Simple proof for Frequent Directions

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Abstract

This paper provides a short proof of the main results in [1] and [2].

Proof of the main results

The main insight needed for producing the simplified proof is changing the item being sketched. We will consider sketching the *covariance* of a stream of matrices rather than the stream itself. The results turns out to be identical.

Let $X_t \in \mathbb{R}^{d \times n_t}$ be a stream of matrices. Let $C = \sum_{t=1}^{T} X_t X_t^T \in \mathbb{R}^{d \times d}$ be their covariance matrix. Frequent Directions [1] maintains a rank deficient approximate covariance matrix $\tilde{C}_t \in \mathbb{R}^{d \times d}$ using Algorithm 1. Set $\tilde{C}_0$ to be the all zeros matrix and compute $\tilde{C}_t = \textsc{Update}(\tilde{C}_{t-1}, X, \ell)$.

Algorithm 1 Frequent Directions Update

```
function UPDATE(C̃, X, ℓ)
    UΛU^T = C̃ + XX^T
    return U · max(Λ − I · λ_ℓ, 0) · U^T
end function
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Note that the rank of $\tilde{C}_t$ is at most $\ell - 1$ for all t by construction. It can therefore be stored in $O(d\ell)$ space. Assuming $n_t < \ell$, the update operation itself also consumes at most $O(d\ell)$ space.

Define $\Delta_t = X_t X_t^T - \tilde{C}_t + \tilde{C}_{t-1}$. Then $\sum_{t=1}^{T} \Delta_t = \sum_{t=1}^{T} X_t X_t^T - \sum_{t=1}^{T}(\tilde{C}_t - \tilde{C}_{t-1}) = C - \tilde{C}$ where $\tilde{C}$ stands for $\tilde{C}_T$, the final sketch.

Moreover, note that the top ℓ eigenvalues of Δ_t are all equal to one another becuase $\Delta_t = U_t \cdot \min(\Lambda_t, I \cdot \lambda_\ell^t) \cdot U_t^T$. As a result $\|\Delta_t\| < \frac{1}{\ell - k} \operatorname{tr}(\bar{P}_k \Delta_t \bar{P}_k)$ for any projection $\bar{P}_k$ having a null space of dimension at most k. Specifically, this holds for $\bar{P}_k$ whose null space contains the eigenvectors of C corresponding to its largest eigenvalues.

$$
\begin{aligned}
\|C - \tilde{C}\| &= \|\sum_{t=1}^{T} \Delta_t\| \leq \sum_{t=1}^{T} \|\Delta_t\| \\
&\leq \frac{1}{\ell - k} \operatorname{tr}\left(\bar{P}_k \left(\sum_{t=1}^{T} \Delta_t\right) \bar{P}_k\right) \\
&\leq \frac{1}{\ell - k} \operatorname{tr}\left(\bar{P}_k C \bar{P}_k\right) = \frac{1}{\ell - k} \sum_{i=k+1}^{d} \lambda_i
\end{aligned}
$$

Here we used that $\operatorname{tr}(\tilde{C_T}) \geq 0$ because $\tilde{C_T}$ is positive semidefinite. This completes the proof of the main claim in [2]. Setting $k = 0$ gives the result in [1].

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References

[1] Edo Liberty. Simple and deterministic matrix sketching. In *The 19th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, KDD 2013, Chicago, IL, USA, August 11-14, 2013*, pages 581–588. ACM, 2013.

[2] Mina Ghashami and Jeff M. Phillips. Relative errors for deterministic low-rank matrix approximations. In Chandra Chekuri, editor, *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014*, pages 707–717. SIAM, 2014.