Real-Space Renormalization Group method

1 Abstract

Our goal is to know very large quantum system's ground state described by Ising model applied to chain of particles with spin $\frac{1}{2}$. In order to achieve this aim we use Real-Space Renormalization Group(RSRG) method and we compare the results with mean field theory's prediction.

2 Theory

A another typical approach when studying a many-body quantum system is to use RSRG method. This approach consists by 4 steps:

- Start with an Hamiltonian H_0 that can be exactly numerical computed.
- Double the system : we pass from N $\rightarrow$ 2N simply taking in consideration other N particles described by the same Hamiltonian. In order to be clear : H_A will be the hamiltonian related to first half of the new system and H_B will be the hamiltonian related to second half.
  Both of them will be matrices with size (d^N, d^N) where d is the number of single particle's physical states (in this case d=2 because we are considering particles with spin $\frac{1}{2}$).Then we compute the new H_{2N} as :

$$H_{2N} = H_A \otimes \mathbb{1}_N + \mathbb{1}_N \otimes H_B + A \otimes B \tag{1}$$

  where A=$\mathbb{1}_{N-1} \otimes \sigma_x$ and B=$\sigma_x \otimes \mathbb{1}_{N-1}$ and tensor product H^{AB} shows the *interation* term between two spin neighbours (the last particle of the first N particle block and the first one of the second block) .
- Now we extract first eigenvalue and the eigenvectors matrix from H_{2N}. Then we project H_{2N} into a more restricted space using only d^N eigenvectors columns that form projection matrix P : in such way we pass from $H_{2N} \rightarrow H_{2N}^{trunc}$.

In particular we have

$$H_{2N}^{trunc} = P^\dagger \cdot H_{2N} \cdot P \tag{2}$$

and

$$H_{next\ step}^{AB} = P^\dagger \left[A \otimes \mathbb{I}^N\right] \cdot P \otimes P^\dagger \cdot \left[\mathbb{I}^N \otimes B\right] \cdot P \tag{3}$$

- We iterate previous two points and ,iteration by iteration, we are keeping constant initial dimensions (and so computational resources) but ,at the same time, we facing bigger and bigger system (exponential growth). Finally when we are satisfied by final ground state energy (or it remains almost constant) we can stop the procedure.

So looking at thermodynamic limit ($N \to \infty$),we recall also the $mean field$ approximation and as consequence of it the *ground state*'s density energy $e = \frac{E_{groundstate}^{MF}}{N}$ will be function of λ and it will follow this behaviour:

$$e(\lambda) = \begin{cases} -1 - \frac{\lambda^2}{4} & \lambda \in [-2:2] \\ -|\lambda| & \lambda \notin [-2:2] \end{cases} \tag{4}$$

3 Code development

My code is composed by two units program : one *module* and one *main program*.

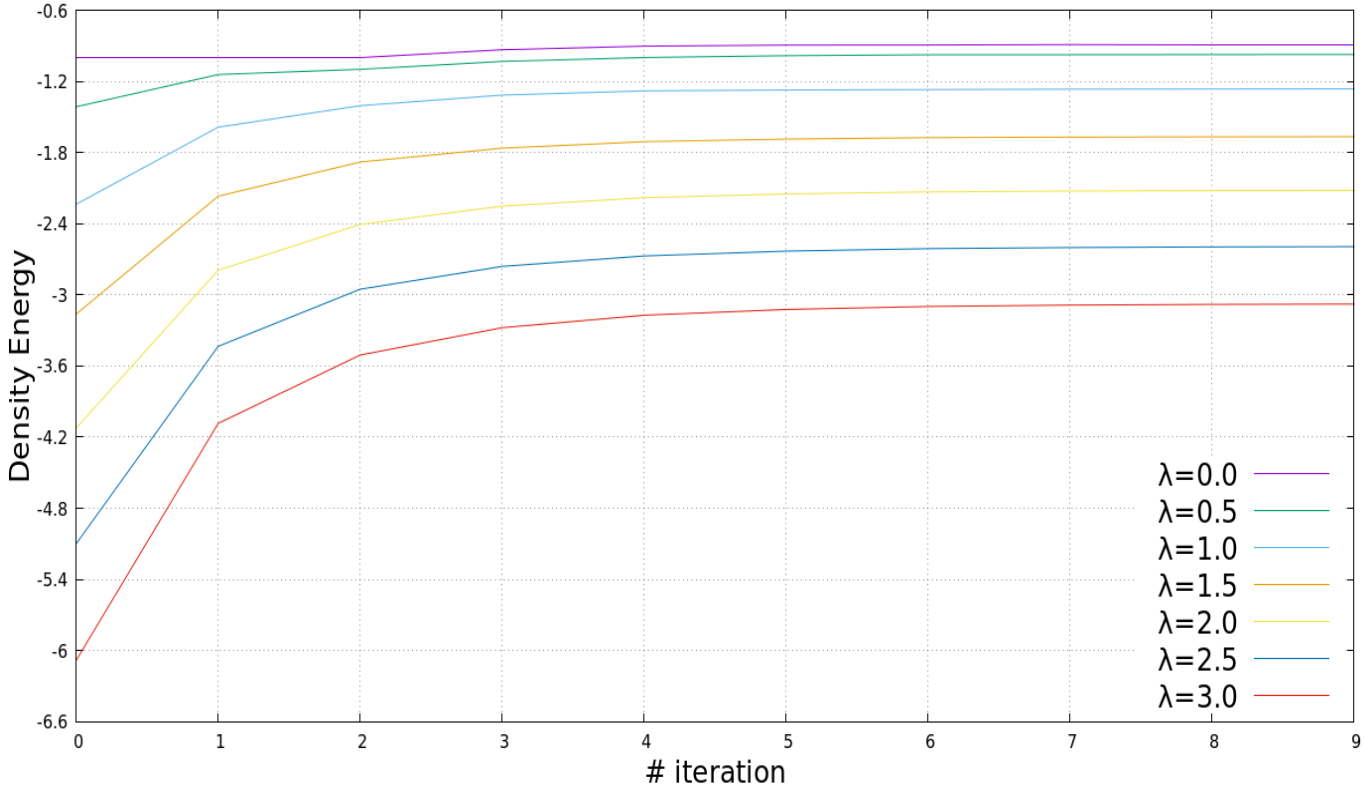
- PROGRAM ***ex10***:

  here the code does all operations needful in order to "make" an Ising quantum model "by hand" and then we extract from it the energy per particle for different λ and different N initial particles.
  At the beginning we impose N_{max} parameter that describes the maximum number of initial particles at the beginning ,$itermax$ parameter that describes how many iterations of RSRG method will be executed and $d\lambda$ parameter that describes how much λ increment will be : the code covers interval from λ=0 to λ=3. Collecting ,for N=2,all the *ground state*'s energy density value found for each iteration and only the $final$ one (that is after *iter max* iterations) for the other N initial particles.

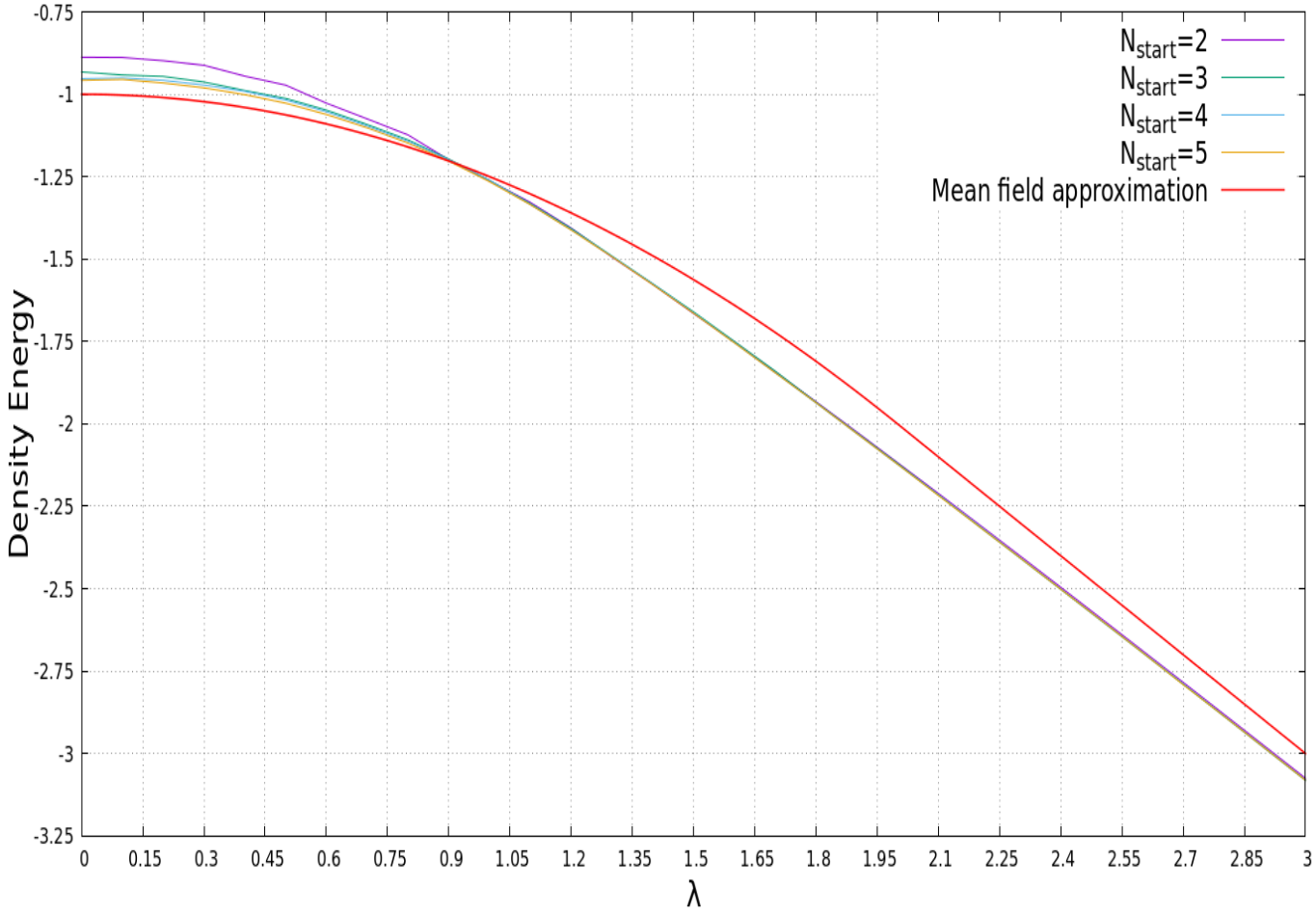
- MODULE ***to-string***: here ,in very simple way, code transforms integer and real number given in input into string and it was very useful in order to optimize writing on files procedure

4 Results

In the following we show results obtained fixing N_{max}=5, $d\lambda = 0.1$ and $itermax = 25$.
In particular as first one we see the ground state's energy density for each iteration fixing N=2 and changing λ's value .



And as the second one curves' behaviour changing N_{start}



5 Comments and Self-Evaluation

We decide to plot energy density $\frac{E}{N-1}$ and not $\frac{E}{N}$ because of,being an open chain,first and last particles lose one neighbour. Anyway as we can see ,looking at previous section, two behaviours can be highlighted:

- Although the code has executed 25 iterations after $\approx$ 10 iterations ground state's density energy remains almost constant (corrections are around 10^{-3})
- Starting with N=2 particles first density energy,that is for λ=0, is quite different from MF prediction but augmenting particles number at the beginning the curves tend to -1.

This is probably related to the fact that we try to extract the very large system's ground state ($2^{25} \approx$ 30 milion particles) starting from only 2 particles and this can bring some approximation errors.
Instead augmenting particles number at the beginning (like N=5 or more) these approximation errors tend to reduce themself.