THE CIRCLE IS NOT THE BOUNDARY OF THOMPSON'S GROUP T AND SPACE OF SUBGROUPS OF WREATH PRODUCTS

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ABSTRACT.

1. Thompson's group T

Thompson's group T acts on the circle S^1, and for any probability measure μ on T there is a unique μ-stationary measure on S^1.are there any assumptions on the measure? non-degenerate probably? some moment assumption?. The objective of this section is to prove that (S^1, ν) is not the Poisson boundary of the μ-random walk on T. This answers a question asked in the second item of Section 6 of [Der13], and which is asked again in [Nav18, Question 19] for finitely supported, non-degenerate and symmetric probability measures on T.

Theorem 1.1. *Let μ be a non-degenerate finitely supported probability measure on Thompson's group T. Let ν be the unique stationary measure on the circle S^1 such that (S^1, ν) is a μ-boundary of T. Then (S^1, ν) is not the Poisson boundary of (T, μ).*

2. The conditional entropy criterion and non-maximality of a μ-boundary

The following criterion goes back to [Kai85, Theorem 2] (see also [Kai00, Theorem 4.6])

Theorem 2.1 (Kaimanovich's conditional entropy criterion)**.** *Let G be a countable group and let μ be a probability measure on G with $H(\mu) < \infty$. Consider a μ-boundary (X, λ) of G. Then (X, λ) is the Poisson boundary of (G, μ) if and only if for λ-almost every $\xi \in X$ we have $h^{\xi}(\mu) = 0$.*

3. Strategy for the proof of Theorem 1.1

(1) Consider a finitely supported non-degenerate probability measure μ on T, and denote by (S^1, ν) the associated μ-boundary given by the circle endowed with the corresponding stationary measure.

(2) We are going to prove the following statement.

 Proposition 3.1. *For λ-almost every $\xi \in S^1$ we have $h^{\xi}(\mu) > 0$.*

(3) We just need to prove that for λ-almost every $\xi \in S^1$ there is $C > 0$ such that for every n sufficiently large, we have $H_{\xi}(w_n) \geq Cn$.

(4) Note that $h^{\xi}(\mu^{*N}) = N h^{\xi}(\mu)$. Hence, since μ is non-degenerate, we may assume that the support of μ contains:
 - The identity element,
 - a non-trivial element supported on $[1/4, 3/4]$,
 - a non-trivial element supported on $[1/2, 1]$,
 - a non-trivial element supported on $[3/4, 1] \cup [0, 1/4]$, and
 - a non-trivial element supported in $[0, 1/2]$.

(5) Let us consider boundary points $\xi \in [3/8, 5/8]$. Let a be the identity element, and b be the non-trivial element supported on $[3/4, 1] \cup [0, 1/4]$.

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(6) Let us consider trajectories of the μ-random walk for which ξ is the boundary point of S^1.

References

[Der13] Bertrand Deroin. The Poisson boundary of a locally discrete group of diffeomorphisms of the circle. *Ergodic Theory Dynam. Systems*, 33(2):400–415, 2013. [Cited on page 1.]

[Kai85] V. A. Kaimanovich. An entropy criterion of maximality for the boundary of random walks on discrete groups. *Dokl. Akad. Nauk SSSR*, 280(5):1051–1054, 1985. [Cited on page 1.]

[Kai00] Vadim A. Kaimanovich. The Poisson formula for groups with hyperbolic properties. *Ann. of Math. (2)*, 152(3):659–692, 2000. [Cited on page 1.]

[Nav18] Andrés Navas. Group actions on 1-manifolds: a list of very concrete open questions. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures*, pages 2035–2062. World Sci. Publ., Hackensack, NJ, 2018. [Cited on page 1.]

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