MATH 11 Problem Set 3.1 Solutions

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1. Recall that the critical numbers of a function $f(x)$ are the values x at which $f'(x)$ does not exist or $f'(x) = 0$.

   (a) The derivative $f'(x)$ exists for all values of x by our theorems for derivatives, and we have $f'(x) = 3x^2 + 2x + 1$. The only possible critical numbers are where $f'(x) = 3x^2 + 2x + 1 = 0$. By the quadratic formula,
   $$x = \frac{-2 \pm \sqrt{(2)^2 - 4(3)(1)}}{6}$$
   which is not possible since $(2)^2 - 4(3)(1) = 4 - 12 = -8$ does not have a square root. So f has no critical numbers.

   (b) We can write
   $$g(t) = \begin{cases} -(3t-4) & 3t-4 < 0 \\ 0 & 3t-4 = 0 \\ 3t-4 & 3t-4 > 0 \end{cases} = \begin{cases} -3t+4 & t < 4/3 \\ 0 & t = 4/3 \\ 3t-4 & t > 4/3 \end{cases}$$
   From the above, you can see that $g'(t)$ exists and equals -3 for $t < 4/3$; $g'(t)$ exists and equals 3 for $t > 4/3$; and a careful investigation using the definition of derivative shows that $g'(t)$ does not exist for $t = 4/3$, so $t = 4/3$ is the only critical number of $g(t)$.

   (c) The derivative of h is
   $$h'(p) = \frac{(p^2+4)(1) - (p-1)(2p)}{(p^2+4)^2} = \frac{-p^2+2p+4}{(p^2+4)^2}$$
   which exists for all numbers p because the denominator is never 0. So all critical points are where the value of $h'(p) = 0$, i.e.,
   $$\frac{-p^2+2p+4}{(p^2+4)^2} = 0 \implies -p^2+2p+4 = 0$$
   giving the roots
   $$p = \frac{-2 \pm \sqrt{(2)^2 - 4(-1)(4)}}{2(-1)} = \frac{-2 \pm \sqrt{4+16}}{-2} = 1 \mp \sqrt{5}$$
   which are the critical numbers of $h(p)$.

   (d) The domain of $g(x)$ is wherever $1 - x^2 \geq 0$, i.e., on the interval $[-1, 1]$. The derivative is
   $$g'(x) = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{1-x^2}}$$
   which exists for x in the open interval $(-1, 1)$. So the critical numbers are where $g'(x)$ does not exist, i.e., -1 and 1, and the numbers where $g'(x) = 0$, i.e., $x = 0$.

2. (a) We assemble a list of numbers of interest including the endpoints and the critical numbers of the function $f(x) = x^3 - 6x^2 + 9x + 2$. We have $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$ so f has critical numbers 1 and 3, both of which are inside the given interval. So the list of numbers of interest is -1, 1, 3, 4. Evaluating f at each of those numbers we have

$$\begin{aligned} f(-1) &= (-1)^3 - 6(-1)^2 + 9(-1) + 2 = -1 - 6 - 9 + 2 = -14 \\ f(1) &= 1^3 - 6(1)^2 + 9(1) + 2 = 1 - 6 + 9 + 2 = 6 \\ f(3) &= 3^3 - 6(3)^2 + 9(3) + 2 = 27 - 54 + 27 + 2 = 2 \\ f(4) &= 4^3 - 6(4)^2 + 9(4) + 2 = 64 - 96 + 36 + 2 = 6 \end{aligned}$$

The smallest of those numbers is -14, attained when $x = -1$, and the largest of those numbers is 6, attained when $x = 4$. So the absolute minumum and absolute maximum values of f on the given interval are -14 and 6, respectively.

(b) We have the critical numbers of f are where $f'(x) = 3(x^2 - 1)^2(2x) = 0$, i.e., where $x^2 - 1 = 0$ or $2x = 0$, i.e., -1, 0, and 1. All of those numbers are inside the given interval. The endpoints are -1 and 2. Evaluating $f(x)$ at that list of numbers, we have

$$\begin{aligned} f(-1) &= ((-1)^2 - 1)^3 = 0 \\ f(0) &= ((0)^2 - 1)^3 = -1 \\ f(1) &= ((1)^2 - 1)^3 = 0 \\ f(2) &= ((2)^2 - 1)^3 = 27 \end{aligned}$$

Therefore the absolute minumum value of f on the given interval is -1 and the absolute maximum value of f on the given interval is 27.

(c) We have

$$f'(x) = \frac{(x^2+4)(2x) - (x^2-4)(2x)}{(x^2+4)^2} = \frac{16x}{(x^2+4)^2}$$

which exists for all values of x and is zero for $x = 0$. We have

$$\begin{aligned} f(-4) &= \frac{(-4)^2 - 4}{(-4)^2 + 4} = \frac{12}{20} = 0.6 \\ f(0) &= \frac{(0)^2 - 4}{(0)^2 + 4} = \frac{-4}{4} = -1.0 \\ f(4) &= \frac{(4)^2 - 4}{(4)^2 + 4} = \frac{12}{20} = 0.6 \end{aligned}$$

so the absolute minimum value of f on the interval is -1.0 and the absolute maximum value is 0.6.

(d) By the product rule,

$$f'(t) = \frac{1}{3}t^{-2/3}(8-t) + t^{1/3}(-1) = \frac{1}{3}t^{-2/3}((8-t) - 3t^{3/3}) = \frac{1}{3}t^{-2/3}(8-4t)$$

which exists for $t \neq 0$ and equals 0 at $t = 2$. Checking $f(t)$ at those points and the endpoints of the domain we have

$$\begin{aligned} f(0) &= \sqrt[3]{0}(8-0) = 0 \\ f(2) &= \sqrt[3]{2}(8-2) = 6\sqrt[3]{2} \\ f(8) &= \sqrt[3]{8}(8-8) = 0 \end{aligned}$$

so the absolute minimum value of $f(t)$ on the interval is 0, and the absolute maximum value is $6\sqrt[3]{2} \approx 7.5595$

3. (a) First, note that f is a continuous function (because it is a polynomial) defined on a closed interval, so the theorems of section 4.1 apply. To find the critical numbers, we find the values of x for which $f'(x)$ does not exist (nowhere) and the values of x for which $f'(x) = 0$. The derivative of f is

$$f'(x) = 4x^3 - 9x^2 + 6x - 1$$

so we must solve the equation $4x^3 - 9x^2 + 6x - 1 = 0$. Normally we would have difficulty solving a cubic equation, but in this case (and most cases you'll see in this course) you can look for roots which are integer factors of the constant term (here -1). So we look for roots of the form $x = \pm 1$. We get lucky right away with the root $x = 1$, so by the factor theorem from algebra we can write

$$4x^3 - 9x^2 + 6x - 1 = (x - 1)(Ax^2 + Bx + C)$$

where we have to figure out what A, B, and C are by polynomial division or otherwise. We obtain

$$4x^3 - 9x^2 + 6x - 1 = (x - 1)(4x^2 - 5x + 1)$$

which further factors to

$$4x^3 - 9x^2 + 6x - 1 = (x - 1)(4x - 1)(x - 1)$$

So the solutions to $f'(x) = 0$ are $x = 1$ and $x = 1/4$, both of which are in the interval $[0, 2]$. The complete set of numbers which we need to investigate is 0 (end number), $1/4 = 0.25$ (critical number), 1 (critical number), and 2 (end number). We evaluate $f(x)$ at each of those numbers, see Table 1, and determine that the minimum value of f on the interval $[0, 2]$ is $f(x) \approx -0.1044$, attained at $x = 0.25$, and the maximum value of f on the interval $[0, 2]$ is $f(x) = 2$ attained at $x = 2$. See Figure 1 for verification. Note that the critical point at $(1, 0)$ is neither a local

x	$f(x)$	comment
0	$f(0.00) = 0.0000$	
0.25	$f(0.25) = -0.1055$	min
1	$f(1.00) = 0.0000$	
2	$f(2.00) = 2.0000$	max

Table 1: Table of important values for $f(x) = x^4 - 3x^3 + 3x^2 - x$ for problem 3a

minimum nor a local maximum, but it is still a critical point because the tangent line at that point is horizontal.

(b) The function $2\cos x$ is continuous everywhere because it is a constant multiple of a continuous function, so the function $f(x) = x - 2\cos x$ is continuous everywhere because it is the difference of two continuous functions. Since the interval $-2 \le x \le 0$ is a closed interval (containing its endpoints), the situation is of the type for which our optimization process works. Calculating the derivative of f,

$$f'(x) = 1 + 2\sin x$$

The critical numbers are where $f'(c)$ does not exist (nowhere) and where $f'(c) = 0$, i.e.,

$$f'(c) = 0 \implies 1 + 2\sin c = 0 \implies \sin c = -\frac{1}{2}$$

Using your knowledge of trigonometry or your calculator, you can determine the principal solution to the above equation to be $-\pi/6$ radians. From the graph of the sine function, we have a whole series of critical numbers $-\pi/6+2k\pi$, $k = 0, \pm1, \pm2, \ldots$, and also another whole series $-5\pi/6+2k\pi$, $k = 0, \pm1, \pm2, \ldots$. However, we can throw away all those roots except $-\pi/6$ because all the others

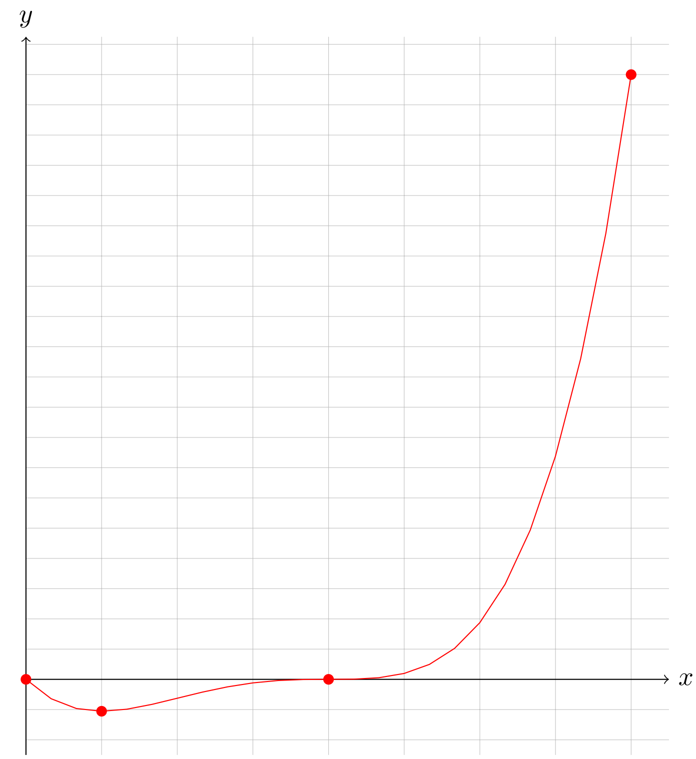

Figure 1: Graph of $f(x) = x^4 - 3x^3 + 3x^2 - x$ for problem 3a

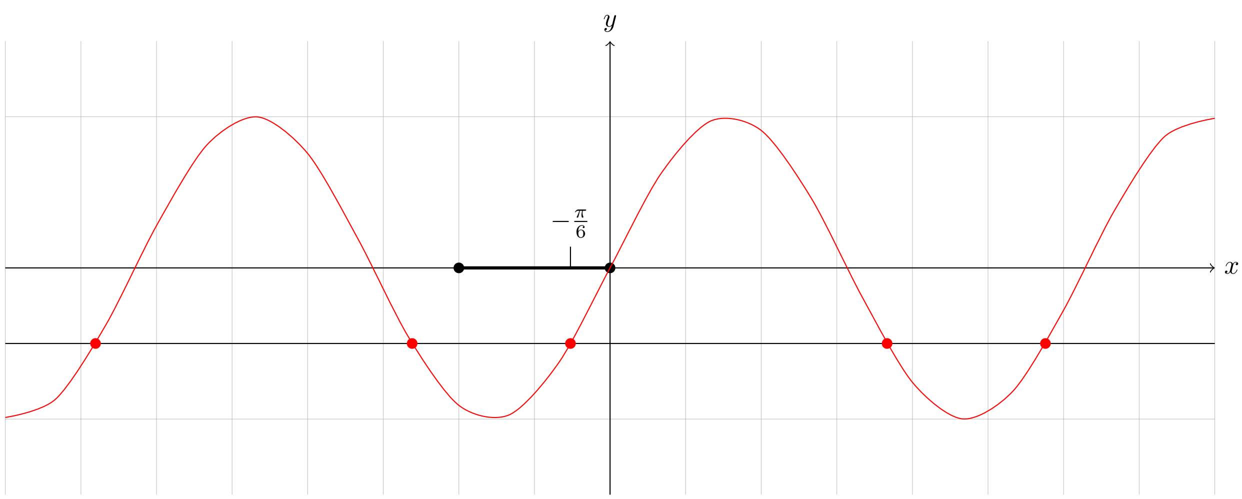

Figure 2: Graph of $y = \sin x$ for problem 3b

lie outside the interval $[-2, 0]$. (The only other feasible contender is $-5\pi/6$, and a short calculation shows it to be -2.6180, definitely outside the interval $[-2, 0]$. See Figure 2 for an illustration of the positions of the solutions of $\sin c = -1/2$. It follows that $f(x) = x - 2\cos x$ has only one critical number in $[-2, 0]$, namely $-\pi/6 \approx -0.5236$. Tabulating the values of f at the critical number and the end numbers we have See Figure 3 for verification.

x	$f(x)$	comment
-2	-1.1677	max
$-\frac{\pi}{6}$	-2.2556	min
0	-2.0000	

Table 2: Table of important values for $f(x) = x - 2\cos x$ for problem 3b

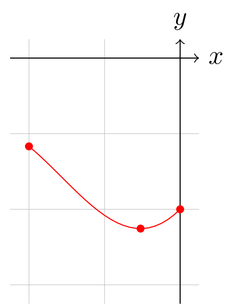

Figure 3: Graph of $f(x) = x - 2\cos x$ for problem 3b

4. Note that μ and W are constants in this problem, so we can write F as a function of θ:

$$F(\theta) = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

In order for our optimization process to work, we need to check that $F(\theta)$ is continuous on the interval $[0, \pi/2]$. F is a quotient of continuous functions, so is continuous unless the denominator is 0. So F is not continous at points where

$$\mu \sin\theta + \cos\theta = 0 \implies \mu \sin\theta = -\cos\theta$$

Rewriting expressions involving sin and cos in terms of tan is a useful trick for solving equations involving trigonometric functions:

$$\implies \frac{\sin\theta}{\cos\theta} = -\frac{1}{\mu} \implies \tan\theta = -\frac{1}{\mu}$$

However (since $\mu > 0$) the equation $\tan\theta = -1/\mu$ has no solutions in the interval $[0, \pi/2]$ because $\tan\theta \geq 0$ for $\theta \in [0, \pi/2]$. So $F(\theta)$ is continuous on that interval. So our optimization process applies to F.

Differentiating F by the quotient rule, we have

$$F'(\theta) = \frac{\left(\frac{d}{d\theta}\mu W\right)(\mu \sin\theta + \cos\theta) - \mu W\left(\frac{d}{d\theta}(\mu \sin\theta + \cos\theta)\right)}{(\mu \sin\theta + \cos\theta)^2} = -\frac{\mu W(\mu \cos\theta - \sin\theta)}{(\mu \sin\theta + \cos\theta)^2}$$

We have already established that the expression in the denominator is never 0 on the interval $[0, \pi/2]$, so $F'(\theta)$ is defined on $[0, \pi/2]$. So the only critical numbers are where $F'(\theta) = 0$, which is only possible if the numerator of $F'(\theta)$ is 0:

$$\mu W(\mu\cos\theta - \sin\theta) = 0 \implies \mu\cos\theta = \sin\theta \implies \tan\theta = \mu$$

So to optimize F, we should consider three numbers: the end numbers $\theta = 0$ and $\theta = \pi/2$ and the critical number θ which satisfies $\tan\theta = \mu$. (That number can be found with your calculator using the $\tan^{-1}$ button: $c = \tan^{-1}\mu$ is the critical number.) We have

$$\begin{aligned} F(0) &= \frac{\mu W}{\mu\sin 0 + \cos 0} = \mu W \\ F(\pi/2) &= \frac{\mu W}{\mu\sin(\pi/2) + \cos(\pi/2)} = W \\ F(c) &= \frac{\mu W}{\mu\sin(c) + \cos(c)} \end{aligned}$$

We can't evaluate $F(c)$ exactly unless we know μ and W, but we can get some estimates. Dividing numerator and denominator by $\cos\theta$ we have

$$F(\theta) = \frac{\mu W\sec\theta}{\mu\tan\theta + 1}$$

At $\theta = c$ we have $\tan\theta = \mu$ and we also have (by the Pythagorean identity)

$$\sec^2 c = 1 + \tan^2 c = 1 + \mu^2 \implies \sec\theta = \sqrt{1+\mu^2}$$

Filling in those values we get

$$F(c) = \frac{\mu W\sqrt{1+\mu^2}}{1+\mu^2} = \frac{\mu W}{\sqrt{1+\mu^2}}$$

No matter what the value of μ, we have

$$1 + \mu^2 > 1 \implies \sqrt{1+\mu^2} > 1 \implies F(c) = \frac{\mu W}{\sqrt{1+\mu^2}} < \mu W = F(0)$$

On the other hand, dividing through by μ, we get

$$F(c) = \frac{W}{\sqrt{(1/\mu^2) + 1}}$$

Similar to the above, we have, no matter what μ is,

$$\left(\frac{1}{\mu}\right)^2 + 1 > 1 \implies \sqrt{\left(\frac{1}{\mu}\right)^2 + 1} > 1 \implies F(c) = \frac{W}{\sqrt{(1/\mu^2) + 1}} < W = F(\pi/2)$$

So even though we could not get numerical values for $F(0)$, $F(c)$, and $F(\pi/2)$, we can still see that $F(c)$ is the smallest of the three. We conclude that $F(\theta)$ is minimized when $\tan\theta = \mu$.