}[|Q_1|^2] < \infty$, *and*
2. $\sup_{m,n\geq 0} \{\mathbb{E}[|\Psi_{n,m}|^2]\} < \infty$.

Then, there exist constants $\ell, \sigma \in \mathbb{R}$ such that the random variables $\frac{1}{\sqrt{n}}(Q_n - \ell n)$ converge in law to a Gaussian random variable with zero mean and variance σ^2.

Furthermore, it is proved in [MS20, Theorem 3.3] that the constant ℓ that appears in the statement of Theorem 2.2 satisfies that $\frac{1}{n}Q_n$ converges to ℓ in L_1 as $n \to \infty$.

We will use this result for the defective adapted cocycle obtained from the word length of the μ-random walk on G at time n. That is, we will consider some word metric d on G, and define $Q_n \coloneqq d(e_G, Z_n)$, for each $n \geq 1$. Since we will be working with finitely supported probability measures, it holds immediately that $\mathbb{E}[|Q_1|^2] < \infty$.

The objective of the following sections of this paper is to prove that there exists a constant $C > 0$ such that

$$\mathbb{E}[d(e_G, Z_{n+m}) - d(e_G, Z_n) - d(e_G, Z_m \circ \theta^n)] = \mathbb{E}[|\Psi_{n,m}|^2] \leq C, \text{ for each } n, m \geq 0,$$

where $G = A \wr H$, d will be the switch-walk-switch word metric and μ is a finitely supported probability measure, as in the hypotheses of Theorem 1.1.

3. CLT for the lamplighter over a hyperbolic group (using pivots)

Let us consider a non-trivial finite group A and a non-elementary hyperbolic group H. Choose an arbitrary symmetric finite generating set S_H of H. Let us denote by d_H (resp. $|\cdot|_H$) the associated word metric (resp. word length) on H, and by d_{sws} (resp. $|\cdot|_{\text{sws}}$) the switch-walk-switch word metric (resp. word length) on $A \wr H$. Additionally, let us consider $\delta \geq 0$ to be a hyperbolicity constant of $\mathrm{Cay}(H, S_H)$.

3.1. Pivots.

Definition 1. Given a path $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ on $\mathrm{Cay}(H, S_H)$ and $g \in H$, let $\pi_\gamma(g)$ be the set of elements visited by γ that minimize the word metric to g. That is, if we denote by $d_H(g, \gamma) \coloneqq \min\{d_H(g, \gamma_i) \mid i = 1, 2 \ldots, k\}$, then

$$\pi_\gamma(g) \coloneqq \{h \in G \mid h = \gamma_i \text{ for some } i \in \{1, \ldots, k\} \text{ and } d_S(g, h) = d_H(g, \gamma)\}. \tag{1}$$

We now introduce the definition of pivots that we will use in the proof of Theorem 1.1. We refer to [Gou22, Section 4A] for details.

Definition 2. Let $C, D > 0$, $L \geq 20C + 100\delta + 1$, and $N \in \mathbb{Z}_{\geq 1}$. Let $\mathbf{w} = \{w_n\}_{n \geq 1} \in \Omega = (A \wr H)^{\mathbb{Z}_+}$, and consider the associated trajectory of the random walk $\{Z_n(w)\}_{n \geq 0}$. To avoid having too much notation we will omit the dependence of Z_n on w. Denote by Z_n^H the projection to H of Z_n, for each $n \geq 0$. A time instant $m \geq 1$ is called a (C, D, L, N)-*pivot* for $\mathbf{w}$ if the following three conditions hold.

(1) $d_H\left(Z_m^H, Z_{m+N}^H\right) \geq L$.

Let γ be an arbitrary geodesic path in $\mathrm{Cay}(H, S_H)$ that connects Z_m^H to Z_{m+N}^H. Then

(2)
$$d_H\left(\pi_\gamma\left(Z_k^H\right), Z_m^H\right) \leq C, \text{ for all } k \in \{0, 1, \ldots, m\},$$

(3) for all $m \leq k \leq m + N$ we have $d_H\left(Z_k^H, \gamma\right) \leq D$, and

(4) for all $k \geq m + N$, we have $d_H\left(\pi_\gamma\left(Z_k^H\right), Z_{m+N}^H\right) \leq C$.

The following lemma will be our main tool.

Lemma 3.1. *For any $C, D, \delta > 0$, and any $L \geq 20C + 100\delta + 1$, there exists $N, R > 0$ large such that*

$$\sup_{i \geq 1} \mathbb{P}\left(\exists m \in [i, i+k] \textit{ such that } m \textit{ is an } (C, D, L, N)\textit{-pivot}\right) \geq 1 - Re^{-k/R}.$$

Sketch of proof. This is proven in proposition 4.11 in Gouëzel's paper, where for Gouëzel's definition of pivots, conditions 1,2, and 4 are met. To see why Gouëzel's proof implies the lemma we state, we observe that for Gouëzel's definition of pivots, the increments $Z_n^{-1} Z_{n+N}$ are drawn from some explicit finite set of isometries $S \subset H$ that receive positive support from μ_H^N. For this finite set of isometries, we can pick some $D > 0$ large enough so that μ^N gives positive mass to each of the sets $\pi_H^{-1}(s) = \{(\varphi, s) \mid \mathrm{supp}(\varphi) \subset B_D([e_H, s])\}$, for $s \in S$. Then tracing through the rest of Gouëzel's proof we have the estimate required. $\square$

3.2. **TSP structure along pivots.**

Proposition 3.2. *Suppose that we are looking at a sample path $\{Z_n\}_{n\geq 0}$ and that we have a pivoting time m. Then the group element $Z_n = (f_n, x_n)$ satisfies the following. The support of f_n can be decomposed as a disjoint union*

$$\mathrm{supp}(f_n) = P_1' \cup P_2' \cup P_3',$$

that satisfies the following properties. Let us denote $P_1 = P_1' \cup \{Z_{m+u}^H\}$, $P_2 = P_2' \cup \{Z_{m+u}^H, Z_{m+u+N}^H\}$ and $P_3 = P_3' \cup \{Z_{m+u+N}^H\}$. Let γ be an arbitrary geodesic path from Z_m^H to Z_{m+N}^H on $\mathrm{Cay}(H, S_H)$*. Then we have*

1. *for all $g \in P_1$, we have $d_H\left(\pi_\gamma(g), Z_m^H\right) \leq C$,*
2. *for all $g \in P_2$ we have $d_H(g, \gamma) \leq D$, and*
3. *for all $g \in P_3$, we have $d_D\left(\pi_\gamma(g), Z_{m+N}^H\right) \leq C$.*

Definition 3. Let $g = (f, x) \in A \wr H$ and suppose that $\mathrm{supp}(f) = P_1 \cup P_2 \cup P_3$. Let η be a path on $\mathrm{Cay}(A \wr H, S_{\mathrm{sws}})$ that realizes $|g|_{S_{\mathrm{sws}}}$. We define the associated *coding* of η as the word u in the alphabet $\{P_1, P_2, P_3\}$, such that $u_i = P_j$ if and only if at the i-th step of η, there is a lamp at a position in P_j which was modified for the first time.

We are going to abuse notation (in this draft) and not make a distinction between the elements visited by a path, and the coding in the alphabet $\{P_1, P_2, P_3\}$ associated with it.

Definition 4. Given a path η, let us call a *backtracking* a subpath of η that is of the form $P_1 P_2^* P_3^\varepsilon P_2^* P_1^{\varepsilon'} P_2^* P_3$ for $\varepsilon, \varepsilon' \geq 1$. Here the $*$ symbolizes 0 or more occurrences.

Lemma 3.3. *Let η be a solution to the TSP for $|Z_n|$. Then the coding of η does not have a subword of the form $P_1 P_3^\varepsilon P_1^{\varepsilon'} P_3$, for $\varepsilon, \varepsilon' \geq 1$.*

Proof. Surgery, meaning that you glue together the excursions to P_1, and you glue together the excursions to P_3, and connect them with any path through P_2. This gives something even shorter than optimal since each gluing strictly reduces the length of the path. $\square$

Corollary 3.4. *Let η be a solution to the TSP for $|Z_n|$. Then the number of backtrackings of η is at most $|P_2|$.*

Proof. Every backtracking must contain at least one element of P_2. (Recall that the path only has an element in its coding if it has not been visited before). $\square$

Lemma 3.5. *Consider a sequence of points $\{Z_n\}_n$ of H, that satisfies the decomposition of* $\mathrm{supp}(f_n)$ *given by the three conditions of Proposition 3.2.*

Let T be the length of a solution to $\mathrm{TSP}(Z_0, Z_n, \mathrm{supp}(f_n)) = \mathrm{TSP}\left(Z_0^H, Z_n^H, P_1 \cup P_2 \cup P_3\right) = |Z_n|_{A\wr H}$*. Then there exists a path η that starts at Z_0^H, finishes at Z_n^H and visits all points in* $\mathrm{supp}(f_n)$ *such that* $\mathrm{length}(\eta) \leq T + 100N(L + 2D)$*, and such that, in the coding of η, all the elements of P_1 appear before any of the elements of P_3.*

Proof. Let us first consider η_0 the optimal solution to the TSP.

Lemma 3.3 implies Corollary 3.4 that the total number of backtrackings is the size of P_2.

Finally, the argument goes as follows: first do all excursions of η_0 on P_1, then visit all elements in P_2, and then do all excursion in P_3. In total we added at most $2D\times$(number of backtrackings)+(length of solution of TSP in P2 that visits all elements in P2). And the number of backtrackings is at most $|P_2|$ by the previous claim.

□

In other words, we are saying that trying to solve the problem by first visiting all elements of P_1, and then visiting all elements of P_3, and crossing the middle section only once, is at a bounded length of being optimal.

Lemma 3.6. *For any $N \in \mathbb{N}$ there exists some $C > 0$ such that the following holds. set $m \in \mathbb{N}$ be an integer and let U be the waiting time until the first pivot after time m. Then we have*

$$\sup_{m\geq 1} \mathbb{E}\left[\left(\sum_{i=m}^{m+U+N} |g_i|\right)^2\right] \leq C.$$

Proof. Recall that the way Gouëzel constructs pivots is as follows: if we let $S \subset H$ be our finite Schottky set, then we can decompose some convolution power μ_H^N as

$$\mu_H^N = \alpha\mu_S + (1-\alpha)\nu$$

for some positive $\alpha > 0$. Then we draw our increments as follows: let $\{\varepsilon_i\}_i$ be i.i.d. Bernoulli(α) random variables. If $\varepsilon_i = 1$, we draw $g_i' = s_i$ according to μ_S. Else we draw $g_i' = w_i$ according to ν. We observe that the sequence $\{g_1'...g_k'\}_k$ has the same distribution as $\{g_1...g_k\}_k$ for $g_i \sim \mu_H^N$.

Now we denote the resampled random walk by $g_1'...g_n' = w_1...w_{k_1}s_1w_{k_1+1}...w_{k_2}s_2...$, where the strings between $s_i's$ may be empty. Now each string $w_{k_{i-1}+1}...w_{k_i}s_i$ is distributed according to $\nu^Z * \mu_S$, where Z is a geometric random variable with parameter α.

Now Gouezel tells us that, conditional on any realization of the increments drawn from ν, the number of μ_S increments ℓ until we see a pivot has an exponential tail. This implies that

$$\mathbb{E}\left[\left(\sum_{i=m}^{m+U+N} |g_i|\right)^2 |\{w_i\}_i\right] \leq \mathbb{E}\left[\left(L\ell + \sum_{i=0}^{\ell-1}\sum_{k=K_i}^{K_{i+1}-1} |w_i|\right)^2 |\{w_i\}_i\right].$$

Now we can integrate over the possible values of w_i and use independence in order to conclude that

$$\mathbb{E}\left[\left(\sum_{i=m}^{m+U+N} |g_i|\right)^2\right]$$

is bounded uniformly over m. □

3.3. **Proof of the CLT.** Let us define $\Phi_{n,m} = |Z_n| - |Z_m^{-1}Z_n| - |Z_m|$. Thanks to Theorem 2.2, it suffices to show that $\sup_{m,n\geq 1}\mathbb{E}(|\Phi_{n,m}|^2)$ is bounded.

We will do the proof for finitely supported μ.

We fix m and n. Let $m+u$ be the first instant after m that you see a pivot.

If $m+u+N > n$, then we use Lemma 3.6

$$\mathbb{E}|\Phi_{n,m}|^2 \leq \sup_{m\geq 1}\mathbb{E}\left[\left(\sum_{i=m}^{m+u}|g_i|\right)^2\right] \leq C.$$

Otherwise, $m+u+N \leq n$ and we do the following.

(1) The three conditions at the beginning of this subsection are satisfies.
(2) Our objective is to get a good upper bound for $\Phi_{n,m}$ in the inequality

$$|Z_n| \geq |Z_m| + |Z_m^{-1}Z_n| - \Phi_{n,m}.$$

(3) We first note that $||Z_m| - |Z_{m+u}||$ has a finite second moment. Indeed, this amount is controlled by the increments done during u steps, and we know the distribution of how large u can be. That is, we use Lemma 3.6 to justify this. The same is true for $\left||Z_m^{-1}Z_n| - |Z_{m+u}^{-1}Z_n|\right|$. Again, this follows from a triangular inequality and Lemma 3.6.

(4) From this, we just need a good upper bound for $\Phi_{n,m}$ in the inequality

$$|Z_n| \geq |Z_{m+u}| + |Z_{m+u}^{-1}Z_n| - \Phi_{n,m}.$$

(5) We note that $\left||Z_{m+u+N}^{-1}Z_n| - |Z_{m+u}^{-1}Z_n|\right|$ is a bounded constant (since it only depends on N), and in particular has a finite second moment.

(6) From this, we just need a good upper bound for $\Phi_{n,m}$ in the inequality

$$|Z_n| \geq |Z_{m+u}| + |Z_{m+u+N}^{-1}Z_n| - \Phi_{n,m}.$$

(7) We look at the TSP between time 0 and n, we use the path η from the previous lemma to get a path which is near optimal and crosses only once the neighborhood of γ.

(8) From this path we obtain near-optimal paths from $|Z_{m+u}|$ and for $|Z_{m+u+N}^{-1}Z_n|$, by doing surgery near the endpoints of γ and possibly adding a constant bounded amount of length.

Indeed, we first take the path from the starting point to the last visit to P_1, and we connect it to Z_{m+u}. This is at most $Optimal + L + 2D$. Similarly we look at the first time we enter P_3, and connect that to a path to Z_{m+u+N}. This again adds at most $Optimal + L + 2D$.

(9) From this, we directly apply Theorem 2.2.

References

[Bar22] Corentin Le Bars. Central limit theorem on CAT(0) spaces with contracting isometries, 2022. [Cited on page 1.]

[BFGK24] Amin Bahmanian, Behrang Forghani, Ilya Gekhtman, and Mallahi-Karai Keivan. A central limit theorem for random walks on horospherical products of Gromov hyperbolic spaces, 2024. [Cited on page 1.]

[Bjö10] Michael Björklund. Central limit theorems for Gromov hyperbolic groups. *J. Theoret. Probab.*, 23(3):871–887, 2010. [Cited on page 1.]

[BQ16a] Y. Benoist and J.-F. Quint. Central limit theorem on hyperbolic groups. *Izv. Math.*, 80(1):3–23, 2016. [Cited on page 1.]

[BQ16b] Yves Benoist and Jean-François Quint. Central limit theorem for linear groups. *Ann. Probab.*, 44(2):1308–1340, 2016. [Cited on page 1.]

[Cho23] Inhyeok Choi. Central limit theorem and geodesic tracking on hyperbolic spaces and Teichmüller spaces. *Adv. Math.*, 431:68, 2023. Id/No 109236. [Cited on page 1.]

[EZ22] Anna Erschler and Tianyi Zheng. Law of large numbers for the drift of the two-dimensional wreath product. *Probab. Theory Related Fields*, 182(3-4):999–1033, 2022. [Cited on page 1.]

[Gil08] L. A. Gilch. Acceleration of lamplighter random walks. *Markov Process. Related Fields*, 14(4):465–486, 2008. [Cited on page 1.]

[Gou17] Sébastien Gouëzel. Analyticity of the entropy and the escape rate of random walks in hyperbolic groups. *Discrete Anal.*, 2017:37, 2017. Id/No 7. [Cited on page 1.]

[Gou22] Sébastien Gouëzel. Exponential bounds for random walks on hyperbolic spaces without moment conditions. *Tunis. J. Math.*, 4(4):635–671, 2022. [Cited on page 3.]

[GTT22] Ilya Gekhtman, Samuel J. Taylor, and Giulio Tiozzo. Central limit theorems for counting measures in coarse negative curvature. *Compos. Math.*, 158(10):1980–2013, 2022. [Cited on page 1.]

[Hor18] Camille Horbez. Central limit theorems for mapping class groups and $\mathrm{Out}(F_N)$. *Geom. Topol.*, 22(1):105–156, 2018. [Cited on page 1.]

[Led01] François Ledrappier. Some asymptotic properties of random walks on free groups. In *Topics in probability and Lie groups: boundary theory*, volume 28 of *CRM Proc. Lecture Notes*, pages 117–152. Amer. Math. Soc., Providence, RI, 2001. [Cited on page 1.]

[MS20] P. Mathieu and A. Sisto. Deviation inequalities for random walks. *Duke Math. J.*, 169(5):961–1036, 2020. [Cited on pages 2 and 3.]

[MSŠ23] Rudi Mrazović, Nikola Sandrić, and Stjepan Šebek. Capacity of the range of random walks on groups. *Kyoto J. Math.*, 63(4):783–805, 2023. [Cited on page 1.]

[Par92] Walter Parry. Growth series of some wreath products. *Trans. Amer. Math. Soc.*, 331(2):751–759, 1992. [Cited on page 2.]

[Sal01] François Salaün. Marche aléatoire sur un groupe libre: théorèmes limite conditionnellement à la sortie. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(4):359–362, 2001. [Cited on page 1.]

[SS87] Stanley Sawyer and Tim Steger. The rate of escape for anisotropic random walks in a tree. *Probab. Theory Related Fields*, 76(2):207–230, 1987. [Cited on page 1.]