ASSIGNMENT

UNIVERSITÀ CATTOLICA DEL SACRO CUORE

PH.D. IN ECONOMICS AND FINANCE

Advanced Asset Pricing and Portfolio Management

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1 Introduction

In this assignment we implement and estimate the baseline Vasicek model as well as an extension, in which we account for time-varying bond risk premium (considered constant in the baseline model). In the extension we propose, the market price of interest rate risk $\lambda_{r,t}$, which contributes to bond risk premium, depends through an affine relation on the current value of the instantaneous interest rate r_t as follows:

$$\lambda_{r,t} = \lambda_0 + \lambda_1 (r_t - \bar{r})$$

with λ_0, λ_1 two real constants and $\bar{r}$ the long-term equilibrium interest rate. In this framework, the market price of interest rate risk is not constant but is rather a function of the spread between the instantaneous interest rate and its long-term equilibrium value. The specification of a linear affine functional form for the market price of interest rate has various consequences on the pricing of bonds and, in general of interest rate derivatives.

2 Vasicek Model

2.1 Instantaneous Interest Rate Dynamics

The standard Vasicek Model assumes that the instantaneous interest rate follows Ornstein-Uhlenbeck dynamics

$$dr_t = \theta(\bar{r} - r_t)dt + \sigma dW_t \tag{1}$$

where $\bar{r}, \theta, \sigma, r_0$ are positive constants and dW_t is the standard brownian increment. The instantaneous rate r_t is mean reverting: as $t \to \infty$, r is attracted towards the value $\bar{r}$. The speed of mean reversion is given by the constant θ, while σ contributes to the volatility of the process.
Notice that solving (1) by integrating on both sides results in rather tedious computations. Instead, the value of r_t can be retrieved using the appropriate tools of stochastic calculus. Consider the process $X_t = e^{\theta t} r_t$. Its stochastic differential can be computed as

$$dX_t = d\left(e^{\theta t} r_t\right) = r_t de^{\theta t} + e^{\theta t} dr_t + d\langle e^{\theta t}, r_t\rangle_t \tag{2}$$

where $\langle e^{\theta t}, r_t\rangle_t$ is the quadratic covariation of $e^{\theta t}$ and r_t. However, since the former is a deterministic function, $\langle e^{\theta t}, r_t\rangle_t = 0$. Hence

$$\begin{aligned}
d\left(e^{\theta t} r_t\right) &= r_t de^{\theta t} + e^{\theta t} dr_t \\
&= r_t \theta e^{\theta t} dt + e^{\theta t} (\theta(\bar{r} - r_t)dt + \sigma dW_t) \\
&= r_t \theta e^{\theta t} dt + e^{\theta t} \theta \bar{r} dt - r_t \theta e^{\theta t} dt + e^{\theta t} \sigma dW_t \\
&= e^{\theta t} \theta \bar{r} dt + e^{\theta t} \sigma dW_t
\end{aligned} \tag{3}$$

Integrating on both sides of (3) yields

$$\begin{aligned}
e^{\theta t} r_t - e^{\theta s} r_s &= \int_s^t e^{\theta v} \theta \bar{r} dv + \int_s^t e^{\theta v} \sigma dW_v \\
e^{\theta t} r_t &= r_s e^{\theta s} + \bar{r}\left(e^{\theta t} - e^{\theta s}\right) + \sigma \int_s^t e^{\theta v} dW_v \\
r_t &= r_s e^{-\theta(t-s)} + \bar{r}\left(1 - e^{-\theta(t-s)}\right) + \sigma \int_s^t e^{-\theta(t-v)} dW_v
\end{aligned} \tag{4}$$

Notice that through (4), r_t can be defined starting from any time $s \in [0, t]$. In fact, as a special case we set $s = 0$ and we obtain

$$r_t = r_0 e^{-\theta(t)} + \bar{r}\left(1 - e^{-\theta(t)}\right) + \sigma \int_0^t e^{-\theta(t-v)} dW_v \tag{5}$$

Notably, if we compute the expectation of (4) conditional on $\mathcal{F}_s$ we get

$$\mathbb{E}\left[r_t|\mathcal{F}_s\right] = r_s e^{-\theta(t-s)} + \bar{r}\left(1 - e^{-\theta(t-s)}\right) \tag{6}$$

It is worth noticing that the expected value of r_t is essentially a convex combination of r_s and $\bar{r}$ with weights $e^{-\theta(t-s)} \in [0,1]$ and $1 - e^{-\theta(t-s)}$. Consequently, the higher the value of θ, the greater the influence of the long-run equilibrium rate $\bar{r}$ on the expected value. Analogous is the impact of the time distance between s and t. The greater the distance, the heavier the weight put on $\bar{r}$.

2.2 Bond Pricing

Consider a zero coupon bond (ZCB) with maturity T. One of the several application of interest rate modelling is the pricing such an instrument. A ZCB is a rather simple instrument in itself, but its price bears many implication on the rest of the instruments' universe. From ZCB of different maturities it is in fact possible to recover the the yield curve that is then used in pricing claims at different times. Hence it is clear why interest rate modelling is pivotal in asset pricing.
We are able to recover the price of a bond using a no-arbitrage argument. Consider a portfolio that holds exactly one zero coupon bond. Naturally, since the portfolio consists of only one instrument, its value at any time $t \in [0,T]$ coincides with the value of the bond B_t. The expected *instantaneous* return of the portfolio (under the historical measure $\mathbb{P}$) equals the risk free rate plus a risk premium for interest rate risk.

$$\mathbb{E}[dB] = B_t r_t dt + B_r \sigma \lambda dt \tag{7}$$

where B_r is defined as the first order derivative of B with respect to r, $\frac{\partial B}{\partial r}$. Let us first work out the expression of dB. Since $B(T-t, r_t)$ is a function of time and a stochastic process r_t, we can apply Ito's Lemma and obtain

$$dB = \frac{\partial B}{\partial t}dt + \frac{\partial B}{\partial r}dr_t + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}d\langle r\rangle_t \tag{8}$$

Let us denote $\frac{\partial B}{\partial t}, \frac{\partial B}{\partial r}$ and $\frac{\partial^2 B}{\partial r^2}$ with B', B_r, B_{rr} respectively. We rewrite (8) as

$$dB = B'dt + B_r dr_t + \frac{1}{2}B_{rr}\sigma^2 dt \tag{9}$$

Notice that we have $d\langle r\rangle_t = \sigma^2 dt$ by Ito Isometry. Substituting in (9) the equation (1) for dr_t and rearranging yields

$$dB = \left(B' + B_r\theta(\bar{r} - r_t) + \frac{1}{2}B_{rr}\sigma^2\right)dt + B_r\sigma dW_t \tag{10}$$

By taking the expectation on both sides we get

$$\mathbb{E}[dB] = \left(B' + B_r\theta(\bar{r} - r_t) + \frac{1}{2}B_{rr}\sigma^2\right)dt \tag{11}$$

We can finally substitute (11) into (7)

$$\left(B' + B_r\theta(\bar{r} - r_t) + \frac{1}{2}B_{rr}\sigma^2\right)dt = B_t r_t dt + B_r\sigma\lambda dt \tag{12}$$

We guess a functional form for $B(T-t, r_t) = \exp\{-a(T-t) - b(T-t)r_t\}$. Recall that, for $B(T-t, r_t)$ to be consistent with financial theory, we need to set the price of the bond at expiry to 1. So $B(0, r_t) = 1$, which in turn implies $a(0) = b(0) = 0$. First of all it is useful to compute the derivatives for $a(t), b(t)$

$$\frac{\partial a}{\partial t} = a'(T-t)(-1) = -a'(\tau) \qquad \frac{\partial b}{\partial t} = b'(T-t)(-1) = -b'(\tau)$$

where we denote $T - t = \tau$. Finally, the partial derivatives of $B(T-t, r_t)$ are

$$\frac{\partial B}{\partial t} = B' = \left(e^{-a(T-t)-b(T-t)r_t}\right)(a'(T-t) + b'(T-t)r_t) = B_t(a'(\tau) + b'(\tau)r_t) \tag{13}$$

$$\begin{aligned}
\frac{\partial B}{\partial r} &= B_r = \left(e^{-a(T-t)-b(T-t)r_t}\right)(b(T-t)) = -B_t b(\tau) &(14)\\
\frac{\partial^2 B}{\partial r^2} &= B_{rr} = \left(e^{-a(T-t)-b(T-t)r_t}\right)(-b(T-t))(-b(T-t)) = B_t b(\tau)^2 &(15)
\end{aligned}$$

Plugging (13), (14) and (15) into (12) yields

$$B_t(a'(\tau)+b'(\tau)r_t) - B_t b(\tau)\theta(\bar{r}-r_t) + \frac{1}{2}B_t b(\tau)^2\sigma^2 = B_t r_t + B_t b(\tau)\sigma\lambda \tag{16}$$

We divide both sides by B_t and separate the terms that multiply r_t from the others to obtain

$$a'(\tau) - b(\tau)\theta\bar{r} + \frac{1}{2}b(\tau)^2\sigma^2 = -b(\tau)\sigma\lambda \tag{17}$$

$$b'(\tau) + b(\tau)\theta = 1 \tag{18}$$

Rearranging the above yields the following ordinary differential equations

$$a'(\tau) = b(\tau)\theta\bar{r} - \frac{1}{2}b(\tau)^2\sigma^2 - b(\tau)\sigma\lambda \tag{19}$$

$$b'(\tau) = 1 - b(\tau)\theta \tag{20}$$

Starting from (20) we multiplying by $e^{\theta\tau}$ and integrating both sides yields

$$e^{\theta\tau}b(\tau) = \frac{1}{\theta}e^{\theta\tau} + c \tag{21}$$

We need to impose the terminal condition $b(0) = 0$, so we find

$$c = -\frac{1}{\theta}$$

and so the first ODE results in

$$b(\tau) = \frac{1-e^{-\theta\tau}}{\theta} \tag{22}$$

Now we need to recover the functional form of $a(\tau)$ by means of (19). We plug in the result for $b(\tau)$ to obtain

$$\begin{aligned}
a'(\tau) &= \frac{1-e^{-\theta\tau}}{\theta}\theta\bar{r} - \frac{1}{2}\left(\frac{1-e^{-\theta\tau}}{\theta}\right)^2\sigma^2 - \frac{1-e^{-\theta\tau}}{\theta}\sigma\lambda\\
&= (1-e^{-\theta\tau})\bar{r} - \frac{(1-e^{-\theta\tau})^2\sigma^2}{2\theta^2} - \frac{\sigma\lambda}{\theta}(1-e^{-\theta\tau}) &(23)
\end{aligned}$$

Integration on both sides result in

$$a(\tau) = \left(\tau + \frac{1}{\theta}e^{-\theta\tau}\right)\bar{r} - \left(\tau + \frac{2}{\theta}e^{-\theta\tau} - \frac{1}{2\theta}e^{-2\theta\tau}\right)\frac{\sigma^2}{2\theta^2} - \left(\tau + \frac{1}{\theta}e^{-\theta\tau}\right)\frac{\lambda\sigma}{\theta} + c \tag{24}$$

Analogously to $b(\tau)$ we shall impose the boundary condition $a(0) = 0$ to recover the integration constant, which boils down to

$$c = -\frac{\bar{r}}{\theta} + \frac{\sigma^2}{2\theta^2}\left(\frac{2}{\theta} - \frac{1}{2\theta}\right) + \frac{\sigma\lambda}{\theta^2}$$

We are then able to conclude

$$a(\tau) = \left(\tau - \frac{1-e^{-\theta\tau}}{\theta}\right)\left(\bar{r} - \frac{\lambda\sigma}{\theta}\right) - \left(\tau - 2\frac{1-e^{-\theta\tau}}{\theta} + \frac{1-e^{-2\theta\tau}}{2\theta}\right)\frac{\sigma^2}{2\theta^2} \tag{25}$$

Lemma 1 (Vasicek Bond Price)
The price of a bond $B(\tau, r_t)$ with time to maturity $\tau = T - t$ given the current instantaneous interest rate r_t is given by

$$B(\tau, r_t) = e^{-a(\tau)-b(\tau)r_t} \tag{26}$$

with

$$\begin{aligned} a(\tau) &= \left(\tau - \frac{1-e^{-\theta\tau}}{\theta}\right)\left(\bar{r} - \frac{\lambda\sigma}{\theta}\right) - \left(\tau - 2\frac{1-e^{-\theta\tau}}{\theta} + \frac{1-e^{-2\theta\tau}}{2\theta}\right)\frac{\sigma^2}{2\theta^2} \\ b(\tau) &= \frac{1-e^{-\theta\tau}}{\theta} \end{aligned}$$

At this point we are interested in making some considerations on the dynamics of the bond price. Through Ito's Lemma we were able to define the differential of the Bond price in (10). Substituting (13), (14), (15) yields

$$dB = \left[B_t(a'(\tau) + b'(\tau)r_t) - B_t b(\tau)\theta(\bar{r} - r_t) + \frac{1}{2}B_t b(\tau)^2\sigma^2\right]dt - B_t b(\tau)\sigma dW_t \tag{27}$$

We plug (19) and (20) into the above and rearrange to obtain

$$dB = B_t\,(r_t - b(\tau)\sigma\lambda)\,dt - B_t b(\tau)\sigma dW_t \tag{28}$$

Finally, dividing both sides of (28) we find

$$\frac{dB}{B_t} = (r_t - b(\tau)\sigma\lambda)\,dt - b(\tau)\sigma dW_t \tag{29}$$

It is worthwhile to analyse the latter equation. The diffusive term is scaled by the volatility of the bond price. In particular, as the sensitivity of the bond price to interest rate variations grows larger in absolute value, so does the impact of volatility on the bond value.
The drift term $(r_t - b(\tau)\sigma\lambda)$, which represents the unconditional expected value of the bond dynamics, consists of the instantaneous interest rate r_t plus a risk premium term $-b(\tau)\sigma\lambda$. Notice in fact that $-b(\tau)\sigma$ is the volatility of B, as we previously remarked, while λ is the market price of risk per unit of volatility.

2.3 Estimation

In order to estimate the model parameters we employ Maximum Likelihood Estimation (MLE). MLE is fairly simple to implement and produce asymptotically efficient estimates. Moreover, it allows us to estimate the market price of risk λ together with the set of model parameters $\theta, \bar{r}, \sigma, \{\nu_j\}_{j=1,\dots,n-1}$, where $\{\nu_j\}_{j=1,\dots,n-1}$ is the standard deviation of the $n-1$ maturities available in the data.
The approach is the following. Given a set of bond yields $y(t,\tau)$, with t the date at which the data is recorded and τ the reference maturity. We can express bond prices as function of the yield as follows

$$B(t,\tau) = e^{-y(t,\tau)\tau} \tag{30}$$

In Lemma 1 we defined the bond price as

$$B(\tau, r_t) = e^{-a(\tau)-b(\tau)r_t}$$

By equating the two former definitions and appropriately rearranging we obtain

$$y(t,\tau) = \frac{a(\tau)}{\tau} + \frac{b(\tau)}{\tau}r_t = \tilde{a}(\tau) + \tilde{b}(\tau)r_t \tag{31}$$

Note that it is possible to rewrite the latter as a function of the yield instead

$$r_t = \frac{y(t,\tau) - \tilde{a}(\tau)}{\tilde{b}(\tau)} \tag{32}$$

The method we rely on rests on the following assumption, *the yield of one specific maturity is perfectly observed*. The remaining $n-1$ maturities are observed with error. It is common practice to choose the shortest yield maturity as perfectly observed yield. However, the method does not rely on this choice and any maturity can be arbitrarily chosen. The previous assumption implies that (32) is exact for a specific τ. Let us denote with τ_1 the maturity of the perfectly observed yield. Given the set of model parameters we can back out the instantaneous interest rate as

$$r_t = \frac{y(t,\tau_1) - \tilde{a}(\tau_1)}{\tilde{b}(\tau_1)} \tag{33}$$

For the other $n-1$ maturities $\{\tau_j\}_{j=2,...,n}$, we can compute the model implied yield rates $\hat{y}$ as

$$\hat{y}(t,\tau_j) = \tilde{a}(\tau_j) + \tilde{b}(\tau_j) r_t \qquad j = 2,...,n \tag{34}$$

The model implied yields will be inherently different from the their observed counterparts. In fact, we postulate that the observations are subject to an error $\eta_{j,t}$. Hence the observed yield can be computed as

$$y(t,\tau_j) = \hat{y}(t,\tau_j) + \eta_{j,t} = \tilde{a}(\tau_j) + \tilde{b}(\tau_j) r_t + \eta_{j,t} \qquad j = 2,...,n \tag{35}$$

We assume that the observation errors are *i.i.d.* Gaussian random variates $\eta_{j,\tau} \sim N(0, v_j^2)$ where v_j is the volatility of the observation error with maturity τ_j. We collect the standard deviations v_j for $j = 2,...,n$ in the vector Σ.

The likelihood function $\mathcal{L}(\Theta,\Sigma)$ is a function of the set of parameters Θ and the set of standard deviations Σ. It is given by

$$\mathcal{L}_t(\Theta,\Sigma) = \Pr(y(t,\tau_1),...,y(t,\tau_n)|y(t-1,\tau_1),...,y(t-1,\tau_n),\Theta,\Sigma) \tag{36}$$

Notice however, that the only source of randomness of the perfectly observed yield $y(t,\tau_1)$ is the dynamics of r_t and in particular the value of the spot rate in the previous period $t-1$. So we can rewrite (36) as

$$\mathcal{L}_t(\Theta,\Sigma) = \Pr(y(t,\tau_1)|r_{t-1},\Theta) \cdot \Pr(y(t,\tau_2),...,y(t,\tau_n)|y(t-1,\tau_1),...,y(t-1,\tau_n),\Theta,\Sigma) \tag{37}$$

Moreover, recalling the result in (35), we observer that for $j > 1$, y_{t,τ_j} depends only on the current instantaneous spot rate and on its observation error, so we shall further simplify to

$$\mathcal{L}_t(\Theta,\Sigma) = \Pr(y(t,\tau_1)|r_{t-1},\Theta) \cdot \Pr(y(t,\tau_2) \cdot ... \cdot y(t,\tau_n)|r_t,\Theta,\Sigma) \tag{38}$$

Finally, by independence of observation errors $\eta_{j,t}$ we can factor the joint probability into the product of the marginals

$$\mathcal{L}_t(\Theta,\Sigma) = \Pr(y(t,\tau_1)|r_{t-1},\Theta) \cdot \Pr(y(t,\tau_2)|r_t,\Theta,\Sigma) \cdot ... \cdot \Pr(y(t,\tau_n)|r_t,\Theta,\Sigma) \tag{39}$$

Without loss of generality, let us rewrite (4) in the form

$$r_{t+\Delta t} = \alpha + \beta r_t + \varepsilon_{t+\Delta t} \tag{40}$$

where

$$\begin{aligned} \alpha &= \bar{r}\left(1 - e^{-\theta \Delta t}\right) \\ \beta &= e^{-\theta(\Delta t)} \\ \varepsilon_{t+\Delta t} &= \sigma \int_t^{t+\Delta t} e^{-\theta(t+\Delta t - v)} dW_v \end{aligned}$$

Being a stochastic integral, we have that $\mathbb{E}[\varepsilon_{t+\Delta t}] = \mathbb{E}\left[\sigma \int_t^{t+\Delta t} e^{-\theta(t+\Delta t-v)} dW_v\right] = 0$. We then compute the variance as

$$\mathbb{E}\left[\varepsilon_{t+\Delta t}^2\right] = \sigma^2 E\left[\left(\int_t^{t+\Delta t} e^{-\theta(t+\Delta t - v)} dW_v\right)^2\right]$$

$$
\begin{aligned}
&= \sigma^2 E\left[\int_t^{t+\Delta t}\left(e^{-\theta(t+\Delta t-v)}\right)^2 dv\right] \\
&= \frac{1-e^{-2\theta\Delta t}}{2\theta}\sigma^2 = \hat{\sigma}^2_{\Delta t}
\end{aligned}
$$

The second equality follows from Ito Isometry. Then, the distribution of r_t conditional on r_{t-1} (it suffices to substitute 1 to Δt in (40) and shift back one period) is

$$
r_t = \alpha + \beta r_{t-1} + \varepsilon_t \tag{41}
$$

with $\varepsilon_t \sim N(0, \hat{\sigma}^2_{\Delta t})$. Naturally, $\alpha = \alpha(\Theta)$ and $\beta = \beta(\Theta)$ are functions of the set of parameters Θ. Plugging the latter into (31) results in

$$
y(t,\tau_1) = \tilde{a}(\tau_1) + \tilde{b}(\tau_1)(\alpha + \beta r_{t-1} + \varepsilon_t) = \tilde{a}(\tau_1) + \tilde{b}(\tau_1)(\alpha + \beta r_{t-1}) + \tilde{b}(\tau_1)\varepsilon_t \tag{42}
$$

We conclude that $y(t,\tau_1)$ is again Gaussian conditional on r_{t-1} with mean $\tilde{a}(\tau_1) + \tilde{b}(\tau_1)(\alpha + \beta r_{t-1})$ and variance $\tilde{b}^2(\tau_1)\hat{\sigma}^2_{\Delta t}$. Therefore, its density is

$$
\Pr(y(t,\tau_1)|r_{t-1},\Theta) = \frac{1}{\sqrt{2\pi\tilde{b}^2(\tau_1)\hat{\sigma}^2_{\Delta t}}}\exp\left\{-\frac{(r_t-(\alpha+\beta r_{t-1}))^2}{2\hat{\sigma}^2_{\Delta t}}\right\} \tag{43}
$$

Turning our attention to those maturities that are not perfectly observed, we can infer their conditional distribution from (35). In fact, as the observation error v_j is Gaussian with zero mean and v_j^2 variance, we deduce that $y(t,\tau_j) \sim N(\tilde{a}(\tau_j) + \tilde{b}(\tau_j)r_t, v_j^2)$ conditional on r_t, Θ and Σ. Therefore, its conditional probability density is given by

$$
\Pr(y(t,\tau_j)|r_t,\Theta,\Sigma) = \frac{1}{\sqrt{2\pi v_j^2}}\exp\left\{-\frac{\left(y(t,\tau_j)-(\tilde{a}(\tau_j)+\tilde{b}(\tau_j)r_t)\right)^2}{2v_j^2}\right\} \tag{44}
$$

Finally, computing the product of $\mathcal{L}_t(\Theta,\Sigma)$ over all t in the estimation period, yields the likelihood of the sets of parameters Θ,Σ. In practice, for computational efficiency, one most often considers the log-likelihood $\ell(\Theta,\Sigma)$, computed as the logarithm of the likelihood function. Analytically

$$
\ell(\Theta,\Sigma) = \log\left(\prod_{t=1}^{T}\mathcal{L}_t(\Theta,\Sigma)\right) = \sum_{t=1}^{T}\log\left(\mathcal{L}_t(\Theta,\Sigma)\right)
$$

The set of parameters of interest, namely $\hat{\Theta}$ and $\hat{\Sigma}$, is the result of $\arg\max_{\Theta,\Sigma}\ell(\Theta,\Sigma)$.

3 Model Extension

In this section we consider a simple yet meaningful extension of the Vasicek model. We consider unitary cost of risk λ, which is constant in the baseline model, that depends on the spread between instantaneous and long-run equilibrium interest rate as follows

$$
\lambda = \lambda_0 + \lambda_1(r_t - \bar{r}) \tag{45}
$$

Note that this extension admits the general Vasicek model as a special case for $\lambda_1 = 0$.
By modelling the price of risk as a function of the *interest rate gap*, we wish to introduce a time-varying perception of interest rate risk by the investor. Indeed, assuming $\lambda_1 < 0$, the effect on the interest-rate-dependent component on the price of risk negatively affects the price of risk whenever $r_t > \bar{r}$. On the other hand, if $r_t < \bar{r}$ the second term on the LHS of (45) is positive and mitigates the impact of the constant component λ_0.
Naturally, the extension to the dynamics of the price of risk does not affect the instantaneous rate dynamics. Therefore all the considerations in Section 2.1 still apply. In what follows, we consider the implications that the newly introduced framework has on the pricing of bonds.

3.1 Bond Pricing in the Extended Model

Analogously to the previous case, we approach the pricing problem by a no-arbitrage argument. Let us recall the no-arbitrage condition we imposed (7)

$$\mathbb{E}[dB] = B_t r_t dt + B_r \sigma \lambda dt$$

In the framework we develop, we substitute λ with the equation for the time-varying interest rate risk market price $\lambda_{r,t}$

$$\mathbb{E}[dB] = B_t r_t dt + B_r \sigma \left(\lambda_0 + \lambda_1 (r_t - \bar{r})\right) dt \tag{46}$$

The derivation traces that developed in Section 2.2 with few differences. Analogously to the previous case, we equate the no-arbitrage expected return in (46) with (11)

$$\left(\frac{\partial B}{\partial t} + B_r \theta(\bar{r} - r_t) + \frac{1}{2} B_{rr} \sigma^2\right) dt = B_t r_t dt + B_r \sigma \left(\lambda_0 + \lambda_1 (r_t - \bar{r})\right) dt \tag{47}$$

Identically to the baseline model, we guess a functional form for the Bond price of the type $B(T-t, r_t) = B(\tau, r_t) = \exp\{-a(\tau) - b(\tau) r_t\}$. We report below the partial derivatives of B that were computed in Section 2.2 for completeness.

$$\begin{aligned}
\frac{\partial B}{\partial t} &= B' = B_t(a'(\tau) + b'(\tau) r_t) \\
\frac{\partial B}{\partial r} &= B_r = -B_t b(\tau) \\
\frac{\partial^2 B}{\partial r^2} &= B_{rr} = B_t b(\tau)^2
\end{aligned}$$

Plugging the partial derivatives in (47) and dividing both sides by dt yields

$$B_t(a'(\tau) + b'(\tau) r_t) - B_t b(\tau) \theta(\bar{r} - r_t) + \frac{1}{2} B_t b(\tau)^2 \sigma^2 = B_t r_t - B_t b(\tau) \sigma \left(\lambda_0 + \lambda_1 (r_t - \bar{r})\right) \tag{48}$$

Up to this step the only difference is that we substituted the parameter λ with the equation of the time varying price of risk $\lambda_{r,t}$. By its very nature, the value of a bond at expiry is equal to the face value, formally $B(0, r_t) = 1$. Notice that, according to the chosen functional form, the latter boundary condition implies $a(0) = b(0) = 0$.
We divide both sides by B_t and we separate the terms that multiply r_t from the others to obtain

$$a'(\tau) - b(\tau)\theta\bar{r} + \frac{1}{2} b(\tau)^2 \sigma^2 = -b(\tau)\sigma(\lambda_0 - \lambda_1 \bar{r}) \tag{49}$$

$$b'(\tau) + b(\tau)\theta = 1 - b(\tau)\sigma\lambda_1 \tag{50}$$

Rearranging the above yields the following couple of ordinary differential equations

$$a'(\tau) = b(\tau)\theta\bar{r} - \frac{1}{2} b(\tau)^2 \sigma^2 - b(\tau)\sigma(\lambda_0 - \lambda_1 \bar{r}) \tag{51}$$

$$b'(\tau) = 1 - b(\tau)(\theta + \sigma\lambda_1) \tag{52}$$

Starting from (52) we set $\tilde{\theta} = \theta + \sigma\lambda_1$ and recover the functional form of $b(\tau)$

$$b(\tau) = \frac{1}{\tilde{\theta}} + \frac{c}{e^{\tilde{\theta}\tau}} \tag{53}$$

where $c \in \mathbb{R}$ is the integration constant. We impose the boundary condition $b(0) = 0$ to back out the value of c as follows

$$b(0) = \frac{1}{\tilde{\theta}} + \frac{c}{e^{\tilde{\theta}(0)}} \implies c = -\frac{1}{\tilde{\theta}} \tag{54}$$

Plugging in the value of c and rearranging concludes our computations for $b(\tau)$

$$b(\tau) = \frac{1 - e^{-\tilde{\theta}\tau}}{\tilde{\theta}} \tag{55}$$

Now we turn our attention to (51). Substituting $b(\tau)$ yields

$$a'(\tau) = \left(\frac{1-e^{-\tilde{\theta}\tau}}{\tilde{\theta}}\right)\theta\bar{r} - \frac{1}{2}\left(\frac{1-e^{-\tilde{\theta}\tau}}{\tilde{\theta}}\right)^2\sigma^2 - \left(\frac{1-e^{-\tilde{\theta}\tau}}{\tilde{\theta}}\right)\sigma(\lambda_0 - \lambda_1\bar{r}) \tag{56}$$

Integrating on both sides we get

$$a(\tau) = \left(\frac{\tilde{\theta}\tau + e^{-\tilde{\theta}\tau}}{\tilde{\theta}^2}\right)\theta\bar{r} - \frac{\sigma^2}{2\tilde{\theta}^2}\left(\tau - \frac{e^{-2\tilde{\theta}\tau}}{2\tilde{\theta}} + \frac{2}{\tilde{\theta}}e^{-\tilde{\theta}\tau}\right) - \left(\frac{\tilde{\theta}\tau + e^{-\tilde{\theta}\tau}}{\tilde{\theta}^2}\right)(\lambda_0 - \lambda_1\bar{r})\sigma + c \tag{57}$$

Finally, imposing the terminal condition $a(0) = 0$ we are able to recover c

$$a(0) = \left(\frac{1}{\tilde{\theta}^2}\right)\theta\bar{r} - \frac{\sigma^2}{2\tilde{\theta}^2}\left(-\frac{1}{2\tilde{\theta}} + \frac{2}{\tilde{\theta}}\right) - \left(\frac{1}{\tilde{\theta}^2}\right)(\lambda_0 - \lambda_1\bar{r})\sigma + c \implies c = -\left(\frac{1}{\tilde{\theta}^2}\right)\left(\theta\bar{r} - \sigma(\lambda_0 - \lambda_1\bar{r})\right) + \frac{\sigma^2}{2\tilde{\theta}^2}\left(-\frac{1}{2\tilde{\theta}} + \frac{2}{\tilde{\theta}}\right) \tag{58}$$

By plugging (58) into (57) we find

$$a(\tau) = \left(\bar{r} - \frac{\sigma\lambda_0}{\tilde{\theta}}\right)\left(\tau - \frac{1-e^{-\tilde{\theta}\tau}}{\tilde{\theta}}\right) - \frac{\sigma^2}{2\tilde{\theta}^2}\left[\tau - 2\frac{1-e^{-\tilde{\theta}\tau}}{\tilde{\theta}} + \frac{1-e^{-2\tilde{\theta}\tau}}{2\tilde{\theta}}\right] \tag{59}$$

Lemma 2 (Extended Vasicek Bond Price)
The price of a bond $B(\tau, r_t)$ with time to maturity $\tau = T - t$ given the current instantaneous interest rate r_t is given by

$$B(\tau, r_t) = e^{-a(\tau) - b(\tau)r_t} \tag{60}$$

with

$$\begin{aligned} a(\tau) &= \left(\bar{r} - \frac{\sigma\lambda_0}{\tilde{\theta}}\right)\left(\tau - \frac{1-e^{-\tilde{\theta}\tau}}{\tilde{\theta}}\right) - \frac{\sigma^2}{2\tilde{\theta}^2}\left[\tau - 2\frac{1-e^{-\tilde{\theta}\tau}}{\tilde{\theta}} + \frac{1-e^{-2\tilde{\theta}\tau}}{2\tilde{\theta}}\right] \\ b(\tau) &= \frac{1-e^{-\tilde{\theta}\tau}}{\tilde{\theta}} \\ \tilde{\theta} &= \theta + \sigma\lambda_1 \end{aligned}$$

It seems appropriate to shed some light on the effect of the introduction of time-varying market price of risk. Recall from (27) the dynamics of the bond price

$$dB = \left[B_t(a'(\tau) + b'(\tau)r_t) - B_t b(\tau)\theta(\bar{r} - r_t) + \frac{1}{2}B_t b(\tau)^2\sigma^2\right]dt - B_t b(\tau)\sigma dW_t$$

substituting to $a'(\tau), b'(\tau)$ their expressions in (51) and (52) respectively and dividing both sides by B_t we obtain

$$\begin{aligned} \frac{dB}{B_t} &= \left[(b(\tau)\theta\bar{r} - \frac{1}{2}b(\tau)^2\sigma^2 - b(\tau)\sigma(\lambda_0 - \lambda_1\bar{r}) + r_t - b(\tau)(\theta + \sigma\lambda_1)r_t) - b(\tau)\theta(\bar{r} - r_t) + \frac{1}{2}b(\tau)^2\sigma^2\right]dt - b(\tau)\sigma dW_t \\ &= [r_t - b(\tau)\sigma(\lambda_0 + \lambda_1(r_t - \bar{r}))]dt - b(\tau)\sigma dW_t \end{aligned} \tag{61}$$

The second equality follows from a mere algebraic rearrangement. The dynamics in (61) are similar to those obtained in the baseline case. The difference lies in the drift parameter. Taking the expectation on both sides results in

$$\mathbb{E}\left[\frac{dB}{B_t}\right] = [r_t - b(\tau)\sigma(\lambda_0 + \lambda_1(r_t - \bar{r}))]dt \tag{62}$$

By setting the market price of risk as a function of the spot interest rate, we introduced a time-varying drift in the process for the bond's price. In fact, the instantaneous expected return of the bond is now dependent on the gap between instantaneous interest rate and long-term equilibrium value. Whenever $r_t > \bar{r}$ the instantaneous expected return is amplified. On the contrary, if $r_t < \bar{r}$, the term $\lambda_1(r_t - \bar{r})$ becomes positive and dampens the overall market price of risk λ.

3.2 Estimation of the Extended Model

The estimation of the extension to the Vasicek model does not differ from the procedure derived in Section 2.3. The only discrepancy lies in the specification of $a(\tau)$ and $b(\tau)$. Although it might seem a minor difference, note that the definition of $a(\tau)$ and $b(\tau)$ has major impact on the estimation procedure. Let us recall that $a(\tau), (\tau)$ are pivotal in the computation of $y(t, \tau)$ as a function of the instantaneous interest rate in (31) and analogously to recover the instantaneous interest rate from the perfectly observed yield through (32).
Furthermore, by defining $a(\tau)$ and $b(\tau)$ as in (59) and (55), we are able to recover the estimates for both λ_0 and λ_1.

4 Application

As complementary to this note, we provide a `Jupyter Notebook` in which we perform a numerical exercise.
We estimate both the standard Vasicek model and the "Extended" Vasicek model on Treasury Inflation-Protected Securities' (TIPS) yields. We then compare the estimates of the two models and make suitable remarks on the implications of the models and on their bond risk premia.