UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

SUBSHIFTS EN LOS GRUPOS DE BAUMSLAG-SOLITAR SOLUBLES NO-ABELIANOS

TESIS PARA OPTAR AL GRADO DE MAGÍSTER EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MATEMÁTICAS APLICADAS

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

EDUARDO ALEJANDRO SILVA MÜLLER

PROFESOR GUÍA:
MICHAEL HEINRICH SCHRAUDNER

MIEMBROS DE LA COMISIÓN:
NATHALIE AUBRUN
SEBASTIÁN DONOSO FUENTES
ALEJANDRO MAASS SEPÚLVEDA

Este trabajo ha sido parcialmente financiado por CONICYT-PFCHA/Magíster Nacional/2019 - 22190176 y CMM Conicyt PIA AFB170001

SANTIAGO DE CHILE
2020

RESUMEN DE LA MEMORIA PARA OPTAR
AL TÍTULO DE MAGÍSTER EN CIENCIAS DE LA INGENIERÍA,
MENCION MATEMÁTICAS APLICADAS
Y AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO
POR: EDUARDO ALEJANDRO SILVA MÜLLER
FECHA: 2020
PROF. GUÍA: MICHAEL HEINRICH SCHRAUDNER

SUBSHIFTS EN LOS GRUPOS DE BAUMSLAG-SOLITAR SOLUBLES NO-ABELIANOS

En este trabajo es de interés el estudio de la dinámica simbólica sobre los grupos de Baumslag-Solitar solubles no-abelianos $\mathrm{BS}(1,N)$, $N \geq 2$, a través de la comprensión de cómo la estructura de dichos grupos fuerza comportamientos sobre las configuraciones del $\mathrm{BS}(1,N)$-full-shift, y el aprovechamiento de construcciones utilizando la geometría del grupo. Se definen sustituciones sobre los grupos de Baumslag-Solitar, logrando dar para una subclase de subshifts sustitutivos una versión del teorema de Mozes [17], que establece condiciones suficientes bajo las cuales éstos son sóficos. Luego se estudia el caso de una familia de subshifts en particular: los llamados "Graph-coloring subshifts", cuyo origen proviene del concepto de coloreos propios en teoría de grafos. Para esta familia se estudia su no-vacuidad, extensibilidad de patrones, cotas sobre su entropía y tipos de mezcla. Posteriormente se estudia una clase de $\mathrm{BS}(1,N)$-subshifts definidos a partir de $\mathbb{Z}$-subshifts, obteniendo resultados sobre la relación entre las entropías de ambos sistemas, y aproximabilidad de la entropía del $\mathrm{BS}(1,N)$-subshift a través de las entropías de $\mathbb{Z}$-subshifts. Finalmente se estudia la subdinámica proyectiva de $\mathrm{BS}(1,N)$-SFTs, basándose en el trabajo de R. Pavlov y M. Schraudner [18], y dando condiciones para la realización de $\mathbb{Z}$-subshifts sóficos como subdinámica proyectiva de un $\mathrm{BS}(1,N)$-SFT.

SUBSHIFTS ON SOLVABLE NON-ABELIAN BAUMSLAG-SOLITAR GROUPS

The objective of this work is to study symbolic dynamics on the solvable non-abelian Baumslag-Solitar groups $\mathrm{BS}(1,N)$, $N \geq 2$, through the comprehension of how the structure of said groups forces some particular behavior of configuration of the $\mathrm{BS}(1,N)$-full-shift, as well as how to take advantage of the geometric structure of the groups in order to make constructions. We define substitutions on Baumslag-Solitar groups, and give a version of Mozes theorem [17], stating sufficient conditions under which a certain subclass of substitutive subshifts are sofic. Then, we study a particular family of subshifts, called the "Graph-coloring subshifts" which are motivated by the concept of a proper coloring in graph theory. For this family we study non-emptiness, extensibility of patterns, bounds on their entropy and mixing properties. Next we study a class of $\mathrm{BS}(1,N)$-subshifts obtained from $\mathbb{Z}$-subshifts, obtaining results relating the entropies of both systems, and the approximability of the $\mathrm{BS}(1,N)$-subshift's entropy through the entropies of a corresponding family of $\mathbb{Z}$-subshifts, called the m-strip subshifts. Finally, we study the projective subdynamics of $\mathrm{BS}(1,N)$-SFTs, based on the work of R. Pavlov and M. Schraudner [18], and giving conditions for the realization of sofic $\mathbb{Z}$-subshifts as projective subdynamics of a $\mathrm{BS}(1,N)$-SFT.

"As long as there are slaughter houses there will always be battlefields"
- Leo Tolstoy

Agradecimientos

En primer lugar, quisiera agradecerle a mis padres, Paulina y Fernando, por el constante apoyo que me han dado, así como a mi hermana Paulina, y mis hermanos Fernando y Rodolfo. Estoy feliz de haber nacido en esta familia, gracias a la cual he superado las barreras que se me han impuesto y logrado superarme continuamente.

Agradezco a mis amigos de la generación, con quienes logramos grandes lazos a lo largo de estos años y nos apoyamos mutuamente en todo momento. A Julio, Diego, Juan Pedro, Freddy, Papitas, Pablo, Maika, Alonso, Bitar y Basti les doy mi más sincera gratitud por todos los buenos recuerdos y sentimientos que dejan en mí. Especial agradecimiento a Bitar y Basti, que siendo mis compañeros de oficina durante el Magíster tuvimos siempre un muy buen ambiente de buena onda y apoyo. Al resto de la gente del DIM que ha estado a mi lado, Nico, Cabezas, Bob, Coba, Rodolfo, Ian, Mariscal, gracias por todo lo que me enseñaron, por los consejos que me dieron y por la amistad durante estos años. Muchas gracias también a la gente que me ha acompañado fuera del DIM: a Bruno, con quien hemos sido grandes amigos desde el primer día que dimos pie en la universidad y de quien siempre aprendo algo nuevo, y a Paola, que es de las amigas más antiguas que tengo y en quien sé que siempre podré confiar.

Quiero también agradecer a los profesores de la carrera, pues es gracias a su apoyo y vocación que estoy aquí. Gracias a Aris Daniilidis, Jaime San Martin, Martin Matamala, Joaquin Fontbona, Alejandro Maass y Michael Schraudner, siendo estos últimos dos profesores quienes me introdujeron al bello mundo de los sistemas dinámicos. Agradezco a Michael por sus útiles consejos y apoyo durante el desarrollo de mi tesis, gracias a los cual aprendí mucho y pudimos llevar a cabo este proceso de una manera amena.

Finalmente quiero agradecer a Natacha por las buenas conversaciones que tuvimos y toda la ayuda que me dio durante la carrera, así como a Karen, Eterin y Óscar que siempre han tenido la mejor disposición del mundo para ayudar, y que gracias a ellos el departamento es lo que es.

Contents

Introduction

Symbolic dynamics over finitely generated groups is a particular topic within the field of dynamical systems. The main object studied in this discipline are **G-subshifts** (or just subshifts if there is no risk of confusion with the group G), that is, closed subsets of the product space $\mathcal{A}^G$, for $\mathcal{A}$ a finite discrete alphabet and G a finitely generated group, which are invariant under the action of the **shift**:

$$\begin{aligned}\sigma : G &\to \mathrm{Homeo}(\mathcal{A}^G, \mathcal{A}^G)\\ g &\mapsto \sigma_g,\end{aligned}$$

where $\mathrm{Homeo}(\mathcal{A}^G, \mathcal{A}^G)$ are the homeomorphisms from $\mathcal{A}^G$ to itself, and for every $x \in \mathcal{A}^G,\ g, h \in G$ we have

$$\sigma_g(x)_h = x_{g^{-1}h}.$$

An important family of subshifts are **subshifts of finite type (SFT)**, which are the ones that can be described completely by a finite set of forbidden patterns. The class of SFTs is interesting since sometimes restricting the attention to it can make it possible to answer a question which in the general case of subshifts is more difficult, or in contrast, a question which is trivial formulated for subshifts becomes complicated when restricting it to SFTs. An example of the latter is the question of whether there exist non-empty strongly aperiodic subshifts (resp. SFTs) on a two symbol alphabet, that is, a subshift (resp. SFT) in which no configuration exhibits a periodic behavior. For general subshifts this question has an affirmative answer, meanwhile restricting the attention to SFTs is more complicated and the answer depends on the group G considered.

The classic theory of symbolic dynamics is the case where $G = \mathbb{Z}$, and it stands out for the large amount of results one can obtain in this context (see [11] and [5]). A remarkable result about $\mathbb{Z}$-SFTs says that all of these systems are conjugated to, and then can be understood through, their presentation as edge-shifts, which are a special case of $\mathbb{Z}$-SFTs described completely by a (finite) directed graph. As a graph is characterized by its adjacency matrix, this implies that we can use the powerful tools of linear algebra in order to understand SFTs over $\mathbb{Z}$.

Throughout the last decades there has been interest in the case $G = \mathbb{Z}^d$, $d \geq 2$, where a more complex structure arises in comparison with the one-dimensional case: questions which have been long answered in the case $d = 1$ become significantly more difficult for $d \geq 2$, sometimes changing its veracity, or remaining as open problems to this day. An example of this contrast is the so called Domino problem, which asks whether, given a finite set of forbidden patterns, the corresponding SFT is empty or not. For $d = 1$ this problem is decidable, meanwhile for $d \geq 2$ it is undecidable.

In recent years interest has grown in studying symbolic dynamics on other finitely generated groups, obtaining results in a general context (e.g. in [3]) as well as in more particular settings, such as free groups or Baumslag-Solitar groups. The latter of these two is precisely the family of groups with which we work in this thesis, and is described below.

Given $m, n \in \mathbb{Z}\backslash\{0\}$, we define the corresponding **Baumslag-Solitar group** as given by its standard presentation

$$\mathrm{BS}(m,n) \coloneqq \langle a, b \mid ba^m b^{-1} = a^n \rangle.$$

These groups arise naturally as HNN-extensions of $\mathbb{Z}$, and were first introduced in [4] by G. Baumslag and D. Solitar, to provide an example of a finitely presented non-Hopfian group, namely, one which is isomorphic to one of its (proper) quotient groups. Since then they have served as useful examples and counterexamples for a wide variety of properties, both in the context of group theory as well as in the context of symbolic dynamics. An example of the latter can be found in [2], where a weakly aperiodic $\mathrm{BS}(m,n)$-subshift is constructed and using this example it is shown that the Domino problem is in fact undecidable in Baumslag-Solitar groups, that is, there does not exist an algorithm which receives as input a finite set of forbidden patterns $\mathcal{F} \subseteq \bigcup_{\substack{F \subseteq \mathrm{BS}(m,n) \\ F \text{ finite}}} \mathcal{A}^F$ and determines whether the corresponding $\mathrm{BS}(m,n)$-SFT $\mathsf{X}_{\mathcal{F}}$ is non-empty.

The subfamily of Baumslag-Solitar groups $\mathrm{BS}(1,N)$, $N \geq 2$, is of particular interest, since it exhibits nice properties. This subfamily actually comprises all the cases for which the Baumslag-Solitar group is solvable, and hence amenable, while still being non-abelian (the case $N = 1$ gives $\mathrm{BS}(1,1) \cong \mathbb{Z}^2$). In particular, this allows us to find a Følner sequence which permits us to define a notion of topological entropy for $\mathrm{BS}(1,N)$-subshifts. Furthermore, the Følner rectangles cam be used to defined substitutions and carry out various constructions in the group. For this reason throughout the rest of this work whenever we talk about Baumslag-Solitar groups we will be referring to the specific case of $\mathrm{BS}(1,N)$, $N \geq 2$.

The thesis is structured in the following way:

- In the first chapter we give the basic notions and definitions necessary for understanding this thesis. We start with the basic facts about presentations of groups and the Cayley graph, to then introduce Baumslag-Solitar groups and prove some useful results about them. We finish the chapter with an introduction to symbolic dynamics on finitely generated groups and a brief introduction to substitutions in $\mathbb{Z}$ and $\mathbb{Z}^d$, $d \geq 2$.

- The second chapter is centered on introducing how the geometry of Baumslag-Solitar groups can force certain rigidity over configurations of the $\mathrm{BS}(1,N)$-full-shift. In particular, we prove that configurations with certain periods in the direction of the generator a must satisfy that any sufficiently high row must be monochromatic.

- The third chapter introduces the concept of rectangles in Baumslag-Solitar groups, which serve as a Følner sequence over which topological entropy can be defined and which at the same time serve as "building blocks" in order to make constructions in the group. In this chapter we define substitutions on Baumslag-Solitar groups, prove generalizations of some classic results about them and obtain a partial version of Mozes theorem, stating conditions under which a certain subclass of substitutive $\mathrm{BS}(1,N)$-subshift is sofic.

- The fourth chapter concerns the family of so-called Graph-Coloring subshifts (GCS), which are defined by local rules such that valid configurations in the subshift give proper colorings of the Cayley graph of $\mathrm{BS}(1,N)$. We study non-emptiness of the GCS depending on the number of colors (symbols) of the alphabet, as well as on the parity of N, and extensibility of locally admissible patterns. We obtain estimates of the GCS' topological entropy and study mixing conditions occurring in these subshifts in terms of the number of colors considered. The chapter finishes showing the (non) existence of frozen colorings in the GCS according to the number of colors considered, using similar methods to the ones in [1].

- In the fifth chapter we define a natural way of extending a $\mathbb{Z}$-subshift to a $\mathrm{BS}(1,N)$-subshift, by considering the family of forbidden patterns defining the former as being forbidden in the direction of both generators a and b in the latter. We focus mainly on the case when the $\mathbb{Z}$-subshift has a mixing symbol, in order to guarantee that the corresponding $\mathrm{BS}(1,N)$-subshift is non-empty and allow us to exploit constructions of configurations using independent copies of small rectangles inside big ones, and find relations between the entropies of both subshifts. Then we define the m-strip subshifts X/R_m associated to a $\mathrm{BS}(1,N)$-subshift X, which allows us to approximate up to any precision the topological entropy of X in terms of the topological entropy of the $\mathbb{Z}$-subshifts X/R_m.

- Finally in the sixth chapter we introduce the concept of the $\langle a\rangle$-projective subdynamics $\mathsf{P}_{\langle a\rangle}(X)$ of a $\mathrm{BS}(1,N)$-subshift X, which intuitively consists in the $\mathbb{Z}$-subshift we see in the a-rows of X. Motivated by the work done in [18], we prove conditions for the realization of sofic $\mathbb{Z}$-subshifts to be realized as $\langle a\rangle$-projective subdynamics of a $\mathrm{BS}(1,N)$-SFT, distinguishing cases according to the positivity of the topological entropy of the former.

Chapter 1

Preliminaries

The purpose of this chapter is to introduce the concepts, constructions and theorems related to the work done in this thesis, as well as to summarize the known results that inspired it. On what follows we assume that the reader has basic knowledge on group theory and topology, particularly that of compact metric spaces.

In the first section we start by defining the concepts of free groups and presentations of groups, which allow us to understand a group in terms of a set of generators and a set of relations that hold on it. We then define normal forms, which allow us to express elements of a group in a prescribed way and with it have a standard way to refer to them. Afterwards we recall the HNN-extension, which is a way of extending a group to a bigger one in which two isomorphic subgroups are conjugate. Finally we introduce the concept of amenability on (countable) groups by using Følner sequences and invariant means, along with their basic properties.

In the second section we introduce Cayley graphs, which are geometric objects associated to a given finitely generated group using a specific set of generators that allow us to interpret the group as the group of symmetries of its Cayley graph.

The third section is devoted to the Baumslag-Solitar groups, which are the most relevant groups in this thesis. We gather some properties of these finitely presented groups, interpret them as HNN-extensions of $\mathbb{Z}$, and explain a normal form as well as the geometry of their Cayley graph.

In the fourth section we state the basic definitions of symbolic dynamics on $\mathbb{Z}$ and more generally on a finitely generated group G, recalling the basic theorems and constructions associated with symbolic dynamics which motivate as well as serve as tools for the work done in this thesis.

Finally in the fifth section we present the concept of substitutions on $\mathbb{Z}$ and on $\mathbb{Z}^d$, with $d \geq 2$. We look at the connection between substitutions and symbolic dynamics given by the shift spaces they generate, and recall the most important theorems from this area.

1.1 Presentations of groups and amenability

The idea behind free groups is to form a group generated by a set that has as few relations between their elements as possible, namely only trivial relations that must hold in any group. With this we are able to obtain any group starting from a free group and forcing some additional set of non trivial relations between its elements. For a more detailed introduction on this topic and for proofs of the mentioned results throughout this section see [12] or [13].

Throughout this chapter we will use the following notation: given a set S we define a new set $S^{-1} \coloneqq \{s^{-1} \mid s \in S\}$ in bijection with S but disjoint from it, where the exponent "$^{-1}$" is purely notational. We write $S^{\pm} \coloneqq S \cup S^{-1}$.

We start by recalling the notion of a generating set for a group.

Definition 1.1 *Let G be a group and $S \subseteq G$ a subset. We say that G is **generated** by S, denoted as $G = \langle S \rangle$, if for every $g \in G$ there exist a finite number of elements $s_1, \dots, s_n \in S^{\pm}$ such that $g = s_1 \cdots s_n$, where we interpret the elements of S^{-1} as the inverses of the elements of S. If there exists a finite subset $S \subseteq G$ such that $G = \langle S \rangle$ we say that G is **finitely generated**.*

A **word** $w = s_1 \dots, s_n$ in $S^{\pm}$ is a finite sequence of symbols of $S^{\pm}$ where we allow the empty word with $n = 0$, which we will denote by ε. For a word w, we say that n is the **length** of w and use the notation $|w| \coloneqq n$. We denote by $(S^{\pm})^n$ the set of words of length n in $S^{\pm}$ and by $(S^{\pm})^* \coloneqq \bigcup_{n\geq 0}(S^{\pm})^n$ the set of all words on $S^{\pm}$. For words $w_1, w_2 \in (S^{\pm})^*$ we define their concatenation $w_1w_2 \in (S^{\pm})^*$ as the sequence of symbols of $S^{\pm}$ formed by those of w_1 followed by those of w_2.

We say that two words w_1, w_2 are related by $\sim$, symbolized by $w_1 \sim w_2$, if w_1 can be obtained from w_2 by adding or deleting from it adjacent pairs of symbols of the form ss^{-1} or $s^{-1}s$, for $s \in S$. One can check that $\sim$ defines an equivalence relation on $(S^{\pm})^*$ and that the quotient set $(S^{\pm})^*/\sim$ has a group structure under the concatenation operation. A word of minimal length within its equivalence class is called a reduced word.

Definition 1.2 *Given a set S we define the **free group** generated by S as the quotient space $F(S) \coloneqq (S^{\pm})^*/\sim$. We also use the notation $F(S) = \langle S \mid \rangle$ to refer to the free group generated by S for reasons that will become clear later. We call $|S|$ the rank of the free group $F(S)$ and if $|S| = n$ for some $n \geq 1$ we use the notation F_n and call this class of groups the finitely generated free groups.*

Free groups are characterized by the following universal property, stating that a group homomorphism from a free group to any other group is completely determined by its values on the elements of the generating set S, and conversely any assignment of values to the generating set extends to a homomorphism defined on the free group $F(S)$. This last statement is of course not true for every group: consider the group $\mathbb{Z}^2$ with generating set $S = \{(0,1),(1,0)\}$, and the map $\varphi : S \to D_8$ from S into the group of symmetries of the regular 4-gon, defined by $\varphi((0,1)) = r$ and $\varphi((1,0)) = s$, where r is a rotation and s a reflection. This map does not extend to a group homomorphism, since a reflection and a rotation do not commute, while the generators of $\mathbb{Z}^2$ do. We conclude that $\mathbb{Z}^2$ is not a free group.

Proposition 1.3 *Let $F(S)$ be the free group generated by S and G any group. Then for every map $\varphi : S \to G$ there exists a unique group homomorphism $\widetilde{\varphi} : F(S) \to G$ such that $\varphi(s) = \widetilde{\varphi}(\iota(s))$, where $\iota : S \to F(S)$ is the embedding of S in $F(S)$. Moreover, if G is a group generated by a set S which satisfies the above property, then $G \cong F(S)$ and we say that G is freely generated by S.*

Example 1.4

1. *The free group of rank 1 is $F_1 = \langle a \,|\, \rangle$, which consists of all elements of the form a^n, for $n \in \mathbb{Z}$. In fact, $F_1 \cong \mathbb{Z}$: this can be seen by defining the map $\varphi : \{a\} \to \mathbb{Z}$ given by $\varphi(a) = 1$ and using the universal property to extend it to a homomorphism $\widetilde{\varphi} : F_1 \to \mathbb{Z}$, which is in fact an isomorphism.*

2. *The free group of rank 2 is $F_2 = \langle a, b \,|\, \rangle$. The words $w_1 = ababba$ and $w_2 = ababaa^{-1}bb^{-1}ba$ on $(\{a, b\}^{\pm})^*$ represent the same element of F_2 since w_1 can be obtained from w_2 by reduction, i.e. by deleting the adjacent pairs aa^{-1} and $b^{-1}b$.*

Using the universal property from Proposition 1.3 one can prove that for two sets S_1, S_2 their generated free groups $F(S_1)$ and $F(S_2)$ are isomorphic as groups if and only if $|S_1| = |S_2|$. Another property (which we will prove due to its importance in what follows) is that any group can be expressed as a quotient of a free group. Moreover, if the group is finitely generated then it is a quotient of a finitely generated free group.

Proposition 1.5 *Let $G = \langle S \rangle$ be a group generated by $S \subseteq G$. Then there exists a normal subgroup N of $F(S)$ such that $G \cong F(S)/N$.*

Proof. Define the map $\varphi : S \to G$ to be the inclusion of S into G. Then by the universal property φ extends to a unique homomorphism $\widetilde{\varphi} : F(S) \to G$. Finally by the first isomorphism theorem we get $G \cong F(S)/\mathrm{Ker}(\widetilde{\varphi})$ and we may set $N := \mathrm{Ker}(\widetilde{\varphi})$. □

From the previous proof we see that the elements of G may be interpreted as reduced words on $F(S)$ where two words are considered equal if they differ by words of $\mathrm{Ker}(\widetilde{\varphi})$. For this reason we will call words in $\mathrm{Ker}(\widetilde{\varphi})$ **relators** of G. To make more precise this idea let us make the following definition.

Definition 1.6 *Given a group G, a set S and a subset $R \subseteq F(S)$ we will say that $\langle S \mid R \rangle$ is a **presentation** of G and write $G = \langle S \mid R \rangle$ if*

$$G \cong F(S)/\langle\langle R \rangle\rangle,$$

where $\langle\langle R \rangle\rangle := \bigcap_{R \subseteq H \trianglelefteq F(S)} H$ *is the normal closure of R in $F(S)$.*

*If there exist finite sets S and R such that $G = \langle S \mid R \rangle$ we say that G is **finitely presented** and that $\langle S \mid R \rangle$ is a **finite presentation** for G. Note that finitely generated (and hence finitely presented) groups have cardinality at most countable.*

Sometimes instead of the relators of R we are going to write $r = e_G$ for $r \in R$, or $r_1 = r_2$ for $r_1 r_2^{-1} \in R$, to emphasize the meaning of the relation inside the group.

Example 1.7 *1. For any set S the group $\langle S \mid \emptyset \rangle$ is isomorphic to the free group of rank $|S|$, which justifies our notation $F(S) = \langle S \,|\, \rangle$.*

2. The group $\langle a, b \mid aba^{-1}b^{-1} \rangle (= \langle a, b \mid ab = ba \rangle)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, the direct product of $\mathbb{Z}$ with itself. Note that in general an element of this group will have many different expressions using reduced words in $\{a, b\}^{\pm}$. For example the words a^2b^2, $abab$, $baba$ and b^2a^2 all evaluate to the same group element.

3. For $n \geq 3$ the dihedral group D_{2n} of symmetries of the regular n-gon has the presentation $\langle s, r \mid s^2, r^n, srsr \rangle$. Again note that an element of this group may have different forms of writing it as a reduced word in terms of its generators. As an example we see that the words r^{n-1} and srs represent the same group element in D_{2n}.

Group presentations are also characterized by a universal property, which basically tells us that a presentation of a group is characterized by extending mappings defined on the generating set to homomorphisms on the whole group, demanding that the relators of the presentation map to the trivial element.

Proposition 1.8 ([12, Chapter 2]) *Let S be a set and $R \subseteq (S^{\pm})^*$. The group $\langle S \mid R \rangle$ together with the canonical map $\pi : S \to \langle S \mid R \rangle$ has the property that for every group G and for every map $\varphi : S \to G$ such that - denoting by φ^* the extension of φ to the set of words $(S^{\pm})^*$ by concatenation - we have $\varphi^*(r) = e_G$ for every $r \in R$, there is a unique group homomorphism $\widetilde{\varphi} : (S^{\pm})^* \to G$ such that $\widetilde{\varphi} \circ \pi = \varphi$. Moreover, $\langle S \mid R \rangle$ together with π are uniquely determined up to isomorphism by this property.*

As we saw in the above examples it is common that for a presentation of a group its elements have different expressions as reduced words on the generators. This complicates the process of understanding the relation between different elements of the group in terms of the generating set. A way of avoiding this confusion is by prescribing a method to express each group element as a unique reduced word in terms of the generators of the group. This leads us to the idea of normal forms.

Definition 1.9 *Let $G = \langle S \mid R \rangle$ be a group, and denote by $\pi : (S^{\pm})^* \to G$ the mapping that evaluates words in $(S^{\pm})^*$ to elements of G (which is certainly surjective since S generates G). A* ***normal form*** *for G is a function $\eta : G \to (S^{\pm})^*$ such that for each $g \in G$ we have $\pi(\eta(g)) = g$.*

We call both the map η as well as its image $\eta(G)$ indistinctly the normal form of G.

Example 1.10 *1. The group $F_2 = \langle a, b \,|\, \rangle$ has a normal form given by the set of all reduced words.*

2. The group $\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid aba^{-1}b^{-1} \rangle$ has a normal form given by $\{a^i b^j \mid i, j \in \mathbb{Z}\}$. Note that we could also define another normal form $\{b^i a^j \mid i, j \in \mathbb{Z}\}$, so we see that in general *there is not a unique way of defining a normal form.*

Now we turn our attention on the construction of a new group from an old one in such a way that it has an additional property: given a group G and two isomorphic subgroups $H, K \leq G$, it may be useful to have a bigger group that contains G but in which H and K are conjugate subgroups. The following construction does precisely this and is called the HNN-extension[1].

Definition 1.11 *Consider a group G with a presentation $\langle S \mid R \rangle$ and two isomorphic subgroups $H, K \leq G$, with $\alpha : H \to K$ an isomorphism. We define the* ***HNN-extension*** *of G with respect to α by*

$$G*_\alpha := \langle S \cup \{t\} \mid R \cup \{t\alpha(h)t^{-1} = h : h \in H\}\rangle.$$

Example 1.12 *Consider the group $G = \mathbb{Z} = \langle a \mid \rangle$, and for $n, m \in \mathbb{Z}\backslash\{0\}$ the isomorphism $\alpha : \langle a^n \rangle \to \langle a^m \rangle$, which maps a^{nk} to a^{mk} for $k \in \mathbb{Z}$. Then the HNN-extension $G*_\alpha$ is isomorphic to $\langle a, t \mid ta^m t^{-1} = a^n \rangle$. This group is called the* ***Baumslag-Solitar group*** $\mathrm{BS}(m, n)$ *and it will be studied in more detail later due to its relevance for the work done in this thesis.*

Note that thanks to the relations added in the construction of $G*_\alpha$, for every $h \in H$ and $k \in K$ we have that $ht = t\alpha(h)$ and $tk = \alpha^{-1}(k)t$, which allows us to choose in which order the elements of G appear with respect to t or t^{-1}. This property allows us to construct a rather simple normal form for the HNN-extension $G*_\alpha$, as the next proposition shows.

Proposition 1.13 ([13, Chapter 1]) *Let G be a group, $H, K \leq G$ and $\alpha : H \to K$ an isomorphism. Choose classes of representatives T_H and T_K of the right cosets for H and K in G, respectively, such that T_H and T_K contain the identity element e_G. Then the HNN-extension $G*_\alpha$ has a normal form given by the set of words of the form $g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \ldots t^{\varepsilon_n} g_n$ for $n \geq 0$, $g_0 \in G$ and for $i = 1, \ldots, n$: $\varepsilon_i \in \{+1, -1\}$ such that*

- *$\varepsilon_i = 1$ implies $g_i \in T_K$,*
- *$\varepsilon_i = -1$ implies $g_i \in T_H$ and*
- *there is no subword of the form $t^\varepsilon e_G t^{-\varepsilon}$, for $\varepsilon \in \{+1, -1\}$.*

The uniqueness of this normal form for each group element in the HNN-extension gives us the following useful proposition, which immediately proves that the canonical homomorphism $G \to G*_\alpha$ is injective and hence the HNN-extension $G*_\alpha$ has a copy of G as a subgroup.

Proposition 1.14 (Britton's lemma) *Consider any group $G = \langle S \mid R \rangle$, an HNN-extension $G*_\alpha$ as above and denote by $\pi : ((S \cup \{t\})^\pm)^* \to G$ the evaluation of words in $((S \cup \{t\})^\pm)^*$ to the group elements. Consider a word w of the form $w = g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \ldots t^{\varepsilon_n} g_n$ with $g_i \in G$ for $i = 0, \ldots, n$ and $\varepsilon_i \in \{+1, -1\}$ for $i = 1, \ldots, n$, such that w has no subwords of the form $t^{-1} g_i t$ for $g_i \in H$, or $t g_i t^{-1}$ for $g_i \in K$. Then $\pi(w) \neq e_G$.*

[1]The letters "HNN" stands for Higman, Neumann and Neumann, who introduced this kind of extension in [9].

The final concept that we introduce in this section is that of amenability. An amenable group G is one that allows a way to average bounded functions defined on G such that this average remains invariant under translations by elements of G. Equivalently, a group G is amenable if it contains a family of finite sets that are almost-invariant by translations of G, in a way that we make precise below. We give the definition only for countable groups since it is the case of interest for this thesis, although it can be extended to arbitrary groups.

Definition 1.15 *A countable group G is said to be* ***amenable*** *if it satisfies one of the following equivalent conditions:*

1. *There exists a sequence of finite subsets of G such that for every $g \in G$:*

$$\lim_{n\to\infty} \frac{|F_n g \triangle F_n|}{|F_n|} = 0.$$

   A sequence $\{F_n\}_{n\in\mathbb{N}}$ with this property is called a (right) Følner sequence.

2. *There exists a G-invariant mean on $\ell^\infty(G,\mathbb{R})$, i.e. an $\mathbb{R}$-linear map $m : \ell^\infty(G,\mathbb{R}) \to \mathbb{R}$ such that $m(\mathbf{1}_G) = 1$, $m(f) \geq 0$ for every $f \in \ell^\infty(G,\mathbb{R})$ such that $f \geq 0$ pointwise, and $m(g \cdot f) = m(f)$ for every $f \in \ell^\infty(G,\mathbb{R})$, where $g \cdot f(t) := f(g^{-1}t)$ is the left action of G on $\ell^\infty(G,\mathbb{R})$.*

The next proposition lists the basic properties of amenable groups.

Proposition 1.16 ([5]) *The following properties hold.*

1. *Every finite group is amenable.*
2. *Every abelian group is amenable.*
3. *Every solvable[2] group is amenable.*
4. *The free group F_2 is not amenable.*
5. *If G is amenable, then every subgroup of G and every quotient of G is amenable.*
6. *For a directed family $\{G_i\}_{i\in I}$ of amenable groups their direct union $G := \bigcup_{i\in I} G_i$ is amenable.*

Proof. We will only prove 1. and 4. Refer to [5, Chapter 4] or [12, Chapter 9] for the proofs of the remaining items.

1. It is straightforward to check that the function $m : \ell^\infty(G,\mathbb{R}) \to \mathbb{R}$ defined by

$$m(f) := \frac{1}{|G|}\sum_{g\in G} f(g)$$

[2] A group G is **solvable** if there are subgroups $\{e_G\} = G_0 \leq G_1 \leq \cdots \leq G_k = G$ such that $G_{j-1} \trianglelefteq G_j$ and G_j/G_{j-1} is an abelian group, for every $j = 1,\ldots,k$.

is a G-invariant mean on $\ell^\infty(G,\mathbb{R})$.

4. For $G = F_2 = \langle a, b\,|\,\rangle$ suppose we have a G-invariant mean defined on $\ell^\infty(G,\mathbb{R})$. Let us consider the set $A \subseteq F_2$ of reduced words that start with a (non-zero) power of a. Then we can write $F_2 = A \cup a^{-1}\cdot A$, where $a^{-1}\cdot A := \{a^{-1}x : x \in A\}$. Denoting by $\mathbf{1}_A \in \ell^\infty(G,\mathbb{R})$ the characteristic function of A, i.e. the function that gives 1 on all the elements of A and 0 on the rest, we have that

$$\mathbf{1}_{F_2} \leq \mathbf{1}_A + \mathbf{1}_{a^{-1}\cdot A},$$

from where

$$1 = m(\mathbf{1}_{F_2}) \leq m(\mathbf{1}_A + \mathbf{1}_{a^{-1}\cdot A}) = m(\mathbf{1}_A) + m(\mathbf{1}_{a^{-1}\cdot A}) = 2m(\mathbf{1}_A),$$

and hence $m(\mathbf{1}_A) \geq \frac{1}{2}$.

Now consider the sets A, $b\cdot A$ and $b^2\cdot A$. These sets are pairwise disjoint and hence

$$\mathbf{1}_{A\cup b\cdot A\cup b^2\cdot A} = \mathbf{1}_A + \mathbf{1}_{b\cdot A} + \mathbf{1}_{b^2\cdot A}.$$

With this:

$$1 = m(\mathbf{1}_{F_2}) \geq m(\mathbf{1}_{A\cup b\cdot A\cup b^2\cdot A}) = m(\mathbf{1}_A) + m(\mathbf{1}_{b\cdot A}) + m(\mathbf{1}_{b^2\cdot A}) = 3m(\mathbf{1}_A) \geq \frac{3}{2},$$

which is a contradiction.

□

Finally, the following lemma tells us that to prove that a sequence is Følner on a finitely generated group it is sufficient to check its almost-invariance with respect to the elements of the generating set.

Lemma 1.17 *Let G be a group generated by a finite set $S \subseteq G$ and let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of finite subsets of G. Then $\{F_n\}_{n\in\mathbb{N}}$ is Følner if and only if for every $s \in S\cup S^{-1}$ we have*

$$\lim_{n\to\infty} \frac{|F_n s\backslash F_n|}{|F_n|} = 0.$$

Proof. The "if" direction follows immediately from the definition of a Følner sequence, so let us see the "only if" direction.

First note that for every $s \in S\cup S^{-1}$ we have

$$|F_n s \triangle F_n| = |F_n s\backslash F_n| + |F_n\backslash F_n s| = |F_n s\backslash F_n| + |F_n s^{-1}\backslash F_n|,$$

from where

$$\lim_{n\to\infty} \frac{|F_n s\triangle F_n|}{|F_n|} = \lim_{n\to\infty} \frac{|F_n s\backslash F_n|}{|F_n|} + \lim_{n\to\infty} \frac{|F_n s^{-1}\backslash F_n|}{|F_n|} = 0 + 0 = 0,$$

and so $\lim_{n\to\infty} \frac{|F_n s\triangle F_n|}{|F_n|} = 0$.

Now if $g \in G$, as S is a generating set we can write $g = s_1 \dots s_m$ for some $s_1, \dots, s_m \in S^{\pm}$. Let us prove by induction on m that $\lim_{n\to\infty} \frac{|F_n s_1 \dots s_m \triangle F_n|}{|F_n|} = 0$. The base case $m = 1$ was done already above, so assume now that the statement holds for $m-1$ and let us prove it for m, $m \geq 2$. Note that

$$\begin{aligned}
|F_n g \triangle F_n| &= |F_n s_1 \dots s_m \triangle F_n| \\
&\leq |F_n s_1 \dots s_m \triangle F_n s_2 \dots s_m| + |F_n s_2 \dots s_m \triangle F_n| \\
&= |F_n s_1 \triangle F_n| + |F_n s_2 \dots s_m \triangle F_n|,
\end{aligned}$$

then use the base case and the induction hypothesis to see that

$$\lim_{n\to\infty} \frac{|F_n g \triangle F_n|}{|F_n|} \leq \lim_{n\to\infty} \frac{|F_n s_1 \triangle F_n|}{|F_n|} + \lim_{n\to\infty} \frac{|F_n s_2 \dots s_m \triangle F_n|}{|F_n|} = 0 + 0 = 0,$$

and the proof is finished. □

1.2 The Cayley graph

We wish to interpret a group as more than a purely algebraic object by viewing it as the group of symmetries of a particular geometric structure. A first result related to this idea is the well known Cayley theorem, which states that every group may be faithfully represented as a subgroup of the symmetric group of some set. By defining Cayley graphs we will be able to see that a finitely generated group can be represented as the symmetry group of its associated Cayley graph, and in that way give a more geometric version of Cayley's theorem. To obtain a more detailed explanation of the concepts and theorems we mention here, we refer to [12] and [14], as this section is partially based on both references.

To be able to talk about Cayley graphs we assume basic knowledge on graph theory, but to clarify the notation used throughout this section we recall the following definition of a directed graph, along with that of a labeled graph.

Definition 1.18 *A* ***directed graph*** *is a 4-tuple $(V, E, \mathfrak{i}, \mathfrak{f})$ where V is called the set of vertices, $E \subseteq V \times V$ is called the set of edges, and the functions $\mathfrak{i} : E \to V$, $\mathfrak{f} : E \to V$ associate to each edge its "initial" and "terminal" vertices, respectively.*

Given an alphabet $\mathcal{A}$, a ***labeled (directed) graph*** *is a 5-tuple $(V, E, \mathfrak{i}, \mathfrak{f}, \lambda)$, where $(V, E, \mathfrak{i}, \mathfrak{f})$ is a directed graph and $\lambda : E \to \mathcal{A}$ is a function which "labels" the edges of the graph.*

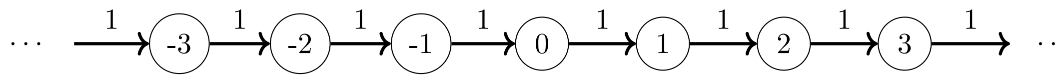
Now we are able to define the Cayley graph associated to a group given a generating set. The vertices of the graph are precisely the group elements and the existence of an edge between two vertices represents that viewing them as words over the generating set these elements differ from each other by a single generator.

Definition 1.19 *Let G be a group generated by a finite set $S \subseteq G$. We define the (right)* ***Cayley graph*** *as the labeled graph $\Gamma(G, S) = (V, E, \mathfrak{i}, \mathfrak{f}, \lambda)$, where $V = G$, $E = \{(g, gs) \in V \times V \mid g \in G, s \in S\}$, $\mathfrak{i}((g, gs)) = g$, $\mathfrak{f}((g, gs)) = gs$ and $\lambda((g, gs)) = s$.*

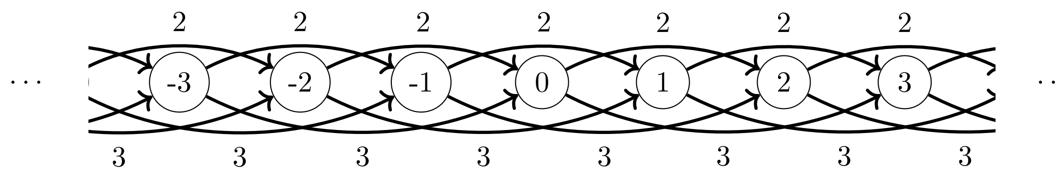
Note that the definition of the Cayley graph depends on the chosen generating set S, although for any finitely generated group most of the geometric properties encoded in the graph remain invariant under changes in the generating set (see [12]). In the following examples we show the Cayley graph of some classic groups.

Example 1.20

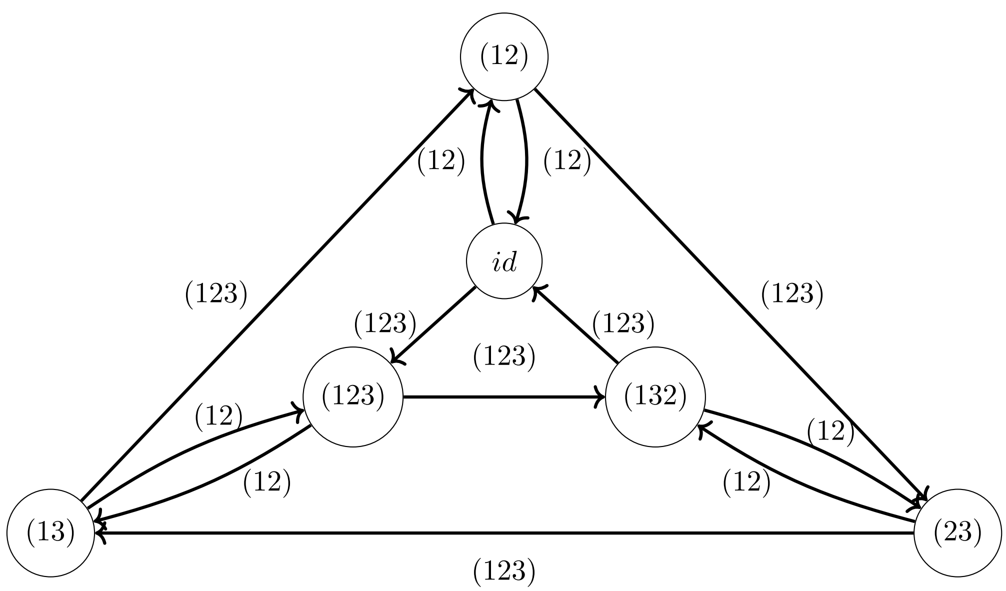
1. *The free group of rank 1 $F_1 \cong \mathbb{Z}$ with generating set $S = \{1\}$ has as its Cayley graph a "discrete line":*



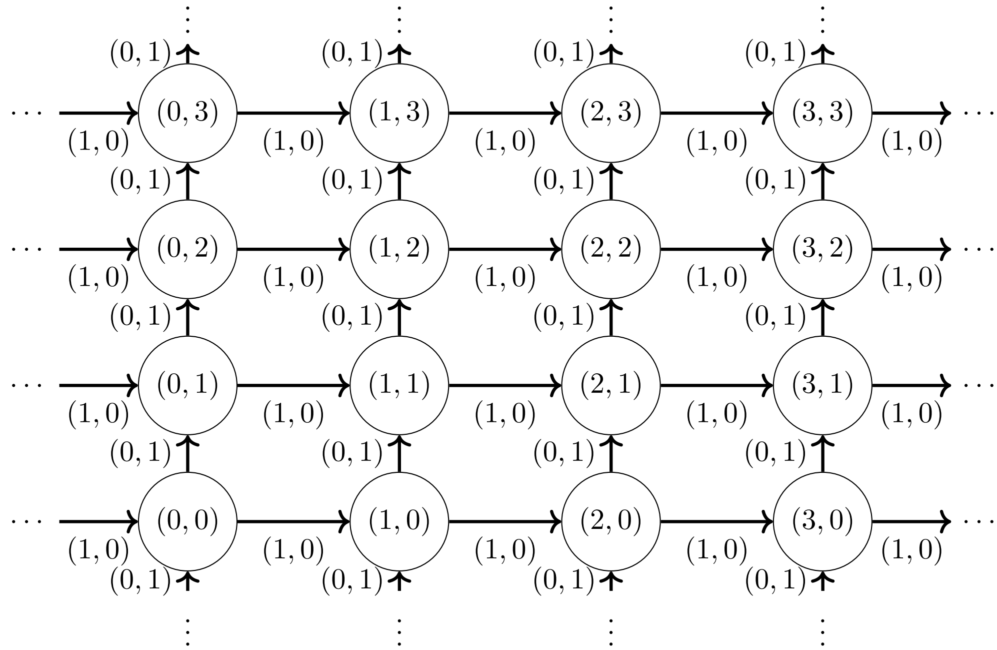
We could also have considered a different generating set like $S = \{2, 3\}$. In that case the Cayley graph would still be "line-like":



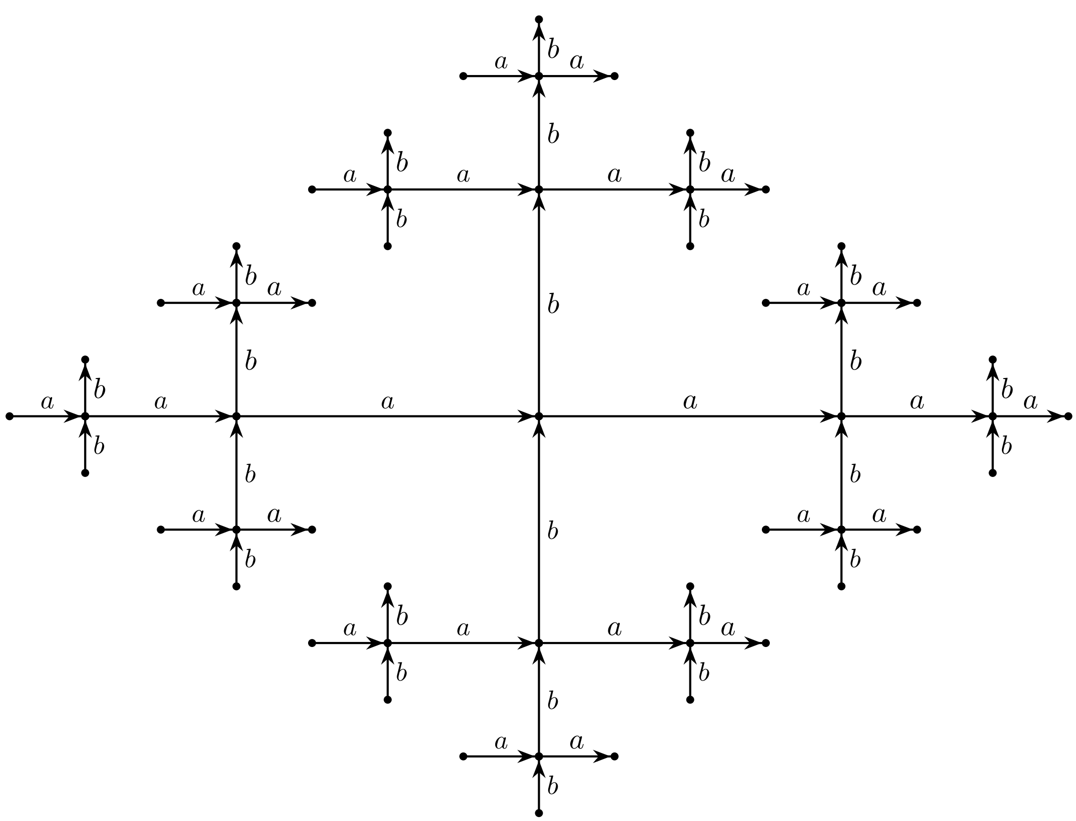
2. *Consider the symmetric group $S_3 := \{\pi : \{1,2,3\} \to \{1,2,3\} \mid \pi$ is a bijection$\}$, with its generating set $S = \{(123), (12)\}$. The corresponding Cayley graph $\Gamma(S_3, S)$ is drawn below:*



3. *The group $\mathbb{Z}^2$ with generating set $S = \{(1,0),(0,1)\}$ has as its Cayley graph the 2-dimensional lattice:*



4. *As a final example, we show (a part of) the Cayley graph of the rank-2 free group $F_2 = \langle a, b \,|\, \rangle$ with generating set $S = \{a, b\}$, which is an infinite 4-regular tree. In this representation of the Cayley graph the names of the vertices are omitted in order to obtain a clearer drawing. They can be recovered by reading off the edge labels following the unique path from the identity (the central vertex) to any other vertex.*



The next theorem states precisely what we said at the beginning about interpreting the group G as the group of symmetries of its associated Cayley graph $\Gamma(G, S)$, and is referred to as "Cayley's better theorem" in [14], in comparision to Cayley's theorem which states that every group is isomorphic to a subgroup of a symmetric group.

Theorem 1.21 ([14, Chapter 1]) *Let G be a group generated by a finite set $S \subseteq G$. Then the Cayley graph $\Gamma(G, S)$ is a connected and locally finite*[3] *labeled graph, on which G acts by bijections that preserve edges along with their orientation and label.*

Now that we have found a way to associate a graph to a given group, we can endow the graph with a metric space structure by standard methods and with it obtain a metric structure on the group G.

Definition 1.22 *Given a group G generated by a finite subset $S \subseteq G$, we define the **word metric** $d_S : G \times G \to \mathbb{R}$ as $d_S(g, h) \coloneqq \inf\{|w| \mid w \in (S^{\pm})^*, \pi(w) = g^{-1}h\}$, where $\pi : (S^{\pm})^* \to G$ is the canonical map which evaluates the word $w \in (S^{\pm})^*$ as an element of the group G, and $|w|$ denotes the length of the word w. It is common to denote $|g|_S \coloneqq d_S(g, e_G)$, for $g \in G$.*

In the context of Cayley graphs the above definition may be interpreted as taking as a generating set $S^{\pm}$ and constructing $\Gamma(G, S^{\pm})$, in which the existence of every directed edge implies the presence of its oppositely oriented counterpart, resulting in a strongly connected (symmetric) Cayley graph. With this the definition of the word metric $d_S(g, h)$ is equivalent to taking the minimal edge-distance between the vertices g and h on the graph $\Gamma(G, S^{\pm})$.

1.3 The Baumslag-Solitar groups

Now we turn our attention to the **Baumslag-Solitar groups**, which were briefly defined in Example 1.12. For any $m, n \in \mathbb{Z}\backslash\{0\}$ the group $\mathrm{BS}(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle$ arose as an HNN-extension of $\mathbb{Z}$ via the isomorphism $\alpha : \langle a^m \rangle \to \langle a^n \rangle$ which mapped $\alpha(a^m) = a^n$.

This family of groups was first introduced (though their origin might be older) by G. Baumslag and D. Solitar in [4], where they used them to provide an example of a group with two generators and one relator which is non Hopfian. Since then these groups have gained attention in the fields of combinatorial group theory and geometric group theory as examples and counterexamples of different properties (see [7] and [15]).

Some basic properties of these groups are listed in the following proposition, showing when two Baumslag-Solitar groups are isomorphic, highlighting some basic cases, characterizing for which values of m and n those Baumslag-Solitar groups are residually finite, solvable and amenable, and finally indicating a non-Hopfian member, which was one of the original motivations for defining these groups as was just said.

[3] A directed graph (V, E) is said to be **strongly connected** if there exists a directed path between any two vertices, and **connected** if there exists an un-directed path between any two vertices. It is said to be **locally finite** if the in-degree and the out-degree at every vertex is finite.

Proposition 1.23 *Baumslag-Solitar groups have the following properties.*

1. *For* $m, n \in \mathbb{Z}\backslash\{0\}$ *we have* $\mathrm{BS}(m,n) \cong \mathrm{BS}(-m,-n) \cong \mathrm{BS}(n,m)$.

2. $\mathrm{BS}(1,1) \cong \mathbb{Z}^2$ *is the fundamental group of the torus, and* $\mathrm{BS}(1,-1)$ *is the fundamental group of the Klein-bottle.*

3. $\mathrm{BS}(m,n)$ *is residually finite*[4] *if and only if* $|m| = |n|$ *or* $|n| = 1$ *or* $|m| = 1$.

4. $\mathrm{BS}(m,n)$ *is solvable if* $|m| = 1$ *or* $|n| = 1$, *and contains a subgroup isomorphic to the free group of rank* 2 *otherwise.*

5. $\mathrm{BS}(m,n)$ *is amenable if and only if* $|m| = 1$ *or* $|n| = 1$.

6. $\mathrm{BS}(2,3)$ *is not Hopfian, that is, there exists an epimorphism* $\gamma : \mathrm{BS}(2,3) \to \mathrm{BS}(2,3)$ *which is not injective.*

Proof.

1. Simply observe that the function $\varphi_1 : \mathrm{BS}(m,n) \to \mathrm{BS}(-m,-n)$ defined by $\varphi_1(a) = a^{-1}$, $\varphi_1(b) = b$ as well as the function $\varphi_2 : \mathrm{BS}(m,n) \to \mathrm{BS}(n,m)$ defined by $\varphi_2(a) = a$, $\varphi_2(b) = b^{-1}$ (and extended properly to the rest of the group using Proposition 1.8) are isomorphisms.

2. We have that $\mathrm{BS}(1,1) = \langle a, b \mid bab^{-1} = a\rangle = \langle a, b \mid ba = ab\rangle$, which is a presentation for $\mathbb{Z}^2$, and this is the fundamental group of the torus. Similarly, $\mathrm{BS}(1,-1) = \langle a, b \mid bab^{-1} = a^{-1}\rangle = \langle a, b \mid ba = a^{-1}b\rangle$, and this is a presentation for the fundamental group of the Klein-bottle.

3. See Theorem 1. of [16].

4. Using 1., the case $|m| = 1$ or $|n| = 1$ is equivalent to working with $\mathrm{BS}(1,n)$. Consider the subgroups of $\mathrm{BS}(1,n)$:
$$K := \langle b\rangle,$$
and
$$H := \langle\{b^{-j}a^kb^j \mid j,k \in \mathbb{Z}\}\rangle.$$
We note that:

   - $HK = G$, since $b \in K \subseteq HK$ and $a = b^{-0}ab^0 \in H \subseteq HK$.

   - H is a normal subgroup. In effect, note that for $j,k \in \mathbb{Z}$ we have
$$bb^{-j}a^kb^jb^{-1} = b^{-(j-1)}a^kb^{j-1} \in H,$$
$$b^{-1}b^{-j}a^kb^jb = b^{-(j+1)}a^kb^{j+1} \in H,$$
and that $ab^{-j}a^kb^ja^{-1} \in H$ and $a^{-1}b^{-j}a^kb^ja \in H$, since $a \in H$ and H is a subgroup.

[4] A group G is said to be **residually finite** if for every $g \in G\backslash\{e_G\}$ there exists a finite group F and a group morphism $\varphi : G \to F$ such that $\varphi(g) \neq e_F$.

- H is abelian. Given $j_1, j_2, k_1, k_2 \in \mathbb{Z}$:

$$\begin{aligned} b^{-j_1}a^{k_1}b^{j_1}b^{-j_2}a^{k_2}b^{j_2}b^{-j_1}a^{-k_1}b^{j_1}b^{-j_2}a^{-k_2}b^{j_2} &= b^{-j_1}a^{k_1}b^{j_1-j_2}a^{k_2}b^{j_2-j_1}a^{-k_1}b^{j_1-j_2}a^{-k_2}b^{j_2} \\ &= e_{\mathrm{BS}(1,n)}. \end{aligned}$$

Hence we see that $\mathrm{BS}(1,n) = H \rtimes K$, and being a semidirect product of two abelian groups we conclude that $\mathrm{BS}(1,n)$ is solvable.

Now if $|m| \neq 1$ and $|n| \neq 1$, we interpret $\mathrm{BS}(m,n)$ as the HNN-extension of $\mathbb{Z}$ via the isomorphism $\alpha : \langle a^m \rangle \to \langle a^n \rangle$ which maps $\alpha(a^m) = a^n$. As $|n| \neq 1$ and $|m| \neq 1$ the element a is neither in the domain nor image of α, and hence by Britton's lemma 1.14 we have that any word on the alphabet $\{b, aba^{-1}\}$ (and its inverses) maps to a non trivial element of the group. From there we see that $F_2 \cong \langle b, aba^{-1} \rangle$ is a subgroup of $\mathrm{BS}(m,n)$, and as F_2 is not solvable we conclude $\mathrm{BS}(m,n)$ cannot be either.

5. The "if" direction follows directly by using that solvable groups are amenable, meanwhile the "only if" part follows from the fact that subgroups of amenable groups are amenable, whereas the free group F_2 is not amenable.

6. Define a group morphism $\gamma : \mathrm{BS}(2,3) \to \mathrm{BS}(2,3)$ by $\gamma(a) = a^2$ and $\gamma(b) = b$, extending it to the rest of the group using Proposition 1.8. This can be done since $\gamma^*(ba^2b^{-1}a^{-3}) = ba^4b^{-1}a^{-6} = ba^2b^{-1}ba^2b^{-1}a^{-6} = a^3a^3a^{-6} = e_{\mathrm{BS}(2,3)}$, i.e. γ respects the only non-trivial relation of $\mathrm{BS}(2,3)$. Moreover γ is surjective since $\gamma(b) = b$ and $\gamma(bab^{-1}a^{-1}) = ba^2b^{-1}a^{-2} = a^3a^{-2} = a$, but is not injective as

$$\begin{aligned} \gamma((a^{-1}bab^{-1})^2a^{-1}) &= (a^{-2}ba^2b^{-1})^2a^{-2} \\ &= (a^{-2}a^3)^2a^{-2} = e_{\mathrm{BS}(2,3)}, \end{aligned}$$

meanwhile $(a^{-1}bab^{-1})^2a^{-1} \neq e_{\mathrm{BS}(2,3)}$ using Britton's lemma as in the proof of item 4.

□

Throughout the rest of this thesis we will focus our attention on the Baumslag-Solitar groups $\mathrm{BS}(1,N)$ for $N \geq 2$, which cover precisely all the cases for which - thanks to the previous proposition - the group is solvable (and amenable) but not abelian. **From now on whenever we talk about "Baumslag-Solitar groups" we will be referring only to the non-abelian solvable case $\mathbf{BS(1,N)}$, unless stated otherwise.**

Before proceeding let us take a look at the Cayley graph of $\mathrm{BS}(1,N)$ with generating set $S = \{a, b\}$. A section of this graph is shown in Figure 1.1, where we see that its structure is that of rows (along the edges labeled by the a-generator) being arranged (in a sideways view) as an N-ary tree. Seen from the front, each row has below it (i.e. in the b^{-1}-direction) a unique a-row, and above it (i.e. in the b-direction) N new a-rows. More formally, defining $A := \{a^k \mid k \in \mathbb{Z}\}$, a set of the form gA for $g \in \mathrm{BS}(1,N)$ will be called an **a-row** of the Cayley graph of $\mathrm{BS}(1,N)$, meanwhile a set of the form $\bigcup_{n=1}^{\infty} gb^{-n}A \cup \bigcup_{n=0}^{\infty}\left(g\prod_{s=1}^{n}(a^{i_s}b)A\right)$ for $g \in \mathrm{BS}(1,N)$ and a sequence $\{i_s\}_{s=1}^{\infty} \in \{0,\dots,N-1\}^{\mathbb{N}}$ will be called a **sheet** of the Cayley graph of $\mathrm{BS}(1,N)$. An example of the latter is illustrated in Figure 1.2.

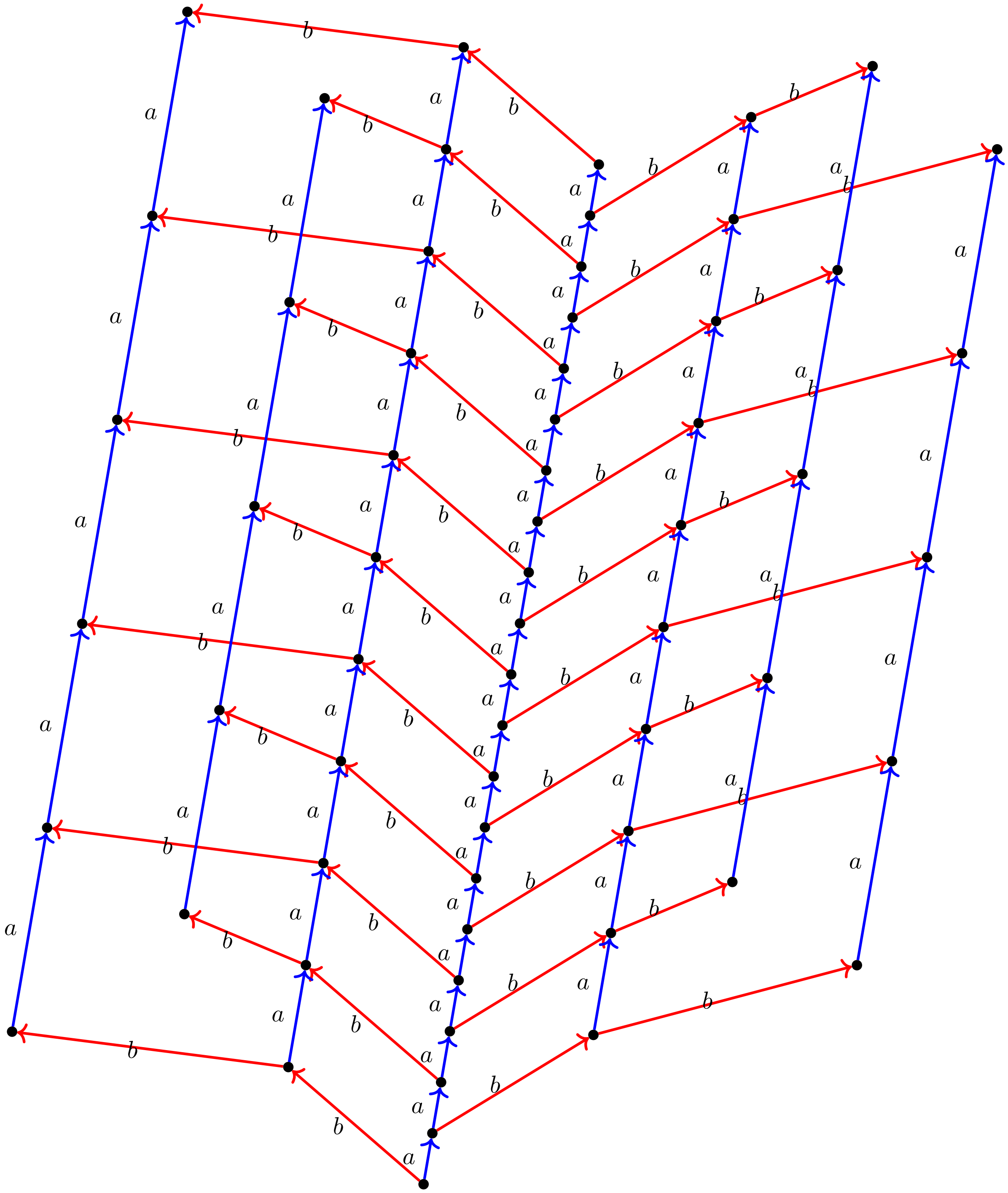


Figure 1.1: Part of the Cayley graph of BS(1, 2). Blue arrows represent the a- generator while red arrows represent the b-generator. Here we see that each a-row has a unique a-row below it and $N = 2$ a-rows above it.

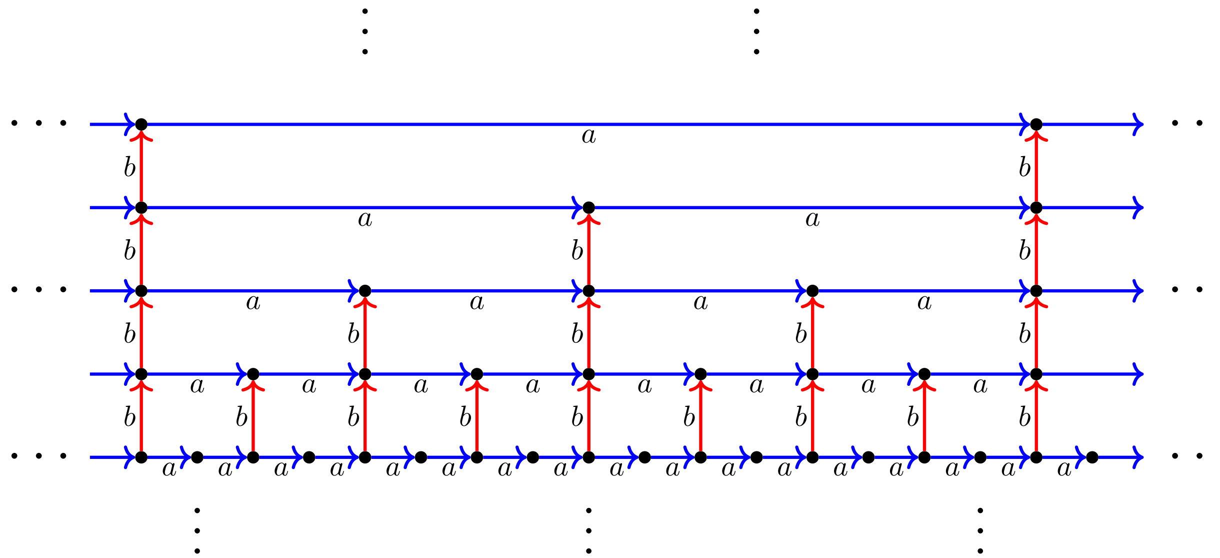


Figure 1.2: Example of (part of) a sheet on the Cayley graph of $BS(1, 2)$.

The next lemma summarizes some of the consequences induced by the defining relation used in the standard presentation of $\mathrm{BS}(1, N)$. Its elementary proof is by induction and is omitted. These additional relations will prove useful in establishing a normal form for $\mathrm{BS}(1, N)$ below.

Lemma 1.24 *The unique relation $bab^{-1} = a^N$ used to define $\mathrm{BS}(1, N)$ forces further identifications: for every $j \geq 0$ and $k \in \mathbb{Z}$ we have $b^j a^k = a^{kN^j} b^j$.*

Proposition 1.25 (Normal form) *Every $g \in \mathrm{BS}(1, N)$ can be decomposed into a normal form $g = b^{-j} a^k b^i$ where $i, j \geq 0$ and $k \in \mathbb{Z}$, and k may be divisible by N only if $ij = 0$. This decomposition is unique.*

Proof. Using that $\mathrm{BS}(1, N)$ is an HNN-extension of $\mathbb{Z}$ we can express any element of this group in the normal form for HNN-extensions from Proposition 1.13. In the notation of this proposition we use $H = \langle a^N \rangle$ and $K = \langle a \rangle = \mathbb{Z}$. The chosen sets of representatives of the right cosets are $T_H = \{e_G, \ldots, a^{N-1}\}$ and $T_K = \{e_G\}$. Hence the normal form is the set of words of the form

$$a^k b^{\varepsilon_1} a^{i_1} b^{\varepsilon_2} a^{i_2} \ldots b^{\varepsilon_n} a^{i_n},$$

for $k \in \mathbb{Z}$, $i_1, \ldots, i_n \in \{0, \ldots, N-1\}$, such that $\varepsilon_j = 1$ implies $i_j = 0$ and there is no subword of the form $b^{\varepsilon} e_{\mathrm{BS}(1,N)} b^{-\varepsilon}$ for any $\varepsilon \in \{+1, -1\}$. In particular these conditions force that if for some $j^* : \ \varepsilon_{j^*} = 1$, then for every $j \geq j^*$ we must have $i_j = 0$ and $\varepsilon_j = 1$. Using this fact we distinguish three different types of words:

The first type are words of the form $a^k b^j$ for $k \in \mathbb{Z}$ and $j \geq 0$. These words are already in the form described in the proposition so we are done in this case.

The second type are words of the form $a^k b^{-1} a^{i_1} \ldots b^{-1} a^{i_n}$ for $k \in \mathbb{Z}$, $n \geq 1$ and $i_1, \ldots, i_n \in \{0, \ldots, N-1\}$. Using the previous lemma we can easily deduce that for every $k' \in \mathbb{Z}$:

$a^{k'}b^{-1} = b^{-1}a^{Nk'}$. By repeatedly applying the last equality on the word we get:

$$\begin{aligned} a^k b^{-1} a^{i_1} \dots b^{-1} a^{i_n} &= b^{-1} a^{Nk+i_1} b^{-1} a^{i_2} \dots b^{-1} a^{i_n} \\ &= b^{-2} a^{N^2 k + N i_1 + i_2} b^{-1} a^{i_3} \dots b^{-1} a^{i_n} \\ &\vdots \\ &= b^{-n} a^{N^n k + \sum_{j=1}^{n} N^{n-j} i_j}, \end{aligned}$$

which also has the form described in the proposition.

Finally the third type are words of the form $a^k b^{-1} a^{i_1} \dots b^{-1} a^{i_n} b^j$ for $k \in \mathbb{Z}$, $n, j \geq 1$, $i_1, \dots, i_n \in \{0, \dots, N-1\}$, and $i_n \neq 0$ (since otherwise a subword of the form $b^{-1} e_G b$ would appear, which is forbidden). Using the same reasoning as in the previous case we obtain

$$a^k b^{-1} a^{i_1} \dots b^{-1} a^{i_n} b^j = b^{-n} a^{N^n k + \sum_{s=1}^{n-1} N^{n-s} i_s + i_n} b^j,$$

which again is of the form described in the proposition except for the condition between the values of the powers of both b's and a. Note that as $n \geq 1$ and $i_n \in \{1, \dots, N-1\}$, the exponent of the a generator is not a multiple of N, so this is only possible if at least one of the powers of the b's is zero, in which case we are in one of the previous two cases.

The uniqueness of this normal form follows immediately from the uniqueness of the normal form for HNN-extensions, and so we finish this proof. $\square$

The normal form for $\mathrm{BS}(1, N)$ described in the previous proposition may be interpreted as following a path in the Cayley graph of $\mathrm{BS}(1, N)$ from the identity $e_{\mathrm{BS}(1,N)}$ to the element $g = b^{-j} a^k b^i$, by first going down (in the direction of b^{-1}) in order to have a common base row with the element g, to then move along the a axis to find the correct sheet, and to finally go upwards (in the direction of b) to arrive at the element g.

To finish this section we prove a lemma in which we give an expression for the normal form of powers of elements of $\mathrm{BS}(1, N)$, also expressed in normal form.

Proposition 1.26 *Let $g = b^{-j} a^k b^i \in \mathrm{BS}(1, N)$ be an element decomposed into its normal form. Then for every $n \geq 1$ the normal form of g^n is given by*

$$g^n = \begin{cases} b^{-j} a^{k \frac{N^{n(i-j)} - 1}{N^{(i-j)} - 1}} b^{ni - (n-1)j}, & \text{if } i > j, \\ b^{-j} a^{nk} b^i, & \text{if } i = j, \\ b^{-nj + (n-1)i} a^{k \frac{N^{n(j-i)} - 1}{N^{(j-i)} - 1}} b^i, & \text{if } i < j. \end{cases}$$

Similarly, the normal form for g^{-n} for $n \geq 1$ is given by

$$g^{-n} = \begin{cases} b^{-ni + (n-1)j} a^{-k \frac{N^{n(i-j)} - 1}{N^{(i-j)} - 1}} b^j, & \text{if } i > j, \\ b^{-i} a^{-nk} b^j, & \text{if } i = j, \\ b^{-i} a^{-k \frac{N^{n(j-i)} - 1}{N^{(j-i)} - 1}} b^{nj - (n-1)i}, & \text{if } i < j. \end{cases}$$

Proof. We prove the stated formulas by induction on $n \geq 1$ and only for g^n, since the normal form of g^{-n} is deduced from the former by taking inverses, or by considering the normal form for $\widetilde{g}^n$ with $\widetilde{g} \coloneqq g^{-1}$. The base case $n = 1$ is obvious, so consider the statement to be true for $n \geq 1$ and let us prove it for $n+1$. Suppose first that $i > j$. Then:

$$\begin{aligned} g^{n+1} &= b^{-j} a^{k \frac{N^{n(i-j)}-1}{N^{(i-j)}-1}} b^{ni-(n-1)j} b^{-j} a^k b^i \\ &= b^{-j} a^{k \frac{N^{n(i-j)}-1}{N^{(i-j)}-1}} b^{n(i-j)} a^k b^i \\ &= b^{-j} a^{k \frac{N^{n(i-j)}-1}{N^{(i-j)}-1}} a^{kN^{n(i-j)}} b^{n(i-j)} b^i \\ &= b^{-j} a^{k \frac{N^{(n+1)(i-j)}-1}{N^{(i-j)}-1}} b^{(n+1)i-nj}. \end{aligned}$$

If $i = j$ then:

$$g^{n+1} = b^{-j} a^{nk} b^i b^{-j} a^k b^i = b^{-j} a^{(n+1)k} b^i.$$

Finally if $i < j$:

$$\begin{aligned} g^{n+1} &= b^{-nj+(n-1)i} a^{k \frac{N^{n(j-i)}-1}{N^{(j-i)}-1}} b^i b^{-j} a^k b^i \\ &= b^{-nj+(n-1)i} a^{k \frac{N^{n(j-i)}-1}{N^{(j-i)}-1}} b^{i-j} a^k b^i \\ &= b^{-nj+(n-1)i} b^{i-j} a^{k \frac{N^{n(j-i)}-1}{N^{(j-i)}-1} N^{(j-i)} + k} b^i \\ &= b^{-(n+1)j+ni} a^{k \frac{N^{(n+1)(j-i)}-1}{N^{(j-i)}-1}} b^i, \end{aligned}$$

finishing the proof. □

1.4 Symbolic dynamics on groups

This section covers the basic definitions and some useful theorems of symbolic dynamics on a finitely generated group G. We define the full G-shift and G-subshifts, morphisms between these spaces, some periodicity conditions, and finally introduce the notion of entropy of a G-subshift in the case where the group G is amenable. A more comprehensive guide on these topics can be found in [11] for the case $G = \mathbb{Z}$, and in [3] and [5] for the more general case where G is a finitely generated group.

Let G be a finitely generated group and $\mathcal{A}$ a finite set, which we call the **alphabet** and refer to its elements as the **symbols**.

Definition 1.27 *Consider the product space $\mathcal{A}^G$. A point $x \in \mathcal{A}^G$ is called a **configuration** or a **coloring**, and we use the notation $x_g \coloneqq x(g)$ for $g \in G$.*

Considering the discrete topology in $\mathcal{A}$, the space $\mathcal{A}^G$ is endowed with the product topology. To be able to describe this topology in more depth we make the following definition.

Definition 1.28 *For a finite subset $F \subseteq G$, an element $p \in \mathcal{A}^F$ is called a **pattern** and the set F is called its **support**. We use the notation* $\operatorname{supp}(p) = F$.

*We say that a pattern p is a **subpattern** of a pattern q, denoted $p \sqsubseteq q$, if there exists $g \in G$ such that $g \cdot \operatorname{supp}(p) \subseteq \operatorname{supp}(q)$ and $p = q|_{g \cdot \operatorname{supp}(p)}$. We also say that the pattern p is a subpattern of a configuration $x \in \mathcal{A}^G$ or that p appears in x, denoted $p \sqsubseteq x$, if there exists $g \in G$ such that $x|_{g \cdot \operatorname{supp}(p)} = p$.*

In the particular case $G = \mathbb{Z}$, patterns are commonly restricted to have a support of the form $\{0, \ldots, n-1\}$ for $n \geq 1$ instead of any finite set as we defined above for a general group.

The topology of $\mathcal{A}^G$ has a base of clopen sets given by the **cylinders** of the form

$$[p]_g \coloneqq \{x \in \mathcal{A}^G \mid x|_{gF} = p\}$$

for some finite subset $F \subseteq G$, $p \in \mathcal{A}^F$ and $g \in G$. We also introduce the notation $[p] \coloneqq [p]_{e_G}$. Note that as we assume that G is finitely generated, this base is countable and hence the product topology is metrizable. A possible metric is given by

$$d(x, y) \coloneqq 2^{-\inf\{|g| \,:\, g \in G \text{ and } x_g \neq y_g\}}, \;\; x, y \in \mathcal{A}^G.$$

In particular, we conclude that $\mathcal{A}^{\mathbf{G}}$ **has the structure of a compact metric space.**

Definition 1.29 *The group G induces a left action by homeomorphisms* [5] *on $\mathcal{A}^G$, called the **shift**, given by*

$$\begin{aligned} \sigma : G &\to \operatorname{Homeo}(\mathcal{A}^G, \mathcal{A}^G) \\ g &\mapsto \sigma_g \end{aligned}$$

where $\operatorname{Homeo}(\mathcal{A}^G, \mathcal{A}^G)$ are the homeomorphisms from $\mathcal{A}^G$ to itself, and for each $x \in \mathcal{A}^G$, $g, h \in G$ we have

$$\sigma_g(x)_h = x_{g^{-1}h}.$$

*The topological dynamical system $(\mathcal{A}^G, \sigma)$ is called the **full G-shift**.*

Now that we have defined the full G-shift we would like to talk about dynamical subsystems of it, which from a topological point of view come to be compact subspaces that are preserved by the action of G. Dynamical subsystems of the full G-shift are called G-subshifts, and are defined below by two equivalent descriptions: the first one arising from the notion of a subsystem of a topological dynamical system and the second one being a combinatorial definition, which is possible thanks to the distinctive combinatorial nature of symbolic dynamics.

Definition 1.30 *A subset $X \subseteq \mathcal{A}^G$ is called a **G-subshift** if it satisfies one of the following equivalent conditions.*

1. *X is closed and G-invariant, that is, for every $g \in G: \;\; \sigma_g(X) \subseteq X$.*

[5] The fact that for every $g \in G$ the map σ_g is a homeomorphism follows from noticing that for every pattern $p \in \mathcal{A}^F$, $F \subseteq G$ finite, we have $\sigma_g([p]_{g^{-1}}) = [p]$, and that σ_g is bijective. Then thanks to the compactness of $\mathcal{A}^G$, σ_g must be a homeomorphism.

2. There exists a family of (forbidden) patterns $\mathcal{F}$ such that

$$X = \mathsf{X}_{\mathcal{F}},$$

where $\mathsf{X}_{\mathcal{F}} := \{x \in \mathcal{A}^G \mid \text{ for every } p \in \mathcal{F} : p \not\sqsubseteq x\}$.

Proof (That these two conditions are indeed equivalent). Note that the condition of being a subpattern of a configuration x is invariant under the action σ, and so the set $\mathsf{X}_{\mathcal{F}}$ is G-invariant. It is also easy to see that $\mathsf{X}_{\mathcal{F}}$ is closed by writing

$$\mathsf{X}_{\mathcal{F}} = \bigcap_{p \in \mathcal{F}} \bigcap_{g \in G} X \backslash [p]_g,$$

with which we have proved 2. $\Rightarrow$ 1.

Now let us prove 1. $\Rightarrow$ 2. Using that X^c is open and that the cylinders form a base of the topology we can write

$$X^c = \bigcup_{p \in \mathcal{F}} [p]$$

for some family of patterns $\mathcal{F} \subseteq \bigcup_{\substack{F \subseteq G \\ F \text{ finite}}} \mathcal{A}^F$. Moreover, using that X is G-invariant we have

$$X^c = \bigcup_{p \in \mathcal{F}} \bigcup_{g \in G} [p]_g,$$

so for this family $\mathcal{F}$: $X = \mathsf{X}_{\mathcal{F}}$ and we have finished the proof. □

Now we proceed to define mappings between subshifts, which respect the dynamical structure of the subshifts together with the shift action of the group G.

Definition 1.31 *Let $\mathcal{A}$, $\mathcal{B}$ be finite sets, and $X \subseteq \mathcal{A}^G$, $Y \subseteq \mathcal{B}^G$ be G-subshifts. A continuous function $\varphi : X \to Y$ that is equivariant, that is, for every $g \in G : \varphi \circ \sigma_g = \sigma_g \circ \varphi$ is called a* ***morphism*** *between X and Y.*

An injective morphism is called an ***embedding*** *of X in Y, meanwhile a surjective morphism is called a* ***factor map*** *between X and Y, and we say that X is an extension of Y and Y is a factor of X. Finally, a bijective morphism is called a* ***conjugacy*** *between X and Y and we say that X and Y are conjugate, denoted by $X \cong Y$. If two G-subshifts are conjugate it means that dynamically they have the same structure.*

Like G-subshifts, morphisms also have a combinatorial definition equivalent to the topological one we have just given, which characterizes a morphism as a map whose values are pointwise determined by looking at a finite set around the coordinate of interest. This characterization was first proven in the case $G = \mathbb{Z}$ by Curtis, Lyndon and Hedlund in [8], but the proof easily generalizes to the case of a finitely generated group G (see [5, Chapter 1]).

Theorem 1.32 (Curtis-Lyndon-Hedlund) *Let $X \subseteq \mathcal{A}^G$, $Y \subseteq \mathcal{B}^G$ be subshifts and $\varphi : X \to Y$ a map. Then φ is a morphism if and only if there exists a finite set $F \subseteq G$ and a map $\Phi : \mathcal{A}^F \to \mathcal{B}$ such that for every $g \in G : \varphi(x)_g = \Phi(x|_{gF})$. In that case φ is called a **sliding block code** and Φ is called its local map.*

Having characterized subshifts in a combinatorial way in terms of a family of forbidden patterns, it is interesting to study which patterns indeed appear in its configurations, which motivates the following definition.

Definition 1.33 *Let $X \subseteq \mathcal{A}^G$ be a subshift. For every finite subset $F \subseteq G$ we define the set of patterns with support F appearing in X by $\mathcal{L}_F(X) := \{x|_F \mid x \in X\}$. We define the **language** of X to be the set of patterns appearing in X, that is,*

$$\mathcal{L}(X) := \bigcup_{\substack{F \subseteq G \\ F \text{ finite.}}} \mathcal{L}_F(X).$$

In the case $G = \mathbb{Z}$ we use the notation $\mathcal{L}_n(X) := \mathcal{L}_{\{0,\dots,n-1\}}(X)$ and $\mathcal{L}(X) := \bigcup_{n \geq 0} \mathcal{L}_n(X)$.

The family of forbidden patterns $\mathcal{F}$ defining the subshift may be necessarily infinite, as we will see below in the example of the even shift S_{even}. Therefore the particular case for which it is possible to choose a finite family of forbidden patterns to describe a subshift gives rise to an important class of subshifts. A subshift X in this class allows to verify whether a given configuration belongs to X by checking locally (i.e. on finite subsets of G) if all appearing patterns are allowed.

Definition 1.34 *A G-subshift $X \subseteq \mathcal{A}^G$ is said to be a **G-subshift of finite type** (SFT) if there exists a finite family of patterns $\mathcal{F}$ such that $X = \mathsf{X}_{\mathcal{F}}$.*

Examples 1.35

1. *On $G = \mathbb{Z}$ and $\mathcal{A} = \{0, 1\}$, the set $\mathsf{X}_{\mathcal{F}}$ defined by the forbidden set $\mathcal{F} = \{11\}$ containing a single word (pattern) is a $\mathbb{Z}$-SFT, called the golden mean shift.*
2. *Again with $G = \mathbb{Z}$ and $\mathcal{A} = \{0, 1\}$, the set $S_{\text{even}} \subseteq \{0, 1\}^{\mathbb{Z}}$ defined by the forbidden set of words $\mathcal{F} = \{10^n 1 \mid n \text{ is odd}\}$ is a $\mathbb{Z}$-subshift, called the even shift. This subshift is not an SFT since it is impossible to verify whether an arbitrarily long word of the form $10^n 1$ appears in a configuration looking only at subwords of a uniformly bounded support.*

A special family of SFTs are **nearest-neighbors** SFTs (denoted NNSFTs): for G a group generated by the finite set $S \subseteq G$, a subshift $X \subseteq \mathcal{A}^G$ is said to be a NNSFT if there exists a (finite) family of patterns $\mathcal{F}$ such that $X = \mathsf{X}_{\mathcal{F}}$ and the support of each pattern $p \in \mathcal{F}$ is of the form $\{e_G, s\}$ for some $s \in S$. Hence NNSFTs is the subclass of SFTs which can be described using a family of forbidden patterns which have the simplest possible shape.

Proposition 1.36 *Every SFT is conjugate to a NNSFT.*

The proof of this fact consists in considering a new alphabet given by the patterns on a sufficiently large (finite) support ocurring on X, and respecting a modified version of the forbidden patterns of X. A complete proof of this fact can be found in [3].

Being an SFT is a property that is invariant under conjugation, in the sense that if two subshifts are conjugate and one of them is an SFT, then the other one must also be an SFT. The same, however, is not true if one only has a factor map instead of a conjugacy, which motivates us to define the smallest class of G-subshifts that is closed by factor maps, and which contains SFTs.

Definition 1.37 *A G-subshift Y is said to be* ***sofic*** *if there exists a G-SFT X and a factor map $\varphi : X \to Y$.*

Example 1.38 *Recall the even shift S_{even} defined in Example 1.35, where we argued that it was not an SFT. Nonetheless, S_{even} is a sofic shift: define a new alphabet $\mathcal{B} \coloneqq \{0_1, 0_2, 1\}$, the SFT $\mathsf{X}_{\mathcal{F}}$ defined by the finite set of forbidden words*

$$\mathcal{F} \coloneqq \{0_1 0_1, 0_2 0_2, 0_1 1, 10_2\} ,$$

and the sliding block code $\varphi : \mathsf{X}_{\mathcal{F}} \to S_{\text{even}}$ given by the local map $\Phi : \mathcal{B} \to \{0, 1\}$ defined by $\Phi(0_1) = \Phi(0_2) = 0$ and $\Phi(1) = 1$. It is easy to see that $\varphi(\mathsf{X}_{\mathcal{F}}) \subseteq S_{\text{even}}$ since the set of forbidden patterns $\mathcal{F}$ forces words in $\mathsf{X}_{\mathcal{F}}$ with 1's on its both ends to be of the form $10_1 0_2 0_1 0_2 \dots 0_1 0_2 1$, hence having an even number of 0's between both 1's. Similarly we see that actually $\varphi(\mathsf{X}_{\mathcal{F}}) = S_{\text{even}}$. Every sequence $s \in S_{\text{even}}$ has a preimage through φ in $\mathsf{X}_{\mathcal{F}}$ constructed by replacing occurrences of 0's in s by alternating 0_1's and 0_2's, which can be done in such a way that no forbidden patterns are produced, since between two 1's in s there is an even number of 0's.

In the case $G = \mathbb{Z}$ sofic subshifts can be understood by studying a particular kind of presentation: let $\Gamma = (V, E, \mathfrak{i}, \mathfrak{f}, \lambda)$ be a labeled graph. We say that $\xi = \{e_n\}_{n\in\mathbb{Z}} \in E^{\mathbb{Z}}$ is a **biinfinite path** in Γ if for every $n \in \mathbb{Z}$ we have $\mathfrak{f}(e_n) = \mathfrak{i}(e_{n+1})$, and we denote by $\lambda_\infty(\xi) \coloneqq \{\lambda(e_n)\}_{n\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ the label of the biinfinite path. With this we can define the $\mathbb{Z}$-subshift

$$\mathsf{S}_\Gamma \coloneqq \left\{ s \in \mathcal{A}^{\mathbb{Z}} \mid s = \lambda_\infty(\xi), \text{for some biinfinite path } \xi \in E^{\mathbb{Z}} \right\} ,$$

which turns out to be a sofic subshift, as it can be seen as a factor of the $\mathbb{Z}$-NNSFT

$$\mathsf{X}_\Gamma \coloneqq \left\{ \xi \in E^{\mathbb{Z}} \mid \xi \text{ is a biinfinite path in } \Gamma \right\} .$$

The fact that all sofic $\mathbb{Z}$-subshifts may be understood through this particular presentation is portrayed in the following proposition.

Proposition 1.39 *([11, Chapter 3]) Let S be a sofic $\mathbb{Z}$-subshift. Then there exists a right-resolving*[6] *labeled graph $\Gamma = (V, E, \mathfrak{i}, \mathfrak{f}, \lambda)$ such that $S \cong \mathsf{S}_\Gamma$.*

On what follows next we describe a useful construction of the labeled graph Γ from the previous proposition.

[6] A labeled graph $\Gamma = (V, E, \mathfrak{i}, \mathfrak{f}, \lambda)$ is called **right-resolving** if for each vertex $v \in V$ all the edges $e \in E$ with $\mathfrak{i}(e) = v$ carry different labels through the labeling function λ.

Definition 1.40 *Let X be a $\mathbb{Z}$-subshift and consider a word $w \in \mathcal{L}(X)$. We define the **follower set** $F_w(X)$ of w in X as the set of words that can follow w in X. That is:*

$$F_w(X) \coloneqq \{v \in \mathcal{L}(X) \mid wv \in \mathcal{L}(X)\}.$$

Definition 1.41 *Suppose X is a $\mathbb{Z}$-subshift with a finite number of distinct follower sets. We define the **follower set graph** of X as the labeled graph $(V, E, \mathfrak{i}, \mathfrak{f}, \lambda)$ whose vertices $V = \{F_w(X)\}_{w \in \mathcal{L}(X)}$ are the (finite) distinct follower sets, and two vertices $F_w(X), F_{w'}(X) \in V$ are connected by an edge if there exists $a \in \mathcal{A}$ such that $F_{w'}(X) = F_{wa}(X)$. In this case we have $\mathfrak{i}((F_w(X), F_{w'}(X))) = F_w(X)$, $\mathfrak{f}((F_w(X), F_{w'}(X))) = F_{w'}(X)$ and $\lambda((F_w(X), F_{w'}(X))) = a$.*

The following theorem characterizes sofic $\mathbb{Z}$-subshifts as those with a finite number of follower sets, providing also a labeled graph-representation of the subshift in terms of these sets. For more details and the proof of this proposition see .

Theorem 1.42 *([11, Chapter 3]) If S is a sofic $\mathbb{Z}$-subshift then S has a finite number of follower sets. Moreover, the follower set graph Γ of S is right-resolving and we have $S \cong \mathsf{S}_\Gamma$.*

It is important to notice that saying "a pattern appears in X" is in general not equivalent to saying "a pattern respects the forbidden family $\mathcal{F}$", as it is possible that the latter cannot be extended consistently (i.e. respecting the forbidden patterns) to the rest of the group to obtain a configuration in X. In the particular case of $\mathbb{Z}$-SFTs both notions are equivalent, but if one changes the group the equivalence is no longer true, as for example with SFTs defined on $G = \mathbb{Z}^2$.

Definition 1.43 *Given a G-subshift X defined by a family of forbidden patterns $\mathcal{F}$, a pattern p is said to be:*

- ***locally admissible*** *if p does not contain an element of $\mathcal{F}$ as a subpattern.*
- ***globally admissible*** *if there exists $x \in X$ such that $x|_F = p$, where F is the support of p.*

Now we proceed to state notions of periodicity, i.e. properties of configurations having patterns repeated in some sense throughout the group, and of mixing, i.e. concepts of how two globally admissible patterns of a subshift may appear simultaneously inside a configuration of this space.

Definition 1.44 *For a configuration $x \in \mathcal{A}^G$ we define its **orbit** $\mathrm{Orb}_G(x) \coloneqq \{\sigma_g(x) \mid g \in G\}$ and its **stabilizer** $\mathrm{Stab}_G(x) \coloneqq \{g \in G \mid \sigma_g(x) = x\}$, which is a subgroup of G. A known result on group actions relates both sets by the formula $|\mathrm{Orb}_g(x)| = |G : \mathrm{Stab}_G(x)|$ for every $x \in \mathcal{A}^G$.*

Definition 1.45 *A configuration $x \in \mathcal{A}^G$ is said to be:*

- ***weakly periodic*** *if $\mathrm{Stab}_G(x)$ contains an infinite cyclic group.*
- ***strongly periodic*** *if $\mathrm{Stab}_G(x)$ is of finite index inside G. By what was said earlier*

this is equivalent to $|\mathrm{Orb}_G(x)| < \infty$*.*

Definition 1.46 *A* G*-subshift* $X \subseteq \mathcal{A}^G$ *is said to be:*

- ***topologically transitive*** *if for every pair of globally admissible patterns* $p, q \subseteq \mathcal{L}(X)$ *there exists* $g \in G$ *such that* $[p]_g \cap [q] \neq \emptyset$*. This is equivalent to the fact that there exists* $x \in X$ *such that* $\overline{\mathrm{Orb}_G(x)} = X$*.*

- ***topologically mixing*** *if for every pair of globally admissible patterns* $p, q \subseteq \mathcal{L}(X)$ *there exists a finite set* $F \subseteq G$ *(depending on* p *and* q*) such that for every* $g \in G \backslash F$: $[p]_g \cap [q] \neq \emptyset$*.*

- ***strongly irreducible*** *if there exists a (non-empty) finite set* $F \subseteq G$ *such that for every pair of globally admissible patterns* $p, q \in \mathcal{L}(X)$ *such that* $\mathrm{supp}(p) \cap \mathrm{supp}(q) \cdot F = \emptyset$: $[p] \cap [q] \neq \emptyset$*.*

- ***minimal*** *if for every* $x \in X$: $\overline{\mathrm{Orb}_G(x)} = X$*. This is equivalent to the property that* X *has no proper non empty closed invariant subset.*

We finish this section by defining the topological entropy of a subshift, which measures the exponential rate at which the cardinality of globally admissible patterns grows in the language of a subshift. To do this in $\mathbb{Z}$ is easy as one can look at the rate of growth of words of length n as n grows, whereas in a general group there is no standard way to do this and it is possible that this notion does not even have sense. For this reason we restrict ourselves to the case of amenable groups where we have a sequence of finite sets, namely a Følner sequence, that allows us to sample throughout the group to look at how patterns grow.

Definition 1.47 *Suppose that the group* G *is amenable and let* $\{F_n\}_{n\in\mathbb{N}}$ *be a (right) Følner sequence on it. We define the* ***topological entropy*** *of the* G*-subshift* X *as*

$$h_{\mathrm{top}}(X) \coloneqq \lim_{n\to\infty} \frac{\log |\mathcal{L}_{F_n}(X)|}{|F_n|}.$$

In the case $G = \mathbb{Z}$ *using the Følner sequence* $F_n \coloneqq \{0, \dots, n-1\}$*, we get*

$$h_{\mathrm{top}}(X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n(X)|.$$

The proof of the existence of this limit for the case $G = \mathbb{Z}$ can be found in [11]. For the general case of G being an amenable group we refer to [10], where it is also shown that the value of $h_{\mathrm{top}}(X)$ does not depend on which Følner sequence one chooses.

1.5 Substitutions

In this section we present the basic notions and properties of substitutions on $\mathbb{Z}$ and $\mathbb{Z}^d$ for $d \geq 2$, starting from the combinatorial definition of substitutions to then explore the dynamical properties of the subshifts they generate. Most of this section is based on [20], [3] and [6], where a more detailed introduction to substitutions can be found.

Let $\mathcal{A}$ be a finite alphabet. Recall that in the previous section we defined for a $\mathbb{Z}$-subshift $\mathcal{L}_n(X)$ to be the set of all words of length n that occurred in X and $\mathcal{L}(X)$ the set of all words occurring in X. In what follows we use the notations $\mathcal{A}^n \coloneqq \mathcal{L}_n(\mathcal{A}^{\mathbb{Z}})$ and $\mathcal{A}^+ \coloneqq \bigcap_{n\geq 1} \mathcal{A}^n$.

Definition 1.48 *A* ***substitution*** *on $\mathbb{Z}$ is a map $\tau : \mathcal{A} \to \mathcal{A}^+$, that is, a function that assigns to each letter a of the alphabet a (non-empty) word $\tau(a)$. This function can be extended to every word on $\mathcal{A}^+$ by concatenation: for a word $w = a_1 \dots a_n \in \mathcal{A}^+$ we define $\tau(w) = \tau(a_1)\dots\tau(a_n)$.*

With the above it makes sense to talk about τ^ℓ for any $\ell \geq 1$.

Examples 1.49 *1. The Fibonacci substitution is defined as $\tau_{\mathrm{Fib}} : \{0,1\} \to \{0,1\}^+$ where $\tau_{\mathrm{Fib}}(0) = 01$ and $\tau_{\mathrm{Fib}}(1) = 0$. We see that $\tau^n_{\mathrm{Fib}}(0)$ converges to the infinite word $\tau^\infty_{\mathrm{Fib}}(0) \coloneqq 0100101001001\dots$. Note that the lengths of the words $\tau^n_{\mathrm{Fib}}(0)$ for $n \geq 1$ satisfy the Fibonacci recurrence relation, whence the name of the substitution.*

2. The Thue-Morse substitution is defined as $\tau_{\mathrm{TM}} : \{0,1\} \to \{0,1\}^+$ where $\tau_{\mathrm{TM}}(0) = 01$ and $\tau_{\mathrm{TM}}(1) = 10$. $\tau^n_{\mathrm{TM}}(0)$ converges to the infinite word $\tau^\infty_{\mathrm{TM}}(0) \coloneqq 0110100110010110\dots$, called the Thue-Morse sequence. Similarly, $\tau^\infty_{\mathrm{TM}}(1)$ is the (infinite) word obtained by exchanging all 0's by 1's and vice versa in $\tau^\infty_{\mathrm{TM}}(0)$.

The first difference one notices between $\mathbb{Z}$ and $\mathbb{Z}^d$ for $d \geq 2$ when trying to define substitutions in the latter, is that the geometric shape of the patterns of a substitution matter, as they need to be glued together in a consistent way in order to be able to iterate the substitution. In this introduction we avoid this trouble by limiting ourselves to substitutions in $\mathbb{Z}^d$ which have as image patterns with support on the same rectangle $[\vec{\mathbb{1}}, \vec{n}] \coloneqq [1, n_1] \times \cdots \times [1, n_d]$, for $\vec{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ such that $n_i \geq 2$ for each $i = 1, \dots, d$. Nonetheless, it is possible to define substitutions with patterns over rectangles of different size, or even more creative shapes as the "chair substitution". See [21, Chapter 1] for an introduction to substitutions and tilings with more general shapes.

Definition 1.50 *A* ***substitution*** *on $\mathbb{Z}^d$ is defined as a map $\tau : \mathcal{A} \to \mathcal{A}^{[\vec{\mathbb{1}}, \vec{n}]}$, which maps each letter of the alphabet to a pattern on the rectangle $[\vec{\mathbb{1}}, \vec{n}]$.*

Example 1.51 *One can easily generalize the Thue-Morse substitution to $\mathbb{Z}^2$ defining $\tau_{\mathrm{TM}} : \{0,1\} \to \{0,1\}^{[\vec{\mathbb{1}}, \vec{2}]^2}$ by*

$$\tau_{\mathrm{TM}}(0) = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array}\,, \quad \tau_{\mathrm{TM}}(1) = \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline\end{array}\,.$$

Figure 1.3 shows a plot of a few iterates of the Thue-Morse substitution on $\mathbb{Z}^2$. The symbol 0 is represented as a white square and the symbol 1 as a black square.

Now we turn our attention to the relation between substitutions and dynamical systems, and their associated properties.

Definition 1.52 *Given a substitution $\tau : \mathcal{A} \to \mathcal{A}^{[\vec{\mathbb{1}},\vec{n}]}$ on $\mathbb{Z}^d$ with $d \geq 1$, we define the subshift associated to τ as*

$$X_\tau := \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} \mid \text{ for every pattern } p : \text{ if } p \sqsubseteq x \text{ then } p \sqsubseteq \tau^\ell(a) \text{ for some } a \in \mathcal{A}, \ell \geq 0 \right\}.$$

It is straightforward to check that this set is indeed a $\mathbb{Z}^d$-subshift.

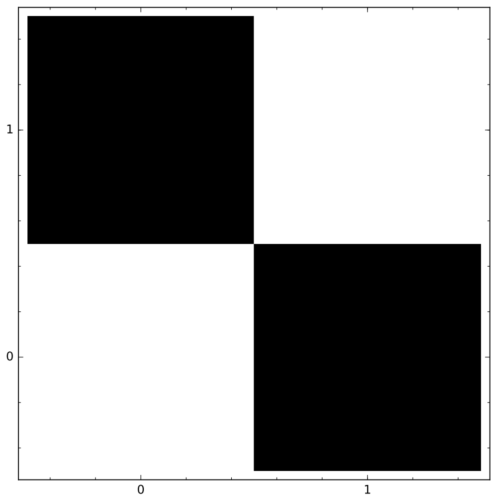
The first result we will mention about these subshifts is a relation between a combinatorial property of the substitution and the dynamical minimality of its associated subshift. We omit the proof of this fact as we will prove it later in the thesis, in the context of the Baumslag-Solitar group $\mathrm{BS}(1, N)$. The proof is adapted immediately for the cases mentioned here.

Definition 1.53 *A substitution τ is said to be* ***primitive*** *if there exists $\ell \geq 1$ such that for every $a, b \in \mathcal{A}$, the symbol b appears in $\tau^m(a)$.*

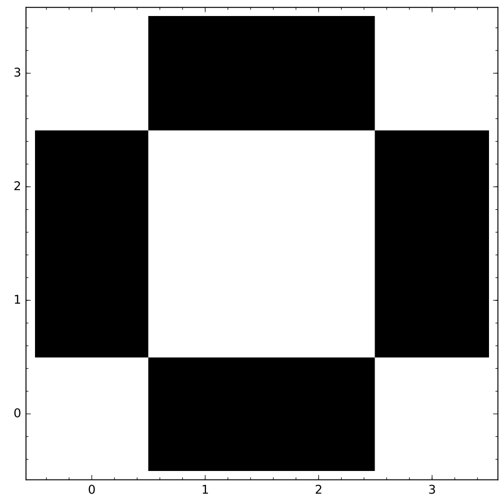
Theorem 1.54 *If τ is a primitive substitution on $\mathbb{Z}^d$, then its associated $\mathbb{Z}^d$-subshift X_τ is minimal.*

To finish this section we state **Mozes Theorem**, originally proven in [17], which provides conditions under which the subshift originating from a $\mathbb{Z}^2$ substitution is image of a $\mathbb{Z}^2$-SFT, that is, is a sofic subshift. Moreover, these techniques generalize to $\mathbb{Z}^d$ for $d \geq 2$ in order to prove the corresponding result on higher dimensions. Under the notion of substitution under which we have worked so far the theorem is stated as follows, though the original statement is more general.

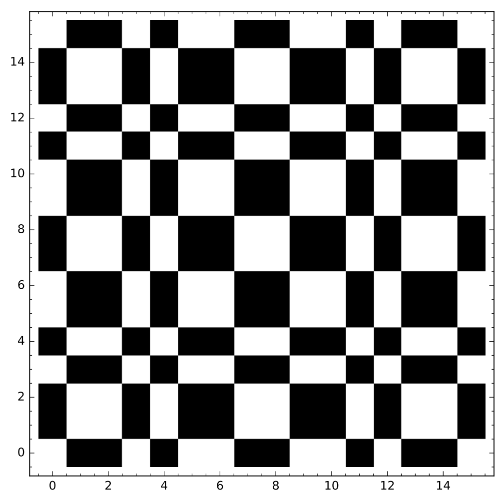
Theorem 1.55 (Mozes) *Let τ be a substitution on $\mathbb{Z}^d$ for $d \geq 2$. Then its associated subshift X_τ is a sofic $\mathbb{Z}^d$-subshift.*



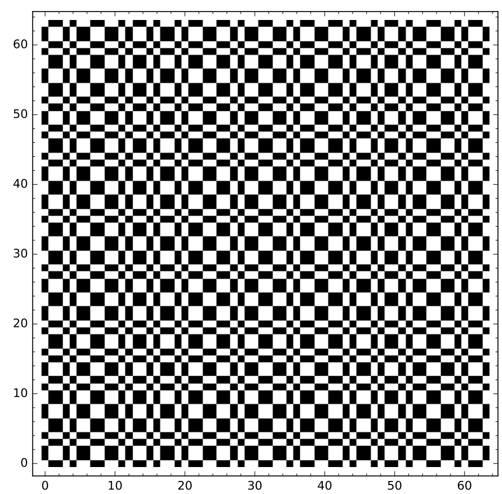
(a) $\tau_{\mathrm{TM}}(0)$.



(b) $\tau_{\mathrm{TM}}^2(0)$.



(c) $\tau_{\mathrm{TM}}^4(0)$.



(d) $\tau_{\mathrm{TM}}^6(0)$.

Figure 1.3: Some iterations of the Thue-Morse substitution τ_{TM} on $\mathbb{Z}^2$.

Chapter 2

The structure of periodic configurations

This chapter is focused on studying how periodic configurations on $\mathcal{A}^{\mathrm{BS}(1,N)}$ exhibit a sort of rigidity when their stabilizer and the structure of the Cayley graph of the group synchronize in some sense. In particular, we will study weak periodicity in the a-direction and obtain results regarding the structure of a-rows of configurations with this periodicity.

We start by giving the following definition fo the a-rows of $\mathrm{BS}(1,N)$, which are the subsets that look like a copy of $\mathbb{Z}$ in the direction of the generator a of the Cayley graph.

Definition 2.1 *For $g \in \mathrm{BS}(1,N)$ we define the **a-row containing** g as*

$$\Gamma_g := \{ga^m \mid m \in \mathbb{Z}\}.$$

*Using the normal form to write $g = b^{-j}a^k b^i$, for $i, j \geq 0$ and $k \in \mathbb{Z}$ such that $k \in N\mathbb{Z}$ is only possible if $ij = 0$, we say that Γ_g is an **a-row at level** $i-j$. Note that since the normal form of an element is unique, the level of the corresponding a-row Γ_g is well defined.*

Remark 2.2 *Given an element $g = b^{-j}a^k b^i \in \mathrm{BS}(1,N)$ written in its normal form, we have that for any $m \in \mathbb{Z}$:*

$$ga^m = b^{-j}a^k b^i a^m = b^{-j}a^{k+mN^i}b^i.$$

Then the a-row Γ_g has an alternative description given by

$$\Gamma_g = \left\{ b^{-j}a^{k+mN^i}b^i \mid m \in \mathbb{Z} \right\},$$

which has the advantage of showing in a clearer way which elements (written in their normal form) are adjacent to each other in the Cayley graph through an edge labeled by the a-generator.

2.1 Weak periodicity in the a-direction

Suppose we have a configuration $x \in \mathcal{A}^{\mathrm{BS}(1,N)}$ such that $a^N \in \mathrm{Stab}(x)$. This periodicity translates to the fact that the configuration stays the same by translating the Cayley graph of $\mathrm{BS}(1,N)$ N steps in the a-direction. Interpreting $x|_{\Gamma_{e_{\mathrm{BS}(1,N)}}}$ as a biinfinite sequence in $\mathcal{A}^{\mathbb{Z}}$ we have that $N \in \mathrm{Stab}(x|_{\Gamma_{e_{\mathrm{BS}(1,N)}}})$, and moreover we must have that $e_{\mathrm{BS}(1,N)} \in \mathrm{Stab}(x|_{\Gamma_b})$ and $e_{\mathrm{BS}(1,N)} \in \mathrm{Stab}(x|_{\Gamma_{ab}})$, from which we see that x can only have one symbol of the alphabet on each of the a-rows Γ_b and Γ_{ab}. Figure 2.1 illustrates this with $N = 2$, where we see that the first a-row exhibits period 2 on its symbols, and with it the next a-rows in the sheets originating from this base are forced to have period 1.

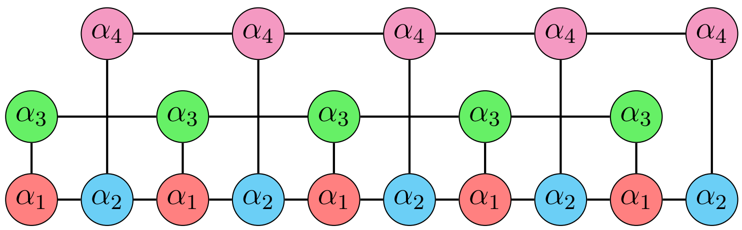


Figure 2.1: Example of a configuration x with $a^N \in \mathrm{Stab}(x)$.

Now suppose instead we have a point $x \in \mathcal{A}^{\mathrm{BS}(1,N)}$ such that $a^{N^2} \in \mathrm{Stab}(x)$. Similarly to the above case, using the relation of the group $\mathrm{BS}(1,N)$ we see that the sequences $x|_{\Gamma_{e_{\mathrm{BS}(1,N)}}}, x|_{\Gamma_b}$ and $x|_{\Gamma_{b^2}}$ must have periods N^2, N and 1, respectively. This is illustrated in Figure 2.2 showing part of a sheet of the Cayley graph of $\mathrm{BS}(1,2)$, where the base a-row has period $4 = 2^2$, the one above it has period $2 = 2^1$ and the next one period $1 = 2^0$.

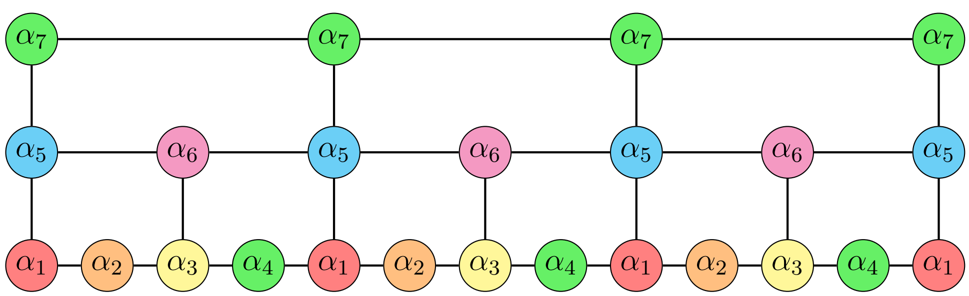


Figure 2.2: Example of a configuration x with $a^{N^2} \in \mathrm{Stab}(x)$.

The behaviour found in the previous examples may be stated as the fact that a configuration having an N-th power of a in its stabilizer is forced to have a-rows sufficiently high in the b-direction which are all monochromatic, in the sense that they have the same symbol repeated over and over again.

Proposition 2.3 *Let $x \in \mathcal{A}^{\mathrm{BS}(1,N)}$ and $\ell \geq 0$ be such that $\sigma_{a^{N^\ell}}(x) = x$. Then every a-row at a level greater or equal than ℓ in x is monochromatically colored. That is, for every $i, j \geq 0$ and $m, k \in \mathbb{Z}$:*

$$x_{b^{-j}a^k b^{j+i+\ell}} = x_{b^{-j}a^k b^{j+i+\ell}a^m}.$$

Moreover, for every $i \geq \ell$ and $h \geq 1$ all the a-rows at level $i+h$ sharing a common base of level i in x are equal. Formally this means that for every $j \geq 0$, $k \in \mathbb{Z}$ and $q \in \{0, \dots, N^h - 1\}$:

$$x_{b^{-j}a^k b^{j+i+h}} = x_{b^{-j}a^k b^{j+i}a^q b^h}$$

Proof. For the first part it suffices to prove the equality for $m = 1$, that is, that for every $i, j \geq 0$ and $k \in \mathbb{Z}$ we have

$$x_{b^{-j}a^k b^{j+i+\ell}} = x_{b^{-j}a^k b^{j+i+\ell}a}.$$

The general case follows directly from the above by induction.

Starting from the group element appearing in right-hand side, we have that

$$\begin{aligned}
b^{-j}a^k b^{j+i+\ell}a &= b^{-j}a^{k+N^{j+i+\ell}}b^{j+i+\ell} \\
&= b^{-j}a^{N^{i+\ell}N^j}a^k b^{j+i+\ell} \\
&= a^{N^i N^\ell}b^{-j}a^k b^{j+i+\ell}.
\end{aligned}$$

Now as $\sigma_{a^{N^\ell}}(x) = x$ we also have that $\sigma_{a^{N^{i+\ell}}}(x) = x$, and so

$$x_{a^{N^i N^\ell}b^{-j}a^k b^{j+i+\ell}} = x_{b^{-j}a^k b^{j+i+\ell}},$$

from which we achieve the claimed equality.

The above proves that each a-row at level greater or equal than ℓ is monochromatically colored by x since such a-rows are precisely Γ_g for $g = b^{-j}a^k b^{i+j+l}$ for arbitrary $i, j \geq 0$ and $k \in \mathbb{Z}$ respecting the conditions of the normal form.

Now let us prove the second statement. Consider $i \geq \ell$, $h \geq 1$, $j \geq 0$, $k \in \mathbb{Z}$ and $q \in \{0, \dots, N^h - 1\}$, and note that

$$\begin{aligned}
b^{-j}a^k b^{j+i}a^q b^h &= b^{-j}a^k b^{j+i}a^q b^h \\
&= b^{-j}a^k a^{qN^{j+i}}b^{j+i+h} \\
&= b^{-j}a^{qN^{j+i}}a^k b^{j+i+h} \\
&= b^{-j}a^{qN^i N^j}a^k b^{j+i+h} \\
&= a^{qN^i}b^{-j}a^k b^{j+i+h} \\
&= a^{qN^{i-\ell}N^\ell}b^{-j}a^k b^{j+i+h}.
\end{aligned}$$

Then as $\sigma_{a^{N^\ell}}(x) = x$ and $i \geq \ell$, we also have that $\sigma_{a^{qN^{i-\ell}N^\ell}}(x) = x$ and hence

$$x_{b^{-j}a^k b^{j+i}a^q b^h} = x_{a^{qN^{i-\ell}N^\ell}b^{-j}a^k b^{j+i+h}} = x_{b^{-j}a^k b^{j+i+h}}.$$

An arbitrary base at level i is represented by the a-row $\Gamma_{b^{-j}a^k b^{j+i}}$ where i, j, k are as above, and the a-rows at level $i+h$ arising from it are precisely the N^h a-rows $\Gamma_{b^{-j}a^k b^{j+i}a^q b^h}$ where $q \in \{0, \dots, N^h - 1\}$. The above together with the fact that each a-row is monochromatically colored by x shows that we have proven the second part of the statement, namely, that all a-rows at level $i+h$ sharing a common base a-row at level i in x are equal. □

Under the further assumption that x is a strongly periodic point we see that the behaviour seen in the previous proposition is extended to the entire group, allowing each a-row to only have one symbol on it.

Corollary 2.4 *Let $x \in \mathcal{A}^{\mathrm{BS}(1,N)}$ be a strongly periodic configuration such that $\exists \ell \geq 0$: $\sigma_{a^{N^\ell}}(x) = x$. Then for every $g \in \mathrm{BS}(1,N)$ the a-row Γ_g is monochromatically colored by x. That is, for every $m_1, m_2 \in \mathbb{Z}$:*

$$x_{ga^{m_1}} = x_{ga^{m_2}}.$$

Proof. To prove the result it suffices to show that for every $g \in \mathrm{BS}(1,N)$ and any $m \in \mathbb{Z}$:

$$x_{ga^m} = x_{ga^{m-1}}.$$

Since x is strongly periodic its orbit under the action of $\mathrm{BS}(1,N)$ is finite, and in particular the set $\{\sigma_{b^{-q}}(x) : q \geq 1\}$ must also be. Hence there exists an increasing sequence of positive integers $\{q_n\}_{n\geq 1}$ such that for every $n \geq 1$

$$\sigma_{b^{-q_1}}(x) = \sigma_{b^{-q_n}}(x),$$

or equivalently

$$\sigma_{b^{q_1-q_n}}(x) = x.$$

By taking n sufficiently large, one can find $k \geq \ell$ and $j \in \{0,\dots,N-1\}^k$ such that $b^{q_n-q_1}g \in \Gamma_k^{(j)}$ and using Proposition 2.3 we see that for $m \geq 1$:

$$\begin{aligned} x_{ga^m} &= (\sigma_{b^{q_1-q_n}}(x))_{ga^m} \\ &= x_{b^{q_n-q_1}ga^m} \\ &= x_{b^{q_n-q_1}ga^{m-1}} \\ &= (\sigma_{b^{q_1-q_n}}(x))_{ga^{m-1}} \\ &= x_{ga^{m-1}}. \end{aligned}$$

Hence every a-row Γ_g in x has the same symbol throughout it. □

Corollary 2.5 *Let $X \subseteq \mathcal{A}^{\mathrm{BS}(1,N)}$ be a $\mathrm{BS}(1,N)$-subshift and $x \in X$ such that there exists $\ell \geq 0$ such that $\sigma_{a^{N^\ell}}(x) = x$. Then there exists a configuration $y \in X$ such that each of its a-rows is monochromatic, and all a-rows of the same level in y are equal. Moreover, if X is an SFT then y can be chosen to be strongly periodic.*

Proof. Thanks to Proposition 2.3 we have that all a-rows at a level greater or equal than ℓ in x are monochromatic. Then we can define for each $n \geq 1$ the configuration $x^n \coloneqq \sigma_{b^{-n}}(x) \in X$ and define $y \in X$ to be a limit point of the sequence $\{x^n\}_{n\in\mathbb{N}}$, which exists by compactness of X.

As each a-row at a sufficiently high level of x is monochromatic and a-rows of the same level sharing a common base of a sufficiently high level in x are equal, the construction of the sequence $\{x^n\}_{n\in\mathbb{N}}$ immediately shows that y satisfies the required properties.

Now suppose that X is an SFT. As periodicity is preserved through a conjugacy map, we may suppose without loss of generality that X is a NNSFT. Considering the configuration

$y \in X$ from above, we see that by the pigeonhole principle there must exist an a-row Γ_g, $g \in \mathrm{BS}(1,N)$, and $h \geq 1$ such that itself together with all of the N^h a-rows $\Gamma_{ga^q b^h}$, for $q \in \{0, \dots, N^h - 1\}$, are equal. Without loss of generality we can assume that

$$y|_{\langle a \rangle} = y|_{\langle a \rangle b^m}.$$

Define a configuration $z \in \mathcal{A}^{\mathrm{BS}(1,N)}$ by

$$z_{b^{-j}a^k b^i} := y_{b^{i-j \pmod h}}, \ i, j \geq 0, k \in \mathbb{Z}.$$

Then z respects the forbidden patterns of X, and hence $z \in X$. It is also clear that $\sigma_a(z) = z$, since

$$z_{a^{-1}b^{-j}a^k b^i} = z_{b^{-j}a^{k-N^j}b^i} = y_{b^{i-j \pmod h}} = z_{b^{-j}a^k b^i}.$$

We also see that for every $i', j', i, j \geq 0$, $k', k \in \mathbb{Z}$:

$$\begin{aligned}
\sigma_{b^{-j'}a^{k'}b^{i'}}(z)_{b^{-j}a^k b^i} &= z_{b^{-i'}a^{-k'}b^{j'}b^{-j}a^k b^i} \\
&= z_{b^{-i'}a^{-k'}b^{j'-j}a^k b^i} \\
&= y_{b^{j'-j+i-i'} \pmod h} \\
&= y_{b^{j'-j+i-i'} \pmod h} \\
&= y_{b^{i-j-w} \pmod h} \quad \text{where } w = j' - i' \pmod h \\
&= z_{b^{-w}b^{-j}a^k b^i} \\
&= \sigma_w(z)_{b^{-j}a^k b^i}.
\end{aligned}$$

Hence we conclude that $\mathrm{Orb}(x) = \{\sigma_{b^w}(z)\}_{w=0}^{h-1}$, and with it z is a strongly periodic configuration. $\square$

Remark 2.6 *Having all a-rows monochromatic is not a sufficient condition for a configuration $x \in \mathcal{A}^{\mathrm{BS}(1,N)}$ to be strongly periodic. One can construct an example of such a configuration by considering a non-periodic sequence $z \in \mathcal{A}^{\mathbb{Z}}$ and define $x \in x \in \mathcal{A}^{\mathrm{BS}(1,N)}$ to consist of monochromatic a-rows whose symbols are defined according to the sequence z. Figure 2.3 shows a sideways view of the Cayley graph of* $\mathrm{BS}(1,N)$*, where each symbol in this figure represents the symbol of the entire a-row at that level in x.*

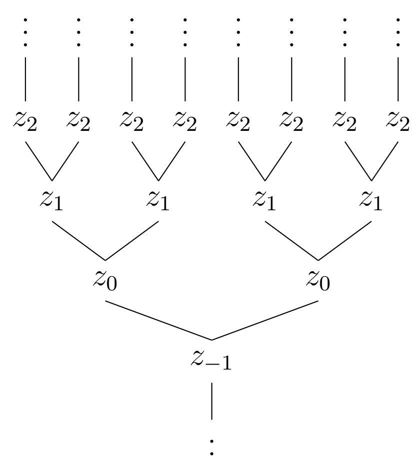


Figure 2.3: Illustration of a configuration in a sideways view of the Cayley graph of $\mathrm{BS}(1,N)$, having all a-rows monochromatic but not being strongly periodic.

Similar results to the ones proved above can be obtained if one considers the broader class of weakly periodic configurations $x \in \mathcal{A}^{\mathrm{BS}(1,N)}$ such that $\sigma_{a^{pN^\ell}}(x) = x$, for $\ell \geq 0$ and $p \notin N\mathbb{Z}$. In the remainder of this section we make this generalization, using the same ideas found in the proofs above.

Proposition 2.7 *Let $x \in \mathcal{A}^{\mathrm{BS}(1,N)}$, $\ell \geq 0$ and $p \notin N\mathbb{Z}$ be such that $\sigma_{a^{pN^\ell}}(x) = x$. Then every a-row of level greater or equal than ℓ is p-periodic, that is, for every $i, j \geq 0$, $k, m \in \mathbb{Z}$:*

$$x_{b^{-j}a^k b^{j+i+\ell} a^{m+p}} = x_{b^{-j}a^k b^{j+i+\ell} a^m}.$$

Moreover, if we further assume that $\gcd(p, N) = 1$, then for every $i \geq \ell$ and $h \geq 0$ all the a-rows of level $i + h$ sharing a common base a-row of level i are equal, up to a translation. More precisely, for every $h \geq 0$ there exists $r \in \mathbb{Z}$ such that for every $j \geq 0$, $i \geq \ell$, $k, m \in \mathbb{Z}$ and $q \in \{0, \dots, N^h - 1\}$:

$$x_{b^{-j}a^k b^{j+i+h} a^m} = x_{b^{-j}a^k b^{j+i} a^q b^h a^{m-qr}}.$$

Proof. Note that for $i, j \geq 0$ and $k, m \in \mathbb{Z}$ we have that

$$\begin{aligned} b^{-j}a^k b^{j+i+\ell} a^{m+p} &= b^{-j}a^k a^{pN^{j+i+\ell}} b^{j+i+\ell} a^m \\ &= b^{-j} a^{pN^{i+\ell}N^j} a^k b^{j+i+\ell} a^m \\ &= a^{pN^{i+\ell}} b^{-j} a^k b^{j+i+\ell} a^m \\ &= a^{N^i pN^\ell} b^{-j} a^k b^{j+i+\ell} a^m. \end{aligned}$$

As $\sigma_{a^{pN^\ell}}(x) = x$ we also have that $\sigma_{a^{N^i pN^\ell}}(x) = x$ and with it

$$\begin{aligned} x_{b^{-j}a^k b^{j+i+\ell} a^{m+p}} &= x_{a^{N^i pN^\ell} b^{-j} a^k b^{j+i+\ell} a^m} \\ &= x_{b^{-j}a^k b^{j+i+\ell} a^m}. \end{aligned}$$

This proves that every a-row of level greater or equal than ℓ is p-periodic, seen as a bi-infinite sequence.

For the second part of the proposition, note that as we assume that $\gcd(p, N) = 1$ then for every $h \geq 1$ we have $\gcd(p, N^h) = 1$ and hence by Bézout's identity there exist $r, s \in \mathbb{Z}$ such that $1 = sp + rN^h$. With this we get that for $j \geq 0$, $i \geq \ell$, $k, m \in \mathbb{Z}$ and $q \in \{0, \dots, N^h - 1\}$:

$$\begin{aligned} b^{-j}a^k b^{j+i} a^q b^h a^{m-qr} &= b^{-j}a^k b^{j+i} a^q a^{-qrN^h} b^h a^m \\ &= b^{-j}a^k b^{j+i} a^{q-qrN^h} b^h a^m \\ &= b^{-j}a^k a^{q(1-rN^h)N^{i+j}} b^{j+i} b^h a^m \\ &= b^{-j} a^{q(1-rN^h)N^i N^j} a^k b^{j+i} b^h a^m \\ &= a^{q(1-rN^h)N^i} b^{-j} a^k b^{j+i} b^h a^m \\ &= a^{qspN^{i-\ell}N^\ell} b^{-j} a^k b^{j+i} b^h a^m \\ &= a^{qsN^{i-\ell}pN^\ell} b^{-j} a^k b^{j+i} b^h a^m. \end{aligned}$$

Then as $\sigma_{a^{pN^\ell}}(x) = x$ we also have that $\sigma_{a^{qsN^{i-\ell}pN^\ell}}(x) = x$ and hence

$$x_{b^{-j}a^k b^{j+i} a^q b^h a^{m-qr}} = x_{a^{qsN^{i-\ell}pN^\ell} b^{-j} a^k b^{j+i} b^h a^m}$$

$$= x_{b^{-j}a^k b^{j+i} b^h a^m}.$$

The above proves that all a-rows $\Gamma_{b^{-j}a^k b^{j+i} a^q b^h}$ (for $q \in \{0, \ldots, N^h - 1\}$) at level $i + h$, originating from the a-row $\Gamma_{b^{-j}a^k b^{j+i}}$ have the same sequence in x, namely $\{x_{b^{-j}a^k b^{j+i} b^h a^m}\}_{m \in \mathbb{Z}}$, up to a translation of qr. $\square$

Remark 2.8 *The second part of the previous propositon still holds if one does not assume that* $\gcd(p, N) = 1$*, with the slight alteration of requiring* $i \geq \ell + 1$ *instead of* $i \geq \ell$*. To prove this it is sufficient to note that* $\sigma_{a^{pN^\ell}}(x) = x$ *and* $d := \gcd(p, N)$*, then we also have* $\sigma_{a^{\frac{N}{d}pN^\ell}}(x) = x$ *and hence*

$$\sigma_{a^{\frac{p}{d}N^{\ell+1}}}(x) = x,$$

from which we can apply Proposition 2.7.

Corollary 2.9 *Let* $x \in \mathcal{A}^{\mathrm{BS}(1,N)}$ *be a strongly periodic configuration such that there exists* $\ell \geq 0$ *such that* $\sigma_{a^{N^\ell}}(x) = x$*. Then for every* $g \in \mathrm{BS}(1, N)$ *the a-row* Γ_g *in* x *has period* p*. That is, for every* $m \in \mathbb{Z}$*:*

$$x_{ga^m} = x_{ga^{m+p}}.$$

Proof. Since x is strongly periodic, it has a finite orbit under the action of $\mathrm{BS}(1, N)$, so in particular the set $\{\sigma_{b^{-q}}(x) : q \geq 1\}$ is finite. Hence there exists a sequence of increasing positive integers $\{q_n\}_{n \geq 1}$ such that for every $n \geq 1$

$$\sigma_{b^{-q_1}}(x) = \sigma_{b^{-q_n}}(x),$$

or equivalently

$$\sigma_{b^{q_1 - q_n}}(x) = x.$$

By taking n sufficiently large, one can find $k \geq \ell$ and $j \in \{0, \ldots, N-1\}^k$ such that $b^{q_n - q_1} g \in \Gamma_k^{(j)}$ and using Proposition 2.3 we see that for $m \geq 1$:

$$\begin{aligned} x_{ga^m} &= (\sigma_{b^{q_1 - q_n}}(x))_{ga^m} \\ &= x_{b^{q_n - q_1} ga^m} \\ &= x_{b^{q_n - q_1} ga^{m+p}} \\ &= (\sigma_{b^{q_1 - q_n}}(x))_{ga^{m+p}} \\ &= x_{ga^{m+p}}. \end{aligned}$$

Hence every a-row Γ_g in x is a sequence of period p. $\square$

Corollary 2.10 *Let* $X \subseteq \mathcal{A}^{\mathrm{BS}(1,N)}$ *be a* $\mathrm{BS}(1, N)$*-subshift and* $x \in X$ *such that there exist* $\ell \geq 0$ *and* $p \notin N\mathbb{Z}$ *such that* $\sigma_{a^{pN^\ell}}(x) = x$*. Then there exists a configuration* $y \in X$ *such that each of its a-rows is p-periodic, and all a-rows of the same level in y are equal (up to translation).*

Proof. Thanks to Proposition 2.7 we have that all a-rows at a level greater or equal than ℓ in x are p-periodic. We define for each $n \geq 1$ the configuration $x^n := \sigma_{b^{-n}}(x) \in X$ and define $y \in X$ to be a limit point of the sequence $\{x^n\}_{n \in \mathbb{N}}$, which exists by compactness of X.

As each a-row at a sufficiently high level of x is p-periodic and a-rows of the same level sharing a common base of a sufficiently high level in x are equal, the construction of the sequence $\{x^n\}_{n\in\mathbb{N}}$ immediately shows that y satisfies the required properties. $\square$

Chapter 3

Substitutions on Baumslag-Solitar groups

As we said in Chapter 1 when we talked about $\mathbb{Z}^d$ substitutions, the main problem that arises when trying to extend the notion of a $\mathbb{Z}$ substitution to higher dimension or to other groups is being able to glue together patterns arising from different letters of the alphabet in a consistent way, allowing us to iterate the substitution indefinitely.

In this chapter we show that it is possible to define substitutions on the groups $\mathrm{BS}(1,N)$ by making use of the structure of rectangles (defined below) and a property of self-similarity of these sets. With this we are able to define a $\mathrm{BS}(1,N)$-subshift associated to a substitution and study its dynamical properties as it has been done in the case of $\mathbb{Z}$ and $\mathbb{Z}^d$ for $d \geq 2$. We finish the chapter proving a version of Mozes theorem for substitutions on $\mathrm{BS}(1,N)$.

3.1 Rectangles on Baumslag-Solitar groups

Inspired by the sets $[1,\ldots,n]^d$ we used in $\mathbb{Z}^d$ to define substitutions, we would like to find a similar structure in $\mathrm{BS}(1,N) = \langle a, b \mid bab^{-1} = a^N \rangle$ that allows us to decompose a bigger rectangle into smaller ones, and consider these sets as the support of the images of a substitution. With this in mind we make the following definition.

Definition 3.1 *For $m \geq 1$ we define the **rectangle of height* $\mathbf{m}$ *of* $\mathrm{BS}(1,N)$ *as*

$$R_m := \left\{ a^j b^k \;\mid\; 0 \leq j < N^m,\ 0 \leq k < m \right\}.$$

That is, the rectangle of height m is formed by considering a base of N^m elements in the a-direction, together with the elements of all the half-sheets that share that common base row up to height m (in the b-direction).

Example 3.2 *The first four rectangles of the group* $\mathrm{BS}(1,2)$ *are pictured below in Figure 3.1.*

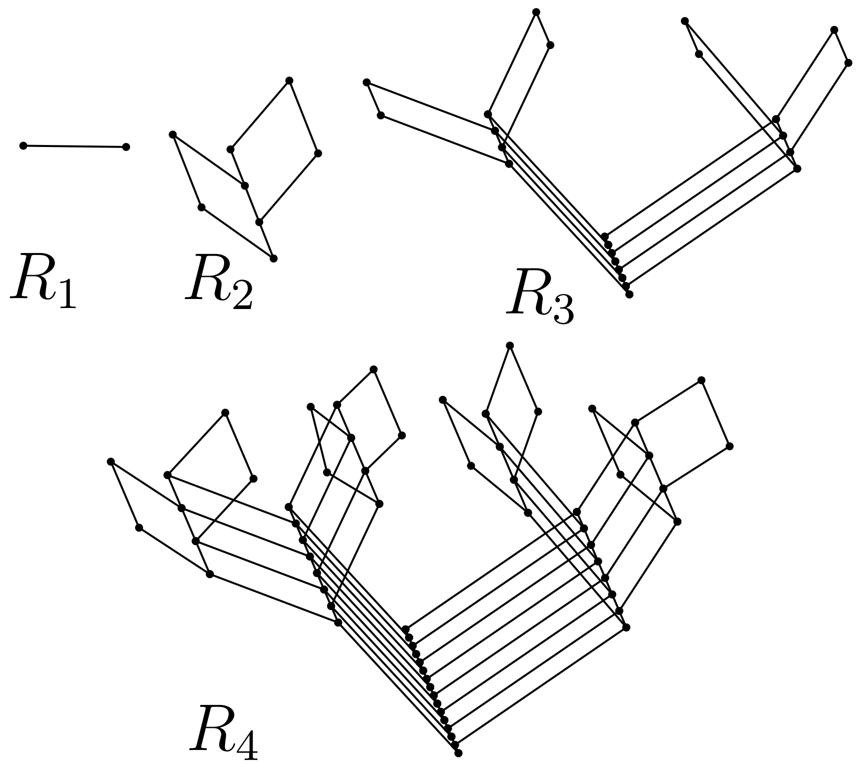


Figure 3.1: The first four rectangles R_m, for N=2.

A nice property of this definition is that thanks to the width of each rectangle in the a-direction in comparision with the height of it in the b-direction, the sequence of rectangles forms a Følner sequence in the group, which gives us a preferred sequence to utilize in constructions, in proofs that require gluing together patterns in order to define configurations in the whole group, and over which to define topological entropy.

Proposition 3.3 *The sequence* $\{R_m\}_{m\geq 1}$ *is a (right) Følner sequence for* $\mathrm{BS}(1,N)$.

Proof. Thanks to Lemma 1.17 it suffices to prove that

$$\lim_{m\to\infty}\frac{|R_m a\backslash R_m|}{|R_m|}=\lim_{m\to\infty}\frac{|R_m b\backslash R_m|}{|R_m|}=\lim_{m\to\infty}\frac{|R_m a^{-1}\backslash R_m|}{|R_m|}=\lim_{m\to\infty}\frac{|R_m b^{-1}\backslash R_m|}{|R_m|}=0.$$

First note that for $m\geq 1$: $|R_m|=mN^m$. With this

$$\begin{aligned}|R_m a\backslash R_m| &= \left|\left\{a^j b^k a | 0\leq j<N^m,\ 0\leq k<m\right\}\backslash R_m\right| \\ &= \left|\left\{a^{N^k+j}b^k\middle| 0\leq j<N^m,\ 0\leq k<m\right\}\backslash R_m\right| \text{ by using Lemma 1.24} \\ &= \left|\left\{a^{N^k+j}b^k\middle| N^m-N^k\leq j<N^m,\ 0\leq k<m\right\}\right| \\ &= \sum_{k=0}^{m-1}N^k=\frac{N^m-1}{N-1}.\end{aligned}$$

and so $\lim\limits_{m\to\infty} \dfrac{|R_m a \backslash R_m|}{|R_m|} = \lim\limits_{m\to\infty} \dfrac{\frac{N^m-1}{N-1}}{mN^m} = 0$. Similarly

$$\begin{aligned}
|R_m a^{-1} \backslash R_m| &= \left|\left\{a^j b^k a^{-1} | 0 \le j < N^m,\ 0 \le k < m\right\} \backslash R_m\right| \\
&= \left|\left\{a^{-N^k+j} b^k \middle| 0 \le j < N^m,\ 0 \le k < m\right\} \backslash R_m\right| \text{ by using Lemma 1.24} \\
&= \left|\left\{a^{-N^k+j} b^k \middle| 0 \le j < N^k,\ 0 \le k < m\right\}\right| \\
&= \sum_{k=0}^{m-1} N^k = \frac{N^m-1}{N-1}.
\end{aligned}$$

With the above we have that $\lim\limits_{m\to\infty} \dfrac{|R_m a \backslash R_m|}{|R_m|} = \lim\limits_{m\to\infty} \dfrac{|R_m a^{-1} \backslash R_m|}{|R_m|} = \lim\limits_{m\to\infty} \dfrac{\frac{N^m-1}{N-1}}{mN^m} = 0$, and so we have established th first and the third limits.

On the other hand,

$$\begin{aligned}
|R_m b \backslash R_m| &= \left|\left\{a^j b^k | 0 \le j < N^m,\ 1 \le k < m+1\right\} \backslash R_m\right| \\
&= \left|\left\{a^j b^m | 0 \le j < N^m\right\}\right| \\
&= N^m,
\end{aligned}$$

and

$$\begin{aligned}
|R_m b^{-1} \backslash R_m| &= \left|\left\{a^j b^k | 0 \le j < N^m,\ -1 \le k < m-1\right\} \backslash R_m\right| \\
&= \left|\left\{a^j b^{-1} | 0 \le j < N^m\right\}\right| \\
&= N^m.
\end{aligned}$$

With this $\lim\limits_{m\to\infty} \dfrac{|R_m b \backslash R_m|}{|R_m|} = \lim\limits_{m\to\infty} \dfrac{|R_m b^{-1} \backslash R_m|}{|R_m|} = \lim\limits_{m\to\infty} \dfrac{N^m}{mN^m} = 0$, and so we also have the second and fourth limits we needed. □

Remark 3.4 *From now on whenever we talk about topological entropy on for Baumslag-Solitar groups* $\mathrm{BS}(1,N)$ *we will be considering Definition 1.47 with the Følner sequence* $\{R_m\}_{m\ge1}$*, that is for any subshift* $X \subseteq \mathcal{A}^{\mathrm{BS}(1,N)}$*:*

$$h_{\mathrm{top}}(X) \coloneqq \lim_{m\to\infty} \frac{1}{|R_m|} \log |\mathcal{L}_{R_m}(X)|.$$

Now we proceed to prove the fundamental property about how rectangles of a particular height can be arranged to form a bigger rectangle, exhibiting a sense of self similarity between them, which will be key to define substitutions on this group. The decomposition shown in the following proposition says that the rectangle R_{2m} decomposes into $2N^m$ copies of the rectangle R_m, N^m of which form the “base” of the rectangle and cover the first m heights in the b-direction, meanwhile the remaining N^m copies distribute along the sheets starting on level m covering the remaining m heights.

Proposition 3.5 *For every $m \geq 1$ the rectangle R_{2m} is a disjoint union of $2N^m$ copies of the rectangle R_m. Moreover, the following decomposition holds:*

$$R_{2m} = \bigcup_{i=0}^{N^m-1}\!\!\!\!\!\cdot\;\; a^{iN^m} R_m \,\dot\cup \bigcup_{(i_1,\ldots,i_m)\in\{0,\ldots,N-1\}^m}\!\cdot\; \left(\prod_{s=1}^{m} a^{i_s} b\right) R_m \tag{3.1}$$

Proof. For $m \geq 1$, consider the decomposition of the rectangle R_{2m} according to heights less than m, and greater than or equal to m:

$$\begin{aligned} R_{2m} = \left\{a^j b^k : 0 \leq j < N^{2m},\ 0 \leq k < 2m\right\} = &\left\{a^j b^k | 0 \leq j < N^{2m},\ 0 \leq k < m\right\} \dot\cup \\ &\dot\cup \left\{a^j b^k \mid 0 \leq j < N^{2m},\ m \leq k < 2m\right\}. \end{aligned} \tag{3.2}$$

The first term of this decomposition gives us

$$\begin{aligned} \left\{a^j b^k \mid 0 \leq j < N^{2m},\ 0 \leq k < m\right\} &= \dot\bigcup_{i=0}^{N^m-1} \left\{a^j b^k \mid iN^m \leq j < (i+1)N^m,\ 0 \leq k < m\right\} \\ &= \dot\bigcup_{i=0}^{N^m-1} \left\{a^{j+iN^m} b^k \mid 0 \leq j < N^m,\ 0 \leq k < m\right\} \\ &= \dot\bigcup_{i=0}^{N^m-1} a^{iN^m} \left\{a^j b^k \mid 0 \leq j < N^m,\ 0 \leq k < m\right\} \\ &= \dot\bigcup_{i=0}^{N^m-1} a^{iN^m} R_m, \end{aligned}$$

from where we identify the first term of the union in Equation (3.1). Regarding the second term of the equation, we will prove that

$$\left\{a^j b^k \mid 0 \leq j < N^{2m},\ m \leq k < 2m\right\} = \dot\bigcup_{(i_1,\ldots,i_m)\in\{0,\ldots,N-1\}^m} \left(\prod_{s=1}^{m} a^{i_s} b\right) R_m.$$

With this the statement of the proposition follows, since we will have identified the second term of Equation (3.2) with the second union of Equation (3.1).

Let us see first the case $m = 1$. we see that

$$\begin{aligned} \{a^j b^k \mid 0 \leq j < N^2, 1 \leq k < 2\} &= \{a^j b \mid 0 \leq j < N^2\} \\ &= \bigcup_{i=0}^{N-1} \{a^{i+Nj} b \mid 0 \leq j < N\} \\ &= \bigcup_{i=0}^{N-1} a^i \{a^{Nj} b \mid 0 \leq j < N\} \\ &= \bigcup_{i=0}^{N-1} a^i \{b a^j \mid 0 \leq j < N\} \end{aligned}$$

$$= \bigcup_{i=0}^{N-1} a^i b R_1,$$

and so it follows. Here we saw that to get the result we had to partition the set $\{0, \dots, N^2-1\}$ in N sets of the form $\{i, i+N, \dots, i+(N-1)N\}$, for $i = 0, \dots, N-1$. The same reasoning allows us to prove this claim for $m \geq 2$: first we make the decomposition

$$\begin{aligned}
\left\{a^j b^k \mid 0 \leq j < N^{2m},\ m \leq k < 2m\right\} &= \dot{\bigcup_{i=0}^{N-1}} \left\{ a^j b^k \mid j \in \left\{i, i+N, \dots, i + \left\lfloor \frac{N^{2m} - i - 1}{N} \right\rfloor N\right\}, \right. \\
&\qquad \left. m \leq k < 2m \right\} \\
&= \dot{\bigcup_{i=0}^{N-1}} \left\{ a^j b^k \mid j \in \left\{i, i+N, \dots, i + N(N^{2m-1} - 1)\right\}, \right. \\
&\qquad \left. m \leq k < 2m \right\} \\
&= \dot{\bigcup_{i=0}^{N-1}} \left\{a^{i+Nj} b^{k+m} \mid 0 \leq j < N^{2m-1},\ 0 \leq k < m\right\}.
\end{aligned}$$

Then we get

$$\begin{aligned}
\left\{a^j b^k \mid 0 \leq j < N^{2m},\ m \leq k < 2m\right\} &= \dot{\bigcup_{i=0}^{N-1}} \left\{a^{i+Nj} b^{k+m} \mid 0 \leq j < N^{2m-1},\ 0 \leq k < m\right\} \\
&= \dot{\bigcup_{i=0}^{N-1}} \left\{a^i a^{Nj} b^m b^k \mid 0 \leq j < N^{2m-1},\ 0 \leq k < m\right\} \\
&= \dot{\bigcup_{i=0}^{N-1}} \left\{a^i b a^j b^{m-1} b^k \mid 0 \leq j < N^{2m-1},\ 0 \leq k < m\right\} \\
&= \dot{\bigcup_{i=0}^{N-1}} a^i b \left\{a^j b^{m-1} b^k \mid 0 \leq j < N^{2m-1},\ 0 \leq k < m\right\} \\
&= \dot{\bigcup_{i=0}^{N-1}} a^i b \left\{a^j b^k \mid 0 \leq j < N^{2m-1},\ m-1 \leq k < 2m-1\right\}.
\end{aligned}$$

By repeating the above argument $m-1$ more times, we arrive at the claimed equality. $\square$

Example 3.6 *In the group* $\mathrm{BS}(1,2)$ *with* $m = 2$ *the rectangle* $R_{2m} = R_4$ *is formed by* $2N^m = 8$ *copies of the rectangle* R_2*, as seen in Figure 3.2.*

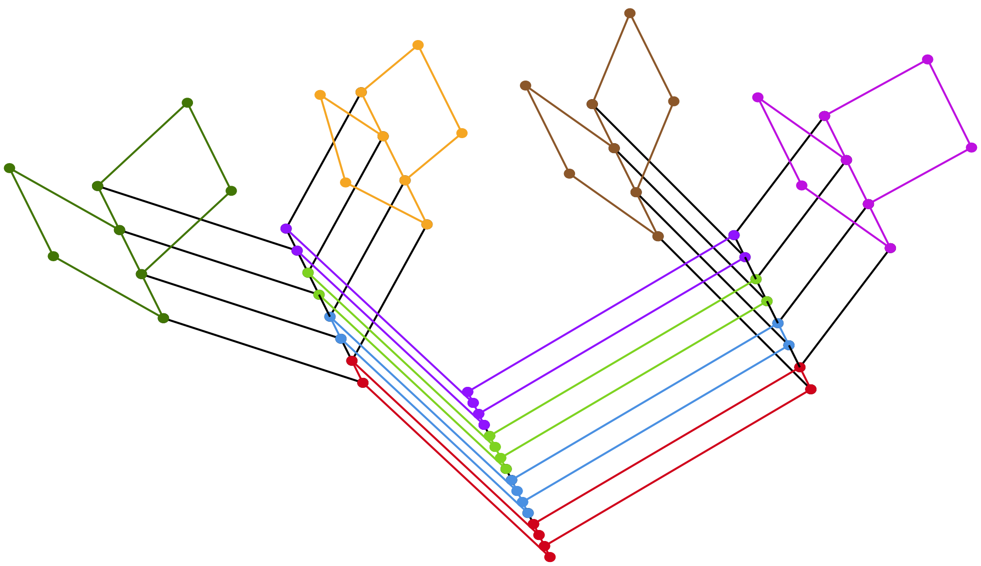


Figure 3.2: In $BS(1,2)$ the rectangle R_4 is formed by 8 copies of R_2, each copy displayed with a different color.

3.2 Substitutions

Now that we have understood the hierarchical structure of rectangles given by Proposition 3.5 we are almost ready to give a notion of substitutions on $\mathrm{BS}(1,N)$. Substitutions will map symbols to patterns on rectangles, and the iteration process will consist in replacing symbols from those rectangular patterns by their image through the substitution thus forming a pattern on a bigger rectangle.

Before making the precise definition we need the following simple lemma for it to make sense.

Lemma 3.7 *For any $m \geq 1$ and $i_1, \ldots, i_m \in \{0, \ldots, N-1\}$ we have that*

$$\left(\prod_{s=1}^{m} a^{i_s} b\right) b^{-1} \in R_m.$$

Proof. We will prove this by induction on m. If $m = 1$, then $\left(\prod_{s=1}^{1} a^{i_s} b\right) b^{-1} = a^{i_1} \in R_1$ because $i_1 \in \{0, \ldots, N-1\}$.

Now assume the lemma is true for m, $m \geq 1$, and let us prove it for $m+1$. We have

$$\left(\prod_{s=1}^{m+1} a^{i_s} b\right) b^{-1} = \left(\prod_{s=1}^{m} a^{i_s} b\right) a^{i_{m+1}} = \left(\prod_{s=1}^{m} a^{i_s} b\right) b^{-1} b a^{i_{m+1}} \tag{3.3}$$

and by the induction hypothesis $\left(\prod_{s=1}^{m} a^{i_s} b\right) b^{-1} \in R_m$, and so for some $0 \leq j < N^m$ and $0 \leq k < m$ the following equality holds:

$$\left(\prod_{s=1}^{m} a^{i_s} b\right) b^{-1} = a^j b^k.$$

Replacing the above in Equation (3.3):

$$\left(\prod_{s=1}^{m+1} a^{i_s} b\right) b^{-1} = a^j b^k b a^{i_{m+1}} = a^j b^{k+1} a^{i_{m+1}} = a^{j+i_{m+1}N^{k+1}} b^{k+1} \text{ using the fact that } ba = a^N b.$$

Noticing that $0 \leq k+1 < m+1$ and that $0 \leq j + i_{m+1}N^{k+1} < N^m + (N-1)N^m = N^{m+1}$ we conclude that

$$\left(\prod_{s=1}^{m+1} a^{i_s} b\right) b^{-1} = a^{j+i_{m+1}N^{k+1}} b^{k+1} \in R_{m+1},$$

finishing the proof. □

Definition 3.8 *Given a (non empty) finite alphabet $\mathcal{A}$, a* ***substitution*** *on* BS$(1,N)$ *is a map $\tau : \mathcal{A} \to \mathcal{A}^{R_m}$, for some fixed $m \geq 1$.*

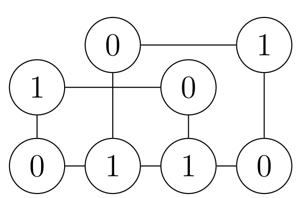
A substitution $\tau : \mathcal{A} \to \mathcal{A}^{R_m}$ can be extended recursively to $\tau^\ell : \mathcal{A} \to \mathcal{A}^{R_{2^{\ell-1}m}}$ for $\ell \geq 1$ by $\tau^1 = \tau$ and $\ell \geq 1$, by setting

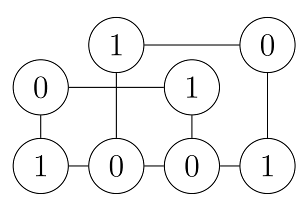
$$\tau^{\ell+1}(\alpha)_{a^{iN^{2^{\ell-1}m}}r} = \tau^{\ell}\left(\tau^{\ell}(\alpha)_{a^i}\right)_r, \text{ for } \alpha \in \mathcal{A}, i \in \left\{0, \dots, N^{2^{\ell-1}m} - 1\right\} \text{ and } r \in R_{2^{\ell-1}m}.$$

$$\tau^{\ell+1}(\alpha)_{\left(\prod_{s=1}^{2^{\ell-1}m} a^{i_s} b\right) r} = \tau^{\ell}\left(\tau^{\ell}(\alpha)_{\left(\prod_{s=1}^{2^{\ell-1}m} a^{i_s} b\right) b^{-1}}\right)_r, \text{ for } \alpha \in \mathcal{A}, i_1, \dots, i_{2^{\ell-1}m} \in \{0, \dots, N-1\}$$
$$\text{and } r \in R_{2^{\ell-1}m}.$$

Note that this definition makes sense thanks to Proposition 3.5 and Lemma 3.7. Intuitively the symbols along the base row of the rectangle $R_{2^{\ell-1}m}$ *blow up* to generate the lower part (the first $2^{\ell-1}m$ heights) of the rectangle $R_{2^\ell m}$ meanwhile the rest of this rectangle (i.e. the remaining $2^{\ell-1}m$ heights) is filled with images of the symbols from the uppermost row on every sheet.

Example 3.9 *Inspired by the Thue-Morse substitutions defined for $\mathbb{Z}$ and $\mathbb{Z}^d$ we define the* BS$(1,2)$*-Thue-Morse substitution to be the one with alphabet $\mathcal{A} := \{0,1\}$ and substitution map $\tau : \mathcal{A} \to \mathcal{A}^{R_2}$ given by*

$\tau(0) =$ 

, $\tau(1) =$ 

.

Its second iteration $\tau^2(0)$ is supported on the rectangle R_4 and shown in Figure 3.3. Note that thanks to how we defined this substitution every word seen on supports of the form $\{g, ga, \ldots, ga^n\}$ or $\{g, gb, \ldots, gb^n\}$, for $g \in G$ and $n \geq 0$ is a subword of the Thue-Morse sequence.

Now that we have defined substitutions on $\mathrm{BS}(1, N)$, we can proceed to study the dynamical properties that arise from them.

Definition 3.10 *Given a substitution τ on $\mathrm{BS}(1, N)$, the* ***$\mathrm{BS}(1, \mathbf{N})$-subshift induced by*** *τ is defined as*

$$X_\tau \coloneqq \left\{x \in \mathcal{A}^{\mathrm{BS}(1,N)} \mid \text{for every pattern } p : p \sqsubseteq x \implies \exists \ell \geq 0 \text{ and } \alpha \in \mathcal{A} \text{ such that } p \sqsubseteq \tau^\ell(\alpha)\right\}.$$

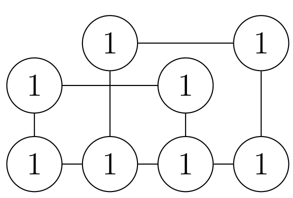
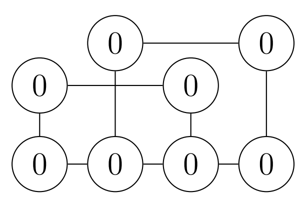
Proposition 3.11 *For every substitution τ its associated subshift X_τ is not empty.*

Proof. Consider a substitution $\tau : \mathcal{A} \to \mathcal{A}^{R_m}$ and fix $\alpha \in \mathcal{A}$. For each $n \geq 1$ define a point $x^n \in \mathcal{A}^{\mathrm{BS}(1,N)}$ such that $x|_{R_{2^n m}} = \tau^n(\alpha)$ and such that on the rest of the elements of the group we have $x_g = \alpha$. Then by compactness of $\mathcal{A}^{\mathrm{BS}(1,N)}$ the sequence $\{x^n\}_{n\in\mathbb{N}}$ subconverges to a point $x \in \mathcal{A}^{\mathrm{BS}(1,N)}$.

Now define another sequence $\{y^n\}_{n\in\mathbb{N}}$ such that $y^n = \sigma_{a^{-n}b^{-n}}(x)$ and again by compactness this sequence subconverges to a point $y \in \mathcal{A}^{\mathrm{BS}(1,N)}$. We claim that this point satisfies $y \in X_\tau$, since if we look at any pattern p we can find n sufficiently large such that $p \sqsubseteq y^n$ and the normal form of all elements of p is of the form $b^{-j}a^k b^i$ with $0 \leq j \leq n-1$, so p is really a subpattern of the portion of a point $x^{n'}$ on the part of the group that was determined by the substitution, from which we conclude. $\square$

Example 3.12 *The subshift associated with the Thue-Morse substitution of Example 3.9 consists of the closure of the orbit of the limit point constructed in the proof of the previous proposition. In effect, consider $x \in X_\tau$. Then for every pattern $p \sqsubseteq x$ there exists $\ell \geq 0$ and $\alpha \in \{0, 1\}$ such that $p \sqsubseteq \tau^\ell(\alpha)$. Using the fact that $0 \sqsubseteq \tau(1)$ and $1 \sqsubseteq \tau(0)$ we see that $p \sqsubseteq \tau^{\ell+1}(1-\alpha)$. Since the pattern p was arbitrary, we have proven that x can be approximated by patterns occurring in the configuration constructed in Proposition 3.11.*

This behaviour does not always happen: if one considers the substitution given by

$\tau(0) =$  , $\tau(1) =$  .

we see that the subshift associated to it has two points: one where each group element sees the symbol 0 *and another one where each element sees the symbol* 1.

The main dynamical difference we see between both subshifts from the previous example is that in the first one the orbit of any configuration is dense while in the second one we have

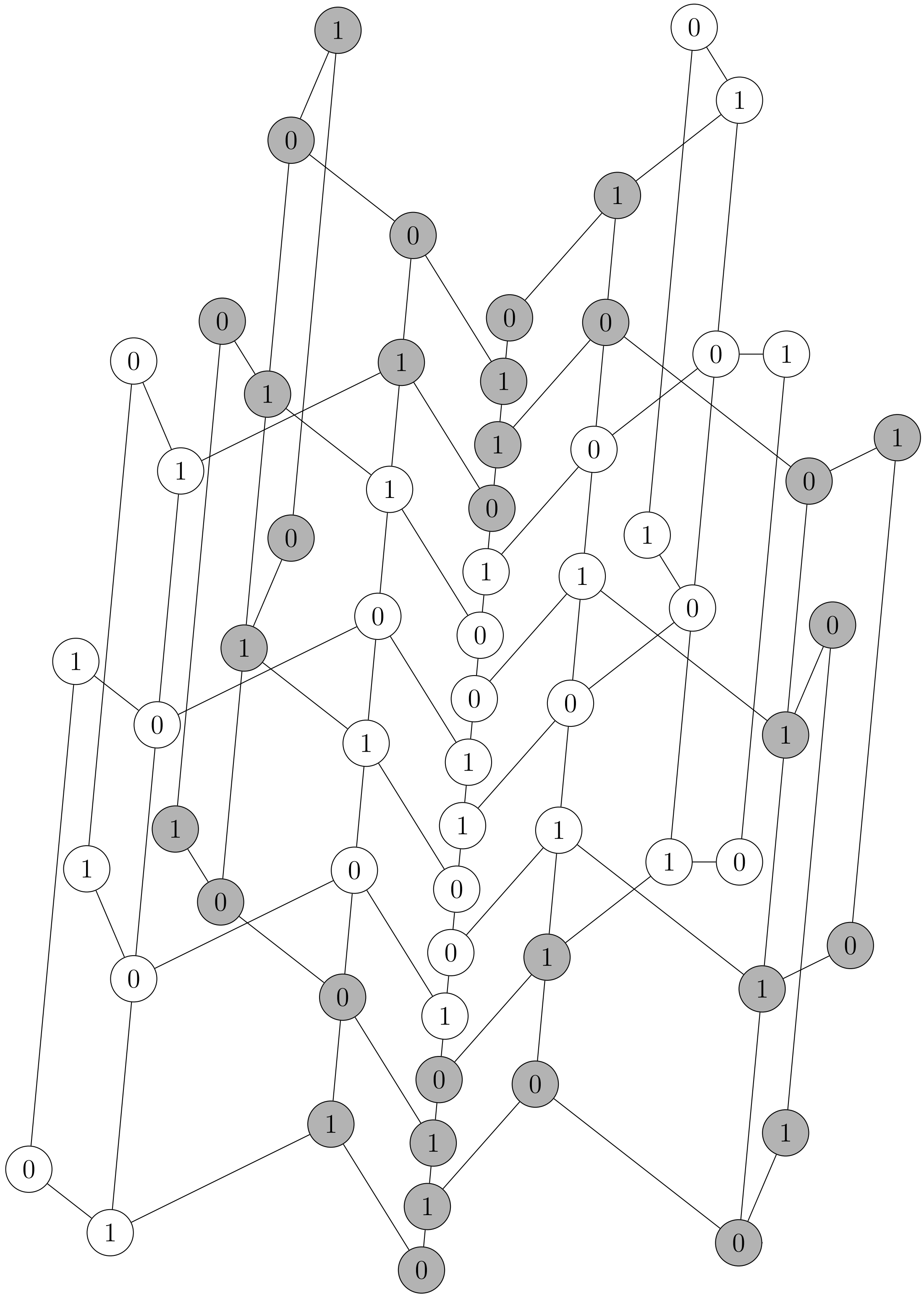


Figure 3.3: The configuration $\tau^2(0)$ for the Thue-Morse substitution defined in Example 3.9. Rectangles R_2 in grey are those arising from $\tau(0)$, meanwhile those in white are arising from $\tau(1)$.

two disjoint subsystems. Hence the first system is minimal whereas the second one is not. Just like in the case of $\mathbb{Z}^d$ subshifts this behaviour is due to a combinatorial property of the substitution, namely, the property of being primitive.

Definition 3.13 *A substitution τ is said to be* ***primitive*** *if there exists $M \geq 1$ such that for every $\alpha, \beta \in \mathcal{A}: \ \beta \sqsubset \tau^M(\alpha)$. That is, there exists an iteration of the substitution such that every symbol of the alphabet appears in the image of every symbol.*

Proposition 3.14 *For every primitive substitution $\tau : \mathcal{A} \to \mathcal{A}^{R_m}$ its associated subshift X_τ is minimal.*

Proof. We will prove that for $x \in X_\tau$ (arbitrary) we have $X_\tau = \overline{\mathrm{Orb}_\sigma(x)}$. Moreover, by the definition of X_τ, to prove this it is sufficient to show that for every $\ell \geq 1$ and for every $\alpha \in \mathcal{A} : \tau^\ell(\alpha) \sqsubseteq x$.

Let $M \geq 1$ be as in the definition of τ being primitive. Then for all $\alpha, \beta \in \mathcal{A}$ we have $\alpha \sqsubset \tau^M(\beta)$ and further, thanks to the definition of the iterates of τ, that for every $\ell \geq 1 : \tau^\ell(\alpha) \sqsubset \tau^{M+\ell}(\beta)$. Take $\ell \geq 1$ and $\alpha \in \mathcal{A}$. Then as $x|_{R_{2(\ell+M+1)-1_m}} \sqsubseteq x$, by the definition of X_τ there must exist $i \geq 1$ and $\gamma \in \mathcal{A}$ such that $x|_{R_{2(\ell+M+1)-1_m}} \sqsubseteq \tau^i(\gamma)$ and of course with $i \geq \ell + M + 1$ for this to make sense. With this there must exist a $\beta \in \mathcal{A}$ such that $\tau^{\ell+M}(\beta) \sqsubseteq x|_{R_{2(\ell+M+1)-1_m}}$ and by what was said earlier this implies $\tau^\ell(\alpha) \sqsubseteq x$, which concludes the proof. □

Example 3.15 *As was shown in Example 3.12, the Thue-Morse substitution is primitive and hence its associated subshift X_τ is minimal.*

Corollary 3.16 *Let X_τ be the shift associated with a primitive substitution τ. Then either X_τ is finite or X_τ contains no (strongly) periodic points.*

Proof. Let us suppose that τ is a primitive substitution and that X_τ has a strongly periodic point x. By the previous proposition X_τ is minimal and so we have we have

$$|X_\tau| = |\overline{\mathrm{Orb}_\sigma(x)}| = |\mathrm{Orb}_\sigma(x)| < \infty,$$

as x was periodic and hence its orbit is finite. □

A last property of substitutions we mention in this section is that under the hypothesis of primitivity, the subshift associated to the substitution remains unchanged if one originally considers an iterate of it.

Proposition 3.17 *Let $\tau : \mathcal{A} \to \mathcal{A}^{R_m}$ be a primitive substitution. Then for every $j \geq 1$: $X_\tau = X_{\tau^j}$.*

Proof. Consider a configuration $x \in X_{\tau^j}$. Then for every pattern $p \sqsubseteq x$ there must exists $\alpha \in \mathcal{A}$ and $\ell \geq 1$ such that $p \sqsubseteq (\tau^j)^\ell(\alpha)$. Then $p \sqsubseteq \tau^{j\ell}(\alpha)$ and hence we conclude that $x \in X_\tau$.

Now suppose we have $x \in X_\tau$ and any pattern $p \sqsubseteq x$. By the definition of X_τ there exists $\alpha \in \mathcal{A}$ and $\ell \geq 1$ such that $p \sqsubseteq \tau^\ell(\alpha)$. As τ is primitive, for every sufficiently large k, $\tau^\ell(\alpha) \sqsubseteq \tau^k(\alpha)$, and in particular for $k \in j\mathbb{N}$. With this $p \sqsubseteq \tau^{rj}(\alpha) = (\tau^j)^r(\alpha)$ for $r \geq 1$ sufficiently large, and as p was arbitrary we have proven that $x \in X_{\tau^j}$. □

3.3 Mozes theorem

In this section we prove a version of Mozes theorem for substitutions of Baumslag-Solitar groups, which roughly states that under appopiate hypothesis the subshift defined by a substitution is sofic. As we said in the first Chapter, the classic version of this theorem refers to substitutions on $\mathbb{Z}^2$ and the ideas quickly generalize to $\mathbb{Z}^d$ for $d \geq 2$. It was originally published in [17] and to be able to prove it in our context we will need analogue results and ideas to the ones found in this paper.

The next theorem asserts that for every configuration $x \in X_\tau$ we can divide the Cayley graph of $\mathrm{BS}(1, N)$ into disjoint copies of R_m, such that in each one of these copies x sees the image of a symbol through τ. This theorem will be essential to prove the surjectivity of the factor map which we will construct later when proving Mozes theorem.

Theorem 3.18 *Let $\tau : \mathcal{A} \to \mathcal{A}^{R_m}$ be a $\mathrm{BS}(1, N)$-substitution and $x \in X_\tau$ a point in its associated $\mathrm{BS}(1, N)$-subshift X_τ. Then there exists a partition of $\mathrm{BS}(1, N)$ into copies of R_m such that the pattern obtained by restricting x to each one of this copies is the pattern $\tau(\alpha)$, where $\alpha \in \mathcal{A}$ is a symbol (depending on the chosen copy of R_m).*

Proof. For each $g \in \mathrm{BS}(1, N)$ we will assign a letter α to g and to the rest of the elements g' of a rectangle R_m containing g. This will allow us to construct a point y satisfying that x can be recovered by replacing each symbol of y by its image through τ, and finally use the construction of this point to prove that $y \in X_\tau$.

Let us consider an order of the group $\mathrm{BS}(1, N)$ given by $\{g_i : i \geq 1\}$, and for each $M \geq 1$ define $P_M := x|_{B(e_G, M)}$, where $B(e_G, M) := \{g \in \mathrm{BS}(1, N) \mid |g| \leq M\}$ and $|g|$ is the length of g in the word metric associated to the generating set $\{a, b\}$ of $\mathrm{BS}(1, N)$ (see Definition 1.22). Note that since $x \in X_\tau$ then for every $M \geq 1$ there must exist $\alpha_M \in \mathcal{A}$ and $\ell_M \in \mathbb{N}$ such that $P_M \sqsubseteq \tau^{\ell_M}(\alpha_M)$, so we can define the set $S^0 := \{(\alpha_M, \ell_M) \in \mathcal{A} \times \mathbb{N} \mid M \geq 1\}$. In what follows we inductively construct a decreasing sequence $\{S^k\}_{k \geq 0}$ of subsets of S^0.

Focus on x_{g_0}: there are infinitely many $M's$ such that $g_0 \in B(e_G, M)$, and thus infinite pairs $(\alpha_M, \ell_M) \in S^0$ such that $x_{g_0} \sqsubseteq p_M \sqsubseteq \tau^{\ell_M}(\alpha_M)$. Moreover, since the alphabet $\mathcal{A}$ is finite, by the pigeonhole principle there must exist $\alpha \in \mathcal{A}$ such that for infinitely many M we have $x_{g_0} \sqsubseteq \tau(\alpha) \sqsubseteq \tau^{\ell_M}(\alpha_M)$, for which x_{g_0} appears in the same position of the rectangle every time. Denote the set of these corresponding pairs (α_M, ℓ_M) by $S^1 \subseteq S^0$, and assign the letter α to g_0 and to all the other elements appearing in the same rectangle $\tau(\alpha)$ from which x_{g_0} is part of.

On the k-th step of this process consider $g_{j_k} \in \{g_i : i \geq 1\}$ the first element (in the order assigned to $\mathrm{BS}(1,N)$ at the beginning of the proof) to which a symbol has not been assigned yet. By definition of S^k there must be infinitely many pairs $(\alpha_M, \ell_M) \in S^k$ defining rectangles containing as subpattern $x_{g_{j_k}}$ and therefore by the pigeonhole principle we can find $\alpha \in \mathcal{A}$ for which infinitely many $(\alpha_M, \ell_M) \in S^k$ satisfy $x_{g_{j_k}} \sqsubseteq \tau(\alpha) \sqsubseteq \tau^{\ell_M}(\alpha_M)$, on which $x_{g_{j_k}}$ appears on the same position of the rectangle $\tau(\alpha)$. Let us define $S^{k+1} \subseteq S^k$ to be the (infinite) set of pairs (α_M, ℓ_M) that satisfy the above, and assign to g_{j_k} as well as to the rest of the elements appearing on the rectangle $\tau(\alpha)$ the letter α.

Doing the above step for every $k \geq 0$ leads to a division of the Cayley graph of $\mathrm{BS}(1,N)$ into rectangles R_m, each of which has a letter α associated to it, and the restriction of x to this rectangle reads $\tau(\alpha)$.

□

The version of Mozes Theorem we state for Baumslag-Solitar groups will be only for substitutions that are *settling*, according to the next definition. These substitutions are those for which every symbol re-appears on the identity position of the rectangle of its image. Hence iterations of the substitution repeat the previous step around the origin of the rectangle, as the iterations grow. This behavior is where the name "settling" comes from.

Definition 3.19 *A substitution $\tau : \mathcal{A} \to \mathcal{A}^{R_m}$ is said to be* ***settling*** *if for every $\alpha \in \mathcal{A}$ we have $\tau(\alpha)_{e_{\mathrm{BS}(1,N)}} = \alpha$.*

Now we proceeed to give the main construction of the extension that will factor onto the substitutive subshift and with it prove its soficity. For simplicity we will have in mind the case $\mathrm{BS}(1,2)$ for figures and when making the construction of the extension, but the ideas generalize to the case $\mathrm{BS}(1,N)$.

Consider a settling substitution $\tau : \mathcal{A} \to \mathcal{A}^{R_m}$ together with its associated subshift X_τ. We will construct a new alphabet $\widehat{\mathcal{A}}$ and a $\mathrm{BS}(1,N)$-SFT $\widehat{X_\tau} \subseteq \widehat{\mathcal{A}}^{\mathrm{BS}(1,N)}$ using as symbols of the alphabet $\mathcal{A}$ tiles with the shape of a cross (to emphasize the dependence on the generators of the group) with information written on them describing the substitution and forcing local rules to be followed as described below.

Each tile of $\widehat{\mathcal{A}}$ will contain:

- a letter of the alphabet $\mathcal{A}$ appearing on the rectangle $\tau(\alpha)$ for some $\alpha \in \mathcal{A}$.

- A position inside R_m, written in the normal form of $\mathrm{BS}(1,N)$.

For example, the information described in these first two items could be as the one seen in Figure 3.4. meaning that this tile represents the position a^0b^1 of the rectangle R_m of $\tau(0)$, which has the symbol 1 on it. We will refer to this information as the **substitution information**, and commonly write it in black in drawings.

- A tile will also contain on its four edges information about the tiles that are allowed to be surrounding it on a bigger pattern, consistent with the letter information described above appearing on itself and on its neighbors. This information can be of two types:

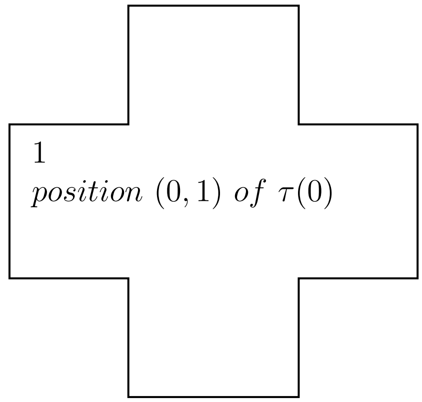


Figure 3.4: Example of the substitution information in a tile.

1. **Inner Joints:** They connect tiles that form part of the same rectangle that is image of a substitution with adjacent positions, for the symbols written on them. This information has the structure

$$\text{“}\textit{connect position}\ (i,j)\ \text{of}\ \tau(\alpha)\ \textit{with position}\ (k,l)\ \text{of}\ \tau(\alpha)\text{”}$$

where $\alpha \in \mathcal{A}$, $k \in \{i+1, i-1\}$ and $l = j$, or $k = i$ and $l \in \{j-1, j+1\}$, $i, k \in \{0, \ldots, m-1\}$ and $j, l \in \{0, \ldots, N^m - 1\}$. We will write the inner joints in purple in drawings.

The local rules we impose for the SFT $\hat{X}_\tau$ force that the inner joints information must be consistent with the substitution information written in each tile. Below in Figure 3.5 we see how two tiles with inner joint information can connect validly.

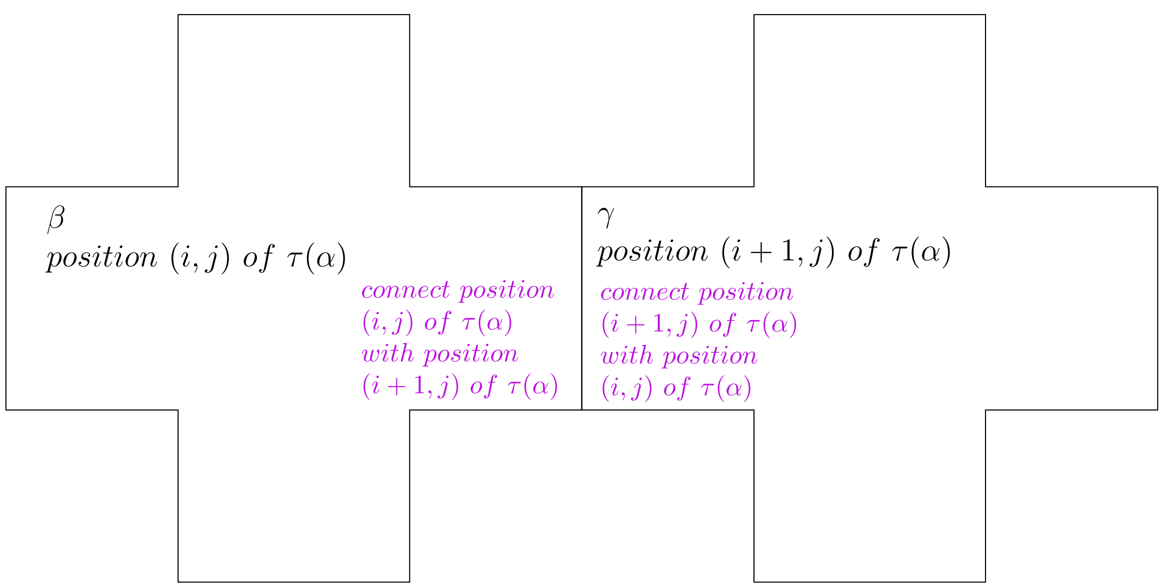


Figure 3.5: Example of the inner joint information in two adjacent tiles.

2. **Outer Joints:** They are similar to inner joints, with the difference that they connect tiles that form part of different rectangles arising from substitutions. We will write the outer joints in red in drawings. The local rules we impose in $\hat{X}_\tau$ force that the outer joint information from two adjacent tiles must be consistent with the substitution information in them. Tiles which have outer joints information are precisely those whose position is in the boundary of rectangle R_m. In Figure 3.6 we show an example of the outer joints information of two adjacent tiles.

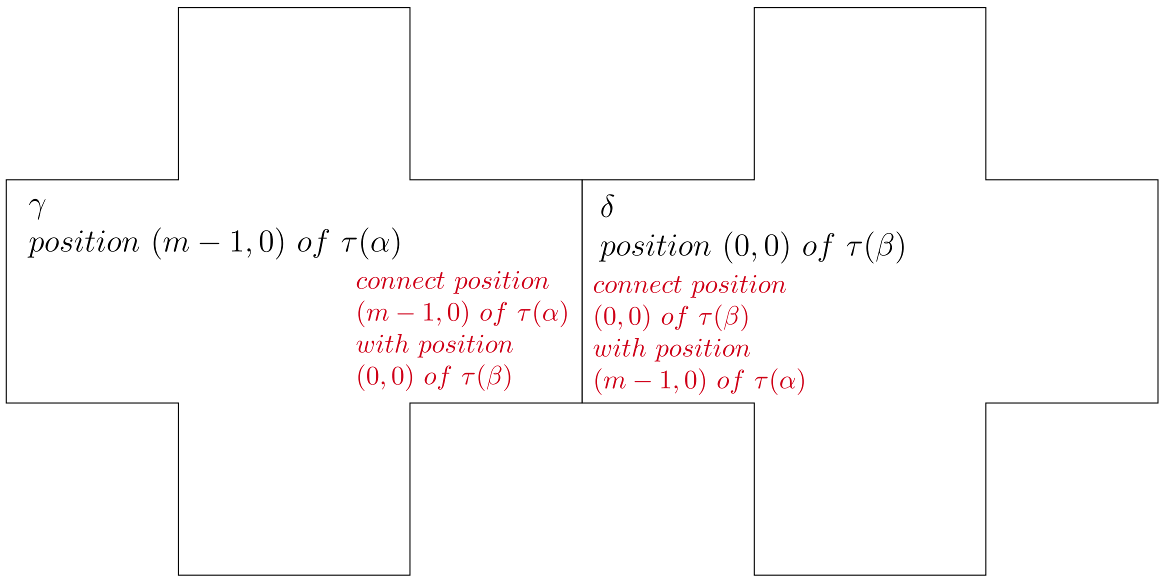


Figure 3.6: Example of the outer joints information in two adjacent tiles.

With this we can refer to a set of tiles forming a copy of R_m as "originating" from the substitution written on it.

- Each tile has an extra information on it called the **Blow up number (BUN)**, which is a number in $\mathbb{Z}/N^m\mathbb{Z}$. This information must be the same on the tiles of a rectangle originating from the same substitution. If the BUN on a rectangle is i then the BUN on the rectangle at its right is $i+1 \pmod{N^m}$, and if a rectangle has BUN$=0$ then the rectangle below it appearing under the $(0,0)$ tile must also have BUN$=0$. This number is used to mark the rectangles from which a new rectangle blows up: rectangles that have BUN=0 must have above them a rectangle originating from the tiles from its upper edge, according to the definition of the iteration of a substitution.

Figure 3.7 shows an example of a tile with all the information described above, and in Figure 3.8 we see a rectangle arranged to form the first iteration of a substitution.

With this, we see that the BS$(1,N)$-subshift $\widehat{X_\tau}$ whose patterns respect the rules described above is a BS$(1,N)$-SFT, since it can be described with forbidden patterns with a support in R_{2m}, checking locally that neighboring tiles respect the joints information, and that the BUN is consistent with adjacent copies of R_m.

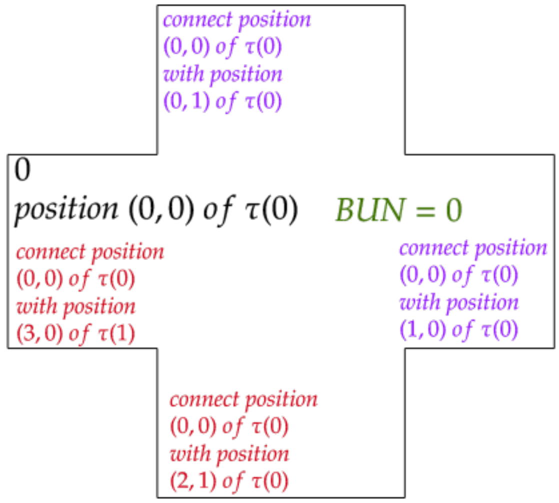


Figure 3.7: Example of a tile for the subshift $\widehat{X_\tau}$. The substitution information is written in black, the inner joints information is in purple and the outer joints information in red.

Now we are ready to state and prove the version of Mozes Theorem for settling substitutions of $\mathrm{BS}(1, N)$, which is the main result of this chapter.

Theorem 3.20 (Mozes) *Let $\tau : \mathcal{A} \to \mathcal{A}^{R_m}$ be a settling substitution on* $\mathrm{BS}(1, N)$. *Then its associated subshift X_τ is sofic.*

Proof. Consider $\widehat{X_\tau}$ the $\mathrm{BS}(1, N)$-SFT constructed above. Denote by $\varphi : \widehat{X_\tau} \to \mathcal{A}^{\mathrm{BS}(1,N)}$ the map that sends a point $\widehat{x} \in \widehat{X_\tau}$ to the point x formed by keeping only the letter information of each tile, that is, forgetting the position written on the tile as well as the junction information and the BUN. In what follows we will prove that φ is a factor map from $\widehat{X_\tau}$ onto X_τ, hence proving that X_τ is sofic.

It is rapidly seen that φ is equivariant, since a shift on the starting point $\widehat{x}$ will lead to a shift of the letters to be kept on the image by φ, and hence a shift from the image point x. We also see that φ is continuous by noting that two points close to each other must be equal in some finite subset of $\mathrm{BS}(1, N)$ around the origin and hence have the same local information written on their tiles, in particular the letters that appear on the image point through φ. With this we already have that φ is a sliding block code and to finish the proof all that is left to see is that $\varphi(\widehat{X_\tau}) = X_\tau$.

Let us show first that $X_\tau \subseteq \varphi(\widehat{X_\tau})$. Given a point $x \in X_\tau$ we can use Theorem 3.18 to find partition of $\mathrm{BS}(1, N)$ into copies of R_m, each one being the support of a pattern showing the image of a symbol through τ. Using this we can construct a valid point $\widehat{y} \in \widehat{X_\tau}$ by filling the tiles with the information of the substitution of each copy of R_m, together with the information of neighboring copies. This new point will satisfy $x = \varphi(\widehat{y}) \in \varphi(\widehat{X_\tau})$ and with it we finish this inclusion.

Now let us see that $\varphi(\widehat{X_\tau}) \subseteq X_\tau$. For $\widehat{x} \in \widehat{X_\tau}$ and $y = \varphi(\widehat{x})$, consider any finite pattern appearing on y. Then we can extend the corresponding pattern in $\widehat{x}$ to make it connected,

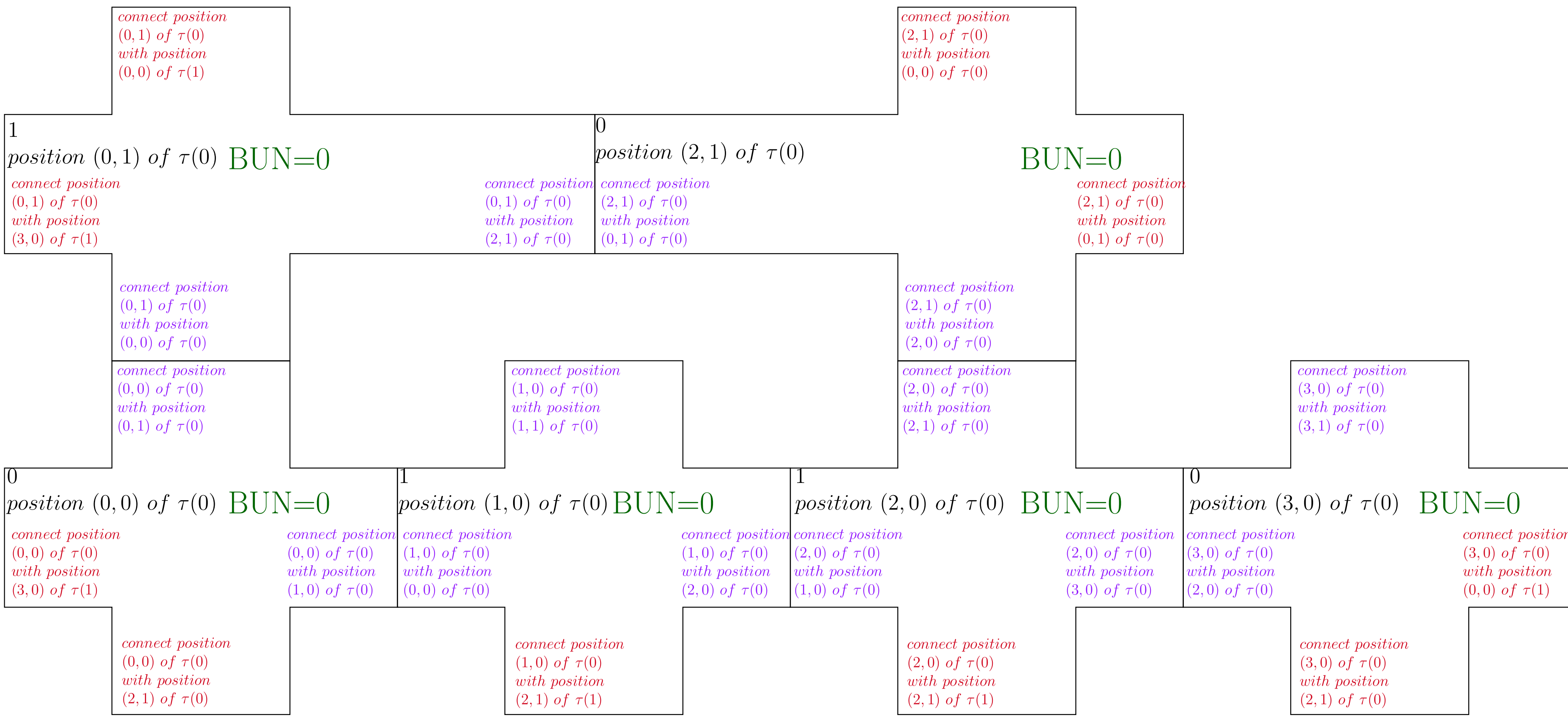


Figure 3.8: Example of an arrangement of the tiles for a point in Y_τ, for the substitution τ defined in Example 3.9.

and if necessary also extend it downwards and leftwards until we reach a tile with BUN=0, forming part (together with its neighbouring tiles) of the image of a substitution $\tau(\alpha)$, being α the symbol appearing in position $(0,0)$ of the normal form of the rectangle. Using the hypothesis that the substitution is settling we see that the image by φ of the (extended) pattern is seen on an iterate of $\tau(\alpha)$, and so we conclude that the original subpattern of y also appears on an iterate of the substitution, finishing the proof. $\square$

We can rapidly extend the previous theorem to a broader class of substitutions by considering a slightly weaker sense of being settling, in exchange for requiring primitivity.

Definition 3.21 *We say that a substitution $\tau : \mathcal{A} \to \mathcal{A}^{R_m}$ is* ***eventually settling*** *if there exists a $j \geq 1$ such that τ^j is settling.*

Corollary 3.22 *Let τ be an eventually settling primitive substitution. Then X_τ is a sofic subshift.*

Proof. Consider $j \geq 1$ such that τ^j is settling. Using Theorem 3.20 we see that X_{τ^j} is sofic, and with Propostion 3.17 we have that $X_\tau = X_{\tau^j}$. Hence X_τ is sofic, as was claimed. $\square$

3.4 An alternative way to iterate substitutions

In this section we provide an alternative definition for the iteration of substitutions on $\mathrm{BS}(1,N)$, in contrast to the one given in Definition 3.8. This second definition is motivated by the idea of *blowing up* each symbol of the substitution in each iteration, and gluing the resulting patterns together in order to form a bigger shape. To state the definition, we first need to generalize Proposition 3.5 to be able to decompose the rectangle $R_{\ell m}$, for $\ell \geq 2$, into copies of R_m glued together.

Proposition 3.23 *For $m \geq 1$ and $\ell \geq 2$, the rectangle $R_{\ell m}$ is a disjoint union of $\ell N^{m(\ell-1)}$ copies of R_m. Moreover, we have the decomposition:*

$$R_{\ell m} = \bigcup_{i=0}^{N^{(\ell-1)m}-1} a^{iN^m} R_m \cup \bigcup_{k=1}^{\ell-1} \bigcup_{q_1,\ldots,q_k=0}^{N^m-1} \bigcup_{i=0}^{N^{(\ell-1-k)m}-1} \left(\prod_{s=1}^{k} a^{q_s} b^m \right) a^{iN^m} R_m.$$

Proof. We have that

$$\begin{aligned} R_{\ell m} &= \left\{ a^j b^k \mid 0 \leq j < N^{\ell m}, 0 \leq k < \ell m \right\} \\ &= \left\{ a^j b^k \mid 0 \leq j < N^{\ell m}, 0 \leq k < m \right\} \cup \left\{ a^j b^k \mid 0 \leq j < N^{\ell m}, m \leq k < \ell m \right\}. \end{aligned}$$

For the first set on the right-hand side,

$$\left\{ a^j b^k \mid 0 \leq j < N^{\ell m}, 0 \leq k < m \right\} = \bigcup_{i=0}^{N^{(\ell-1)m}-1} \left\{ a^j b^k \mid iN^m \leq j < (i+1)N^m, 0 \leq k < m \right\}$$

$$= \bigcup_{i=0}^{N^{(\ell-1)m}-1} a^{iN^m} R_m.$$

Now for the second set,

$$\begin{aligned}
\left\{a^j b^k \mid 0 \leq j < N^{\ell m}, m \leq k < \ell m\right\} &= \left\{a^j b^{k+m} \mid 0 \leq j < N^{\ell m}, 0 \leq k < (\ell-1)m\right\} \\
&= \bigcup_{q=0}^{N^m-1} \left\{ a^j b^m b^k \mid j \in \left\{q, q+N^m, \ldots, q + \left\lfloor \frac{N^{\ell m}-q+1}{N^m} \right\rfloor N^m \right\}, \right. \\
&\qquad \left. 0 \leq k < (\ell-1)m \right\} \\
&= \bigcup_{q=0}^{N^m-1} \left\{ a^j b^m b^k \mid j \in \left\{q, q+N^m, \ldots, q + (N^{(\ell-1)m}-1)N^m\right\}, \right. \\
&\qquad \left. 0 \leq k < (\ell-1)m \right\} \\
&= \bigcup_{q=0}^{N^m-1} \left\{a^{q+jN^m} b^m b^k \mid 0 \leq j < N^{(\ell-1)m}, 0 \leq k < (\ell-1)m\right\} \\
&= \bigcup_{q=0}^{N^m-1} a^q b^m \left\{a^j b^k \mid 0 \leq j < N^{(\ell-1)m}, 0 \leq k < (\ell-1)m\right\} \\
&= \bigcup_{q=0}^{N^m-1} a^q b^m R_{(\ell-1)m}.
\end{aligned}$$

With this we conclude that

$$R_{\ell m} = \bigcup_{i=0}^{N^{(\ell-1)m}-1} a^{iN^m} R_m \cup \bigcup_{q=0}^{N^m-1} a^q b^m R_{(\ell-1)m}. \tag{3.4}$$

We prove the proposition by induction. If $\ell = 2$, then Equation (3.4) is precisely what we wanted. Now if $\ell > 2$, using Equation (3.4) and the induction hypothesis:

$$\begin{aligned}
R_{\ell m} &= \bigcup_{i=0}^{N^{(\ell-1)m}-1} a^{iN^m} R_m \cup \bigcup_{q=0}^{N^m-1} a^q b^m R_{(\ell-1)m} \\
&= \bigcup_{i=0}^{N^{(\ell-1)m}-1} a^{iN^m} R_m \cup \bigcup_{q=0}^{N^m-1} a^q b^m \left(\bigcup_{p=0}^{N^{(\ell-2)m}-1} a^{pN^m} R_m \cup \right. \\
&\qquad \left. \cup \bigcup_{k=1}^{\ell-2} \bigcup_{q_1,\ldots,q_k=0}^{N^m-1} \bigcup_{i=0}^{N^{(\ell-2-k)m}-1} \left(\prod_{s=1}^{k} a^{q_s} b^m \right) a^{iN^m} R_m \right) \\
&= \bigcup_{i=0}^{N^{(\ell-1)m}-1} a^{iN^m} R_m \cup \bigcup_{q=0}^{N^m-1} \bigcup_{p=0}^{N^{(\ell-2)m}-1} a^q b^m a^{pN^m} R_m \cup \\
&\qquad \cup \bigcup_{k=1}^{\ell-2} \bigcup_{q_1,\ldots,q_{k+1}=0}^{N^m-1} \bigcup_{i=0}^{N^{(\ell-2-k)m}-1} \left(\prod_{s=1}^{k+1} a^{q_s} b^m \right) a^{iN^m} R_m
\end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{i=0}^{N^{(\ell-1)m}-1} a^{iN^m} R_m \cup \bigcup_{q=0}^{N^m-1} \bigcup_{p=0}^{N^{(\ell-2)m}-1} a^q b^m a^{pN^m} R_m \cup \\
&\qquad \cup \bigcup_{k=2}^{\ell-1} \bigcup_{q_1,\ldots,q_k=0}^{N^m-1} \bigcup_{i=0}^{N^{(\ell-1-k)m}-1} \left(\prod_{s=1}^{k} a^{q_s} b^m \right) a^{iN^m} R_m \\
&= \bigcup_{i=0}^{N^{(\ell-1)m}-1} a^{iN^m} R_m \cup \bigcup_{k=1}^{\ell-1} \bigcup_{q_1,\ldots,q_k=0}^{N^m-1} \bigcup_{i=0}^{N^{(\ell-1-k)m}-1} \left(\prod_{s=1}^{k} a^{q_s} b^m \right) a^{iN^m} R_m,
\end{aligned}
$$

finishing the proof. □

The new definition for the iterates of a substitution should satisfy that for every $\ell \geq 1$ and $\alpha \in \mathcal{A}$, each element of $\tau^\ell(\alpha)$ at level j, $0 \leq j < m^\ell$, blows up into a rectangle covering levels jm up until $(j+1)m - 1$, in some sheet of $R_{m^{\ell+1}}$. This process should also guarantee that all of the sheets are covered in this process. These conditions translate in an important difference between the iteration of substitutions defined in Definition 3.8 and the definition we will give later: in the latter the support of the patterns seen as the images of the iterations will not be entire rectangles, but rather a subset of them. Below we define $S(m, \ell) \subseteq R_{m^\ell}$, which will be the support of the ℓ-th iteration of a substitution $\tau : \mathcal{A} \to \mathcal{A}^{R_m}$.

Definition 3.24 *Let $m \geq 2$. For any $\ell \geq 1$ define the set*

$$
S(m, \ell) := R_{m\ell} \cup \bigcup_{k=\ell}^{m^{\ell-1}-1} \left\{ a^i b^j \mid 0 \leq i < N^{(k+1)m}, km \leq j < (k+1)m \right\} \subseteq R_{m^\ell}.
$$

Now we are ready to give the alternative definition of the iterates of a substitution. This definition is given recursively, separated in four cases according to the proportion between elements to be blown up in $S(m, \ell)$, and the amount of sheets occurring at a particular height in $S(m, \ell+1)$. We first give an intuitive description in each chase, and then state the formal definition below.

- The first case is for elements of height 0. In this case there are more elements to be blown up than sheets at this height (there is only one sheet at height 0), so the elements blow up into rectangles glued together sideways in the base of $S(m, \ell + 1)$.

- The second case is for elements of level $j \in \{1, \ldots, \ell - 1\}$: in this case there are still more elements to blow up than sheets at each height, so we use the first j coefficients of the power of a of the element in base N^m to choose a sheet at height jm, and the remaining elements blow up in copies of R_m glued sideways to the copies described before.

- The third and fourth case are elements of $S(m, \ell)$ for which at each height $j \in \{\ell, \ldots, m^\ell - 1\}$ there are less elements to be blown up than sheets to be covered, so it becomes necessary to repeat the blowing up of some elements throughout some sheets in order to cover all of them.

Definition 3.25 *Given a (finite) alphabet $\mathcal{A}$ and some fixed $m \geq 2$, consider a substitution $\tau : \mathcal{A} \to \mathcal{A}^{R_m}$. Given $\ell \geq 1$, we define (recursively) the $(\ell+1)$-iterate of the substitution $\tau^{\ell+1} : \mathcal{A} \to \mathcal{A}^{S(m,\ell+1)}$. For each $\alpha \in \mathcal{A}$:*

- *For $i \in \{0, \dots, N^{\ell m} - 1\}$,*

$$\tau^{\ell+1}(\alpha)|_{a^{iN^m} R_m} = \tau(\tau^{\ell}(\alpha)_{a^i}).$$

- *For $d_0, \dots, d_{\ell-1} \in \{0, \dots, N^m - 1\}$ and $j \in \{1, \dots, \ell - 1\}$,*

$$\tau^{\ell+1}(\alpha)|_{\left(\prod_{s=0}^{j-1} a^{d_s} b^m\right) a^{d_j N^{jm} + \cdots + d_{\ell-1} N^{(\ell-1)m}} R_m} = \tau\left(\tau^{\ell}(\alpha)_{a^{d_0 + d_1 N^m + \cdots + d_{\ell-1} N^{(\ell-1)m}} b^j}\right).$$

- *For $d_0, \dots, d_{\ell-1} \in \{0, \dots, N^m - 1\}$ and $j \in \{\ell, \dots, m\ell - 1\}$,*

$$\tau^{\ell+1}(\alpha)|_{\left(\prod_{s=1}^{j-\ell} a^{q_s} b^m\right)\left(\prod_{s=0}^{\ell-1} a^{d_s} b^m\right) R_m} = \tau\left(\tau^{\ell}(\alpha)_{a^{d_0 + d_1 N^m + \cdots + d_{\ell-1} N^{(\ell-1)m}} b^j}\right),$$

  for every $q_1, \dots q_{j-\ell} \in \{0, \dots, N^m - 1\}$.

- *For $k \in \{\ell, \dots, m^{\ell-1} - 1\}$, $j \in \{km, \dots, (k+1)m - 1\}$ and $d_0, \dots, d_k \in \{0, \dots, N^m - 1\}$,*

$$\tau^{\ell+1}(\alpha)|_{\left(\prod_{s=1}^{j-k-1} a^{q_s} b^m\right)\left(\prod_{s=0}^{k} a^{d_s} b^m\right) R_m} = \tau\left(\tau^{\ell}(\alpha)_{a^{d_0 + d_1 N^m + \cdots + d_k N^{km}} b^j}\right),$$

  for every $q_1, \dots q_{j-k-1} \in \{0, \dots, N^m - 1\}$.

Chapter 4

Graph-coloring subshifts

Given a group G generated by a finite subset $S \subseteq G$ it is interesting to study the properties of proper colorings (in the sense used commonly in graph theory) of its Cayley graph. In this spirit we define for $n \geq 2$ the **graph-coloring subshift** (GCS)

$$\mathcal{C}_n := \left\{ x \in \{0, \dots, n-1\}^G \mid (\forall g \in G)(\forall s \in S)\ x_g \neq x_{gs} \right\},$$

that is, for each configuration $x \in \mathcal{C}_n$ we have that x describes a proper coloring of the Cayley graph $\Gamma(G, S)$ of G with respect to the generator S. The subshift $\mathcal{C}_n$ is always an SFT, since it can be described by the finite set of forbidden patterns

$$\mathcal{F} := \{\{x_{e_G} = i, x_s = i\} \mid i \in \{0, \dots, n-1\}, s \in S\}.$$

The easiest case to study is $G = \mathbb{Z}$: since the Cayley graph of $\mathbb{Z}$ with respect to the generator $S = \{1\}$ is bipartite (see Example 1.20), then $\mathcal{C}_n \neq \emptyset$ for every $n \geq 2$ and the number of words of length m of $\mathcal{C}_n$ is $|\mathcal{L}_m(\mathcal{C}_n)| = n(n-1)^{m-1}$. From this last equality we see that the topological entropy of $\mathcal{C}_n$ is

$$h_{\text{top}}(\mathcal{C}_n) = \lim_{m \to \infty} \frac{1}{m} \log(n(n-1)^{m-1}) = \log(n-1).$$

Note that in the case $n = 2$ the subshift $\mathcal{C}_2$ only has two configurations. Nonetheless, starting from 3 colors the dynamics of this subshift becomes more interesting: for $n \geq 3$ the subshift $\mathcal{C}_n$ is topologically mixing. In fact, we can prove that given any two words $w_1, w_2 \in \mathcal{L}(\mathcal{C}_n)$ and $M \geq 1$, we can find a pattern $u \in \mathcal{L}(\mathcal{C}_n)$ with $|u| = M + |w_1| + |w_2|$, $u|_{[1,|w_1|]} = w_1$ and $u|_{[M+|w_1|+1, M+|w_1|+|w_2|]} = w_2$: let i and j be the last symbols appearing in w_1 and w_2, respectively. If $i = j$, choose $k, l \in \{0, \dots, n-1\} \setminus \{i\}$, $k \neq l$, and define $u := w_1(kl)^{\frac{M}{2}} w_2$ if M is even, and $u := w_1(kl)^{\left\lfloor \frac{M}{2} \right\rfloor} k w_2$ if M is odd. In the other case, that is if $i \neq j$, choose $k \in \{, \dots, n-1\} \setminus \{i, j\}$ and define $u := w_1(ki)^{\frac{M}{2}} w_2$ if M is even and $u := w_1(ki)^{\left\lfloor \frac{M}{2} \right\rfloor} k w_2$ if M is odd. It is straightforward to check that said u satisfies the claimed property in each case.

Another case of interest is $G = \mathbb{Z}^2$, or more generally $G = \mathbb{Z}^d$ for $d \geq 2$. Proper colorings of these groups have been studied recently, for example in [1] and [19]. The latter of these two papers concerns mixing properties for colorings of $\mathbb{Z}^d$, where the authors define the notion of a frozen q-coloring of $\mathbb{Z}^d$ as a coloring with q colors such that it cannot be modified on any finite subset to create a different coloring, and prove that $\mathbb{Z}^d$ admits frozen q-colorings if and only if $2 \leq q \leq d+1$. They also prove that for $q \geq d+2$ any q-coloring of the boundary of the rectangle $\{1, \ldots, n\}^d$ for $n \geq d+2$ can be extended to a q-coloring of the entire box, and that if $q \geq 2d+1$ this can be done for every $n \geq 1$, therefore classifying all q-colorings of $\mathbb{Z}^d$ in terms of the properties with respect to the extensibility of patterns they exhibit.

The purpose of this chapter is to study the case $G = \mathrm{BS}(1, N)$. We start by addressing the (non-)emptiness of the GCS $\mathcal{C}_n$ depending on the values of N and n. Then we study the extensibility of locally admissible patterns, and later use it to give bounds for the topological entropy of $\mathcal{C}_n$. Finally we ask and answer partially the question of which mixing properties these $\mathrm{BS}(1, N)$-subshifts possess.

4.1 Non-emptiness of the GCS

The first question one should ask after making the definition of a GCS is whether this subshift is empty or not, that is, for which $n \geq 2$ the Cayley graph of $\mathrm{BS}(1, N)$ admits a proper n-coloring. As we see in the next proposition, the answer to this question in the case of two colors depends on the parity of N, meanwhile for a number of colors $n \geq 3$ we have non-emptiness for every $N \geq 2$.

Proposition 4.1 *If N is odd $\mathcal{C}_2 \neq \emptyset$, and if N is even then $\mathcal{C}_2 = \emptyset$ meanwhile $\mathcal{C}_3 \neq \emptyset$. With this for every $N \geq 2$ and $n \geq 3$ we have $\mathcal{C}_n \neq \emptyset$.*

Proof. Let us see first that if N is even then $\mathcal{C}_2 = \emptyset$. To see this suppose we have $x \in \mathcal{C}_2$ and without loss of generality let us suppose that $x_{e_G} = 0$. Then by the coloring rules we must have $x_{a^N} = 0$, and $x_b = x_{a^N b} = 1$, but this cannot be since $x_{a^N b} = x_{ba}$, so the neighboring vertices b and ba have the same color and this contradicts the GCS's definition.

Now let us show that if N is odd then $\mathcal{C}_2 \neq \emptyset$. Moreover, we will show that in fact $|\mathcal{C}_2| = 2$. To create a point $x \in \mathcal{C}_2$ let us impose that $x_{e_{\mathrm{BS}(1,N)}} = 0$, and define for every $g \in \mathrm{BS}(1, N)$ expressed in its normal form $g = b^{-j} a^k b^i$ with $i, j \geq 0$ and $k \in \mathbb{Z}$: $x_g = i + j + k \mod 2$. We check that this provides a consistent coloring of the Cayley graph: for g as above and a generator $s \in \{a, b\}$, the normal form of gs is given by

$$ga = b^{-j} a^{k+N^i} b^i,$$

if $s = a$ and

$$gb = b^{-j} a^k b^{i+1},$$

if $s = b$. Then $x_{ga} = i + j + k + N^i \mod 2$ and $x_{gb} = i + 1 + j + k \mod 2$, and as N is odd we see that $x_{ga} \neq x_g$ and $x_{gb} \neq x_g$. With this, neighboring elements in the graph have different colors and hence x forms a valid configuration in $\mathcal{C}_2$. Also note that this point is completely determined by our choice of $x_{e_{\mathrm{BS}(1,N)}} = 0$. If we had chosen instead $x_{e_{\mathrm{BS}(1,N)}} = 1$, we would have obtained the same configuration as above, but with 0's and 1's interchanged, thus we conclude that $|\mathcal{C}_2| = 2$.

Let us see now that for N even we have $\mathcal{C}_3 \neq \emptyset$. For this notice first that if we have an a-row $\Gamma_g := \{ga^k : k \in \mathbb{Z}\}, g \in G$ colored consistently with the coloring rules using two colors, i.e. the two colors strictly alternate along Γ_g, then we can color any row directly above it respecting the coloring rules: as Γ_g is colored with two colors and N is even, then any row directly above it only sees one color through its edges connecting it to Γ_g, so if we choose the two remaining colors we can color this new row consistently. On the other hand note that if Γ_g is colored consistently, we can also color the row exactly below it: as Γ_g is colored with two colors we can color the vertices below it with the remaining color in $\{0, 1, 2\}$, and as N is even we can choose any other color and fill the gaps consistently between the already colored vertices on this new row, alternating those two colors. With the two previous facts it is easy to see that one can construct inductively a point in $\mathcal{C}_3$. Hence $\mathcal{C}_3 \neq \emptyset$. The final statement of the proposition follows from the fact that if $n > 3$, then $\mathcal{C}_n$ contains a copy of $\mathcal{C}_{n-1}$ by simply considering colorings with a smaller palette of colors. □

Remark 4.2 *Thanks to the previous proposition we see that the interesting cases of GCS's start at $n = 3$, since $|\mathcal{C}_2| \in \{0, 2\}$.*

4.2 Extension of patterns and topological entropy

As the last remark says, in the case of 2-colorings of $\mathrm{BS}(1, N)$ the subshift $\mathcal{C}_2$ is finite so the configurations appearing in it are somewhat rigid and admit very few different ways of coloring the Cayley graph. This leads us to ask how many ways to color are there in the case of $n \geq 3$, and with it be able to estimate the degree of disorder that can be present in $\mathcal{C}_n$. A way of doing this is estimating the topological entropy $h_{\mathrm{top}}(\mathcal{C}_n)$, which as we commented in the first chapter measures the rate at which patterns in the language of the subshift grow as the support of the patterns becomes bigger.

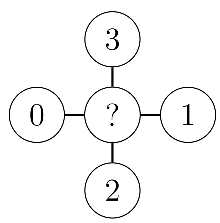
To be able to give estimates of the topological entropy we will encounter the challenge of extending colorings of a finite subset of the Cayley graph to bigger patterns containing it. In the language of symbolic dynamics this question is the same as asking if locally admissible patterns are globally admissible, or under which extra conditions a locally admissible pattern is globally admissible. A first result in this direction answers the question affirmatively, subject to having a minimum of available colors.

Proposition 4.3 *For $n \geq 5$, every locally admissible pattern p of $\mathcal{C}_n$ is globally admissible.*

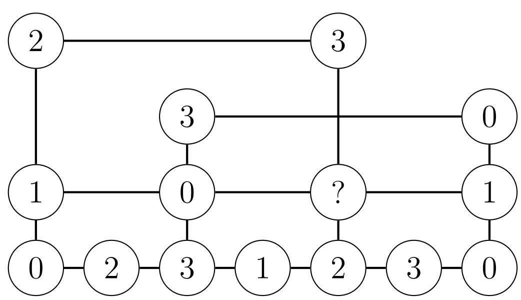
Proof. This follows from the fact that for a finite graph, its chromatic number is less than its maximum degree. Since every finite subgraph of the Cayley graph of $\mathrm{BS}(1, N)$ has maximum degree at most 4, then the considered pattern can be extended to any finite graph containing it by choosing for each vertex a color that none of its neighbors has, and thus preserving the coloring rules. This process can then be iterated indefinitely to finally obtain a configuration $x \in \mathcal{C}_n$ such that $x|_{\mathrm{supp}(p)} = p$.

More formally, given a locally admissible pattern p with support S we define a point $x^0 \in \{0,\dots,n-1\}^{\mathrm{BS}(1,N)}$ such that $x^0|_S = p$. We define S_1 to be S together with all vertices that are adjacent to it, and define $x^1 \in \{0,\dots,n-1\}^{\mathrm{BS}(1,N)}$ such that $x^1|_S = x^0|_S$ and the rest of the vertices of S_1 are colored respecting the coloring rules as said above, that is, using the fact that each vertex $S_1 \backslash S$ has at most 4 neighbors already colored, so there is always an available color. Inductively, for $j \geq 1$, to define x^{j+1} we define S_{j+1} to be S_j together with all vertices that are adjacent to it, and define $x^{j+1} \in \{0,\dots,n-1\}^{\mathrm{BS}(1,N)}$ such that $x^{j+1}|_{S_j} = x^j|_{S_j}$ and the rest of the vertices of S_{j+1} are colored as said above. By compactness we find a limit point $\overline{x}$ of $\{x^j\}_{j\geq 0}$, and by how we constructed this sequence we see that each vertex is properly colored and so $\overline{x} \in \mathcal{C}_n$. □

Remark 4.4 *The previous proposition does not hold for $n \in \{3,4\}$. For example the locally admissible pattern $p : \{a^{-1}, a, b^{-1}, b\} \to \{0,1,2,3\}$ given by $p_{a^{-1}} = 0, p_a = 1, p_{b^{-1}} = 2, p_b = 3$ cannot be realized within a point $x \in \mathcal{C}_4$ since no color can be assigned to x_{e_G} respecting the coloring rules.*



Moreover, one can consider a connected (as subgraph of the Cayley graph) pattern that has the same behaviour:



The previous example showed a locally admissible pattern that could not be extended to form a valid configuration of the group, and the main property of this example is that although the support of the chosen pattern can be taken to be a connected subgraph of the Cayley graph, it has a "gap" which allows us to surround a position in such a way that the pattern could not be extended. A way of avoiding this type of pathological patterns is to require the supports of the patterns to only be rectangles, and in this case we see that the coloring can indeed be extended as the following lemma and proposition show.

Lemma 4.5 *For every $n \geq 3$ any n-coloring of the rectangle R_m can be extended to a proper n-coloring of the rectangle R_{m+1}.*

Proof. For every $k \in \{0, \dots, m\}$ we define

$$H_k \coloneqq (R_{m+1} \backslash R_m) \cap \{a^i b^k \mid i \geq 0\},$$

which represents the elements of R_{m+1} of height k, outside of the rectangle R_m. Note that by definition we have

$$R_{m+1} = R_m \cup \bigcup_{k=0}^{m} H_k.$$

Then we can extend the pattern p on R_m by successive colorings (respecting the condition of being a proper coloring) coloring first H_0, then H_1, and so until we color H_m. In each step of this process each vertex has at most 2 neighbors already colored, so having at least 3 available colors is enough for this process to be carried out. □

By using the same ideas present in the proof of the previous lemma we get the following proposition. Indeed, one can inductively color the rows of the Cayley graph one by one, so that each vertex has at most two colored neighbors at the moment it must choose its color.

Proposition 4.6 *For $n \geq 3$ every locally admissible pattern for $\mathcal{C}_n$ with support a rectangle is globally admissible.*

Now we can proceed to give estimates for the entropy of the GCS.

Proposition 4.7 *For $n \geq 3$ we have the following estimate for the topological entropy of the GCS $\mathcal{C}_n$:*

$$\log(n-2) \leq h_{\text{top}}(\mathcal{C}_n) \leq \log(n-1).$$

Proof. Let us see that $h_{\text{top}}(\mathcal{C}_n) \leq \log(n-1)$: a coloring of the rectangle R_m may be extended to a coloring of the rectangle R_{m+1} by coloring the remaining vertices having on each one at most $n-1$ options. With this

$$|\mathcal{L}_{R_{m+1}}(\mathcal{C}_n)| \leq |\mathcal{L}_{R_m}(\mathcal{C}_n)|(n-1)^{|R_{m+1} \backslash R_m|}.$$

A simple calculation shows that $|R_{m+1} \backslash R_m| = N^m(mN + N - m)$. In effect, it suffices to notice that

$$\begin{aligned} R_{m+1} \backslash R_m &= \left\{a^i b^j \mid 0 \leq i < N^{m+1},\ 0 \leq k < m+1\right\} \setminus \left\{a^i b^j \mid 0 \leq i < N^m,\ 0 \leq k < m\right\} \\ &= \left\{a^i b^j \mid N^m \leq i < N^{m+1},\ 0 \leq k < m\right\} \cup \left\{a^i b^m \mid 0 \leq i < N^{m+1}\right\}, \end{aligned}$$

and that this union is disjoint. With this:

$$\begin{aligned} \frac{1}{|R_{m+1}|} \log |\mathcal{L}_{R_{m+1}}(\mathcal{C}_n)| &= \frac{1}{(m+1)N^{m+1}} \log |\mathcal{L}_{R_{m+1}}(\mathcal{C}_n)| \\ &\leq \frac{1}{(m+1)N^{m+1}} \log |\mathcal{L}_{R_m}(\mathcal{C}_n)| + \frac{N^m(mN+N-m)}{(m+1)N^{m+1}} \log(n-1) \\ &= \frac{1}{N} \frac{m}{m+1} \frac{1}{mN^m} \log |\mathcal{L}_{R_m}(\mathcal{C}_n)| + \frac{mN+N-m}{m+1} \frac{1}{N} \log(n-1) \end{aligned}$$

$$= \frac{1}{N}\frac{m}{m+1}\frac{1}{|R_m|}\log|\mathcal{L}_{R_m}(\mathcal{C}_n)| + \frac{m(N-1)+N}{m+1}\frac{1}{N}\log(n-1).$$

Taking the limit $m \to \infty$ we get

$$\begin{aligned} h_{\text{top}}(\mathcal{C}_n) &= \lim_{m\to\infty}\frac{1}{|R_{m+1}|}\log|\mathcal{L}_{R_{m+1}}(\mathcal{C}_n)| \\ &\leq \lim_{m\to\infty}\frac{1}{N}\frac{m}{m+1}\frac{1}{|R_m|}\log|\mathcal{L}_{R_m}(\mathcal{C}_n)| + \lim_{m\to\infty}\frac{m(N-1)+N}{m+1}\frac{1}{N}\log(n-1) \\ &= \frac{1}{N}h_{\text{top}}(\mathcal{C}_n) + \frac{N-1}{N}\log(n-1), \end{aligned}$$

from where

$$\frac{N-1}{N}h_{\text{top}}(\mathcal{C}_n) \leq \frac{N-1}{N}\log(n-1),$$

and one arrives at $h_{\text{top}}(\mathcal{C}_n) \leq \log(n-1)$.

Now let us see that $h_{\text{top}}(\mathcal{C}_n) \geq \log(n-2)$: the rectangle R_m can be colored starting from the upper levels to the lower levels, ensuring at least $n-2$ color options at each element, since in this way each vertex of the Cayley graph has at most two neighbors already colored. Then this coloring can be extended to the whole graph by using Proposition 4.6 and hence forming a globally admissible configuration for $\mathcal{C}_n$. From this we get that

$$|\mathcal{L}_{R_m}(\mathcal{C}_n)| \geq (n-2)^{|R_m|},$$

and so $h_{\text{top}}(\mathcal{C}_n) \geq \log(n-2)$. □

In particular, the lower bound from the previous proposition shows us that $h_{\text{top}}(\mathcal{C}_n) > 0$ for every $n \geq 4$, but gives us no new information for the case of three colors $\mathcal{C}_3$, since by definition $h_{\text{top}}(\mathcal{C}_3) \geq 0$. In the case of odd N we can exploit the fact that the Cayley graph of $\text{BS}(1,N)$ is bipartite (we know this since we have already constructed a 2-coloring of it in Proposition 4.1, or equivalently observing that it has no odd cycles) to prove that the topological entropy of $\mathcal{C}_3$ is positive.

Proposition 4.8 *If N is odd, then the GCS $\mathcal{C}_3 \subseteq \{0,1,2\}^{\text{BS}(1,N)}$ has positive topological entropy. Moreover,*

$$h_{\text{top}}(\mathcal{C}_3) \geq \frac{1}{2}\log(2).$$

Proof. As N is odd, the Cayley graph of $\text{BS}(1,N)$ is bipartite and hence so is every rectangle R_m, for $m \geq 1$. Consider a *partition* of R_m into two sets A and B, meaning that all edges of the graph are composed of a vertex in A and a vertex in B. Then one of them, which we take to be A without loss of generality, must have cardinality at least $\frac{1}{2}|R_m|$. Then we can create proper colorings of R_m by coloring the vertices of B with one color and have the freedom to choose between two colors for every vertex of A, and then extend this pattern on R_m to the rest of the group as was said earlier.

With the above we can estimate a lower bound for the number of proper colorings of R_m:

$$|\mathcal{L}_{R_m}(\mathcal{C}_3)| \geq 2^{|A|} \geq 2^{\frac{1}{2}|R_m|}.$$

Then taking logarithm and dividing by $|R_m|$ we get

$$\frac{1}{|R_m|}\log|\mathcal{L}_{R_m}(\mathcal{C}_3)| \geq \frac{1}{2}\log(2),$$

to finally take limit as $m \to \infty$ and obtain

$$h_{\text{top}}(\mathcal{C}_3) \geq \frac{1}{2}\log(2) > 0,$$

which is what we wanted. □

Remark 4.9 *The question of whether $h_{\text{top}}(\mathcal{C}_3)$ is positive remains open when N is even, since in this case the Cayley graph of $\mathrm{BS}(1,N)$ is not bipartite and hence the method used above to color sets of vertices independently cannot be applied.*

4.3 Mixing properties

Thanks to Proposition 4.3 we know that for $n \geq 5$ we can extend any admissible pattern to a proper coloring of the Cayley graph of $\mathrm{BS}(1,N)$, from which we obtain the following mixing property for $\mathcal{C}_n$.

Theorem 4.10 *Let $n \geq 5$. Then for every finite subsets $F_1, F_2 \subseteq \mathrm{BS}(1,N)$ such that $2 \leq d(F_1,F_2) := \inf_{f_1\in F_1, f_2\in F_2} |f_1 f_2^{-1}|$ (where $|\cdot|$ denotes the word metric described in Definition 1.22) and for every choice of (locally) admissible patterns $p_i \in \{0,\dots,n-1\}^{F_i}, i = 1,2$, there exists $x \in \mathcal{C}_n$ such that $x|_{F_1} = p_1$ and $x|_{F_2} = p_2$. With this we have that for $n \geq 5$ the GCS $\mathcal{C}_n$ is strongly irreducible.*

Proof. Let $F_1, F_2 \subseteq \mathrm{BS}(1,N)$ be finite subsets with $d(F_1,F_2) \geq 2$ and consider two locally admissible patterns $p_i \in \{0,\dots,n-1\}^{F_i}, i = 1,2$. Then we can define a new pattern $p \in \{0,\dots,n-1\}^{F_1\cup F_2}$ such that $p(f) = p_i(f)$ for $f \in F_i, i = 1,2$ consistently, since $F_1 \cap F_2 = \emptyset$. This pattern is also locally admissible thanks to the distance existing between F_1 and F_2, as between every vertex of F_1 and every vertex of F_2 there is at least one uncolored vertex. Then, according to Proposition 4.3, this pattern is globally admissible and hence there exists $x \in \mathcal{C}_n$ such that $x|_{F_1\cup F_2} = p$. This x satisfies $x|_{F_1} = p_1$ and $x|_{F_2} = p_2$ and so we have found the desired point. □

The previous proposition tells us that for $n \geq 5$ the subshift $\mathcal{C}_n$ has a strong mixing property. On the other side, we can see that $\mathcal{C}_2$ has no type of mixing behavior: this $\mathrm{BS}(1,N)$-subshift is either empty if N is even, or if N is odd then there is no way to assign the same color to two elements of the group $g, h \in \mathrm{BS}(1,N)$ such that $g^{-1}h \in \{a^{2m+1} \mid m \in \mathbb{Z}\}$, which forbids the gluing of patterns at arbitrarily large distances. This contrast between $\mathcal{C}_2$ and $\mathcal{C}_n$ for $n \geq 5$ raises the question of what kind of mixing behaviors does $\mathcal{C}_n$ have for $n \in \{3,4\}$. To study this we will use a similar approach as that of [1], by introducing the concept of a frozen coloring.

Definition 4.11 *Let $n \geq 3$. A configuration $x \in \mathcal{C}_n$ is called a* ***frozen coloring*** *if for every $y \in \mathcal{C}_n$ such that there exists a non-empty finite subset $F \subseteq \mathrm{BS}(1,N)$ with $x|_{F^c} = y|_{F^c}$, then $x = y$. That is, no coloring of $\mathrm{BS}(1,N)$ other than x can coincide with it outside of any finite set.*

Frozen colorings are configurations in which the neighboring vertices of every finite subset of the Cayley graph determine unequivocally how this subset must be colored, in order to obtain a proper coloring. This behavior is the reason frozen colorings are closely related to the (lack of) mixing properties of the GCS, as the next proposition shows.

Proposition 4.12 *If $\mathcal{C}_n$ has a frozen coloring, then it is not strongly irreducible.*

Proof. Looking for a contradiction, suppose that $\mathcal{C}_n$ is strongly irreducible and $x \in \mathcal{C}_n$ is a frozen coloring.

Consider any other configuration $y \in \mathcal{C}_n$ such that $y_{e_G} \neq x_{e_G}$. As $\mathcal{C}_n$ is strongly irreducible, there exists $F \subseteq G$ finite such that for any two patterns $p, q \in \mathcal{L}(\mathcal{C}_n)$ with $\mathrm{supp}(p) \cap \mathrm{supp}(q) \cdot F = \emptyset$ we have $[p] \cap [q] \neq \emptyset$. Considering the patterns $y|_{e_G}$ and $x|_{\partial B_M}$, where $\partial B_M \coloneqq \{g \in \mathrm{BS}(1,N) : |g| = M\}$, for M sufficiently large we have $\partial B_M \cap F = \emptyset$. Then there must exist a coloring $z \in \mathcal{C}_n$ such that $z_{e_G} = y_{e_G} \neq x_{e_G}$, and $z|_{\partial B_M} = x|_{\partial B_M}$. Moreover, we can assume that $z|_{B^c_{M-1}} = x|_{B^c_{M-1}}$, where $B_{M-1} \coloneqq \{g \in \mathrm{BS}(1,N) : |g| \leq M-1\}$, since z and x coincide on ∂B_M and hence re-coloring z as x outside of this ball gives us a proper coloring.

With this we have found a configuration $z \in \mathcal{C}_n$ which coincides with x outside of a finite set but is different from x inside it, so we have a contradiction with the fact that x is a frozen coloring. Hence we conclude that any strongly irreducible subshift $\mathcal{C}_n$ cannot have a frozen coloring. □

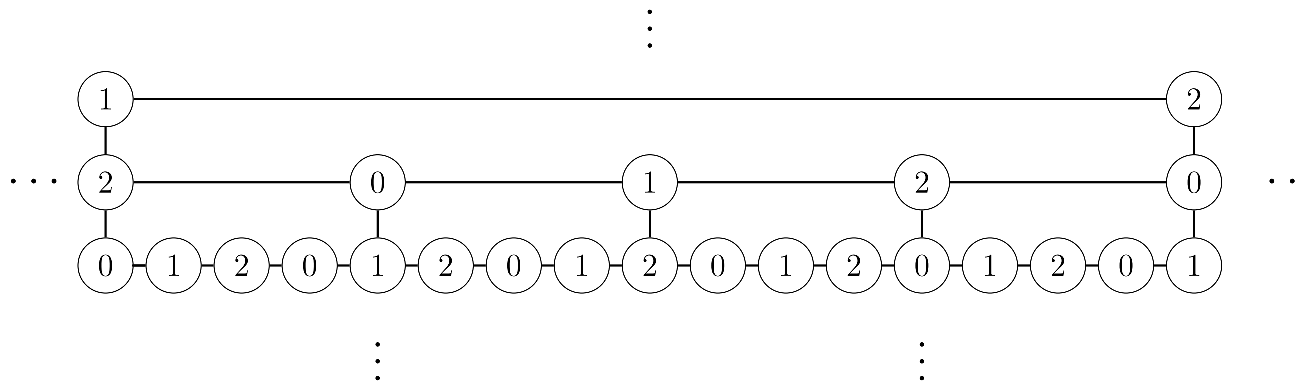
We already know by Theorem 4.10 that the GCS $\mathcal{C}_n$ for $n \geq 5$ is strongly irreducible, and hence by the above proposition it cannot have a frozen coloring. On the other hand, as the case $n = 3$ is the first non-finite subshift of the GCS's $\mathcal{C}_n$ for any $N \geq 2$, it is reasonable to conjecture that its properties must still be somewhat rigid. The next proposition confirms this by showing that $\mathcal{C}_3$ possesses a frozen coloring, and with it exhibits the lack of a strong mixing behavior of the GCS with three colors.

Theorem 4.13 *For every $N \geq 2$ the GCS $\mathcal{C}_3 \subseteq \{0,1,2\}^{\mathrm{BS}(1,N)}$ admits a frozen coloring, and hence is not strongly irreducible.*

Proof. The proof will be divided in three cases, depending on the value of $N \pmod 3$. For each one of these cases we will construct explicitly the claimed frozen coloring.

Let us suppose first that $N = 1 \pmod 3$, and with it for every $i \geq 0$ we have $N^i = 1 \pmod 3$. Let us define the configuration $x \in \{0,1,2\}^{\mathrm{BS}(1,N)}$ such that for $g = b^{-j}a^k b^i \in \mathrm{BS}(1,N)$ written in its normal form:

$$x_g \coloneqq 2(i-j) + k \pmod 3.$$



Note that $gb = b^{-j}a^kb^{i+1}$ and $ga = b^{-j}a^kb^ia = b^{-j}a^{k+N^i}b^i$. With this, remembering that $N^i = 1 \pmod 3$ we have

$$\begin{aligned} x_{gb} &= x_g + 2 \pmod 3, \text{ and} \\ x_{ga} &= x_g + N^i \pmod 3 = x_g + 1 \pmod 3. \end{aligned}$$

Therefore $x_{gb} \neq x_g$ and $x_{ga} \neq x_g$, and so $x \in \mathcal{C}_3$ defines a proper coloring. Let us see that x defines a frozen coloring: looking for a contradiction let us suppose $y \in \mathcal{C}_3$ is such that there exists a finite subset $F \subseteq \mathrm{BS}(1, N)$ with $y|_{F^c} = x|_{F^c}$ and for every $f \in F :\ x_f \neq y_f$. By using the fact that F is finite, and shifting both configurations if necessary, we may assume that

$$F \subseteq \{g \in \mathrm{BS}(1, N) \mid g = a^kb^i, \text{ for some } k \in \mathbb{Z}, i \geq 0\}.$$

Geometrically, this means that g is in the "upper" section of the Cayley graph of $\mathrm{BS}(1, N)$, that is, in a sheet having as a base the subgroup $\langle a \rangle$. Now consider g in F which maximizes the value of $i + k$ in its normal form. In other words,

$$g \coloneqq \operatorname{argmax}\{i + k \mid g = a^kb^i \in F,\ k \in \mathbb{Z}, i \geq 0\}.$$

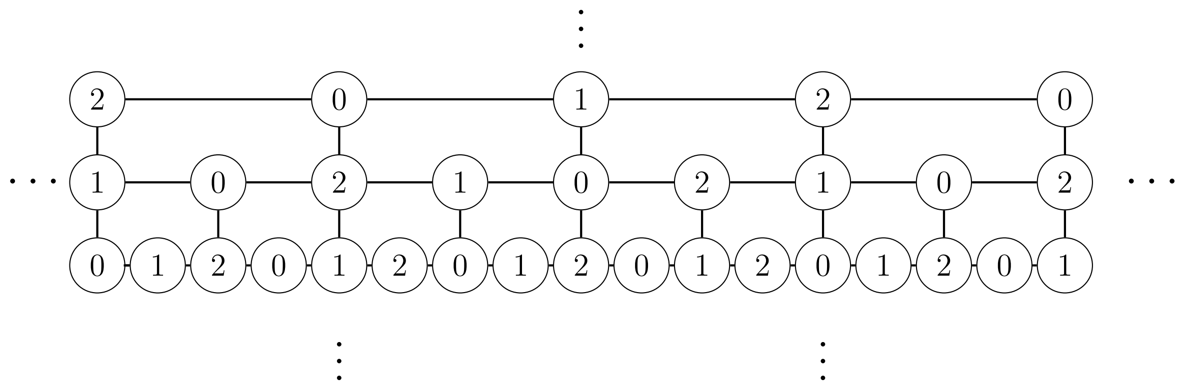
Then $y_{gb} = x_{gb} = x_g + 2 \pmod 3$, and $y_{ga} = x_{ga} = x_g + 1 \pmod 3$, and so as y defines a proper coloring we must have $y_g = x_g$, but this gives a contradiction since as $g \in F$ we should have $y_g \neq x_g$.

Now let us see the case $N = 2 \pmod 3$. Note that now we have that for $i \geq 0$:

$$N^i = \begin{cases} 1 \pmod 3 \text{ if } i \text{ is even,} \\ 2 \pmod 3 \text{ if } i \text{ is odd.} \end{cases}$$

Define a configuration $x \in \{0, 1, 2\}^{\mathrm{BS}(1,N)}$ such that for $g = b^{-j}a^kb^i \in \mathrm{BS}(1, N)$ written in its normal form:

$$x_g \coloneqq i - j + k \pmod 3.$$



Then as before we have

$$x_{gb} = x_g + 1 \pmod 3, \text{ and}$$
$$x_{ga} = x_g + N^i \pmod 3,$$

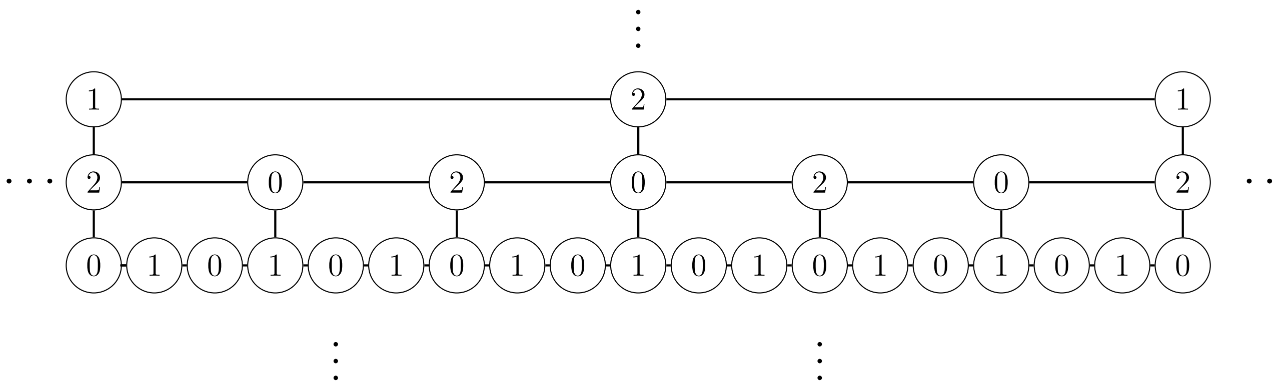
and so $x_{gb} \neq x_g$ and $x_{ga} \neq x_g$ (since N^i can be either 1 or 2). With this we have that $x \in \mathcal{C}_3$ defines a proper coloring, so it only remains to prove that x defines a frozen coloring. Looking for a contradiction, suppose there exists $y \in \mathcal{C}_3$ and a finite set $F \subseteq \mathrm{BS}(1,N)$ such that $x|_{F^c} = y|_{F^c}$ and for every $f \in F : x_f \neq y_f$. Again, by shifting both configurations if necessary, we may assume that $F \subseteq \{g \in \mathrm{BS}(1,N) \mid g = a^k b^i, \text{ for some } k \in \mathbb{Z}, i \geq 0\}$. Define $i_{max} \coloneqq \max\{i \geq 0 \mid a^k b^i \in F, \text{ for some } k \in \mathbb{Z}\}$ and $B_{max} \coloneqq \{g = a^k b^i \in F \mid i = i_{max}\}$.

If i_{max} is even, let us take $g = a^k b^i \in B_{max}$ which minimizes the value of k. Then as i_{max} is even we have $N^{i_{max}} = 1$ and with it $y_{gb} = x_{gb} = x_g + 1 \pmod 3$, and $y_{ga^{-1}} = x_{ga^{-1}} = x_g - N^{i_{max}} = x_g + 2 \pmod 3$. But then we must have $y_g = x_g$ and this gives a contradiction since $g \in F$ and therefore $x_g \neq y_g$.

Now if i_{max} is odd, let us consider the element $g = a^k b^i \in B_{max}$ which maximizes the value of k. Then as i_{max} is odd we have $N^{i_{max}} = 2$ and with it $y_{gb} = x_{gb} = x_g + 1 \pmod 3$, and $y_{ga} = x_{ga} = x_g + N^{i_{max}} = x_g + 2 \pmod 3$. But then we must have $y_g = x_g$ and this gives a contradiction since $g \in F$ and therefore $x_g \neq y_g$, proving that x defines a frozen coloring.

Finally suppose that $N = 0 \pmod 3$. The method used on the previous two cases to find a frozen coloring on $\mathcal{C}_3$ was to construct a configuration using a function $f(j,k,i)$ of the coefficients of the normal form of every element of the group, which $\mod 3$ had period 3 on the variable k. This cannot be done in the case $N \in 3\mathbb{Z}$ since here the configuration constructed would satisfy $x = \sigma_{a^3}(x)$ and hence $x = \sigma_{a^N}(x)$ (since $N \in 3\mathbb{Z}$). But then by Proposition 2.3 the coloring x would have to have monochromatic rows appearing on it, which contradicts the fact that $x \in \mathcal{C}_3$ defines a proper coloring. Nonetheless, using a similar but different kind of function we can define a configuration $x \in \{0,1,2\}^{\mathrm{BS}(1,N)}$ such that for $g = b^{-j} a^k b^i \in \mathrm{BS}(1,N)$ written in its normal form:

$$x_g \coloneqq (k \pmod 2) + 2(i - j) \pmod 3.$$



We see that since N^i is odd for every $i \geq 0$ we have:

$$x_{gb} = x_g + 2 \pmod 3, \text{ and}$$
$$x_{ga} = (k + N^i \pmod 2) + 2(i - j) \pmod 3 = x_g + 1 \pmod 3,$$

from which $x \in \mathcal{C}_3$ defines a proper coloring, and we can proceed as we did previously in the case $N^i = 1 \pmod 3$. That is, to prove that x defines a frozen coloring we suppose $y \in \mathcal{C}_3$ is such that there exists a finite subset $F \subseteq \mathrm{BS}(1, N)$ with $y|_{F^c} = x|_{F^c}$ and for every $f \in F: \ x_f \neq y_f$. By shifting both configurations if necessary, we may assume that $F \subseteq \{g \in \mathrm{BS}(1, N) \mid g = a^k b^i, \text{ for some } k \in \mathbb{Z}, i \geq 0\}$. Now let us consider g in F which maximizes the value of $i + k$ in its normal form. Then $y_{gb} = x_{gb} = x_g + 2 \pmod 3$, and $y_{ga} = x_{ga} = x_g + 1 \pmod 3$, and so as y defines a proper coloring we must have $y_g = x_g$, but this gives a contradiction since as $g \in F$ we should have $y_g \neq x_g$. Hence we see that x a frozen coloring.

To finish the proof we simply use the previous proposition to see that $\mathcal{C}_3$ cannot be strongly irreducible, as we have constructed a frozen coloring in it. □

This last theorem together with the previous comments settle the existence of frozen colorings for $n = 3$ and $n \geq 5$. To tackle the case of four colors, and in the process give an alternative proof of the lack of frozen colorings for $n \geq 5$, we will use a proposition from [1] used in that paper to prove that $\mathbb{Z}^d$ does not admit frozen q-colorings for $q \geq d + 2$, whose proof we include here for the sake of completeness.

Proposition 4.14 ([1]) *For a graph Γ let us define its **edge-isoperimetric constant** by*

$$i_e(\Gamma) := \inf_{F \subseteq \Gamma \text{ finite}} \frac{|E(F, \Gamma \backslash F)|}{|F|},$$

where $E(F, \Gamma\backslash F)$ are the edges of Γ connecting vertices from F to $\Gamma\backslash F$. Denote by Δ the maximum degree of Γ. Then for every $q > \frac{1}{2}\Delta + \frac{1}{2} i_e(\Gamma) + 1$ there do not exist frozen q-colorings of Γ.

Proof. We denote $\Gamma = (V, E)$, and for $F \subseteq V$ we use the notation $E(F, F)$ for the edges of E that connect vertices from F with each other, and $E(F, \Gamma\backslash F)$ for the edges of E that connect vertices from F with vertices from the rest of the graph.

Given a subset of vertices $F \subseteq V$ we say that a q-coloring x of Γ is frozen on F if for every subset $F' \subseteq F$ and every q-coloring y of Γ such that $x|_{F'^c} = y|_{F'^c}$, then $x = y$. Hence x is a frozen q-coloring if and only if x is a frozen q-coloring of F, for every finite subset $F \subseteq V$.

We are going to prove the following claim: for every $F \subseteq V$ finite, if

$$(q-1)|F| \geq |E(F,F)| + |E(F,\Gamma\backslash F)|,$$

then no q-coloring of Γ is frozen on F. In particular, as $2|E(F,F)| + |E(F,\Gamma\backslash F)|$ equals the sum of the degrees of vertices in F, we conclude that if $(q-1-\frac{\Delta}{2})|F| > \frac{1}{2}|E(F,\Gamma\backslash F)|$ then

$$(q-1)|F| > \frac{\Delta}{2}|F| + \frac{1}{2}|E(F,\Gamma\backslash F)| \geq |E(F,F)| + |E(F,\Gamma\backslash F)|,$$

and hence no q-coloring of Γ is frozen of F. With this the proposition follows immediately, since if $q > \frac{1}{2}\Delta + \frac{1}{2}i_e(\Gamma) + 1$ then there must exist some finite subset $F \subseteq V$ such that

$$q > \frac{1}{2}\Delta + \frac{1}{2}\frac{|E(F,\Gamma\backslash F)|}{|F|} + 1,$$

from which $(q-1-\frac{\Delta}{2})|F| > \frac{1}{2}|E(F,\Gamma\backslash F)|$ and so no q-coloring of Γ is frozen on F, and in particular no q-coloring of Γ can be a frozen coloring.

We now proceed to prove the claim. Suppose there exists a q-coloring x which is frozen on F. Consider the subgraph $\Gamma' = (F \cup N(F), E(F) \cup E(F,\Gamma\backslash F))$ where $N(F)$ is the neighborhood of F on Γ. For distinct colors i, j denote by $\Gamma'_{i,j}$ the subgraph of Γ' consisting of edges between vertices which x colors with i and j, and call the connected components of these subgraphs the bi-color components. Note that the family of bi-color components $\mathcal{A}$ partition the edges of Γ', so we have $\sum_{A\in\mathcal{A}} |E(A)| = |E(F)| + |E(F,\Gamma\backslash F)|$. Note also that each $A \in \mathcal{A}$ must contain a vertex of $N(F)$ since otherwise we could interchange both colors appearing on the vertices of A and obtain a valid coloring, contradicting the fact that x is frozen on F. With this $|A \cap F| \leq |E(A)|$. Note also that each vertex of F has $q-1$ colors as neighbours, again since otherwise we could change the color of the vertex to obtain a new q-coloring and contradict that x is frozen on F. With this we see that $\sum_{A\in\mathcal{A}} |A \cap F| = (q-1)|F|$, and so getting all this together:

$$(q-1)|F| = \sum_{A\in\mathcal{A}} |A \cap F| \leq \sum_{A\in\mathcal{A}} |E(A)| = |E(F)| + |E(F,\Gamma\backslash F)|,$$

which contradicts the hypothesis of the claim. $\square$

Using the above proposition we can prove that for $n \geq 4$ the GCS $\mathcal{C}_n$ does not admit a frozen coloring.

Theorem 4.15 *For $n \geq 4$ the GCS $\mathcal{C}_n$ does not admit a frozen coloring.*

Proof. Denoting by Γ the Cayley graph of $\mathrm{BS}(1,N)$ its maximum degree is $\Delta = 4$, and if we prove that $i_e(\Gamma) = 0$ we will have that, using proposition 4.14, for $q > \frac{1}{2}\cdot 4 + \frac{1}{2}\cdot 0 + 1 = 3$ this graph does not admit frozen q-colorings, proving the statement.

Let us see that $i_e(\Gamma) = 0$. Consider for every $k \geq 1: \ \gamma_k \coloneqq E(R_k, \Gamma \backslash R_k)$. We see that $\gamma_1 = 2N + 2$, and that for every $k \geq 2$:

$$\gamma_k = N^k + 2 + N(\gamma_{k-1} - N^{k-1}) = 2 + N\gamma_{k-1},$$

by using the fact that the N sheets arising from the base of the rectangle R_k are copies of the rectangle R_{k-1}. With this we have that

$$\gamma_{k+1} - \gamma_k = N(\gamma_k - \gamma_{k-1}), \ \gamma_2 - \gamma_1 = 2N^2,$$

and so $\gamma_k - \gamma_{k-1} = 2N^k$, for every $k \geq 2$. Now summing this equality from $k = 2$ to $k = m$ we see that

$$\begin{aligned}\gamma_m - (2N + 2) = \gamma_m - \gamma_1 = \sum_{k=2}^{m} \gamma_k - \gamma_{k-1} &= 2\sum_{k=2}^{m} N^k \\ &= 2\frac{N^{m+1} - N^2}{N - 1},\end{aligned}$$

and hence $\gamma_m = 2\frac{N^{m+1}-1}{N-1}$. From this calculation we can estimate the edge-isoperimetric constant:

$$i_e(\Gamma) \leq \liminf_{m\to\infty} \frac{|E(R_m, \Gamma\backslash R_m|)}{|R_m|} = \liminf_{m\to\infty} \frac{\gamma_m}{|R_m|} = \liminf_{m\to\infty} \frac{2}{mN^m}\frac{N^{m+1} - 1}{N - 1} = 0,$$

and so $i_e(\Gamma) = 0$ as we had claimed at the beginning of the proof. $\square$

Chapter 5

Relationship between 1-dimensional and Baumslag-Solitar subshifts

In this chapter we study how a $\mathbb{Z}$-subshift can be extended into a $\mathrm{BS}(1,N)$-subshift in such a way that they share properties, or that certain behavior of one of them forces similar behavior in the other. We start by seeing examples in which the induced $\mathrm{BS}(1,N)$-subshift is empty and find a sufficient condition which guarantees non-emptiness, for the case of vertex $\mathbb{Z}$-subshifts. We then study the Fibonacci $\mathrm{BS}(1,N)$-subshift and define the notion of having a mixing symbol, which permits us to exploit the relations between the original $\mathbb{Z}$-subshift and the induced $\mathrm{BS}(1,N)$-subshift. We finish the chapter by defining the m-strip subshifts for a $\mathrm{BS}(1,N)$-subshift, which gives us a way of approximating its entropy at any degree using the entropies of 1-dimensional subshifts.

5.1 Baumslag-Solitar subshifts arising from 1-dimensional subshifts

We start by defining a way to extend a $\mathbb{Z}$-subshift into a $\mathrm{BS}(1,N)$-subshift, by considering the forbidden patterns of the former as forbidden in the latter along both generators of the group $\mathrm{BS}(1,N)$. This in particular will imply that for any configuration x in the extension and any $g \in \mathrm{BS}(1,N)$ the restrictions $x|_{g\langle a\rangle}$ and $x|_{g\langle b\rangle}$ may be identified with a configuration in the original $\mathbb{Z}$-subshift.

Definition 5.1 *For a finite alphabet $\mathcal{A}$, let $\mathsf{X}_{\mathcal{F}} \subseteq \mathcal{A}^{\mathbb{Z}}$ be the $\mathbb{Z}$-subshift defined by a list of forbidden patterns $\mathcal{F} \subseteq \mathcal{A}^*$. We define its corresponding $\mathrm{BS}(1,N)$-subshift as the one with forbidden patterns $\mathcal{F}$ along both generators a and b. More formally, the corresponding subshift is $\mathsf{X}_{\hat{\mathcal{F}}} \subseteq \mathcal{A}^{\mathrm{BS}(1,N)}$ defined by the forbidden patterns*

$$\hat{\mathcal{F}} := \{\hat{p}_a \mid p \in \mathcal{F}\} \cup \{\hat{p}_b \mid p \in \mathcal{F}\},$$

where for $s \in \{a,b\}$ the pattern $\hat{p}_s$ is defined by $\hat{p}_s : \{s^t \mid t \in \mathrm{supp}(p)\} \to \mathcal{A}$ given by $\hat{p}_{s^t} = p_t$ for every $t \in \mathrm{supp}(p)$.

We say that the $\mathbb{Z}$*-subshift* $\mathsf{X}_{\mathcal{F}}$ ***induces*** *the* $\mathrm{BS}(1,N)$*-subshift* $\mathsf{X}_{\hat{\mathcal{F}}}$*, and that the latter arises or originates from the former.*

Remark 5.2 *If one chooses two sets of forbidden patterns* $\mathcal{F}_1, \mathcal{F}_2$ *such that* $\mathsf{X}_{\mathcal{F}_1} = \mathsf{X}_{\mathcal{F}_2}$ *then their induced* $\mathrm{BS}(1,N)$*-subshifts are equal, i.e.* $\mathsf{X}_{\hat{\mathcal{F}}_2} = \mathsf{X}_{\hat{\mathcal{F}}_1}$*. In particular we see that* $\mathsf{X}_{\mathcal{F}}$ *is an SFT if and only if so is* $\mathsf{X}_{\hat{\mathcal{F}}}$*.*

It is important to note that it may happen that the induced $\mathrm{BS}(1,N)$-subshift $\mathsf{X}_{\hat{\mathcal{F}}}$ is empty even though the $\mathbb{Z}$-subshift $\mathsf{X}_{\mathcal{F}}$ is not. In fact, we have already seen an example of this in Chapter 4, when we saw that the GCS $\mathcal{C}_2$ is empty for even N (see Proposition 4.1). Moreover, it is possible to construct positive entropy $\mathbb{Z}$-subshifts inducing empty $\mathrm{BS}(1,N)$-subshifts, as the following example shows.

Example 5.3 *Consider the* $\mathbb{Z}$*-SFT on* $\{0,1,2\}$ *defined by the list of forbidden patterns* $\mathcal{F} = \{00, 11, 12, 21, 22\}$*, which can be interpreted as the edge-shift from the graph in Figure 5.1. The Perron eigenvalue of this graph is* $\lambda_P = \sqrt{2}$*, and so* $h_{\mathrm{top}}(X_{\mathcal{F}}) = \frac{1}{2}\log(2) > 0$*. We claim that if* N *is even then* $X_{\hat{\mathcal{F}}} = \emptyset$*.*

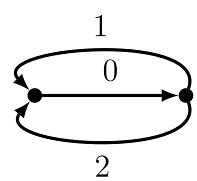
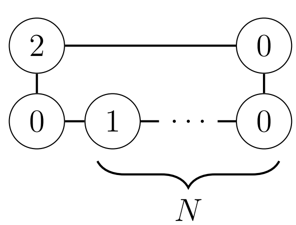


Figure 5.1: Example of a graph whose associated vertex-shift has positive entropy, while it induces an empty $\mathrm{BS}(1,N)$-subshift for even N.

To see this suppose there exists $x \in X_{\hat{\mathcal{F}}}$*, and without loss of generality we may assume that* $x_{e_{\mathrm{BS}(1,N)}} = 0$ *(if not, then the forbidden patterns force that* $x_a = 0$ *and we instead consider* $\sigma_{a^{-1}}(x)$*). Then, as* N *is even and the forbidden patterns only allow a* 0 *to be continued by a* 1 *or* 2*, and any of those only by a* 0*, we must have* $x_{a^N} = 0$*. But then, again by the forbidden patterns, we must have* $x_b = x_{e_G b} \in \{1,2\}$ *and so in order to respect the forbidden patterns* $x_{a^N b} = 0$*. But then the word "00" appears in the direction of the* b *generator, which is not allowed.*



We conclude that the $\mathrm{BS}(1,N)$*-subshift* $\mathsf{X}_{\hat{\mathcal{F}}}$ *is empty, for even* N*. For odd* N*, similar examples can be created.*

The examples given above force emptiness of the induced $\mathrm{BS}(1,N)$-subshift by considering $\mathbb{Z}$-SFTs represented by a labeled graph over which one can find cycles of lengths not found in the cycles of the Cayley graph of $\mathrm{BS}(1,N)$. With this idea in mind, in the next section we give a sufficient condition which guarantees non-emptiness of the $\mathrm{BS}(1,N)$-subshift induced by a vertex shift.

5.2 Non-emptiness of Baumslag-Solitar subshifts arising from vertex-shifts

As we said above, we are going to restrict ourselves to 1-step subshifts of finite type on $\mathbb{Z}$, and see them as vertex-shifts defined by a matrix A with entries in $\{0,1\}$.

In this context we have the following characterization of non-emptiness of the associated $\mathrm{BS}(1,N)$-subshift, which roughly says that this subshifts is non-empty if and only if there exists a family of configurations in the $\mathbb{Z}$-subshift whose elements can be stacked on top of each other to cover the upper half of the Cayley graph of $\mathrm{BS}(1,N)$ while respecting the forbidden patterns in the direction of the generator b.

Lemma 5.4 *Let $\mathsf{X}_{\mathcal{F}}$ be a 1-step $\mathbb{Z}$-SFT, whose list of forbidden patterns $\mathcal{F}$ consists of patterns of length 2. Denote by $X_{\hat{\mathcal{F}}}$ the $\mathrm{BS}(1,N)$-subshift induced by $\mathsf{X}_{\mathcal{F}}$. Then $\mathsf{X}_{\hat{\mathcal{F}}} \neq \emptyset$ if and only if there exists $x \in \mathsf{X}_{\mathcal{F}}$ such that for every $n \geq 1$ and for every n-tuple $(i_1,\dots,i_n) \in \{0,\dots,N-1\}^n$ there exists $x^{(i_1,\dots,i_n)} \in \mathsf{X}_{\mathcal{F}}$ such that*

1. *for every $i_1 \in \{0,\dots,N-1\}$ and for every $k \in \mathbb{Z}$: $x_{i_1+Nk}x_k^{(i_1)} \notin \mathcal{F}$, and*
2. *for every $n \geq 1$, every $(i_1,\dots,i_n,i_{n+1}) \in \{0,\dots,N-1\}^{n+1}$ and $k \in \mathbb{Z}$:*
$$x^{(i_1,\dots,i_n)}_{i_{n+1}+Nk}x_k^{(i_1,\dots,i_n,i_{n+1})} \notin \mathcal{F}.$$

Proof. Suppose $\mathsf{X}_{\hat{\mathcal{F}}} \neq \emptyset$. We are going to construct inductively the family

$$x \cup \bigcup_{n\geq 1}\{x^{(i_1,\dots,i_n)} \mid (i_1,\dots i_n) \in \{0,\dots,N-1\}^n\}$$

that satisfies Conditions 1. and 2.

Consider an arbitrary configuration $y \in \mathsf{X}_{\hat{\mathcal{F}}}$. Define $x \in \mathsf{X}_{\mathcal{F}}$ as $x \coloneqq y|_{\langle a\rangle}$, and for $i_1 \in \{0,\dots,N-1\}$ define $x_k^{(i_1)} \coloneqq y_{a^{i_1}ba^k}$ for $k \in \mathbb{Z}$, that is $x^{(i_1)} = \sigma_{b^{-1}a^{-i_1}}(y)|_{\{a^k:k\in\mathbb{Z}\}}$. By definition of the $\mathrm{BS}(1,N)$-subshift we have that $x^{(i_1)} \in \mathsf{X}_{\mathcal{F}}$ and Condition 1. holds.

Using the same idea as above, for $n \geq 2$ and $(i_1,\dots,i_n) \in \{0,\dots,N-1\}^n$ we define $x_k^{(i_1,\dots,i_n)} \coloneqq y_{(\prod_{j=1}^n a^{i_j}b)a^k}$, that is $x^{(i_1,\dots,i_n)} = \sigma_{(\prod_{j=1}^n a^{i_j}b)^{-1}}(y)|_{\{a^k:k\in\mathbb{Z}\}}$. With this we have that Condition 2. holds, since for every $k \in \mathbb{Z}$

$$x^{(i_1,\dots,i_n)}_{i_{n+1}+Nk}x_k^{(i_1,\dots,i_n,i_{n+1})} = y_{(\prod_{j=1}^n a^{i_j}b)a^{i_{n+1}+Nk}}y_{(\prod_{j=1}^{n+1} a^{i_j}b)a^k}$$

$$= y_{\left(\prod_{j=1}^{n} a^{i_j} b\right) a^{i_n+1} a^{Nk}} y_{\left(\prod_{j=1}^{n} a^{i_j} b\right) a^{i_n+1} b a^k}$$
$$= y_{\left(\prod_{j=1}^{n} a^{i_j} b\right) a^{i_n+1} b a^k b^{-1}} y_{\left(\prod_{j=1}^{n} a^{i_j} b\right) a^{i_n+1} b a^k},$$

which is a pattern not in $\mathcal{F}$ by the definition of $X_{\hat{\mathcal{F}}}$, using that y belongs to $X_{\hat{\mathcal{F}}}$.

Now suppose we have the configurations $x, x^{(i_1,\dots,i_n)} \in \mathsf{X}_{\mathcal{F}}$ for every $n \geq 1$ and $(i_1, \dots, i_n) \in \{0, \dots, N-1\}^n$ satisfying Conditions 1. and 2. We will construct a configuration $y \in \mathsf{X}_{\hat{\mathcal{F}}}$.

Start by defining $y^0 \in \mathcal{A}^{\mathrm{BS}(1,N)}$ by $y^0_{a^k} = x_k$ for every $k \in \mathbb{Z}$, and for $n \geq 1$ and $(i_1, \dots, i_n) \in \{0, \dots, N-1\}^n$ by $y^0|_{\left(\prod_{j=1}^{n} a^{i_j} b\right) a^k} = x_k^{(i_1,\dots,i_n)}$, for $k \in \mathbb{Z}$. With this we have defined the configuration y^0 for every $g \in \mathrm{BS}(1,N)$ whose normal form is of the type $g = a^k b^i$, with $k \in \mathbb{Z}$ and $i \geq 0$. For the rest of the elements g (those whose normal form begins with a negative power of the generator b) assign $y^0_g = \alpha$ any letter of the alphabet $\mathcal{A}$. With this we have defined $y^0 \in \mathcal{A}^{\mathrm{BS}(1,N)}$.

Now define the sequence $y^n \coloneqq \sigma_{b^n}(y^0)$, for $n \geq 1$, and consider any accumulation point z of it (which exists by compactness of $\mathcal{A}^{\mathrm{BS}(1,N)}$). Using the fact that $\mathsf{X}_{\mathcal{F}}$ is a 1-step $\mathbb{Z}$-SFT and the construction done above, we see that no forbidden patterns can occur in z, and hence we have that $z \in X_{\hat{\mathcal{F}}}$. □

By using the previous lemma we are able to give a sufficient condition for the matrix that defines a vertex-shift such that the $\mathrm{BS}(1,N)$-subshift it induces is not empty. This theorem roughly says that a sufficient condition for non-emptiness is that the graph defining the vertex-shift contains an essential subgraph on which the existence of a path of length N between two vertices implies that they are connected by an edge.

Theorem 5.5 *Let $X_\Gamma \subseteq V^{\mathbb{Z}}$ be the vertex shift associated to an essential (directed) graph $\Gamma = (V, E)$ and denote by X the $\mathrm{BS}(1,N)$-SFT it induces. Suppose Γ contains an essential subgraph $\widetilde{\Gamma} = (\widetilde{V}, \widetilde{E})$ whose associated matrix $\widetilde{A}$ satisfies:*

$$\forall i, j \in \widetilde{V} : \widetilde{A}^N_{ij} > 0 \implies \widetilde{A}_{ij} > 0.$$

Then $X \neq \emptyset$.

Proof. We will prove that $X \neq \emptyset$ by using Lemma 5.4.

As $\widetilde{\mathcal{G}}$ is essential its associated vertex-shift $X_{\widetilde{\mathcal{G}}}$ is not empty and hence contains a point $x \in X_{\widetilde{\mathcal{G}}}$. The condition satisfied by the matrix $\widetilde{A}$ means that if there is a path of length N between two vertices of $\widetilde{\mathcal{G}}$, then those vertices are already connected by an edge of $\widetilde{\mathcal{G}}$. Let us define the family of configurations from Lemma 5.4. First for $i_1 \in \{0, \dots, N-1\}$ set

$$x^{(i_1)}_k \coloneqq x_{i_1+Nk+1}.$$

With this we have that

$$x_{i_1+Nk} x^{(i_1)}_k = x_{i_1+Nk} x_{i_1+Nk+1}$$

which is an allowed pattern, and that for every $k \in \mathbb{Z}$ we have

$$x_k^{(i_1)} x_{k+1}^{(i_1)} = x_{i_1+Nk+1} x_{i_1+N(k+1)+1} = x_{i_1+Nk+1} x_{i_1+Nk+1+N}.$$

Since any two symbols separated by a path of length N must be adjacent, this is also an allowed pattern and hence $x^{(i_1)}$. With this we have Condition 1. of the previous lemma.

Having already defined $x^{(i_1,\dots,i_n)} \in X_{\widetilde{\mathcal{G}}}$ for every $(i_1,\dots,i_n) \in \{0,\dots,N-1\}^n$, let us consider $(i_1,\dots,i_n,i_{n+1}) \in \{0,\dots,N-1\}^{n+1}$ and for $k \in \mathbb{Z}$ define

$$x_k^{(i_1,\dots,i_n,i_{n+1})} := x_{i_{n+1}+Nk+1}^{(i_1,\dots,i_n)}.$$

Then by the same argument as above we have $x^{(i_1,\dots,i_{n+1})} \in X_{\widetilde{\mathcal{G}}}$ and for every $k \in \mathbb{Z}$ the pattern

$$x_{i_{n+1}+Nk}^{(i_1,\dots,i_n)} x_k^{(i_1,\dots,i_n,i_{n+1})}$$

is allowed.

We have thus confirmed the hypothesis for Lemma 5.4, and hence we conclude that the $\mathrm{BS}(1,N)$-subshift induced by $X_{\widetilde{\mathcal{G}}}$ is non empty. □

5.3 The Fibonacci subshift and subshifts with mixing symbols

In this section we study another way of assuring non-emptiness of the induced $\mathrm{BS}(1,N)$-subshift, by requiring a strong mixing behavior of the $\mathbb{Z}$-subshift, namely having a *mixing symbol*. Before making the corresponding definitions, let us study first a particular example: the Fibonacci subshift, which generalizes the definitions made for $\mathbb{Z}$ and $\mathbb{Z}^d$ (where it is sometimes called the hard-core or hard-square subshift). In what follows we prove that this $\mathrm{BS}(1,N)$-subshift is non-empty and moreover give a construction for a non-trivial strongly periodic point, that is, one for which its stabilizer has finite index yet it is not the whole group.

Define the Fibonacci $\mathrm{BS}(1,N)$-subshift as $X_{\mathrm{Fib}} := \mathsf{X}_{\widehat{\mathcal{F}}} \subseteq \mathcal{A}^{\mathrm{BS}(1,N)}$, the $\mathrm{BS}(1,N)$-subshift induced by the golden-mean $\mathbb{Z}$-subshift $\mathsf{X}_{\mathcal{F}} \subseteq \{0,1\}^{\mathbb{Z}}$, where $\mathcal{F} = \{11\}$. Of course this subshift is not empty since it contains the trivially strongly periodic point $(x_g = 0)_{g \in \mathrm{BS}(1,N)} \in X_{\mathrm{Fib}}$. Does it contain any strongly periodic points that are more interesting than this one? The answer to this question is affirmative and next proposition gives an explicit construction for such a configuration.

Proposition 5.6 *The Fibonacci* $\mathrm{BS}(1,N)$*-subshift has a non-trivial strongly periodic point.*

Proof. Let us define a configuration $x \in \mathcal{A}^{\mathrm{BS}(1,N)}$ by setting for each $g = b^{-j}a^kb^i \in \mathrm{BS}(1,N)$, $i,j \geq 0,\ k \in \mathbb{Z}$ written in its normal form:

$$x_{b^{-j}a^kb^i} = \begin{cases} 1, & \text{if } i+j+k \in (N+1)\mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that by definition we have that if $x_{b^{-j}a^kb^i} = 1$, then $i + j + k \in (N+1)\mathbb{Z}$, and so

$$\begin{aligned} x_{b^{-j}a^kb^ib} &= 0, \text{ since } i + j + k + 1 \notin (N+1)\mathbb{Z}, \\ x_{b^{-j}a^kb^ia} &= x_{b^{-j}a^{k+N^i}b^i} = 0, \text{ since } i + j + k + N^i \notin (N+1)\mathbb{Z}. \end{aligned}$$

With this we have proved that in x there are no two adjacent 1's and so $x \in X_{\text{Fib}}$.

Now let us see that x is a strongly periodic configuration. Note that for any $g = b^{-j}a^kb^i \in \text{BS}(1,N)$ written in its normal form:

$$x_{a^{-(N+1)}b^{-j}a^kb^i} = x_{b^{-j}a^{k-(N+1)N^j}b^i} = x_{b^{-j}a^kb^i},$$

and that

$$x_{b^{-(N+1)}b^{-j}a^kb^i} = x_{b^{-(N+1)-j}a^kb^i} = x_{b^{-j}a^kb^i}.$$

Hence $a^{N+1} \in \text{Stab}(x)$ and $b^{N+1} \in \text{Stab}(x)$, and with it $H \coloneqq \langle a^{N+1}, bN+1 \rangle \leq \text{Stab}(x)$. It suffices to prove that H has finite index in $\text{BS}(1,N)$ to finish the proof.

We will prove that for each $g \in \text{BS}(1,N)$, there exists $r \in R_{N+2}$ such that $gH = rH$, thus proving that $[\text{BS}(1,N) : H] \leq |R_{N+2}| < \infty$. Given any $g = b^{-j}a^kb^i \in \text{BS}(1,N)$ written in its normal form, consider integers $i^*, w^*, j^*, k^* \in (N+1)\mathbb{Z}$ (whose existence we will justify later) such that:

$$\begin{cases} i - i^* \leq 0, \\ \exists m \in \mathbb{Z} : \; kN^{i^*-i} + w^* = mN^{j+i^*-i}, \\ 0 \leq \ell^* \coloneqq j^* - (j + i^* - i) \leq N, \text{ and} \\ 0 \leq m + k^*N^{\ell^*} < N^{N+2}. \end{cases}$$

Then defining $h = a^{-k^*}b^{-j^*}a^{-w^*}b^{i^*} \in H$ we have that

$$\begin{aligned} gh^{-1} &= b^{-j}a^kb^i \cdot b^{-i^*}a^{w^*}b^{j^*}a^{k^*} \\ &= b^{-j}a^kb^{i-i^*} \cdot a^{w^*}b^{j^*}a^{k^*} \\ &= b^{-(j+i^*-i)}a^{kN^{i^*-i}} \cdot a^{w^*}b^{j^*}a^{k^*} \\ &= b^{-(j+i^*-i)}a^{kN^{i^*-i}+w^*} \cdot b^{j^*}a^{k^*} \\ &= b^{-(j+i^*-i)}a^{mN^{j+i^*-i}} \cdot b^{j^*}a^{k^*} \\ &= a^mb^{-(j+i^*-i)} \cdot b^{j^*}a^{k^*} \\ &= a^mb^{j^*-(j+i^*-i)} \cdot a^{k^*} \\ &= a^mb^{\ell^*} \cdot a^{k^*} \\ &= a^{m+k^*N^{\ell^*}}b^{\ell^*} \in R_{N+2}, \end{aligned}$$

and so we have found $r \in R_{N+2}$ such that $gH = rH$, as was claimed. Now let us prove that we can choose such integers as above: i^* can be any multiple of $N+1$ bigger than i. The existence of w^* follows from the fact that $N+1$ and N^j are coprime. The existence of j^* is just noting that we want to translate $j + i^* - i$ into an interval of length $N+1$, and the existence of k^* follows from this same idea but with an interval of length at most $N^{N+1} + N^N$. $\square$

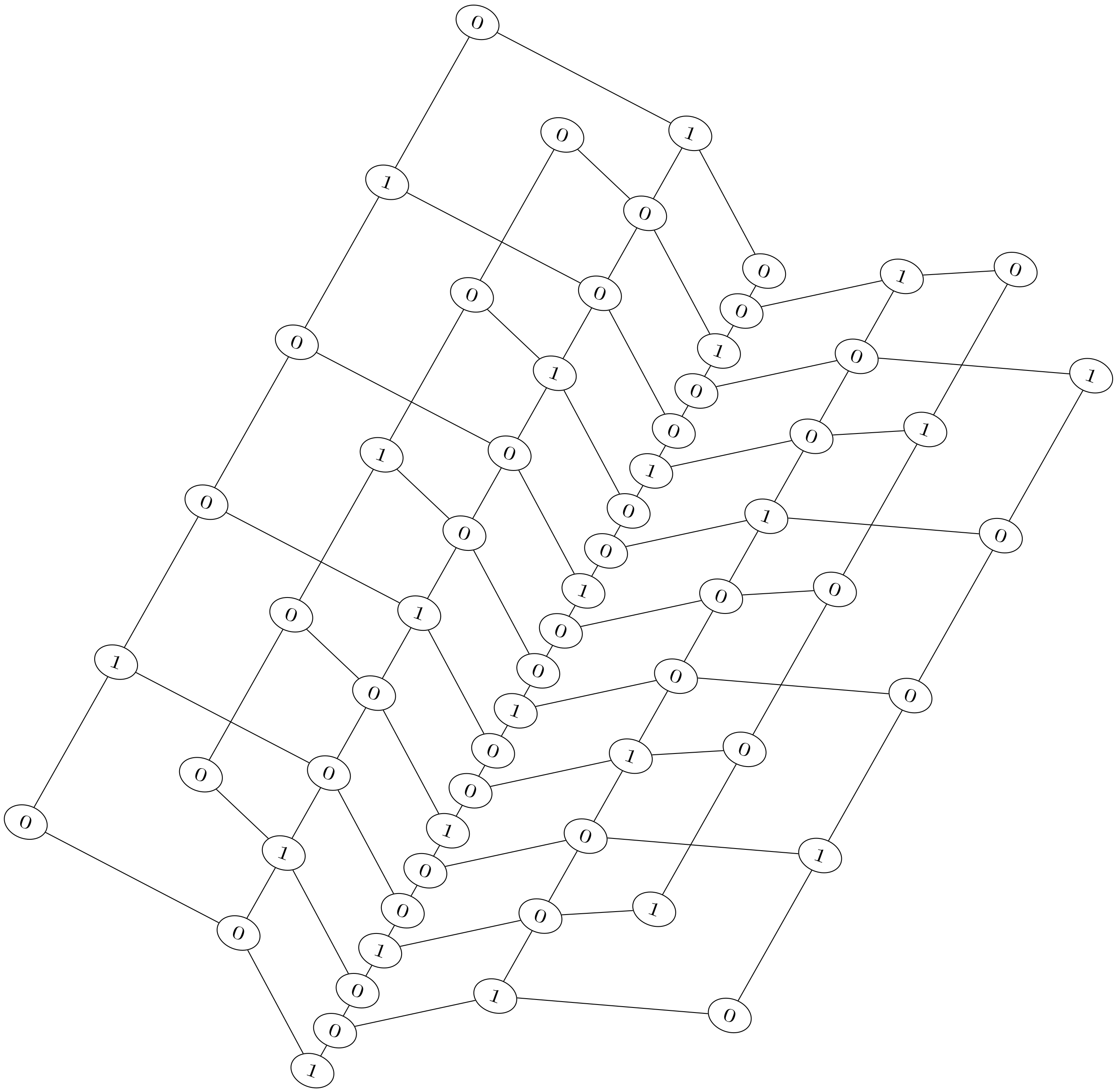


Figure 5.2: A part of the strongly periodic configuration of X_{Fib} constructed in the proof of Proposition 5.6.

In the next section we will study this type of subshifts in a more general case. A specific property present in the Fibonacci subshift is having a symbol in the alphabet which can be used to surround two admissible patterns and glue them together to form a globally admissible pattern (in the case of the Fibonacci subshift this symbol is 0). By centering our attention into subshifts with such a particular symbol will allow us to understand more in depth how patterns in a configuration behave and in particular we will be able to compare the topological entropy of a $\mathbb{Z}$-subshift with the one of the $\mathrm{BS}(1,N)$-subshift it induces.

Definition 5.7 *A subshift $X \subseteq \mathcal{A}^G$ is said to have a* ***mixing symbol*** *if there exists a symbol in the alphabet $\alpha_M \in \mathcal{A}$ such that for any pair of globally admissible patterns p_1, p_2 in X such that $d(\mathrm{supp}(p_1), \mathrm{supp}(p_2)) \geq 2$, there exists $x \in X$ such that $x|_{\mathrm{supp}(p_1)} = p_1$, $x|_{\mathrm{supp}(p_2)} = p_2$ and $x_h = \alpha_M$ for every $h \in G$ with $d(h, \mathrm{supp}(p_1)) = 1$. In the case of $\mathbb{Z}$-subshifts this means that the mixing symbol is allowed to precede or follow any symbol while respecting the forbidden patterns describing the subshift.*

Let $X \subseteq \mathcal{A}^{\mathrm{BS}(1,N)}$ be a $\mathrm{BS}(1,N)$-subshift with a mixing symbol. This property allows us to construct a pattern over a big rectangle by first defining patterns over "independent" copies of smaller rectangles and then filling the gaps with the mixing symbol. Using constructions of this nature we obtain the following two properties, the first of which relates the topological entropy of the subshift X with the allowed patterns on a rectangle R_m, and the second one showing the density of weakly periodic points of X.

Proposition 5.8 *Consider a subshift $X \subseteq \mathcal{A}^{\mathrm{BS}(1,N)}$ with a mixing symbol. Then for every $m \in \mathbb{N}$ we have*

$$\frac{N}{N+1}\frac{m}{m+1}\frac{1}{mN^m}\log|\mathcal{L}_{R_m}(X)| \leq h_{\mathrm{top}}(X) \leq \frac{1}{mN^m}\log|\mathcal{L}_{R_m}(X)|.$$

Proof. The inequality $h_{\mathrm{top}}(X) \leq \frac{1}{mN^m}\log|\mathcal{L}_{R_m}(X)|$ comes from the fact that

$$h_{\mathrm{top}}(X) = \inf_{m\geq 1}\frac{1}{mN^m}\log|\mathcal{L}_{R_m}(X)|.$$

The proof of this fact follows the spirit of the subadditive lemma: define

$$L := \inf_{m\geq 1}\frac{1}{mN^m}\log|\mathcal{L}_{R_m}(X)|,$$

and for arbitrary $\varepsilon > 0$ take $m \geq 1$ such that $\frac{1}{mN^m}\log|\mathcal{L}_{R_m}(X)| \leq L + \varepsilon$. Note that for any $k \geq 1$ we have that

$$|\mathcal{L}_{R_{2k}}| \leq |\mathcal{L}_{R_k}|^{2N^k},$$

as by Proposition 3.5 the rectangle R_{2k} is composed of $2N^k$ disjoint copies of the rectangle R_k. Taking logarithm we obtain

$$\log|\mathcal{L}_{R_{2k}}| \leq 2N^k\log|\mathcal{L}_{R_k}|.$$

Now for $n \geq 1$ we use the above to estimate $\log|\mathcal{L}_{R_{2^n m}}|$:

$$\begin{aligned}\log|\mathcal{L}_{R_{2^n m}}| &= \log|\mathcal{L}_{R_{2 2^{n-1} m}}| \\ &\leq 2N^{2^{n-1}m}\log|\mathcal{L}_{R_{2^{n-1}m}}| \\ &= 2N^{2^{n-1}m}\log|\mathcal{L}_{R_{2 2^{n-2} m}}| \\ &\leq 2N^{2^{n-1}m}\cdot 2N^{2^{n-2}m}\log|\mathcal{L}_{R_{2^{n-2}m}}|\end{aligned}$$

and iterating this process we arrive at

$$\log|\mathcal{L}_{R_{2^n m}}| \leq \prod_{i=1}^{n}(2N^{2^{n-i}m})\log|\mathcal{L}_{R_m}|$$

$$
\begin{aligned}
&= 2^n N^{m \sum_{i=1}^{n} 2^{n-i}} \log |\mathcal{L}_{R_m}| \\
&= 2^n N^{m \sum_{i=0}^{n-1} 2^{i}} \log |\mathcal{L}_{R_m}| \\
&= 2^n N^{m(2^n-1)} \log |\mathcal{L}_{R_m}|.
\end{aligned}
$$

With this

$$L \leq \frac{1}{2^n m N^{2^n m}} \log |\mathcal{L}_{R_{2^n m}}| \leq \frac{1}{2^n m N^{2^n m}} 2^n N^{m(2^n-1)} \log |\mathcal{L}_{R_m}| = \frac{1}{mN^m} \log |\mathcal{L}_{R_m}| \leq L + \varepsilon,$$

and taking $n \to \infty$ yields $L \leq h_{\text{top}}(X) \leq L + \varepsilon$. Finally as ε was arbitrary the claimed equality follows.

Now we proceed to prove that

$$\frac{N}{N+1} \frac{m}{m+1} \frac{1}{mN^m} \log |\mathcal{L}_{R_m}(X)| \leq h_{\text{top}}(X).$$

To see this, note that given fixed $m \geq 1$ and some $k \geq 1$, we can subdivide the rectangle R_{m+k} with disjoint copies of R_m leaving some separation between them. Using the mixing symbol $\alpha_M \in \mathcal{A}$ to fill those gaps we can globally admissible patterns of shape R_k independently, producing a globally admissible pattern on R_{m+k}.

More specifically, note that two copies of R_m positioned at the same height of the rectangle R_{m+k} and separated by at least N^{m-1} elements can be filled by two arbitrary with the mixing symbol α_M between their bases for them to be independent. On the other hand, copies of the rectangle R_m at different heights need to be separated by at least one row filled with the mixing symbol α_M for them to be independent. In this way we see that each copy of the rectangle R_m uses a base of length $N^m + N^{m-1}$ and a height of $m+1$. A representation of the structure of independent copies of R_m is shown in Figure 5.3.

Now we can calculate how many independent copies of R_m fit inside R_{m+k} as

$$\#\{\text{independent copies of } R_m \text{ inside } R_{m+k}\} = \sum_{i=0}^{\lfloor \frac{k}{m+1} \rfloor} \left\lfloor \frac{N^{m+k-i(m+1)}}{N^m + N^{m-1}} \right\rfloor N^{i(m+1)},$$

where the index of this sum symbolizes the heights at which we put the copies of R_m, and the factor $N^{i(m+1)}$ comes from the amount of sheets we have at that height.

To simplify calculations, we will assume $k = \ell(m+1)$, for some $\ell \geq 1$. Then we obtain:

$$
\begin{aligned}
\sum_{i=0}^{\lfloor \frac{k}{m+1} \rfloor} \left\lfloor \frac{N^{m+k-i(m+1)}}{N^m + N^{m-1}} \right\rfloor N^{i(m+1)} &= \sum_{i=0}^{\ell} \left\lfloor \frac{N^{m+(\ell-i)(m+1)}}{N^m + N^{m-1}} \right\rfloor N^{i(m+1)} \\
&\geq \sum_{i=0}^{\ell} \left(\frac{N^{m+(\ell-i)(m+1)}}{N^m + N^{m-1}} - 1 \right) N^{i(m+1)}
\end{aligned}
$$

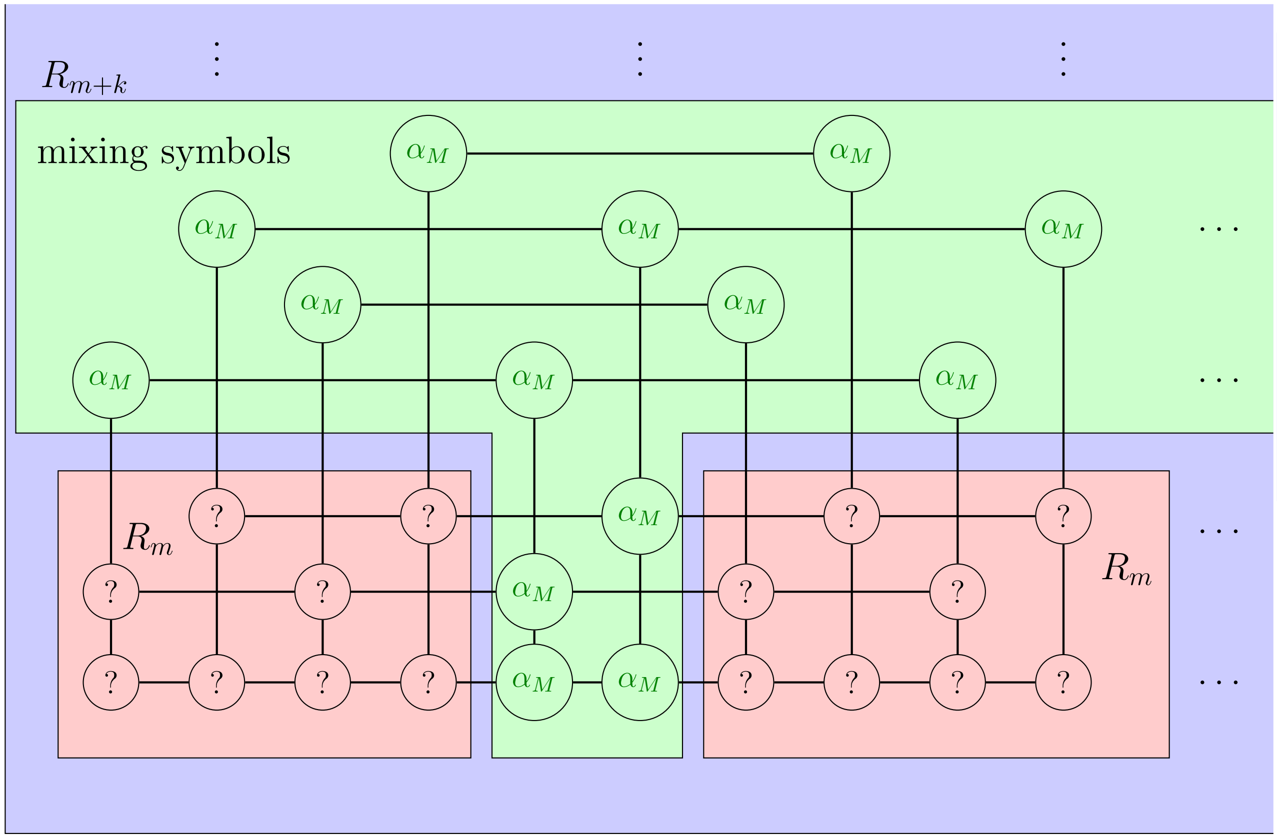


Figure 5.3: Copies of the rectangle R_m separated from each other using the mixing symbol α_M.

$$
\begin{aligned}
&= \sum_{i=0}^{\ell} \left(\frac{N^{m+\ell(m+1)}}{N^m + N^{m-1}} - N^{i(m+1)} \right) \\
&= \sum_{i=0}^{\ell} \left(\frac{N^{\ell(m+1)+1}}{N+1} - N^{i(m+1)} \right) \\
&= \frac{N^{\ell(m+1)+1}}{N+1} (\ell+1) - \frac{N^{(m+1)(\ell+1)} - 1}{N^{m+1} - 1}.
\end{aligned}
$$

Using this to estimate the amount of patterns on the rectangle $R_{m+\ell(m+1)}$, and along with it give a lower bound for the entropy:

$$
|\mathcal{L}_{R_{m+\ell(m+1)}}(X)| \geq |\mathcal{L}_{R_m}(X)|^{\frac{N^{\ell(m+1)+1}}{N+1}(\ell+1) - \frac{N^{(m+1)(\ell+1)}-1}{N^{m+1}-1}},
$$

from where

$$
\begin{aligned}
\frac{1}{|R_{m+\ell(m+1)}|} \log |\mathcal{L}_{R_{m+\ell(m+1)}}(X)| &= \frac{1}{(m+\ell(m+1))N^{m+\ell(m+1)}} \log |\mathcal{L}_{R_{m+\ell(m+1)}}(X)| \\
&\geq \frac{\left(\frac{N^{\ell(m+1)+1}}{N+1} (\ell+1) - \frac{N^{(m+1)(\ell+1)}-1}{N^{m+1}-1} \right)}{(m+\ell(m+1))N^{m+\ell(m+1)}} \log |\mathcal{L}_{R_m}(X)|
\end{aligned}
$$

$$= \left(\frac{\frac{N}{N+1}(\ell+1)}{(m+\ell(m+1))N^m} - \frac{\frac{N^{\ell(m+1)+m+1}-1}{N^{m+1}-1}}{(m+\ell(m+1))N^{m+\ell(m+1)}} \right) \log |\mathcal{L}_{R_m}(X)|.$$

Taking $\ell \to \infty$ the second term inside the parenthesis vanishes and we obtain

$$h_{\mathrm{top}}(X) \geq \frac{N}{N+1} \frac{m}{m+1} \frac{1}{mN^m} \log |\mathcal{L}_{R_m}(X)|,$$

as was claimed.

□

The property of having a mixing symbol also implies the density of the set of weakly periodic points of the subshift. The proof consists of using the mixing symbol to glue the same pattern to itself repeatedly in order to construct a weakly periodic point which contains a particular pattern within it.

Proposition 5.9 *Let $X \subseteq \mathcal{A}^{\mathrm{BS}(1,N)}$ be a* BS$(1, N)$*-SFT with a mixing symbol. Then the set of weakly periodic points of X is dense in X.*

Proof. We will prove that for every $m \geq 1$ and for every globally admissible pattern $P \in \mathcal{A}^{R_m}$ whose support is the rectangle R_m we can find a weakly periodic point $x \in X$ such that $x|_{R_m} = P$. With this we will have proven approximate every point by a sequence of weakly periodic points, and thus proving the density of the set of weakly periodic points of X.

First we construct a point $x^0 \in X$ by concatenating copies of P surrounded by the mixing symbol along the generator a direction, and repeating this pattern through the generator b direction. Let us denote $\alpha_M \in \mathcal{A}$ the mixing symbol and define $x^0_g = \alpha_M$ for every $g \in \mathrm{BS}(1, N)$ whose normal form is $g = b^{-j}a^k b^i$ with $j \geq 1$. Thus, we are forcing this configuration to be equal to the mixing symbol everywhere outside the "main half-sheet" $H := \{a^k b^i \mid k \in \mathbb{Z}, i \geq 0\}$.

Define $x^0|_{R_{m+1}}$ by setting $x^0|_{R_m} = P$ and for every $g \in R_{m+1} \backslash R_m$ set $x^0_g = \alpha_M$. By copying the pattern seen on rectangle R_{m+1} to the left and to the right (in the direction of the a-generator) we define configuration x^0 on the set $R_{m+1}\langle a \rangle$. This configuration is illustrated in Figure 5.4.

We can now copy this pattern upwards, to define the symbol x^0_g for every $g \in H$, and hence finish defining x_g for every $g \in \mathrm{BS}(1, N)$. Let us do this by imposing that for every $r \geq 1$ and $i \in \{0, \dots, N^{r(m+1)} - 1\}$ define

$$x^0|_{a^i b^{r(m+1)}\langle a \rangle} := x^0|_{R_{m+1}\langle a \rangle}.$$

Note that by construction we have that x^0 satisfies $\sigma_{b^{-r(m+1)}}(x^0)|_H = x^0|_H$ for all $r \geq 1$, where H is the "main half-sheet" described above.This construction is shown in Figure 5.5.

Finally we define for every $n \geq 1$ the point $x^n := \sigma_{b^{-(n(m+1))}}(x^0) \in X$. By the construction made above we see that this sequence converges to a point $x \in X$ which satisfies $\sigma_{b^{-(m+1)}}(x) = x$ and that $x|_{R_m} = P$, so x is a weakly periodic point (since the subgroup generated by $b^{-(m+1)}$ is contained in its stabilizer) which sees the pattern P in the rectangle R_m. □

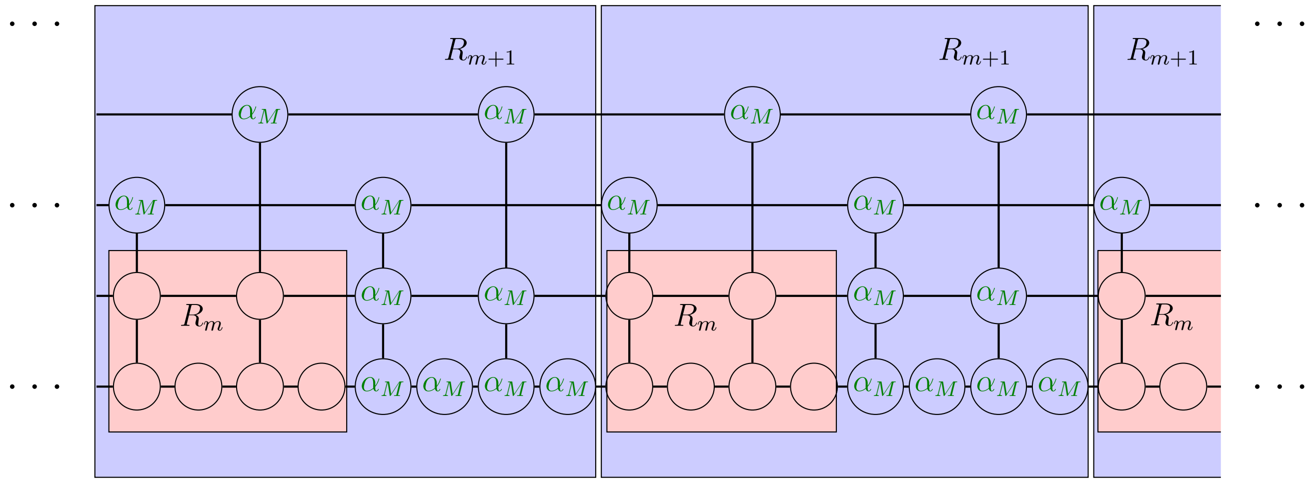


Figure 5.4: Periodic configuration on $R_{m+1}\Gamma_a$, from the proof of Proposition 5.9.

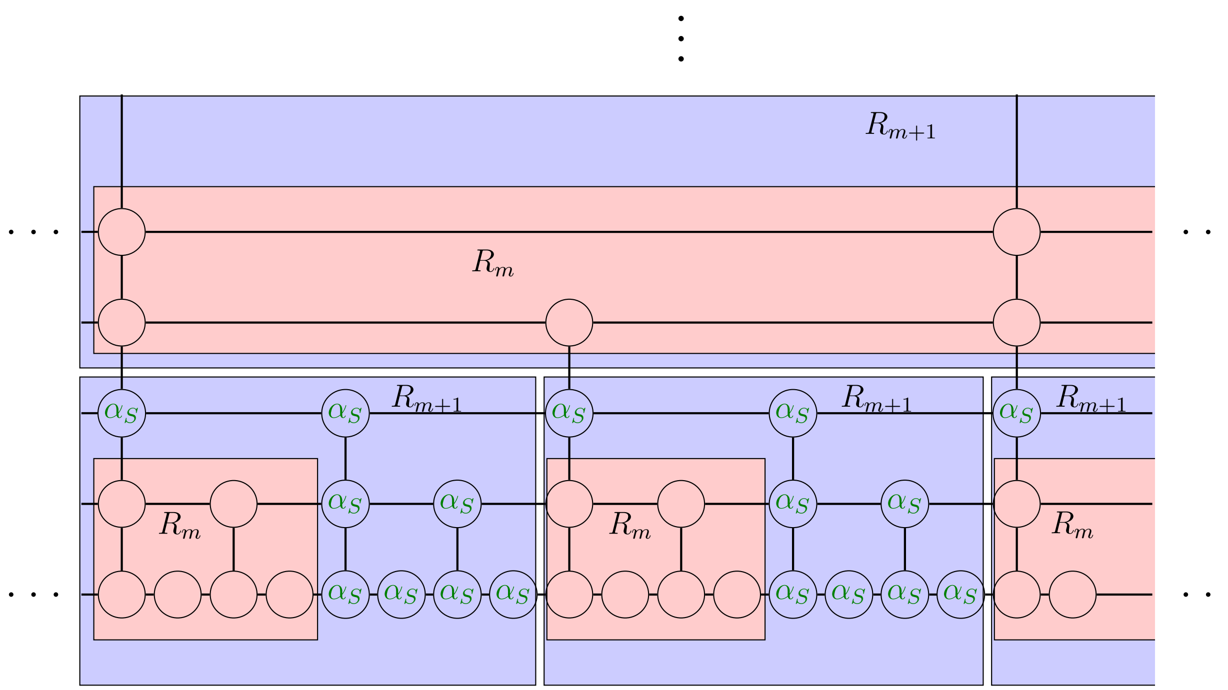


Figure 5.5: Extending the periodic configuration upwards, from the proof of Proposition 5.9.

5.4 Topological entropy and the m-strip subshift

The relation between the forbidden patterns of a $\mathbb{Z}$-subshift and the $\mathrm{BS}(1,N)$-subshift it induces forces that looking at supports of the form $\{gs^k : k \in \mathbb{Z}\}$, for $g \in \mathrm{BS}(1,N)$ and $s \in \{a,b\}$, for any configuration in the $\mathrm{BS}(1,N)$-subshift we will see a configuration from the $\mathbb{Z}$-subshift. This relation allows us to compare the allowed patterns of both subshifts and with them their entropies, as the next proposition shows.

Proposition 5.10 *Let* $\widehat{X} \subseteq \mathcal{A}^{\mathrm{BS}(1,N)}$ *be a* $\mathrm{BS}(1,N)$*-subshift induced by a* $\mathbb{Z}$*-subshift* $X \subseteq \mathcal{A}^{\mathbb{Z}}$ *according to Definition 5.1. Then*

$$h_{\mathrm{top}}(\widehat{X}) \leq h_{\mathrm{top}}(X).$$

Proof. The amount of globally admissible patterns appearing on the rectangle R_m on $\widehat{X}$ are at most the number of possible words of X appearing on the rows of the rectangle. There is 1 row of length N^m, N rows of length N^{m-1}, and in general N^i rows of length N^{m-i} for $i = 0, \dots, m-1$. With this we have

$$|\mathcal{L}_{R_m}(X)| \leq \prod_{i=0}^{m-1} |\mathcal{L}_{N^{m-i}}(X)|^{N^i}.$$

Taking logarithm and dividing by $|R_m| = mN^m$ we get

$$\begin{aligned}
\frac{1}{mN^m} \log |\mathcal{L}_{R_m}(X)| &\leq \frac{1}{mN^m} \sum_{i=0}^{m-1} N^i \log |\mathcal{L}_{N^{m-i}}(X)| \\
&\leq \frac{1}{mN^m} \sum_{k=1}^{m} N^{m-k} \log |\mathcal{L}_{N^k}(X)| \\
&= \frac{1}{mN^m} \sum_{k=1}^{m} N^{m-k} N^k \frac{1}{N^k} \log |\mathcal{L}_{N^k}(X)| \\
&= \frac{1}{m} \sum_{k=1}^{m} \frac{1}{N^k} \log |\mathcal{L}_{N^k}(X)|,
\end{aligned}$$

then taking $m \to \infty$ and using the fact that if a sequence converges to a number then its Cesàro means also converge to it, we get the desired inequality. □

Nonetheless, the inequality from Proposition 5.10 may be strict. Moreover, it may be possible that a positive entropy $\mathbb{Z}$-SFT gives rise to an empty $\mathrm{BS}(1,N)$-subshift, as was shown in Example 5.3.

$\mathsf{X}_{\mathcal{F}}$ having a mixing symbol implies that the $\mathrm{BS}(1,N)$-subshift it induces $X_{\hat{\mathcal{F}}}$ also has a mixing symbol, and this in particular allows us to prove that this subshift is not empty and to give a lower bound on its topological entropy, by constructing patterns with independent words from $\mathsf{X}_{\mathcal{F}}$ with a separation of mixing symbols between them. In particular, we see that a $\mathbb{Z}$-SFT of positive entropy with a mixing symbol gives rise to a $\mathrm{BS}(1,N)$-SFT of positive entropy.

Proposition 5.11 *Let $\widehat{X} \subseteq \mathcal{A}^{\mathrm{BS}(1,N)}$ be a nearest-neighbors* $\mathrm{BS}(1,N)$*-SFT induced by a $\mathbb{Z}$-subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, with a mixing symbol. Then $\widehat{X} \neq \emptyset$ and*

$$h_{\mathrm{top}}(\widehat{X}) \geq \frac{1}{2} h_{\mathrm{top}}(X).$$

Proof. We will define patterns with support on the set R_{2m}, which can then be extended to the rest of the group using the mixing symbol. To give a lower bound of the amount of patterns we can get with this support, note that we can fill the even rows of R_{2m} using words from $\mathcal{L}(X)$, while assigning to the odd rows the mixing symbol to obtain globally admissible patterns. With this, and remembering that in R_{2m} there are N^{2i} rows of length N^{2m-2i}, for $0 \leq i \leq m-1$, we get

$$|\mathcal{L}_{R_{2m}}(\widehat{X})| \geq \prod_{i=0}^{m-1} |\mathcal{L}_{N^{2m-2i}}(X)|^{N^{2i}}.$$

Taking logarithm and dividing by $|R_{2m}| = 2mN^{2m}$ we obtain

$$\begin{aligned}
\frac{1}{2mN^{2m}} \log |\mathcal{L}_{R_{2m}}(\widehat{X})| &\geq \frac{1}{2mN^{2m}} \sum_{i=0}^{m-1} N^{2i} \log |\mathcal{L}_{N^{2m-2i}}(X)| \\
&= \frac{1}{2mN^{2m}} \sum_{k=1}^{m} N^{2(m-k)} N^{2k} \frac{1}{N^{2k}} \log |\mathcal{L}_{N^{2k}}(X)| \\
&= \frac{1}{2} \frac{1}{m} \sum_{k=1}^{m} \frac{1}{N^{2k}} \log |\mathcal{L}_{N^{2k}}(X)|.
\end{aligned}$$

Taking $m \to \infty$ we get

$$h_{\mathrm{top}}(\widehat{X}) = \lim_{m\to\infty} \frac{1}{2mN^{2m}} \log |\mathcal{L}_{R_{2m}}(\widehat{X})| \geq \frac{1}{2} \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \frac{1}{N^{2k}} \log |\mathcal{L}_{N^{2k}}(X)| = \frac{1}{2} h_{\mathrm{top}}(X),$$

as was claimed. □

In particular, the previous proposition allows us to estimate the topological entropy of the Fibonacci $\mathrm{BS}(1,N)$-subshift, defined in the previous section.

Corollary 5.12 *For the Fibonacci* $\mathrm{BS}(1,N)$*-subshift X_{Fib} we have*

$$\frac{1}{2} \log(\varphi) \leq h_{\mathrm{top}}(X_{\mathrm{Fib}}) \leq \log(\varphi),$$

where $\varphi := \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Inspired by the ideas of the proof of Proposition 5.11 we can find a new way to approximate the entropy of a $\mathrm{BS}(1,N)$-subshift (with a mixing symbol) using a sequence of $\mathbb{Z}$-subshifts. Intuitively the definition below consists of, for a fixed $m \geq 1$, creating a $\mathbb{Z}$-subshift using as alphabet the patterns with support R_m appearing in X, and neighboring rules according to the allowed patterns of X.

Definition 5.13 *Let $X \subseteq \mathcal{A}^{\mathrm{BS}(1,N)}$ be a subshift and $m \geq 1$. We define the alphabet*

$$\mathcal{A}_m := \{x|_{R_m} | x \in X\},$$

and interpret a configuration $y \in \mathcal{A}_m^{\mathbb{Z}}$ as an infinite pattern in $\mathcal{A}^{R_m^\infty}$, where $R_m^\infty := R_m \Gamma_a$ and $\Gamma_a = \{a^k \mid k \in \mathbb{Z}\}$, through the concatenation of its symbols. We define the ***m-strip subshift*** *by*

$$X/R_m := \{y \in \mathcal{A}_m^{\mathbb{Z}} \mid \text{using the above interpretation, } y \text{ contains no forbidden patterns of } X\}.$$

Remark 5.14 *From the definition it is immediate that if X is an SFT then so is X/R_m, and that if X has a mixing symbol then so does X/R_m.*

The relation between the allowed patterns of X and X/R_m gives us the following proposition, which provides lower and upper bounds for the topological entropy of X using the topological entropy of the m-strip subshift X/R_m.

Proposition 5.15 *Let $X \subseteq \mathcal{A}^{\mathrm{BS}(1,N)}$ be a $\mathrm{BS}(1,N)$-subshift with a mixing symbol, and for $m \geq 1$ consider $X/R_m \subseteq \mathcal{A}_m^{\mathbb{Z}}$ the $\mathbb{Z}$-subshift defined above. Then*

$$\frac{1}{(m+1)N^m} h_{\mathrm{top}}(X/R_m) \leq h_{\mathrm{top}}(X) \leq \frac{1}{mN^m} h_{\mathrm{top}}(X/R_m).$$

Proof. Let us prove first that

$$h_{\mathrm{top}}(X) \leq \frac{1}{mN^m} h_{\mathrm{top}}(X/R_m).$$

For this note that if for any $n \geq 1$ we look at the rectangle R_{mn} and separate it in heights of length m, in each one of these blocks we will be looking at a word of X/R_m. More specifically, looking at height jm up to $(j+1)m - 1$ we will see a total of N^{jm} copies of words of X/R_m of length $N^{nm-m-jm}$, for $j = 0, \ldots, n-1$. This allows us to give an upper bound for the amount of different patterns appearing on R_{mn}:

$$|\mathcal{L}_{R_{mn}}(X)| \leq \prod_{j=0}^{n-1} |\mathcal{L}_{N^{m(n-1-j)}}(X/R_m)|^{N^{jm}}.$$

Taking logarithm and dividing by mnN^{mn} we get

$$\begin{aligned}
\frac{1}{mnN^{mn}} \log |\mathcal{L}_{R_{mn}}(X)| &\leq \sum_{j=0}^{n-1} \frac{N^{jm}}{mnN^{mn}} \log |\mathcal{L}_{N^{m(n-1-j)}}(X/R_m)| \\
&\leq \sum_{j=0}^{n-1} \frac{N^{jm} N^{m(n-j-1)}}{mnN^{mn}} \frac{1}{N^{m(n-1-j)}} \log |\mathcal{L}_{N^{m(n-1-j)}}(X/R_m)| \\
&\leq \sum_{j=0}^{n-1} \frac{1}{mnN^m} \frac{1}{N^{m(n-1-j)}} \log |\mathcal{L}_{N^{m(n-1-j)}}(X/R_m)| \\
&= \frac{1}{mN^m} \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{N^{m(n-1-j)}} \log |\mathcal{L}_{N^{m(n-1-j)}}(X/R_m)|
\end{aligned}$$

$$= \frac{1}{mN^m}\frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{N^{jm}}\log|\mathcal{L}_{N^{jm}}(X/R_m)|,$$

from where taking limit as $n \to \infty$ we obtain $h_{\text{top}}(X) \leq \frac{1}{mN^m}h_{\text{top}}(X/R_m)$.

Now let us show that

$$h_{\text{top}}(X) \geq \frac{1}{(m+1)N^m}h_{\text{top}}(X/R_m),$$

by using an argument similar to the one above. If we look at the rectangle $R_{n(m+1)}$ we can form allowed patterns by inserting words of X/R_m at heights separated by $m+1$ elements, where the extra row is filled with the mixing symbol to gain independence between different copies of words of X/R_m. With this we have that for every $n \geq 1$

$$|\mathcal{L}_{R_{(m+1)n}}(X)| \geq \prod_{j=0}^{n-1}|\mathcal{L}_{N^{(m+1)n-m-j(m+1)}}(X/R_m)|^{N^{j(m+1)}} = \prod_{j=0}^{n-1}|\mathcal{L}_{N^{(m+1)(n-j)-m}}(X/R_m)|^{N^{j(m+1)}},$$

and taking logarithm and dividing both sides by $(m+1)nN^{(m+1)n}$

$$\begin{aligned}
\frac{1}{(m+1)nN^{(m+1)n}}\log|\mathcal{L}_{R_{(m+1)n}}(X)| &\geq \sum_{j=0}^{n-1}|\frac{N^{j(m+1)}}{(m+1)nN^{(m+1)n}}\log|\mathcal{L}_{N^{(m+1)(n-j)-m}}(X/R_m)| \\
&= \sum_{j=0}^{n-1}\frac{N^{j(m+1)}N^{(m+1)(n-j)-m}}{(m+1)nN^{(m+1)n}}\frac{\log|\mathcal{L}_{N^{(m+1)(n-j)-m}}(X/R_m)|}{N^{(m+1)(n-j)-m}} \\
&= \frac{1}{(m+1)N^m}\frac{1}{n}\sum_{j=0}^{n-1}\frac{\log|\mathcal{L}_{N^{(m+1)(n-j)-m}}(X/R_m)|}{N^{(m+1)(n-j)-m}} \\
&\geq \frac{1}{(m+1)N^m}\frac{1}{n}\sum_{j=0}^{n-1}h_{\text{top}}(X/R_m) \\
&= \frac{1}{(m+1)N^m}h_{\text{top}}(X/R_m),
\end{aligned}$$

where we used that $h_{\text{top}}(X/R_m) = \inf_{k\geq 1}\frac{1}{k}\log|\mathcal{L}_k(X/R_m)|$. Taking limit as $n \to \infty$ yields

$$h_{\text{top}}(X) \geq \frac{1}{(m+1)N^m}h_{\text{top}}(X/R_m),$$

which is what we wanted. Note that we also could have taken limit directly on the penultimate inequality instead of using the lower bound given by $h_{\text{top}}(X/R_m)$, which would still have converged to the final result. $\square$

Remark 5.16 *In the case that X is a* $\mathrm{BS}(1,N)$*-subshift arising from a $\mathbb{Z}$-subshift Y with a mixing symbol, the rectangle R_1 is just one row of N symbols so the $\mathbb{Z}$-subshift X/R_1 is conjugate to the higher-power shift Y^N. As we have the relation $h_{\text{top}}(Y^N) = Nh_{\text{top}}(Y)$ between the topological entropies of Y and its higher power Y^N, the bounds given by the previous proposition become*

$$\frac{1}{2}h_{\text{top}}(Y) = \frac{1}{(1+1)N}h_{\text{top}}(X/R_1) \leq h_{\text{top}}(X) \leq \frac{1}{N}h_{\text{top}}(X/R_1) = h_{\text{top}}(Y),$$

which are the estimates we had deduced in Propositions 5.10 and 5.11.

Remark 5.17 *By using Proposition 5.15 we see that the sequence* $\{\frac{1}{mN^m}h_{\mathrm{top}}(X/R_m)\}_{m\geq 1}$ *converges to* $h_{\mathrm{top}}(X)$ *and furthermore we can estimate the rate of convergence:*

$$\begin{aligned}\left|h_{\mathrm{top}}(X) - \frac{1}{mN^m}h_{\mathrm{top}}(X/R_m)\right| &\leq \frac{1}{mN^m}h_{\mathrm{top}}(X/R_m) - \frac{1}{(m+1)N^m}h_{\mathrm{top}}(X/R_m) \\ &= \frac{1}{m(m+1)N^m}h_{\mathrm{top}}(X/R_m) \\ &\leq \frac{\log|\mathcal{A}|}{m+1}.\end{aligned}$$

Chapter 6

Projective subdynamics

This chapter is motivated by the work done in [18] with respect to projective subdynamics of multidimensional subshifts, where the realization of $\mathbb{Z}$-subshifts as projective subdynamics of a $\mathbb{Z}^d$-subshift is studied. The main result of this paper characterizes under which conditions it is possible to realize a sofic $\mathbb{Z}$-subshift as the (stable or unstable) projective subdynamics of a $\mathbb{Z}^d$-SFT.

In what follows we start by giving the corresponding definition of $\langle a \rangle$-projective subdynamics in $\mathrm{BS}(1,N)$ and study the dichotomy between it being stable and unstable. The rest of the chapter focuses in studying the realization of sofic $\mathbb{Z}$-subshifts as $\langle a \rangle$-projective subdynamics of a $\mathrm{BS}(1,N)$-SFT, separating the case in which the $\mathbb{Z}$-subshift has positive topological entropy from the case in which it has zero entropy.

6.1 Definition and first consequences

We start by defining the concept of projective subdynamics on $\mathrm{BS}(1,N)$, analogously to how it is defined for $\mathbb{Z}^d$. Intuitively the $\langle a \rangle$-projective subdynamics of a $\mathrm{BS}(1,N)$-subshift consists of the $\mathbb{Z}$-subshift we see appearing in the rows (along the a-direction) of the group in the $\mathrm{BS}(1,N)$-subshift considered.

Definition 6.1 *Let $X \subseteq \mathcal{A}^{\mathrm{BS}(1,N)}$ be a $\mathrm{BS}(1,N)$-subshift. We define its* $\langle \mathbf{a} \rangle$***-projective subdynamics*** *as*

$$\mathsf{P}_{\langle a \rangle}(X) \coloneqq \{x|_{\langle a \rangle} \mid x \in X\}.$$

The fact that $\mathsf{P}_{\langle a \rangle}(X)$ can be interpreted as a $\mathbb{Z}$-subshift is straightforward: for every $x \in \mathsf{P}_{\langle a \rangle}(X)$ we can define a corresponding configuration $\tilde{x}_i \coloneqq x_{a^i}$, for $i \in \mathbb{Z}$. The idea behind this definition is that the extra space of the group $\mathrm{BS}(1,N)$ can be used to create restrictions about the sequences appearing on the row $\langle a \rangle$, and with it force some particular kind of $\mathbb{Z}$-subshift to occur as $\langle a \rangle$-projective subdynamics. In the next definition we divide projective subdynamics in two types, depending on whether a finite or infinite number of rows are needed to obtain the $\langle a\rangle$-projective subdynamics of the $\mathrm{BS}(1,N)$-subshift.

Definition 6.2 *Let $X = \mathsf{X}_{\mathcal{F}} \subseteq \mathcal{A}^{\mathrm{BS}(1,N)}$ be a $\mathrm{BS}(1,N)$-subshift defined by a family of forbidden patterns $\mathcal{F}$. We associate to it a decreasing sequence of $\mathbb{Z}$-subshifts $(X_{\langle a\rangle,M})_{M\geq 0}$ defined by*

$$X_{\langle a\rangle,M} \coloneqq \{x|_{\langle a\rangle} \mid x \in \mathcal{A}^{\langle a\rangle^{[M]}} \text{ and for every } F \subseteq \langle a\rangle^{[M]} \text{ finite: } x|_F \notin \mathcal{F}\},$$

where $\langle a\rangle^{[M]} \coloneqq \{b^{-M}a^ib^j \mid i \in \mathbb{Z}, 0 \leq j \leq 2M\} \subseteq \mathrm{BS}(1,N)$ is an infinite strip of $\mathrm{BS}(1,N)$ starting at height $-M$ and covering every sheet originating from it up to height M. Intuitively, the sequences from $X_{\langle a\rangle,M}$ are those which can be extended to the strip $\langle a\rangle^{[M]}$ without forcing the appearance of a forbidden pattern of the family $\mathcal{F}$.

For every $M \geq 0$ we have the inclusion $X_{\langle a\rangle,M+1} \subseteq X_{\langle a\rangle,M}$: any sequence defined on $\langle a\rangle$ that can be extended to the strip $\langle a\rangle^{[M+1]}$ without creating forbidden patterns can be extended to the strip $\langle a\rangle^{[M]}$ with the same restrictions. We also see that the $\langle a\rangle$-projective subdynamics is precisely the decreasing intersection of these sets, that is,

$$\mathsf{P}_{\langle a\rangle}(X) = \bigcap_{M\geq 0} X_{\langle a\rangle,M}.$$

The equality between $\mathsf{P}_{\langle a\rangle}(X)$ and the intersection from the right hand side allows us to distinguish the case where this intersection stabilizes after a finite number of sets $X_{\langle a\rangle,M}$, from the case where it is really a proper infinite intersection. The $\langle a\rangle$-projective subdynamics $\mathsf{P}_{\langle a\rangle}(X)$ is said to be ***stable*** *if there exists $M' \geq 0$ such that for every $M \geq M'$: $X_{\langle a\rangle,M} = X_{\langle a\rangle,M'}$, that is, the intersection stabilizes at some moment. On the other hand, $\mathsf{P}_{\langle a\rangle}(X)$ is said to be* ***unstable*** *if for every $M' \geq 0$ there exists an $M > M'$ such that $X_{\langle a\rangle,M} \subsetneq X_{\langle a\rangle,M'}$.*

In the next proposition we state some basic facts about how (un)stability of the $\langle a\rangle$-projective subdynamics relates the properties of being an SFT and being sofic between the subshift X and its $\langle a\rangle$-projective subdynamics $\mathsf{P}_{\langle a\rangle}(X)$.

Proposition 6.3 *Let X be a $\mathrm{BS}(1,N)$-subshift and $\mathsf{P}_{\langle a\rangle}(X)$ its $\langle a\rangle$-projective subdynamics.*

1. *If X is an SFT and the $\langle a\rangle$-projective subdynamics is stable, then $\mathsf{P}_{\langle a\rangle}(X)$ is sofic.*
2. *If $\mathsf{P}_{\langle a\rangle}(X)$ is an SFT then the $\langle a\rangle$-projective subdynamics is stable.*

Proof. 1. Let $S = \mathsf{P}_{\langle a\rangle}(X)$ be the stable $\langle a\rangle$-projective subdynamics of the $\mathrm{BS}(1,N)$-SFT X determined by the finite set of forbidden patterns $\mathcal{F}$, and choose $M \geq 1$ sufficiently large such that $\mathcal{F} \subseteq \mathcal{A}^{R_{M-1}}$ and $S = X_{\langle a\rangle,M}$. The idea to prove that S is sofic is to view it as a factor of a $\mathbb{Z}$-SFT defined by a construction similar to that of a higher-block shift, on the strip $\langle a\rangle^{[M]}$.

Note that we can view the strip $\langle a\rangle^{[M]}$ as the union of translates of the rectangle $R \coloneqq \{b^{-M}a^ib^j \mid 0 \leq i < N^{2M+1}, 0 \leq j \leq 2M\}$ of height $2M+1$, with its base in b^{-M}. We define a new alphabet

$$\mathcal{B} \coloneqq \{p \in \mathcal{A}^R \mid p \text{ does not contain patterns of } \mathcal{F}\},$$

where the symbols are patterns on R, and we define the $\mathbb{Z}$-subshift

$$H_M \coloneqq \{x \in \mathcal{B}^{\mathbb{Z}} \mid \forall i \in \mathbb{Z} : \ (x_i)|_{aR\cap R} = (x_{i+1})|_{R\cap a^{-1}R}\}.$$

Moreover, this subshift is a $\mathbb{Z}$-SFT, since we are only requiring local rules to be satisfied: that in each point two adjacent symbols of the alphabet, which in this case are patterns over R, coincide restricted to a subset.

We can now define the map $\pi : H_M \to S$ which given a point $x \in H_M$, interprets it as a configuration on the strip $\langle a\rangle^{[M]}$ (keeping in mind the overlapping between adjacent symbols occurring in x), and returns its restriction to $\langle a\rangle$, that is, for each $i \in \mathbb{Z}$ we have $\pi(x)_{a_i} = (x_i)_{e_{\mathrm{BS}(1,N)}}$. This map commutes with the shift since for each $i \in \mathbb{Z}$

$$\pi(\sigma(x))_{a^i} = (\sigma(x)_i)_{e_{\mathrm{BS}(1,N)}} = (x_{i+1})_{e_{\mathrm{BS}(1,N)}} = \pi(x)_{a^{i+1}} = \sigma_{a^{-1}}(\pi(x))_{a^i},$$

and it is continuous: if $(x^n)_{n\in\mathbb{N}} \subseteq H_M$ is a sequence such that $x^n \xrightarrow[n\to\infty]{} x \in H_M$, then for every $L \geq 1$ we can find n^* sufficiently large such that for all $n \geq n^*$ we have $x^n|_{\{-L,L\}} = x|_{\{-L,\dots,L\}}$ and so $\pi(x^n)|_{[a^{-L},a^L]} = \pi(x)|_{[a^{-L},a^L]}$, and as L was arbitrary we see that $\pi(x^n) \xrightarrow[n\to\infty]{} \pi(x)$. With this we have proven that π is a sliding block code.

Finally we show that π is surjective. By the definition of $X_{\langle a\rangle,M}$ we have that for each $s \in S = X_{\langle a\rangle,M}$ there exists $y \in \mathcal{A}^{\langle a\rangle^{[M]}}$ such that $y|_{\langle a\rangle} = s$. Then we define $x \in H_M$ by setting

$$x_i = y|_{a^iR},$$

which satisfies

$$\pi(x)_{a^i} = (x_i)_{e_{\mathrm{BS}(1,N)}} = y_{a^i} = s_{a^i}, \ i \in \mathbb{Z},$$

from where $\pi(x) = s$.

With this we have seen that π is a factor code from the SFT H_M to S, and hence the latter is a sofic $\langle a\rangle$-subshift.

2. As $\mathsf{P}_{\langle a\rangle}(X)$ is an $\langle a\rangle$-SFT we can find $L \geq 0$ sufficiently large such that for every $x \in \mathcal{A}^{\mathbb{Z}}$ the following property holds: if for every $i \in \mathbb{Z}$ $x|_{[a^{i+1},a^{i+L}]} \in \mathcal{L}_L(\mathsf{P}_{\langle a\rangle}(X))$, then $x \in \mathsf{P}_{\langle a\rangle}(X)$.

   Remember that $\mathsf{P}_{\langle a\rangle}(X) = \bigcap_{M\geq 0} X_{\langle a\rangle,M}$ and that for each $M \geq 0$ we have the inclusion $X_{\langle a\rangle,M+1} \subseteq X_{\langle a\rangle,M}$, so in order to prove that the $\langle a\rangle$-projective subdynamics is stable it is enough to see that for some $M \geq 0$ $X_{\langle a\rangle,M} \subseteq \mathsf{P}_{\langle a\rangle}(X)$. By using that $X_{\langle a\rangle,M}$ is a subshift and what was said above it suffices to prove that for every $x \in X_{\langle a\rangle,M}$: $x|_{[a,a^L]} \in \mathcal{L}_L(\mathsf{P}_{\langle a\rangle}(X))$.

   Looking for a contradiction let us suppose that for each $M \geq 0$ there exists $x^M \in X_{\langle a\rangle,M}$ such that $x|_{[a,a^L]} \notin \mathcal{L}(\mathsf{P}_{\langle a\rangle}(X))$. By using compactness we can extract a subsequence of the sequence $\{x^M\}_{M\geq 0}$, convergent to a configuration $x \in \mathcal{A}^{\langle a\rangle}$ such that $x|_{[a,a^L]} \notin \mathcal{L}(\mathsf{P}_{\langle a\rangle}(X))$, and hence $x \notin \mathsf{P}_{\langle a\rangle}(X)$. However, we had that for every $M \geq M' \geq 0$: $x^{M'} \in X_{\langle a\rangle,M'} \subseteq X_{\langle a\rangle,M}$, and by the closedness of these subshifts $\forall M \geq 0 : \ x \in X_{\langle a\rangle,M}$. But this means that $x \in \bigcap_{M\geq 0} X_{\langle a\rangle,M} = \mathsf{P}_{\langle a\rangle}(X)$, which is a contradiction as we had already seen that $x \notin \mathsf{P}_{\langle a\rangle}(X)$.

$\square$

If we interpret the stability (resp. unstability) of the $\langle a\rangle$-projective subdynamics as the necessity of finite (resp. infinite) rows of extra space needed to check that the sequence appearing in the row $\langle a\rangle$ belongs to $\mathsf{P}_{\langle a\rangle}(X)$, the previous proposition is somewhat intuitive: the stable $\langle a\rangle$-projective subdynamics of an SFT is sofic since it suffices to use a finite number of rows to check if a sequence belongs to $\mathsf{P}_{\langle a\rangle}(X)$. Similarly, if $\mathsf{P}_{\langle a\rangle}(X)$ is an SFT then it means that to force a biinfinite sequence to belong to it we need to check local validity only, and this is done using only finite rows.

In the next proposition we see a class of $\mathrm{BS}(1,N)$-subshifts whose $\langle a\rangle$-projective subdynamics is always stable, thanks to the existence of a mixing symbol (see Definition 5.7) which allows to extend locally admissible patterns to global configurations.

Proposition 6.4 *If X is a $\mathrm{BS}(1,N)$-subshift with a mixing symbol, then its $\langle a\rangle$-projective subdynamics is stable. In particular if X is an SFT then $\mathsf{P}_{\langle a\rangle}(X)$ is sofic.*

Proof. We will prove that $\mathsf{P}_{\langle a\rangle}(X) = X_{\langle a\rangle,0}$. In effect, denoting by $\alpha_M \in \mathcal{A}$ the mixing symbol, if $x \in X_{\langle a\rangle,0}$ we can define a point $x' \in \mathcal{A}^{\mathrm{BS}(1,N)}$ by $x'|_{\langle a\rangle} = x$ and $x'_g = \alpha_S$ for all $g \notin \langle a\rangle$. By the definition of a mixing symbol and that $x \in X_{\langle a\rangle,0}$ we see that x' contains no forbidden patterns of the subshift X, and hence $x' \in X$. As this configuration satisfies $x'|_{\langle a\rangle} = x$, we have finished the proof.

Now if X is an SFT it suffices to use the previous proposition together with what we just proved to see that $\mathsf{P}_{\langle a\rangle}(X)$ must be sofic. $\square$

The main question we address in what follows asks which (sofic) $\mathbb{Z}$-subshifts may be realized as $\langle a\rangle$-projective subdynamics of $\mathrm{BS}(1,N)$-SFTs. In particular it is immediate that any $\mathbb{Z}$-SFT can be obtained as $\langle a\rangle$-projective subdynamics of the $\mathrm{BS}(1,N)$-SFT constructed by considering the forbidden patterns of the original $\mathbb{Z}$-SFT as forbidden patterns in $\mathcal{A}^{\mathrm{BS}(1,N)}$ along the a-direction and putting no restrictions along the b-direction.

Proposition 6.5 *For every $\mathbb{Z}$-SFT Y there exists a $\mathrm{BS}(1,N)$-SFT X such that $Y = \mathsf{P}_{\langle a\rangle}(X)$ and the $\langle a\rangle$-projective subdynamics is stable.*

A more interesting case to study is that of proper sofic $\mathbb{Z}$-subshifts. In the following example we see a non-trivial construction of a $\mathrm{BS}(1,N)$-SFT whose $\langle a\rangle$-projective subdynamics is a proper sofic $\mathbb{Z}$-subshift, namely the sunny-side up shift.

Example 6.6 *We will prove that the $\mathbb{Z}$-subshift*

$$S_{\leq 1} := \left\{x \in \{0,1\}^{\mathbb{Z}} \mid |\{n \in \mathbb{Z} : x_n = 1\}| \leq 1\right\}$$

of sequences with at most one 1 can be realized as the unstable $\langle a\rangle$-projective subdynamics of a $\mathrm{BS}(1,N)$-SFT, for any $N \geq 2$.

Define a $\mathrm{BS}(1,N)$-SFT $\mathsf{X}_{\mathcal{F}}$ using the forbidden patterns $\mathcal{F} := \{p_n\}_{n=0}^{N-2} \cup \{q_n\}_{n=0}^{N-1}$, where

$p_n \in \{0,1\}^{\{a^i | i=0,\dots,n+1\}}$ is defined by

$$p_n = 10^n 1,$$

and $q_n \in \{0,1\}^{\{a^i | i=0,\dots,N-1\} \cup \{a^n b\}}$ is defined by

$$q_n|_{\{a^i | i=0,\dots,N-1\}} = 0^{N-1}1$$
$$q_n(a^n b) = 0$$

These forbidden patterns force that there cannot be two 1's at distance less than $N-1$ in the direction of the generator a, and that above a pattern in the direction of the generator a showing a word $0^{N-1}1$, there must be 1's. We will prove that these local constraints are sufficient to guarantee that $S_{\leq 1} = \mathsf{P}_{\langle a \rangle}(\mathsf{X}_{\mathcal{F}})$.

To see that $S_{\leq 1} \subseteq \mathsf{P}_{\langle a \rangle}(\mathsf{X}_{\mathcal{F}})$, consider $s \in S_{\leq 1}$. We will construct a configuration $x \in \mathsf{X}_{\mathcal{F}}$ such that $x|_{\langle a \rangle} = s$. If $s = 0^\infty$, it suffices to consider $x := (x_g = 0)_{g \in \mathrm{BS}(1,N)} \in \mathsf{X}_{\mathcal{F}}$. Now suppose s has one 1, and without loss of generality we may assume that $s_1 = 1$, $s_n = 0$ for $n \neq 1$. Define a configuration $x \in \{0,1\}^{\mathrm{BS}(1,N)}$ by

$$\begin{cases} x_g = 1, \text{ for } g \in \bigcup_{j \geq 0} \left\{ a^{1-\sum_{r=0}^{j-1} \varepsilon_r N^r} b^j \mid \varepsilon_r \in \{0,\dots,N-1\}, \text{ for } r = 0,\dots,j-1 \right\}, \text{ and} \\ x_g = 0, \text{ in the rest of the group.} \end{cases}$$

This definition forces the configurationx to respect the forbidden patterns in $\mathcal{F}$. In effect, there are no 1's in the direction of generator a at a distance closer than $N-1$, since if $x_g = 1$ then $g = a^{1-\sum_{r=0}^{j-1} \varepsilon_r N^r} b^j$ for some $j \geq 0$ and $\varepsilon_r \in \{0,\dots,N-1\}$, for $r = 0,\dots,j-1$, and so for $i \in \{-(N-2),\dots,-1\}$:

$$g a^i = a^{1-\sum_{r=0}^{j-1} \varepsilon_r N^r + iN^j} b^j,$$

whence $x_{ga^i} = 0$. We also see that for $i \in \{-(N-1),\dots,-1,0\}$

$$g a^i b = a^{1-\sum_{r=0}^{j-1} \varepsilon_r N^r + i \cdot N^j} b^{j+1},$$

and so $x_{ga^ib} = 1$. Hence $x \in \mathsf{X}_{\mathcal{F}}$ and $x|_{\langle a \rangle} = s$, from which we conclude that $s \in \mathsf{P}_{\langle a \rangle}(\mathsf{X}_{\mathcal{F}})$.

Now let us see that $\mathsf{P}_{\langle a \rangle}(\mathsf{X}_{\mathcal{F}}) \subseteq S_{\leq 1}$. We will prove by induction that for every $n \geq 0$: $10^n 1 \notin \mathcal{L}(\mathsf{P}_{\langle a \rangle}(\mathsf{X}_{\mathcal{F}}))$. The forbidden patterns p_n from the family $\mathcal{F}$ already show this for $n \leq N-2$, so now let's prove it for $n \geq N-1$. By contradiction, suppose $10^n 1 \in \mathcal{L}(\mathsf{P}_{\langle a \rangle}(\mathsf{X}_{\mathcal{F}}))$, and choose $x \in \mathsf{X}_{\mathcal{F}}$ such that $x|_{\{a^i | i=0,\dots,n+1\}} = 10^n 1$. Then by the induction hypothesis we see that $x|_{\{a^i | i=-(N-1),\dots,n+1\}} = 0^{N-1}10^n 1$, and thanks to the family $\mathcal{F}$ we also see that $x_{a^{n+1}b} = 1$, and $x_{a^{n+1-kN}b} = 1$, where $k = \left\lfloor \frac{n+N}{N} \right\rfloor$. Since $a^{n+1}b$ and $a^{n+1-kN}b = a^{n+1}ba^{-k}$ are two elements at distance k through the generator a, with $k < n$, this contradicts the induction hypothesis.

From the construction we also see that the $\langle a \rangle$-projective subdynamics is unstable, since arbitrarily large heights are needed to forbid two 1's separated by a long block of 0's from appearing on the row $\langle a \rangle$.

Remark 6.7 *The previous example shows a surprising property of* $\mathrm{BS}(1,N)$ *in contrast with* $\mathbb{Z}^2$*: as shown in [18] the sofic* $\mathbb{Z}$*-subshift* $S_{\leq 1}$ ***cannot*** *be realized as the* $\mathbb{Z}$*-projective subdynamics of a* $\mathbb{Z}^2$*-SFT. Nonetheless, we've just shown that it can be realized as the unstable* $\langle a \rangle$*-projective subdynamics of a* $\mathrm{BS}(1,N)$*-SFT, for every* $N \geq 2$*. This shows a remarkable difference between* $\mathbb{Z}^2$ *and* $\mathrm{BS}(1,N)$ *with respect to the nature of the* $\mathbb{Z}$*-subshifts which can arise as their projective subdynamics.*

6.2 Realization of positive entropy sofic subshifts

One of the first results proven in [18] assures that any sofic $\mathbb{Z}$-subshift with positive topological entropy may be realized as $\langle a \rangle$-projective subdynamics of a $\mathbb{Z}^2$-SFT, and the same ideas used in this proof can be modified to prove the corresponding result on $\mathrm{BS}(1,N)$. To do this we first need to state the same lemma used to prove the result on $\mathbb{Z}^2$:

Lemma 6.8 ([18]) *Let* S *be a sofic* $\mathbb{Z}$*-subshift with positive topological entropy* $h_{\mathrm{top}}(S) > 0$*. Then for any* $K \geq 1$ *there exists an* $M \geq 1$ *and a set of words* $W = \{w_1, \ldots, w_K\} \subseteq \mathcal{L}_M(S)$ *such that if we denote by* Y *the subshift consisting of all biinfinite concatenations of words from* W*, then* $Y \subseteq S$*, and for any word* $v \in \mathcal{L}_{3M-1}(Y)$ *there exists a unique* $1 \leq m \leq M$ *so that* $v|_{[m,m+M-1]}, v|_{[m+M,m+2M-1]} \in W$*. In particular, this implies that* Y *is a* $\mathbb{Z}$*-SFT of step* $3M-1$*.*

Now we are ready to state and prove the main result regarding the realization of positive entropy sofic $\mathbb{Z}$-subshifts as $\langle a \rangle$-projective subdynamics of a $\mathrm{BS}(1,N)$-SFT.

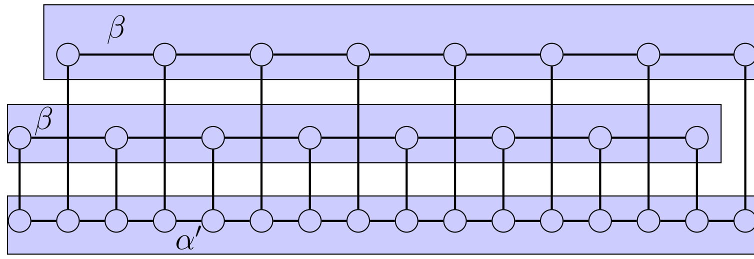
Theorem 6.9 *For every sofic* $\mathbb{Z}$*-subshift of positive entropy there exists a* $\mathrm{BS}(1,N)$*-SFT* X *such that* $S = \mathsf{P}_{\langle a \rangle}(X)$*.*

Proof. We are going to prove the result for $\mathrm{BS}(1,2)$ since in this case it is simpler to define the local rules of the SFT X. The ideas for the proof in the general case $N \geq 2$ naturally extrapolate from the case $N = 2$.

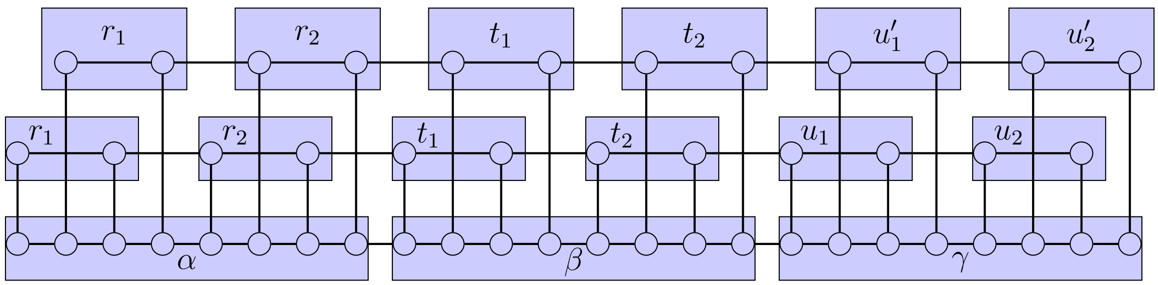
Let us choose a labeled graph $(V, E, \mathfrak{i}, \mathfrak{f}, \lambda)$ such that S is the subshift whose points are given as labels of biinfinite paths along the edges E, and let us assume $V = \{0, 1, \ldots, K-1\}$. Choose $M \geq 1$ and pick a set of words $W = \{w_0, \ldots, w_{2K-1}\} \subseteq \mathcal{L}_M(S)$ satisfying the previous lemma. Denote by Y the $\mathbb{Z}$-SFT formed by biinfinite concatenations of words from W.

Define a $\mathrm{BS}(1,2)$-SFT X by the following local rules:

1. For any pattern $\alpha \notin \mathcal{L}(Y)$ with support $\{e_G, a, \ldots a^{10M-1}\}$, there exists a subpattern $\alpha' \notin \mathcal{L}(Y)$ of length $8M$ such that the first row on both sheets immediately above it contain the same pattern $\beta \in \mathcal{L}(Y)$ (of length $4M$) which is the concatenation of 4 words of W.

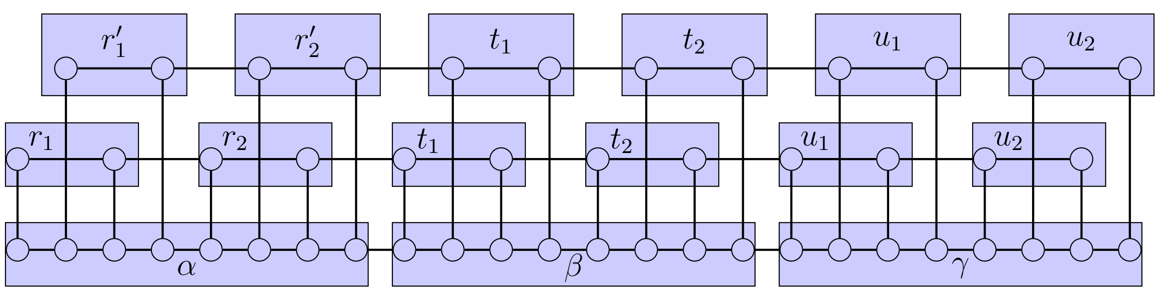


2. Suppose we have a pattern in X of the shape



with α, β, γ of length $4M$, $r_1, r_2, t_1, t_2, u_1, u_2, u_1', u_2'$ of length M, and $t_1, t_2 \in W$. Then we must have $u_1 = u_1'$, $u_2 = u_2'$, and $r_1, r_2, u_1, u_2 \in W$. Aditionally, as $r_1, t_1, u_1 \in W$ we can find $i, j, k \in \{0, \ldots, 2K-1\}$ such that $r_1 = w_i$, $t_1 = w_j$ and $u_1 = w_k$. For the pattern to be valid we require that there must be a (directed) path in G between $i \pmod K$ and $j \pmod K$ with label α and a (directed) path between $j \pmod K$ and $k \pmod K$ with label β.

3. Suppose we have a pattern in X of the shape



with α, β, γ of length $4M$, $r_1, r_2, r_1', r_2', t_1, t_2, u_1, u_2$ of length M, and $t_1, t_2 \in W$. Then we must have $r_1 = r_1'$, $r_2 = r_2'$, and $r_1, r_2, u_1, u_2 \in W$. Aditionally, as $r_1, t_1, u_1 \in W$ we can find $i, j, k \in \{0, \ldots, 2K-1\}$ such that $r_1 = w_i$, $t_1 = w_j$ and $u_1 = w_k$. For the pattern to be valid we require that there must be a path in G between $i \pmod K$ and $j \pmod K$ with label α and a path between $j \pmod K$ and $k \pmod K$ with label β.

Let us prove that $S = \mathsf{P}_{\langle a \rangle}(X)$:

First we will show that $S \subseteq \mathsf{P}_{\langle a \rangle}(X)$. For arbitrary $s \in S$, we can find a biinfinite sequence of edges of G such that its label reads s, i.e. $\{e_n\}_{n \in \mathbb{Z}} \in E^{\mathbb{Z}}$ such that $s = \{\lambda(e_n)\}_{n \in \mathbb{Z}}$. For every $n \in \mathbb{Z}$ let us define $v_n := \mathfrak{i}(e_n)$ the initial vertex of e_n, and define a biinfinite sequence y by choosing for each $j \in \mathbb{Z}$ the word $y_{[jM,jM+M-1]} \in \{w_{v_{2jM}}, w_{v_{2jM}+K}\} \subseteq W$, such that $y_{[jM,jM+M-1]} \neq y_{[(j+1)M,(j+1)M+M-1]}$ for every $j \in \mathbb{Z}$. Now define $x \in E^{\mathrm{BS}(1,2)}$ by $x|_{\langle a\rangle} = s$, $x|_{b\langle a\rangle} = y$, $x|_{ab\langle a\rangle} = y$ and fill the rest of the rows of the group such that the two rows originating from a same base are copies of y shifted by M between them (this is precisely to force that no pattern of the shape of rules 2 and 3 may occur outside the base row $\langle a\rangle$). That is, for every $g \in \mathrm{BS}(1,2)\backslash\langle a\rangle$ such that $gb\langle a\rangle \neq \langle a\rangle$ and $gab\langle a\rangle \neq \langle a\rangle$: $x_{gb\langle a\rangle} = y = x_{gab\langle a\rangle}$ (equal up to a translation) and if $x_{gba^i} = y_k$, then $x_{gaba^i} = y_{k\pm M}$. In the particular case in which $gb\langle a\rangle = \langle a\rangle$ (resp. $gab\langle a\rangle = \langle a\rangle$) we only require that $x|_{gab\langle a\rangle}$ (resp. $x|_{gb\langle a\rangle}$) is a translate of y, because as we said in the beginning we are forcing that $x|_{\langle a\rangle} = s$.

The first rule of the subshift X is satisfied by x, since any pattern not in $\mathcal{L}(Y)$ must have support on $\langle a\rangle$ and we constructed both rows originating from this base to follow the first rule, as the two rows originating from this base have the sequence y on them. We also see that by construction a pattern of the shape as indicated in the second and third rule can appear in x only with its base on the set $\langle a\rangle$, since on all other a-rows of the group the pattern seen on the two rows of both sheets originating from it have a copy of y shifted by M. Therefore for a pattern with the shape and structure of the second or third rule with a base on the set $\langle a\rangle$ the construction of y tells us that both upper rows originating from this base have the same patterns and the condition of paths in G is fulfilled precisely because we chose $y_{[jM,(j+1)M+M-1]}$ such that if $w_i = y_{[jM,(j+1)M+M-1]} \in W$ then $i = v_{2jM} \pmod K$. With this we see that x satisfies the three local rules, so $x \in X$ and this point satisfies $x|_{\langle a\rangle} = s$, from which we conclude the first inclusion.

Now we will prove that $\mathsf{P}_{\langle a\rangle}(X) \subseteq S$. Suppose there exists $x \in X$ such that $x|_{\langle a\rangle} \notin S$. As $Y \subseteq S$ this in particular implies that $x|_{\langle a\rangle} \notin Y$, so we can find on this sequence a pattern of size $3M$ not in $\mathcal{L}(Y)$. We can extend this pattern to one of length $10M$ by considering $x|_{\{a^{-3M+1},\dots,a^{7M}\}}$, and using the first rule we can find a subpattern $\alpha' \notin \mathcal{L}(Y)$ such that the rows originating from it have the same pattern $\beta \in Y$. Now we see that the base α' together with both rows β originating from it can be extended to a pattern of the shape appearing in Rules 2 and 3. Using that x must satisfy rules 2 and 3 we can extend this pattern and recursively apply both rules infinitely to the left and right, finding in this way a biinfinite sequence of edges $\{e_n\}_{n\in\mathbb{Z}}$ which read $x|_{\langle a\rangle}$, that is $x|_{\langle a\rangle} = \{\lambda(e_n)\}_{n\in\mathbb{Z}} \in S$. But now this contradicts the fact that this point was precisely one that satisfied that this sequence was not in S, so we finally conclude that $\mathsf{P}_{\langle a\rangle}(X) \subseteq S$. $\square$

6.3 Restrictions of the zero entropy case

The construction made in the proof of the previous theorem relied completely on S having positive entropy, since it is the fundamental hypothesis for the lemma used throughout the proof. This indicates that the zero entropy case may be more restrictive, and in what follows we see that indeed zero entropy sofic subshifts occurring as $\langle a\rangle$-projective subdynamics of a $\mathrm{BS}(1,N)$-SFT need to satisfy a particular condition related to their periodic points.

Definition 6.10 *A zero entropy sofic $\mathbb{Z}$-subshift S is said to have a* ***$\mathbf{BS(1,}\boldsymbol{N}\mathbf{)}$-good set***

of periods *if it admits a right-resolving presentation by a labeled graph* Γ *with* $S = \mathsf{S}_\Gamma$*, such that for every cycle* c *in* Γ *with no proper subcycles there exists a finite set* $Q \subseteq \mathrm{Per}(S)$ *of periodic points and non-negative integers* $\{i_q | q \in Q\}$ *such that the least common multiple* $\text{l.c.m.}\{N^{i_q}|\mathrm{Orb}(q)|\ |q \in Q\}$ *is a multiple of* $|c|$*, the length of the cycle* c*.*

Remark 6.11 *This definition is tailor-made for zero entropy sofic subshifts, since it is precisely in this case that the required condition is not trivially fulfilled. Note that SFTs can be represented by right-resolving labeled graphs whose label function is injective, so the cycle* c *itself gives rise to a periodic point whose least period is the length of the cycle. Similarly, consider a sofic subshift of positive entropy represented by a right-resolving labeled graph* Γ*. Positive entropy implies that there must be a vertex which forms part of two distinct cycles* c_1, c_2 *labeling two different periodic points* u^∞ *and* v^∞*, respectively. Then for every* $r \geq 1$ *we see that the periodic point* $(u^r v^r)^\infty$ *has as least period a multiple of* r *(since in the middle of* $u^r v^r$ *we see a* uv *which by how we chose* u *and* v *does not appear anywhere else in this word). In particular, the above is satisfied for any length of a cycle* c *in* Γ *with no proper subcycles.*

In the following example we see a zero entropy proper sofic subshift that does not have a $\mathrm{BS}(1,2)$-good set of periods.

Example 6.12 *The zero entropy sofic* $\mathbb{Z}-$*subshift given by the labeled graph from Figure 6.1 does not have a* $\mathrm{BS}(1,2)$*-good set of periods, since its only periodic configurations are the fixed points* 0^∞ *and* 1^∞*, meanwhile the cycle of length* 3 *in the middle must appear having odd length in any right-resolving presentation of the subshift. This comes from the fact that in any such presentation we must be able to find paths which read the words* 10^n2 *for each* $n \in 3\mathbb{N}$*.*

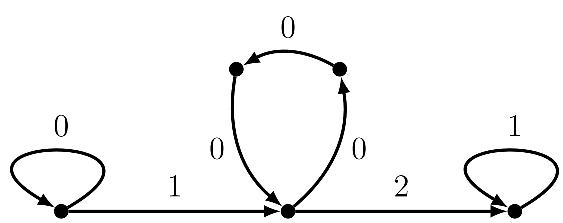


Figure 6.1: Example of a zero entropy sofic $\mathbb{Z}$-subshift that does not have a $\mathrm{BS}(1,2)$-good set of periods.

Moreover, this example can be generalized to provide, for each $N \geq 2$*, a right-resolving labeled graph whose associated* $\mathbb{Z}$*-sofic shift does not have a* $\mathrm{BS}(1,N)$*-good set of periods. This can be done by the same argument given above, considering the graph from Figure 6.1 and changing the cycle in the middle for one with length coprime to* N*.*

Nonetheless, note that given a right-resolving labeled graph Γ, one can always find N sufficiently large such that S_Γ has a $\mathrm{BS}(1,N)$-good set of periods. In effect, it suffices to consider N to be the product of the lengths of all cycles in Γ.

Now we prove that having a $\mathrm{BS}(1,N)$-good set of periods is a necessary condition for a zero entropy proper $\mathbb{Z}$-sofic subshift to be the stable $\langle a \rangle$-projective subdynamics of a $\mathrm{BS}(1,N)$-

SFT.

Theorem 6.13 *Let S be a zero entropy proper sofic $\mathbb{Z}$-subshift realized as the stable $\langle a\rangle$-projective subdynamics of a* $\mathrm{BS}(1,N)$*-SFT. Then S has a* $\mathrm{BS}(1,N)$*-good set of periods.*

Proof. Suppose that X is a $\mathrm{BS}(1,N)$-SFT defined by a (finite) family of forbidden patterns $\mathcal{F} \subseteq \mathcal{A}^{R_m}$, for some $m \geq 1$, let S be its stable $\langle a\rangle$-projective subdynamics, which we already know forces S to be sofic, and suppose S has zero entropy. Let us prove that S must have a $\mathrm{BS}(1,N)$-good set of periods.

Since S is the stable $\langle a\rangle$-projective subdynamics, we can find $M \geq 0$ such that $S = X_{\langle a\rangle,M}$, and furthermore we can assume $M \geq m$. By using the fact that S is sofic, we introduce its follower set presentation $\Gamma = (V, E, \mathfrak{i}, \mathfrak{f}, \lambda)$ (defined in Proposition 1.42). Consider any cycle with no proper subcycles $c \in \Gamma$, with label $u := \lambda(c) \in \mathcal{L}(S)$ and with length $|c|$. If $|c| = 1$, then choosing any periodic point $q \in \mathrm{Per}(S)$ (which exists since all non-empty $\mathbb{Z}$-sofic shifts contain periodic configurations) we get that $N^0|\mathrm{Orb}(q)|$ is a multiple of $1 = |c|$. Now for the rest of the proof let us suppose that $|c| > 1$, and write $|c| = \prod_{j=1}^{J} p_j^{i_j}$, where $J \geq 1$, p_j is a prime number and $i_j \geq 1$ for each $j \in \{1, \ldots, J\}$.

We will prove that for each $j \in \{1, \ldots, J\}$ there exists a finite set of periodic points of $Q_j \subseteq \mathrm{Per}(S)$ and non-negative integers $\{l_q | q \in Q_j\}$ such that the least common multiple $\mathrm{l.c.m.}\{N^{l_q}|\mathrm{Orb}(q)| \; | q \in Q_j\}$ is a multiple of $p_j^{i_j}$. Having done this for each j, we will have that the finite set of periodic points $Q := \bigcup_{j=1}^{J} Q_j \subseteq \mathrm{Per}(S)$ together with the non-negative integers $\bigcup_{j=1}^{J}\{l_q | q \in Q_j\}$ satisfies that the least common multiple $\mathrm{l.c.m.}\{N^{l_q}|\mathrm{Orb}(q)| \; | q \in Q\}$ is a multiple of $|c|$, hence proving that S has a $\mathrm{BS}(1,N)$-good set of periods.

Fix $j \in \{1, \ldots, J\}$. Define the vertices $v_0, v_1 \ldots, v_{|c|-1} \in V$ to be the vertices of the cycle c, ordered such that $(v_i, v_{i+1}) \in E$ for each $i \in \{0, \ldots, |c|-2\}$, and $(v_{|c|-1}, v_0) \in E$. In particular we will consider the initial vertex $v_0 := \mathfrak{i}(c) \in V$ of the cycle c, and for $\ell_j := \frac{|c|}{p_j} \in \mathbb{N}$ the vertex $v_{\ell_j} \in V$ at distance ℓ_j of v_0 through the cycle. Finally define $u_{(\ell_j)} := u|_{[1,\ell_j]} \in \mathcal{L}(S)$.

As Γ is the follower-set presentation of S we can find a word $w \in \mathcal{L}(S)$ such that $v_0 = F(w)$, and moreover as c is a cycle and $\lambda(c) = u$ we have that $v_0 = F(wu^k)$ for every $k \geq 0$. Furthermore by the definition of $u_{(\ell_j)}$ we have that $v_{\ell_j} = F(wu_{(\ell_j)})$ and again as c is a cycle, that $v_{\ell_j} = F(wu^k u_{(\ell_j)})$ for every $k \geq 0$.

Since c is a cycle in Γ with no proper subcycles, the vertices v_0 and v_{ℓ_j} are different, and hence $F(wu^k) \neq F(wu^k u_{(\ell_j)})$, for all $k \geq 0$. This gives rise to two cases specifying the relation between these two follower-sets:

Case 1: There exists $t \in \mathcal{L}(S)$ such that $\forall k \geq 0 : t \in F(wu^k)$ and $t \notin F(wu^k u_{(\ell_j)})$.

Let us consider $k > N^m \left(|\mathcal{A}|^{N^m N^{4M}(4M+1)} + 1\right)$, and fix a locally admissible configuration $C \in \langle a\rangle^{[2M]}$ with

$$C|_{\{a^{-|w|}, \ldots, a^{N^{2M}k|c|+|t|-1}\}} = wu^{N^{2M}k}t,$$

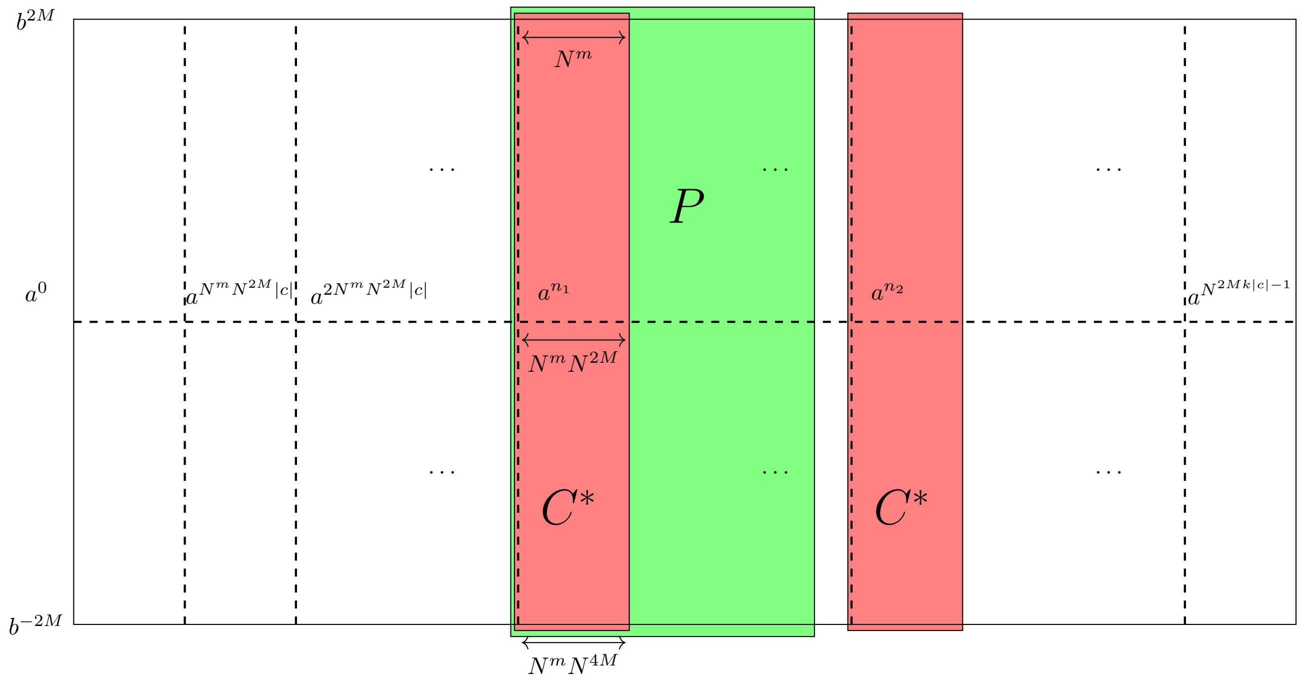


Figure 6.2: Schematic description of the patterns used in the proof. The red patterns are equal, and the green pattern P covers all the first red pattern up until just before reaching the second red pattern.

where the subword $u^{N^{2M}k}$ occupies coordinates $\{a^0, \dots a^{N^{2M}k|c|-1}\}$ of the pattern. By our choice of k we have that $k|c| > N^m|c|\left(|\mathcal{A}|^{N^m N^{4M}(4M+1)} + 1\right)$, and so by the pigeonhole principle we can find two different coordinates $n_1, n_2 \in N^m|c|N^{2M}\mathbb{N}_0$ with $0 \leq n_1 < n_2 \leq N^m|c||\mathcal{A}|^{N^m N^{4M}(4M+1)}$ such that we can extract from C two equal, non-overlapping patterns

$$C^* \coloneqq C|_{a^{n_1}b^{-2M}T} = C|_{a^{n_2}b^{-2M}T},$$

where $T \coloneqq \{a^i b^l \mid 0 \leq i \leq N^m N^{4M} - 1,\ 0 \leq l \leq 4M\}$. Finally, denote by $P \coloneqq C|_{a^{n_1}b^{-2M}\{a^i b^l | 0 \leq i \leq (n_2 - n_1)N^{2M}-1,\ 0 \leq l \leq 4M\}}$. A schematic description of these patterns is shown in Figure 6.2. Although the drawing is planar, it symbolizes every sheet originating from the height b^{-2M} up until the a-row at height height b^{2M}. The red patterns are those that were found using the pigeonhole principle, and the green pattern is P, covering the strip starting from the first pattern and stopping just before reaching the second one.

Now we can write C as the concatenation of the pattern P together with the infinite patterns appearing on C to the left and to the right of P, which we will denote as $C = C^- P C^+$. Thanks to how we defined P we see that the configuration $C^- P^m C^+$ is locally admissible in $\langle a \rangle^{[2M]}$ for each $m \geq 1$, and so is the pattern P^∞ formed by concatenating copies of P infinitely to both sides. This is because the forbidden patterns of the subshift X have support R_m and we chose n_1, n_2 and P in such a way that this pattern can be glued to itself respecting these rules.

Now let us define $\widetilde{C} \coloneqq C|_{\langle a \rangle^{[M]}}$, $\widetilde{C}^- \coloneqq C^-|_{\langle a \rangle^{[M]}}$, $\widetilde{C}^+ \coloneqq C^+|_{\langle a \rangle^{[M]}}$ and $\widetilde{P} \coloneqq P|_{\langle a \rangle^{[M]}}$. Note that as above, the configurations $\widetilde{C}^- \widetilde{P}^m \widetilde{C}^+$ for $m \geq 1$ are locally admissible on $\langle a \rangle^{[M]}$, and so is $\widetilde{P}^\infty$. In particular as each row of $\widetilde{P}^\infty$ can be seen in P^∞ as the central row of a strip $\langle a\rangle^{[M]}$, we conclude that every row of the pattern $\widetilde{P}^\infty$ is in $X_{\langle a\rangle,M} = S$. These rows will form the periodic points from the set $Q_j \subseteq \mathrm{Per}(S)$ we talked about at the beginning of the proof.

Choose $n^* \geq 1$ minimum such that the pattern

$$R \coloneqq \widetilde{P}|_{a^{n_1}b^{-M}\{a^i b^l | 0\leq i< N^M n^*,\ 0\leq l<2M+1\}}$$

satisfies $R^\infty = \widetilde{P}^\infty$, and so for any $\widetilde{k}$ sufficiently large the configuration $\widetilde{C}^- R^{\widetilde{k}} \widetilde{C}^+$ is admissible in $\langle a\rangle^{[M]}$. This n^* must then respect the periods of each of the periodic points found on the rows of the pattern $\widetilde{P}^\infty$, taking into account that due to the geometry of the group the periodic points from rows with positive b height see their least periods in the direction of the generator a multiplied by N raised to the height in the direction of the b generator. In particular as n^* is minimal, it must be the least common multiple of these numbers, and so we only need to prove that n^* is a multiple of $p_j^{i_j}$ by the argument given at the start of the proof.

Suppose n^* is not a multiple of $p_j^{i_j}$. Then there are infinite positive integers $\widetilde{k}'$ such that $\widetilde{k}' n^* = \frac{|c|}{p_j} = \ell_j \pmod{|c|}$, since the prime decomposition of n^* can only contain powers of p_j with exponent less or equal to $i_j - 1$. But then as the row $\widetilde{P}^\infty|_{\langle a\rangle}$ has period n^* and is composed of concatenated copies of u thanks to the choice of P, we must have that $u = (u_{(\ell_j)})^{p_j}$ and that for $\widetilde{k}'$ large enough $\widetilde{C}^- R^{\widetilde{k}'} \widetilde{C}^+$ is a locally admissible configuration on the strip $\langle a\rangle^{[M]}$, which restricted to $\langle a\rangle$ has as subword $wu^{k^*}u_{(\ell_j)}t \in \mathcal{L}(S)$ as the pattern restricted to this row is a point from S. This implies that $t \in F(wu^{k^*}u_{(\ell_j)})$ and contradicts the hypothesis that began this case.

Case 2: There exists $t \in \mathcal{L}(S)$ such that $\forall k \geq 1 : t \notin F(wu^k)$ and $t \in F(wu^k u_{(\ell_j)})$.

This case is very similar to the previous one, so we will omit some of the details while making the necessary adjustments for it to work.

Again let us consider any $k > N^m \left(|\mathcal{A}|^{N^m N^{4M}(4M+1)} + 1\right)$, and fix a locally admissible configuration $C \in \langle a\rangle^{[2M]}$ with

$$C|_{\{a^{-|w|},\ldots,a^{N^{2M}k|c|+\ell_j+|t|-1}\}} = wu^{N^{2M}k}u_{(\ell_j)}t,$$

where the subword $u^{N^{2M}k}$ occupies coordinates $\{a^0, \ldots a^{N^{2M}k|c|-1}\}$ of the pattern. We have that thanks to the choice of $k : k|c| > N^m|c| \left(|\mathcal{A}|^{N^m N^{4M}(4M+1)} + 1\right)$, and so by the pigeonhole principle we can find two different coordinates $n_1, n_2 \in N^m|c|N^{2M}\mathbb{N}_0$ with $0 \leq n_1 < n_2 \leq N^m|c||\mathcal{A}|^{N^m N^{4M}(4M+1)}$ such that we can extract from C two equal, non-overlapping patterns

$$C|_{a^{n_1}b^{-2M}T} = C|_{a^{n_2}b^{-2M}T},$$

where $T \coloneqq \{a^i b^l \mid 0 \leq i \leq N^m N^{4M} - 1,\ 0 \leq j \leq 4M\}$. Following the same definitions for P, C^- and C^+ as on the previous case, we again see that $C^- P^m C^+$ is a locally admissible configuration in $\langle a\rangle^{[2M]}$ for every $m \geq 1$, and that so is the configuration P^∞. Now we define $\widetilde{C} := C|_{\langle a\rangle^{[M]}}$, $\widetilde{C}^- := C^-|_{\langle a\rangle^{[M]}}$, $\widetilde{C}^+ := C^+|_{\langle a\rangle^{[M]}}$ and $\widetilde{P} := P|_{\langle a\rangle^{[M]}}$. The configurations $\widetilde{C}^-\widetilde{P}^m\widetilde{C}^+$ for $m \geq 1$ are locally admissible on $\langle a\rangle^{[M]}$, and so is $\widetilde{P}^\infty$. In particular as each row of $\widetilde{P}^\infty$ can be seen in P^∞ as the central row of a strip $\langle a\rangle^{[M]}$, we conclude that every row of the pattern $\widetilde{P}^\infty$ is a periodic point from the subshift $X_{\langle a\rangle,M} = S$. In the same way as on the previous case, these rows will form the periodic points from the set $Q_j \subseteq \mathrm{Per}(S)$ we talked about at the beginning of the proof.

Let us choose $n^* \geq 1$ minimum such that the pattern

$$R := \widetilde{P}|_{a^{n_1}b^{-M}\{a^ib^l|0\leq i<N^Mn^*,\ 0\leq l<2M+1\}}$$

satisfies $R^\infty = \widetilde{P}^\infty$, and so for any $\widetilde{k}$ sufficiently large the configuration $\widetilde{C}^-R^{\widetilde{k}}\widetilde{C}^+$ is admissible in $\langle a\rangle^{[M]}$. By the same reason given in the previous case it suffices to show that n^* is a multiple of $p_j^{i_j}$ to finish the proof. If this were not the case, then as showed before there are infinite positive integers $\widetilde{k}'$ such that $\widetilde{k}'n^* = \frac{|c|}{p_j} = \ell_j \pmod{|c|}$. But then as the row $\widetilde{P}^\infty|_{\langle a\rangle}$ has period n^* and is composed of concatenated copies of u thanks to the choice of P, we must have that $u = (u_{(\ell_j)})^{p_j}$ and that for $\widetilde{k}'$ large enough $\widetilde{C}^-R^{\widetilde{k}'}\widetilde{C}^+$ is a locally admissible configuration on the strip $\langle a\rangle^{[M]}$, which restricted to $\langle a\rangle$ has as subword $wu^{k^*}t \in \mathcal{L}(S)$. But this implies that $t \in F(wu^{k^*})$ and contradicts the hypothesis of this case. □

Remark 6.14 *With the above theorem we showed that the condition of having a* $\mathrm{BS}(1,N)$*-good set of periods is necessary for a* $\mathbb{Z}$*-subshift* Y *to occur as the stable* $\langle a\rangle$*-projective subdynamics of a* $\mathrm{BS}(1,N)$*-SFT, which reproduces the analogue result for* $\mathbb{Z}^2$ *shown in [18]. However, in the context of* $\mathbb{Z}^2$*-subshifts a complete characterization is given: a* $\mathbb{Z}$*-sofic shift can be realized as stable* $\mathbb{Z}$*-projective subdynamics of a* $\mathbb{Z}^2$*-SFT if and only if it has a good set of periods and does not have* ***a universal period****, that is, there cannot exist* $p \in \mathbb{N}$ *and* $M \geq 1$ *such that for every* $y \in Y$ *there exists a finite set of coordinates* $F_y \subseteq \mathbb{Z}$ *with* $|F_y| \leq M$ *and a point* $z \in Y$ *periodic with period* p *such that* $y|_{\mathbb{Z}\setminus F_y} = z|_{\mathbb{Z}\setminus F_y}$*. Moreover, this last condition also applies for unstable* $\mathbb{Z}$*-projective subdynamics.*

As we commented after Example 6.6, it is here that we see a fundamental difference between the nature of projective subdynamics of $\mathbb{Z}^2$ *with that of* $\mathrm{BS}(1,N)$*: we showed that the sunny-side up shift, which has a universal period, can be realized as unstable* $\langle a\rangle$*-projective subdynamics of the latter, while it cannot be realized as* $\mathbb{Z}$*-projective subdynamics of the former. This illustrates that the family of sofic* $\mathbb{Z}$*-subshifts which can be realized as* $\mathbb{Z}$*-projective subdynamics of a* $\mathbb{Z}^2$*-SFT is different from those who can be realized as* $\langle a\rangle$*-projective subdynamics of a* $\mathrm{BS}(1,N)$*-SFT. Nonetheless, we have found some similarities between both cases: both families contain positive entropy sofic* $\mathbb{Z}$*-subshifts, and we found an analogous of the* $\mathbb{Z}^2$ *condition of "having a good set of periods" as a sufficient condition for realizability of zero entropy sofic* $\mathbb{Z}$*-subshifts. However, unlike the* $\mathbb{Z}^2$ *case, we are far from giving a complete characterization of said family yet.*

Conclusion

In this thesis we tackled the (rather general) question of how well we can understand subshifts defined on solvable (non-abelian) Baumslag-Solitar groups, inspired by results and ideas present in the study of $\mathbb{Z}^2$-subshifts. In particular, we searched for similitudes and differences between these cases in order to understand which complications appearing are characteristic of Baumslag-Solitar groups, and understand them through a geometric perspective by looking at the Cayley graph of $\mathrm{BS}(1,N)$. We studied substitutions in $\mathrm{BS}(1,N)$ and proved a partial version of Mozes theorem, investigated some basic dynamical properties of graph-coloring subshifts, how $\mathbb{Z}$-subshifts can be extended to $\mathrm{BS}(1,N)$-subshifts together with properties of these extensions, and finally studied partial results regarding realization of sofic $\mathbb{Z}$-subshifts as $\langle a\rangle$-projective subdynamics of a $\mathrm{BS}(1,N)$-SFT.

This investigation had as one powerful tool the sheets structure of $\mathrm{BS}(1,N)$, which allowed us to resemble the Cayley graph of the group to the lattice $\mathbb{Z}^2$ and adapt constructions made in the latter in order to prove analogue results. Nonetheless, the "independence" between sheets gained when moving in the direction of the generator b in the Cayley graph causes completely different behavior than that of the lattice for some properties and this of course also poses obstructions when trying to adapt techniques from $\mathbb{Z}^2$. Another important tool were the rectangles of $\mathrm{BS}(1,N)$ (see Definition 3.1): this family of subsets arose as a natural generalization of rectangles in $\mathbb{Z}^2$, forming a Følner sequence for $\mathrm{BS}(1,N)$ and hence giving us a preferred family over which to compute the topological entropy of a $\mathrm{BS}(1,N)$-subshift. They were also essential when trying to extend constructions done in $\mathbb{Z}^2$ to $\mathrm{BS}(1,N)$, taking advantage of the similitude between rectangles of both groups, and the fact that the whole group may be decomposed as a disjoint union of translates of rectangles.

Nonetheless, the results obtained throughout this thesis leave some unanswered questions and space for future research in this topic, which we list below separated by each chapter.

- **Chapter 2: The structure of periodic configurations.**
  Are there any other (families of) directions of periodicity which force some kind of rigidity in the structure of a periodic configuration? We studied the case of periods in the family $\{a^{pN^k} \mid k \in \mathbb{N},\ p \notin N\mathbb{Z}\}$, but it remains open what can be said about periodicity along the b direction. An interesting starting point may be studying configurations which are periodic through every sheet up to some height, that is, a configuration $x \in \mathcal{A}^{\mathrm{BS}(1,N)}$ with

$$\{a^j b^m \mid 0 \le j < N^m\} \subseteq \mathrm{Stab}(x),\ \text{for some } m \ge 1.$$

- **Chapter 3: Substitutions on Baumslag-Solitar groups.**
  The most important question arising from this chapter is whether we can find a version of Mozes' theorem without the hypothesis of an (eventually) settling substitution. The intuition says that this should be the case, but when trying to prove the theorem technical details become complex and cumbersome really quickly.

- **Chapter 4: Graph-coloring subshifts.**
  While this was answered affirmatively for the case of odd N, it remains an open question whether or not $h_{\text{top}}(\mathcal{C}_3) > 0$ for even N. Answering it may require more delicate understanding about how patterns can repeat with a determined frequency throughout the Cayley graph of $\text{BS}(1, N)$, in contrast to the rather simple proof for odd N, which exploits the bipartitity of the graph in this case.

  Another interesting topic to look into is what kind of mixing conditions we can find in $\mathcal{C}_n$ for $n \in \{3, 4\}$. We know that $\mathcal{C}_2$ is finite, whereas for $n \geq 5$ the subshift $\mathcal{C}_n$ exhibits a really strong mixing condition, so one would hope to find an intermediate mixing property satisfied in the two remaining cases.

- **Chapter 5: Relationship between 1-dimensional and Baumslag-Solitar subshifts.**
  One of the key results of this chapter was to show that if $\widehat{X}$ is a $\text{BS}(1, N)$-subshift induced by a 1-step $\mathbb{Z}$-SFT X having a mixing symbol, then

  $$\frac{1}{2} h_{\text{top}}(X) \leq h_{\text{top}}(\widehat{X}) \leq h_{\text{top}}(X).$$

  A full-shift is a trivial example of a subshift attaining the upper bound, but we did not provide examples where this inequality is strict. Finding such examples is an interesting task. On the contrary, one could try to improve the lower bound in order to understand how small an entropy gap between $h_{\text{top}}(\widehat{X})$ and $h_{\text{top}}(X)$ can be achieved. A possible approach should take advantage of having a mixing symbol in order "not lose" too many patterns when passing from $\mathbb{Z}$ to $\text{BS}(1, N)$.

- **Chapter 6: Projective subdynamics.**
  In the final chapter of this thesis we defined projective subdynamics for $\text{BS}(1, N)$-subshifts and gave partial results of realizability of sofic $\mathbb{Z}$-subshifts as $\langle a \rangle$-projective subdynamics of a $\text{BS}(1, N)$-SFT. In particular for sofic $\mathbb{Z}$-subshifts of zero entropy we were only able to give a necessary condition for it to be realized in a stable way. The complete characterization obtained for $\mathbb{Z}^2$ suggests that we might be able to find similar (sufficient) conditions for the realizability of these $\mathbb{Z}$-subshifts. Especially one would hope to find an analogous property to having a universal period which limits a $\mathbb{Z}$-subshift from being realized as $\langle a \rangle$-projective subdynamics inside a $\text{BS}(1, N)$-SFT.

A final yet completely unexplored direction is what can be said in the case of subshifts defined on an arbitrary Baumslag-Solitar group $\text{BS}(m, n)$, for $m, n \in \mathbb{Z} \backslash \{0, 1\}$. Things seem to become more complicated and considerably different: we no longer have such a clean normal form as that of Proposition 1.25, we will have trouble finding a family of sets similar to the rectangles over which to make constructions, and as these groups are no longer amenable (see Proposition 1.23) the concept of ordinary topological entropy does not apply.

Nonetheless, the work done in this thesis studying the solvable case is a first approach to the general question and may give ideas on how to proceed.

Bibliography

[1] Noga Alon, Raimundo Briceño, Nishant Chandgotia, Alexander Magazinov, and Yinon Spinka. Mixing properties of colorings of the $\mathbb{Z}^d$ lattice. *arXiv preprint arXiv:1903.11685*, 2019.

[2] Nathalie Aubrun and Jarkko Kari. Tiling problems on baumslag-solitar groups. *Electronic Proceedings in Theoretical Computer Science*, 128:35–46, 2013.

[3] Sebastian Andres Barbieri Lemp. *Shift spaces on groups : computability and dynamics*. Theses, Université de Lyon, June 2017.

[4] Gilbert Baumslag and Donald Solitar. Some two-generator one-relator non-hopfian groups. *Bulletin of the American Mathematical Society*, 68(3):199–202, 1962.

[5] Tullio Ceccherini-Silberstein and Michel Coornaert. *Cellular Automata and Groups*. Springer Berlin, 2013.

[6] N. Pytheas Fogg, Valéré Berthé, Sébastien Ferenczi, Christian Mauduit, and Anne Siegel, editors. *Substitutions in Dynamics, Arithmetics and Combinatorics*. Springer Berlin Heidelberg, 2002.

[7] Pierre de La Harpe. *Topics in geometric group theory*. Univ. of Chicago Press, 2003.

[8] G. A. Hedlund. Endomorphisms and automorphisms of the shift dynamical system. *Mathematical systems theory*, 3:320–375, 1969.

[9] Graham Higman, B. H. Neumann, and Hanna Neuman. Embedding theorems for groups. *Journal of the London Mathematical Society*, s1-24(4):247–254, 1949.

[10] David Kerr and Hanfeng Li. *Ergodic Theory*. Springer International Publishing, 2016.

[11] Douglas Lind and Brian Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, 1995.

[12] Clara Löh. *Geometric group theory: an introduction*. Springer International Publishing, 2017.

[13] R. C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Springer, 1977.

[14] John Meier. *Groups, Graphs and Trees an Introduction to the Geometry of Infinite Groups*. Cambridge University Press, 2008.

[15] Stephen Meskin. Nonresidually finite one-relator groups. *Transactions of the American Mathematical Society*, 164:105, 1972.

[16] David Moldavanskii. On the residual properties of baumslag–solitar groups. *Communications in Algebra*, 46(9):3766–3778, February 2018.

[17] Shahar Mozes. Tilings, substitution systems and dynamical systems generated by them. *Journal d'Analyse Mathématique*, 53(1):139–186, Dec 1989.

[18] Ronnie Pavlov and Michael Schraudner. Classification of sofic projective subdynamics of multidimensional shifts of finite type. *Transactions of the American Mathematical Society*, 367, 05 2014.

[19] Ron Peled and Yinon Spinka. Rigidity of proper colorings of $\mathbb{Z}^d$. *arXiv preprint arXiv:1808.03597*, 2018.

[20] Martine Queffélec. *Substitution Dynamical Systems - Spectral Analysis*. Springer Berlin Heidelberg, 2010.

[21] Lorenzo Adlai. Sadun. *Topology of tiling spaces*. American Mathematical Society, 2008.