THE CENTRAL LIMIT THEOREM FOR LAMPLIGHTER RANDOM WALKS ON HYPERBOLIC GROUPS

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Abstract. We prove a central limit theorem for random walks on wreath products $A \wr H = \bigoplus_H F \wr H$, where A is a non-trivial group and H is a non-elementary hyperbolic group, for step distributions with a finite second moment and bounded lamp range. Additionally, we prove a central limit theorem for the range of a random walk on a non-elementary hyperbolic group H, for step distributions with a finite second moment.

1. Introduction

(1) Talk about random walks on groups; trying to prove limit laws in general
(2) Make a list of things known about lamplighters and why they are relevant
(3) Say that in this paper we concentrate on the CLT

Here are some things that we should cite... I am missing many more but this is a good start. [BQ16b, BQ16a, Bjö10, Cho23a, EZ22, GTT22, Gil08, Gou17, Hor18, Bar22, Led01, MSv23, Sal01, SS87].

2. Main results

Consider the switch-walk-switch word length $|\cdot|$ on $A \wr H$.

Theorem 2.1. *Let A be a non-trivial group and H a non-elementary hyperbolic group. Consider a probability measure μ on $A \wr H$ such that μ_H is non-elementary and has a finite second moment, and such that μ has bounded lamp range. Denote by $\{w_n\}_{n\geq 0}$ the μ-random walk on $A \wr H$, and let $C = \lim_{n\to\infty} \frac{\mathbb{E}(|w_n|)}{n}$ be the drift of the μ-random walk on $A \wr B$. Then the sequence of normalized random variables $\frac{|w_n|-Cn}{\sqrt{n}}$, $n \geq 1$, converges in law to a non-degenerate gaussian law.*

We prove a central limit theorem for the range of random walks with a finite second moment on hyperbolic groups.

Theorem 2.2. *Let H be a non-elementary hyperbolic group and let μ be a non-elementary probability measure on H with a finite second moment. Let C be the probability that the μ-random walk on H starting at e_H never returns to e_H. Then the sequence of normalized random variables $\frac{|R_n|-Cn}{\sqrt{n}}$, $n \geq 1$, converges in law to a non-degenerate gaussian law.*

3. Preliminaries

3.1. Hyperbolic groups.
Say basic things about hyperbolicity; explain pivots

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3.2. **Lamplighter groups.** Say basic things about wreath products and lamplighter groups. Give the word metric formula for the sws generating set.

If we are going to work with infinite lamp groups, it is maybe better to work with the following standard word length (if the lamp group is finite, one can instead consider switch-walk-switch metric and for this one the first term below does not appear.) Here for any $g = (f, x) \in A \wr H$ the word length $|g|$ is

$$|g| = \sum_{b \in \mathrm{supp}(f)} |f(b)|_A + \mathrm{TSP}(e_H, x, \mathrm{supp}(f)).$$

3.3. **Random walks.**

(1) Recall basic concepts of random walks on groups.
(2) Introduce all necessary results and definitions of Mathieu-Sisto.
(3) Introduce pivots and the results of Gouëzel that we will use.

We will use the following general criterion of Mathieu-Sisto.

Definition 3.1 (Defective adapted cocycle)**.**

Theorem 3.2 ([MS20])**.** *Suppose that Q_n is a defective adapted cocycle with defect*

$$\Phi_{m,n} := Q_n - (Q_m + w_m Q_{m-n}).$$

Suppose that there exists $C > 0$ such that

$$\sup_{m,n \geq 1} \mathbb{E}[|\Phi_{m,n}|^2] \leq C.$$

Then the CLT holds for Q_n.

4. CLT for the lamplighter over a hyperbolic group

4.1. **Pivots.** Let δ be the hyperbolicity constant of H. Denote by d_H the word metric on H.

Given a path γ on the Cayley graph of H and $g \in H$, denote by $\pi_\gamma(g)$ the element of γ that minimizes word metric to g.

Definition 4.1. Let $C, D, L > 0$ where L is sufficiently large depending on C and δ, and let $N \in \mathbb{Z}_+$. A time $m \geq 1$ is a *pivot* for the sample path $\{w_n\}_n$ if

(1) $d_H(w_m^H, w_{m+N}^H) \geq L$
(2) Denote γ any geodesic in H connecting w_m to w_{m+N}. We want:
$$\forall k \leq n, \pi_\gamma(w_k) \in \mathcal{N}_C(w_m),$$
(3) for all $n \leq k \leq n + N$ we have $w_k \in \mathcal{N}_D(\gamma)$, and
(4) for all $k \geq m + N$, $\pi_\gamma(w_k) \in \mathcal{N}_C(w_{m+N})$.

In the previous definition one can just take $L \geq 20C + 100\delta + 1$ (see Section 4A of Gouezels paper for details).

The following lemma will be our main tool

Lemma 4.2. *For any $C, D, \delta > 0$, and any $L > 0$ sufficiently large (depending on C, D, δ), there exists $R > 0$ large such that*

$$\sup_{m \geq 1} \mathbb{P}(\exists i \in [m, m+k] \text{ such that } i \text{ is a pivot}) \geq 1 - Re^{-k/R}.$$

Sketch of proof. This is proven in proposition 4.11 in Gouezel's paper, where for Gouezel's definition of pivots, conditions 1,2, and 4 are met. To see why Gouezel's proof implies the lemma we state, we observe that for Gouezel's definition of pivots, the increments $w_n^{-1} w_{n+N}$ are drawn from some explicit finite set of isometries $S \subset H$ that receive positive support from μ_H^N. For this finite set of isometries, we can pick some $D > 0$ large enough so that μ^N gives positive mass to each of the sets $\pi_H^{-1}(s) = \{(\varphi, s), \mathrm{supp}\varphi \subset B_D([e, s])\}$. Then tracing through the rest of Gouezel's proof we have the estimate required. $\square$

4.2. **TSP structure along pivots.** Suppose that we are looking at a sample path $\{w_n\}_{n\geq 0}$ and that we have a pivoting time m. Then the group element $w_n = (f_n, x_n)$ looks like this:

$$\mathrm{supp}(()f_n) = P_1' \cup P_2' \cup P_3'$$

is a disjoint union. Let us denote $P_1 = P_1' \cup \{w_{m+u}\}$, $P_2 = P_2' \cup \{w_{m+u}, w_{m+u+N}\}$ and $P_3 = P_3' \cup \{w_{m+u+N}\}$ such that

(1) for all $g \in P_1$, we have $\pi_\gamma(g) \in \mathcal{N}_C(w_m)$,
(2) for all $g \in P_2$ we have $g \in \mathcal{N}_D(\gamma)$, and
(3) for all $g \in P_3$ $\pi_\gamma(g) \in \mathcal{N}_C(w_{m+N})$.

Here γ represents a geodesic from w_m to w_{m+N}.

We are going to abuse notation (in this draft) and not make a distinction between the elements visited by a path, and the coding in the alphabet $\{P_1, P_2, P_3\}$ associated with it.

Definition 4.3. Given a path η, let us call a *backtracking* a subpath of η that is of the form $P_1 P_2^* P_3^\varepsilon P_2^* P_1^{\varepsilon'} P_2^* P_3$ for $\varepsilon, \varepsilon' \geq 1$. Here the $*$ symbolizes 0 or more occurrences.

Lemma 4.4. *Let η be a solution to the TSP for $|w_n|$. Then the coding of η does not have a subword of the form $P_1 P_3^\varepsilon P_1^{\varepsilon'} P_3$, for $\varepsilon, \varepsilon' \geq 1$.*

Proof. Surgery, meaning that you glue together the excursions to P_1, and you glue together the excursions to P_3, and connect them with any path through P_2. This gives something even shorter than optimal since each gluing strictly reduces the length of the path. □

Corollary 4.5. *Let η be a solution to the TSP for $|w_n|$. Then the number of backtrackings of η is at most $|P_2|$.*

Proof. Every backtracking must contain at least one element of P_2. (Recall that the path only has an element in its coding if it has not been visited before). □

Lemma 4.6. *Consider a sequence of points $\{w_n\}_n$ of H, that satisfies the decomposition of $\mathrm{supp}(()f_n)$ given by the three conditions above, such that $\pi_\gamma(w_0)$ is within distance C of the beginning of γ, and $\pi_\gamma(w_n)$ is within distance C of the end of γ. Also we have that the length of γ is at least L (where L is the large constant from the definition of pivots). (this is equivalent to satisfying the three properties above).*

Let T be the length of a solution to $\mathrm{TSP}(w_0, w_n, \mathrm{supp}(()f_n)) = \mathrm{TSP}(w_0, w_n, P_1 \cup P_2 \cup P_3) = |w_n|_{\mathbb{Z}/2\mathbb{Z}\wr H}$. Then there exists a path η that starts at w_0, finishes at w_n and visits all points in $\mathrm{supp}(()f_n)$ such that length$(\eta) \leq T + 100N(L + 2D)$, and which satisfies the following:

The path η induces a linear order of $\mathrm{supp}(()f_n)$. What we require is that in the coding of η, all of the elements of P_1 appear before any of P_3. (i.e. the coding does not have a subsequence of the form $P_3 P_1$.

Proof. Let us first consider η_0 the optimal solution to the TSP.

Lemma 4.4 implies Corollary 4.5 that the total number of backtrackings is the size of P_2.

Finally, the argument goes as follows: first do all excursions of η_0 on P_1, then visit all elements in P_2, and then do all excursion in P_3. In total we added at most $2D\times$(number of backtrackings)+(length of solution of TSP in P2 that visits all elements in P2). And the number of backtrackings is at most $|P_2|$ by the previous claim.

□

In other words, we are saying that trying to solve the problem by first visiting all elements of P_1, and then visiting all elements of P_3, and crossing the middle section only once, it's at a bounded length of being optimal.

Lemma 4.7. *For any $N \in \mathbb{N}$ there exists some $C > 0$ such that the following holds. set $m \in \mathbb{N}$ be an integer and let U be the waiting time until the first pivot after time m. Then we have*

$$\sup_{m\geq 1} \mathbb{E}\left[\left(\sum_{i=m}^{m+U+N} |g_i|\right)^2\right] \leq C.$$

Proof. I'll go into the construction of pivots and explain why this is true. Maybe there's a simpler reasoning using only the exponential estimates on U.

Recall that the way Gouezel constructs pivots is as follows: if we let $S \subset H$ be our finite Schottky set, then we can decompose some convolution power μ_H^N as

$$\mu_H^N = \alpha\mu_S + (1-\alpha)\nu$$

for some positive $\alpha > 0$. Then we draw our increments as follows: let $\{\varepsilon_i\}_i$ be i.i.d. Bernoulli(α) random variables. If $\varepsilon_i = 1$, we draw $g_i' = s_i$ according to μ_S. Else we draw $g_i' = w_i$ according to ν. We observe that the sequence $\{g_1'...g_k'\}_k$ has the same distribution as $\{g_1...g_k\}_k$ for $g_i \sim \mu_H^N$.

Now we denote the resampled random walk by $g_1'...g_n' = w_1...w_{k_1}s_1w_{k_1+1}...w_{k_2}s_2...$, where the strings between $s_i's$ may be empty. Now each string $w_{k_{i-1}+1}...w_{k_i}s_i$ is distributed according to $\nu^Z * \mu_S$, where Z is a geometric random variable with parameter α.

Now Gouezel tells us that, conditional on any realization of the increments drawn from ν, the number of μ_S increments ℓ until we see a pivot has an exponential tail. This implies that

$$\mathbb{E}\left[\left(\sum_{i=m}^{m+U+N} |g_i|\right)^2 |\{w_i\}i\right] \leq \mathbb{E}\left[\left(L\ell + \sum_{i=0}^{\ell-1}\sum_{k=K_i}^{K_{i+1}-1} |w_i|\right)^2 |\{w_i\}_i\right].$$

Now we can integrate over the possible values of w_i and use independence in order to conclude that

$$\mathbb{E}\left[\left(\sum_{i=m}^{m+U+N} |g_i|\right)^2\right]$$

is bounded uniformly over m. $\square$

4.3. **Proof of the CLT.** Let us define $\Phi_{n,m} = |w_n| - |w_m^{-1}w_n| - |w_m|$. It suffices to show that $\mathbb{E}(|\Phi_{n,m}|^2)$ is finite (uniformly on n and m).

We will do the proof for finitely supported μ.

We fix m and n. Let $m+u$ be the first instant after m that you see a pivot.

If $m+u+N > n$, then we use Lemma 4.7

$$\mathbb{E}|\Phi_{n,m}|^2 \leq \sup_{m\geq 1} \mathbb{E}\left[\left(\sum_{i=m}^{m+u} |g_i|\right)^2\right] \leq C.$$

Otherwise, $m+u+N \leq n$ and we do the following.

(1) The three conditions at the beginning of this subsection are satisfies.
(2) Our objective is to get a good upper bound for $\Phi_{n,m}$ in the inequality

$$|w_n| \geq |w_m| + |w_m^{-1}w_n| - \Phi_{n,m}.$$

(3) We first note that $||w_m| - |w_{m+u}||$ has a finite second moment. Indeed, this amount is controlled by the increments done during u steps, and we know the distribution of how large u can be. That is, we use Lemma 4.7 to justify this. The same is true for $\left||w_m^{-1}w_n| - |w_{m+u}^{-1}w_n|\right|$. Again, this follows from a triangular inequality and Lemma 4.7.

(4) From this, we just need a good upper bound for $\Phi_{n,m}$ in the inequality

$$|w_n| \geq |w_{m+u}| + |w_{m+u}^{-1} w_n| - \Phi_{n,m}.$$

(5) We note that $\left||w_{m+u+N}^{-1} w_n| - |w_{m+u}^{-1} w_n|\right|$ is a bounded constant (since it only depends on N), and in particular has a finite second moment.

(6) From this, we just need a good upper bound for $\Phi_{n,m}$ in the inequality

$$|w_n| \geq |w_{m+u}| + |w_{m+u+N}^{-1} w_n| - \Phi_{n,m}.$$

(7) We look at the TSP between time 0 and n, we use the path η from the previous lemma to get a path which is near optimal and crosses only once the neighborhood of γ.

(8) From this path we obtain near-optimal paths from $|w_{m+u}|$ and for $|w_{m+u+N}^{-1} w_n|$, by doing surgery near the endpoints of γ and possibly adding a constant bounded amount of length.

Indeed, we first take the path from the starting point to the last visit to P_1, and we connect it to w_{m+u}. This is at most $Optimal + L + 2D$. Similarly we look at the first time we enter P_3, and connect that to a path to w_{m+u+N}. This again adds at most $Optimal + L + 2D$.

(9) From this, we directly apply 3.2.

5. The CLT for the range of random walks on hyperbolic groups

We borrow the framework from [MS20] for proving a CLT - We observe the following trivial fact that whenever $1 \leq m \leq n$ and denoting $R_{m,n}$ for the range between times m and n we have

$$|R_n| = |R_m| + |R_{m,n}| - |R_m \cap R_{m,n}|.$$

In the language of [MS20], we say that $\{|R_n|\}_{n\geq 1}$ is a *defective adapted cocycle* with defect $\Phi_{m,n} := |R_m \cap R_{m,n}|$.

By theorem 4.2 in [MS20], to prove a CLT for the sequence $|R_n|$ it is enough to show a second-moment deviation inequality: that

$$\mathbb{E}[\Phi_{m,n}^2] \leq C.$$

For some C not depending on m, n. We instead prove a stronger version of the deviation inequality:

Proposition 5.1. *There exists $C > 0$ such that for any $1 \leq m \leq n$.*

$$\mathbb{P}(\Phi_{m,n} \geq k) \leq Ce^{-k/C},$$

Proof. Let $\hat{R}_n$ denote the range of the reversed random walk - that is, the random walk driving by $\hat{\mu}$.' If is enough to show that for any $n, n' \in \mathbb{N}$ we have

$$\mathbb{P}(\sup_{n,n'} |\hat{R}_n \cap R_n| \geq k) \leq Ce^{-k/C}.$$

This is an immediate consequence of lemma 5.3 of [Cho23b]. (maybe this is actually Lemma 4.9 of the arxiv version of [Cho23b]?) ∎

This concludes the proof of Theorem 2.2.

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