CLT FOR RANGE OF RANDOM WALKS ON HYPERBOLIC GROUPS

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Abstract. We prove a central limit theorem for random walks with a finitely supported step distribution on wreath products of the form $A \wr H = \bigoplus_H A \wr H$, where A is a non-trivial finite group and H is a non-elementary hyperbolic group.

1. Introduction

In this paper we consider wreath products where the group of lamps is a finite group, and the base group is a hyperbolic group. We will consider a word length in these wreath products given by the switch-walk-switch generating set with respect to some fixed generating set of H (see Subsection 2.3.1).

Theorem 1.1. *Let A be a non-trivial finite group and let H be a non-elementary hyperbolic group. Endow $A \wr H$ with the switch-walk-switch word metric $|\cdot|_{\text{sws}}$ associated with some arbitrary finite generating set of H. Consider a finitely supported probability measure μ on $A \wr H$ such that $\text{supp}(\mu_H)$ is non-elementary. Denote by $\{w_n\}_{n\geq 0}$ the μ-random walk on $A \wr H$, and let $C = \lim_{n\to\infty} \frac{\mathbb{E}(|w_n|_{\text{sws}})}{n}$ be the drift of the μ-random walk on $A \wr H$. Then the sequence of normalized random variables $\frac{|w_n|_{\text{sws}} - Cn}{\sqrt{n}}$, $n \geq 1$, converges in law to a non-degenerate gaussian random variable.*

This result is new even for simple random walks on $\mathbb{Z}/2\mathbb{Z} \wr F_2$.

1.1. Background.

- The CLT for non-abelian free groups is due to [SS87] and [Led01]. Then for non-elementary hyperbolic groups with a finite exponential moment is due to [Bjö10]. This was generalized for any finite second moment measure in [BQ16a]. The last two results hold more generally for group acting on a Gromov hyperbolic space by isometries. [BQ16b] show a CLT for random walks on $\text{GL}_d(\mathbb{R})$ with a finite second moment. See also [Gou17].
- [EZ22] prove a law of large numbers for random walks on $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^2$ with a finite $(2+\varepsilon)$-moment, for some $\varepsilon > 0$. They also discuss limit laws in other wreath products.
- [BFGK24] prove a central limit theorem for random walks on the group of affine transformations of a horospherical product of Gromov hyperbolic spaces.
- [Cho23] proves a central limit theorem for groups acting with contracting elements.
- [GTT22] prove a CLT with respect to the counting measure on the Cayley graph of a group acting on a hyperbolic space.
- [Hor18] proves a CLT for random walks on mapping class groups and $\text{Out}(F_n)$.
- [Bar22] proves a CLT for groups acting on a CAT(0) space.
- [Gil08] proves that the drift of $\mathbb{Z}/2\mathbb{Z} \wr G$ is strictly larger than that of its projection to G.
- [MSŠ23] prove a LLN and CLT for the capacity of the range of a random walk on a group.
- [Sal01] proves a LLN and CLT for a simple random walk on a free group, conditioned on the boundary point.

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2. Preliminaries

2.1. **Notation for graphs and paths.**

- We will work with undirected, unlabeled graphs (V, E).
- A path $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n)$ is an ordered sequence of vertices in the graph.

2.2. **Hyperbolic groups.** Say basic things about hyperbolicity; explain pivots

2.3. **Wreath products.** We consider the wreath products $A \wr H$, where A is a finite non-trivial group and H is a finitely generated group. Let S_H be a finite and symmetric generating set of H.

2.3.1. *The switch-walk-switch word metric.* We consider the *switch-walk-switch* S_{sws} generating set of $A \wr H$, given by

$$S_{\text{sws}} \coloneqq \left\{ (\delta_a, 0)(\mathbf{0}, s)(\delta_{a'}, 0) \middle| a, a' \in A \text{ and } s \in S_H \right\}.$$

The following goes back to [Par92, Theorem 1.2].

Proposition 2.1. *For any $g = (f, x) \in A \wr H$, the word length of g with respect to the standard generating set is*

$$|g|_{\text{sws}} = \mathrm{TSP}(e_H, x, \mathrm{supp}(f)).$$

2.4. **Random walks on groups.** Let G be a countable group and consider a probability measure μ on G. Consider the product space $\Omega \coloneqq G^{\mathbb{Z}_+}$ endowed with the product σ-field. For each $n \geq 1$ we denote by

$$X_n : \Omega \to G$$
$$w \coloneqq (w_1, w_2, \cdots) \mapsto X_n(w) \coloneqq w_n$$

the n-th coordinate map. We endow Ω with the product probability measure $\mu^{\mathbb{Z}_+}$.

We denote by

$$\theta : \Omega \to \Omega$$
$$w \coloneqq (w_1, w_2, \cdots) \mapsto \theta(w) \coloneqq (w_2, w_3, \dots)$$

the shift map in the space of increments.

Now we define the μ-random walk $\{Z_n\}_{n\geq 0}$ on G as follows. We define $Z_0(w) = e_G$ for each $w \in \Omega$, and for each $n \geq 1$ we define

$$Z_n(w) \coloneqq Z_{n-1}(w) \cdot X_n(w).$$

We remark that $Z_n(w)(Z_m \circ \theta^n)(w) = Z_{n+m}(w)$, for each $w \in \Omega$ and $n, m \geq 1$.

2.5. **Defective adapted cocycles and the central limit theorem.** A sequence $\mathcal{Q} = \{Q_n\}_{n\geq 1}$ of maps $Q_n : \Omega \to \mathbb{R}$ such that Q_n is measurable with respect to $\sigma(X_1, \dots, X_n)$, for each $n \geq 1$, is called a *defective adapted cocycle.* We will use the convention $Q_0 \equiv 0$. The *defect of* $\mathcal{Q}$ is the collection of maps $\Psi = \{\Psi_{n,m}\}_{n,m\geq 0}$ defined by

$$\Psi_{n,m}(w) = Q_{n+m}(w) - Q_n(w) - (Q_m \circ \theta^n)(w), \text{ for each } w \in \Omega \text{ and } n, m \geq 0.$$

The following result states that the central limit theorem holds for defective adapted cocycles that satisfy a second-moment deviation inequality.

Theorem 2.2 ([MS20, Theorem 4.2]). *Let G be a countable group endowed with a probability measure μ. Consider $\mathcal{Q}$ a defective adapted cocycle on $\Omega = G^{\mathbb{Z}_+}$, and denote by $\{\Psi_{n,m}\}_{n,m\geq 0}$ its defect. Suppose that*

1. $\mathbb{E}[|Q_1|^2] < \infty$, *and*
2. $\sup_{m,n\geq 0} \left\{ \mathbb{E}\left[|\Psi_{n,m}|^2 \right] \right\} < \infty$.

Then, there exist constants $\ell, \sigma \in \mathbb{R}$ such that the random variables $\frac{1}{\sqrt{n}}(Q_n - \ell n)$ converge in law to a Gaussian random variable with zero mean and variance σ^2.

Furthermore, it is proved in [MS20, Theorem 3.3] that the constant ℓ that appears in the statement of Theorem 2.2 satisfies that $\frac{1}{n}Q_n$ converges to ℓ in L_1 as $n \to \infty$.

We will use this result for the defective adapted cocycle obtained from the word length of the μ-random walk on G at time n. That is, we will consider some word metric d on G, and define $Q_n := d(e_G, Z_n)$, for each $n \geq 1$. Since we will be working with finitely supported probability measures, it holds immediately that $\mathbb{E}[|Q_1|^2] < \infty$.

The objective of the following sections of this paper is to prove that there exists a constant $C > 0$ such that

$$\mathbb{E}[d(e_G, Z_{n+m}) - d(e_G, Z_n) - d(e_G, Z_m \circ \theta^n)] = \mathbb{E}[|\Psi_{n,m}|^2] \leq C, \text{ for each } n, m \geq 0,$$

where $G = A \wr H$, d will be the switch-walk-switch word metric and μ is a finitely supported probability measure, as in the hypotheses of Theorem 1.1.

3. CLT for the lamplighter over a hyperbolic group (using pivots)

Let us consider a non-trivial finite group A and a non-elementary hyperbolic group H. Choose an arbitrary symmetric finite generating set S_H of H. Let us denote by d_H (resp. $|\cdot|_H$) the associated word metric (resp. word length) on H, and by d_{sws} (resp. $|\cdot|_{\mathrm{sws}}$) the switch-walk-switch word metric (resp. word length) on $A \wr H$. Additionally, let us consider $\delta \geq 0$ to be a hyperbolicity constant of $\mathrm{Cay}(H, S_H)$.

3.1. **Pivots.**

Definition 1. Given a path $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ on $\mathrm{Cay}(H, S_H)$ and $g \in H$, let $\pi_\gamma(g)$ be the set of elements visited by γ that minimize the word metric to g. That is, if we denote by $d_H(g, \gamma) := \min\{d_H(g, \gamma_i) \mid i = 1, 2 \ldots, k\}$ the minimal distance between g and any element of the path γ, then

$$\pi_\gamma(g) := \{h \in G \mid h = \gamma_i \text{ for some } i \in \{1, \ldots, k\} \text{ and } d_S(g, h) = d_H(g, \gamma)\}. \qquad (1)$$

We now introduce the definition of pivots that we will use in the proof of Theorem 1.1. We refer to [Gou22, Section 4A] for details.

Definition 2. Let $C, D > 0$, $L \geq 20C + 100\delta + 1$, and $N \in \mathbb{Z}_{\geq 1}$. Let $\mathbf{w} = \{w_n\}_{n\geq 1} \in \Omega = (A \wr H)^{\mathbb{Z}_+}$, and consider the associated trajectory of the random walk $\{Z_n(\mathbf{w})\}_{n\geq 0}$. To avoid having too much notation we will omit the dependence of Z_n on $\mathbf{w}$. Denote by Z_n^H the projection of Z_n to H, for each $n \geq 0$. A time instant $m \geq 1$ is called a (C, D, L, N)-*pivot* for $\mathbf{w}$ if the following conditions hold.

(1) $d_H\left(Z_m^H, Z_{m+N}^H\right) \geq L$.

Let γ be an arbitrary geodesic path in $\mathrm{Cay}(H, S_H)$ that connects Z_m^H to Z_{m+N}^H. Then

(2) for all $0 \leq k \leq m$ we have $d_H\left(\pi_\gamma\left(Z_k^H\right), Z_m^H\right) \leq C$,
(3) for all $m \leq k \leq m+N$ we have $d_H\left(Z_k^H, \gamma\right) \leq D$, and
(4) for all $k \geq m+N$, we have $d_H\left(\pi_\gamma\left(Z_k^H\right), Z_{m+N}^H\right) \leq C$.

The following lemma will be our main tool.

Lemma 3.1. *For any $C, D, \delta > 0$, and any $L \geq 20C + 100\delta + 1$, there exists $N, R > 0$ such that for all $k \geq 1$ we have*

$$\sup_{i\geq 1} \mathbb{P}\,(\textit{there is no } (C, D, L, N)\textit{-pivot between instants } i \textit{ and } i+k) \leq Re^{-k/R}.$$

3.2. **The structure of the TSP along pivot times.** Suppose that we are looking at a sample path $\{Z_n\}_{n\geq 0}$ of the μ-random walk on $A \wr H$, and that we have a (C, D, L, N)-pivot time m. Denote by $Z_n = (\varphi_n, Z_n^H)$ the lamp configuration and projection to H of Z_n, respectively. Let γ be an arbitrary geodesic path from Z_m^H to Z_{m+N}^H on $\mathrm{Cay}(H, S_H)$. Then, using that the support of μ is finite, we can find constants $C, D \geq 1$ such that the support of φ_n can be decomposed as a disjoint union

$$\mathrm{supp}(\varphi_n) = P_1' \cup P_2' \cup P_3',$$

that satisfies the following properties. Let us denote $P_1 = P_1' \cup \{Z_{m+u}^H\}$, $P_2 = P_2' \cup \{Z_m^H, Z_{m+N}^H\}$ and $P_3 = P_3' \cup \{Z_{m+N}^H\}$. Then we have $d_H(Z_m^H, Z_{m+N}^H) \geq L$, together with

(1) for all $g \in P_1$, we have $d_H\left(\pi_\gamma(g), Z_m^H\right) \leq C$,
(2) for all $g \in P_2$ we have $d_H(g, \gamma) \leq D$, and
(3) for all $g \in P_3$, we have $d_H\left(\pi_\gamma(g), Z_{m+N}^H\right) \leq C$.

Definition 3. Let $g = (f, x) \in A \wr H$ and let η be a path on $\mathrm{Cay}(A \wr H, S_{\mathrm{sws}})$ that realizes $|g|_{S_{\mathrm{sws}}}$. We will identify η with a path of minimal length on $\mathrm{Cay}(H, S_H)$ that starts at e_H, visits all elements in $\mathrm{supp}(f)$, and finishes at x. Consider the finite subsets $P_1, P_2, P_3 \subseteq H$ as above. We define the associated *coding* of η as the word u in the alphabet $\{P_1, P_2, P_3\}$, such that $u_i = P_j$ if and only if at the i-th step of η, there is a lamp at a position in P_j which was modified for the first time.

Lemma 3.2. *Let η be a solution to the TSP problem associated with Z_n (i.e. it visits the support of φ_n, begins at $Z_0^H = e_H$ and finishes at Z_n^H). Then the coding of η does not have a subword of the form $P_1 P_3^{\varepsilon} P_1^{\varepsilon'} P_3$, for $\varepsilon, \varepsilon' \geq 1$.*

Proof. Suppose that the coding of η contains a subword of the form $P_1 P_3^{\varepsilon} P_1^{\varepsilon'} P_3$, for $\varepsilon, \varepsilon' \geq 1$. In particular, during this coding the path does not modify any lamps in the set P_2, and for each transition in the coding between P_1 and P_3, or vice-versa, it must cross P_2. Each of these crossings contributes at least $L - 2C$ units to the length of the path.

Let us decompose the path η as a concatenation of paths $\eta = \eta_1\eta_2\eta_3\eta_4\eta_5\eta_6$ such that

- the path η_1 is the path η until the last time it visits an element of P_1, before the first occurrence of P_3 in the subword of the coding that we are considering,
- the coding of the path η_2 is empty,
- the coding of the path η_3 starts with P_3, and η_3 follows η until the last time it visits an element of P_3 before the next occurrence of P_1 in the subword of the coding that we consider,
- the coding of the path η_4 is empty,
- the coding of the path η_5 starts with P_1 and it follows η until the first time it visits an element of P_3, and
- the path η_6 follows η starting from the last point visited by η_5.

Now we consider the following modification of the path η, which we call $\widetilde{\eta} = \eta_1\xi_1\eta_4\eta_5\xi_2\eta_3\xi_3\eta_6$ where

- the path ξ_1 connects geodesically the last element of η_1 with the first element of η_4,
- the path ξ_2 connects geodesically the last element of η_5 with the first element of η_3, and
- the path ξ_3 connects geodesically the last element of η_3 with the first element of η_6.

From this, we removed the paths η_2 and η_4, and added the paths ξ_1, ξ_2, ξ_3. Then, we have

$$\begin{aligned}\text{length}(\widetilde{\eta}) &= \text{length}(\eta) - \text{length}(\eta_2) - \text{length}(\eta_4) + \text{length}(\xi_1) + \text{length}(\xi_2) + \text{length}(\xi_3) \\ &\le \text{length}(\eta) - 2(L - 2C) + 2(C+2) + C + 3 \\ &= \text{length}(\eta) - 2L + 7C + 7 < \text{length}(\eta),\end{aligned}$$

since $L > 20C + 100\delta + 1$. This is a contradiction, since the path $\widetilde{\eta}$ is strictly shorter than η and it is also a solution to the TSP associated with Z_n. □

Definition 4. Given a path η in $\text{Cay}(A \wr H, S_{\text{sws}})$, let us call a *backtracking* a subpath of η that has an associated coding of the form $P_1 P_2^* P_3^\varepsilon P_2^* P_1^{\varepsilon'} P_2^* P_3$ for $\varepsilon, \varepsilon' \ge 1$. Here the $*$ symbolizes 0 or more occurrences of the symbol.

Lemma 3.3. *Let η be a solution of the TSP associated with Z_n. Then the number of backtrackings of η is at most $2|P_2|$.*

Proof. Indeed, thanks to Lemma 3.2, if η has a backtracking, then during this subpath there must be at least one lamp in a position on P_2 that is modified. Since an optimal path needs to modify a lamp only once, and there are $|P_2|$ lamp positions in P_2, one obtains the result (the extra factor of 2 comes from the fact that consecutive backtrackings may share a subpath). □

Lemma 3.4. *Consider the trajectory $\{Z_k\}_{k=0}^n$ of the μ-random walk up to time n, together with the sets P_1, P_2 and P_3 as above. Let T be the length of a solution to the TSP problem associated with Z_n. Then there exists a path η that starts at Z_0^H, finishes at Z_n^H and visits all points in $\text{supp}(\varphi_n)$ such that $\text{length}(\eta) \le T + 100N(L + 2D)$, and such that, in the coding of η, all instances of P_1 appear before any instance of P_3.*

Proof. Let us denote by η_0 the solution to the TSP associated with Z_n. Note that thanks to Lemma 3.3, the number of backtrackings in η_0 is at most $2|P_2|$. We modify the path η_0 as follows: first do all excursions of η_0 on P_1 concatenating them in the boundary between P_1 and P_2. This adds at most $\#|\text{excursions in } P_1| \cdot 2CD$ units of length. Then visit all elements of P_2, and finish at Z_{m+N}^H. This adds $K(C, D, \ell(\gamma), N)$ units of length. Then do all excursion in P_3. This adds again at most $\#|\text{excursions in } P_3| \cdot 2CD$ units of length. Note that $\#|\text{excursions in } P_1| + \#|\text{excursions in } P_3| \le 4\#|\text{backtrackings}| \le 8\#|P_2|$. Hence, the new path has a larger length only by a constant written in terms of the parameters of the pivot time.

□

In other words, we are saying that trying to solve the problem by first visiting all elements of P_1, and then visiting all elements of P_3, and crossing the middle section only once, is at a bounded length of being optimal.

Lemma 3.5. *For any $N \in \mathbb{N}$ there exists some $C \ge 1$ such that the following holds. Let $m \in \mathbb{N}$ be an integer and let U be the waiting time until the first pivot after time m. Then we have*

$$\sup_{m \ge 1} \mathbb{E}\left[\left(\sum_{i=m}^{m+U+N} |g_i|\right)^2\right] \le C.$$

Proof. Recall that the way Gouëzel constructs pivots is as follows: if we let $S \subset H$ be our finite Schottky set, then we can decompose some convolution power μ_H^{*N} as

$$\mu_H^{*N} = \alpha\mu_S + (1 - \alpha)\nu$$

for some $\alpha > 0$. Then we draw our increments as follows: let $\{\varepsilon_i\}_i$ be i.i.d. Bernoulli(α) random variables. If $\varepsilon_i = 1$, we draw $g_i' = s_i$ according to μ_S. Else we draw $g_i' = w_i$

according to ν. We observe that the sequence $\{g'_1 ... g'_k\}_k$ has the same distribution as $\{g_1 ... g_k\}_k$ for $g_i \sim \mu_H^N$.

Now we denote the resampled random walk by $g'_1 ... g'_n = w_1 ... w_{k_1} s_1 w_{k_1+1} ... w_{k_2} s_2 ...$, where the strings between $s'_i s$ may be empty. Now each string $w_{k_{i-1}+1} ... w_{k_i} s_i$ is distributed according to $\nu^Z * \mu_S$, where Z is a geometric random variable with parameter α.

Now Gouezel tells us that, conditional on any realization of the increments drawn from ν, the number of μ_S increments ℓ until we see a pivot has an exponential tail [Gou22, Lemma 4.9]. This implies that

$$\mathbb{E}\left[\left(\sum_{i=m}^{m+U+N} |g_i|\right)^2 \Big| \{w_i\}_i\right] \leq \mathbb{E}\left[\left(L\ell + \sum_{i=0}^{\ell-1} \sum_{k=k_i}^{k_{i+1}-1} |w_i|\right)^2 \Big| \{w_i\}_i\right].$$

Now we can integrate over the possible values of w_i and use independence in order to conclude that

$$\mathbb{E}\left[\left(\sum_{i=m}^{m+U+N} |g_i|\right)^2\right]$$

is bounded uniformly over m. □

3.3. **Proof of the CLT.** Let us define $\Psi_{n,m} = |Z_n| - |Z_m^{-1} Z_n| - |Z_m|$. Thanks to Theorem 2.2, it suffices to show that $\sup_{m,n \geq 1} \mathbb{E}(|\Psi_{n,m}|^2)$ is bounded.

We will do the proof for finitely supported μ.

We fix m and n. Let $m + u$ be the first instant after m that you see a pivot.

If $m + u + N > n$, then we use Lemma 3.5

$$\mathbb{E}|\Psi_{n,m}|^2 \leq \sup_{m \geq 1} \mathbb{E}\left[\left(\sum_{i=m}^{m+u} |g_i|\right)^2\right] \leq C.$$

Otherwise, $m + u + N \leq n$ and we do the following.

(1) The three conditions at the beginning of this subsection are satisfies.
(2) Our objective is to get a good upper bound for $\Phi_{n,m}$ in the inequality
$$|Z_n| \geq |Z_m| + |Z_m^{-1} Z_n| - \Phi_{n,m}.$$
(3) We first note that $||Z_m| - |Z_{m+u}||$ has a finite second moment. Indeed, this amount is controlled by the increments done during u steps, and we know the distribution of how large u can be. That is, we use Lemma 3.5 to justify this. The same is true for $\left||Z_m^{-1} Z_n| - |Z_{m+u}^{-1} Z_n|\right|$. Again, this follows from a triangular inequality and Lemma 3.5.
(4) From this, we just need a good upper bound for $\Phi_{n,m}$ in the inequality
$$|Z_n| \geq |Z_{m+u}| + |Z_{m+u}^{-1} Z_n| - \Phi_{n,m}.$$
(5) We note that $\left||Z_{m+u+N}^{-1} Z_n| - |Z_{m+u}^{-1} Z_n|\right|$ is a bounded constant (since it only depends on N), and in particular has a finite second moment.
(6) From this, we just need a good upper bound for $\Phi_{n,m}$ in the inequality
$$|Z_n| \geq |Z_{m+u}| + |Z_{m+u+N}^{-1} Z_n| - \Phi_{n,m}.$$
(7) We look at the TSP between time 0 and n, we use the path η from the previous lemma to get a path which is near optimal and crosses only once the neighborhood of γ.

(8) From this path we obtain near-optimal paths from $|Z_{m+u}|$ and for $|Z_{m+u+N}^{-1} Z_n|$, by doing surgery near the endpoints of γ and possibly adding a constant bounded amount of length.

Indeed, we first take the path from the starting point to the last visit to P_1, and we connect it to Z_{m+u}. This is at most $Optimal + L + 2D$. Similarly we look at the first time we enter P_3, and connect that to a path to Z_{m+u+N}. This again adds at most $Optimal + L + 2D$.

(9) From this, we directly apply Theorem 2.2.

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