Reifenberg's Topological Disc Theorem

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Here $B_\rho = \{x \in \mathbf{R}^n : |x| \le \rho\}$ and $B_\rho(y) = \{x \in \mathbf{R}^n : |x - y| \le \rho\}$.

First we introduce Reifenberg's ϵ-approximation property for subsets of $\mathbf{R}^n$.

Definition: If $\epsilon > 0$ and if S is a closed subset of the ball B_2, we say that S, containing 0, has the m-dimensional ϵ-Reifenberg approximation property in B_1 if for each $y \in S \cap B_1$ and for each $\rho \in (0, 1]$, there is an m-dimensional subspace $L_{y,\rho}$ such that $d_{\mathcal{H}}(S \cap B_\rho(y),\, y + L_{y,\rho} \cap B_\rho(y)) < \epsilon$.

Here $d_{\mathcal{H}}(A_1, A_2)$ is the Hausdorff distance between A_1, A_2; thus $d_{\mathcal{H}}(A_1, A_2) = \inf\{\epsilon > 0 : A_1 \subset B_\epsilon(A_2) \ \&\ A_2 \subset B_\epsilon(A_1)\}$.

Now we can state the main theorem.

Theorem (Reifenberg's disc theorem). *There is a constant $\epsilon = \epsilon(n) > 0$ such that if S, containing 0, is a closed subset of the ball B_2 which satisfies the above ϵ-Reifenberg approximation property in B_1, then $B_1 \cap S$ is homeomorphic to the closed unit ball in $\mathbf{R}^m$.*

In fact, there is a closed subset $M \subset \mathbf{R}^n$ such that $M \cap B_1 = S \cap B_1$ and such that is homeomorphic to a subspace T_0 of $\mathbf{R}^n$ via a homeomorphism $\tau : T_0 \to M$ with $|\tau(x) - x| \le C(n)\epsilon$ for each $x \in T_0$, and $\tau(x) = x$ for each $x \in T_0 \setminus B_2$. For any given $\alpha \in (0, 1)$ we can additionally arrange that τ and τ^{-1} are Hölder continuous with exponent α provided S satisfies the ϵ-Reifenberg condition with suitable $\epsilon = \epsilon(n, \alpha)$.

We'll need the following lemma in the proof of the above theorem.

Lemma 1 (Extension Lemma). *Let $\epsilon, r > 0$, let $y_1, \ldots, y_Q$ be a finite collection of points in $\mathbf{R}^n$ with $|y_i - y_k| \ge r$ for each $i \ne k$, and assume that $f : \{y_1, \ldots, y_Q\} \to \mathbf{R}^N$ is given such that $|f(y_i) - f(y_k)| \le \epsilon$ whenever $|y_i - y_k| \le 6r$. Then there is an extension $\overline{f} : \cup_i B_{2r}(y_i) \to \mathbf{R}^N$ such that $|\nabla \overline{f}| \le C(n)\epsilon r^{-1}$ and $|\overline{f}(x) - f(y_i)| \le C(n)\epsilon$ for $x \in B_{2r}(y_i)$, $i = 1, \ldots, Q$.*

Furthermore there is $\epsilon = \epsilon(n) > 0$ such that if $N = n^2$ (where $\mathbf{R}^{n^2}$ is identified with the set of $n \times n$ matrices in the usual way) and if each $f(y_i)$ is the matrix of an orthogonal projection of $\mathbf{R}^n$ onto some m-dimensional subspace $L_i \subset \mathbf{R}^n$, then we can

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choose the extension $\overline{f}$ such that each $\overline{f}(x)$ is the matrix of an orthogonal projection of $\mathbf{R}^n$ onto some m-dimensional subspace L_x.

Proof: The proof uses a partition of unity $\{\psi_j\}$ for $\cup_i B_{2r}(y_i)$ of special type. Indeed we claim that there is a partition of unity for $\cup_i B_{2r}(y_i)$ with $\psi_i \in C_c^\infty(\mathbf{R}^n)$, $\psi_i \equiv 0$ outside $B_{3r}(y_i)$, $\psi_i(y_i) = 1$, and $\sup|\nabla\psi_i| \le C(n)r^{-1}$.

We see this as follows: first let ψ^0 be a $C^\infty(\mathbf{R}^n)$ function with $\psi^0(x) \equiv 1$ for $|x| < \frac{1}{3}$, $0 < \psi^0(x) < 1$ for $< \frac{1}{3}|x| \le \frac{5}{2}$, and $\psi^0(x) \equiv 0$ for $|x| \ge \frac{5}{2}$. For each $i = 1, \ldots, Q$ let $\psi_i^0(x) = \psi^0(\frac{x-y_i}{r})$, $\widetilde{\psi}_i^0(x) = \psi_i^0 \,\Pi_{k\ne i}(1 - \psi_k^0(x))$, and $\psi_i(x) = \frac{\widetilde{\psi}_i^0(x)}{\sum_k \widetilde{\psi}_k^0(x)}$. This evidently gives a partition of unity with the stated properties.

It is now straightforward to check that

$$\overline{f}(x) = \sum_{i=1}^{Q} \psi_i(x) f(y_i).$$

is a suitable extension.

For the second part of the lemma we recall that the orthogonal projections onto m-dimensional subspaces of $\mathbf{R}^n$ form a smooth (in fact real-analytic) compact submanifold $\mathcal{P}$ of $\mathbf{R}^{n^2}$, and hence there is a $\delta = \delta(n) > 0$ such that there is a smooth nearest-point projection map Ψ of the δ-neighbourhood $\mathcal{N}_\delta$ of $\mathcal{S}$ onto $\mathcal{S}$.

Now by the first part of the lemma we have an extension $\overline{f}^0$ such that $|f(y_i) - \overline{f}^0(x)| \le C(n)\epsilon$ for each $x \in B_{2r}(y_i)$; but by definition $f(y_i) \in \mathcal{S}$, so this means that if ϵ is small enough (depending only on n) we have $\overline{f}^0(x) \in \mathcal{N}_{\delta/2}$ and hence we can define $\overline{f} = \Psi \circ \overline{f}^0$. Evidently then $\overline{f}$ has the correct properties.

The second lemma involves a simple observation about the subspaces $L_{y,\rho}$ appearing in the ϵ-Reifenberg condition; in particular it shows that these must vary quite slowly (up to tilts of order ϵ) as y and ρ vary.

Lemma 2. *If $\epsilon > 0$ and if S satisfies the ϵ-Reifenberg condition above, then $\|L_{y_1,\sigma} - L_{y_2,\rho}\| \le 32\epsilon$ and dist$(y_1, y_2 + L_{y_2,\rho}) \le 32\epsilon\rho$ whenever y_1, $y_2 \in S \cap B_1$ and $0 < \frac{\rho}{8} \le \sigma \le \rho \le 1$.*

The proof, which involves only the definition of the ϵ-Reifenberg condition and the triangle inequality for $d_{\mathcal{H}}$, is left as an exercise for the reader.

Finally, we need the following "squash lemma":

Lemma 3 ("Squash Lemma"). *There is a constant $\epsilon_0 = \epsilon_0(n)$ such that the following holds. If $\epsilon \in (0, \epsilon_0]$, $\rho > 0$, L is an m-dimensional subspace of $\mathbf{R}^n$,*

$$\Phi(x) = p_L(x) + e(x), \qquad x \in B_{3\rho},$$

where p_L is orthogonal projection onto L and $\rho^{-1}|e(x)| + |\nabla e(x)| \le \epsilon$ for all $x \in B_{3\rho}$, and if

$$G = \{x + g(x) \,:\, x \in B_{3\rho} \cap L\}$$

is the graph of a C^1 function $g : B_{3\rho} \cap L \to L^\perp$ with $\rho^{-1}|g(x)| + |\nabla g(x)| \le 1$ at each point x of $B_{3\rho} \cap L$, then $\Phi(G \cap B_{3\rho})$ is the graph of a C^1-function $\widetilde{g} : U \to L^\perp$ over some domain U with $B_{11\rho/4} \cap L \subset U \subset L$ and with $\rho^{-1}|\widetilde{g}| + |\nabla \widetilde{g}(x)| \le 4\epsilon$ on $B_{11\rho/4} \cap L$.

Proof of the squash lemma: All hypotheses are written in "scale invariant" form, so there is no loss of generality in taking $\rho = 1$, which we do. Now by definition

$$\Phi(x + g(x)) = x + e(x + g(x)) \tag{1}$$

for $x \in B_2 \cap L$, and, if $h(x) = e(x + g(x))$, by the chain rule we have $|d_x h| \le 2\epsilon$ at each point x of $L \cap B_2$. Now we can write $h = h^\perp + h^T$, where $h^\perp = p_L^\perp \circ h$ and $h^T = p_L \circ h$. Then (1) says

$$\Phi(x + g(x)) = x + h^T(x) + h^\perp(x), \quad x \in B_2 \cap L. \tag{2}$$

Now let

$$Q(x) = x + h^T(x), \quad x \in B_2 \cap L,$$

and observe that

$$|d\,Q - \mathrm{id}| \le 2\epsilon, \quad |Q - \mathrm{id}| \le \epsilon \quad \text{on } B_2 \cap L,$$

and hence, for small enough $\epsilon \in (0, \frac{1}{6})$, by the inverse function theorem Q is a diffeomorphism of $B_2 \cap L$ onto a subset U where $L \cap B_{11/4} \subset U \subset L$ and $|d\,Q^{-1} - \mathrm{id}| \le 2\epsilon(1 + 2\epsilon) \le 3\epsilon$. Thus (2) can be written

$$\Phi(x + g(x)) = Q(x) + \widetilde{g}(Q(x)), \qquad x \in B_{11/4} \cap L,$$

where $\widetilde{g} = p_L^\perp \circ h \circ Q^{-1}$ on U, and, since $|d\,h \circ Q^{-1}| \le 2\epsilon(1 + 3\epsilon) \le 3\epsilon$, we have $|d\,\widetilde{g}| \le 3\epsilon$ and the proof is complete.

Proof of the Reifenberg disc theorem: The proof is based on an inductive procedure, making successive approximations to $S_* = S \cap B_1$ by C^∞ embedded submanifolds.

Let $T_0 = L_{0,1}$ (which without loss of generality we could take to be $\mathbf{R}^m \times \{0\}$) be an m-dimensional subspace such that $d_{\mathcal{H}}(S \cap B_1, T_0 \cap B_1) < \epsilon$, and let $r_j = \left(\frac{1}{8}\right)^j$, $j = 0, 1, \ldots$. The quantity r_j is going the be the "scale" used at the j^{th} step of the inductive process.

We in fact define maps $\sigma_j : \mathbf{R}^n \to \mathbf{R}^n$ and subsets $M_j \subset \mathbf{R}^n$ for $j = 0, 1, \ldots,$ as follows:

For $j \ge 1$, let $B_{r_j/2}(y_{ji})$, $i = 1, \ldots, Q_j$, be a maximal pairwise disjoint collection of balls centered in $S_* = B_1 \cap S$. Then evidently $S_* \subset \cup_{i=1}^{Q_j} B_{r_j}(y_{ji})$ and also $\operatorname{dist}(S_*, \mathbf{R}^n \setminus (\cup_{i=1}^{Q_j} B_{3r_j/2}(y_{ji}))) \ge r_j/2$. When $j = 0$ we take $Q_0 = 1$, $y_{0\,1} = 0$, and $M_0 = T_0$, $\sigma_0 =$ the orthogonal projection of $\mathbf{R}^n$ onto T_0.

For $j \geq 1$ and for each $i = 1, \ldots, Q_j$ let L_{ji} be one of the m-dimensional subspaces $L_{y_{ji}, 8r_j}$ (corresponding to $y = y_{ji}$ and $\rho = 8r_j$ in the ϵ-Reifenberg condition). Thus

$$d_{\mathcal{H}}(S \cap B_{8r_j}(y_{ji}),\, (y_{ji} + L_{ji}) \cap B_{8r_j}(y_{ji})) < 8\epsilon\, r_j, \quad i = 1, \ldots, Q_j.$$

For $j \geq 1$ we have by Lemma 2 that

$$d_{\mathcal{H}}((y_{ji} + L_{ji}) \cap B_{r_j}(y_{ji}),\, (y_{\ell k} + L_{\ell k}) \cap B_{r_j}(y_{ji})) \leq 264\epsilon r_j \tag{1}$$

for any pair $y_{j\, i}, y_{\ell\, k}$ with $|y_{j\, i} - y_{\ell\, k}| \leq 6r_{j-1}$, where either $\ell = j-1$ and $k \in \{1, \ldots, Q_{j-1}\}$ or $\ell = j$ and $k \in \{1, \ldots, Q_j\}$. Notice of course that (1) implies

$$|p_{ji} - p_{\ell k}| < 264\epsilon, \quad \operatorname{dist}(y_{ji}, y_{\ell k} + L_{\ell k}) < 264\epsilon r_j \tag{2}$$

for such j, ℓ, i, k, where p_{ji} denotes the orthogonal projection of $\mathbf{R}^n$ onto L_{ji}.

In view of the inequalities (2) (together with the fact that $|y_{ji} - y_{jk}| \geq r_j$ for each $i \neq k$), we can apply the extension lemma with $r = r_j$, with y_{ji} in place of y_i and with the orthogonal projection p_{ji} in place of $f(y_i)$, to give orthogonal projections $p_{j,x}$ of $\mathbf{R}^n$ onto m-dimensional subspaces $L_{j,x}$ such that $p_{j,x} = p_{ji}$ when $x = y_{ji}$ and

$$\begin{aligned} &\left|\frac{\partial p_{j,x}}{\partial x^\ell}\right| \leq \frac{C(n)\epsilon}{r_j}, \quad x \in \cup_{i=1}^{Q_j} B_{2r_j}(y_{ji}),\ \ell = 1, \ldots, n, \\ &|p_{j,x} - p_{ji}| \leq C(n)\epsilon, \qquad x \in B_{2r_j}(y_{ji}),\ i = 1, \ldots, Q_j. \end{aligned} \tag{3}$$

Next let ψ_{ji} be a partition of unity for $\cup_{i=1}^{Q_j} B_{3r_j/2}(y_{ji})$ such that $|\nabla \psi_{ji}| \leq C(n)/r_j$ and support $\psi_{ji} \subset B_{2r_j}(y_{ji})$ for each $i = 1, \ldots, Q_j$. (This is constructed in precisely the same way as our partition of unity for the extension lemma, except that we start with a smooth function φ with support in $B_2(0)$ rather than in $B_3(0)$ as before; actually the construction can be simplified here because we do not need $\psi_{ji}(y_{ji}) = 1$ and $\psi_{jk}(y_{ji}) = 0$ for $i \neq k$.)

Now we can define σ_j and M_j for $j \geq 1$. First we define [1]

$$\sigma_j(x) = x - \sum_{i=1}^{Q_j} \psi_{ji}(x) p_{j,x}^{\perp}(x - y_{ji}), \qquad x \in \mathbf{R}^n, \tag{4}$$

and then we take

$$M_j = \sigma_j(M_{j-1}). \tag{5}$$

First note that, since $\sigma_j(x) \equiv x$ for $x \in \mathbf{R}^n \setminus (\cup_{i=1}^{Q_j} B_{2r_j}(y_{ji}))$, we have

$$M_j \setminus (\cup_{i=1}^{Q_j} B_{2r_j}(y_{ji})) = M_{j-1} \setminus (\cup_{i=1}^{Q_j} B_{2r_j}(y_{ji})) \tag{6}$$

[1] of course it doesn't matter that the $p_{j,x}$ are not defined outside $\cup_{i=1}^{Q_j} B_{2r_j}(y_{ji})$ because the ψ_{ji} vanish identically there. (If you wish to be pedantic, you can define e.g. $p_{j,x}$ to be the orthogonal projection onto T_0 for $x \in \mathbf{R}^n \setminus (\cup_{i=1}^{Q_j} B_{2r_j}(y_{ji}))$.)

for each $j \geq 1$.

We claim that each M_k is a properly embedded C^∞ m-dimensional submanifold of $\mathbf{R}^n$ and that for each $k \geq 1$ and each $i \in \{1, \ldots, Q_k\}$

$$\begin{aligned} &M_k \cap B_{2r_k}(y_{ki}) = \operatorname{graph} g_{ki} \\ &\sup |\nabla g_{ki}| \leq \gamma\, \epsilon, \qquad \sup |g_{ki}| \leq \gamma\, \epsilon\, r_k. \end{aligned} \tag{7}$$

where $\gamma \geq 1$ is a constant (to be specified as a function of n alone) and where g_{ki} is a C^∞ function over a domain in the affine space $y_{ki} + L_{ki}$ with values normal to L_{ki}.

We want to inductively to check this. Observe that if $j \geq 1$ and if M_{j-1} is a smooth embedded submanifold satisfying (7) with $k = j-1$, then by the definition (4) we have

$$\begin{aligned} \sigma_j(x) - x &= -\textstyle\sum_{k=1}^{Q_j} \psi_j(x) p^\perp_{j,x}(x - y_{jk}) \\ &= -\textstyle\sum_{k=1}^{Q_j} \psi_j(x) p^\perp_{jk}(x - y_{jk}) + \textstyle\sum_{k=1}^{Q_j} \psi_j(x)(p^\perp_{jk} - p^\perp_{j,x})(x - y_{jk}). \end{aligned} \tag{8}$$

Now for each $i \in \{1, \ldots, Q_j\}$, we can pick an $i_0 \in \{1, \ldots, Q_{j-1}\}$ such that $y_{ji} \in B_{r_{j-1}}(y_{j-1\, i_0})$. Then, assuming that (7) holds with $k = j-1$ and with some constant $\gamma = \gamma_{j-1}$, for $x \in B_{2r_j}(y_{ji}) \cap M_{j-1} (\subset B_{2r_{j-1}}(y_{j-1\, i_0}) \cap M_{j-1})$ we can write $x = \xi + g_{j-1}(\xi)$, with $g_{j-1}(\xi) \in L^\perp_{j-1\, i_0}$, $\xi \in (y_{j-1\, i_0} + L_{j-1\, i_0}) \cap B_{2r_{j-1}}(y_{j-1\, i_0})$ and with $r_{j-1}^{-1}|g_{j-1}(\xi)| + |\nabla g_{j-1}(\xi)| \leq \gamma_{j-1}\epsilon$. Then we have, for each $k \in \{1, \ldots, Q_j\}$,

$$\begin{aligned} p^\perp_{jk}(x - y_{jk}) = {} & p^\perp_{j-1\, i_0}(\xi + g_{j-1}(\xi) - y_{j-1\, i_0}) \\ & + p^\perp_{j-1\, i_0}(y_{jk} - y_{j-1\, i_0}) + (p^\perp_{jk} - p^\perp_{j-1\, i_0})(\xi + g_{j-1}(\xi) - y_{jk}), \end{aligned}$$

and using (2), (3) together with the fact that $p^\perp_{j-1\, i_0}(\xi - y_{j-1\, i_0}) = 0$ (because $\xi - y_{j-1\, i_0} \in L_{j-1\, i_0}$), we have clearly then that

$$|p^\perp_{jk}(x - y_{jk})| \leq C(n)\epsilon(1 + \gamma_{j-1})r_j, \quad x \in B_{2r_j}(y_{ji}) \cap M_{j-1}, \quad |y_{jk} - y_{ji}| \leq 6r_j.$$

Using this in (8), and keeping in mind that for any $i \in \{1, \ldots, Q_j\}$ and for any $x \in B_{2r_j}(y_{ji})$, we have that at most $C(n)$ terms in the sums on the right of (8) can be non-zero, and that these terms correspond to the indices k such that $|y_{ji} - y_{jk}| \leq 6r_j$, hence, using also (3), we again deduce from (8) that

$$|\sigma_j(x) - x| \leq C(n)(1 + \gamma_{j-1})\epsilon r_j, \quad x \in \cup_{i=1}^{Q_j} B_{2r_j}(y_{ji}) \cap M_{j-1}. \tag{9}$$

By first differentiating in (8) and using similar considerations on the right side, we also conclude

$$\sup_{x \in M_{j-1}} |\nabla'(\sigma_j(x) - x)| \leq C(n)(1 + \gamma_{j-1})\epsilon r_j, \tag{9$'$}$$

where ∇' denotes gradient taken on the submanifold M_{j-1}.

We refer to (9) and (9)′ subsequently as "the coarse estimates" for $|\sigma_j(x)-x|$, because, although useful, they are insufficient in themselves to complete that inductive proof that there is a fixed constant $\gamma = \gamma(n)$ such that (7) holds for all k; indeed after k applications of this coarse inequality, we will only have established that (7) holds with $\gamma = C(n)^k$.

Now assume that $j \geq 2$ and that (7) holds for $k = 1, \ldots, j-1$, take an arbitrary $i_0 \in \{1, \ldots, Q_j\}$, and write $y_0 = y_{j\,i_0}$, $p_0 = p_{j\,i_0}$, and $L_0 = L_{j\,i_0}$. Since $\sum_{i=1}^{Q_j} \psi_{ji} \equiv 1$ in $U_j \equiv \cup_{i=1}^{Q_j} B_{3r_j/2}(y_{ji})$ we can rearrange the defining expression for σ_j to give

$$\sigma_j(x) = y_0 + p_0(x - y_0) + e(x), \quad x \in U_j, \tag{10}$$

where e is given by

$$e(x) \equiv \sum_{i=1}^{Q_j} \psi_{ji}(x) p_0^{\perp}(y_{ji} - y_0) - \sum_{i=1}^{Q_j} \psi_{ji}(x)(p_{j,x}^{\perp} - p_0^{\perp})(x - y_{ji}), \quad x \in \mathbf{R}^n. \tag{11}$$

Now observe that by (2) and (3) we have $|p_{j,x} - p_0| \leq C(n)\epsilon\, r_j$ for $x \in B_{6r_j}(y_0)$. Using additionally the first inequality in (3) and the fact that $|\nabla\psi_{ji}| \leq C(n)/r_j$, it then follows easily that

$$r_j^{-1}|e(x)| + |\nabla e(x)| \leq C(n)\epsilon, \text{ if } x \in B_{3r_j/2}(y_0), \tag{12}$$

where $C(n)$ is a fixed constant determined by n alone (and which is independent of any properties of M_{j-1}; in particular it is independent of whatever constant γ appears in (7)).

But now we can apply the Squash Lemma with $\widetilde{\sigma}_j(x) \equiv \sigma_j(x + y_0) - y_0$ in place of Φ, $2\,r_j$ in place of ρ, and $C(n)\epsilon$ in place of ϵ. Assuming that (7) holds with γ, ϵ such that $\gamma\epsilon \leq \frac{1}{2}$, we thus conclude

$$\sigma_j(M_{j-1} \cap B_{3r_j/2}(y_0)) = G_j, \tag{13}$$

where $G_j = \{x + g_j(x) : x \in \Omega_j\}$ is the graph of a C^∞ function g_j defined over a domain Ω_j contained in the affine space $y_0 + L_0$ with $B_{11r_j/8}(y_0) \cap (y_0 + L_0) \subset \Omega_j$ and with

$$r_j^{-1}|g_j| + |\nabla g_j| \leq C(n)\epsilon, \quad x \in B_{11r_j/8}(y_0) \cap (y_0 + L_0), \tag{14}$$

with $C(n)$ not depending on γ. Of course since $|\sigma_j(x) - x| < C(n)\gamma\epsilon$ (by (8)), we thus have, so long as $C(n)\gamma\epsilon \leq \frac{1}{32}$ that $\sigma_j(M_{j-1} \cap B_{3r_j/2}(y_0)) \supset \sigma_j(M_{j-1}) \cap B_{11r_j/8}(y_0)$, and hence (13) and (14) imply

$$M_j \cap B_{11r_j/8}(y_0)) = G_j, \tag{15}$$

with G_j still as in (14).

Now we actually need to establish a result like this over the ball $B_{2r_j}(y_0)$ rather than merely over $B_{11r_j/8}(y_0)$; to achieve this, we observe that each y_{ji} is contained in one of the balls $B_{r_{j-1}}(y_{j-1\,i_0})$ for some $i_0 \in \{1, \dots, Q_{j-1}\}$, and so $B_{r_{j-1}/4}(y_{ji}) \subset B_{5r_{j-1}/4}(y_{j-1\,i_0})$. Also, by using the above argument with $j-1$ in place of j and with i_0 in place of i, we deduce that

$$(15)' \qquad M_{j-1} \cap B_{11r_{j-1}/8}(y_{j-1\,i_0})) = G_{j-1},$$

where $G_{j-1} = \{x + g_{j-1}(x) : x \in \Omega_{j-1}\}$ is the graph of a C^∞ function g_{j-1} defined over a domain Ω_{j-1} contained in the affine space $y_{j-1\,i_0} + L_{j-1\,i_0}$ with $B_{11r_{j-1}/8}(y_{j-1\,i_0}) \cap (y_{j-1\,i_0} + L_{j-1\,i_0}) \subset \Omega_{j-1}$ and with

$$(14)' \qquad r_{j-1}^{-1}|g_{j-1}| + |\nabla g_{j-1}| \le C(n)\epsilon, \quad x \in B_{11r_{j-1}/8}(y_{j-1\,i_0}) \cap (y_{j-1\,i_0} + L_{j-1\,i_0}).$$

But then by using the coarse estimates (9), (9)$'$ we deduce that in fact (7) holds with $k = j$ and a fixed constant γ which depending only on n and not on γ.

Notice that since $S_* \subset \cup_{i=1}^{Q_j} B_{r_j}(y_{ji})$ it is clear from (7) and the ϵ-Reifenberg condition in the ball $B_{2r_j}(y_{ji})$, that

$$(16) \qquad S_* \subset B_{C(n)\epsilon r_j}(M_j), \quad j \ge 0.$$

Notice also that (7) tells us that for $j \ge 2$

$$M_j \cap (\cup_{i=1}^{Q_j} B_{2r_j}(y_{ji})) \subset (\cup_{i=1}^{Q_j} B_{C(n)\epsilon r_j}(y_{ji} + L_{ji})) \subset B_{C(n)\epsilon r_j}(S),$$

and hence, since $M_j \setminus (\cup_i B_{2r_j}(y_{ji})) = M_{j-1} \setminus (\cup_i B_{2r_j}(y_{ji}))$ by mathematical induction it follows that

$$(17) \qquad M_j \cap B_{1+r_j/2} \subset B_{C(n)\epsilon\, r_j}(S)$$

for each $j = 0, 1, \dots$, provided $\epsilon \le \epsilon_0$, where $\epsilon_0 = \epsilon_0(n)$.

Next we want to show that the sequence $\tau_j = \sigma_j \circ \sigma_{j-1} \circ \cdots \sigma_0 | T_0$ is a sequence of C^∞ diffeomorphisms of T_0 onto M_j which converge uniformly on T_0 to a homeomorphism τ of T_0 onto a closed set M. In fact notice that by (8) we have

$$|\tau_j(x) - \tau_{j-1}(x)| \le C(n)\epsilon \left(\frac{1}{8}\right)^j, \quad j \ge 1, \ x \in T_0,$$

and hence by iterating we get

$$(18) \qquad |\tau_{j+k}(x) - \tau_j(x)| \le C(n)\epsilon \left(\frac{1}{8}\right)^j, \quad j \ge 0, \ k \ge 1, \ x \in T_0,$$

which shows that τ_j is Cauchy with respect to the uniform norm on T_0, and hence τ_j converges uniformly to a continuous map $\tau : T_0 \to \mathbf{R}^n$. Of course τ is the identity outside B_2 because each σ_j is the identity outside B_2. We let $M = \tau(T_0)$, so that M is a closed subset of $\mathbf{R}^n$ and in fact is the Hausdorff limit (with respect to the Hausdorff metric $d_{\mathcal{H}}$) of the sequence $M_j = \tau_j(T_0)$. Notice in particular that setting $j = 0$ and taking limit as $k \to \infty$ in the above inequality, we get

$$|\tau(x) - x| \le C(n)\epsilon, \quad x \in T_0. \tag{19}$$

(Thus τ is in the distance sense quite close to the identity if ϵ is small.)

Next we want to discuss injectivity of τ_j, τ; in fact we'll show that τ_j, τ are injective and that both τ and τ^{-1} are Hölder continuous.

To establish this, we first claim

$$(1 - C(n)\epsilon)|x - y| \le |\sigma_j(x) - \sigma_j(y)| \le (1 + C(n)\epsilon)|x - y|, \quad x,\, y \in M_{j-1}, \tag{20}$$

or equivalently

$$|\sigma_j(x) - \sigma_j(y) - (x - y)| \le C(n)\epsilon|x - y|, \quad x,\, y \in M_{j-1}. \tag{20$'$}$$

To prove this, note that if $|x - y| \ge r_j$ with $x,\, y \in M_{j-1}$, we can write

$$\begin{aligned}|\sigma_j(x) - \sigma_j(x) - (x - y)| &= |(\sigma_j(x) - x) - (\sigma_j(y) - y)| \\ &\le |\sigma_j(x) - x| + |\sigma_j(y) - y| \\ &\le C(n)\epsilon\, r_j \le C(n)\epsilon|x - y|,\end{aligned}$$

where we used (8) in the second inequality.

Now if $|x - y| < r_j$ we use the definition (4) to write

$$\begin{aligned}(\sigma_j(x) - \sigma_j(y)) - (x - y) = \textstyle\sum_{i=1}^{Q_j}(&\psi_{ji}(x)p^{\perp}_{j,x}(x - y_{ji}) \\ &- \psi_{ji}(y)p^{\perp}_{j,y}(y - y_{ji})), \qquad x,\, y \in \mathbf{R}^n,\end{aligned}$$

and note that we can rearrange the sum here to give

$$\begin{aligned}(\sigma_j(x) - \sigma_j(y)) - (x - y) = \textstyle\sum_{i=1}^{Q_j}\big(&\psi_{ji}(x)(p^{\perp}_{j,x}(x - y) \\ &+ \psi_{ji}(x)(p^{\perp}_{j,x} - p^{\perp}_{j,y})(y - y_{ji}) + (\psi_{ji}(x) - \psi_{ji}(y))p^{\perp}_{j,y}(y - y_{ji})\big)\,.\end{aligned}$$

Now the second group of terms is (by (3)) trivially $\le C(n)\epsilon\,|x - y|$ in absolute value for any $x,\, y \in \mathbf{R}^n$ with $|x - y| \le r_j$. Further if $x,\, y \in M_{j-1}$, then by virtue of (7) (used with y in place of z) we see that the first and third group of terms on the right is $\le C(n)\epsilon\,|x - y|$ in absolute value. Thus we again get (20).

Now it is easy to establish the required injectivity and continuity of τ. In fact by iterating the inequality (20) we get

$$|\tau_j(x) - \tau_j(y)| \le (1 + C\epsilon)^j|x - y|, \quad x,\, y \in T_0,\ j \ge 1, \tag{21}$$

and by (8) we have

$$|\tau_j(x) - \tau_{j-1}(x)| \le C\epsilon r_j, \quad x \in T_0,\ j \ge 1,$$

and so (Cf. the discussion of uniform convergence of the τ_j above)

$$|\tau_j(x) - \tau(x)| \le C\epsilon r_j. \tag{22}$$

Then by the triangle inequality, for any $j \ge 0$ we have

$$\begin{aligned}|\tau(x) - \tau(y)| &\le |\tau(x) - \tau_j(x)| + |\tau_j(x) - \tau_j(y)| + |\tau_j(y) - \tau(y)| \\ &\le 2C(n)\epsilon r_j + (1 + C(n)\epsilon)^j |x - y| \\ &\le r_j + (1 + C(n)\epsilon)^j |x - y| \text{ if } 2\epsilon C(n) \le 1.\end{aligned}$$

Now let $\alpha \in (0,\,1)$ be arbitrary and take $x,\, y \in T_0$ with $0 < |x-y| < \frac{1}{2}$. Choose j such that $r_j \le |x-y|^\alpha$ and $(1+C(n)\epsilon)^j \le |x-y|^{-(1-\alpha)}$; thus we need $j \ge \frac{\alpha}{\log 8} \log\left(\frac{1}{|x-y|}\right)$ and also $j \le \frac{(1-\alpha)}{\log(1+C(n)\epsilon)} \log\left(\frac{1}{|x-y|}\right)$. Since $\log(1 + C(n)\epsilon) \to 0$ as $\epsilon \downarrow 0$, we see that such a choice of $j \in \{1, 2, \dots\}$ exists provided $\epsilon \le \epsilon_0$, where $\epsilon_0 = \epsilon_0(n, \alpha)$. Then the above inequality gives

$$|\tau(x) - \tau(y)| \le 2|x - y|^\alpha, \quad x,\, y \in T_0 \text{ with } |x - y| < \tfrac{1}{2}.$$

Thus we can arrange for Hölder continuity with any exponent $\alpha < 1$. Similarly we have from the first inequality in (20) and (22) that

$$\begin{aligned}|x - y| &\le (1 + C\epsilon)^j |\tau_j(x) - \tau_j(y)| \\ &\le (1 + C\epsilon)^j \left(|\tau_j(x) - \tau(x)| + |\tau_j(y) - \tau(y)| + |\tau(x) - \tau(y)|\right) \\ &\le (1 + C(n)\epsilon)^j \left(C(n)\epsilon r_j + |\tau(x) - \tau(y)|\right)\end{aligned}$$

and j is again at our disposal. We in fact first choose ϵ such that $C(n)\epsilon \le 1$, so that

$$|x - y| \le (1 + C(n)\epsilon)^j \left(r_j + |\tau(x) - \tau(y)|\right),$$

and then choose j such that $\alpha \in (0,\,1)$

$$4^{-j} \le \frac{1}{2}|x - y| \text{ and } (1 + C(n)\epsilon)^j \le |x - y|^{-(\alpha/(1-\alpha)}.$$

Notice that this requires $j \ge \log(2/|x-y|)/\log\left(\frac{8}{1+C(n)\epsilon}\right)$ and $j \le \alpha^{-1}(1-\alpha)\log(1/|x-y|)/\log(1+C(n)\epsilon)$, and again certainly such a choice of j exists provided $0 < |x-y| < \frac{1}{2}$ and provided we take $\epsilon \le \epsilon_0$ for suitable $\epsilon_0 = \epsilon_0(n, \alpha)$. In this case the above inequality gives

$$\tfrac{1}{2}|x - y| \le |x - y|^{-\alpha/(1-\alpha)}|\tau(x) - \tau(y)|, \quad |x - y| < \tfrac{1}{2},$$

which of course gives

$$|x - y|^\alpha \le 2|\tau(x) - \tau(y)|, \quad |x - y| < \tfrac{1}{2}.$$

Thus τ is injective, and the inverse is Hölder continuous with exponent α, for any given $\alpha \in (0,\, 1)$, provided the ϵ-Reifenberg condition holds with $\epsilon \le \epsilon_0$, where $\epsilon_0 = \epsilon_0(n,\, \alpha)$.

Now the proof of the Reifenberg inequality is complete, because we have shown that τ maps T_0 Hölder continuously onto M with Hölder continuous inverse, and by (16) and (17) we have

$$M \cap B_1 = S_*,$$

because (by (19)) M_j converges to M with respect to the Hausdorff distance metric.