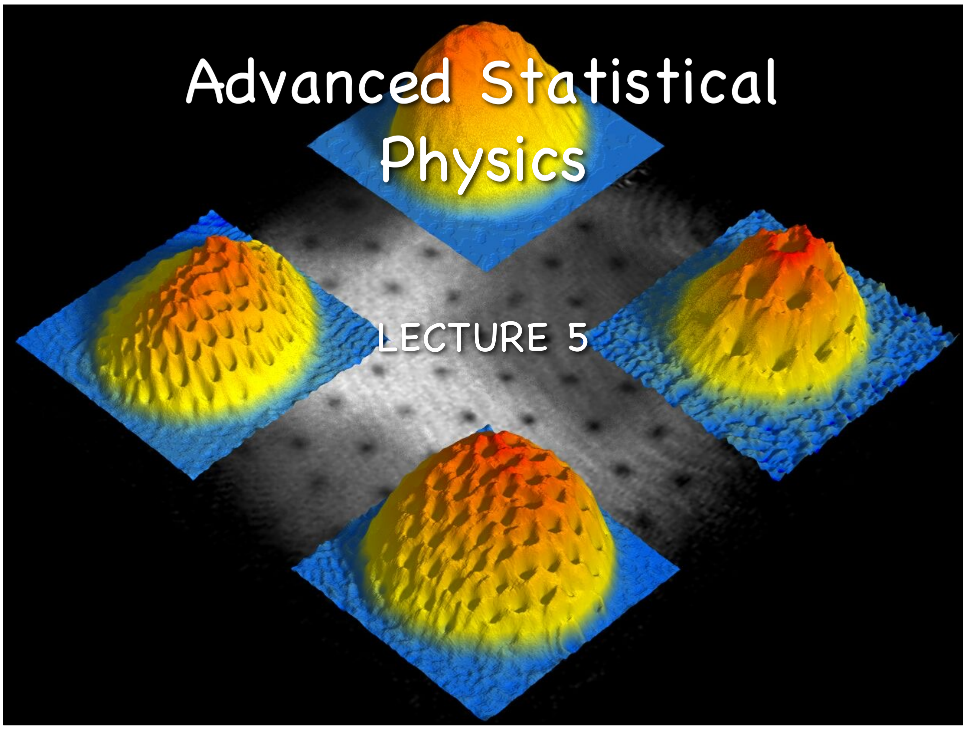


1

Outline

- Goldstone modes
- Mermin-Wagner theorem
- Topological defects, winding number
- Generalized rigidity
- Metastability
- Critical point: scale free phenomena & universality
- Universality classes

- Scope: to learn when a continuous symmetry can be broken and, when it happens, what kind of consequences derive for the dynamical behaviour of the new phase; to deepen the concept of universality classes and its connection with the critical point

2

2

in un solido se lo sposto con un dito il sistema tutto [illegible],

Broken symmetries

3

- Consider a system which below a critical temperature breaks the symmetry of the Hamiltonian. The Hamiltonian (microscopic model) in our macroscopic system still obeys the initial symmetry group, whatever it may have been, but the state of the matter involved does not at low enough temperatures
- Do any consequences flow from that? The answer is that some of the most important and interesting properties of matter follow from precisely this fact
- Spontaneously broken symmetry has three main consequences:

(a) it affects the spectrum of elementary excitations and fluctuations → per es. C_V nei solidi ≠ gas

per via dei fononi

(b) it causes new static/response properties ("generalized rigidity") solido rigido, acqua no

(c) it permits certain special soliton-like objects/excitations called "order-parameter defects"

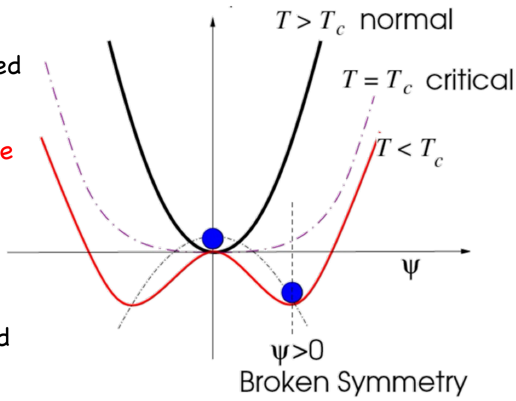
- Broken symmetry, however, is not the only way in which cold matter may change its behavior qualitatively. Although the cases in which the other alternative—a continuous phase transition to a qualitatively different behavior without change of symmetry—occurs are relatively rare, they are quite fascinating (topological order in quantum phases)

3

Broken symmetries & Landau theory

4

- The simplest state of broken symmetry is a uniform stationary solution that corresponds to a minimum of the potential but does not possess all symmetries of the free energy functional
- Suppose that $f(\Psi,T)$ is invariant to certain transformations G_i in the order parameter space that belong to a transformation group G, i.e. $f(G_i\Psi,T)=f(\Psi,T)$ when $G_i \in G$
- It follows that if Ψ_e is an equilibrium state the transformed state is also an equilibrium, and all equilibrium states related by symmetry transformations G_i are degenerate
- An almost trivial example is broken symmetry in 1D order parameter space. The order parameter Ψ can be represented then by a real number. $G=Z_2$



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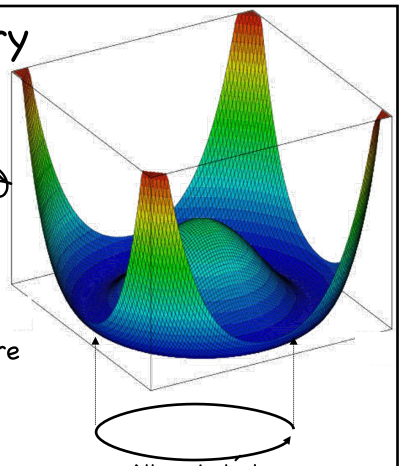
Continuous broken symmetry

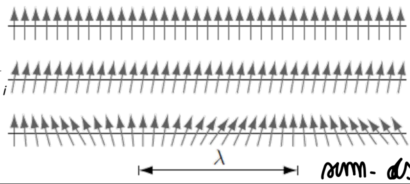
- When a continuous symmetry is spontaneously broken interesting consequences follow
- In this case, there is a non-countable infinity of degenerate equilibirum states. The system must choose one of these
- As a consequence of the degeneracy, there emerges a type of excited state in which the local ground state changes very gradually over space, so as to form a "wave" of very long wavelength



All equivalent equilibrium states

anche se 1D ricordo non lo trovo da fare

1D XY model example:



Ψ_0 — $f(\Psi_0, T) = f_0$ invariante sotto rotazione

variazione angolo $G_i \Psi_0 = \Psi_i$ — $f(\Psi_i, T) = f_0$

$\Psi(\vec{r})$ — $f(\Psi_i, T) \cong f_0$

λ

simm. discreta ora

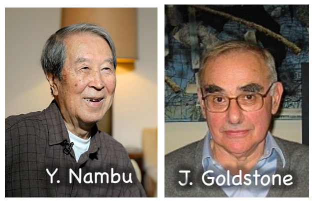
5

5

varia l'angolo su una scala λ (ovv $\lambda \to \infty$ $f \to f_0$)

quindi se rompo una simm. continua posso sempre avvicinarmi in modo continuo al g.s

- Such a state is orthogonal to any one ground/equilibrium state, with an energy approaching the ground/equilibrium state energy when the wavelength approaches infinity



i fononi sono una rottura della simm continua

- This is called a "Nambu-Goldstone excitation"
- In the first example, the broken symmetry is discrete, and there are no gapless excitations in the symmetry-broken phase. In other cases, like in magnets and crystal lattices, the broken symmetry, since it is continuous, leads to gapless excitations (for example: spin waves and phonons)
- It can be proved that the number of Goldstone modes is at most (dim G - dim H) if G is the symmetry group of the theory and H is subgroup of G which is left unbroken
- A crystal has three Goldstone modes, the longitudinal and the two transverse phonon modes, while G is the group of translations and rotations, dim G = 6, and H is a discrete subgroup, dim H = 0. For superfluid ^{4}He, as we shall see, G=U(1) which is completely broken: (dim G - dim H) = 1 longitudinal phonon mode

6

6

infatti in Ising 1D non avevo la soluzione $m = \pm m_0$ che avrebbe portato a rottura di simmetria

The Mermin-Wagner theorem

7

occhio a non confondere rottura simm. con transiz. di fase

- Let us now discuss the difficulties involved in breaking a symmetry in a low dimensional system. In QFT and statistical mechanics, the Mermin-Wagner theorem states that continuous (as in the XY and Heisenberg models) symmetries cannot be spontaneously broken at finite temperature in 1D and 2D theories (proved by Coleman in QFT and by Mermin, Wagner and Hohenberg in statistical mechanics)
- This is because if such a spontaneous symmetry breaking occurred, then the long-range fluctuations, corresponding to the Goldstone bosons, can be created with little energy cost and since they increase the entropy they are favored
- As the spatial dimensionality decreases, such fluctuations dominate and the stability of the low-temperature ordered state deteriorates.
- We will see, however, that the XY model nevertheless undergoes an unusual phase transition without an onset of long-range order in 2D, which is known as the Kosterlitz-Thouless transition.
- The dimensionality where long-range order disappears is known as lower critical dimension. The Ising model in 1D does not display long-range order at finite temperatures, in 2D it has an ordered phase below T_C.⇒ lower critical dimension for discrete broken symmetry = 1 (but beware: for short range interactions!)

7

XY model

- We will discuss the XY model on a (hyper)cubic lattice as a concrete example. The XY model has Hamiltonian:

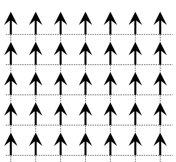
$$H = -J_{XY}\sum_{\langle i<j\rangle} \vec{s}_i\cdot\vec{s}_j - \sum_i \vec{h}\cdot\vec{s}_j \qquad \vec{s}_i = (\cos\vartheta_i ; \sin\vartheta_i) \quad \vartheta_i \in [0, 2\pi] \quad |\vec{s}_i| = 1 \ \forall i$$

→ primi vicini

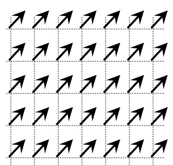
at each site of the lattice we put a classic spin that can freely rotate around a fixed direction and has no component along such direction.

- H has U(1) symmetry: invariant for the rotation of all the spins by the same angle

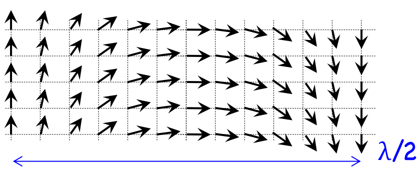
a T = 0



have the same energy



- Goldstone: continuous broken symmetry ⇒ existence of infinite ground states ⇒ excited states with very low energy (spin waves) for large wavelength λ



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Lower critical dimension, XY model

- This XY Hamiltonian can be rewritten in terms of the angles ϑ_i with $\mathbf{h}$=(h,0)

$$H = -J_{XY}\sum_{\langle i<j\rangle}\cos\vartheta_i\cos\vartheta_j + \sin\vartheta_i\sin\vartheta_j - \sum_i h\cos\vartheta_i =$$

$$= -J_{XY}\sum_{\langle i<j\rangle}\cos\left(\vartheta_i - \vartheta_j\right) - \sum_i h\cos\vartheta_i$$

- Suppose that the temperature is very low and neighboring spins are aligned almost parallel to each other, then, the argument $(\vartheta_i - \vartheta_j)$ of the cosine is very small compared to π, and it would be a good approximation to expand the cosine to second order
- Assuming h=0, we construct an effective Hamiltonian, valid at low temperatures

$$H = -J_{XY}\sum_{\langle i<j\rangle}\cos\left(\vartheta_i - \vartheta_j\right) \cong -J_{XY}\sum_{\langle i<j\rangle}\left[1 - \tfrac{1}{2}\left(\vartheta_i - \vartheta_j\right)^2\right] = E_0 + \frac{J_{XY}}{2}\sum_{\langle i<j\rangle}\left(\vartheta_i - \vartheta_j\right)^2$$

from now on we will drop the ground state energy E_0 that does not play any significant role in our discussion

- Since we are interested in the behavior of the system over a large spatial scale, we are allowed and going to ignore discreteness of the spatial coordinates of the lattice

$\theta_i \mapsto \theta(r)$

9

9

Spin-wave approximation

- Thus, consider a continuous function $\vartheta(\mathbf{r})$, which would represent the analogue of ϑ_i in the continuous limit: $i \to r_i$; we can write

$$H \approx \frac{J_{XY}}{2}\sum_{\langle i<j\rangle}\left(\vartheta_i - \vartheta_j\right)^2 = \frac{J_{XY}}{2}\sum_{\vec r}\left[\vartheta(\vec r + \Delta\vec x) - \vartheta(\vec r)\right]^2 + \left[\vartheta(\vec r + \Delta\vec y) - \vartheta(\vec r)\right]^2 =$$

$$\approx \frac{J_{XY}}{2}\sum_{\vec r}\Delta x^2\left[\frac{\Delta_x\vartheta(\vec r)}{\Delta x}\right]^2 + \Delta y^2\left[\frac{\Delta_y\vartheta(\vec r)}{\Delta y}\right]^2 \underset{|\Delta\vec x| = |\Delta\vec y|}{=} \frac{J_{XY}}{2}\sum_{\vec r}\Delta x\Delta y\left\{\left[\frac{\Delta_x\vartheta(\vec r)}{\Delta x}\right]^2 + \left[\frac{\Delta_y\vartheta(\vec r)}{\Delta y}\right]^2\right\}$$

tanto $\Delta x = \Delta y$

- Replacing the finite differences by derivatives and the sum over lattice sites by an integral, we obtain the expression

$$\Rightarrow H_{SW} = \frac{J}{2}\int d\vec r\left[\nabla\vartheta(\vec r)\right]^2$$

- This is the spin-wave approximation around an assumed ordered state, a quadratic Hamiltonian, which is expected to be valid at low temperatures.

↳ possibile perchè a T piccole θ_i e θ_j vicini
poco per termini vicini

10

10

- Using the Divergence Theorem:

$$\int_V d\vec{r}\, \vec{\nabla} \cdot \vec{F}(\vec{r}) = \int_S \vec{F}(\vec{r}) \cdot d\vec{S}$$

with $\vec{F}(\vec{r}) = \vartheta(\vec{r})\vec{\nabla}\vartheta(\vec{r})$,
we obtain (note that we can freely choose S)

$$\int_V d\vec{r} \left[\vec{\nabla}\vartheta(\vec{r})\right]^2 + \int_V d\vec{r}\, \vartheta(\vec{r})\nabla^2\vartheta(\vec{r}) = \int_S \vartheta(\vec{r})\vec{\nabla}\vartheta(\vec{r}) \cdot d\vec{S}$$

- By taking S such that $\vec{\nabla}\vartheta(\vec{r}) \cdot d\vec{S} = 0$,

  we obtain an equivalent expression for the Hamiltonian in the spin-wave approximation:

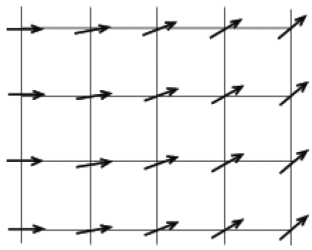
$$\Rightarrow \boxed{H_{sw} = -\frac{J}{2}\int d\vec{r}\, \vartheta(\vec{r})\nabla^2\vartheta(\vec{r})}$$

- To understand what type of spin configurations are stable under the spin wave approximation, we adopt the variational principle with respect to a local angle variable (or a scalar field) $\vartheta(\mathbf{r})$, $\delta H/\delta\vartheta(\mathbf{r})=0$

- The functional variation of the Hamiltonian yields the following Laplace equation

$$\nabla^2\vartheta(\vec{r}) = 0$$

- We solve this Laplace equation under the boundary condition that the left boundary (x=0) has $\vartheta(x=0,y)=0$ and the right (x=L) has $\vartheta(x=L,y)=\vartheta_0$, along the chosen x direction (note that with $\vartheta_0 \neq 0$ it is not a Goldstone mode).



- The solution is the uniformly rotated state, $\vartheta(x,y)=x\vartheta_0/L$, as depicted in figure

- By inserting $\nabla\vartheta=(\vartheta_0/L, 0, 0, \ldots)$ into the Hamiltonian, the energy of this configuration has the value

$$E = \frac{J}{2}\int d\vec{r} \left[\nabla\vartheta(\vec{r})\right]^2 = \frac{J}{2}\left(\frac{\vartheta_0}{L}\right)^2 L^d = \frac{J}{2}\vartheta_0^2 L^{d-2}$$

- This result shows that energy increases indefinitely as $L \rightarrow \infty$ for $d>2$.

- A very large energy is needed to twist both sides of the system by a finite angle, suggesting that the system is robust against the change of boundary conditions

per $d>2$
ruotare l'angolo
in modo uniforme
su una scala grande
costo energia
infinito
↳ sistema rigido

- The system is thus considered to have rigid long-range order (d >2).
- On the contrary, if d<2, the twist energy $E = \frac{J}{2}\vartheta_0^2 L^{d-2}$

does not increase with system size and the effects of the boundaries do not propagate deep into the system. Hence no long-range order exists.

- This simple argument therefore illustrates that the stability of the long-range order in the XY model changes when the dimension of the system passes through d=2. The case d=2 is marginal and needs more careful scrutiny.
- Considering qualitatively the effects of temperature, the energy cost for a fluctuation of linear size L we will have $|\nabla\Psi|^2 \propto 1/L^2$ and thus

$$\int_{L^d} d^d\vec{r}\ \tfrac{1}{2}\left|\nabla\Psi(\vec{r})\right|^2 \approx L^{d-2}$$

hence, $P_{fluc} \propto \exp\left[-\beta(const.)L^{d-2}\right]$

- We conclude that a continuous symmetry can be broken for T > 0 in d > 2, but not for d=2.
- For T > 0, the lower critical dimension is 2 for continuous symmetries and 1 for discrete symmetries
- These qualitative considerations can be made rigorous and have important consequences, for example, for magnetization in a 2D XY model and for superfluidity in 2D

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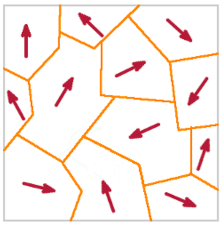
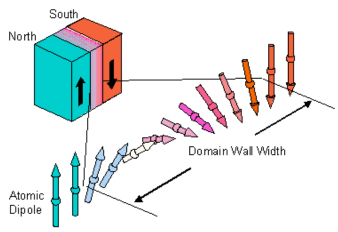
Order parameter heterogeneities

- Although the energy of a macroscopic phase is minimized when the symmetry is broken uniformly throughout the system, it turns out that the symmetry may be broken differently in different parts of the sample due to a variety of reasons
- Under those circumstances, defects will appear, for instance, in the boundary separating those spatial regions characterized by states or configurations with different values of the order parameter.
- Thus, another consequence of the spontaneous breaking of a symmetry is the emergence of defect structures such as vortices in superfluids, domain walls in ferromagnets, dislocations in crystals, or disclinations in nematic liquid crystals
- These defects are responsible for determining important mechanical/response properties connected with the "generalized rigidity" of the macroscopic phase





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Rigidity

- When you kick a stone, no doubt remains in your mind that it possesses a property that we call rigidity, which we recognize as the very essence of broken translational invariance: translational symmetry is broken everywhere within it, in that the individual atoms all occupy fixed positions
- We are so accustomed to the rigidity of solid bodies that it is hard to realize that such action at a distance is not built into the laws of nature (except in the case of the long-range forces such as gravity or electrostatics)
- It is strictly a consequence of the fact that free energy is minimized when symmetry is broken in the same way throughout the sample: the orientation and position of the lattice is the same everywhere. Of course, in general they are not quite the same, since the lattice can deform elastically: there is the elastic energy that is proportional to $\nabla\cdot\mathbf{u}$ ($\mathbf{u}$ being the lattice displacement), which will allow the lattice to deform slightly under a force. But nonetheless the lattice transmits that force from one end to the other

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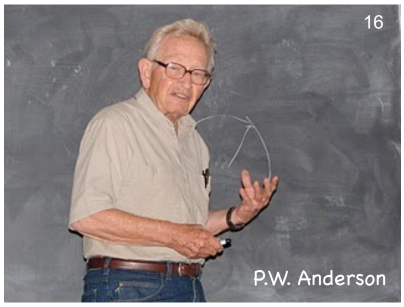
"Generalized rigidity"



- So, another consequence of broken symmetry concept is the notion of "generalized rigidity"
- P.W. Anderson noted that we can generalize this concept to a wide variety of broken symmetries, each with their own type of rigidity/superflow

Table of some broken-symmetry phenomena in condensed matter physics:

High phase	Low phase	Order parameter	Broken symmetry	Goldstone or Higgs Boson	Rigidity / superflow	Singularities or defects
liquid	solid	displacement vectors	Translational and rotational	longitudinal and transverse phonons	sheer stress	Dislocations Grain boundaries Vacancies Interstitials
liquid ^{4}He	He II	$\langle\psi\rangle$	Gauge U(1)	phonon	superfluidity	Vortex lines
normal metal	superconductor	$\langle\psi\uparrow\psi\downarrow\rangle$	Gauge U(1)	Maissner effect	super-conductivity	Flux lines
paramagnet	ferromagnet	**M**	Rotational SO(3)	Spin waves	permanent magnetism	Domain walls

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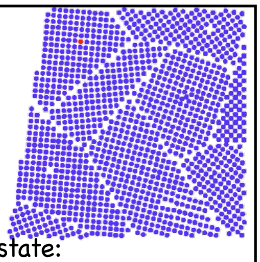
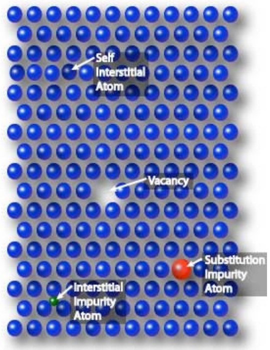
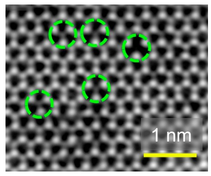
Crystal Defects



- A region in a solid, where the microscopic arrangement of atoms differs drastically from that of a perfect crystal is called a defect.
- Defects are particularly important in the dynamics of a phase transition since they are readily formed at the initial stages of evolution towards an ordered state: defects emerge due to incompatibility of local ordering
- Cooling rapidly through a phase transition can leave behind the so-called topological defects, i.e. defects that cannot be fixed by any local rearrangement
- The two most important kinds of point defects are vacancies and interstitials.
- If a vacancy occurs then a regular lattice position remains empty. In contrast to that, if an interstitial is present, an extra atom is inserted in the lattice





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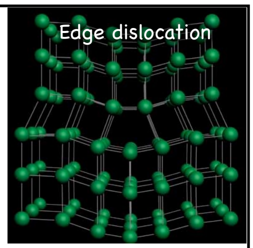
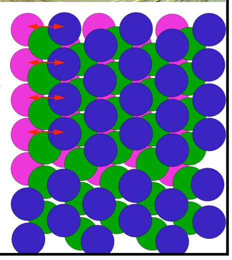
Extended solid defects

- Some examples of extended defects are dislocations (line defects in 3D but point defects in 2D), grain boundaries (planar defects in 3D but line defects in 2D) and stacking faults.
- An edge dislocation can be depicted by cutting into the crystal and inserting an extra lattice plane into the crystal.
- A screw dislocation is produced by cutting into the crystal and displacing the two planes against each other by one lattice spacing.
- Stacking faults occur in a number of crystal structures, but the common example is in close-packed structures. They are formed by a local deviation of the stacking sequence of layers in a crystal.



FCC
ABC

HCP
ABA



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Topological defects & order parameter

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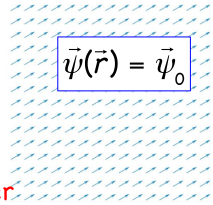
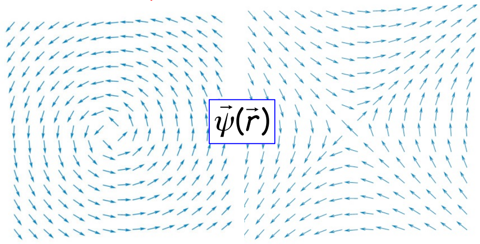
- Many physical systems with broken continuous symmetry can sustain topological defects.
- These typically can be described by the variation of some elastic variable in some region of space. We already have this macroscopic variable, it is the order parameter.
- Let's consider the XY model and its order parameter ψ which represent the magnetization.
- When topological defects aren't present the order parameter is homogeneous



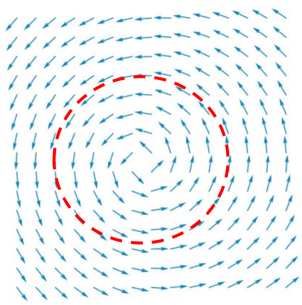
- When topological defects are present the order parameter is no more homogeneous; but a topological defect is more than a nonhomogeneous order parameter
- Both these configurations are "smooth": each spin is pointing in nearly the same direction as its neighbors. However, there seems to be a single discontinuity at the center of each configuration, a point defect in this case



19 nel centro ho una singolarità

20

- To demonstrate, let's draw a loop around the first point defect.
- As we cross each arrow with our loop, take note of the direction of that arrow. You'll note that the arrows change direction very smoothly.
- Once you get back to the beginning of the loop, the direction of the arrow should be the same as where you started. However, in the meantime, the direction of the arrow has gone around completely in a circle

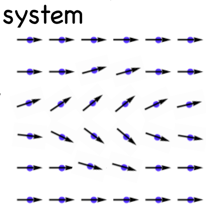


- There's no way to deform smoothly and locally the spins in a way they don't go around in a circle: ↳ riesco solo globalmente

❖ Trying with a local deformation you will simply displace the discontinuity

❖ Trying with a smooth deformation you will end with no defect only when your deformation affects the whole configuration of the system

- On the contrary, it is very easy to imagine a smooth and local deformation of ψ obtained from the homogeneously ordered (ground state) configuration … this is not a topological defect



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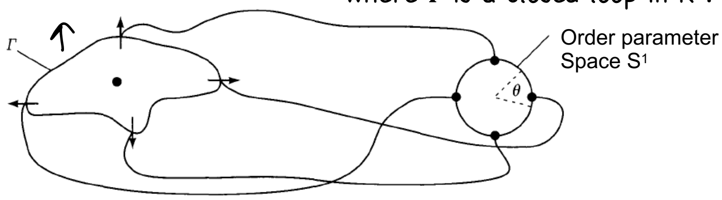
Defects & the order parameter space

- In the following we will show how concepts borrowed from topology provide the necessary tools to characterize, classify, and combine elementary defects.
- Let us start from the concept of the order-parameter space we already encountered in the last lecture. The order-parameter space is the set of all possible values of the order parameter
- A simple example is the XY model in 2D. Suppose that low-temperature spin configurations are mapped to a set of points in the space S^1 (the unit circle) by the rule

$$\frac{\vec{\Psi}(\vec{r})}{\left|\vec{\Psi}(\vec{r})\right|} = \left(\cos\vartheta(\vec{r}); \sin\vartheta(\vec{r})\right) \in S^1 \qquad \forall \vec{r} \in \Gamma \in R^2$$

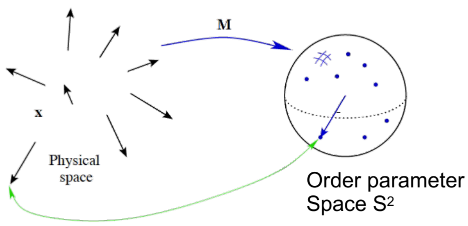
spazio fisico

where $\mathbf{\Gamma}$ is a closed loop in R^2.

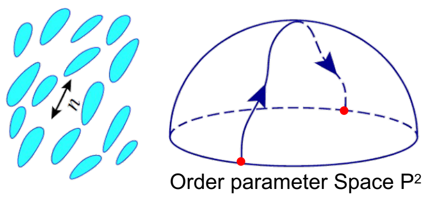


- In this example, the arrow may point in any direction in a 360 degree circle. Therefore, the order parameter space is a circle.

- Another example is the Heisenberg model, in which the spin orientation at $\mathbf{r} \in R^d$ is specified by a three-dimensional unit vector.
- The order-parameter space is then S^2, the surface of the unit sphere.
- Notice that the magnitude of the local variable/parameter is ignored and only its direction is considered in the analysis of topological defects.

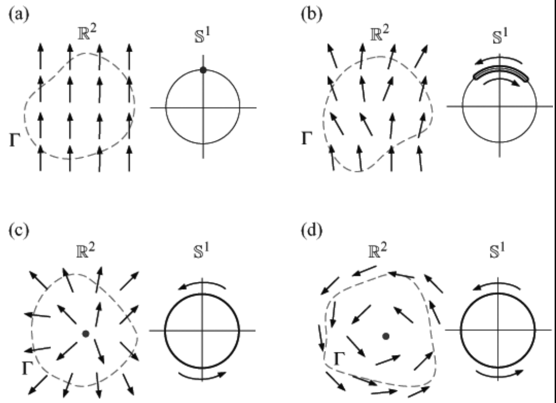


- The last example corresponds to a nematic liquid crystal in d=3, in which rod-like molecules are oriented as in the Heisenberg model but without the direction of the arrows



- Thus, the up and down orientations are identified, and consequently only the upper half of a sphere constitutes the order parameter space known as P^2, a special hemisphere: the two points marked in red dots are identical and the curve drawn around the half sphere is a closed loop.

- One of the reasons to introduce the order-parameter space is its advantage in the classification of topological defects and their stabilities.



- Consider again the XY model in 2D. As shown in figure, the existence of a vortex, a typical topological defect, significantly influences the image in the order-parameter space.
- In panels (a) and (b), there is no vortex surrounded by the loop $\boldsymbol{\Gamma}$, and the images in S^1, drawn bold, are essentially the same
- in the sense that we can continuously deform the image in (b) into a single point as in (a).
- In contrast, the images in (c) and (d) are equivalent to each other but cannot be continuously reduced to a point as in (a) since those in (c) and (d) wind the circumference of S^1

quando ho difetti topologici su una linea chiuso percorro tutto lo spazio del parametro d'ordine

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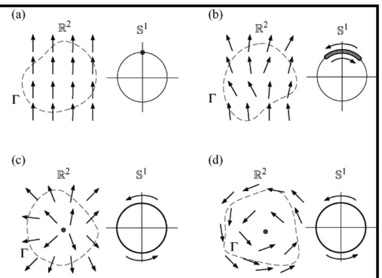
non ho modo di andare da $a/b \mapsto c/d$

The winding number



- We say in the latter case that the winding number is W=1.
- Also, in the real space R^2, we can continuously change the spin configuration of (b) into (a) and also (c) changes into (d), the latter by rotating each spin roughly by 90° .
- Such a continuous rotation never succeeds in changing (a) to (c)
- The two configurations (a) and (b) are said to belong to the same homotopy class. Similarly the homotopy class of (c) and (d) is the same.
- In general, two configurations are equivalent and belong to the same homotopy class if one of them can be deforme continuously to the other.
- The winding number, which is formally defined as

$$W = \frac{1}{2\pi}\oint_\Gamma \vec{\nabla}\vartheta(\vec{r})\cdot d\vec{r}$$ Nobile analisi III

INSIGHT: lo quadratico potrebbe nascere fuori dalla topologia

quantitatively characterizes the homotopy class and is an example of a topological invariant. The latter name comes from the (topological) stability of the winding number under continuous deformation. Here we refer to the topological stability, not the thermodynamics stability. Nevertheless, the former may leads to the latter (e.g. Kosterlitz-Thouless transition)

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i difetti topologici hanno massa

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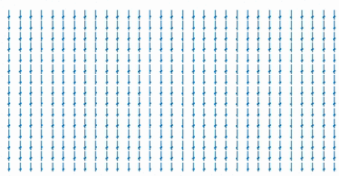
- Note that it is possible to create a vortex in the XY model with winding numbers other than 0 (no vortex) or 1 by the field configuration:

$$\vartheta(\vec{r}) = k\theta + \varphi_0$$

where θ is the polar angle of the position vector **r**, φ_0 a constant and k is an integer, k∈Z. In fact:

$$W = \frac{1}{2\pi}\oint_\Gamma \left[\frac{\partial}{\partial r};\frac{1}{r}\frac{\partial}{\partial \theta}\right]\left(k\theta+\varphi_0\right)\cdot r\left[dr;d\theta\right] = \frac{1}{2\pi}\oint_\Gamma \left[0;\frac{k}{r}\right]\cdot r\left[dr;d\theta\right] = \frac{k}{2\pi}\oint_\Gamma d\theta = k$$

- The image of spin configurations in S^1 winds the circumference k times in this case.
- The addition (or the aggregation) of two vortices of winding numbers W_1 and W_2 realizes a single vortex with winding number $W=W_1+W_2$.
- One can have a defect which corresponds to going around the circle clockwise and the other corresponds to going around the circle anti-clockwise. If these two point defects meet, they cancel each other out: they behave like particles and antiparticles, annihilating when they encounter each other.



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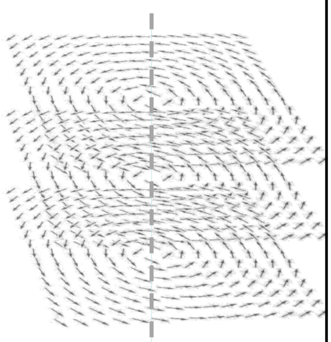
Vortex lines

- These discussions can be generalized to arbitrary types of topological defects, not just in spin systems but also in solids and liquid crystals.
- We, however, restrict ourselves to spin systems for simplicity of presentation and consider next the XY model in three spatial dimensions R^3.
- The basic topological defect is a line of vortices created by stacking vortices, each on a two-dimensional cross-section.
- In the case of R^3, the order parameter space is again S^1 because the orientation of a spin can be specified by a point on S^1.
- The homotopy group is also the same, each element being specified by the winding number counted along a loop surrounding a vortex line.



- Now let's consider the Heisenberg model where each spin may point in any direction in 3D

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Defects in other spin models

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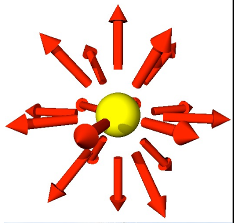
- As already mentioned, the order parameter space for the Heisenberg model is not a circle, but the surface of a sphere
- That makes a big difference, because if you draw a loop around the sphere, now it is possible to continuously deform the path into a single point. (In mathematical terms, every loop is "homotopic" to a point).



- The basic topological defect is a hedgehog structure, and in its most basic form, all spins point outward on the surface of S^2.
- The topological invariant is the wrapping number. The simplest hedgehog has the wrapping number 1.

- In the case of an ordered medium with a discrete symmetry group, such as the Z_2 symmetry of the Ising model, topological defects have the dimensionality d-1.
- They are always generalized domain walls separating regions with different values of the order parameter.

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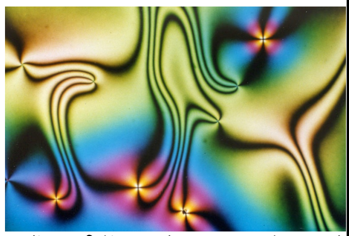
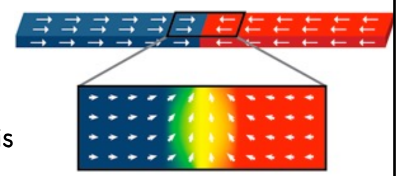
Topological defects "mass"

- Topological defects of the order parameter costs an amount of energy proportional to their extension; i.e. they are gapped (i.e they have "mass") soliton/localized-like excitations
- When the broken symmetry is continuous, a discontinuity in the order parameter would make the distortion energy in free energy density infinite (unless ψ=0 at the discontinuity); therefore the real order parameter in the above example must vary smoothly
- Any deviation from the equilibrium value incurs energetic penalty, as it rises the value of the potential and therefore has to be restricted to a narrow interval. This is the defect core; its size is determined by a balance between potential and distortion energy

$$f\left(\Psi,T\right) = \left[\tfrac{1}{2}\left|\vec{\nabla}\Psi(\vec{r})\right|^2 + V\left(\Psi(\vec{r})\right)\right]$$



Bending of the order parameter and topological defects in liquid crystals



Bending of the order parameter neara domain wall in a ferromagnet

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Metastability

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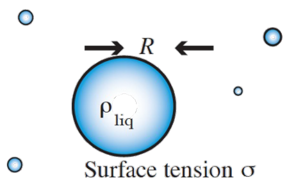
- In a first order phase transition the system can easily be found out of equilibrium, in a metastable state; for example:

1. Different substances when quickly cooled down from the liquid phase do not became crystalline solids (stable phase) but they form glasses (metastable phase) which are disordered "solid" states even at atomic level and where the slowing down of diffusive motion prevents the reaching of the equilibrium state for thousand of years

2. Supersaturated vapour: we consider the condensation of a gas; by compressing a gas at constant temperature we reach the coexistence curve and see that liquid drops are not readily present in the system. The gas remains stable for long periods even inside the coexistence curve. This is because gas molecules have to condense to form a very small liquid drop, but if the drop radius is very small the energetic cost of the surface is bigger than the energetic gain coming from the volume of the liquid phase: there is a free energy barrier to reach nucleation



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Critical point: loss of a scale parameter

- The phase transitions we're most used to are water boiling to steam or freezing to ice

- Water is, symmetrically, very different from ice. So to go from one to the other you need to start building an interface and then slowly grow your new phase (crystal growth). This is a first order phase transition and it's the only way to make ice

- Water and steam are, symmetrically, the same. At most pressures the transition still goes the same way - build an interface and grow. However, if one increases the pressure enough there comes a special point (the critical point) where the distinction between the two phases becomes a bit indistinct. The cost of building an interface goes to zero so there's no need to grow anything. You just smoothly change between the two. This is a continuous phase transition

- One of the consequences of criticality is a loss of a length scale parameter. This is why, for instance, a critical fluid looks cloudy (critical opalescence). Light is being scattered by structure at every scale

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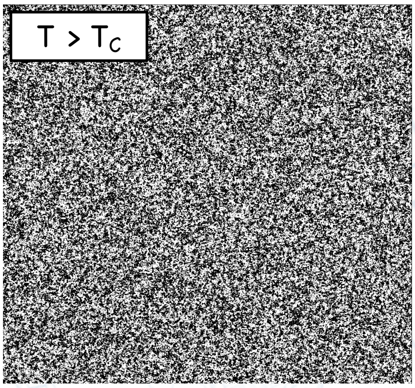
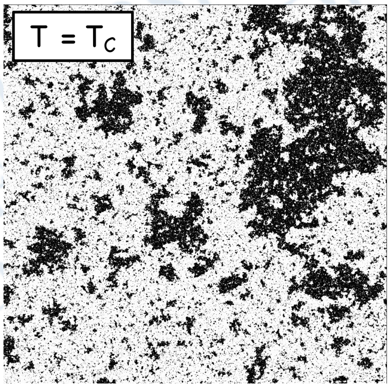
Scale-free phenomena

Clustering of spins in a 2D Ising model at $T>T_C$ and at $T=T_C$



Above T_C we see clusters of some characteristic size determining the scale of the cluster size distribution



At T_C we see clusters of all sizes giving a scale-free cluster size distribution apart from effects from the finite lattice size

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Scale free phenomena

Clustering of spins in a 2D Ising model at $T=T_C$

The following pictures show a critical Ising model at length scales ranging from $L = 2^{17}$ to $L = 2^{11}$

Can you tell which is which?

At T_C we see clusters of all sizes giving a scale-free cluster size distribution: this is the meaning of a diverging correlation length; this is the origin of universality …

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Universality

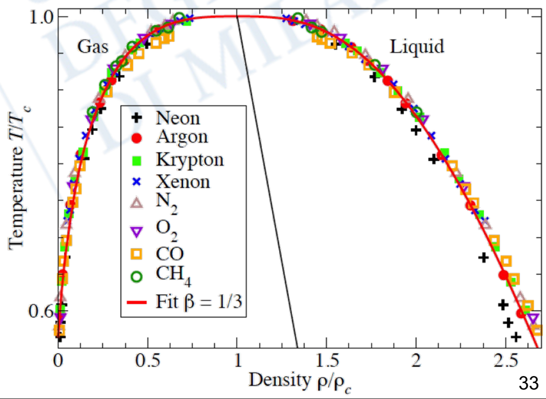
- The behavior near continuous transitions is unusually independent of the microscopic details of the system, so much so that we give a new name to it, universality

- This figure shows that the liquid-gas coexistence lines for a variety of atoms and small molecules appear quite similar near their critical points (T_C, ρ_C) when rescaled to the same critical density and temperature.
- The curve is a fit to the Argon data, with the critical exponent β=1/3

- Hence argon and carbon monoxide satisfy

$$\rho^{CO}(T) = A\rho^{Ar}(BT)$$

- for some overall changes of scale A, B.



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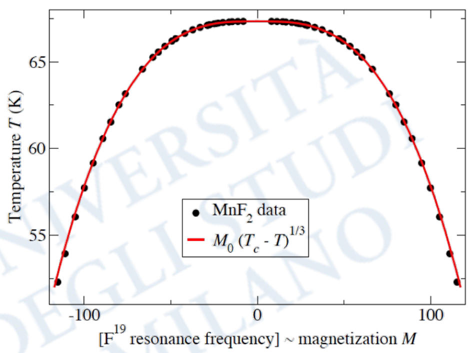
- This figure shows a completely different physical system: interacting electronic spins in manganese fluoride, going through a ferromagnetic transition

- This magnet and the previous liquid-gas transitions all behave the same at their critical points: the jumps in magnetization and density near T_C both vary as $(T_C-T)^\beta$ with the same exponent $\beta\approx0.325$ [β=0.325±0.005]



- Also, there are many other properties (susceptibility, specific heat, correlation lengths) which have power-law singularities at the critical point, and all of the exponents of these power laws for the liquid-gas systems agree with the corresponding exponents for the magnets. This is universality

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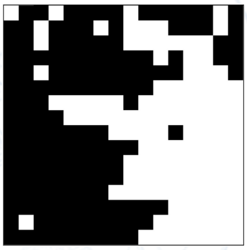
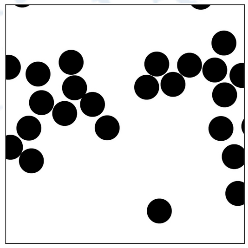
- Thus the second feature of critical phenomena is universality. Close to the critical point it turns out that the physics of a system doesn't depend on the exact details of what the microscopic degrees of freedom are doing, but only on broad characteristics such as dimension, symmetry or whether the interaction is long or short ranged. Two systems that share these properties are in the same universality class and will behave identically around the critical point

- To get a more clear feeling about how universality arises, consider this two pictures which show two systems at criticality



Ising model for a magnet: each site can be up or down and neighbouring sites like to line up. The two phases at the critical point are the opposite magnetisations represented here by black and white squares



Lennard-Jones fluid: atoms are attracted to one another at close enough range but a strong repulsion prevents overlap. The two phases in this case are a dense liquid and a sparse gas

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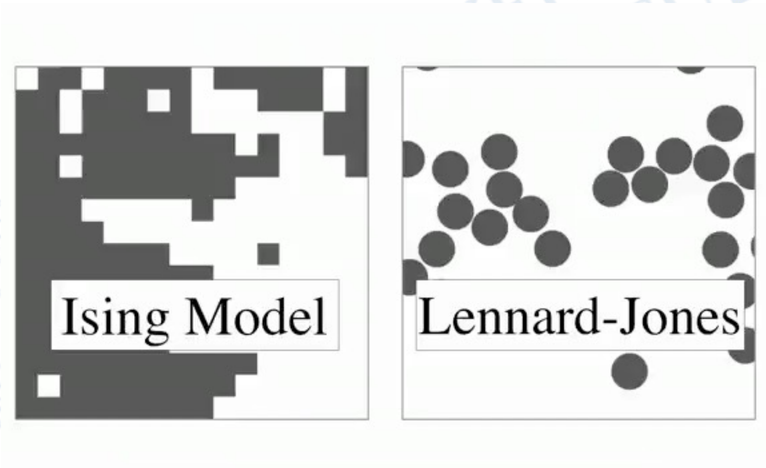
Universality at the critical point

- At the critical point, by looking directly to the microscopic degrees of freedoms it is possible to see the differences in the two models, but…



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Beyond Mean-Field approximation

- We find that, while the observed values of an exponent differ very little as one goes from system to system within a given category (or even from category to category), these values are considerably different from the ones following from the mean field approximation: MFA leads to sufficiently accurate results for most of the thermostatic properties of these systems away from critical points, most often it fails miserably close to the critical points (wrong critical exponents)

- Clearly, we need a theory of phase transitions which is basically different from the mean field theory

- To begin with, some questions arise:

(i) Are these critical exponents completely independent of one another or are they mutually related? In the latter case, how many of them are truly independent?

(ii) On what characteristics of the given system do they depend? This includes the question why, for systems differing so much from one another, they differ so little

(iii) How can they be evaluated from first principles?

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- The answer to question (i) is YES: the various exponents do obey certain relations which turn up as equalities, and the number of these (restrictive) relations is such that only two of the exponents are truly independent

- As regards question (ii), it turns out that our exponents depend upon a very small number of characteristics, or parameters, of the problem, which explains why they differ so little from one system to another in a given category of systems

- The characteristics that seem to matter are:

(a) the dimensionality, d, of the space in which the system is embedded,

(b) the number of components, n, of the order parameter of the problem, and

(c) the range of microscopic interactions in the system

(d) The presence of disorder

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Universality hypothesis

Thus in absence of disorder and for short-range interactions, systems with the same values of the dimensionality of space, d, and of the order parameter, n, are members of the same universality class and share the same critical exponents

UNIVERSALITY CLASS		THEORETICAL MODEL	PHYSICAL SYSTEM	ORDER PARAMETER
$d = 2$	$n = 1$	Ising model in two dimensions	Adsorbed films	Surface density
	$n = 2$	XY model in two dimensions	Helium-4 films	Amplitude of superfluid phase
	$n = 3$	Heisenberg model in two dimensions		Magnetization
$d = 3$	$n = 1$	Ising model in three dimensions	Uniaxial ferromagnet	Magnetization
			Fluid near a critical point	Density difference between phases
			Mixture of liquids near consolute point	Concentration difference
			Alloy near order-disorder transition	Concentration difference
	$n = 2$	XY model in three dimensions	Planar ferromagnet	Magnetization
			Helium 4 near superfluid transition	Amplitude of superfluid phase
	$n = 3$	Heisenberg model in three dimensions	Isotropic ferromagnet	Magnetization

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Lecture 5: Suggested books

- K. Huang, "Statistical Mechanics", II edition, Wiley
- H. Nishimori and G. Ortiz, "Elements of Phase Transitions and Critical Phenomena", Oxford
- P.M. Chaikin and T.C. Lubensky, "Principles of condensed matter physics", Cambridge
- P.W. Anderson, "Basic notions in condensed matter theories", Benjamin

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