Université Sorbonne Paris Nord

Mémoire M2 Mathématiques Fondamentales

Self-similar groups and simple amenable groups of Burnside type

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Abstract

In this work we are interested in showing V. Nekrashevych's construction for the first known examples of simple amenable groups of Burnside type, that is, which are finitely generated, infinite and periodic. In the process, we study self-similar groups and the notion of Schreier graphs, which form a fundamental part of the proof of Nekrashevych's result of periodic groups arising from fragmentations of dihedral actions. This theorem provides a family of solutions to Burnside's problem, containing some already known ones as is the case of the Grigorchuk group. Another important tool are topological full groups, which allows us to obtain a family of simple amenable groups starting from the above family. We finish with examples of Nekrashevych's result, namely some groups generated by piecewise isometries of a polygon.

Résumé

Dans ce travail, nous nous intéressons à montrer la construction de V. Nekrashevych pour les premiers exemples connus de groupes simples moyennables qui soient groupes de Burnside, c'est-à-dire qui soient de type fini, infinis et périodiques. Dans le processus, nous étudions les groupes auto-similaires et la notion de graphes de Schreier, qui constituent une partie fondamentale de la preuve du résultat de Nekrashevych des groupes périodiques résultant de la fragmentation des actions diédraux. Ce théorème fournit une famille de solutions au problème de Burnside, contenant certaines déjà connues comme c'est le cas du groupe de Grigorchuk. Un autre outil important sont les groupes pleins topologiques, qui nous permettent d'obtenir des exemples de groupes simples moyennables à partir de la famille ci-dessus. Nous terminons par des exemples du résultat de Nekrashevych, à savoir des groupes générés par des isométries par morceaux d'un polygone.

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Introduction

One of the most influential questions in group theory is Burnside's problem, originally asked by W. Burnside in 1902 [Bur02]: *does there exist a finitely generated infinite group such that all of its elements are of torsion?*. Let us say that a group G is *periodic* if for any element $g \in G$, there exists $n \in \mathbb{N}$ with $g^n = 1$. Of course, all finite groups are periodic. Similarly, it is easy to find examples of infinite periodic groups, as is the case of an infinite direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$. Nonetheless, in the previous case it happens that any generating set for the group must be infinite. Thus Burnside's question becomes relevant: can we find such an example of an infinite periodic group, among finitely generated groups? We call such a group a *group of Burnside type*.

A related question to approach is the *bounded Burnside problem*: does there exist a finitely generated, infinite group, whose elements are periodic, and whose orders are uniformly bounded by a constant? Such a group is said to be *of bounded exponent*. A family of groups where to search for solutions to this question is the following. Let $m, n \geq 1$, and define the *free m-generated Burnside group of exponent n* as the group

$$B(m,n) \coloneqq F_m/F_m^n,$$

where F_m is the free group on n generators, and F_m^n is the subgroup of F_m generated by elements of the form g^n, for $g \in F_m$. That is, $B(m,n)$ is in some sense the "largest" group with m generators, such that any of its elements g satisfies $g^n = 1$. More precisely, given any group G which is m-generated and such that $g^n = 1$ for any $g \in G$, there is a unique homomorphism from $B(m,n)$ to G, up to a reordering of the chosen generating sets. Hence some new questions to ask are: for which values of m and n is $B(m,n)$ finite? For which ones is it infinite?

In his 1902 paper, W. Burnside showed that $B(1,n) \cong \mathbb{Z}/n\mathbb{Z}$ for any $n \geq 1$, that $B(m,2)$ is a direct product of m copies of $\mathbb{Z}/2\mathbb{Z}$, that $B(m,3)$ is finite of order at most 3^{2m-1}, and that $B(2,4)$ is finite of order at most 2^{12}. Soon it was proven that a positive answer to Burnside's problem could not be found among linear groups: in 1905 W. Burnside [Bur05] proved that any subgroup of $\mathrm{GL}_n(\mathbb{C})$ with bounded exponent is finite, and moreover I. Schur [Sch11] proved in 1911 that every finitely generated periodic subgroup of $\mathrm{GL}_n(\mathbb{C})$ must be finite.

It has also been proven by I. Sanov [San40] in 1940 that $B(m,4)$ is finite and by M. Hall [Hal58] in 1958 that $B(m,6)$ is finite, for any $m \geq 1$. On the contrary, it is currently an open problem whether $B(2,5)$ is a finite group.

Burnside's problem remained unsolved until 1964 when E. Golod and I. Shafarevich [Gol64] answered it affirmatively. Soon after, the bounded Burnside problem was also answered affirmatively by S. Adian and P. Novikov [NA68] in 1968, where they proved that for every odd integer $n > 4381$ there exist infinite, finitely generated groups of exponent n. Later S. Adian [Adi75] improved the bound to n odd and $n > 665$, and recently S. Adian [Adi15] improved the result for n odd, $n > 101$. A restricted version of the bounded Burnside problem, where the group is required to moreover be residually finite, was answered negatively by E. Zelmanov [Zm90] in 1989.

Multiple examples of groups of Burnside type have been described since the original answer to Burnside's problem in 1964, and they can be split into three classes.

The first one are solutions of the bounded Burnside problem, which are constructed using small cancellation theory. Groups in this class come from the original proof of S. Adyan and P. Novikov [Adi79], A. Olshanskii [Os91, Iva94, Lys96], E. Rips [Rip82], and also via the theory of hyperbolic groups developed by M. Gromov [Gro87, Os93]. The second class comes from the original solution to Burnside's problem by E. Golod and I. Shafarevich [Gol64], who proved that for each prime p there exists an infinite group generated by three elements, in which each elements has order a power of p. Such groups are called Golod-Shafarevich groups. Finally, the third class are groups generated by automata and by actions on rooted trees, which are the ones that interest us the most for this work. The first examples of groups in this class were constructed by S. Aleshin [Ale72], V. Sushchanskii [Sk79], R. Grigorchuk [Gri80] and N. Gupta and S. Sidki [GS83].

Note that groups of the two latter classes are residually finite, and hence they cannot be simple. On the other hand, groups of the first class can be simple, as is the case of Tarski monsters [Os82].

One of the main results presented in this work is due to V. Nekrashevych [Nek18], where a new family of groups of Burnside type is described, which we proceed to explain now. Let X be a Cantor set, and $\langle a, b\rangle$ an infinite dihedral group generated by two involutions a and b, acting by homeomorphisms on X. We say that a finite subgroup A of homeomorphisms of X is a *fragmentation* of the involution a if for every $x \in X$ and $g \in A$ we have that $g(x) \in \{x, a(x)\}$, and for every $x \in X$ there exists $g \in A$ such that $g(x) = a(x)$. Now consider the group $G = \langle A \cup B\rangle$ generated by fragmentations A and B of the involutions a and b, respectively. We call G a *fragmentation* of the dihedral group. Suppose there exists a point $\xi \in X$ such that $a(\xi) = \xi$ and such that for every $g \in G$ with $g(\xi) = \xi$, the set of fixed points of g accumulates on ξ. Then V. Nekrashevych's Theorem 5.1 states that G must be a periodic group. In fact such hypothesis about the fixed point ξ is not so restrictive, in the sense that any action of the D_∞ on the Cantor space can be *fragmented* in order to satisfy it (see Proposition 4.17).

The family of periodic groups provided by Nekrashevych's result intersects with some of the above classes, as it contains Grigorchuk's group for example, but it also contains many simple groups. As we mention at the end of Section 1.3, groups of Burnside type were the first examples of non-amenable groups without free subgroups, namely free Burnside groups and Tarski monsters [Ov80, Adi82].

The Grigorchuk groups [Gri83, Gri84] were the first examples of amenable groups which are not elementary amenable. In fact, most examples of non elementary amenable groups are either based on Grigorchuk groups, or are defined by actions on rooted trees [BV05, BKN10, AAV13], so they are in particular residually finite.

The question of the existence of finitely generated infinite simple amenable groups was open for a long time, until it was answered by K. Juschenko and N. Monod in [JM13]. They showed that the *topological full group* (defined in Section 1.4) of a minimal homeomorphism of the Cantor set is amenable. If τ is a minimal homeomorphism of a Cantor space, then the topological full group of $\langle\tau\rangle$ has simple derived subgroup, and if the homeomorphism is expansive, then the derived subgroup is finitely generated [Mat06, BM08]. Moreover, the method used in [JM13] were generalized in [JNdlS16] to cover a wider class of non-elementary amenable groups.

For arbitrary fragmentations A, B of the generators a, b of a minimal action of the dihedral group, the group $\langle A \cup B\rangle$ can be embedded into the topological full group of a minimal subshift, as is proven by N. Matte Bon in [MB15]. Hence it follows immediately that all groups generated by fragmentations a minimal action of the dihedral group are amenable. Hence V. Nekrashevych's result implies the first examples of simple amenable groups of Burnside type (see Theorem 5.4). Up until this point, the only previously known simple groups of Burnside type were of bounded exponents and non-amenable, as for example it is the case of Tarski monsters [Os82].

We now proceed to explain the organization of this work. In Chapter 1 we introduce notation, basic definitions and make comments about the fundamental concepts we will use. In particular we discuss amenable and elementary amenable groups, and we briefly introduce topological full groups and explain their importance for Nekrashevych's Theorem of simple amenable groups of Burnside type. We also mention the topic of growth of groups, introduce Grigorchuk groups as the first examples of groups of intermediate growth, and their relevance with Burnside's problem. We also mention Nekrashevych's theorem about existence of simple groups of intermediate growth [Nek18], although it is not the topic of this work. Next, in Chapter 2 we define rooted trees and study their automorphisms groups. We define self-similar groups and in particular we show that Grigorchuk's group is periodic. Then in Chapter 3 we introduce Schreier graphs as a generalization of Cayley graphs, and explain their connection with the proof of Theorem 5.1 through the concept of orbital graphs. We also show a result which is of interest by its own, namely the fact that any regular connected graph can be realized as the Schreier graph of a free group, when given the correct orientation and labeling to its edges (see Theorem 3.2). Afterwards, in Chapter 4 we study in detail fragmentations of dihedral actions, and introduce its corresponding orbital graphs and graphs of germs. We finish the chapter by characterizing the structure of the graph of germs of a fixed point of one of the involutions in terms of its orbital graph, and show most of the results needed in order to prove 5.1. This theorem is proved in Chapter 5, after which it is explained how to use K. Juschenko and N. Monod's result about amenability of the topological full group of a minimal homeomorphism of a Cantor space in order to prove Theorem 5.4, proving the existence of simple amenable groups of Burnside type.

Chapter 1

Generalities

In this chapter we present all preliminary concepts used throughout the rest of the work, as well as some historical background and motivation for the most important results. Namely, V. Nekrashevych's result about periodic simple amenable groups arising from dihedral actions on Cantor spaces. In Section 1.1 we present the notation we will use for groups and actions of groups, together with the notion of residually finite groups.

In Section 1.2 we introduce Cayley graphs as a motivation for our later discussion on Schreier graphs in Chapter 3. Then, in Section 1.3 we present the notion of amenability together with a pertinent discussion on non-elementary amenability and examples of amenable non-elementary amenable groups. Afterwards we introduce topological full groups in Section 1.4, which are a fundamental part of Nekrashevych's result. Next in Section 1.5 we introduce the notion of growth of groups, show how subexponential growth implies amenability, and discuss Milnor's problem about groups of intermediate growth. We briefly mention Nekrashevych's result about simple groups of intermediate growth, although it is not our main focus for this work.

1.1 Definitions and notation

Throughout this work we will be working with finitely generated groups, usually denoted by G or H, and actions of them on a space X (usually a rooted tree, the boundary of a rooted tree, or a topological space), meaning homomorphisms $G \to \mathrm{Aut}(X)$, and denoted $G \curvearrowright X$. When there is no confusion about the considered action, given $g \in G$ and $x \in X$ we denote $g(x) \in X$ the action of g on x. We denote by ε or 1 the identity element of a group G, depending on the context.

We now state some basic group properties which will be mentioned multiple times on this work.

Definition 1.1. A group G is said to be *residually finite* if for every $g \in G\backslash\{1\}$, there exists a finite group F and a homomorphism $\varphi : G \to F$ such that $\varphi(g) \neq 1$.

Note that Definition 1.1 is equivalent to saying that for any $g \in G\backslash\{1\}$, there exists a normal subgroup of finite index (The kernel of φ) which does not contain g.

Lemma 1.2. *Let G be a group and $H \leqslant G$ a subgroup of finite index. Then there exists a normal subgroup of finite index $K \lhd G$ contained in H.*

Proof. Consider the finite group $\mathrm{Sym}(G/H)$ of permutations of the finite set of left cosets G/H, and the action $\rho : G \to \mathrm{Sym}(G/H)$ given by the action of G by left multiplication. That is, for

$g, x \in G$ we have $\rho(g)(xH) = gxH$. Then the stabilizer of an element $xH \in G/H$ is the subgroup xHx^{-1}. Denote $K := \ker(\rho)$, which by construction is a normal subgroup of G contained in H. Moreover, as we have that G/K is isomorphic to $\mathrm{Im}(\rho) \subseteq \mathrm{Sym}(G/H)$ and the latter is a finite group, we conclude that K has finite index in G. □

From the comments after Definition 1.1 and Lemma 1.2 we get the following proposition.

Proposition 1.3. *Let G be a group. Then the following are equivalent.*

1. *G is residually finite.*

2. *The intersection of all subgroups of finite index of G is trivial.*

3. *The intersection of all normal subgroups of finite index of G is trivial.*

A final remark to do is that the class of residually finite groups intersects trivially with the class of simple groups.

Proposition 1.4. *Let G be a simple, non-trivial, group. Then G is not residually finite.*

1.2 Cayley graphs

We wish to interpret a group as more than a purely algebraic object by viewing it as the group of symmetries of a particular geometric structure. A first result related to this idea is the well known Cayley theorem, which states that every group may be faithfully represented as a subgroup of the symmetric group of some set. By defining Cayley graphs we will be able to see that a finitely generated group can be represented as the symmetry group of its associated Cayley graph, and in that way give a more geometric version of Cayley's theorem. To obtain a more detailed explanation of the concepts and theorems we mention here, we refer to [LÏ7] and [Mei08], as this section is partially based on both references.

To be able to talk about Cayley graphs we assume basic knowledge on graph theory, but to clarify the notation used throughout this section we recall the following definition of a directed graph, along with that of a labeled graph.

Definition 1.5. A *directed graph* is a 4-tuple $(V, E, \mathfrak{i}, \mathfrak{f})$ where V is called the set of vertices, $E \subseteq V \times V$ is called the set of edges, and the functions $\mathfrak{i} : E \to V$, $\mathfrak{f} : E \to V$ associate to each edge its "initial" and "terminal" vertices, respectively. If the orientation is not relevant, we omit the information of the functions ι and η and simply refer to the graph by the tuple (E, V).

Given an alphabet $\mathcal{A}$, a *labeled directed graph* is a 5-tuple $(V, E, \mathfrak{i}, \mathfrak{f}, \lambda)$, where $(V, E, \mathfrak{i}, \mathfrak{f})$ is a directed graph and $\lambda : E \to \mathcal{A}$ is a function which "labels" the edges of the graph.

If we omit the initial and terminal functions, we will talk about *undirected graphs* and *undirected labeled graphs*, respectively.

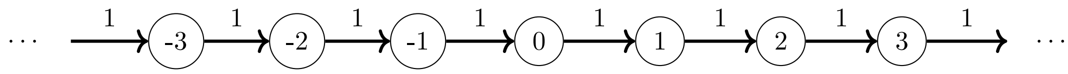
Now we are able to define the Cayley graph associated to a group given a generating set. The vertices of the graph are precisely the group elements and the existence of an edge between two vertices represents that viewing them as words over the generating set these elements differ from each other by a single generator.

Definition 1.6. Let G be a group generated by a finite set $S \subseteq G$. We define the (right) **Cayley graph** as the labeled graph $\Gamma(G, S) = (V, E, \mathfrak{i}, \mathfrak{f}, \lambda)$, where $V = G$, $E = \{(g, gs) \in V \times V \mid g \in G, s \in S\}$, $\mathfrak{i}((g, gs)) = g$, $\mathfrak{f}((g, gs)) = gs$ and $\lambda((g, gs)) = s$.

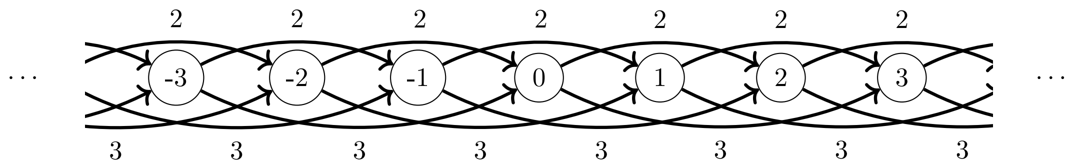
Note that the definition of the Cayley graph depends on the chosen generating set S, although for any finitely generated group most of the geometric properties encoded in the graph remain invariant under changes in the generating set. Below we show some Cayley graphs for some groups.

Example 1.7.

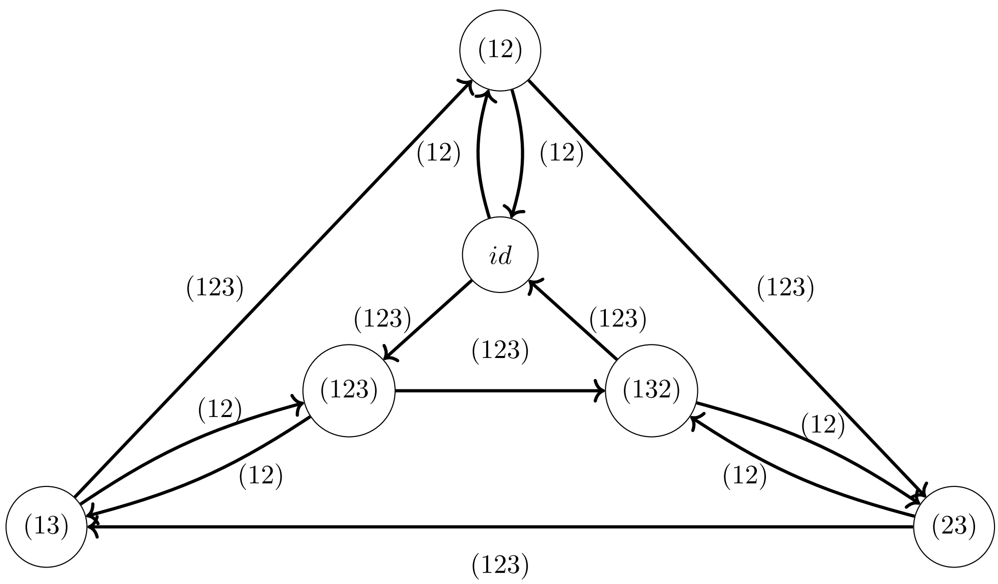
1. The free group of rank 1 $F_1 \cong \mathbb{Z}$ with generating set $S = \{1\}$ has as its Cayley graph a "discrete line": We could also have considered a different generating set like $S = \{2, 3\}$.



   In that case the Cayley graph would still be "line-like":

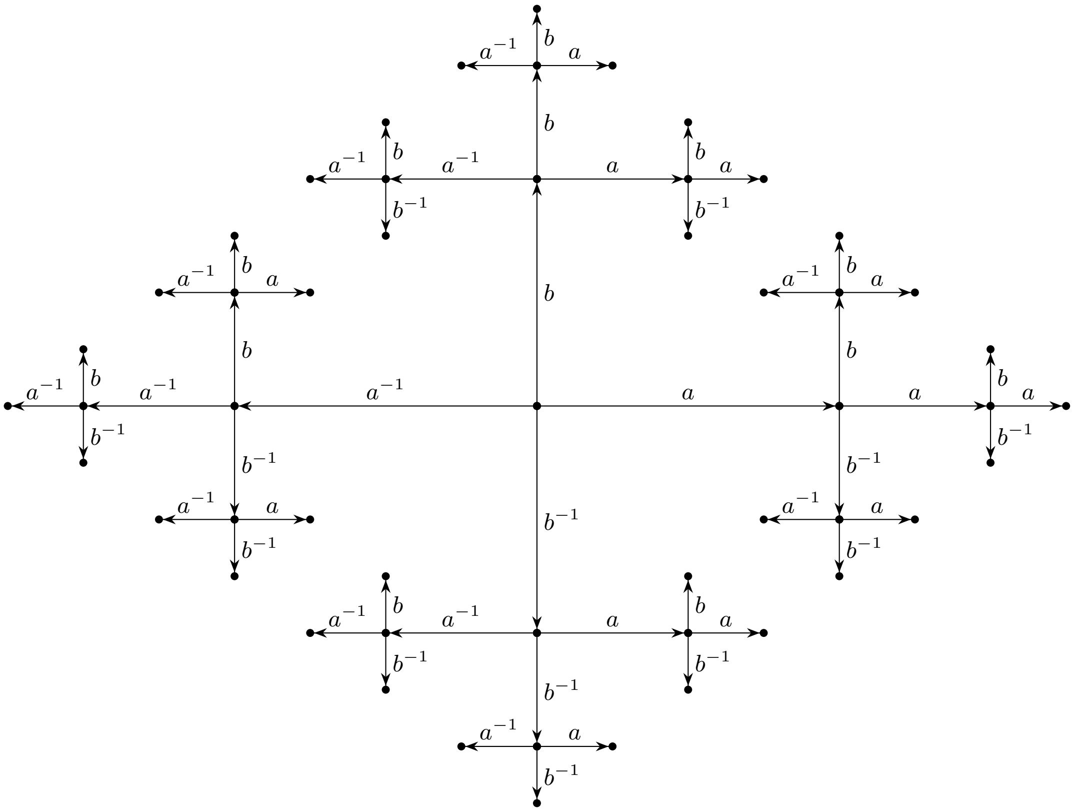


2. Consider the symmetric group $S_3 := \{\pi : \{1, 2, 3\} \to \{1, 2, 3\} \mid \pi \text{ is a bijection}\}$, with its generating set $S = \{(123), (12)\}$. The corresponding Cayley graph $\Gamma(S_3, S)$ is drawn below:



3. As a final example, we show (a part of) the Cayley graph of the rank-2 free group $F_2 = \langle a, b \,|\, \rangle$ with generating set $S = \{a, b\}$, which is an infinite 4-regular tree. In this representation of the Cayley graph the names of the vertices are omitted in order to obtain a clearer drawing. They can be recovered by reading off the edge labels following the unique path from the identity (the central vertex) to any other vertex.

The next theorem states precisely what we said at the beginning of this section about interpreting the group G as the group of symmetries of its associated Cayley graph $\Gamma(G, S)$, and is



referred to as “Cayley’s better theorem” in [Mei08], in comparison to Cayley’s theorem which states that every group is isomorphic to a subgroup of a symmetric group.

Theorem 1.8 ([Mei08, Chapter 1]). *Let G be a group generated by a finite set $S \subseteq G$. Then the Cayley graph $\Gamma(G, S)$ is a connected and locally finite*[1] *labeled graph, on which G acts by bijections that preserve edges along with their orientation and label.*

Now that we have found a way to associate a graph to a given group, we can endow the graph with a metric space structure by standard methods and with it obtain a metric structure on the group G.

Definition 1.9. Given a group G generated by a finite subset $S \subseteq G$, we define the **word metric** $d_S : G \times G \to \mathbb{R}$ as $d_S(g, h) \coloneqq \inf\{|w| \mid w \in (S^{\pm})^*, \pi(w) = g^{-1}h\}$, where $\pi : (S^{\pm})^* \to G$ is the canonical map which evaluates the word $w \in (S^{\pm})^*$ as an element of the group G, and $|w|$ denotes the length of the word w. It is common to denote $|g|_S \coloneqq d_S(g, e_G)$, for $g \in G$.

In the context of Cayley graphs the above definition may be interpreted as taking as a generating set $S^{\pm}$ and constructing $\Gamma(G, S^{\pm})$, in which the existence of every directed edge implies the presence of its oppositely oriented counterpart, resulting in a strongly connected (symmetric) Cayley graph. With this the definition of the word metric $d_S(g, h)$ is equivalent to taking the minimal edge-distance between the vertices g and h on the graph $\Gamma(G, S^{\pm})$.

In Chapter 3 we define Schreier graphs, which are a generalization of Cayley graphs and will allow us to understand geometrically the action of a group on a space.

[1]A directed graph (V, E) is said to be **strongly connected** if there exists a directed path between any two vertices, and **connected** if there exists an un-directed path between any two vertices. It is said to be **locally finite** if the in-degree and the out-degree at every vertex is finite.

1.3 Amenability

An amenable group G is one that allows a way to average bounded functions defined on G such that this average remains invariant under translations by elements of G. Equivalently, a group G is amenable if it contains a family of finite sets that are almost-invariant by translations of G, in a way that we make precise below. We give the definition only for countable groups since it is the case of interest for this thesis, although it can be extended to arbitrary groups.

Definition 1.10. A countable group G is said to be **amenable** if it satisfies one of the following equivalent conditions:

1. There exists a sequence of finite subsets of G such that for every $g \in G$:
$$\lim_{n\to\infty} \frac{|F_n g \triangle F_n|}{|F_n|} = 0.$$
A sequence $\{F_n\}_{n\in\mathbb{N}}$ with this property is called a (right) Følner sequence.

2. There exists a G-invariant mean on $\ell^\infty(G,\mathbb{R})$, i.e. an $\mathbb{R}$-linear map $m : \ell^\infty(G,\mathbb{R}) \to \mathbb{R}$ such that $m(\mathbf{1}_G) = 1$, $m(f) \geq 0$ for every $f \in \ell^\infty(G,\mathbb{R})$ such that $f \geq 0$ pointwise, and $m(g \cdot f) = m(f)$ for every $f \in \ell^\infty(G,\mathbb{R})$, where $g \cdot f(t) \coloneqq f(g^{-1}t)$ is the left action of G on $\ell^\infty(G,\mathbb{R})$.

An important case is the one of finitely generated groups. On this context it is straightforward to prove that it suffices to verify the condition of being a Følner sequence only for elements on the finite generating set.

Proposition 1.11. *Let G be a group generated by a finite set S. Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of finite subsets of G. Then $\{F_n\}_{n\in\mathbb{N}}$ is a Følner sequence if and only if for any $s \in S \cup S^{-1}$ we have*
$$\lim_{n\to\infty} \frac{|F_n s \triangle F_n|}{|F_n|} = 0.$$

The next proposition lists the basic properties of amenable groups.

Proposition 1.12 ([CSC18]). *The following properties hold.*

1. *Every finite group is amenable.*
2. *Every abelian group is amenable.*
3. *Every solvable group is amenable.*
4. *If G is amenable, then every subgroup of G and every quotient of G is amenable.*
5. *A group extension of an amenable group by an amenable group is amenable.*
6. *A direct limit of amenable groups is amenable.*
7. *The free group F_2 is not amenable.*

Denote by AG the class of amenable groups. It follows from Proposition 1.12 there is a large class of groups which are "clearly" amenable. Namely, groups which are subgroups, quotients, extensions and direct limits of abelian and finite groups. Following the notation of M. Day [Day57] we call this the class of *elementary amenable groups* and denote it by EG. On the other hand, again thanks to Proposition 1.12 groups which contain free subgroups are "clearly" non-amenable, so denoting by NF the class of groups which contain no free subgroups we have that AG is contained in NF. J. von Neumann asked in 1929 whether AG=NF [Neu29], and M. Day made in 1957 the remark that up to that point it was not known whether EG=AG or even if EG=NF.

C. Chou proved in 1980 that the class of elementary amenable groups EG can in fact be constructed from abelian and finite groups by only using the operations of group extensions and direct limits, and using the existence of groups of Burnside type, that is, finitely generated infinite periodic groups, that the class EG is strictly contained in NF [Cho80]. In other words, groups of Burnside type provide examples of groups which do not contain free subgroups and at the same time are not elementary amenable. We develop further into results connecting the existence of groups of Burnside type with questions about amenability in the Introduction, where Burnside's problem is discussed in more detail. For more results on amenable groups see [Pat88, Gre69, CSC18, LÏ7].

1.4 Topological full groups

Topological full groups were introduced in [GPS99]. They will play a fundamental role in this work, since they are the main tool used to pass from Nekrashevych's Theorem of periodic groups (Theorem 5.1), to the stronger result about simple amenable period groups (Theorem 5.4). The most important result we will use is Theorem 1.15. The idea is that, although a group may be complicated to study by itself, the orbits of its (non-free) actions may be easier to understand and still contain useful information about the group.

Suppose G is a group acting on a topological space X. We define a *germ* to be an equivalence class of an element $(g, x) \in G \times X$, where the equivalence relation identifies (g_1, x_1) with (g_2, x_2) if and only if $x_1 = x_2$ and there exists a neighborhood U of x_1 such that $g_1|_U = g_2|_U$. We define the *grupoid of germs* to be the set of all germs of the action, and endow it with the quotient topology of $G \times X$. A base for this topology is given by the sets $W_{g,u} = \{(g, x) \text{ germ } \mid x \in U\}$, for $g \in G$, U a neighborhood of X.

It is natural to identify points $x \in X$ with germs of the identity $(1, x)$. It is important to notice that it may happen that the topology on the groupoid of germs is not Hausdorff. Moreover, these non-Hausdorffness will play an essential role as hypothesis for the main theorems presented (see Definition 4.4, Theorem 5.1 and Theorem 5.4).

As we said, we want to study the group by looking at its orbits. In particular, two groups which have the same grupoid of germs will share the properties we can find through this method. We now define the group which is in some sense the biggest one which shares the same groupoid of germs with a fixed given group acting on a topological space X. We restrict the definition to the case of a countable group acting on a Cantor space, since is the relevant one to us.

Definition 1.13. Let G be a group acting on a Cantor set X. We define the *topological full group* $\mathrm{F}(G, X)$ as the group of all homeomorphisms $h : X \to X$ such that for every $\zeta \in X$ there exists a neighborhood U of ζ and an element $g \in G$ such that $h|_U = g|_U$.

Consider a non-empty clopen set $U \subseteq X$ and $g_1, \dots, g_n \in G$ such that $U_1 \coloneqq g_1(U), \dots, U_n \coloneqq g_n(U)$ are pairwise disjoint. We can use these sets to find a copy of the symmetric group $\mathrm{S_n}$ inside the topological full group $\mathrm{F}(G, X)$, by considering homeomorphisms which permute these disjoint subsets. In effect, for a permutation $\alpha \in \mathrm{S}_n$, we define the corresponding element $h_\alpha \in \mathrm{F}(G, X)$ by

$$h_\alpha(\zeta) = \begin{cases} g_j g_i^{-1}(\zeta), \text{ if } \zeta \in U_i, \text{ and } \alpha(i) = j, \\ \zeta \text{ if } \zeta \in X \setminus \cup_{i=1}^n U_i. \end{cases}$$

This gives us an embedding of S_n, and hence also of the alternating group A_n on the topological full group $\mathrm{F}(G, X)$

Definition 1.14. We define $\mathrm{A}(G, X)$ to be the subgroup of $\mathrm{F}(G, X)$ generated by all embeddings A_n on the topological full group, as described above.

Recall that the action of G on X is called *expansive* if there exists a constant $c > 0$ such that for any two distinct points $\zeta_1, \zeta_2 \in X$ there exists a $g \in G$ such that $d(g(\zeta_1), g(z_2)) \geq c$. The action of G on X is called *minimal* if all orbits are dense.

Theorem 1.15 ([Nek19])**.** *Suppose the action of G on X is minimal. Then $\mathsf{A}(G, X)$ is simple and is contained in every non-trivial normal subgroup of the topological full group $\mathsf{F}(G, X)$. If the action of G on X is expansive and has infinite orbits, then $\mathsf{A}(G, X)$ is finitely generated.*

We will not mention more results about topological full groups, since the above is all we need for our objetive. For more properties related to this work see [Nek19] and for a deeper study of topological full groups see [Mat06, Mat13, dC14].

1.5 Growth of groups

Let G be a group, and $S \subseteq G$ a finite generating set, which we will assume to be symmetric ($S = S^{-1}$). For every element $g \in G$ we define its S-length $\ell_S(g)$ as the minimal $n \geq 0$ such that g can be expressed as a product of n elements in S, that is,

$$\ell_S(g) \coloneqq \min\{n \geq 0 \mid g = s_1 \cdots, s_n, \text{ for } s_1, \ldots s_n \in S\}. \tag{1.1}$$

Definition 1.16. Let G be a finitely generated group and $S \subseteq G$ a finite symmetric generating set. We define the *growth function of G with respect to S* as the map $\gamma_{G,S} : \mathbb{N} \to \mathbb{N}$ given by

$$\gamma_{G,S}(n) \coloneqq |\, \{g \in G \mid \ell_S(g) \leq n\} \,|. \tag{1.2}$$

Definition 1.17. Given two functions $\gamma, \gamma' : \mathbb{N} \to \mathbb{N}$, we say that γ *dominates* γ', denoted by $\gamma' \preccurlyeq \gamma$, if there exists a constant $C > 0$ such that $\gamma'(n) \leq \gamma(Cn)$, for all $n \in \mathbb{N}$.

If $\gamma' \preccurlyeq \gamma$ and $\gamma \preccurlyeq \gamma'$ we say that γ and γ' are *equivalent*, and we write $\gamma \sim \gamma'$.

Proposition 1.18. *If G is a finitely generated group and $S, S' \subseteq G$ are two finite symmetric generating sets, then $\gamma_{G,S} \sim \gamma_{G,S'}$.*

Example 1.19.
1. For $G = \mathbb{Z}$ and the generating set $S = \{1, -1\}$, the growth function is given by $\gamma_{G,S}(n) = 2n + 1$. More generally, it is possible to see that for $G = \mathbb{Z}^d$, $d \geq 1$, we have $\gamma_{G,S}(n) \sim d$.
2. When $G = F_2 = \langle a, b \mid \rangle$ the free group of rank 2 and the generating set $\{a, b, a^{-1}, b^{-1}\}$, we have 1 element of S-length 0, and for $k \geq 1$ we have $4 \cdot 3^{k-1}$ elements of S-length k. After a calculation, we arrive at the growth function $\gamma_{G,S}(n) = 1 + 2(3^n - 1)$.

Consider a group G generated by a finite set S. It is simple to see that the function $\gamma_{G,S}$ is semi-multiplicative, that is $\gamma_{G,S}(n + m) \leq \gamma_{G,S}(n)\gamma_{G,S}(m)$. Thus the limit

$$\omega(G) \coloneqq \lim_{n \to \infty} \gamma_{G,S}(n)^{1/n}$$

exists, and takes values in $[1, +\infty)$.

Definition 1.20. We say that G has *exponential growth* if $\omega(G) > 1$, and *subexponential growth* otherwise. In the latter case, we say that G has *polynomial growth* of there exists $d \geq 1$ such that $\gamma_{G,S} \preccurlyeq n^d$, and *intermediate growth* otherwise.

Proposition 1.21. *Suppose G is a finitely generated group with subexponential growth. Then G is amenable.*

Proof. Suppose that G is not amenable, fix a finite symmetric generating set S with $1 \in S$, and denote by $F_n \coloneqq B_S(1,n)$ the ball in G of radius n centered at the identity. As G is not amenable, the sequence $\{F_n\}_{n\in\mathbb{N}}$ is not a Følner sequence. Hence thanks to Proposition 1.11 there must exist $\varepsilon > 0$ and a subsequence $\{F_{n_i}\}_{i\in\mathbb{N}}$ such that

$$\frac{|F_{n_i}s \triangle F_{n_i}|}{|F_{n_i}|} > \varepsilon, \ \forall s \in S \cup S^{-1}.$$

With this, we see that

$$|F_{n_{i+1}}| - |F_{n_i}| \geq \frac{1}{|S|}\sum_{s\in S} |F_{n_i}s \triangle F_{n_i}| \geq \varepsilon |F_{n_i}|.$$

Hence $|F_{n_{i+1}}| \geq (1+\varepsilon)|F_{n_i}|$ and by induction we have $|F_{n_{i+1}}| \geq (1+\varepsilon)^i |F_{n_1}$, so that $\omega(G) \geq 1+\varepsilon$. That is, G has exponential growth. □

Note that it is easy to find examples of groups of polynomial of exponential growth, as we saw in Example 1.19. On the other hand, the existence of groups of intermediate growth is not clear and was first asked by J. Milnor in [Mil68b]. In fact, there are classes of groups whose growth is always either polynomial or exponential, as is the case for solvable groups according to the Milnor-Wolf theorem [Wol68, Mil68a]. A negative answer to this question was given by R. Grigorchuk [Gri84] with an explicit construction of a family of groups of *intermediate growth*, creating in the process a new area of research in the field of growth of groups. Originally, R. Grigorchuk introduced the first example of these groups in [Gri80] as a solution to Burnside's problem, that is, an example of an infinite finitely generated periodic group. This topic will be covered in more detail in the Introduction. The family of Grigorchuk groups is defined by choosing a sequence $\omega \in \{0,1,2\}^{\mathbb{N}}$, and defining the group $\mathcal{G}_\omega = \langle a, b_\omega, c_\omega, d_\omega\rangle$ by the action of the generators $a, b_\omega, c_\omega, d_\omega$ on the unit interval: a interchanges both halves of the interval, while the elements $b_\omega, c_\omega, d_\omega$ interchange some dyadic subintervals according to rules dictated by the sequence ω. In this notation, the original group $\mathcal{G}$ constructed by Grigorchuk corresponds to the sequence $\omega = (012)^\infty$, and its intermediate growth behavior follows by proving that its growth function satisfies the bounds

$$e^{n^{1/2}} \preccurlyeq v_{\mathcal{G}}(n) \preccurlyeq e^{n^\beta}, \quad \text{where } \beta = \log_{32} 31 < 1.$$

Currently, the best known bounds for the Grigorchuk group are

$$e^{n^{\lambda-\varepsilon}} \preccurlyeq v_{\mathcal{G}}(n) \preccurlyeq e^{n^\lambda},$$

for any $\varepsilon > 0$ and where $\lambda \approx 0.7674\ldots$ is the positive root of the polynomial $X^3 - X^2 - 2X - 4$. The upper bound is due to L. Bartholdi [Bar98] and the lower bound is due to A. Erschler and T. Zheng [EZ20].

For further reading about groups of groups, the solution to Milnor's problem and related problems and questions, see [Gri14].

It is important to mention that V. Nekrashevych's publication [Nek18] also provides the first examples of finitely generated simple groups of intermediate growth, solving an open question (see [Gri14, Problem 15], [Man12, page 132], [MK95, Problem 15.17] and [BM07, BE17]).

Since we are concentrating on Burnside's problem, we will not explain in detail Nekrashevych's construction of a simple group of intermediate growth, although both solutions are extremely related. See [Nek18] for further details.

Chapter 2

Groups acting on trees

In this chapter we define rooted trees and their automorphisms groups, in particular proving the equivalence between finitely generated residually finite groups and some particular group actions on rooted trees in Proposition 2.8. We also study equivalent ways to present actions on rooted trees via automata and wreath products, and we give some examples. We finish this chapter by proving in Theorem 2.20 the fact that the Grigorchuk group is periodic.

2.1 Rooted trees and their boundaries

A *tree* is a connected undirected graph with no cycles. A *rooted tree* is a tree with a special vertex, called the *root* of the tree. The choice of a root induces a partial order on the vertices, according to their distance to the root. We say that a vertex at distance n from the root is *of level* n, and refer to the set of vertices of level n as the *nth* level. An example of a rooted tree is presented in Figure 2.1

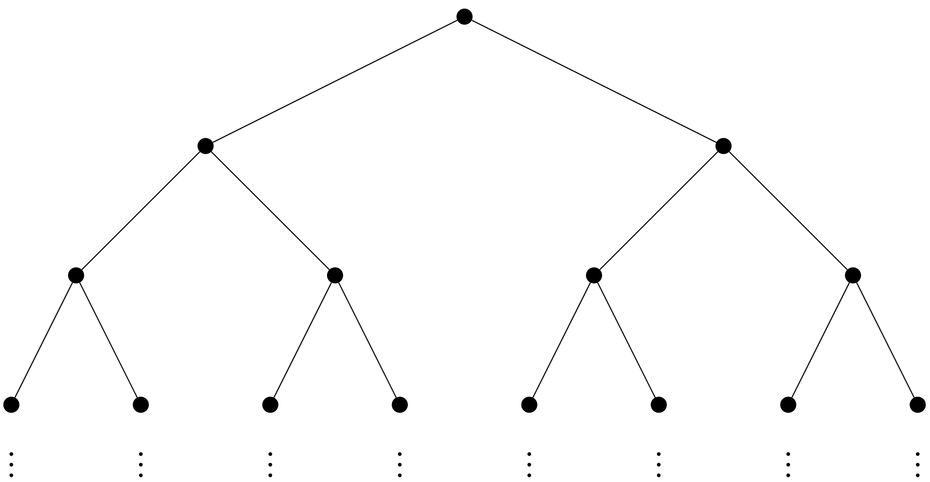


Figure 2.1 – Example of a rooted tree. The uppermost vertex is the root, and the rest of the graph is organized by levels according to the distance of the vertices to the root.

Any sequence of positive integers $\overline{m} = \{m_n\}_{n=1}^{\infty}$ (called the *branching sequence*) defines a rooted tree $\mathsf{T}_{\overline{m}}$ as follows: on level 0 we just have one vertex (the root of the tree), and to every vertex on level $n-1$ we attach m_n new vertices downwards, thus forming level n, for each $n \geq 1$. The norm $|u|$ of a vertex u is the level to which it belongs. In this case, the rooted tree is *spherically homogeneous*, meaning that all vertices of the same level have the same degree. We will often suppose that this condition holds for the rooted trees we consider. If the sequence $\overline{m}$ is constant, say equal to $d \geq 2$, we call $\mathsf{T}_{\overline{m}}$ the *regular rooted tree of degree* d, and we denote it by T_d.

When there is no risk of confusion, we will refer to a rooted tree simply as a *tree*, and use the notation T to denote the tree itself, as well as the collection of its vertices. We will denote by T^n the vertices of level n, and given any vertex $v \in \mathsf{T}$ we will denote by T_v the rooted tree with root v, whose vertices are all vertices of T whose geodesics to the root of T pass through v.

Let T be a rooted tree. We say that a function $\varphi : \mathsf{T} \to \mathsf{T}$ is an *automorphism* of T if φ is a bijection which preserves the edges between vertices of T, and which maps the root to itself. By definition, for any $n \in \mathbb{N}: \ v \in \mathsf{T}^n$ if and only if $\varphi(v) \in \mathsf{T}^n$. We denote by $\mathrm{Aut}(\mathsf{T})$ the group formed by all automorphisms of a tree T, with respect to the operation of composition.

Given an (infinite) tree T, we define its boundary $\partial\mathsf{T}$ as the set of rays $\gamma : \mathbb{N} \to \mathsf{T}$, such that for any $i \geq 0: \ \gamma(i)$ is a vertex of level i, adjacent to the vertex $\gamma(i+1)$. The ray γ is an infinite geodesic which starts at the root of T and goes to infinity. On $\partial\mathsf{T}$ there is a natural topology, in which two paths are close to each other if they have a large common beginning. This topology is metrizable: given any two rays $\gamma, \gamma' \in \partial\mathsf{T}$, define their distance to be 2^{-n} if n is the first level at which $\gamma(n) \neq \gamma'(n)$. The space $\partial\mathsf{T}$ endowed with this topology is a Cantor space, that is, a non-empty totally disconnected compact metrizable space with no isolated points.

Suppose G is a group acting by automorphisms on the tree T. Then for any geodesic ray γ, its image $g(\gamma)$ is also a geodesic ray on T. Hence, the action of G on T naturally induces an action of G on the boundary $\partial\mathsf{T}$. The relation between both actions will be useful in some contexts below.

Finally, we describe an alternative way of defining a rooted tree which will be sometimes more convenient. Let X be a finite set, which we will call from now on the *alphabet*. We denote X^* the set of finite words over the alphabet X, including the empty word. For a word $w \in \mathsf{X}^*$ such that $w = x_1 \cdots x_n$, for some letters $x_1, \ldots, x_n \in \mathsf{X}$, we say that w has length n and note $|w| = n$. We associate to X^* a rooted tree, whose vertices are the elements of X^*, the root being the empty word, and where two words $v, w \in \mathsf{X}^*$ are connected by an edge if and only if there exists $x \in \mathsf{X}$ with $v = wx$ or $w = vx$. By an abuse of notation, we make the natural identification between X^* and the rooted tree associated to it. For any word $v \in \mathsf{X}^*$, we denote by $v\mathsf{X}^*$ the rooted subtree of X^* with root v. We denote by X^ω the boundary of the tree X^*.

2.2 Groups acting on rooted trees

Recall that we say that a group G *acts* on a rooted tree T, denoted $G \curvearrowright \mathsf{T}$, if there exists a homomorphism $G \to \mathrm{Aut}(\mathsf{T})$, and we identify G with its image as a subgroup of automorphisms of T. In this section we study basic properties of such an action on spherically homogeneous trees, that is, rooted trees where all vertices of the same level have the same degree.

Definition 2.1. An action of a group G on a spherically homogeneous tree T is said to be *level-transitive* if it is transitive on every level. That is, for any pair of vertices $u, v \in \mathsf{T}$ of the same level, there exists $g \in G$ such that $g(u) = v$.

Note that level-transitivity of the action can be understood in terms of the action of G on the boundary $\partial\mathsf{T}$: The action $G \curvearrowright \mathsf{T}$ is level-transitive if and only if the corresponding action $G \curvearrowright \partial\mathsf{T}$ is minimal (meaning that the orbit of each point is dense on $\partial\mathsf{T}$).

Definition 2.2. Let $G \curvearrowright \mathsf{T}$ be an action of the group G on the rooted tree T. We define for every vertex $v \in \mathsf{T}$:

1. the *vertex stabilizer* as the subgroup $G_v := \{g \in G \mid g(v) = v\}$ of elements fixing v, and

2. the *rigid stabilizer* as $G[v] := \{g \in G \mid g(u) = u \text{ for all } u \notin \mathsf{T}_v\}$, that is, the subgroup formed by automorphisms of G acting trivially on the vertices outside of the subtree T_v.

We define for every $n \in \mathbb{N}$:

3. the *nth level stabilizer* as the subgroup $\mathrm{St}_G(n) := \bigcap_{v \in \mathsf{T}^n} G_v$, and

4. the *nth level rigid stabilizer* as $\mathrm{RiSt}_G(n) := \langle G[v] \mid v \in \mathsf{T}^n \rangle$, the subgroup generated by all rigid stabilizers of vertices of level n.

Proposition 2.3. *Let G be a level-transitive automorphism subgroup of the rooted tree $\mathsf{T}_{\overline{m}}$, with branching sequence $\overline{m} = \{m_n\}_{n=1}^{\infty}$. Then*

1. *For each $v \in \mathsf{T}^n$, the vertex stabilizer G_v has index $m_1 \cdots m_n$ in G.*

2. *For every $v \in \mathsf{T}$ and $g \in G$ we have $g \cdot G_v \cdot g^{-1} = G_{g(v)}$ and $g \cdot G[v] \cdot g^{-1} = G[g(v)]$.*

3. *For every $n \geq 1$, the stabilizer $\mathrm{St}_G(n)$ is a normal subgroup of finite index of G, and $\bigcap_{n \in \mathbb{N}} \mathrm{St}(n) = \{1\}$.*

4. *For any $n \geq 1$ the level rigid stabilizer $\mathrm{RiSt}_G(n)$ is a normal subgroup, equal to the direct product $\prod_{v \in \mathsf{T}^n} G[v]$.*

Consider two vertices $u, v \in \mathsf{T}$. We say that v is a prefix of u if the geodesic connecting u to the root of T passes through v.

5. *If v is a prefix of u, then $G_u \leqslant G_v$ and $G[u] \leqslant G[v]$.*

6. *If v is not a prefix of u and u is not a prefix of v, then $G[v] \cap G[u] = [G[v], G[u]] = \{1\}$.*

Proof.

1. As G is level transitive, for each $w \in \mathsf{T}^n$ there exists $g_w \in G$ such that $g_w(v) = w$. Hence the family $\{g_w G_v\}_{w \in \mathsf{T}^n}$ is a disjoint collection of left cosets of G_v. Moreover, their union is G since for any $g \in G$, there exists $w \in \mathsf{T}^n$ such that $g(w) = v$, and hence $g_w^{-1} g \in G_v$. To finish, it suffices to notice that $|\mathsf{T}^n| = m_1 \cdots m_n$.

2. This follows directly from the definitions of the stabilizer and the rigid stabilizer of a vertex.

3. Consider $n \geq 1$. Then thanks to 2. we have for any $g \in G$ that $g\mathrm{St}_G(n)g^{-1} = \bigcap_{v \in \mathsf{T}^n} g G_v g^{-1} = \bigcap_{v \in \mathsf{T}^n} G_{g(v)} = \mathrm{St}_G(n)$, since G is level-transitive. Hence $\mathrm{St}_G(n)$ is a normal subgroup. Moreover, it is of finite index thanks to the fact that for any subgroups $H, K \leqslant G$ we have $[G : H \cap K] \leq [G : H][G : K]$. The last part of the proposition follows from the fact that any non-trivial automorphism of T must act non-trivially in at least one level.

4. The fact that $\mathrm{RiSt}_G(n)$ is a normal subgroup is analogous to the previous item. For the second statement, note that the family $\{G[v]\}_{v \in \mathsf{T}^n}$ is a collection of subgroups of $\mathrm{RiSt}_G(n)$, which intersect pairwise at the identity and whose product forms G. Hence G is the (inner) direct product of these subgroups.

5. Let $u, v \in \mathsf{T}$. If v is a prefix of u and $g \in G$ fixes u, then it fixes any subword of u and in particular v. Hence $G_u \leqslant G_v$. An analogous argument works for the rigid stabilizer.

6. Let $u, v \in \mathsf{T}$. Suppose u is not a prefix of v and v is not a prefix of u, and let n be the lowest level between the levels of u and v. Without loss of generality, say that v is of level n. Then any automorphism $g \in G[v]$ must be the identity outside of the subtree rooted at v, and as v is not a prefix of u we conclude that g is the identity on u. Hence $G[v] \cap G[u] = \emptyset$.

   Consider $g \in G[v]$, $h \in G[u]$. Then g acts as the identity outside T_v and h acts as the identity outside T_u. Hence to prove that $g^{-1}h^{-1}gh = 1$, it suffices to show that it acts as the identity on T_v and T_u. In effect, for any $x \in \mathsf{T}_v$ we have

$$g^{-1}h^{-1}gh(x) = g^{-1}h^{-1}g(x) = g^{-1}(g(x)) = x,$$

   since u is not a prefix of v and hence h acts as the identity on T_v. A similar calculation holds for $x \in \mathsf{T}_u$.

□

It follows that any level-transitive subgroup $G \leqslant \mathrm{Aut}(\mathsf{T})$ satisfies the following dichotomy: either all but a finite number of level rigid stabilizers $\mathrm{RiSt}_G(n)$ are trivial, or all rigid stabilizers $G[v]$ and $\mathrm{RiSt}_G(n)$ are infinite.

Definition 2.4. Let G act on a spherically homogeneous tree T. We say that the action $G \curvearrowright \mathsf{T}$ is

1. of *branch type* if it is level-transitive and for every $n \geq 1$ the nth level rigid stabilizer $\mathrm{RiSt}_G(n)$ has finite index in G,

2. of *weakly branch type* if it is level-transitive and for every $n \geq 1$ the nth level rigid stabilizer $\mathrm{RiSt}_G(n)$ is infinite (or equivalently, if for any vertex $v \in \mathsf{T}$, the rigid stabilizer $G[v]$ is non-trivial),

3. of *weakly branch non-branch type* if, starting from a certain level, the rigid stabilizers are trivial,

4. of *non-branch type* if the rigid stabilizers of all levels are trivial, and

5. locally trivial if there exists a vertex $v \in \mathsf{T}$ such that the stabilizer G_v acts trivially on the subtree T_v with root v. Otherwise, we say that the action is locally non-trivial.

Definition 2.5. We say that a group G is a *(weakly) branch group* if it has a faithful action on a tree T of (weakly) branch type.

Note that the automorphisms group $\mathrm{Aut}(\mathsf{T})$ is always a branch group, since the rigid stabilizer of any vertex $v \in \mathsf{T}$ coincides with $\mathrm{Aut}(\mathsf{T}_v)$. However, this group is not finitely generated. We will give many examples of finitely generated branch groups as examples of self-similar groups in Section 2.5.

Proposition 2.6. *Let G be a group acting on the tree T in a weakly branch way. Then for any two different points $\zeta, \eta \in \partial\mathsf{T}$, the stabilizers G_ζ and G_η are different.*

Proof. We start by proving a lemma.

Lemma 2.7. *Let G be a group acting on T in a weakly branch way. Consider a vertex $v \in \mathsf{T}$ and a ray $\xi \in \partial\mathsf{T}$ passing through v. Then the orbit of ξ under the action of the group $G[v]$ is infinite.*

Proof. As the rigid stabilizer $G[v]$ is non-trivial, there exists $g \in G_{[}v]$ which acts non trivially on some vertex $u \in \mathsf{T}_v$. Let w be a vertex of the ray ξ of the same level as u. As G is level-transitive, there must exist $h \in G_v$ such that $h(w) = u$. Thus $h^{-1}gh(w) \neq w$ and the orbit of ξ has at least two points. By repeating the argument, we can find more points by studying the action on even lower levels of the tree, so by induction the orbit is infinite. □

Now we can finish the proof. As the rays ξ and η are different, we can find $u \in \mathsf{T}$ belonging to ξ and not to η. Then $G[u] \leqslant G_\eta$. Thanks to Lemma 2.7 the $G[u]$-orbit of the point η is infinite, while as u is not in ξ we have that the $G[u]$-orbit of ξ is a single point. We conclude that $G_\xi \neq G_\eta$. □

Recall the definition of a residually finite group from Definition 1.1. Thanks to 3. in Proposition 2.3 any group G acting with a level-transitive action on a rooted tree is residually finite, since the intersection of all normal subgroups of finite index of G must be trivial. We finish this section by showing the converse: any residually finite group acts via a level-transitive action on a rooted tree.

Proposition 2.8. *Every finitely generated residually finite group has a faithfully spherically transitive action of nonbranch type on a spherically homogeneous rooted tree. More precisely, we have the following assertions.*

1. *Let $\{H_n\}_{n\in\mathbb{N}}$, $H_1 = G$, be a decreasing sequence of finite index subgroups of a group G, such that the only normal subgroup of $\bigcap_{n\in\mathbb{N}} H_n$ is trivial. Let $m_n := [H_n : H_{n+1}]$ be the sequence of indices of the subgroups. Then there exists a rooted tree $T_{\overline{m}}$ with branch index $\overline{m} = \{m_n\}_{n\in\mathbb{N}}$ and a canonically defined faithful action of the group G on it.*

2. *If $\{H_n\}_{n\in\mathbb{N}}$ is a sequence of normal subgroups, then the construction of the action of the previous item leads to a nonbranch action. Moreover, the stabilizer of any vertex in this case coincides with the stabilizer of the level to which the vertex belongs.*

Proof.
1. Consider a rooted tree T whose vertices of level n are the left cosets with respect to the n-th subgroup H_n, and such that two vertices of consecutive levels gH_n and fH_{n+1} are connected by an edge if and only if $fH_{n+1} \subseteq gH_n$. The left-regular action by multiplication of G on its left cosets induces an action of G on T, which is faithful since the intersection $\bigcap_{n\in\mathbb{N}} H_n$ is trivial, and which is level-transitive. By definition, the subgroup H_n is the stabilizer of the vertex H_n and $\bigcap_{g\in G} g^{-1}H_n g$ is the nth level stabilizer.

2. Since all subgroups of the sequence of 1. are of finite index, we can replace them by the intersection of their conjugates by all elements of G, so that the sequence from 1. is composed of normal subgroups, and so H_n is the stabilizer of level n. This action is of nonbranch type. In effect, since the action is transitive on the levels, it suffices to show that for any $n \geq 1$ the rigid stabilizers of some vertex of level n is trivial.

   Consider $v_n = H_n \in T^n$, and let $g \in G$, $g \notin H_n$. Suppose the rigid stabilizer of the vertex $u = gH_n$ is non-trivial. Consider $f \in G[u]$ non-trivial and $h \in G$ such that $hH_n \neq gH_n$. Then for any $k \geq n$ we have that the vertex hH_k is fixed by f, hence $f \in \mathrm{St}_G(k)$ and hence the intersection of the stabilizers is trivial, which leads to a contradiction.

□

2.3 Automata

In this section we give the basic definitions of automata, which will prove itself to be a useful language to define automorphisms of rooted trees and be able to do computations with them.

Let X be a finite set and consider the associated rooted tree X^* whose vertices are the words on the alphabet X, where the root is the empty word.

Let $g \in \mathrm{Aut}(\mathsf{X}^*)$ be any automorphism of X^*, let $v \in \mathsf{X}^*$ be a vertex, and consider the subtrees $v\mathsf{X}^*$ and $g(v)\mathsf{X}^*$. Then by definition the map $g : v\mathsf{X}^* \to g(v)\mathsf{X}^*$ is again a morphism of rooted trees.

Both of these subtrees are naturally isomorphic to X^*, by deleting from each vertex the prefix associated to the root. Thus we can identify $v\mathsf{X}^*$ with $g(v)\mathsf{X}^*$ and get an automorphism $g|_v : \mathsf{X}^* \to \mathsf{X}^*$, which is uniquely determined by the condition

$$g(vw) = g(v)g|_v(w). \tag{2.1}$$

We call $g|_v$ the *restriction* of g to v. It is immediate to check that it satisfies the following properties.

$$g|_{vw} = g|_v g|_w,\ \forall g \in \mathrm{Aut}(\mathsf{X}^*), \forall v, w \in \mathsf{X}^*,\ \text{and} \tag{2.2}$$

$$(g_1 \cdot g_2)|_v = g_1|_{g_2(v)} \cdot g_2|_v,\ \forall g_1, g_2 \in \mathrm{Aut}(\mathsf{X}^*), \forall v \in \mathsf{X}^*. \tag{2.3}$$

Now we proceed to define the notion of automaton. Intuitively, the automaton represents an object which can be in certain specific states, and which can read from a tape a letter, express an output letter and change its state.

Definition 2.9. An *automaton* A over the alphabet X is given by

1. a set of states, usually denoted by A, and
2. a map $\tau : \mathsf{A} \times \mathsf{X} \to \mathsf{X} \times \mathsf{A}$.

If $\tau(q, x) = (y, p)$, then $y \in \mathsf{X}$ and $p \in \mathsf{A}$ as functions of $(q, x) \in \mathsf{A} \times \mathsf{X}$ are called *output* and *transition functions*, respectively.

We will use the notation A for the automaton as well as for the set of its states. We will refer to the automaton as the tuple (A, X) if it is relevant to explicit the alphabet X. If the set A is finite, we say that the automaton is finite.

According to the intuition prior to the definition, the automaton in state q reading the letter x gives as output letter y, and moves to state p.

Given an automorphism g of the rooted tree X^* denote by $Q(g) = \{g|_v \mid v \in \mathsf{X}^*\}$ the set of its restrictions. Then $Q(g)$ is the set of states for an automaton which being in a state $g|_v$ and reading as input a letter $x \in \mathsf{X}$, gives as output the letter $g|_v(x)$ and goes to state $g|_v|_x = g|_{vx}$. This motivates the following notation.

Given any automaton A, if $\tau(q, x) = (y, p)$, we will write

$$q \cdot x = y \cdot p, \tag{2.4}$$

and

$$y = q(x),\ p = q|x.$$

Let $g \in \mathrm{Aut}(\mathsf{X}^*)$ be any automorphism, and denote by $Q(g) = \{g|_v \mid v \in \mathsf{X}^*\}$ the set of its restrictions to the vertices of X^*.

We will commonly define automata via their *Moore diagrams*: a directed labeled graph whose vertices are identified with the states of the automatons, and such that if $q \cdot x = y \cdot p$, then we have an edge starting at q, ending in p and labeled by (x, y). In Figure 2.2 we see an example of a Moore diagram for an automaton.

The above discussion shows that every automorphism of X^* is defined by a finite automaton $(\mathsf{A}, \mathsf{X}^*)$, for some appropriate set of states A and map τ.

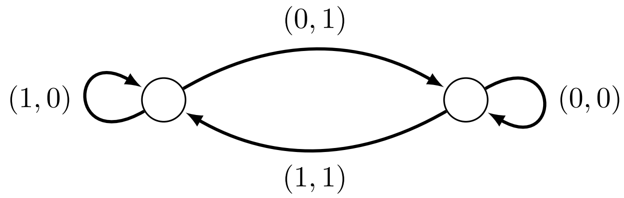


Figure 2.2 – Example of an automaton

2.4 Permutational wreath products

Let X be a finite set of cardinality $d \geq 2$, and consider the set of words X^*, with its natural identification as a rooted tree.

In what follows we introduce the notion of a permutational wreath product, which allows us to express more clearly the automorphisms group of X^*.

Definition 2.10. Let H be a group acting by permutations on a set X, and let G be an arbitrary group. The *permutational wreath product* $H \wr_{\mathsf{X}} G$ is the semi-direct product $H \ltimes \oplus_{\mathsf{X}} G$, where H acts on the direct sum by the respective permutations of the direct factors.

Every element of a permutational wreath product $H \wr_{\mathsf{X}} G$ can be written in the form $h \cdot g$, where $h \in H$ and $g \in \oplus_X G$. If we fix some indexing of $\{x_1, \ldots, x_d\}$ of the set X, then g can be written as $(g_1, \ldots, g_d)$, for $g_i \in G$ the coordinate of g corresponding to x_i. Under this notation, the group operation on the group is given by

$$\alpha(g_1, \ldots, g_d) \cdot \beta(f_1, \ldots, f_d) = \alpha\beta(g_{\beta(1)} f_1, \ldots, g_{\beta(d)} f_d), \tag{2.5}$$

where $g_i, f_i \in G$, $\alpha, \beta \in H$ and $\beta(i)$ is the image of i under the action of α, that is, the index such that $\beta(x\beta_i) = x_{\beta(i)}$.

Proposition 2.11. *Let X be a finite set of cardinality d, with a fix indexing $\{x_1, \ldots, x_d\}$, and consider the associated tree X^* and the group $\mathrm{Sym}(\mathsf{X})$ of permutations of X. Then we have an isomorphism*

$$\psi : \mathrm{Aut}(\mathsf{X}^*) \to \mathrm{Sym}(X) \wr_{\mathsf{X}} \mathrm{Aut}(\mathsf{X}^*),$$

given by

$$\psi(g) = \alpha(g|_{x_1}, \ldots, g|_{x_d}),$$

where $\alpha \in \mathrm{Sym}(\mathsf{X})$ is the permutation equal to the action of g on X, identified with the first level of X^*.

Proof. It is immediate that ψ is a bijection, hence it is sufficient to check that it is an homomorphism. But this follows from Equation (2.5). In effect, for $g, h \in \mathrm{Aut}(\mathsf{X}^*)$ we have that

$$\begin{aligned} \psi(g)\psi(h) &= \alpha(g|_{x_1}, \ldots, g|_{x_d})\beta|(h|_{x_1}, \ldots, h|_{x_d}) \\ &= \alpha\beta(g|_{h(x_1)} h|_{x_1}, \ldots g|_{h(x_d)} h|_{x_d}) \\ &= \alpha\beta((gh)|_{x_1}, \ldots (gh)|_{x_d}) \\ &= \psi(gh). \end{aligned}$$

□

We will identify $g \in \mathrm{Aut}(\mathsf{X}^*)$ with its image $\psi(g) \in \mathrm{Sym}(\mathsf{X}) \wr_{\mathsf{X}} \mathrm{Aut}(\mathsf{X}^*)$, so that we write

$$g = \sigma_g(g|_{x_1}, \dots g|_{x_d}),$$

where $\sigma_g \in \mathrm{Sym}(\mathsf{X})$ is the permutation defined by g on the first level X of the tree X^*.

The subgroup $(\mathrm{Aut}(\mathsf{X}^*))^{\mathsf{X}}\mathrm{Sym}(\mathsf{X}) \wr_{\mathsf{X}} \mathrm{Aut}(\mathsf{X}^*)$ is the first level stabilizer $\mathrm{St}(1)$. It acts on the tree X^* as

$$(g_1, \dots, g_d)(x_i w) = x_i g_i(w), \tag{2.6}$$

that is, the ith coordinate of $(g_1, \dots, g_d)$ acts on the ith subtree $x_i\mathsf{X}^*$.

On the other hand, the subgroup $\mathrm{Sym}(\mathsf{X}) \leqslant \mathrm{Sym}(\mathsf{X}) \wr_{\mathsf{X}} \mathrm{Aut}(\mathsf{X}^*)$ is identified with the group of *rooted automorphisms* $\alpha = \alpha(1, \dots, 1)$, acting as

$$\alpha(xw) = \alpha(x)w.$$

The relations given by equation (2.6) is called *wreath recursions* and they are a compact way of defining recursively an automorphism of the rooted tree X^*. For example, consider the case $\mathsf{X} = \{0, 1\}$, so that X^* is a binary rooted tree. Denote by $\sigma \in \mathrm{Sym}(\{0,1\})$ the transposition $(0, 1)$. Then the element

$$a = \sigma(1, a)$$

defines an automorphism of the binary rooted tree X^* which permutes the vertices of level 1, acts as the identity on the subtree rooted at 0, and as a on the subtree rooted at 1.

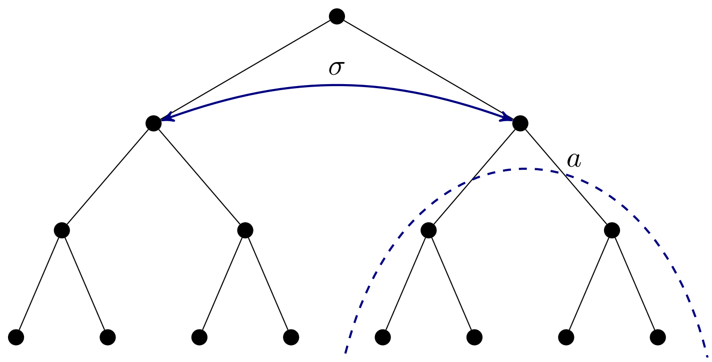


Figure 2.3 – The wreath recursion $a = \sigma(1, a)$ acting on the binary tree.

2.5 Self-similar actions

Definition 2.12. A faithful action of a group G on X^* (resp. on X^ω), is said to be *self-similar* if for every $g \in G$ and every $x \in \mathsf{X}$, there exists $h \in G$ and $y \in \mathsf{X}$ such that for every $w \in \mathsf{X}^*$ (resp. on X^ω) we have

$$g(xw) = yh(w). \tag{2.7}$$

We will denote self-similar actions as pairs (G, X), where G is the group acting on the rooted tree X^* (or its boundary X^ω), of alphabet X.

Given a self-similar action (G, X) and $g, h \in G$, $x, y \in \mathsf{X}$ as in Equation (2.7), the elements h and y are uniquely determined by the elements g and x, since the action is faithful. Thus, we have a map $\tau(g, x) = (y, h)$, defining an automaton with set of states G such that $y = g(x)$ and

$h = g|_x$. This automaton is called the *complete automaton of the self-similar action*. We now give an equivalent definition to Definition 2.12, based on this automata nature of the action.

Definition 2.13. A faithful action of a group G on X^* is *self-similar* if there exists an automaton (G, X) such that the action of $g \in G$ on X^* coincides with the action of the state g of the automaton.

For any faithful action of G on X^*, G is isomorphic to a subgroup of $\mathrm{Aut}(\mathsf{X}^*)$. We will still denote by G this subgroup. In this context we will talk about *self-similar subgroups* of $\mathrm{Aut}(\mathsf{X}^*)$, or *self-similar automorphism groups* of the tree X^*. We give another alternative definition to Definition 2.12, in the above terms.

Definition 2.14. An automorphism subgroup $G \leqslant \mathrm{Aut}(\mathsf{X}^*)$ is said to be *self-similar* (or *state-closed*) if for every $g \in G$ and $v \in \mathsf{X}^*$ we have $g|_v \in G$.
We say that a self-similar action of a group G is *finite-state* if every element $g \in G$ is finite-state as an automorphism of X^*, that is, if the set $\{g|_v \mid v \in \mathsf{X}^*\}$ is finite.

Self-similar groups, according to Definition 2.14, can be understood in terms of permutational wreath products.

Definition 2.15. An automorphism group $G \leqslant \mathrm{Aut}(\mathsf{X}^*)$ is said to be *self-similar* if

$$G \leqslant \mathrm{Sym}(\mathsf{X}) \wr_{\mathsf{X}} G.$$

Recall that $G \leqslant \mathrm{Sym}(\mathsf{X}) \wr_{\mathsf{X}} G$ is an abuse of notation for saying that $\psi(G) \leqslant \mathrm{Sym}(\mathsf{X}) \wr_{\mathsf{X}} G$, where $\psi : \mathrm{Aut}(\mathsf{X}^*) \to \mathrm{Sym}(\mathsf{X}) \wr_{\mathsf{X}} \mathrm{Aut}(\mathsf{X}^*)$ is the wreath recursion from Proposition 2.11. This wreath recursion restricted to G actually determines the action of G on X^*.

Proposition 2.16. *Let G be a group and suppose that we have a homomorphism $\psi : G \to \mathrm{Sym}(\mathsf{X}) \wr_{\mathsf{X}} G$. Let $(\mathsf{A}_\psi, \mathsf{X})$ be the automaton whose output and transition functions are given by*

$$\psi(g) = \alpha(g|_{x_1}, \ldots, g|_{x_d}),$$

where $\alpha \in \mathrm{Sym}(\mathsf{X})$ is such that $g(x) = \alpha(x)$, for all $x \in \mathsf{X}$. Then the transformations of X^* defined by the states of A_ψ give an action of the group G on X^*.

Suppose that G is a finitely generated group, and fix a finite generating set $\{g_1, \ldots, g_n\}$. Then the wreath recursion $\psi : G \to \mathrm{Sym}(\mathsf{X}) \wr_{\mathsf{X}} G$ is uniquely determined by its action on the generating set. This means that ψ is completely characterized by a system of equations of the form

$$\begin{cases} \psi(g_1) & = \tau_1(h_{11}, h_{12}, \ldots, h_{1d}) \\ \psi(g_2) & = \tau_2(h_{21}, h_{22}, \ldots, h_{2d}) \\ \quad\vdots & \qquad\quad \vdots \\ \psi(g_n) & = \tau_n(h_{n1}, h_{n2}, \ldots, h_{nd}), \end{cases} \tag{2.8}$$

where $\tau_i \in \mathrm{Sym}(\mathsf{X})$ and $h_{ij} \in G$, for each $i, j = 1, \ldots, n$.

In particular, this implies that the set of all finitely-generated self-similar groups acting on X^* is countable.

If we have in Equation (2.8) that all elements h_{ij} belong to $\{g_1, \ldots, g_n\}$, then the generating set $\{g_1, \ldots, g_n\}$ is a finite sub-automaton A of the complete automaton of the action, and we say that the group G is *generated* by the automaton A. Below we give examples of groups acting on regular (binary) trees.

Example 2.17. Consider the finite set $\mathsf{X} = \{0, 1\}$, and the action on the binary tree X^* given by

$$a(0w) = 1w \qquad\qquad a(1w) = 0a(w),$$

for $w \in \mathsf{X}^*$. The corresponding action on the binary tree is represented in Figure 2.4, and an automaton defining this group is represented in Figure 2.5.

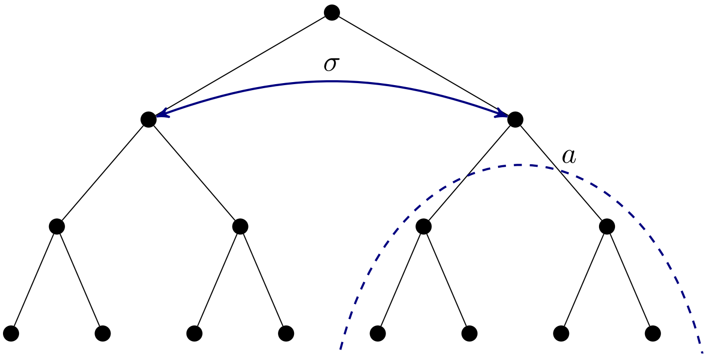


Figure 2.4 – Action of the odometer on the binary tree.

Figure 2.5 – Automaton representing the odometer

The group described by this action is isomorphic to $\mathbb{Z}$, and is commonly called the *dyadic odometer* or *adding machine*, because under the natural identification of infinite dyadic sequences with the corresponding 2-adic numbers, the action of a corresponds to the operation of addition by 1. In effect, it is straightforward to prove by induction that

$$a^n(x_1x_2\dots x_m) = y_1y_2\dots y_m$$

if and only if

$$y_1 + y_2 \cdot 2 + y_3 \cdot 2^2 + \cdots + y_m \cdot 2^{m-1} = \\ (x_1 + x_2 \cdot 2 + x_3 \cdot 2^2 + \cdots + x_m \cdot 2^{m-1}) + n \pmod{2^m}.$$

Identify the boundary of the binary tree with the set of infinite sequences on the alphabet $\{0, 1\}$, and now identify this set with the dyadic integers via the bijection

$$x_1x_2x_3\dots \mapsto \sum_{k\geq 1} x_k \cdot 2^{k-1}.$$

Thus the above calculation shows that the action of a on the dyadic integers is the addition by 1.

Example 2.18. For the alphabet $\mathsf{X} = \{0, 1\}$, consider the automorphisms of X^* described by

$$a(0w) = 1a(w) \qquad a(1w) = 0a(w)$$
$$b(0w) = 0a(w), \qquad b(1w) = 1b(w),$$

for $w \in \mathsf{X}^*$.

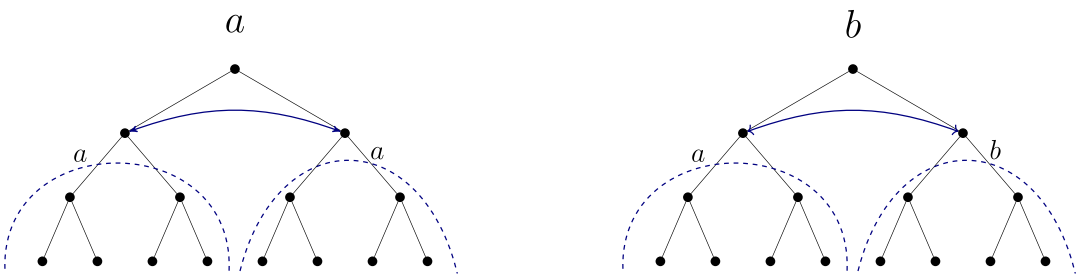


Figure 2.6 – Action of the generators of lamplighter group on the binary tree.

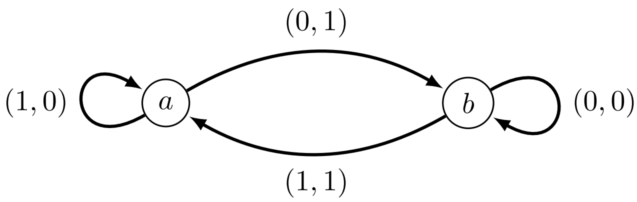


Figure 2.7 – Automaton representing the lamplighter group.

The group generated by a and b is called the *lamplighter group* L_2. The action of L_2 on the binary tree is represented in Figure 2.6, and its representation via an automaton in Figure 2.7.

It is possible to prove that $L_2 = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$, that is, $L_2 = \bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on the direct sum by translation of the coordinates. In particular, L_2 is a metabelian group of exponential growth, and although finitely generated it is not finitely presented.

Example 2.19. Consider the alphabet $\mathsf{X} = \{0, 1\}$. *Grigorchuk's group* $\mathcal{G}$ is generated by four automorphisms $a, b, c, d \in \mathsf{X}^*$, defined recursively by

$$\begin{aligned}
a(0w) &= 1w & a(1w) &= 0w \\
b(0w) &= 0a(w) & b(1w) &= 1c(w) \\
c(0w) &= 0a(w) & c(1w) &= 1d(w) \\
d(0w) &= 0w & d(1w) &= 1b(w).
\end{aligned}$$

Its action on the binary tree is represented in Figure 2.8, and an automaton representing $\mathcal{G}$ is given in Figure 2.9. We will talk about more properties of this group in Section 2.6.

2.6 The Grigorchuk group

This group was first defined by R. Grigorchuk in [Gri80] as an example of an infinitely generated 2-group, that is, a group of Burnside type in which the order of every element is a power of 2. Later, it was shown to be the first example of a finitely generated group of intermediate growth, as is explained at the end of Section 1.5.

We finish this chapter by proving the fact that Grigorchuk's group $\mathcal{G}$ is in fact a torsion group.

Theorem 2.20. *Grigorchuk's group $\mathcal{G}$ is an infinite 2-group.*

Proof. We will do the proof in three steps.

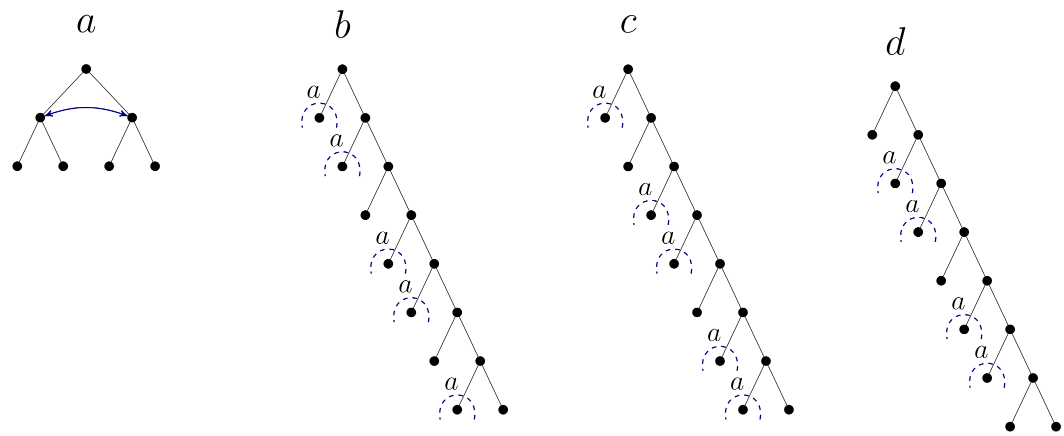


Figure 2.8 – Action of Grigorchuk's group on the binary tree.

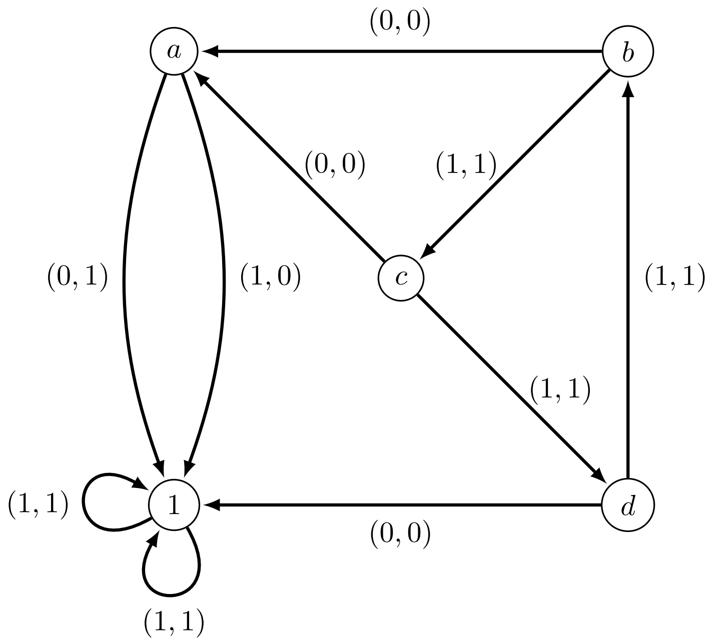


Figure 2.9 – Automaton representing the Grigorchuk group.

First step. Note that $a^2 = 1$, and

$$b^2 = (a^2, c^2) = (1, c^2),\ c^2 = (a^2, d^2) = (1, d^2),\ d^2 = (1, b^2) = (1, b^2).$$

The above implies that $\{1, b^2, c^2, d^2\}$ is an automaton in which every state acts trivially on X, and hence by induction acts trivially on the whole tree X^*. This is the same as saying that $b^2 = c^2 = d^2 = a^2 = 1$.

On the other hand, we have the relations

$$\begin{aligned} bc &= (a, c)(a, d) = (1, cd) \\ cd &= (a, d)(1, b) = (a, db) \\ db &= (1, b)(a, c) = (a, bc). \end{aligned}$$

Hence the elements $\{bc, cd, db\}$ satisfy the same recurrence relations as the elements $\{d, b, c\}$. As such relations determine the automorphisms uniquely, we conclude that $bc = d$, $cd = b$ and $db = c$. Since b, c, d are involutions, we also have $cb = d$, $dc = b$ and $bd = c$. This proves that we have an isomorphism $\{1, b, c, d\} \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. Hence the Grigorchuk group is a quotient of the free product

$$(\mathbb{Z}/2\mathbb{Z}) * ((\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})).$$

Using the normal form for writing elements of free products, we see that any element g of the Grigorchuk group $\mathcal{G}$ can be written in the form

$$g = s_0 a s_1 a s_2 a \cdots s_{m-1} a s_m, \text{ for } s_i \in \{b, c, d\} \text{ for } i = 1, \ldots, m-1, \; s_0, s_m \in \{1, b, c, d\}. \tag{2.9}$$

Second step. Let $\mathrm{St}_{\mathcal{G}}(1)$ be the stabilizer of the first level of the tree X^*. By using the wreath recursion decomposition, any element $g \in \mathrm{St}_{\mathcal{G}}(1)$ can be written as $g = (g_0, g_1)$ for $g_0 = g|_{0\mathsf{X}^*}$ and $g_1 = g|_{1\mathsf{X}^*}$. The maps

$$\begin{aligned} \varphi_i : \mathrm{St}_{\mathcal{G}}(1) &\to \mathcal{G} \\ g &\mapsto \varphi_i(g) = g_i, \end{aligned}$$

$i = 0, 1$, are called the *virtual homomorphisms* from $\mathrm{St}_{\mathcal{G}}(1)$ to $\mathcal{G}$. Note that these virtual homomorphisms are surjective. In effect, we have that

$$\begin{aligned} b &= (a, c), \; aba = (c, a) \\ c &= (a, d), \; aca = (d, a) \\ d &= (1, b), \; ada = (b, 1), \end{aligned}$$

and so

$$\begin{aligned} \varphi_0(b) = a, \; \varphi_1(b) = c, \; \varphi_0(aba) = c, \; \varphi_1(aba) = a \\ \varphi_0(c) = a, \; \varphi_1(c) = d, \; \varphi_0(aca) = d, \; \varphi_1(aca) = a \\ \varphi_0(d) = 1, \; \varphi_1(d) = b, \; \varphi_0(ada) = b, \; \varphi_1(ada) = 1. \end{aligned}$$

Hence $\varphi_0(\mathrm{St}_{\mathcal{G}}(1)) = \mathcal{G}$ and $\varphi_1(\mathrm{St}_{\mathcal{G}}(1)) = \mathcal{G}$, hence proving surjectivity. In particular, as we have a map from a proper subgroup of $\mathcal{G}$ onto $\mathcal{G}$, we get that the Grigorchuk group is infinite.

Third step. Now we will prove that for any $g \in \mathcal{G}$, there exists $n \geq 1$ such that $g^{2^n} = 1$.

Write $g = s_0 a s_1 a s_2 a \cdots s_{m-1} a s_m$ as in Equation 2.9. We will do induction on the length of g with respect to the generating set $\{a, b, c, d\}$.

We already proved in Step 1. the case when the length of g is 1. It is also simple to show that the claimed property holds for elements of length 2, since

$$\begin{aligned} (ad)^4 &= (\sigma(1, b)\sigma(1, b))^2 = (b, b)^2 = 1 \\ (ac)^8 &= (\sigma(a, d)\sigma(a, d))^4 = (da, ad)^4 = 1 \\ (ab)^{15} &= (\sigma(a, c)\sigma(a, c))^8 = (ca, ca)^8 = 1, \end{aligned}$$

and noting that the elements da, ac, ba are conjugate to ad, ac, ab, respectively, and hence have the same order.

Suppose now that for any $g \in \mathcal{G}$ of length less than k there is $n \geq 1$ such that $g^{2^n} = 1$. Write $g = s_0 a s_1 a s_2 a \cdots s_{m-1} a s_m$ as the shortest word representing g. If $s_0 = s_m = 1$, then g starts and ends with a, and then aga is a shorter element with the same order as g (since they are conjugate to each other), hence proving the statement for g. If $s_0 \neq 1$ and $s_m \neq 1$, then we can also find an element $w \in \{b, c, d\}$ so that wgw is a shorter word than g, and the same reasoning applies.

Hence the remaining cases to prove are covered (after possibly conjugating g by b, c or d) by supposing that g is of the form

$$g = as_1as_2 \cdots as_{k/2},$$

for $s_i \in \{b, c, d\}$, $i = 1, \ldots, k/2$.

If $k/2$ is even, then we can write

$$g = (as_1a) \cdot s_2 \cdot (as_3a)c \cdots s_{k/2} = (g_0, g_1).$$

But then the equations written in Step 2. imply that $as_ia \in \{b, c, d\} \times \{a, 1\}$ and $s_i \in \{a, 1\} \times \{b, c, d\}$. Therefore, the lengths of g_0 and g_1 are at most $k/2$, and by the induction hypothesis we can find n big enough such that $g_0^{2^n} = g_1^{2^n} = 1$. This of course implies that $g^{2^n} = 1$.

Finally, suppose that $k/2$ is odd. Then we can write

$$g^2 = (as_1a) \cdot s_2 \cdot (as_3a) \cdots (as_{k/2}a) \cdot s_1 \cdot (as_2a) \cdots s_{k/2} = (h_0, h_1).$$

We make the final argument depending on which one of three possible cases we are in.

Case 1. Suppose there exists j such that $s_j = d$. Then we will have $s_j = (1, b)$ and $as_ja = (b, 1)$ in the above product, so that the length of each h_i is at most $k-1$. But then the induction hypothesis gives us the existence of $n \geq 1$ such that $h_0^{2^n} = h_1^{2^n} = 1$ and so $g^{2^{n+1}} = 1$.

Case 2. Suppose there exists j such that $s_j = c$. Then $s_j = (a, d)$ and $as_ja = (d, a)$, so that each h_i either has length less than k or is equal to a word of length k involving d. In the first case we can apply the induction hypothesis, while on the second case we are in Case 1.

Case 3. The remaining case is when g is of the form $g = abab \cdots ab$, which is of order 16.

□

Chapter 3

Schreier graphs

We begin this chapter by introducing Schreier graphs, together with orbital graphs and mentioning their relevance in the proof of Theorem 5.1. In Theorem 3.2 we prove a result of universality of the Schreier graphs of free groups, namely, that this class of families covers all regular connected graphs, up to a choice of orientation and labeling of the edges. Regarding level-transitive tree actions, we prove in Proposition 3.4 that the orbital graphs of the levels of a tree converge to the orbital graph of an element of the boundary of the tree. We finish the chapter by showing some explicit examples in Section 3.4. For a more detailed description of Schreier graphs see the publication by R. Grigorchuk [Gri11], on which this chapter is partially based on.

3.1 Basic definitions

Let G be a finitely generated group. We have seen already in Section 1.2 the concept of a *Cayley graph* with respect to a finite generating set, which captures in a geometrical object the action of the generators on the elements of the group. Now we will define a more general type of graphs associated to a group with respect to a finite generating set, but which now represents the action of the generators on the cosets of some fixed subgroup.

Definition 3.1. Let G be a group together with a finite generating set S, and a subgroup $H \leqslant G$. We define the corresponding *(left) Schreier graph* as the directed labeled graph whose vertices are the left cosets gH, $g \in G$, and such that any $g \in G$ and $s \in S$ define an edge starting from gH and ending in sgH, labeled by s.

When $s \in S$ is an involution, that is, $s^2 = 1$, we will omit the orientation of the edges labeled by s. Thus, if the generating set has i involutions and j non-involutions, the Schreier graph always has regular degree $i + 2j$. Commonly we will see Schreier graphs as rooted graphs, the root being the vertex associated to the coset $1 \cdot H$.

An important subclass of Schreier graphs are *orbital graphs*, which capture the action of the group on a space. Suppose G is a finitely generated group acting on a set X, S a finite generating set for G. For any $x \in X$, we define the orbital graph Γ_x as the Schreier graph of G with respect to the stabilizer subgroup G_x. When X is a topological space, it is possible to make the analogous definition replacing G_x by the subgroup of elements fixing a neighborhood of x. This is called the *graph of germs*. These graphs will be further studied in Chapter 4, on the case when G is a group originating from the fragmentation of a dihedral action. Later, they will play an essential role in the proof of Theorem 5.1 in Chapter 5, where the dynamics of the group on some particular orbital graphs will be used to prove that under some reasonable conditions such groups are periodic.

Another essential concept for the proof of Theorem 5.1 is the convergence of rooted graphs. We now proceed to define a topology on the space of rooted graphs which allows us to formalize that idea. Denote by $\mathcal{X}_d$ the space of connected d-regular rooted graphs. We define a topology on $\mathcal{X}_d$ by choosing as a base of open sets the sets $B_{(\Gamma,o)}(o,n) \coloneqq \{v \in \Gamma \mid d(v,o) \leq n\}$, that is, the set of vertices of Γ such that the distance from these vertices to the root o in Γ is at most n. This topology is metrizable: given two regular rooted graphs (Γ_1, o_1), (Γ_2, o_2) define their distance to be $d((\Gamma_1, o_1), (\Gamma_2, o_2)) = 2^{-n}$, where n is the least positive integer such that $B_{(\Gamma_1,o_1)}(o_1, n)$ is not isomorphic to $B_{(\Gamma_2,o_2)}(o_2, n)$.

Similarly, we will denote by $\mathcal{X}_{2m}^{\mathrm{Cay}}$ the space of $2m$-regular rooted non-oriented Cayley graphs and by $\mathcal{X}_{2m}^{\mathrm{Sch}}$ the space of $2m$-regular rooted non-oriented Schreier graphs of an m-generated group G (whose generators do not contain involutions). Of course we can consider analogous definitions on the case when there are generators of G which are involutions.

3.2 Realization of graphs as Schreier graphs

In this section we present a theorem stating that the Schreier graphs of free groups are universal, in the sense that any $2m$-regular connected non-oriented graph is contained in this family (by giving a proper orientation to its edges).

Theorem 3.2. *Let Γ be a connected non-oriented $2m$-regular graph. Then there exists a subgroup $H \leqslant F_m$ of the free group F_m with basis A, such that the Schreier graph $\Gamma(F_m, H, A)$ with removed labels and orientation is isomorphic to the graph Γ.*

Proof. We start by proving the following Lemma.

Lemma 3.3. *Every finite non-oriented $2m$-regular graph Δ possesses a 2-regular subgraph with the same vertex set as Δ. We call such a subgraph a 2-factor.*

Proof. Let us prove that Δ has a subgraph that is a union of pairwise disjoint cycles. We will prove this by induction on $m \geq 1$, and on the number of vertices n of Δ. When $m = 1$ or $n = 1$ the result is immediate, and if Δ is disconnected we can apply the induction hypothesis on each connected component. Thus, we may suppose that Δ is connected.

As the degree of each vertex is $2m$, it is in particular an even number, and hence there exists an *Eulerian cycle*. That is, a closed path γ in Δ that passes through each edge exactly one time.

Choosing a sense of motion along γ induces an orientation of the edges of Δ, by declaring each edge to be oriented according to the sense in which γ traverses it. The number of edges starting each vertex is the same as the number of edges finishing on each vertex, and is equal to m.

Let us construct a new graph $\widetilde{\Delta}$ via the following procedure. For each vertex $v \in \Delta$ we will consider two vertices v^- and v^+ in $\widetilde{\Delta}$, such that the edges finishing in v turn into edges finishing in v^-, and the edges starting from v turn into edges starting from v^+. Thus we have transformed Δ into a bipartite graph $\widetilde{\Delta}$, whose pieces are $V^- = \{v^- \mid v \in \Delta\}$ and $V^+ = \{v^+ \mid v \in \Delta\}$. Thanks to Hall's theorem, $\widetilde{\Delta}$ has a *perfect matching*: a set of edges such that each vertex of the graph is incident to one and only one edge. We now proceed to prove that such a perfect matching gives rise to a 2-factor of Δ.

For any vertex $u \in \Delta$ choose an edge starting from u which belongs to the perfect matching of $\widetilde{\Delta}$. Denote by u_1 the vertex at which it finishes. Now repeat the same procedure: take an edge starting from u_1 which belongs to the perfect matching, and call its finishing vertex u_2. Proceeding in this way we will eventually complete a cycle, since the graph is finite. Moreover, by the property satisfied by the perfect matching, this cycle is completed for the first time with the initial vertex u. We have thus constructed a cycle c_1 that does not pass mote than once through any vertex. Nonetheless, there could be vertices of Δ which are not covered by c_1. We simply need to repeat the same procedure (with new starting points), and we will finish with a partition of the vertex set of Δ into a pairwise disjoint family of cycles passing precisely once through each of their vertices. This partition defines the 2-factor we were looking for. $\square$

Now we can prove the proposition.

First suppose that Γ is finite. Then we can apply directly Lemma 3.3 to $\Delta_1 \coloneqq \Gamma$ in order to obtain a 2-factor δ_1, which is oriented by our construction. Let $S = \{s_1, \ldots, s_m\}$. Label the edges of δ_1 with s_1, and remove them from the graph Δ_1. We thus obtain a $(2m-2)$-regular graph Δ_2, for which Lemma 3.3 implies the existence of a 2-factor δ_2. We label the edges of δ_2 with s_2, remove them from Δ_2 to obtain the graph Δ_3 and repeat. We finish with a chain of subgraphs Δ_i with respective 2-factors δ_i whose edges are labeled by s_i, for $i = 1, \ldots, m$. The union of the edges of these 2-factors coincides with the set of edges of Γ, and hence we obtain a labeling of the edges of γ with the elements of S. Now note that a graph labeled by the elements of S is a Schreier graph of F_m if and only if exactly for every vertex v and every $s \in S$, exactly one edge labeled by s starts and exactly one edge labeled by s finishes in v. Hence the labeling on the edges of Γ induced by the labels of the 2-factors determines the structure of a Schreier graph of a free group on Γ, as was claimed.

Now we will explain how to proceed when Γ is infinite.

We say that an oriented graph Σ labeled by the elements of a finite set S is a *pre-Schreier graph of degree* $m = |S|$ if, for any $s \in S$ and any vertex $v \in \Sigma$, at most one edge finishes in v and is labeled by s, and at most one edge starts in v and is labeled by s. Such labelings are said to be *admissible*.

Note that every finite graph with maximum degree at most $2m$ can be extended to a finite $2m$-regular graph. In effect, we can connect any two vertices u, v whose degrees are less than $2m$ by an edge, hence increasing their degrees by 1. Repeating this operation until no such vertices exist gives a regular graph of even degree, which may be lower than $2m$. Then we can simply add the necessary number of loops to each vertex in order to attain degree $2m$.

Consider Γ an infinite $2m$-regular graph, choose an arbitrary vertex $o \in \Gamma$ to be the root, and consider the sequence of finite subgraphs $\{\Delta_n\}_{n\geq 1}$, where $\Delta_n = B_\Gamma(o, n)$ is the induced subgraph formed by vertices at distance at most n from the root o. On each one of these subgraphs we introduce an orientation and a labeling of the edges by the elements of S, making the graph into a pre-Schreier graph of degree m. Then we construct an infinite rooted tree T whose vertices are pairs (Δ_n, R), where R is an admissible labeling of the graph Δ_n. The root of T is defined to be the graph Δ_0 consisting of a single vertex, and two pairs (Δ_n, R), (Δ_{n+1}, Q) are connected by and edge if and only if the restriction of Q to Δ_n coincides with R.

Since every finite graph can be labeled by only a finite number of different ways, the tree T is locally finite. Moreover, since each of the graphs Δ_n has at least one appropriate labeling, the tree is infinite. Thus we can find an infinite path in T connecting its root to infinity. Such path determines a labeling on Γ, which is consistent with the labeling on all the subgraphs Δ_n, thus making Γ into a Schreier graph. $\square$

3.3 Substitution rules

In this section we show some examples of Schreier graphs that are constructed by means of self-similar groups. Let G be a group acting on a d-regular rooted tree T, $d \geq 2$, in a level transitive way. Define for each $n \in \mathbb{N}$ the graph $\Gamma_n \coloneqq \Gamma(G, G_u, S)$ as the Schreier graph of G with respect to the stabilizer G_u of any vertex u of level n. The choice of u is not relevant, precisely because we are supposing the action to be level-transitive. By construction, for each $n \geq 1$,the projection map which takes a vertex of level $n+1$ to its predecessor at level n is a covering map from Γ_{n+1} into Γ_n . In addition, we also have the Schreier graphs of the natural extension of the action of G to the boundary $\partial\mathsf{T}$ of the tree. That is, the graphs Γ_ξ, $\xi \in \partial\mathsf{T}$, where $\Gamma_\xi = \Gamma(G, G_\xi, S)$ is the Schreier graph of G with respect to the stabilizer G_ξ, or equivalently, the orbital graph for the action on the orbit of a point ξ. We call Γ_x a *boundary Schreier graph* for the action of G on T.

There is a close relationship between the described sequence $\{\Gamma_n\}_{n\in\mathbb{N}}$ and the boundary Schreier graphs Γ_ξ, $\xi \in \partial\mathsf{T}$. Let $\xi \in \partial\mathsf{T}$ be an infinite ray represented by the sequence $\{u_n\}_{n=1}^{\infty}$ of vertices of T. Denote by $P = G_\xi$ the stabilizer of ξ, and $P_n = G_{u_n}$ the stabilizer of u_n, $n \in \mathbb{N}$. Then $\{P_n\}_{n\in\mathbb{N}}$ is a decreasing sequence, and the following relation holds

$$P = \bigcap_{n\in\mathbb{N}} P_n. \tag{3.1}$$

Then it is possible to show that we actually have convergence of the graphs Γ_n to the respective boundary Schreier graph, for the topology described in Section 3.1.

Proposition 3.4. *We have*

$$(\Gamma(G, P, S), P) = \lim_{n\to\infty} (\Gamma(G, P_n, S), P_n), \tag{3.2}$$

where the convergence is with respect to the topology of the space $\mathcal{X}_{2m}^{\mathrm{Sch}}$.

Proof. Note that in the Schreier graph $\Gamma(G, P, S)$, a ball of radius n and center v is completely determined by such vertex, and by the set of words of length at most $2n$ over the alphabet S that define elements of the subgroup P. Thanks to Equation (3.1), for any $k \geq 1$ there exists $N \geq 1$ such that for any $n \geq N$ the set of words of length at most $2k$ that define elements in P and in P_n coincide. Hence, by the above remark, for any $k \geq 1$ there exists $N \geq 1$ such that for any $n \geq N$ the neighborhoods of radius k centered at the root in $\Gamma(G, P, S)$ and $\Gamma(G, P_n, S)$ are isomorphic. By definition, this means that we have the convergence stated in Equation (3.2). $\square$

When G is an automata group and the sequence $\{\Gamma_n\}_{n\in\mathbb{N}}$ is associated to the set of generators that correspond to the set of states of the finite automaton, it is possible to prove that said sequence satisfies the following recurrence property.

Definition 3.5. Let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of finite Schreier graphs associated with a group G generated by a finite set S, and acting on a regular rooted tree defined by an alphabet $\mathsf{X} = \{x_1, \ldots, x_d\}$.

We say that the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ is *recurrent* if there exists a rule according to which every edge (u, v, s), $u, v \in \mathsf{X}^n$, $v = s(u)$, $s \in S$, of the graph Γ_n and every symbol $x \in \mathsf{X}$ defines an edge (ux, wy, s), for some $y \in \mathsf{X}$ and some $w \in \mathsf{X}^n$, of the graph Γ_{n+1} so that the graph obtained from Γ_n by this substitution applied to all edges is isomorphic to the graph Γ_{n+1}, and such that this is valid for all $n \geq 1$.

Proposition 3.6. *Consider an automata group G and let the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ be the sequence of Schreier graphs on each level of the rooted tree on which G acts, as defined at the beginning of this section. Then the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ is recurrent.*

Proof. The automaton defines self-similarity relations $s(xu) = ys'(u)$, for $x, y \in \mathsf{X}$, $s, s' \in S$, $u \in \mathsf{X}^*$. This shows that if $w = s'(u)$, then $yw = s(xu)$. $\square$

3.4 Examples

We finish this chapter by giving examples of Schreier graphs for some self-similar groups.

Example 3.7. Recall Grigorchuk's group $\mathcal{G}$ defined in Example 2.19. In Figure 3.1 we see the orbital graph of the Grigorchuk group on each level of the binary tree, and in Figure 3.2 we see the same graphs, but with a different order of the vertices. We can also see the orbital graphs of

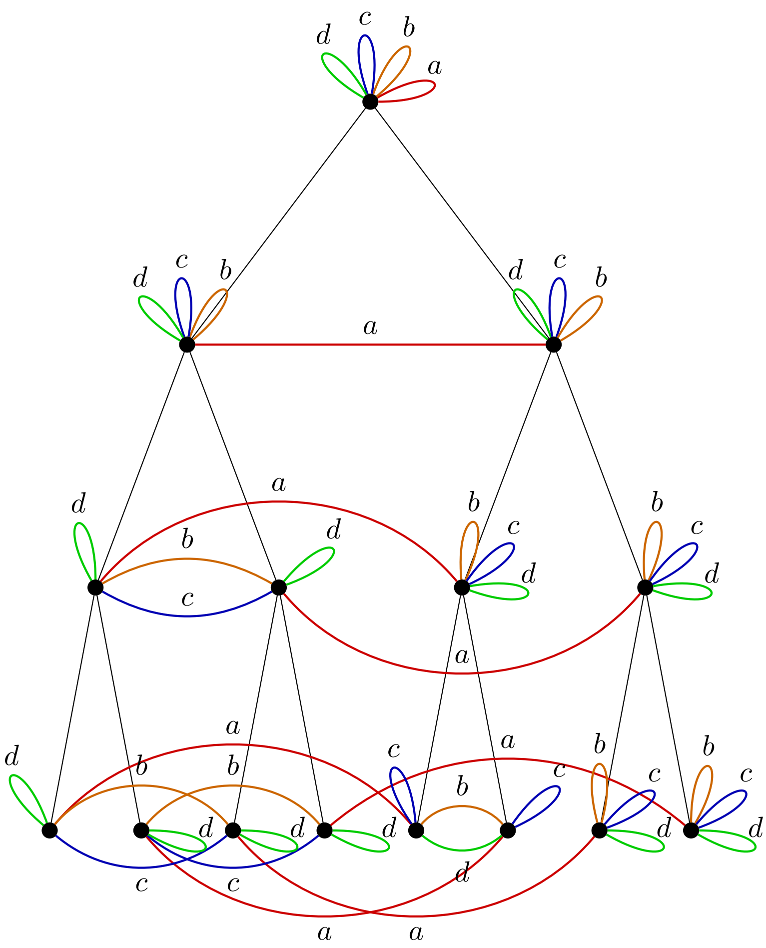


Figure 3.1 – Orbital graphs of the action of the Grigorchuk group on levels $1, 2$ and 3, drawn inside of the binary tree.

the points in the boundary of the binary tree. These are of course limits of the graphs described above, as follows from Proposition 3.4. Note that we have two distinct behaviors. The point 1^∞ is a fixed point for the action of b, c, and d, and hence the corresponding orbital graph only has one end (one direction going to infinity), as is illustrated in Figure 3.3. On the other hand, points which are not fixed points for b, c and d, as for example 0^∞, give rise two orbital graphs with two ends. This situation is shown in Figure 3.4.

The above situation may seem puzzling at first: both graphs arising as limit of the same finite orbital graphs of the levels of the binary tree, one of them is one-ended while the other one is two-ended. The important remark is that they are limits as *rooted* orbital graphs: in the case of the one-ended orbital graph of 1^∞ the root of the finite graphs is always chosen to be one of the extremities, so in the limit this extremity is preserved. On the other hand, the case of two-ended orbital graphs hides the root at the middle, so that the extremities get lost in the limit.

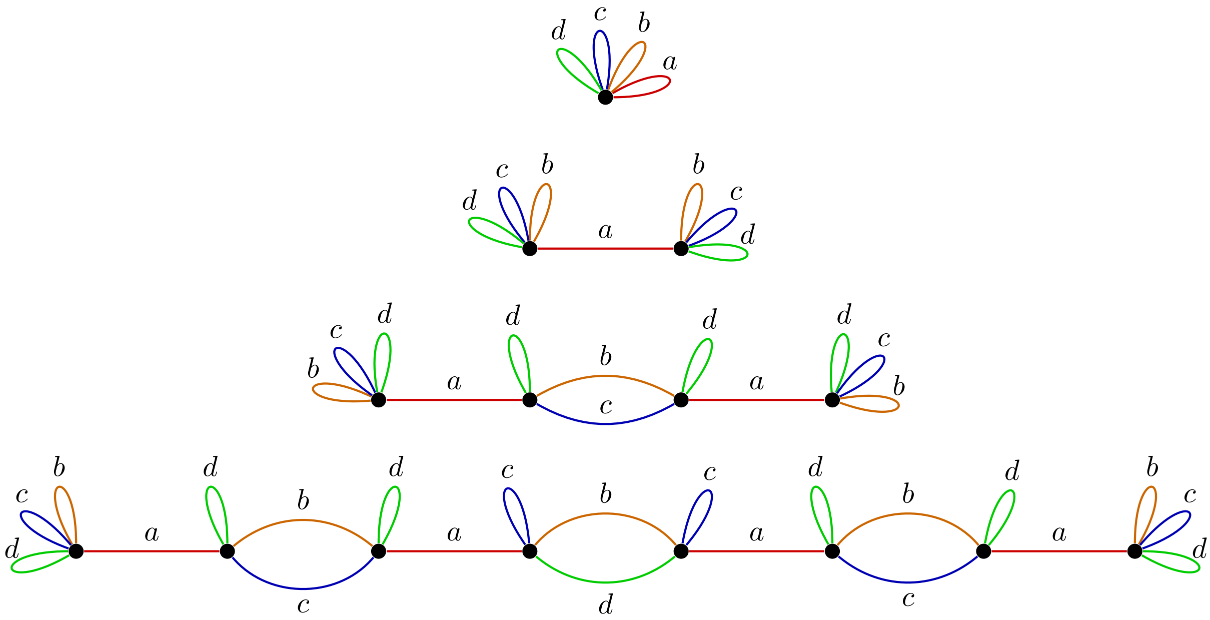


Figure 3.2 – Orbital graphs of the action of the Grigorchuk group on levels $1, 2$ and 3, with the vertices ordered in a simpler looking way.

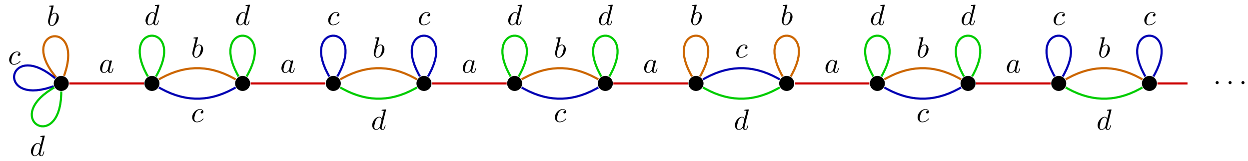


Figure 3.3 – Orbital graph of the point 1^∞ for the Grigorchuk group. Note that 1^∞ is a fixed point of b, so the orbital graph has one end.

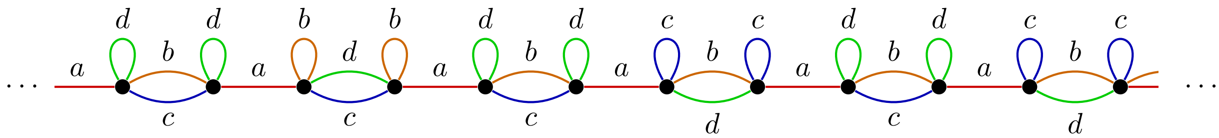


Figure 3.4 – Orbital graph of a non-fixed point for the Grigorchuk group. In this case the orbital graph has two ends.

Example 3.8. Consider the group generated by the automaton represented in Figure 3.7. Its orbital graphs on the levels of the binary tree are represented in Figure 3.6, and in Figure 3.5 where the vertices are in a different order.

Consider Γ to be the infinite Schreier graph associated to the sequence 0^∞. Then it is possible to prove that the set of vertices can be identified with the integers, while the set of edges is $E = \bigcup_{k\geq 0} E_k$, where $E_0 = \{(i, i+1) \mid i \in \mathbb{Z}\}$ and $E_k = \{(2^k(n-1/2), 2^k(n+1/2)) \mid n \in \mathbb{Z}\}$ for $k > 0$. This graph Γ has intermediate growth, and in fact its type of growth has been shown to be $n^{\log_4 n}$ [BH05].

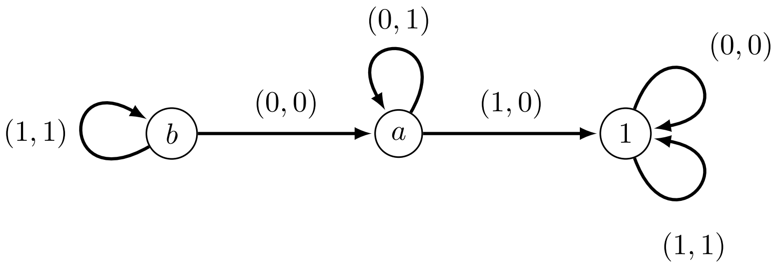


Figure 3.5 – Automaton defining the group G.

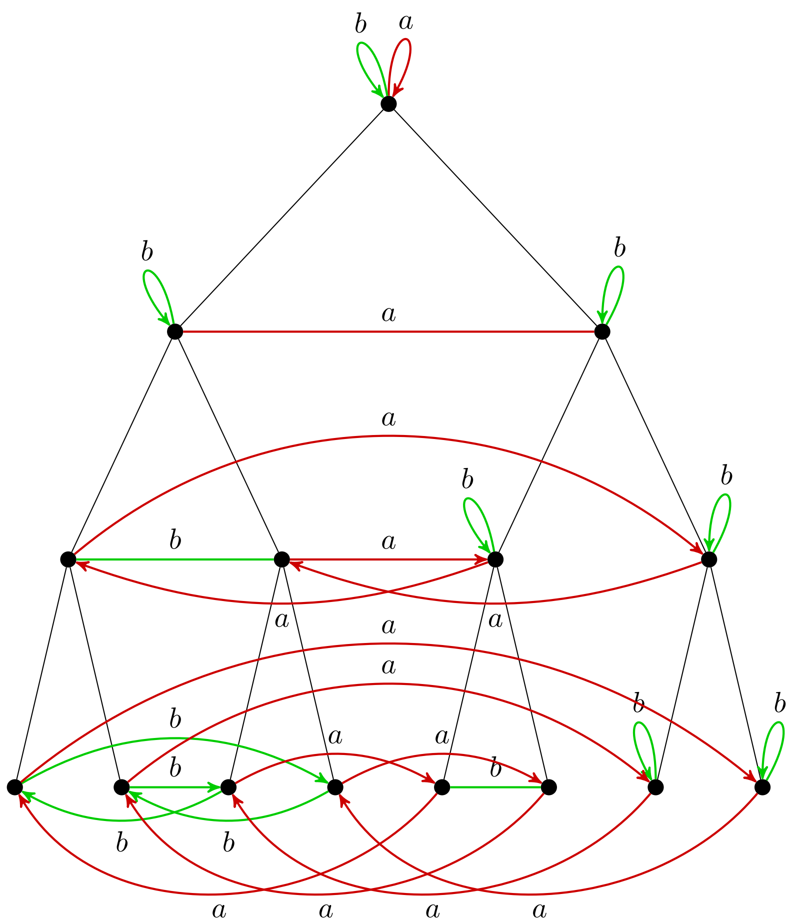


Figure 3.6 – Schreier graphs of the group G for the first three levels of the binary tree.

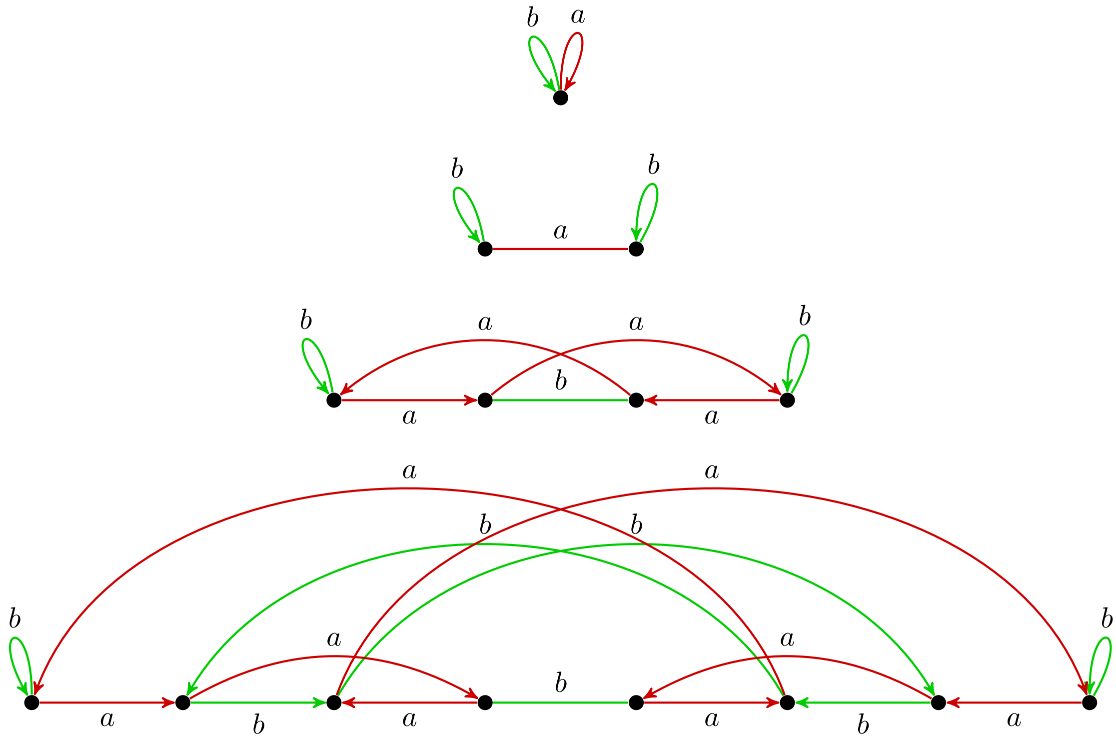


Figure 3.7 – Schreier graphs of the group G for the levels of the tree.

Chapter 4

Dihedral actions

We start this chapter by reviewing basic facts about orbital graphs of groups acting on a topological space. We are mainly concerned with minimal actions, which imply some repetitiveness properties on the associated orbital graphs, and which will be key for the main results of Chapter 5. We then introduce dihedral actions and mention some models for them according to the different possible dynamical behaviors. Finally, we define fragmentations of a dihedral action and discuss how these new groups can be understood starting from the orbital graphs of the original dihedral action.

4.1 Orbital graphs

From now on and unless stated otherwise, all considered graphs will be directed and labeled. We allow loops as well as multiple edges. Given such a graph Γ, a vertex $v \in \Gamma$, and $n \geq 0$, recall that we define the ball $B(v, r)$ as the (induced) subgraph of Γ whose vertex set are vertices from Γ at distance at most n from v.

Throughout the rest of this section we will consider a finitely generated group G, together with a finite generating set S, which acts by homeomorphisms on a compact metrizable space X. We will denote the identity element of G by ε.

Similar to Definition 2.2 of stabilizers and rigid stabilizers for actions on rooted trees, we define the analogous concepts for actions on a topological space.

Definition 4.1. Given a point $\zeta \in X$ we define

1. the *stabilizer* G_ζ as the subgroup of G formed by elements $g \in G$ such that $g(\zeta) = \zeta$, and
2. the *local stabilizer* $G_{(\zeta)}$ as the subgroup of G formed by elements $g \in G$ such that g acts trivially on a neighborhood of ζ.

It is a well-known fact that the G-orbit of an element ζ is in correspondence with the set of left-cosets of the stabilizer G_ζ.

We now recall that given an action of G on a space X, we can define the orbital graph of a point by using the notion of a Schreier graph (see Definition 3.1).

Definition 4.2. Consider $\zeta \in X$. We define the *orbital graph* Γ_ζ of ζ to be the Schreier graph of G modulo G_ζ, and the *graph of germs* $\widetilde{\Gamma}_\zeta$ to be the Schreier graph of G modulo $G_{(\zeta)}$.

Note that the vertices of the orbital graph Γ_ζ are identified with the G-orbit of ζ, while the vertices of the graph of germs $\widetilde{\Gamma}_\zeta$ identify with *germs*: equivalence classes of pairs (g, ζ), where we identify two pairs (g_1, ζ) and (g_2, ζ) if there exists a neighborhood U of ζ such that $g_1|_U = g_2|_U$.

It follows from the definition that $G_{(\zeta))} \leqslant G_\zeta$. Moreover, it is a normal subgroup: consider $h \in G_\zeta$ and $g \in G_{(\zeta)}$. Then g fixes a neighborhood U of ζ, and so the element $h^{-1}gh$ fixes the neighborhood $h^{-1}(U)$; thus proving that $h^{-1}gh \in G_{(\zeta)}$. The next proposition shows the relation between the corresponding orbital graph and the graph of germs, and follows easily from the construction of such graphs and subgroups.

Proposition 4.3. *For any $\zeta \in X$ we have $G_{(\zeta)} \lhd G_\zeta$. The map $\widetilde{\Gamma}_\zeta \to \Gamma_\zeta$ is a Galois covering and the group of Deck transformations is (isomorphic to) $G_\zeta/G_{(\zeta)}$. We call this group the group of germs of ζ.*

Definition 4.4. Consider a point $\zeta \in X$. We say that ζ is

1. *regular* if its group of germs $G_\zeta/G_{(\zeta)}$ is trivial,
2. *singular* if it is not regular, and
3. a *purely non-Hausdorff singularity* if for every $g \in G_\zeta$, the interior set of fixed points of g accumulates on ζ.

Remark 4.5. It is important to note that when G is countable the set of singular points is nowhere dense. In effect, for any $g \in G$ the set $\{\zeta \in X \mid g(\zeta) = \zeta \text{ and } g \notin G_{(\zeta)}\}$ is precisely the boundary of the set of fixed points of g, so it is a closed set with empty interior.

Now we pass to define a notion regarding the dynamics of the action of G on X, namely minimality. Intuitively, a minimal action is one which cannot be decomposed into strictly smaller dynamics. We make the following definition in terms of the orbit of

Definition 4.6. We say that the action of G on X is *minimal* if the G-orbit of every point is dense in X.

Minimal actions are characterized by the fact that the G-orbit of any point ζ comes back close to it really often. More precisely, say that a subset $S \subseteq G$ is *syndetic* if there exists a finite set $K \subseteq G$ such that $G = K \cdot S := \{ks \mid \ k \in K, \ s \in S\}$. Then the action of G on X is minimal if and only if for every point $\zeta \in X$ and any open neighborhood U of ζ, the set of *return times* $T_U(\zeta) := \{g \in G \mid \ g(\zeta) \in U\}$ is syndetic. The idea is that ζ has uniformly bounded gaps between consecutive visits to the neighborhood U.

By taking advantage of this repetitiveness behavior of minimal actions we get the following proposition, which intuitively states that all orbital graphs of the action locally look like the orbital graph of a regular point.

Proposition 4.7. *Suppose the action of G on X is minimal. Let $\zeta \in X$ be a G-regular point. Then for every ball $B(\zeta, r)$ of Γ_ζ there exists $R(r) > 0$, independent of ζ, such that for every $\eta \in X$ there exists a vertex η' of Γ_η such that $d(\eta, \eta') \leq R(r)$, and the rooted balls $B(\zeta, r)$ and $B(\eta', r)$ are isomorphic.*

Proof. Thanks to the definition of the orbital graph Γ_ζ, the elements of the ball $B(\zeta, r)$ are characterized by satisfying a system of equations of the form $g_1(\zeta) = g_2(\zeta)$, or $g_2(\zeta) \neq g_2(\zeta)$, for elements $g_1, g_2 \in G$. Moreover this system is finite since we only need to look between elements $g \in G$ whose word length in terms of the finite generating set S is at most r. Since ζ is a regular point, such system of equations is valid in a neighborhood N of ζ. We conclude that for any $\zeta' \in N$, the balls $B(\zeta', r)$ and $B(\zeta, r)$ are isomorphic.

To finish the proof, we use minimality of the action: fixing an element $\eta \in X$, we know that its orbit is dense in X, so in particular it must intersect N. This proves that we can write $X = \bigcup_{g \in G} g^{-1}(N)$, and in fact thanks to compactness there must exist $g_1, \ldots, g_k \in G$ such that $X = \bigcup_{i=1}^{k} g_i^{-1}(N)$.

Define $\widetilde{R}(r)$ to be the maximum word length of the elements $g_1, \ldots, g_k$ in terms of the finite generating set S. Then for any $\eta \in X$, we can find $1 \leq i \leq k$ such that $\eta' := g_i(\eta) \in N$, and $B(\eta', r)$ is isomorphic to $B(\zeta, r)$. Note that by the definition of $\widetilde{R}(r)$, we indeed have $d(\eta, \eta') \leq \widetilde{R}(r)$.

To finish, it suffices to note that as S is finite, there are finitely many different balls $B(\zeta, r)$ of a fixed radius $r > 0$, when varying ζ. This allows us to choose $R(r)$ big enough such that the above holds independently of the choice of $\zeta \in X$. □

4.2 Minimal actions of the dihedral group

Note that a minimal action of G on X can have a finite orbit only if X is finite, since such an orbit would be dense in X. Using this idea we can understand how orbital graphs and stabilizers of a minimal dihedral action look like.

Proposition 4.8. *Let a, b be homeomorphisms of period two of a Cantor set X such that the dihedral group $\langle a, b \rangle$ acts minimally on X. Then*

1. *The orbital graphs of $\langle a, b \rangle$ are either two ended or one ended infinite chains. The graphs of germs are two ended infinite chains.*

2. *If the stabilizer $\langle a, b \rangle_\xi$ is non-trivial, then there exists a point ξ' in the orbit of ξ such that $\langle a, b \rangle_{\xi'}$ is equal to either $\langle a \rangle$ or $\langle b \rangle$.*

Proof.

1. Consider an orbital graph Γ. Its vertices are in correspondence with an orbit of an element $\xi \in X$, and by the remark made at the start of this section there must be infinitely many of them.

   For the second part of the statement suppose we have a one-ended graph of germs $\widetilde{\Gamma}$, so that it is a one-ended chain. Denote by $\zeta \in X$ the root of this chain, which must be fixed by one of the generators. By definition of the graph of germs, this generator must also fix a neighborhood of ζ, and by minimality of the action we conclude that it must fix a different vertex of $\widetilde{\Gamma}$. This gives a contradiction, since then $\widetilde{\Gamma}$ would be finite.

2. Suppose $\xi \in X$ is a point with non-trivial stabilizer $\langle a, b \rangle_\xi$. Then the orbital graph Γ_ξ is a one-ended chain, and choosing $\xi' \in X$ to be the root of Γ_ξ gives a point in the orbit of ξ whose stabilizer is equal to one of the generators.

□

Note that in fact from the proof of the second statement, it follows that any non-trivial stabilizer is conjugate to te group generated by one of the generators of D_∞.

Now we explore how the minimality of the dihedral action can be expressed in terms of the minimality of the infinite cyclic subgroup of D_∞.

Proposition 4.9. *Let a and b be homeomorphisms of period two of a Cantor set X. If the action of the dihedral group $\langle a, b \rangle$ is minimal, then*

- *either the action of $\langle ab \rangle$ is minimal, or*

- *X can be decomposed as a disjoint union of two clopen $\langle ab\rangle$-invariant sets S_1, S_2 such that the action of $\langle ab\rangle$ on each of these sets is minimal, and $a(S_1) = b(S_1) = S_2$, $a(S_2) = b(S_2) = S_1$.*

Proof. Consider any non-empty closed $\langle ab\rangle$-invariant subset $A \subseteq G$. Then $a(A)$ is also a closed ab-invariant subset, since $ab(a(A)) = a((ab)^{-1}A) = a(A)$.

The above implies that both $A \cup a(A)$ and $A \cap a(A)$ are closed $\langle a, b\rangle$-invariant subsets. By minimality of the action of the dihedral group, we conclude that $A \cup a(A) = X$, and that either $A \cap a(A) = X$ (in which case the action of $\langle ab\rangle$ is minimal), or $A \cap a(A) = \emptyset$. In the latter case, defining $S_1 \coloneqq A$ and $S_2 \coloneqq a(A)$ shows the second possibility from the proposition holds. □

In particular, if the action of D_∞ is non-free, then the D_∞-minimality is equivalent to the $\mathbb{Z}$-minimality.

Now we proceed to give two universal models for dihedral actions on cantor spaces. The first one refers to expansive actions, which intuitively are the ones that let us uniformly distinguish arbitrarily close points via the action of a group element.

Definition 4.10. An action of a group G on a metric space X is said to be *expansive* if there exists a constant $c > 0$ such that for every pair of distinct points $x, y \in X$, there exists $g \in G$ with $d(gx, gy) > c$.

Proposition 4.11. *Let a and b be homeomorphisms of period two of a Cantor set X. Suppose that they generate an expansive action of D_∞. Then there exists a finite alphabet A, a permutation $\iota : A \to A$ such that $\iota^2 = \varepsilon$, and a $\mathbb{Z}$-subshift $S \subseteq A^{\mathbb{Z}}$ (that is, a closed shift-invariant subset S) such that there exists a homeomorphism $X \to S$ conjugating the action of a and b with the homeomorphisms of S given by:*

$$a(w)(n) = \iota(w(-n)),\ \ b(w)(n) = \iota(w(1-n)),$$

$w \in S,\ n \in \mathbb{Z}$.

Proof. Thanks to the expansiveness assumption, it is possible to find a partition $\mathcal{U} = \{U_1, \ldots, U_n\}$ of X into clopen sets such that every point $\zeta \in X$ is determined by the map $I_\zeta : D_\infty \to \mathcal{U}$ defined as $I_\zeta(g) = U_i$ if and only if $g(\zeta) \in U_i$, which we call the *itinerary* of ζ. Up to a refinement of the partition $\mathcal{U}$, we may suppose that for any $U \in \mathcal{U}$ we have $a(U) \in \mathcal{U}$.

Thus for any $\zeta \in X$ and any $g \in D_\infty$ we have $I_\zeta(g) = a(I_\zeta(ag))$. Thus the itinerary of ζ, and hence ζ itself, is completely determined by the sequence $I_\zeta((ab)^n)$, $n \in \mathbb{Z}$. Denote $J_\zeta(n) = I_\zeta((ab)^n)$. Consider now $S \subseteq \mathcal{U}^{\mathbb{Z}}$ to be the set of sequences $\{J_\zeta(n)\}_{n\in\mathbb{Z}}$, for $\zeta \in X$. It is clear that S is a $\mathbb{Z}$-subshift. Moreover, we have

$$\begin{aligned} J_{a(\zeta)}(n) &= I_\zeta((ab)^n a) = I_\zeta(a(ba)^n) = a(I_\zeta((ba)^n)) = a(J_\zeta(-n)), \text{ and} \\ J_{b(\zeta)}(n) &= I_\zeta((ab)^n b) = I_\zeta(a(ba)^{n-1}) = a(I_\zeta(ba)^{n-1}) = a(J_\zeta(1-n)). \end{aligned}$$

Finally, it suffices to define ι to be the permutation of $\mathcal{U}$ equal to the action of a on $\mathcal{U}$. □

The second type of dihedral actions we are interested in are residually finite minimal actions. Recall that in Proposition 2.8 we showed that residually finite groups can be seen as subgroups of automorphisms of spherically homogeneous rooted trees. The next proposition shows that if we add the restriction of being a minimal action, we can only find odometers, up to conjugation.

Proposition 4.12. *Every minimal residually finite action of $\mathbb{Z}$ on a Cantor set is topologically conjugate to an odometer, i.e. the map $\alpha : x \to x + 1$ on the projective limit $\overline{\mathbb{Z}}$ of a sequence $\mathbb{Z}/(d_1 d_2 \ldots d_n)\mathbb{Z}$ of finite cyclic groups.*

The sequence $\{d_i\}_{i\in\mathbb{Z}}$ is of course the sequence defining the branching of the tree on which the group acts.

Proposition 4.13. *Consider a minimal residually finite action of the dihedral group D_∞ on a Cantor set X. Suppose that the restriction of the action to the infinite cyclic subgroup of D_∞ is minimal. Then there exists a homeomorphism of X with a projective limit $\overleftarrow{\mathbb{Z}}$ of finite cyclic groups conjugating the action of D_∞ with the action generated by the homeomorphisms*

$$a(x) = 1 - x,\ \ b(x) = -x.$$

Proof. Let a and b be the involutions generating D_∞. Then the element $\alpha = ab$, which generates the infinite cyclic subgroup of D_∞, has a minimal action by hypothesis. Hence thanks to Proposition 4.12 it is conjugate to the action $x \mapsto x+1$ on some projective limit $\overleftarrow{\mathbb{Z}}$ of finite cyclic groups.

We have $b\alpha b^{-1} = \alpha^{-1}$. Note that for any $b_1, b_2 \in D_\infty$ such that $b_i\alpha b_i = \alpha^{-1}$, we have that b_1b_2 commutes with α. Hence, by minimality and continuity of the action of α, b_1b_2 commutes with any transformation of the form $x \mapsto x+k$, for $k \in \overleftarrow{\mathbb{Z}}$. In particular, $b_1b_2(k) = b_1b_2(k+0) = k + b_1b_2(0)$ for any $k \in \overleftarrow{\mathbb{Z}}$. Hence defining $g = b_1b_2(0)$ we have that b_1b_2 is the action $x \mapsto x+g$.

The transformation $b_0(x) = -x$ satisfies $b_0\alpha b_0 = \alpha^{-1}$, so we conclude that any involution b such that $b\alpha b = \alpha^{-1}$ is of the form $b(x) = -x+g$, for some $g \in \overleftarrow{\mathbb{Z}}$. In particular, $a = a\alpha b$ is given by $a(x) = -x+g+1$.

If all the cyclic groups defining the projective limite have odd order, then the equation $2x = g$ has a solution in $\overleftarrow{\mathbb{Z}}$ for every $g \in \overleftarrow{\mathbb{Z}}$. On the other case, either the equation $2x = g$, or the equation $2x = g+1$ has a solution. Thus, in the odd case both involutions a and b have fixed points, while on the other case only one of them has fixed points.

Suppose that b has a fixed point $\xi \in \overleftarrow{\mathbb{Z}}$. Up to conjugating by the shift $x \mapsto x - \xi$, we may assume that $\xi = 0$. This finishes the proof, since we have $a(x) = -x+1$ and $b(x) = -x$. $\square$

Example 4.14. Consider X the ring of dyadic integers, that is, the projective limit of the cyclic groups $\mathbb{Z}/2^n\mathbb{Z}$, and the dihedral action $a : x \mapsto x+1$, $b \mapsto -x+1$. Such action is conjugate to a dihedral action by automorphisms of the boundary of a two rooted tree $\{0,1\}^\infty$, given by

$$a(0w) = 1w, \quad a(1w) = 0w$$
$$b(0w) = 0a(w), \quad b(1w) = 1b(w).$$

The action of a and b on the infinite binary tree is illustrated in Figure 4.1.

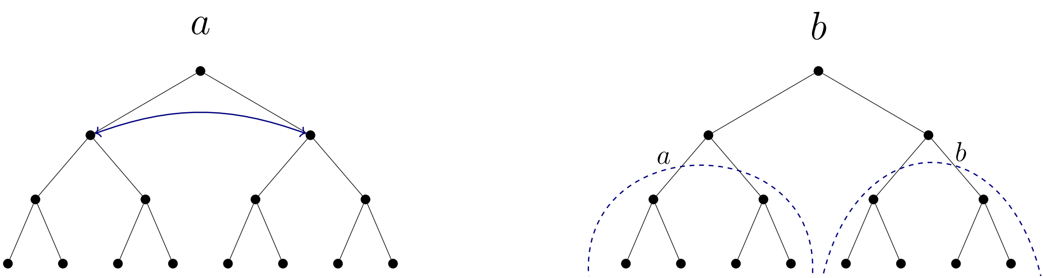


Figure 4.1 – Action of the Gray code the binary tree.

4.3 Fragmentation of dihedral groups

4.3.1 Fragmentation of an involution

We start by defining fragmentations of an involution a acting on X as a finite group of homeomorphisms which on each point can act either as a or as the identity, but in such a way that the action of a is not lost for any point.

Definition 4.15. Let a be a homeomorphism of period two of a Cantor set X. A *fragmentation of a* is a finite group A of homeomorphisms of X such that

1. for every $h \in A$ and $\zeta \in X$, either $h(\zeta) = \zeta$ or $h(\zeta) = a(\zeta)$, and
2. for every $\zeta \in X$ there exists $h \in A$ such that $h(\zeta) = a(\zeta)$.

Given A a fragmentation of the involution a, we can partition X according to the points on which the elements of A act as a or the identity. Define for each $h \in A$ the sets

$$E_{h,\varepsilon} = \{\zeta \in X \mid \ h(\zeta) = \zeta\}, \text{ and } E_{h,a} = \{\zeta \in X \mid \ h(\zeta) = a(\zeta)\}.$$

It is clear that both sets are closed, a-invariant, that $E_{h,a} \cup E_{h,\varepsilon}$ is the set of fixed points of a and that $X = E_{h,\varepsilon} \cup E_{h,a}$. In out context we will always work with minimal actions of $D_\infty = \langle a, b\rangle$, and hence the set of fixed points of a must have empty interior. In effect, the graph of germs of a point in such interior would not be two-ended and this contradicts Proposition 4.8. We will be working under this hypothesis from now on, which in particular implies that $\mathrm{int}(E_{h,\varepsilon}) \cap \mathrm{int}(E_{h,a}) = \emptyset$.

We define the set of *pieces* $\mathcal{P}$ of the fragmentation A to be the set formed by the subsets of X obtained by taking intersections of the form $\bigcap_{h\in A} Q_h$, where for each $h \in A$ we have $Q_h \in \{\mathrm{int}(E_{h,\varepsilon}), \mathrm{int}(E_{h,a})\}$. Each piece of the fragmentation satisfies that the action of each element of a on it is well defined to be either the identity, or A.

Define for every piece $P \in \mathcal{P}$ the epimorphism $\pi_P : A \to \mathbb{Z}/2\mathbb{Z}$ given by

$$\pi_P(h) = \begin{cases} 0, & \text{if } P \subseteq E_{h,\varepsilon}, \\ 1, & \text{if } P \subseteq E_{h,a}. \end{cases}$$

Then the mapping $\{\pi_P\}_{P\in\mathcal{P}} : A \to (\mathbb{Z}/2\mathbb{Z})^{\mathcal{P}}$ defines an embeding, and we can identify A with a subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\mathcal{P}}$.

The main hypothesis we will have for the theorems of Chapter 5 is that G contains a purely non-Hausdorff singularity, as defined in Definition 4.4. In fact, any involution acting non-freely on a Cantor space can be fragmented in such a way that we obtain a purely non-Hausdorff singularity, as we explain below.

Lemma 4.16. *Suppose $\xi \in X$ is a fixed point of the involution a. Then for any $n \geq 1$ we can find a partition of $X\backslash\{\xi\}$ consisting of open a-invariant subsets $P_1, \ldots, P_n$, each one of them accumulating on ξ.*

Proof. Since X is a Cantor space we can find a decreasing sequence $\{U_k\}_{k\geq 0}$ of clopen neighborhoods of ξ such that $U_0 = X$ and $\bigcap_{k\geq 0} U_k = \{\xi\}$. Define $V_k = U_k \cap a(U_k)$, for each $k \geq 0$, so that in particular V_k is an a-invariant subset. Then the sequence $\{V_k\}_{k\geq 0}$ is also decreasing and formed by clopen sets, and as ξ is a fixed point for a it also satisfies $\bigcap_{k\geq 0} V_k = \{\xi\}$. Moreover, up to passing to a subsequence, we can suppose that for each $k \geq 0$ we have $V_k \neq V_{k+1}$.

Now it suffices to consider any partition of the integers into n infinite sets $N = I_1 \cup \cdots I_n$ and define $P_j = \bigcup_{k\in I_j} V_k \backslash V_{k+1}$. ∎

Proposition 4.17. *Suppose a is an involution acting non-freely by homeomorphisms on a Cantor space X. The there exists A a fragmentation of a with a purely non-Hausdorff singularity.*

Proof. Take $\xi \in X$ a fixed point of a, and denote by $\mathcal{P} = \{P_1, \dots, P_n\}$ the partition of $X \backslash \{\xi\}$ into open a-invariant subsets accumulating on ξ, given by Lemma 4.16. Now it suffices to choose a subgroup $A \leqslant (\mathbb{Z}/2\mathbb{Z})^{\mathcal{P}}$ such that each homomorphism $\pi_{P_i} : A \to \mathbb{Z}/2\mathbb{Z}$ is surjective, and such that there is no element $h \in A$ such that $\pi_{P_i}(h) = 1$ for every $1 \leq i \leq n$. Such a subgroup always exists for $n \geq 3$.

Finally, as we explained above such a subgroup is isomorphic to a finite subgroup A of homeomorphisms of X, which by construction is precisely a fragmentation of A with ξ as a purely non-Hausdorff singularity. □

Example 4.18. Recall the example of the Gray code dihedral action from Example 4.14. Note that the tree automorphism b has as unique fixed point the sequence $\xi = 111 \cdots \in X = \{0, 1\}^\infty$. To define a fragmentation of b, we will first define the pieces of the fragmentation. Note that the sets W_n of sequences starting with exactly n 1's is a partition of $X \backslash \{\xi\}$, and define $P_0 = \bigcup_{n \geq 0} W_{3n}$, $P_1 = \bigcup_{n \geq 0} W_{3n+1}$ and $P_2 = \bigcup_{n \geq 0} W_{3n+2}$. Now consider the homeomorphisms of X: b_1 acting as b on $P_0 \cup P_1$ and the identity on P_2, b_2 acting as b on $P_0 \cup P_2$ and the identity on P_1, and b_3 acting as b on $P_1 \cup P_2$ and as the identity of P_0. Then $B = \{\varepsilon, b_1, b_2, b_3\}$ is a fragmentation of b. The action of these elements on the infinite binary tree can be seen on Figure 4.2. The group generated by a and B is precisely *Grigorchuk's group*, studied in further detail in Section 2.6.

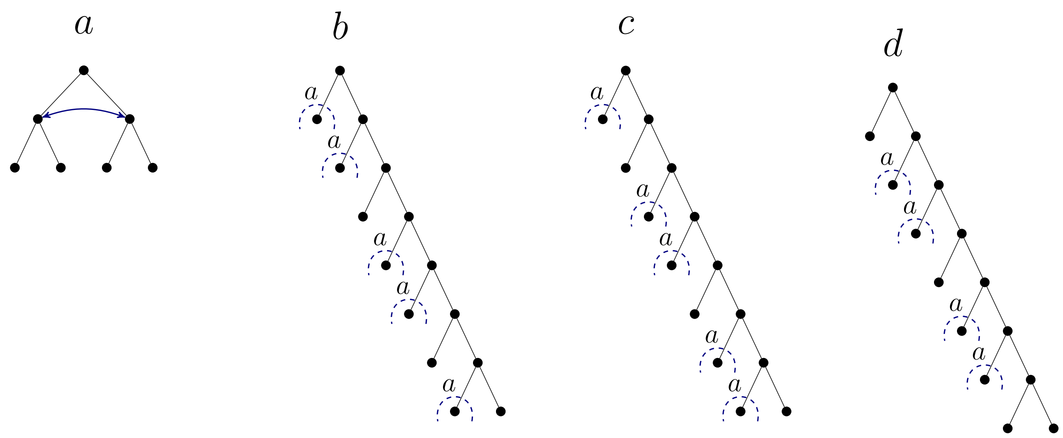


Figure 4.2 – The action of a, b_1, b_2 and b_3 on (part of) the infinite binary tree. This is known as the Grigorchuk group.

4.3.2 The orbital graph of fragmented dihedral group

Definition 4.19. Let $\langle a, b \rangle$ be a dihedral group acting on a Cantor set X, and consider fragmentations A and B of a and b, respectively. We say that the group $G = \langle A \cup B \rangle$ is a *fragmentation of the dihedral action.*

Consider $G = \langle A \cup B \rangle$ a fragmentation of a minimal action of a dihedral group $\langle a, b \rangle$.

Note that two vertices of the same orbital graph of $\langle a, b \rangle$ connected by an edge labeled by a must belong to the same piece $P \in \mathcal{P}_A$. These vertices are thus connected in the corresponding orbital graph of G, now by edges labeled by all elements $h \in A$ which act as a on them. Of course, the same holds for the involution b together with its fragmentation B.

We conclude that the orbital graphs of G can be obtained from the orbital graphs of $\langle a, b\rangle$, replacing edges labeled by a by (possibly multiple) edges labeled by elements of A, and edges labeled by b by (possibly multiple) edges labeled by elements of B.

Definition 4.20. Consider an orbital graph Γ_ζ. A finite connected subgraph $\Sigma \subseteq \Gamma_\zeta$ is called a *segment* if for every $v_1, v_2 \in \Sigma$, then all edges connecting v_1 to v_2 in Γ_ζ are also edges in Σ. However, we will omit the loops of the endpoints of the segment Σ.

As we saw earlier, every stabilizer $\langle a, b\rangle_\zeta$ of a point $\zeta \in X$ is conjugate to a stabilizer equal to $\langle a\rangle$ or $\langle b\rangle$. It follows that every stabilizer G_ζ of a singular point is conjugate to the stabilizer of a fixed point of a or b.

Lemma 4.21. *Suppose that $\xi \in X$ is a fixed point of a, and define H_ξ as the subgroup of A consisting of elements $h \in A$ fixing ξ and acting non trivially on every neighborhood of ξ. Then there is a natural isomorphism between H_ξ and $G_\xi/G_{(\xi)}$.*

Proof. It suffices to prove that any germ (g, ξ) is equal (as germs) to a germ (h, ξ), for $h \in H_\xi$. In other words, for every $g \in G_\xi$, there exists an $h \in H_\xi$ which coincides which g in a neighborhood of ξ.

Using the generating set $A \cup B$ of G, we can write $g = b_1 a_1 \cdots b_n a_n$, where $a_i \in A$, $b_i \in B$ for $i = 1, \ldots, n$, and they are all non-trivial, except possibly for b_1 or a_n.

As the action is minimal and ξ is a fixed point of a, there cannot be fixed points of b in the orbit of ξ, nor other fixed points of a different from ξ. This in particular implies that points in the orbit of ξ distinct from ξ are not in the boundary of the piece of A and B to which they belong.

From the above, we conclude that the germs $(b_n, a_n(\xi))$, $(a_{n-1}, b_n a_n(\xi))$,$\ldots$, $(b_1, a_1 \cdots b_n(\xi))$ are equal either to germs of the identity, a or b. Hence, the germ (g, ξ) is equal to a germ of the form (ha_n, ξ) for $h \in \langle a, b\rangle$. As ξ is a fixed point of a (and not of b), the germ (g, ξ) is equal to (a_n, ξ), $a_n \in A$. $\square$

Now we define an auxiliary graph W_ξ, formed by $|H_\xi|$ copies of Γ_ξ, whose roots ξ are connected by edges labeled by the elements of H_ξ.

Definition 4.22. Given $\xi \in X$ a fixed point of a, we define the graph W_ξ as the one whose set of vertices is $H_\xi \times \Gamma_\xi$, and such that two vertices $(h_1, v_1), (h_2, v_2) \in W_\xi$ are connected by an edge labeled by $h \in A \cup B$ if

1. $h_1 = h_2$ and v_1 and v_2 are connected in Γ_ξ by an edge labeled by h, or
2. if $v_1 = v_2 = \xi$ and $h = h_1 h_2$.

Proposition 4.23. *The graph of germs $\widetilde{\Gamma}_\xi$ is naturally isomorphic to W_ξ. The action of the group of deck transformations $G_\xi/G_{(\xi)} \cong H_\xi$ of the covering $\widetilde{\Gamma}_\xi \to \Gamma_\xi$ coincides with the natural action of H_ξ on W_ξ.*

Proof. By the reasoning done in the proof of Lemma 4.21, any germ (g, ξ) is equal to a germ of the form $(g'h, \xi)$, where $g \in \langle a, b\rangle$, $g \neq b$, and $h \in H_\xi$. We identify $(g'h, \xi)$ to the vertex $(h, g'(\xi))$ of W_ξ. This gives an isomorphism of graphs, and the second part of the statement follows from the isomorphism between $G_\xi/G_{(\xi)}$ and H_ξ. $\square$

Denote $P_1, \ldots, P_n \in \mathcal{P}_A$ the pieces of the fragmentation A which accumulate on ξ. Note that the map $\{\pi_{P_i}\}_{i=1}^n : H_\xi \to (\mathbb{Z}/2\mathbb{Z})^n$ is an isomorphic embedding.

Denote $\Lambda_i = \widetilde{\Gamma}_\xi/\mathrm{Ker}(\pi_{P_i})$. Thanks to Proposition 4.23, Λ_i is isomorphic to the quotient $W_\xi/\mathrm{Ker}(\pi_{P_i})$, where two copies of Γ_ξ indexed by h_1 and h_2, respectively, will be identified if

$h_1|_{P_i} = h_2|_{P_i}$. As there are only two possibilities (elements act either as the identity or as a), we conclude that Λ_i is (identified with) the graph obtained by taking two copies $\{0\} \times \Gamma_\xi$ and $\{1\} \times \Gamma_\xi$ of Γ_ξ and connecting the points $(0, \xi)$ and $(1, \xi)$ by edges labeled by the elements $h \in A$ which act as a in P_i.

Denote by $\lambda_i : \widetilde{\Gamma}_\xi \to \Gamma_i$ the natural covering map. In terms of W_ξ and Γ_i, it is given by the formula

$$\lambda_i(h, v) = (\pi_{P_i}(h), v).$$

explain a little what follows, and define convergence of graphs and show main properties.

Proposition 4.24. *If $\zeta_n \in P_i$, $n \geq 1$, is a sequence of regular points converging to ξ, then the rooted orbital graphs Γ_{ζ_n} converge to Γ_i, in the space of rooted labeled graphs.*

Proof. Let $r \geq 0$ and consider the ball $B(\xi, r)$ of the graph of germs $\widetilde{\Gamma}_\xi$. This ball is completely characterized by a finite number of equalities and inequalities of germs (g, ξ), which also hold in a neighborhood of ξ (by definition of being a germ). We claim that for a sufficiently small neighborhood U of ξ, and any regular point $\zeta \in U \cap P_i$, the ball $B(\zeta, r)$ of the orbital graph Γ_ζ is equal to the quotient of the ball $B(\xi, r)$ of the graph of germs $\widetilde{\Gamma}_\xi$ by $\mathrm{Ker}(\pi_{P_i})$.

Suppose we have two distinct germs $(g_1, \xi) \neq (g_2, \xi)$ with $g_1(\xi) = g_2(\xi)$. Then there exists $g \in \langle a, b \rangle$ and $h_1, h_2 \in H_\xi$ such that $(g_1, \xi) = (gh_1, \xi)$ and $(g_2, \xi) = (gh_2, \xi)$. There are two cases:

If $\pi_{P_i}(h_1 h_2) = 0$, then $h_1|_{P_i} = h_2|_{P_i}$, and so there exists a neighborhood U of ξ such that $g_1|_{U \cap P_i} = g_2|_{U \cap P_i}$. On the other hand, if $\pi_{P_i}(h_1 h_2) = 1$ then $h_1 h_2|_{P_i} = a|_{P_i}$. As the set of fixed points of a is nowhere dense, there must exist a neighborhood U of ξ such that for any regular point $\zeta \in U \cap P_i$ we have $g_1(\zeta) \neq g_2(\zeta)$. This concludes the proof of the proposition. □

Corollary 4.25. *Every segment of an orbital graph of G is isomorphic to a segment of the orbital graph of a regular point. In particular, for every segment Σ of an orbital graph of G an isomorphic copy of Σ is contained in every orbital graph of G, on some bounded distance $R(\Sigma)$ from every vertex of the orbital graph.*

Proof. Suppose we have a singular point $\xi \in X$ and consider a segment Σ of its orbital graph Γ_ξ. Thanks to Proposition 4.24, there exists a sequence of orbital graphs Γ_{ζ_n} converging to a graph Λ_i associated to Γ_ξ. Noticing that Σ is also a segment of Λ_i, we conclude that for n big enough we can find an isomorphic copy of Σ on the orbital graph of the regular point ζ_n.

The second part of the proposition follows by combining the first part together with Proposition 4.7. □

Corollary 4.26. *For every oriented segment Σ of an orbital graph of G, there exist isomorphic copies of Σ and Σ^{-1} in every oriented orbital graph.*

Proof. Using Proposition 4.25, we can suppose that Σ is a segment of an orbital graph Γ_ξ of a singular point ξ. For any graph Λ_i as defined above, a copy of Σ appears in $\{0\} \times \Gamma_\xi$, while a copy of Σ^{-1} appears in $\{1\} \times \Gamma_\xi$. Choosing a subsegment Σ' of Λ containing both copies and repeating the argument of the proof of Proposition 4.25 shows that we can find a copy of Σ', and hence copies of Σ and Σ^{-1}, in every orbital graph. □

Chapter 5

Simple groups of Burnside type

The goal of this chapter is to show how the dihedral fragmentation groups from Chapter 4 can be used to construct the first known examples of simple amenable groups of Burnside type. In Section 5.1 we prove a first key result regarding periodicity of minimal actions of dihedral fragmentations, and then in Section 5.2 explain how the concept of the topological full group allows us to use the obtained result in order to get a simple amenable group of Burnside type.

5.1 From dihedral actions to periodic groups

The following theorem is key for connecting actions of fragmentations of dihedral actions and Burnside's problem.

Theorem 5.1. *Let G be a fragmentation of a minimal dihedral group action on a Cantor set X. If there exists a purely non-Hausdorff singularity $\xi \in X$, then G is periodic.*

Proof. Thanks to Proposition 4.8, we can suppose that ξ is in fact a fixed point of a.

For the sake of contradiction, suppose there exists an element $g \in G$ of infinite order. Let us denote $\ell_{A\cup B}(g) := m \geq 1$.

The main tool for the proof will be the following lemma. Intuitively, it says that for any vertex v of a segment Δ of an orbital graph of edge-length m, it is possible to find a copy of Δ in an orbital graph of a regular point where the g-orbit of v returns to Δ.

Lemma 5.2. *Given any segment Σ of an orbital graph of G, a subsegment $\Delta \subseteq \Sigma$ of edge-length m, and a vertex v of Δ, there exists an embedding φ of Σ into an orbital graph of a regular point and an integer $k \geq 1$ such that $g^k(\varphi(v)) \in \varphi(\Delta)$.*

Proof. Looking for a contradiction, let us suppose that we can find $v \in \Delta \subseteq \Sigma$, Δ of edge-length m, such that for every orbital graph Γ of a regular point and any embedding $\varphi : \Sigma \to \Gamma$ the sequence $\{g^k(\varphi(v))\}_{k\geq 1}$ does no intersect $\varphi(\Delta)$.

By definition of an orbital graph and since $\ell_{A\cup B}(g) = m$, we know that any vertex $u \in \Gamma$ is at distance at most m from $g(u)$. Since the edge-length of Δ is m, this implies that the sequence $\{g^k(\varphi(v))\}_{k\geq 1}$ is contained in exactly one of the two connected components of $\Gamma\backslash\varphi(\Delta)$, and as the order of g is infinite this sequence converges to one of the two ends of Γ.

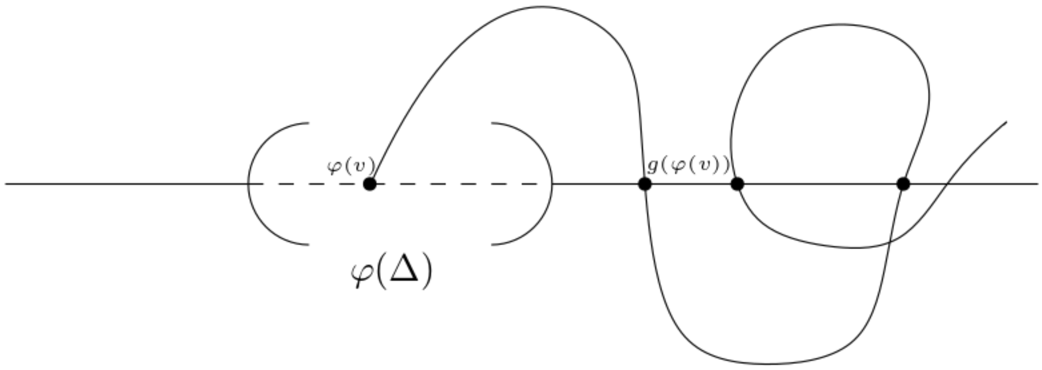


Figure 5.1 – The g-orbit of $\varphi(v)$ stays on one connected component of $\Gamma\backslash\varphi(v)$.

Corollary 4.26 tells us that there exist embeddings of Σ into the orbital graph Γ_ξ of the purely non-Hausdorff singularity ξ in both possible orientations. In particular, we can find an embedding $\varphi : \Sigma \to \Gamma_\xi$ such that $g(\varphi(v))$ is on the same side of $\varphi(\Delta)$ as ξ.

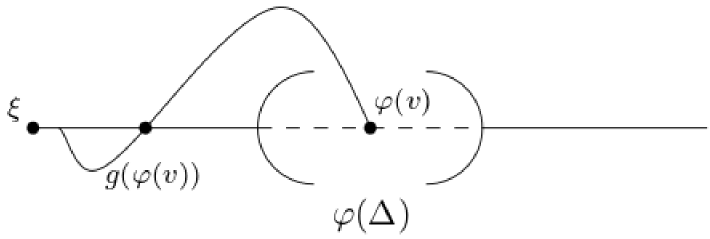


Figure 5.2 – We can find a copy of Δ where $g(\phi(v))$ is at the same side of $\varphi(\Delta)$ as ξ.

This does not give an immediate contradiction with the previous paragraph, since that situation was only true for orbital graphs of regular points. Nonetheless, we will still be able to reduce to that case by lifting the g-orbit of v to the graph of germs and projections into graphs Λ_i, as explained below.

Consider the corresponding copy $\varphi_0 : \Sigma \to \{\varepsilon\} \times \Gamma_\xi$ of Σ in the ray $\{\varepsilon\} \times \Gamma_\xi$ of the graph of germs $\widetilde{\Gamma}_\xi = W_\xi$. Using the same notation as in Section 4.3 we denote $\mathcal{P}_1, \dots, \mathcal{P}_n$ the pieces of the fragmentation A which accumulate on ξ.

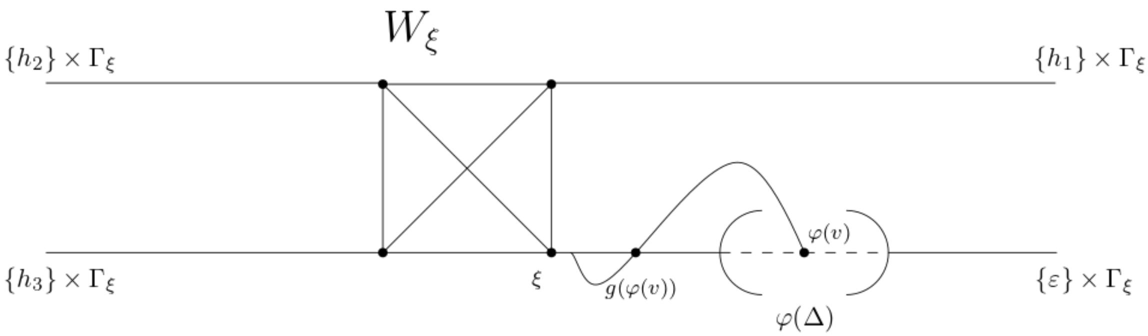


Figure 5.3 – The copy of Δ in the graph of germs W_ξ.

For any $i = 1, \dots, n$, consider the image $\lambda_i \circ \varphi_0(\Sigma)$ in Λ_i under the covering λ_i. By definition, it will belong to the ray $\{0\} \times \Gamma_\xi$ of Λ_i, and since $g(\varphi_0(v))$ is on the same side of this ray as $(0, \xi)$, with respect to $\varphi_0(\Delta)$, we conclude that the sequence $\{g^k(\lambda_i \circ \varphi_0(v))\}_{k\geq 1}$ converges to the end of the ray $\{1\} \times \Gamma_\xi$ of Λ_i.

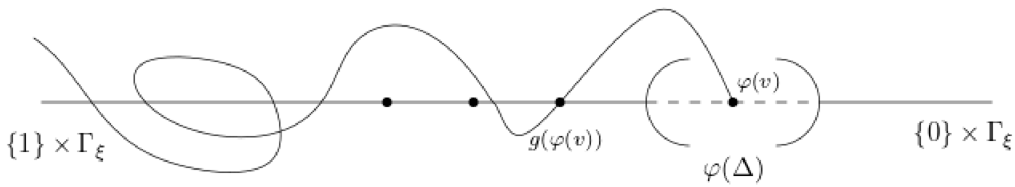


Figure 5.4 – The g-orbit of v converges to the end of the ray $\{1\} \times \Gamma_\xi$.

Again lifting this orbit to the graph of germs, we see that the sequence $\{g^k(\varphi_0(v))\}_{k\geq 1}$ must converge in $\widetilde{\Gamma}_\xi$ to an end $\{h\}\times\Gamma_\xi$, for some $h \neq \varepsilon$. Here is where we need to use that ξ is a purely non-Hausdorff singularity: thanks to this property, as $h \in G_\xi$, then ξ can be approximated by interior points of the set of fixed points of h. This implies that there must exist a $j \in \{1,\dots,n\}$ such that h acts as the identity on P_j.

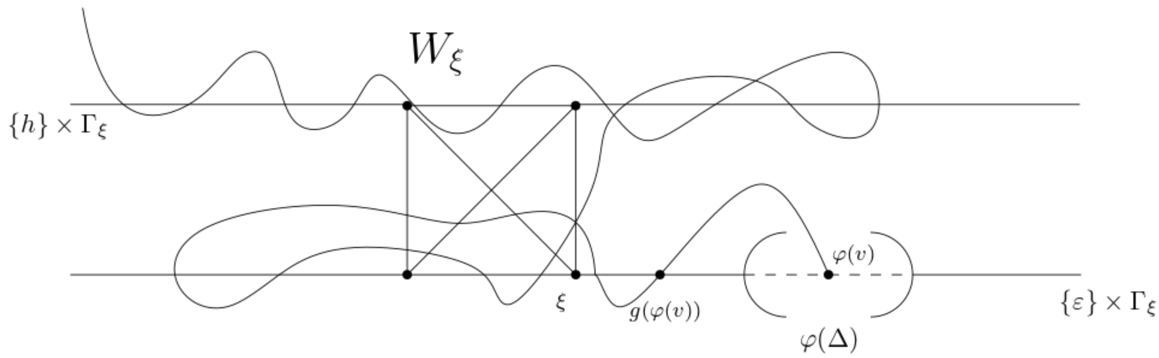


Figure 5.5 – The g-orbit of v converges to the end of the a copy of the orbital graph $\{h\} \times \Gamma_\xi$.

Finally, consider the projection $\lambda_j : \widetilde{\Gamma}_\xi \to \Lambda_j$, so that $\lambda_j(\{h\}\times\Gamma_\xi) = \lambda_j(\{\varepsilon\}\times\Gamma_\xi) = \{0\}\times\Gamma_\xi$.

This implies that on Λ_j the sequence $\lambda_j(g^k(\varphi_0(v)))$ will move from one connected component of $\Lambda_j\backslash\lambda_j(\varphi_0(\Delta))$ to another, which is a contradiction.

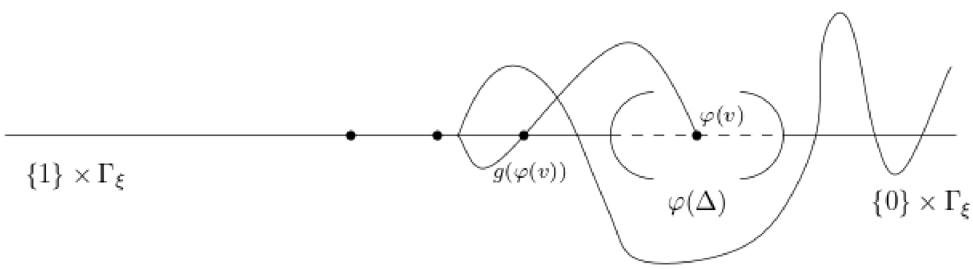


Figure 5.6 – We can find a projection such that the g-orbit of v converges to the end of $\{0\}\times\Gamma_\xi$.

□

To avoid excessive notation, from now on we will omit the embedding function φ, and the distinction between Δ (resp. a vertex v) and their copies on different orbital graphs will be clear from context.

The above lemma tells us that the g-orbit of v comes back to Δ in a particular copy of Σ on an orbital graph of a regular point. We will now show that we can in fact always find another segment Σ' containing Δ such that the above holds on any copy of Σ' on any orbital graph of a regular point.

Consider any segment Δ of edge-length m and denote its vertices by $v_0, \dots, v_m$. Thanks to Lemma 5.2 we can find a copy of Δ in an orbital graph Γ of a regular point, with $g^{k_0}(v_0) \in \Delta$, for some $k_0 \geq 1$. Choose Σ_0 a sufficiently large segment of Γ, containing Δ as well as the $(m+1)$-neighborhood of $\{g^k(v_0)\}_{k=0}^{k_0}$. Then we have that $g^{k_0}(v_0) \in \Delta$ in every copy of Σ_0, in every orbital graph.

Now apply Lemma 5.2 to $v_1 \in \Delta \subseteq \Sigma_0$. We find an orbital graph and a copy of Σ_0 on it, on which both sequences $g^k(v_0)$ and $g^k(v_1)$ come back to Δ. We again consider a (bigger) segment Σ_1 containing Δ such that $g^k(v_0)$ and $g^k(v_1)$ returns to Δ in every orbital graph containing Σ_1.

Proceeding in the same way, at the end we arrive at a segment Σ_m such that every vertex of Δ returns inside Σ_m back to Δ, under some positive power of g.

It follows that the g-orbit of every vertex of $\Delta \subseteq \Sigma_m$ is finite, and contained in Σ_m.

With the last remark we are ready to finish the proof. Let Γ be an orbital graph of a regular point. Thanks to Corollary 4.25 we can find $R > 0$ such that for any vertex $u \in \Gamma$, there exists a copy of Σ_m on both sides of u, at distance at most R. This implies that for every vertex u of Γ, the sequence $g^k(u)$ either includes a point of one of the neighboring copies of Δ, or always stays between them. Denote by $|\Sigma_m|$ the number of vertices of Σ_m. On the first case the length of the g-orbit of u is not more than M, while on the second case, it is less than $2R + 2M$. As neither of this bounds depends on u, it follows that the size of all g-orbits of vertices of Γ is uniformly bounded.

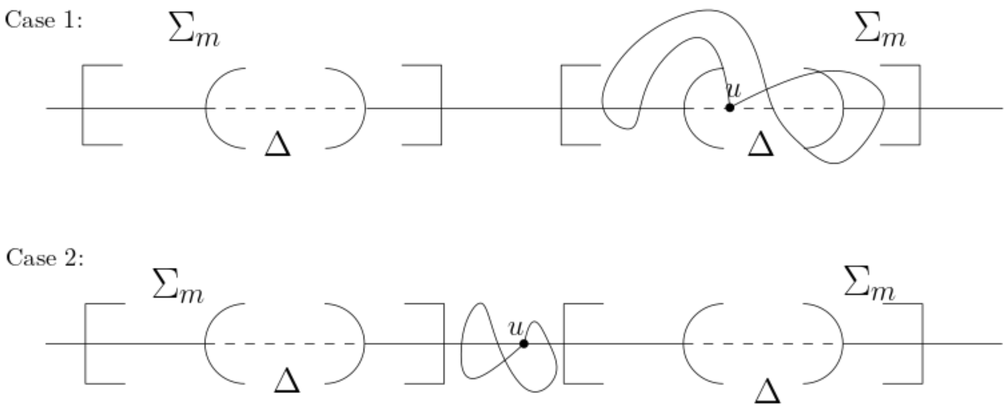


Figure 5.7 – Two possibilities for the g-orbit of u. On the first case it stays inside a copy of Σ_m, on the second one it bounces between both orbits.

We conclude that there exists n such that g^n acts trivially on the vertices of Γ. Moreover, by minimality of the action, the set of vertices of Γ is dense in X. We conclude that $g^n = \varepsilon$, and this gives us a contradiction since we had supposed that g has infinite order. $\square$

5.2 Simple amenable periodic groups

Recall the basic concepts about topological full groups introduced in Section 1.4. K. Juschenko and N. Monod have proven that the topological full group of a minimal homeomorphism on a Cantor space is amenable [JM13]. This allows us to prove that fragmentations of dihedral actions acting minimally on a Cantor space are amenable, by showing that in fact they embed into the topological full group of a minimal action on a Cantor space. This was first proven for the Grigorchuk group by N. Matte Bon [MB15], and the same idea works in our context, as we show below.

Theorem 5.3. *Let G be a fragmentation of a minimal action of the dihedral group on a cantor space X. Then G can be embedded into the topological full group of a minimal action of $\mathbb{Z}$ on a Cantor space, and hence it is amenable.*

Proof. The idea is to use the partition provided by the fragmentation in order to code G.

Consider $\zeta \in X$ a regular point. Then its orbital graph Γ_ζ is a bi-infinite chain, and we can identify its vertices with the integers $\mathbb{Z}$, so that adjacent vertices in Γ_ζ correspond to consecutive integers.

Denote by $\mathcal{P}_A$ and $\mathcal{P}_B$ the associated partitions of the fragmentations A and B of the involutions a and b, respectively. Note that two adjacent vertices of Γ_ζ are connected by edges labeled by elements of one of the two fragmentations A or B. Without loss of generality suppose it is A. Then these two adjacent vertices are in the same piece P of $\mathcal{P}_A$, and are connected by edges labeled by all the homeomorphisms $h \in A$ such that h acts as a on P. Using this property, we can code the orbital graph Γ_ζ by defining the sequence $w_\zeta = (a_n)_{n\in\mathbb{Z}}$ be the sequence of elements of $\mathcal{P}_A \cup \mathcal{P}_B$ which assigns to each vertex te partition to which it belongs.

Denote by W the set of bi-infinite sequences on the finite alphabet $(\mathcal{P}_A \cup \mathcal{P}_B)$ such that every finite subword belongs to the sequence w_ζ. Denote by $\sigma : (\mathcal{P}_A \cup \mathcal{P}_B)^{\mathbb{Z}} \to (\mathcal{P}_A \cup \mathcal{P}_B)^{\mathbb{Z}}$ the shift. By construction this set is closed and σ-invariant, so it is a subshift. Moreover, it is minimal. In effect, it suffices to check that for any sequence $x \in W$, its orbit $\{\sigma^n(x)\}_{n\in\mathbb{Z}}$ intersects al cylinder sets $C(i, u) := \{y \in W \mid y|_{[i,i+|u|-1]} = u\}$, where $i \in \mathbb{Z}$ and u ranges over all finite subwords of w_ζ. We can further reduce the problem to proving that for any sequence $x \in W$ and any subword u of w_ζ, u is a subword of x. Finally, this follows from Proposition 4.7.

Given any homeomorphism $s \in A \cup B$, its action on a vertex of Γ_ζ is uniquely determined by the labels of the edges connecting it to its two neighbors. This allows us to define an action of s on W by

$$s(w) = \begin{cases} \sigma(w), & \text{if } \pi_{w(0)}(s) = 1, \\ \sigma^{-1}(w), & \text{if } \pi_{w(-1)}(s) = 1, \\ w & \text{otherwise.} \end{cases}$$

This extends to an embedding of $G = \langle A \cup B \rangle$ into the topological full group of the shift σ acting on W.

Finally, as topological full groups of a minimal homeomorphism of a Cantor space is amenable by [JM13], we conclude that G is also amenable. $\square$

Theorem 5.4. *Suppose that the action of $\langle a, b \rangle$ on X is expansive. Let G be a fragmentation of the dihedral group. Then the action of G on X is also expansive, and the group $\mathsf{A}(G, X)$ is simple and finitely generated.*

Proof. It suffices to show that the expansiveness of D_∞ implies the expansiveness of G. The rest of the theorem follows from 1.15.

Consider $c > 0$ as in the definition of expansiveness. That is, such that for any pair of distinct points $\zeta_1, \zeta_2 \in X$ we can find $g \in \langle a, b \rangle$ with $d(g(\zeta_1), g(\zeta_2)) \geq c$.

To prove that $G = \langle A \cup B \rangle$ is expansive, consider two different points $\zeta_1, \zeta_2 \in X$. Suppose there exist pieces $P_1, P_2 \in \mathcal{P}_A$ such that $\zeta_i \in P_i$, $i = 1, 2$ (this is true up to boundary points of the pieces of the fragmentation $\mathcal{P}_A$). Consider elements $a_1, a_2 \in A$ with $a_i|_{P_i} = a|_{P_i}$, $i = 1, 2$. We have that either $a_1|_{P_2} = a|_{P_2}$ or $a_1|_{P_2} = \varepsilon|_{P_2}$, and similarly for a_2 and P_1. Then there exists an element $a' \in \{a_1, a_2, a_1a_2\}$ such that $a'|_{P_1\cup P_2} = a|_{P_1\cup P_2}$. Thus we have $a'(\zeta_1) = a(\zeta_1)$ and $a'(\zeta_2) = a(\zeta_2)$. Moreover this holds even for elements which are not contained in the pieces of the fragmentation $\mathcal{P}_A$, by continuity.

By making the analogous argument with b and B, we conclude that for any pair of distinct points $\zeta_1, \zeta_2 \in X$ we can find elements of the fragmentations $a' \in A$ and $b' \in B$ such that

$a'(\zeta_i) = b(\zeta_i)$ and $b'(\zeta_i) = b(\zeta_i)$, $i = 1, 2$. We conclude that for any $g \in \langle a, b\rangle$, there exists $g' \in G = \langle A \cup B\rangle$ such that $g'(\zeta_i) = g(\zeta_i)$, for $i = 1, 2$. From this, it is immediate that the expansiveness of $\langle a, b\rangle$ implies the expansiveness of G. □

In the case where $\mathrm{A}(G, X)$ is finitely generated, then by its definition it must be a subgroup of a fragmentation of the dihedral group with the same group of germs. In particular, we conclude that if G has a purely non Hausdorff singularity, then Theorem 5.1 implies $\mathrm{A}(G, X)$ is periodic, and Theorem 5.3 implies it is amenable.

Example 5.5. Consider the torus $\mathbb{R}^2/\mathbb{Z}^2$, and a vector $v \in \mathbb{R}^2/\mathbb{Z}^2$. Define the central symmetries a and b, where for each $x \in \mathbb{R}^2/\mathbb{Z}^2$ we have $a(x) = -x + v$ and $b(x) = -x$. If v is chosen appropriately, the transformation a is minimal in $\mathbb{R}^2/\mathbb{Z}^2$ (we need that the coordinates of v together with 1 form a linearly independent set over $\mathbb{Q}$).

Consider a partition of the torus into three b-invariant subsets P_1, P_2, P_3 such that the fixed point 0 belongs to the boundary of each one. An example of this is illustrated in Figure 5.8.

Figure 5.8 – A partition of the torus on three sets which have 0 at each one of their boundaries.

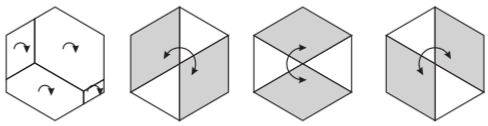


Figure 5.9 – A group of Burnside type arising from piecewise isometries of a polygon.

Now define the transformations b_1, b_2 and b_3 which act trivially on P_1, P_2 and P_3, respectively, and as b on the rest of the torus. This defines a fragmentation of the dihedral action of a and b on the torus. We may now lift this action to an action by homeomorphisms on the Cantor set (we can do this by the universal property of the Cantor set among compact metric spaces), and the conditions of Theorem 5.1 are satisfied, so we have found a group of Burnside type arising from piecewise isometries of a polygon.

Conclusion

Burnside's problem together with its related questions has become one of the most important problems in group theory, due to the development in the field it has inspired. Examples are the theory of self-similar groups and small cancellation theory.

In the case presented in this work, the main tool used to approach Burnside's problem are topological full groups. The idea behind this technique is that while the group itself may be complicated, it is possible that studying the orbits of a (non-free) action on a topological space can give useful information about algebraic properties of the group. This case is seen throughout this work, when we showed and used the fact that the orbital graphs of minimal fragmented dihedral actions on Cantor sets are simply infinite chains. This orbit manipulation allowed us to prove Nekrashevych's Theorem 5.1, providing a class of groups of Burnside type which arise from a dynamical origin.

Having proven the previous theorem, we can now use the theory of full topological groups to our advantage. To obtain simplicity it suffices to use Theorem 1.15, which states that under a expansiveness hypothesis of the action, we can find a simple finitely generated subgroup of the topological full group. Then by a technique of N. Matte Bon illustrated in Theorem 5.3 it will also be amenable. We have thus constructed simple amenable groups of Burnside type by means of the dynamics of an appropriate action on a Cantor set and the use of topological full groups.

Finally, it is interesting to look at specific groups where we can apply the proven theorems. Examples of Theorem 5.1 are not hard to visualize. Among them we find the already well-known Grigorchuk group, but also some new families as are some groups of piecewise isometries of polygons, as explained in Example 5.5, as well as some substitutive (palindromic) subshifts, which are explained in Nekrashevych's original publication [Nek18]. On the other hand, looking for explicit examples of Theorem 5.4 may be a little harder, since passing to the alternating subgroup $\mathcal{A}(G, X)$ of the topological full group makes the visualization of the obtained group more abstract. An outstanding example is given in Nekrashevych's publication [Nek18], where it is shown a group F which is a fragmentation of a dihedral action and whose associated alternating subgroup coincides with the derived subgroup $[F, F]$.

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