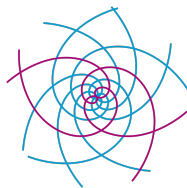# Continuity of asymptotic entropy on groups

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**MM**
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# Harmonic functions and the Poisson boundary

$G$ a countable group, $\mu$ a probability measure on $G$.

A function $f : G \to \mathbb{R}$ is called $\mu$-harmonic if $f(g) = \sum_{h \in G} f(gh)\mu(h)$ for all $g \in G$.

$$H^\infty(G, \mu) := \{f : G \to \mathbb{R} \mid f \text{ bounded and } \mu\text{-harmonic}\}$$

The Poisson boundary $(B, \nu)$ of $(G, \mu)$ is a probability space endowed with a $G$-action such that $\nu = \mu * \nu$ (i.e., $\nu$ is $\mu$-stationary) and

$$H^\infty(G, \mu) \cong L^\infty(B, \nu).$$

**Example 1:** $G = F_2$ **the free group of rank** $2$**.**

For any (non-degenerate) $\mu \in \mathrm{Prob}(F_2)$ there is a unique $\mu$-stationary prob. measure $\nu$ on $\partial F_2$.

If $H(\mu) = -\sum_{g \in F_2} \mu(g) \log \mu(g) < \infty$, then $(\partial F_2, \nu)$ is the Poisson boundary of $(G, \mu)$. [Dynkin-Maljutov '61, Derriennic '75, Ancona '87, Kaimanovich '94, Chawla-Forghani-Frisch-Tiozzo '22]

(true more generally for hyperbolic groups)

## Examples of Poisson boundaries

**Example 2:** $G = \mathbb{Z}/2\mathbb{Z} \wr B$ **the wreath product with** $\mathbb{Z}/2\mathbb{Z}$ **lamps and base group** $B$**.**

$$\mathbb{Z}/2\mathbb{Z} \wr B := \bigoplus_B \mathbb{Z}/2\mathbb{Z} \rtimes B.$$

Consider $\mu \in \mathrm{Prob}(\mathbb{Z}/2\mathbb{Z} \wr B)$ non-degenerate with $H(\mu) < \infty$.

Kaimanovich-Vershik '83, Erschler '04: the Poisson boundary of $(\mathbb{Z}/2\mathbb{Z} \wr B, \mu)$ is non-trivial if and only if the projection to $B$ is transient.

Under general hypotheses on $\mu$ (e.g., a finite first moment), there is a $\mu$-stationary prob. measure $\nu$ on $\prod_B \mathbb{Z}/2\mathbb{Z}$ such that $\left(\prod_B \mathbb{Z}/2\mathbb{Z}, \nu\right)$ is the Poisson boundary of $(\mathbb{Z}/2\mathbb{Z} \wr B, \mu)$. [Kaimanovich '00, Erschler '11, Lyons-Peres '21, Frisch-S. '24]

⚠ In general there will be multiple $\mu$-stationary measures (e.g., if $B$ is amenable).

Consider $G$ a countable group and $\mu \in \mathrm{Prob}(G)$ non-degenerate with $H(\mu) = -\sum_{g \in G} \mu(g) \log \mu(g) < \infty$.

The asymptotic entropy of $(G, \mu)$ is defined as

$$h(\mu) := \lim_{n \to \infty} \frac{H(\mu^{*n})}{n}.$$

**The entropy criterion** [Avez '72, Derriennic '81, Kaimanovich-Vershik '83]

The Poisson boundary of $(G, \mu)$ is non-trivial $\iff h(\mu) > 0$.

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**Question:** Fix $G$ and consider $\mu, \{\mu_k\}_{k \geq 1} \in \mathrm{Prob}(G)$ with finite entropy such that:

- $\lim_{k \to \infty} \mu_k(g) = \mu(g)$ for each $g \in G$ and
- $\lim_{k \to \infty} H(\mu_k) = H(\mu)$.

Is it true that $\lim_{k \to \infty} h(\mu_k) = h(\mu)$?

## Discontinuity of asymptotic entropy

**Example:** $\mathrm{FSym}(\mathbb{N}) := \{\sigma : \mathbb{N} \to \mathbb{N} \mid \sigma \text{ bijective and finitely supported}\}$

- Kaimanovich '83: there are infinitely supported probability measures $\mu$ on $\mathrm{FSym}(\mathbb{N})$ with $H(\mu) < \infty$ and $h(\mu) > 0$.

- Any finitely supported probability measure $\mu$ on $\mathrm{FSym}(\mathbb{N})$ has $h(\mu) = 0$.

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- Any finitely supported probability measure $\mu$ on $\mathrm{FSym}(\mathbb{N})$ has $h(\mu) = 0$.

**Example:** $\mathbb{Z}/2\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z})$

- Gilch '08: any non-degenerate finitely supported prob. measure on $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is recurrent.
- $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ has $\mathbb{Z}$ as a finite index subgroup (that admits finitely supported transient probability measures).
- There are $\mu, \{\mu_k\}_{k\geq 1} \in \mathrm{Prob}(\mathbb{Z}/2\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}))$ finitely supported such that $h(\mu) > 0$ and $h(\mu_k) = 0$ for all $k \geq 1$, with $\mu_k \xrightarrow[k\to\infty]{} \mu$.

**Theorem [S. '25]**

Let $B$ be a virtually nilpotent group of at least cubic growth (e.g. $B = \mathbb{Z}^d$, $d \geq 3$). For any non-degenerate probability measures $\mu, \{\mu_k\}_{k \geq 1}$ on $\mathbb{Z}/2\mathbb{Z} \wr B$ such that

- $\mu_k(g) \xrightarrow[k \to \infty]{} \mu(g)$ for all $g \in \mathbb{Z}/2\mathbb{Z} \wr B$ and
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we have $h(\mu_k) \xrightarrow[k \to \infty]{} h(\mu)$.

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**Intuition**: Entropy at time $n$ on $\mathbb{Z}/2\mathbb{Z} \wr B \longleftrightarrow$ number of distinct points visited by the projection to the base $B$ up to time $n$.

**Definition**

Let $\mu$ be a prob. measure on a countable group $G$. Denote by $(X_n)_{n \geq 0}$ the $\mu$-random walk on $G$. We define the escape probability

$$p_{\mathrm{esc}}(\mu) := \mathbb{P}(X_n \neq e_G \text{ for all } n \geq 1).$$

It holds almost surely that $\displaystyle\lim_{n \to \infty} \frac{\#|\{X_0, X_1, \ldots, X_n\}|}{n} = p_{\mathrm{esc}}(\mu)$.

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Uses (a uniform version of) a comparison lemma of heat kernels of symmetric and non-symmetric Markov operators due to Coulhon and Saloff-Coste (late 80's).

**Theorem [S. '25]**

Let $G$ be a countable group and consider non-degenerate prob. measures $\mu$, $\{\mu_k\}_{k \geq 1}$ on $G$ with finite entropy and such that $\mu_k \xrightarrow[k \to \infty]{} \mu$. Suppose that there is a Polish space $X$ such that

► there are prob. measures $\nu$ (resp. $\nu_k$) on $X$ such that $(X, \nu)$ (resp. $(X, \nu_k)$) is the Poisson boundary of $(G, \mu)$ (resp. $(G, \mu_k)$).

If $\nu_k \xrightarrow[k \to \infty]{} \nu$ weakly, then $h(\mu_k) \xrightarrow[k \to \infty]{} h(\mu)$.

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**Starting point:** Kaimanovich-Vershik '83:

$$h(\mu) = -\sum_{g\in G} \mu(g) \int_X \log\left(\frac{dg_*^{-1}\nu}{d\nu}(\xi)\right) d\nu(\xi).$$

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If $X$ is furthermore compact and admits a unique $\mu$-stationary prob. measure, then the weak convergence $\nu_k \xrightarrow[k \to \infty]{} \nu$ always holds.

## Uniqueness of stationary measures on Poisson boundaries

**Applications**:

- ▶ Hyperbolic groups (Previously proved with different methods by Erschler-Kaimanovich '13, Gouëzel-Mathéus-Maucourant '18, Choi '24).
- ▶ Acylindrically hyperbolic groups [Choi '24] (e.g. mapping class groups, $\mathrm{Out}(F_n)$, etc...)
- ▶ $\mathrm{SL}_d(\mathbb{Z})$ for $d \geq 3$.
- ▶ Many groups acting on $\mathrm{CAT}(0)$-spaces.