MATH 110 Lecture 3.3

How Derivatives Affect the Shape of a Graph

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Tuesday, March 10, 2026

Department of Indigenous Knowledge and Science
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Contents

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Obtaining Information about f from f'

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- In this section we package the MVT by applying it in some very common general situations.
- That leads to simple rules for interpreting f' to obtain information about f.

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- The MVT then says that $f'(c) < 0$ for some c in (a_1, b_1), contradiction.
- Our assumption that f is not increasing is false. Therefore f is increasing.

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- We then apply the Increasing/Decreasing test on each of those intervals.
- If f' is continuous, the locations at which we split the x-axis are the critical numbers for f.

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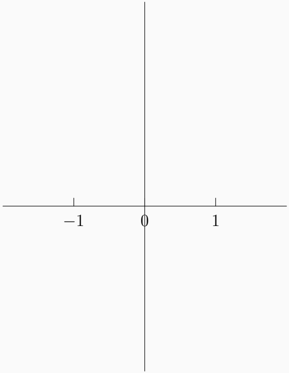

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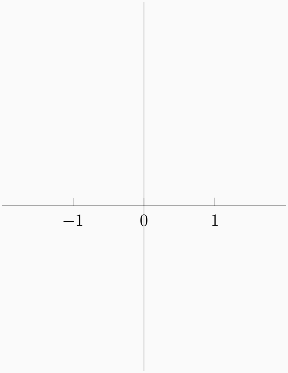

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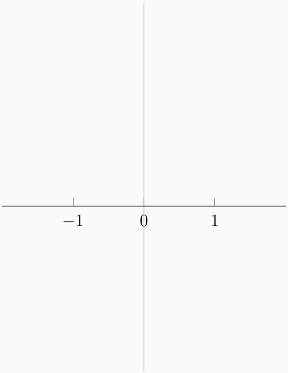

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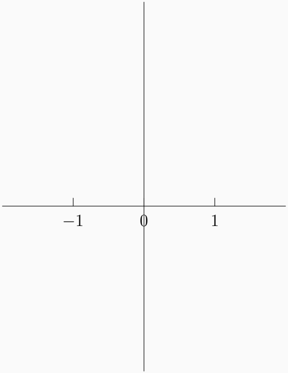

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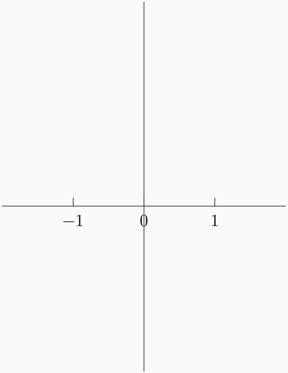

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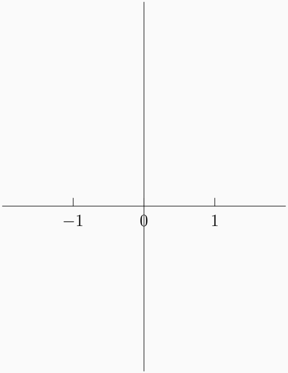

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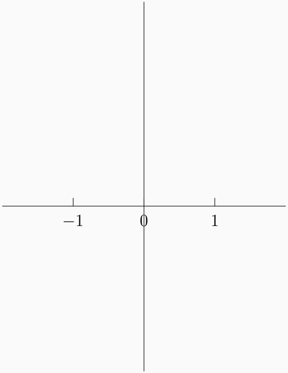

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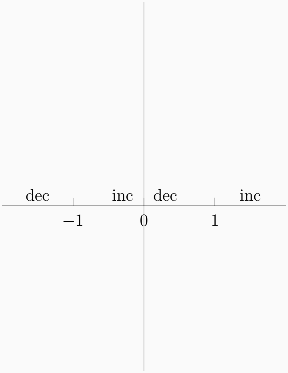
- So f decreases on $(-\infty, -1)$ and $(0, 1)$ and increases on $(-1, 0)$ and $(1, \infty)$.



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- We conclude f has a local min at -1, a local max at 0, and a local min at 1.

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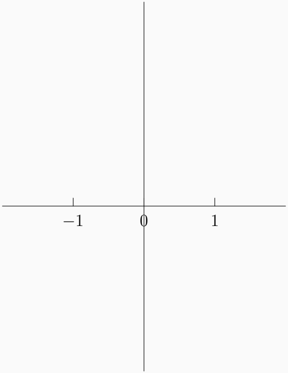

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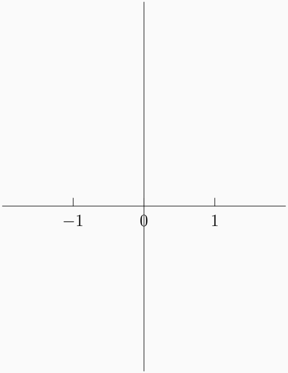

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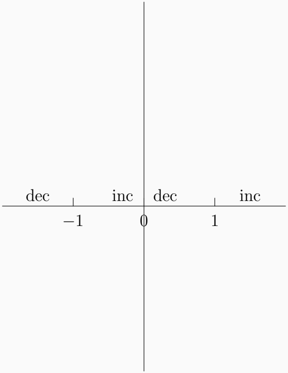

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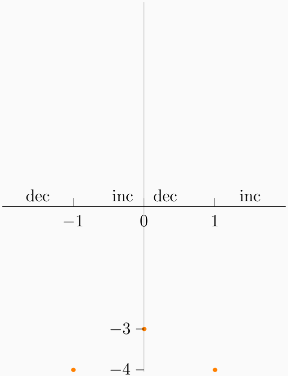

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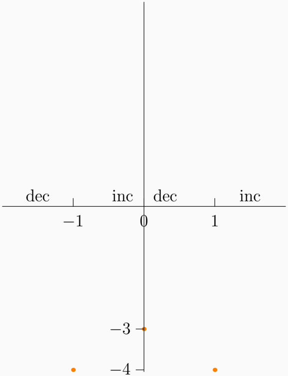
- We know that f is decreasing on $(0, 1)$ and increasing on $(1, \infty)$, so the critical point at 1 is a local min.



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- Concave upward means it is shaped something like a bowl.
- Concave downward is like an upside-down bowl.
- **Definition:** If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph lies below all of its tangents on I, it is called concave downward on I.

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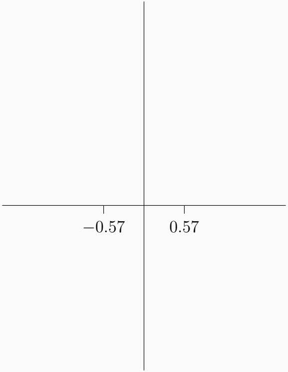

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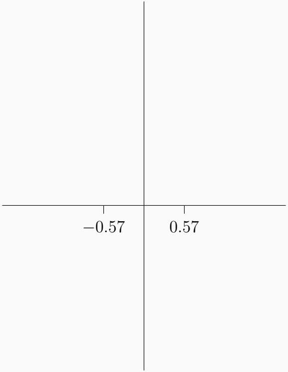

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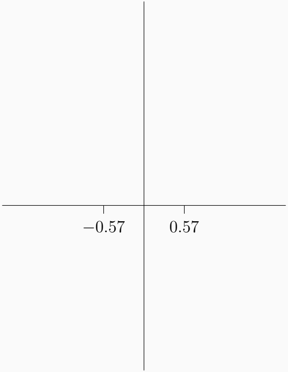
$(-\infty, -\frac{1}{\sqrt{3}})$	-	-	+



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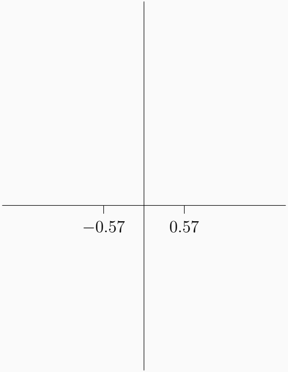

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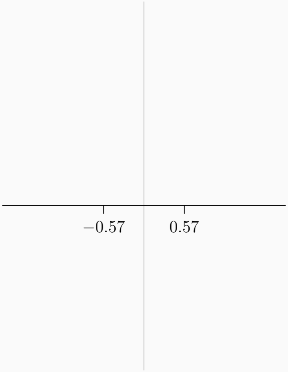
$(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$	+	-	-
$(\frac{1}{\sqrt{3}}, \infty)$	+	+	+



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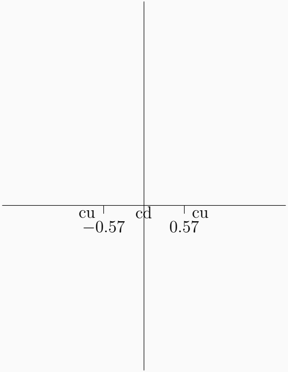
- We would like to determine where $f(x) = x^4 - 2x^2 - 3$ is concave up/down.
- The second derivative: $f''(x) = 12x^2 - 4 = 12(x + 1/\sqrt{3})(x - 1/\sqrt{3})$.
- We have

I	$x + \frac{1}{\sqrt{3}}$	$x - \frac{1}{\sqrt{3}}$	f''
$(-\infty, -\frac{1}{\sqrt{3}})$	-	-	+
$(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$	+	-	-
$(\frac{1}{\sqrt{3}}, \infty)$	+	+	+

- So f is concave up on $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$, and concave down on $(-1/\sqrt{3}, 1/\sqrt{3})$.



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- Finding inflection points is similar to finding extrema Split the x-axis where $f''(x) = 0$ or D.N.E. The points at which f'' changes sign are inflection points.

Example: $f(x) = x^4 - 2x^2 - 3$ Inflection Points

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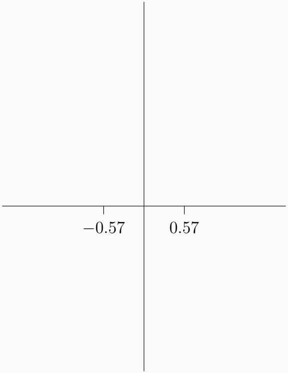

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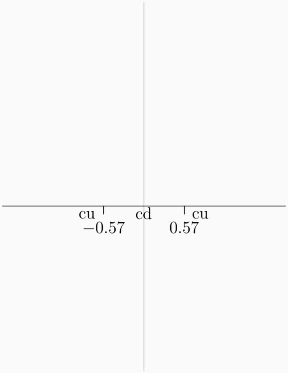

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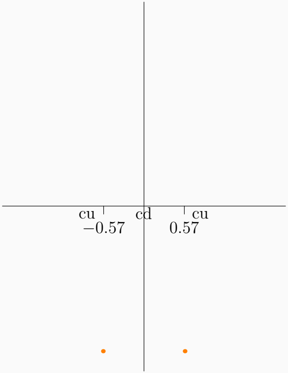

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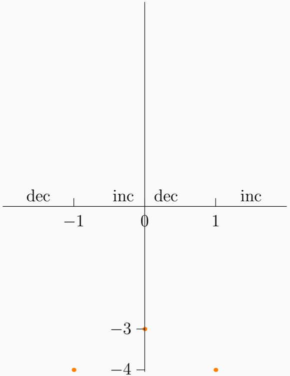

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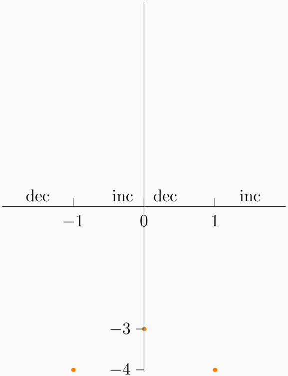

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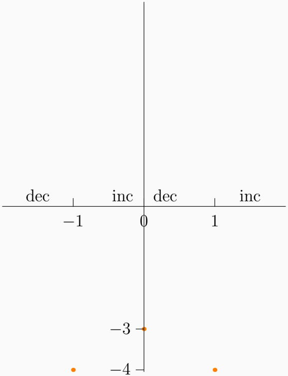

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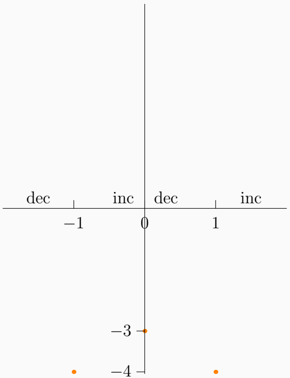

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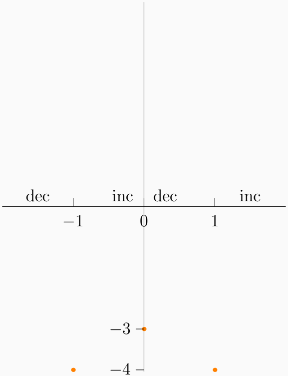

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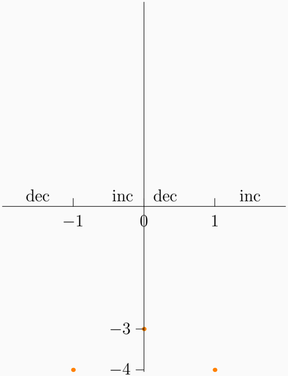

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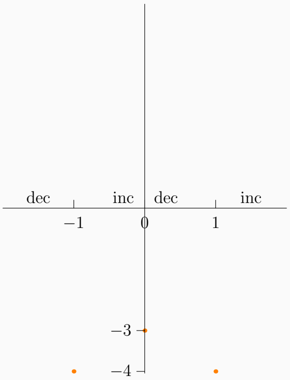
- Finally, $f''(0) = -4 < 0$, so the $c = 0$ must be a local max.



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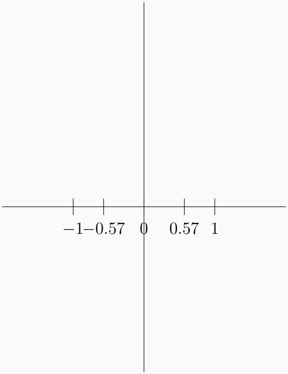

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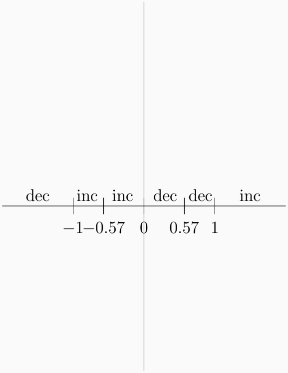

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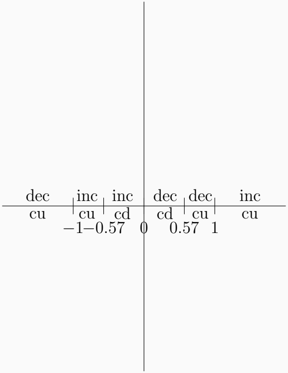

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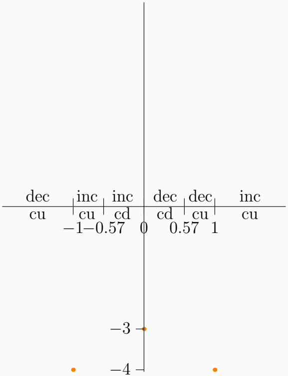

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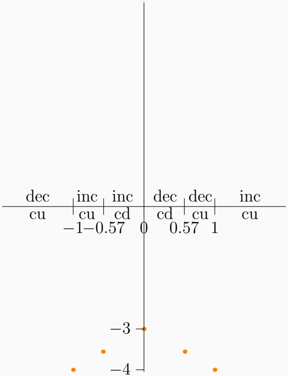

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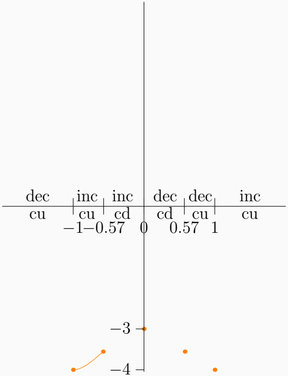

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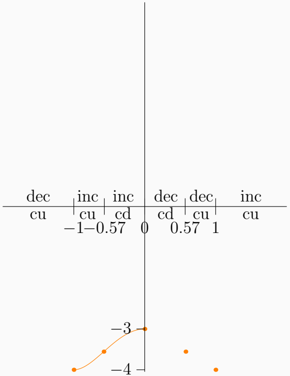

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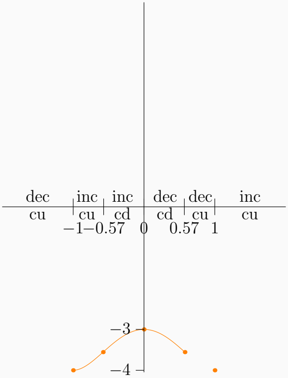

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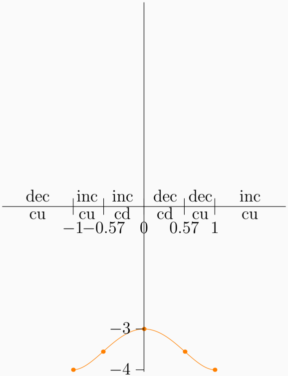

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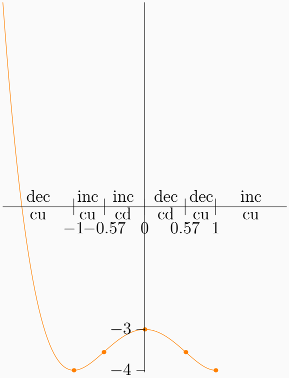

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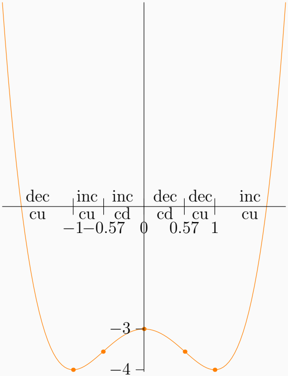

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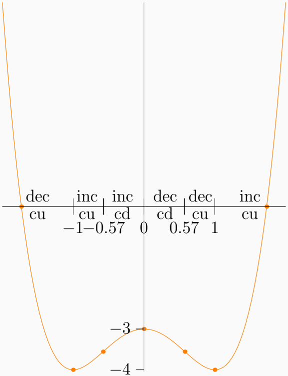

Examples and Exercises

Examples

1. Sketch the graph of a function that satisfies all of the conditions $f'(0) = f'(2) = f'(4) = 0, f'(x) > 0$ if $x < 0$ or $2 < x < 4$, $f'(x) < 0$ if $0 < x < 2$ or $x > 4, f''(x) > 0$ if $1 < x < 3, f''(x) < 0$ if $x < 1$ or $x > 3$.
2. For the function $f(x) = x^4 - 6x^2 - 8x + 15$
   2.1 Find the intervals of increase or decrease.
   2.2 Find the local maximum and minimum values.
   2.3 Find the intervals of concavity and the inflection points.
   2.4 Use the above information to sketch the graph.
3. The same for the function $f(\theta) = 2\cos\theta + \cos^2\theta$, $0 \leq \theta \leq 2\pi$.
4. The same for the function $f(x) = x^{2/3}(6-x)^{1/3}$

Exercises

Now you should work on Problem Set 3.3. After you have finished it, you should try the following additional exercises from Section 3.3:

3.3 C-level: 1–44, 53;
B-level: 45–46, 54–57, 59–62, 67–68, 72–73;
A-level: 58, 63–66, 69–71, 74–77