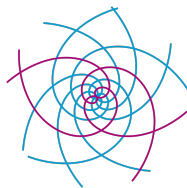


# Continuity of asymptotic entropy on groups

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**MM**  
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$G$  a countable group,  $\mu$  a probability measure on  $G$ .

A function  $f : G \rightarrow \mathbb{R}$  is called  **$\mu$ -harmonic** if  $f(g) = \sum_{h \in G} f(gh)\mu(h)$  for all  $g \in G$ .

$$H^\infty(G, \mu) := \{f : G \rightarrow \mathbb{R} \mid f \text{ bounded and } \mu\text{-harmonic}\}$$

The **Poisson boundary**  $(B, \nu)$  of  $(G, \mu)$  is a probability space endowed with a  $G$ -action such that  $\nu = \mu * \nu$  (i.e.,  $\nu$  is  **$\mu$ -stationary**) and

$$H^\infty(G, \mu) \cong L^\infty(B, \nu).$$

**Example 1:  $G = F_2$  the free group of rank 2.**

For any (non-degenerate)  $\mu \in \text{Prob}(F_2)$  there is a unique  $\mu$ -stationary prob. measure  $\nu$  on  $\partial F_2$ .

If  $H(\mu) = -\sum_{g \in F_2} \mu(g) \log \mu(g) < \infty$ , then  $(\partial F_2, \nu)$  is the Poisson boundary of  $(G, \mu)$ . [Dynkin-Maljutov '61, Derriennic '75, Ancona '87, Kaimanovich '94, Chawla-Forghani-Frisch-Tiozzo '22]

(true more generally for hyperbolic groups)

**Example 2:**  $G = \mathbb{Z}/2\mathbb{Z} \wr B$  the wreath product with  $\mathbb{Z}/2\mathbb{Z}$  lamps and base group  $B$ .

$$\mathbb{Z}/2\mathbb{Z} \wr B := \bigoplus_B \mathbb{Z}/2\mathbb{Z} \rtimes B.$$

where  $\bigoplus_B \mathbb{Z}/2\mathbb{Z} = \{f : B \rightarrow \mathbb{Z}/2\mathbb{Z} \mid f \text{ with finite support}\}$  and  $B$  acts on  $f \in \bigoplus_B \mathbb{Z}/2\mathbb{Z}$  by:

$$(b \cdot f)(x) = f(b^{-1}x), \quad x, b \in B.$$

### Lamplighter interpretation

Multiplying an element  $(f, x)$  on the right by elements of  $\mathbb{Z}/2\mathbb{Z}$  changes the **lamp configuration**  $f$  at the **current position**  $x$ , while multiplying by elements of  $B$  changes said current position.

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$$\mathbb{Z}/2\mathbb{Z} \wr B := \bigoplus_B \mathbb{Z}/2\mathbb{Z} \rtimes B.$$

Consider  $\mu \in \text{Prob}(\mathbb{Z}/2\mathbb{Z} \wr B)$  non-degenerate with  $H(\mu) < \infty$ .

Kaimanovich-Vershik '83, Erschler '04: the Poisson boundary of  $(\mathbb{Z}/2\mathbb{Z} \wr B, \mu)$  is non-trivial if and only if the projection to  $B$  is transient.

Under general hypotheses on  $\mu$  (e.g., a finite first moment), there is a  $\mu$ -stationary prob. measure  $\nu$  on  $\prod_B \mathbb{Z}/2\mathbb{Z}$  such that  $(\prod_B \mathbb{Z}/2\mathbb{Z}, \nu)$  is the Poisson boundary of  $(\mathbb{Z}/2\mathbb{Z} \wr B, \mu)$ . [Kaimanovich '00, Erschler '11, Lyons-Peres '21, Frisch-S. '24]

⚠ In general there will be multiple  $\mu$ -stationary measures (e.g., if  $B$  is amenable).

Consider  $G$  a countable group and  $\mu \in \text{Prob}(G)$  non-degenerate with  $H(\mu) = -\sum_{g \in G} \mu(g) \log \mu(g) < \infty$ .

The **asymptotic entropy** of  $(G, \mu)$  is defined as

$$h(\mu) := \lim_{n \rightarrow \infty} \frac{H(\mu^{*n})}{n}.$$

**The entropy criterion [Avez '72, Derriennic '81, Kaimanovich-Vershik '83]**

The Poisson boundary of  $(G, \mu)$  is non-trivial  $\iff h(\mu) > 0$ .

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**The entropy criterion [Avez '72, Derriennic '81, Kaimanovich-Vershik '83]**

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**Question:** Fix  $G$  and consider  $\mu, \{\mu_k\}_{k \geq 1} \in \text{Prob}(G)$  with finite entropy such that:

- ▶  $\lim_{k \rightarrow \infty} \mu_k(g) = \mu(g)$  for each  $g \in G$  and
- ▶  $\lim_{k \rightarrow \infty} H(\mu_k) = H(\mu)$ .

Is it true that  $\lim_{k \rightarrow \infty} h(\mu_k) = h(\mu)$ ?

**Example:**  $\text{FSym}(\mathbb{N}) := \{\sigma : \mathbb{N} \rightarrow \mathbb{N} \mid \sigma \text{ bijective and finitely supported}\}$

- ▶ Kaimanovich '83: there are infinitely supported probability measures  $\mu$  on  $\text{FSym}(\mathbb{N})$  with  $H(\mu) < \infty$  and  $h(\mu) > 0$ .
- ▶ Any finitely supported probability measure  $\mu$  on  $\text{FSym}(\mathbb{N})$  has  $h(\mu) = 0$ .



## Discontinuity of asymptotic entropy

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**Example:**  $\mathbb{Z}/2\mathbb{Z} \wr D_\infty$ , where  $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$

- ▶ Gilch '08: any non-degenerate finitely supported prob. measure on  $D_\infty$  is recurrent.
- ▶  $D_\infty$  has  $\mathbb{Z}$  as a finite index subgroup (that admits finitely supported transient probability measures).
- ▶ There are  $\mu, \{\mu_k\}_{k \geq 1} \in \text{Prob}(\mathbb{Z}/2\mathbb{Z} \wr D_\infty)$  finitely supported such that  $h(\mu) > 0$  and  $h(\mu_k) = 0$  for all  $k \geq 1$ , with  $\mu_k \xrightarrow[k \rightarrow \infty]{} \mu$ .

### Theorem [S. '25]

Let  $B$  be a virtually nilpotent group of at least **cubic growth** (e.g.  $B = \mathbb{Z}^d$ ,  $d \geq 3$ ). For any non-degenerate probability measures  $\mu, \{\mu_k\}_{k \geq 1}$  on  $\mathbb{Z}/2\mathbb{Z} \wr B$  such that

- ▶  $\mu_k(g) \xrightarrow[k \rightarrow \infty]{} \mu(g)$  for all  $g \in \mathbb{Z}/2\mathbb{Z} \wr B$  and
- ▶  $H(\mu_k) \xrightarrow[k \rightarrow \infty]{} H(\mu)$

we have  $h(\mu_k) \xrightarrow[k \rightarrow \infty]{} h(\mu)$ .

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**Intuition:** Entropy at time  $n$  on  $\mathbb{Z}/2\mathbb{Z} \wr B \longleftrightarrow$  number of distinct points visited by the projection to the base  $B$  up to time  $n$ .

### Definition

Let  $\mu$  be a prob. measure on a countable group  $G$ . Denote by  $(X_n)_{n \geq 0}$  the  $\mu$ -random walk on  $G$ . We define the **escape probability**

$$p_{\text{esc}}(\mu) := \mathbb{P}(X_n \neq e_G \text{ for all } n \geq 1).$$

It holds almost surely that  $\lim_{n \rightarrow \infty} \frac{\#\{X_0, X_1, \dots, X_n\}}{n} = p_{\text{esc}}(\mu)$ .

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Uses (a uniform version of) a comparison lemma of heat kernels of symmetric and non-symmetric Markov operators due to Coulhon and Saloff-Coste (late 80's).

### Theorem [S. '25]

Let  $G$  be a countable group and consider non-degenerate prob. measures  $\mu, \{\mu_k\}_{k \geq 1}$  on  $G$  with finite entropy and such that  $\mu_k \xrightarrow[k \rightarrow \infty]{} \mu$ . Suppose that there is a Polish space  $X$  such that

- ▶ there are prob. measures  $\nu$  (resp.  $\nu_k$ ) on  $X$  such that  $(X, \nu)$  (resp.  $(X, \nu_k)$ ) is the Poisson boundary of  $(G, \mu)$  (resp.  $(G, \mu_k)$ ).

If  $\nu_k \xrightarrow[k \rightarrow \infty]{} \nu$  weakly, then  $h(\mu_k) \xrightarrow[k \rightarrow \infty]{} h(\mu)$ .

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**Starting point:** Kaimanovich-Vershik '83:

$$h(\mu) = - \sum_{g \in G} \mu(g) \int_X \log \left( \frac{dg_*^{-1} \nu}{d\nu}(\xi) \right) d\nu(\xi).$$

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If  $X$  is furthermore **compact and admits a unique  $\mu$ -stationary prob. measure**, then the weak convergence  $\nu_k \xrightarrow[k \rightarrow \infty]{} \nu$  always holds.



## Applications:

- ▶ Hyperbolic groups (Previously proved with different methods by Erschler-Kaimanovich '13, Gouëzel-Mathéus-Maucourant '18, Choi '24).
- ▶ Acylindrically hyperbolic groups [Choi '24] (e.g. mapping class groups,  $\text{Out}(F_n)$ , etc...)
- ▶  $\text{SL}_d(\mathbb{Z})$  for  $d \geq 3$ .
- ▶ Many groups acting on  $\text{CAT}(0)$ -spaces.