

CLT FOR RANGE OF RANDOM WALKS ON HYPERBOLIC GROUPS

MAKSYM CHAUDKHARI, KUNAL CHAWLA, CHRISTIAN GORSKI, EDUARDO SILVA

ABSTRACT. We prove a central limit theorem for random walks with a finitely supported step distribution on wreath products of the form $A \wr H = \bigoplus_H A \wr H$, where A is a non-trivial finite group and H is a non-elementary hyperbolic group.

1. INTRODUCTION

In this paper we consider wreath products where the group of lamps is a finite group, and the base group is a hyperbolic group. We will consider a word length in these wreath products given by the switch-walk-switch generating set with respect to some fixed generating set of H (see Subsection 2.3.1).

Theorem 1.1. *Let A be a non-trivial finite group and let H be a non-elementary hyperbolic group. Endow $A \wr H$ with the switch-walk-switch word metric $|\cdot|_{\text{sws}}$ associated with some arbitrary finite generating set of H . Consider a finitely supported probability measure μ on $A \wr H$ such that $\text{supp}(\mu_H)$ is non-elementary. Denote by $\{w_n\}_{n \geq 0}$ the μ -random walk on $A \wr H$, and let $C = \lim_{n \rightarrow \infty} \frac{\mathbb{E}(|w_n|_{\text{sws}})}{n}$ be the drift of the μ -random walk on $A \wr H$. Then the sequence of normalized random variables $\frac{|w_n|_{\text{sws}} - Cn}{\sqrt{n}}$, $n \geq 1$, converges in law to a non-degenerate gaussian random variable.*

This result is new even for simple random walks on $\mathbb{Z}/2\mathbb{Z} \wr F_2$.

1.1. Background.

- The CLT for non-abelian free groups is due to [SS87] and [Led01]. Then for non-elementary hyperbolic groups with a finite exponential moment is due to [Bjö10]. This was generalized for any finite second moment measure in [BQ16a]. The last two results hold more generally for group acting on a Gromov hyperbolic space by isometries. [BQ16b] show a CLT for random walks on $\text{GL}_d(\mathbb{R})$ with a finite second moment. See also [Gou17].
- [EZ22] prove a law of large numbers for random walks on $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^2$ with a finite $(2 + \varepsilon)$ -moment, for some $\varepsilon > 0$. They also discuss limit laws in other wreath products.
- [BFGK24] prove a central limit theorem for random walks on the group of affine transformations of a horospherical product of Gromov hyperbolic spaces.
- [Cho23] proves a central limit theorem for groups acting with contracting elements.
- [GTT22] prove a CLT with respect to the counting measure on the Cayley graph of a group acting on a hyperbolic space.
- [Hor18] proves a CLT for random walks on mapping class groups and $\text{Out}(F_n)$.
- [Bar22] proves a CLT for groups acting on a $\text{CAT}(0)$ space.
- [Gil08] proves that the drift of $\mathbb{Z}/2\mathbb{Z} \wr G$ is strictly larger than that of its projection to G .
- [MSŠ23] prove a LLN and CLT for the capacity of the range of a random walk on a group.
- [Sal01] proves a LLN and CLT for a simple random walk on a free group, conditioned on the boundary point.

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2. PRELIMINARIES

2.1. Notation for graphs and paths.

- We will work with undirected, unlabeled graphs (V, E) .
- A path $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n)$ is an ordered sequence of vertices in the graph.

2.2. Hyperbolic groups. Say basic things about hyperbolicity; explain pivots

2.3. Wreath products. We consider the wreath products $A \wr H$, where A is a finite non-trivial group and H is a finitely generated group. Let S_H be a finite and symmetric generating set of H .

2.3.1. The switch-walk-switch word metric. We consider the *switch-walk-switch* S_{sws} generating set of $A \wr H$, given by

$$S_{\text{sws}} := \left\{ (\delta_a, 0)(\mathbf{0}, s)(\delta_{a'}, 0) \mid a, a' \in A \text{ and } s \in S_H \right\}.$$

The following goes back to [Par92, Theorem 1.2].

Proposition 2.1. *For any $g = (f, x) \in A \wr H$, the word length of g with respect to the standard generating set is*

$$|g|_{\text{sws}} = \text{TSP}(e_H, x, \text{supp}(f)).$$

2.4. Random walks on groups. Let G be a countable group and consider a probability measure μ on G . Consider the product space $\Omega := G^{\mathbb{Z}^+}$ endowed with the product σ -field. For each $n \geq 1$ we denote by

$$X_n : \Omega \rightarrow G$$

$$w := (w_1, w_2, \dots) \mapsto X_n(w) := w_n$$

the n -th coordinate map. We endow Ω with the product probability measure $\mu^{\mathbb{Z}^+}$.

We denote by

$$\theta : \Omega \rightarrow \Omega$$

$$w := (w_1, w_2, \dots) \mapsto \theta(w) := (w_2, w_3, \dots)$$

the shift map in the space of increments.

Now we define the μ -random walk $\{Z_n\}_{n \geq 0}$ on G as follows. We define $Z_0(w) = e_G$ for each $w \in \Omega$, and for each $n \geq 1$ we define

$$Z_n(w) := Z_{n-1}(w) \cdot X_n(w).$$

We remark that $Z_n(w)(Z_m \circ \theta^n)(w) = Z_{n+m}(w)$, for each $w \in \Omega$ and $n, m \geq 1$.

2.5. Defective adapted cocycles and the central limit theorem. A sequence $\mathcal{Q} = \{Q_n\}_{n \geq 1}$ of maps $Q_n : \Omega \rightarrow \mathbb{R}$ such that Q_n is measurable with respect to $\sigma(X_1, \dots, X_n)$, for each $n \geq 1$, is called a *defective adapted cocycle*. We will use the convention $Q_0 \equiv 0$. The *defect* of \mathcal{Q} is the collection of maps $\Psi = \{\Psi_{n,m}\}_{n,m \geq 0}$ defined by

$$\Psi_{n,m}(w) = Q_{n+m}(w) - Q_n(w) - (Q_m \circ \theta^n)(w), \text{ for each } w \in \Omega \text{ and } n, m \geq 0.$$

The following result states that the central limit theorem holds for defective adapted cocycles that satisfy a second-moment deviation inequality.

Theorem 2.2 ([MS20, Theorem 4.2]). *Let G be a countable group endowed with a probability measure μ . Consider \mathcal{Q} a defective adapted cocycle on $\Omega = G^{\mathbb{Z}^+}$, and denote by $\{\Psi_{n,m}\}_{n,m \geq 0}$ its defect. Suppose that*

- (1) $\mathbb{E}[|Q_1|^2] < \infty$, and
- (2) $\sup_{m,n \geq 0} \{\mathbb{E}[|\Psi_{n,m}|^2]\} < \infty$.

Then, there exist constants $\ell, \sigma \in \mathbb{R}$ such that the random variables $\frac{1}{\sqrt{n}}(Q_n - \ell n)$ converge in law to a Gaussian random variable with zero mean and variance σ^2 .

Furthermore, it is proved in [MS20, Theorem 3.3] that the constant ℓ that appears in the statement of Theorem 2.2 satisfies that $\frac{1}{n}Q_n$ converges to ℓ in L_1 as $n \rightarrow \infty$.

We will use this result for the defective adapted cocycle obtained from the word length of the μ -random walk on G at time n . That is, we will consider some word metric d on G , and define $Q_n := d(e_G, Z_n)$, for each $n \geq 1$. Since we will be working with finitely supported probability measures, it holds immediately that $\mathbb{E}[|Q_1|^2] < \infty$.

The objective of the following sections of this paper is to prove that there exists a constant $C > 0$ such that

$$\mathbb{E}[d(e_G, Z_{n+m}) - d(e_G, Z_n) - d(e_G, Z_m \circ \theta^n)] = \mathbb{E}[|\Psi_{n,m}|^2] \leq C, \text{ for each } n, m \geq 0,$$

where $G = A \wr H$, d will be the switch-walk-switch word metric and μ is a finitely supported probability measure, as in the hypotheses of Theorem 1.1.

3. CLT FOR THE LAMPLIGHTER OVER A HYPERBOLIC GROUP (USING PIVOTS)

Let us consider a non-trivial finite group A and a non-elementary hyperbolic group H . Choose an arbitrary symmetric finite generating set S_H of H . Let us denote by d_H (resp. $|\cdot|_H$) the associated word metric (resp. word length) on H , and by d_{sws} (resp. $|\cdot|_{\text{sws}}$) the switch-walk-switch word metric (resp. word length) on $A \wr H$. Additionally, let us consider $\delta \geq 0$ to be a hyperbolicity constant of $\text{Cay}(H, S_H)$.

3.1. Pivots.

Definition 1. Given a path $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_k)$ on $\text{Cay}(H, S_H)$ and $g \in H$, let $\pi_\gamma(g)$ be the set of elements visited by γ that minimize the word metric to g . That is, if we denote by $d_H(g, \gamma) := \min\{d_H(g, \gamma_i) \mid i = 1, 2, \dots, k\}$ the minimal distance between g and any element of the path γ , then

$$\pi_\gamma(g) := \{h \in G \mid h = \gamma_i \text{ for some } i \in \{1, \dots, k\} \text{ and } d_S(g, h) = d_H(g, \gamma)\}. \quad (1)$$

We now introduce the definition of pivots that we will use in the proof of Theorem 1.1. We refer to [Gou22, Section 4A] for details.

Definition 2. Let $C, D > 0$, $L \geq 20C + 100\delta + 1$, and $N \in \mathbb{Z}_{\geq 1}$. Let $\mathbf{w} = \{w_n\}_{n \geq 1} \in \Omega = (A \wr H)^{\mathbb{Z}_+}$, and consider the associated trajectory of the random walk $\{Z_n(\mathbf{w})\}_{n \geq 0}$. To avoid having too much notation we will omit the dependence of Z_n on \mathbf{w} . Denote by Z_n^H the projection of Z_n to H , for each $n \geq 0$. A time instant $m \geq 1$ is called a (C, D, L, N) -pivot for \mathbf{w} if the following conditions hold.

$$(1) \quad d_H(Z_m^H, Z_{m+N}^H) \geq L.$$

Let γ be an arbitrary geodesic path in $\text{Cay}(H, S_H)$ that connects Z_m^H to Z_{m+N}^H . Then

$$(2) \quad \text{for all } 0 \leq k \leq m \text{ we have } d_H(\pi_\gamma(Z_k^H), Z_m^H) \leq C,$$

$$(3) \quad \text{for all } m \leq k \leq m+N \text{ we have } d_H(Z_k^H, \gamma) \leq D, \text{ and}$$

$$(4) \quad \text{for all } k \geq m+N, \text{ we have } d_H(\pi_\gamma(Z_k^H), Z_{m+N}^H) \leq C.$$

The following lemma will be our main tool.

Lemma 3.1. *For any $C, D, \delta > 0$, and any $L \geq 20C + 100\delta + 1$, there exists $N, R > 0$ such that for all $k \geq 1$ we have*

$$\sup_{i \geq 1} \mathbb{P}(\text{there is no } (C, D, L, N)\text{-pivot between instants } i \text{ and } i+k) \leq R e^{-k/R}.$$

why do we need to multiply the exponential by R ?

3.2. The structure of the TSP along pivotal times. Suppose that we are looking at a sample path $\{Z_n\}_{n \geq 0}$ of the μ -random walk on $A \wr H$, and that we have a (C, D, L, N) -pivot time m . Denote by $Z_n = (\varphi_n, Z_n^H)$ the lamp configuration and projection to H of Z_n , respectively. Let γ be an arbitrary geodesic path from Z_m^H to Z_{m+N}^H on $\text{Cay}(H, S_H)$. Then, since μ is finitely supported, there exist constants $\tilde{C}, \tilde{D} > 0$ such that the support of φ_n can be decomposed as a disjoint union

$$\text{supp}(\varphi_n) = P'_1 \cup P'_2 \cup P'_3,$$

that satisfies the following properties. Let us denote $P_1 = P'_1 \cup \{Z_{m+u}^H\}$, $P_2 = P'_2 \cup \{Z_{m+u}^H, Z_{m+u+N}^H\}$ and $P_3 = P'_3 \cup \{Z_{m+u+N}^H\}$. Then we have

- (1) for all $g \in P_1$, we have $d_H(\pi_\gamma(g), Z_m^H) \leq C$,
- (2) for all $g \in P_2$ we have $d_H(g, \gamma) \leq D$, and
- (3) for all $g \in P_3$, we have $d_H(\pi_\gamma(g), Z_{m+N}^H) \leq C$.

Definition 3. Let $g = (f, x) \in A \wr H$ and suppose that $\text{supp}(f) = P_1 \cup P_2 \cup P_3$. Let η be a path on $\text{Cay}(A \wr H, S_{\text{sws}})$ that realizes $|g|_{S_{\text{sws}}}$. We define the associated *coding* of η as the word u in the alphabet $\{P_1, P_2, P_3\}$, such that $u_i = P_j$ if and only if at the i -th step of η , there is a lamp at a position in P_j which was modified for the first time.

Lemma 3.2. *Let η be a solution to the TSP for the support of φ_n , that begins at $Z_0^H = e_H$ and finishes at Z_n^H . Then the coding of η does not have a subword of the form $P_1 P_3^\varepsilon P_1^{\varepsilon'} P_3$, for $\varepsilon, \varepsilon' \geq 1$.*

Proof. Surgery, meaning that you glue together the excursions to P_1 , and you glue together the excursions to P_3 , and connect them with any path through P_2 . This gives something even shorter than optimal since each gluing strictly reduces the length of the path. \square

Definition 4. Given a path η in $\text{Cay}(H, S_H)$, let us call a *backtracking* a subpath of η that has an associated coding of the form $P_1 P_2^\varepsilon P_3^\varepsilon P_2^{\varepsilon'} P_1^{\varepsilon'} P_2^* P_3$ for $\varepsilon, \varepsilon' \geq 1$. Here the $*$ symbolizes 0 or more occurrences of the symbol.

Corollary 3.3. *Let η be a solution to the TSP for $|Z_n|$. Then the number of backtrackings of η is at most $|P_2|$.*

Proof. Every backtracking must contain at least one element of P_2 . (Recall that the path only has an element in its coding if it has not been visited before). \square

Lemma 3.4. *Consider a sequence of points $\{Z_n\}_n$ of H , that satisfies the decomposition of $\text{supp}(\varphi_n)$ given by the three conditions above.*

Let T be the length of a solution to $\text{TSP}(Z_0^H, Z_n^H, \text{supp}(\varphi_n)) = \text{TSP}(Z_0^H, Z_n^H, P_1 \cup P_2 \cup P_3) = |Z_n|_{A \wr H}$. Then there exists a path η that starts at Z_0^H , finishes at Z_n^H and visits all points in $\text{supp}(\varphi_n)$ such that $\text{length}(\eta) \leq T + 100N(L + 2D)$, and such that, in the coding of η , all the elements of P_1 appear before any of the elements of P_3 .

Proof. Let us first consider η_0 the optimal solution to the TSP.

Lemma 3.2 implies Corollary 3.3 that the total number of backtrackings is the size of P_2 .

Finally, the argument goes as follows: first do all excursions of η_0 on P_1 , then visit all elements in P_2 , and then do all excursion in P_3 . In total we added at most $2D \times (\text{number of backtrackings}) + (\text{length of solution of TSP in } P_2 \text{ that visits all elements in } P_2)$. And the number of backtrackings is at most $|P_2|$ by the previous claim. \square

In other words, we are saying that trying to solve the problem by first visiting all elements of P_1 , and then visiting all elements of P_3 , and crossing the middle section only once, is at a bounded length of being optimal.

Lemma 3.5. *For any $N \in \mathbb{N}$ there exists some $C > 0$ such that the following holds. set $m \in \mathbb{N}$ be an integer and let U be the waiting time until the first pivot after time m . Then we have*

$$\sup_{m \geq 1} \mathbb{E} \left[\left(\sum_{i=m}^{m+U+N} |g_i| \right)^2 \right] \leq C.$$

Proof. Recall that the way Gouëzel constructs pivots is as follows: if we let $S \subset H$ be our finite Schottky set, then we can decompose some convolution power μ_H^N as

$$\mu_H^N = \alpha \mu_S + (1 - \alpha) \nu$$

for some positive $\alpha > 0$. Then we draw our increments as follows: let $\{\varepsilon_i\}_i$ be i.i.d. Bernoulli(α) random variables. If $\varepsilon_i = 1$, we draw $g'_i = s_i$ according to μ_S . Else we draw $g'_i = w_i$ according to ν . We observe that the sequence $\{g'_1 \dots g'_k\}_k$ has the same distribution as $\{g_1 \dots g_k\}_k$ for $g_i \sim \mu_H^N$.

Now we denote the resampled random walk by $g'_1 \dots g'_n = w_1 \dots w_{k_1} s_1 w_{k_1+1} \dots w_{k_2} s_2 \dots$, where the strings between s'_i s may be empty. Now each string $w_{k_{i-1}+1} \dots w_{k_i} s_i$ is distributed according to $\nu^Z * \mu_S$, where Z is a geometric random variable with parameter α .

Now Gouëzel tells us that, conditional on any realization of the increments drawn from ν , the number of μ_S increments ℓ until we see a pivot has an exponential tail. This implies that

$$\mathbb{E} \left[\left(\sum_{i=m}^{m+U+N} |g_i| \right)^2 \mid \{w_i\}_i \right] \leq \mathbb{E} \left[\left(L\ell + \sum_{i=0}^{\ell-1} \sum_{k=K_i}^{K_{i+1}-1} |w_i| \right)^2 \mid \{w_i\}_i \right].$$

Now we can integrate over the possible values of w_i and use independence in order to conclude that

$$\mathbb{E} \left[\left(\sum_{i=m}^{m+U+N} |g_i| \right)^2 \right]$$

is bounded uniformly over m . □

3.3. Proof of the CLT. Let us define $\Psi_{n,m} = |Z_n| - |Z_m^{-1} Z_n| - |Z_m|$. Thanks to Theorem 2.2, it suffices to show that $\sup_{m,n \geq 1} \mathbb{E}(|\Psi_{n,m}|^2)$ is bounded.

We will do the proof for finitely supported μ .

We fix m and n . Let $m+u$ be the first instant after m that you see a pivot.

If $m+u+N > n$, then we use Lemma 3.5

$$\mathbb{E} |\Psi_{n,m}|^2 \leq \sup_{m \geq 1} \mathbb{E} \left[\left(\sum_{i=m}^{m+u} |g_i| \right)^2 \right] \leq C.$$

Otherwise, $m+u+N \leq n$ and we do the following.

- (1) The three conditions at the beginning of this subsection are satisfied.
- (2) Our objective is to get a good upper bound for $\Phi_{n,m}$ in the inequality

$$|Z_n| \geq |Z_m| + |Z_m^{-1} Z_n| - \Phi_{n,m}.$$

- (3) We first note that $||Z_m| - |Z_{m+u}||$ has a finite second moment. Indeed, this amount is controlled by the increments done during u steps, and we know the distribution of how large u can be. That is, we use Lemma 3.5 to justify this. The same is true for $||Z_m^{-1} Z_n| - |Z_{m+u}^{-1} Z_n||$. Again, this follows from a triangular inequality and Lemma 3.5.

- (4) From this, we just need a good upper bound for $\Phi_{n,m}$ in the inequality

$$|Z_n| \geq |Z_{m+u}| + |Z_{m+u}^{-1} Z_n| - \Phi_{n,m}.$$

- (5) We note that $\left| |Z_{m+u+N}^{-1}Z_n| - |Z_{m+u}^{-1}Z_n| \right|$ is a bounded constant (since it only depends on N), and in particular has a finite second moment.
- (6) From this, we just need a good upper bound for $\Phi_{n,m}$ in the inequality

$$|Z_n| \geq |Z_{m+u}| + |Z_{m+u+N}^{-1}Z_n| - \Phi_{n,m}.$$

- (7) We look at the TSP between time 0 and n , we use the path η from the previous lemma to get a path which is near optimal and crosses only once the neighborhood of γ .
- (8) From this path we obtain near-optimal paths from $|Z_{m+u}|$ and for $|Z_{m+u+N}^{-1}Z_n|$, by doing surgery near the endpoints of γ and possibly adding a constant bounded amount of length.

Indeed, we first take the path from the starting point to the last visit to P_1 , and we connect it to Z_{m+u} . This is at most $Optimal + L + 2D$. Similarly we look at the first time we enter P_3 , and connect that to a path to Z_{m+u+N} . This again adds at most $Optimal + L + 2D$.

- (9) From this, we directly apply Theorem 2.2.

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