Thompson's group T, groups acting on the circle and Poisson boundaries

based on joint work with **Martín Gilabert Vio** and **Cosmas Kravaris**

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Harmonic functions on groups

Let G be a countable group and let μ be a probability measure on G. A function $f : G \to \mathbb{R}$ is called **μ-harmonic** if $f(g) = \sum_{h \in G} f(gh)\mu(h)$ for all $g \in G$. The **Poisson boundary** (B, ν) of (G, μ) is a probability G-space such that ν **is μ-stationary** (i.e. $\nu = \mu * \nu$), and provides an isomorphism of Banach algebras

$$L^{\infty}(B, \nu) \to \{f : G \to \mathbb{R} \mid f \text{ bounded and } \mu\text{-harmonic}\}$$

$$F \mapsto \left(f(g) = \int_B F(gx)d\nu(x), \text{ for } g \in G\right)$$

Problem: Describe the Poisson boundary in terms of the geometric properties of G.

One can often identify **G-equivariant quotients of** (B, ν), called **μ-boundaries**.

Knowing a μ-boundary corresponds to finding a subspace of bounded μ-harmonic functions. Saying that it is the Poisson boundary means that there are none of them missing.

Examples of Poisson boundaries

Gromov-hyperbolic groups. Let G be a non-elementary Gromov hyperbolic group, and denote by ∂G its Gromov boundary. Then for any non-elementary $\mu \in \mathrm{Prob}(G)$ there is a unique μ-stationary probability measure ν on ∂G. If $H(\mu) < \infty$, then $(\partial G, \nu)$ is the Poisson boundary of (G, μ) (Kaimanovich '94, Chawla-Forghani-Frisch-Tiozzo '22, and many more).

Wreath products. Consider the lamplighter groups $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d := \left(\bigoplus_{\mathbb{Z}^d} \mathbb{Z}/2\mathbb{Z}\right) \rtimes \mathbb{Z}^d$, $d \geq 3$. Let $\mu \in \mathrm{Prob}(\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d)$ be a non-degenerate finitely supported probability measure. Then there is a μ-stationary probability measure ν on $\prod_{\mathbb{Z}^d} \mathbb{Z}/2\mathbb{Z}$ such that $(\prod_{\mathbb{Z}^d} \mathbb{Z}/2\mathbb{Z}, \nu)$ is the Poisson boundary of $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$ (Erschler '11, Lyons-Peres '21).

Groups of homeomorphisms of the circle

The action $G \curvearrowright S^1$ is called **proximal** if for every proper interval $I \subset S^1$ and every $\epsilon > 0$ there is $g \in G$ with $\mathrm{diam}(g(I)) < \epsilon$.

Theorem [Deroin-Kleptsyn-Navas '07]

Let $G \curvearrowright S^1$ by orientation-preserving homeomorphisms with no invariant probability measure on S^1, and let $\mu \in \mathrm{Prob}(G)$ be non-degenerate. Suppose that $G \curvearrowright S^1$ is proximal. Then there is a **unique μ-stationary probability measure** ν on S^1 and (S^1, ν) is a μ-boundary of G.

The proof goes as follows: one shows that for almost every sample path $\mathbf{w} = (w_n)_{n \geq 0} \in G^{\mathbb{N}}$ of the μ-random walk on G there exists a point $\xi(\mathbf{w}) \in S^1$ such that $\lim_{n\to\infty}(w_n)_*\nu = \delta_{\xi(\mathbf{w})}$ in the weak-$*$ topology. The measure ν is the distribution of $\xi(\mathbf{w})$ on S^1.

Theorem [Deroin '13]

Suppose furthermore that If the action of $G \curvearrowright S^1$ is **strongly discrete** and sufficiently regular, and that μ is finitely supported. Then (S^1, ν) is the Poisson boundary of (G, μ).

This is satisfied in particular by cocompact lattice in $\mathrm{PSL}_2(\mathbb{R})$. The groups covered by the above result fall within a family that is conjectured to be composed only of Gromov-hyperbolic groups, and hence their Poisson boundaries could alternatively be described using their Gromov boundaries.

Question [Deroin '13, Navas '17]: Is (S^1, ν) always the Poisson boundary of (G, μ)?

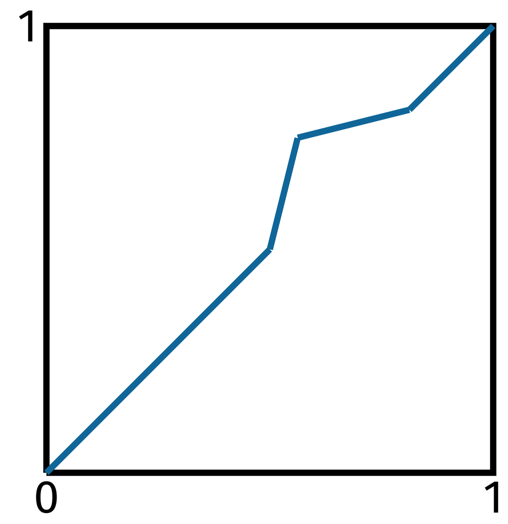
Main Theorem [Gilabert - Kravaris - S. '25]

Let $G \leq \mathrm{Homeo}_+(S^1)$ be a countable group acting proximally, minimally and topologically non-freely on S^1. Let μ be a non-degenerate probability measure on G with $-\sum_{g\in G}\mu(g)\log(\mu(g)) < \infty$. Then (S^1, ν) is not the Poisson boundary of (G, μ).

The proof for Thompson's group T and finitely supported μ

- Thompson's group T is the group of **dyadic piecewise affine homeomorphisms of the circle**: that is, T is the group of orientation-preserving homeomorphisms $g : S^1 \to S^1$ such that the derivative of g is defined outside a finite subset of the dyadic rationals $\mathbb{Z}[1/2]/\mathbb{Z}$ and takes values in $\{2^k\}_{k\in\mathbb{Z}}$.
- For each $g \in T$, define a finitely supported function $C_g \colon \mathbb{Z}[1/2]/\mathbb{Z} \to \mathbb{R}$ by setting
$$C_g(x) = \log_2\left((g^{-1})'(x^+)\right) - \log_2\left((g^{-1})'(x^-)\right), \text{ for } x \in \mathbb{Z}[1/2]/\mathbb{Z},$$
where $(g^{-1})'(x^+)$ (resp. $(g^{-1})'(x^-)$ is the left (resp. right) derivative of g^{-1} at x. That is, $C_g(x)$ is the derivative jump of g^{-1} at x.
- Denote the set of all (not necessarily finitely supported) functions $\mathbb{Z}[1/2]/\mathbb{Z} \to \mathbb{R}$ by $\mathbb{R}^{\mathbb{Z}[1/2]/\mathbb{Z}}$.
- For almost every trajectory $\mathbf{w} = (w_n)_n$ $in T^{\mathbb{N}}$ of the μ-random walk, the configurations $(C_{w_n})_{n\geq 0}$ converge pointwise to a map $C_\infty(\mathbf{w}) \in \mathbb{R}^{\mathbb{Z}[1/2]/\mathbb{Z}}$. The hitting measure λ is a μ-stationary prob. measure on $\mathbb{Z}^{\mathbb{Z}[1/2]/\mathbb{Z}}$ such that **the space** $(\mathbb{Z}^{\mathbb{Z}[1/2]/\mathbb{Z}}, \lambda)$ **is a μ-boundary of** T.
- Since the measure λ is nontrivial, there exists $y \in \mathbb{Z}[1/2]/\mathbb{Z}$ and $k \in \mathbb{Z}$ such that $f \colon G \to [0,1]$ defined by
$$f(g) = \mathbb{P}_g\left[\mathbf{w} \in G^{\mathbb{N}} \mid C_\infty(\mathbf{w})(y) = k\right], \text{ for } g \in G$$
satisfies $f(e_T) > 0$. The function f is bounded and μ-harmonic.
- There exists a sequence $\{g_n\}_{n\geq 0} \subseteq T$ such that $\mathrm{supp}(g_n)$ are closed intervals containing y and such that $\mathrm{diam}(\mathrm{supp}(g_n)) \xrightarrow[n\to\infty]{} 0$ and $f(g_n) \xrightarrow[n\to\infty]{} 0$.

  Indeed, one constructs a sequence such that:
  - $g_n(y) = y$,
  - $\mathrm{supp}(g_n)$ is a dyadic interval containing y and of length $2^{-n} + 2^{-2n}$, and
  - the derivative jump of g_n at y is equal to 2^n.



  Clearly $\mathrm{diam}(\mathrm{supp}(g_n)) \xrightarrow[n\to\infty]{} 0$. Moreover, if $g \in T$ fixes y we have
$$f(g) = \mathbb{P}_g\left[C_\infty(\mathbf{w})(y) = k\right] = \mathbb{P}\left[C_\infty(\mathbf{w})(y) = k - \log_2(g')^+(y) + \log_2(g')^-(y)\right]$$
  so that in particular $f(g_n) = \mathbb{P}\left[C_\infty(\mathbf{w})(y) = k - n\right] \xrightarrow[n\to\infty]{} 0.$
- If (S^1, ν) were the Poisson boundary of (T, μ), then there would exist $h \in L^\infty(S^1, \nu)$ such that
$$f(g) = \int_{S^1} h(gx)d\nu(x), \text{ for all } g \in G.$$
  Set $I_n = \mathrm{supp}(g_n)$ for each $n \geq 1$. The equality
$$f(g_n) = \int_{S^1} h(g_n x)d\nu(x) = \int_{S^1\setminus I_n} h(g_n x)d\nu(x) + \int_{I_n} h(g_n x)d\nu(x)$$
  and the fact that ν is non-atomic imply that $\int_{S^1\setminus I_n} h(x)d\nu(x) = \int_{S_1\setminus I_n} h(g_n x)d\nu(x) \xrightarrow[n\to\infty]{} 0$. This would imply that $f(e_T) = \int_{S^1} h(x)d\nu(x) = 0$, which is a contradiction.

⚠This approach only works for groups of piecewise affine transformations of S^1. The general proof is based on conditional entropy techniques (cf. Kaimanovich, Erschler).

Our main theorem is related to the well-known open problem on whether Thompson's group F, the group of dyadic piecewise affine homeomorphisms of the interval $[0, 1]$, **is amenable**. Indeed, the action of a countable group G on its Poisson boundary $(\partial_\mu G, \nu)$ is amenable, and hence for ν-almost every $x \in \partial_\mu G$ the stabilizer subgroup $G_x \leq G$ is amenable. If the circle were the Poisson boundary of T then we would conclude that F is amenable, since for each $x \in S^1$ the stabilizer $T_x \leq T$ contains a copy of F. Our theorem implies that this strategy does not work for μ with finite entropy.

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