MATH 110 Problem Set 2.8 Solutions

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1. Differentiating both sides of the relationship between x and y with respect to time,

$$\begin{aligned} \frac{d}{dt}(x^2+y^2) &= \frac{d}{dt}25 \\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \end{aligned}$$

We are given that $dy/dt = 6$ and $y = 4$, so

$$\begin{aligned} 2x\frac{dx}{dt} + 2(4)(6) &= 0 \\ \frac{dx}{dt} &= -\frac{24}{x} \end{aligned}$$

To find dx/dt, we still need to know x. Fortunately, we can figure out x from $y = 4$ and the relationship $x^2 + y^2 = 25$: $x^2 + 16 = 25$, $x^2 = 9$, $x = \pm 3$, so $dx/dt = \mp 8$.

2. Differentiating both sides of the given relationship with respect to t,

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Dividing through by 2 and substituting everything we know,

$$z\frac{dz}{dt} = 5(2) + 12(3) = 46$$

We still need z in order to figure out dz/dt; but from the relationship $z^2 = x^2 + y^2$ and $x = 5$ and $y = 12$ we have $z^2 = 5^2 + 12^2 = 25 + 144 = 169$ so $z = \pm 13$ and so $dz/dt = \pm 46/13$.

3. The words "rate" and "how fast" flag this as a related rates question. We need to find a relationship between the non-time variables. The volume of a sphere in terms of its radius is $V = (4/3)\pi r^3$. Differentiating the relationship with respect to time we have

$$\begin{aligned} \frac{d}{dt}V &= \frac{d}{dt}\frac{4}{3}\pi r^3 \\ \frac{dV}{dt} &= \frac{4}{3}\pi \cdot 3r^2\frac{dr}{dt} \end{aligned}$$

where the chain rule was used in the last step. The latter expression can be simplified to

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

We are given that $dr/dt = 4$ and $r = 40$ so

$$\frac{dV}{dt} = 4\pi(40)^2 \cdot 4 \approx 80424$$

When the radius of the sphere is 40 mm and is increasing at 4 mm/s, the volume is increasing at a rate of about 80424 mm^3/s.

4. We need a relationship connecting the surface area S of a sphere to its diameter D. We have $S = 4\pi r^2$ and $r = D/2$ so $S = 4\pi(D/2)^2 = \pi D^2$. Differentiating with respect to time,

$$\frac{dS}{dt} = 2\pi D \frac{dD}{dt}$$

We are given that $dS/dt = -1$ (negative because S is decreasing) and $D = 10$, so

$$-1 = 2\pi(10)\frac{dD}{dt} \implies \frac{dD}{dt} = -\frac{1}{20\pi} \approx -0.0159$$

So under the given conditions, the diameter is decreasing at a rate of approximately 0.0159 cm/min.

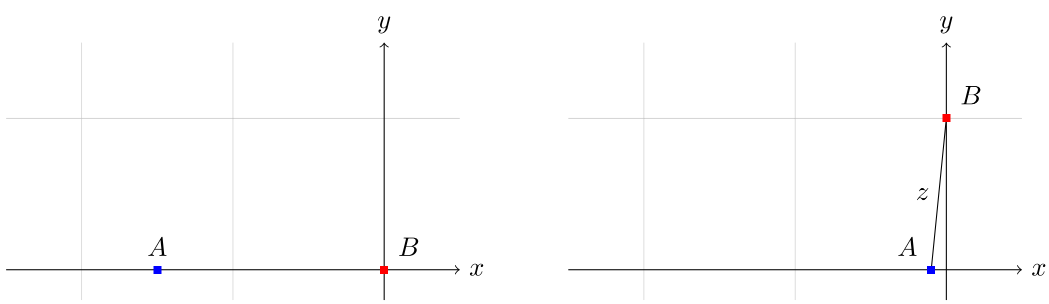
5. Call the location of ship B at noon the point O. Call the position of ship A with respect to O on the east-west axis through O $x(t)$; we are given $x(t) = -150$ when $t = 0$ is noon. Similarly call the position of ship B with respect to O on the north-south axis through O $y(t)$; we are given $y(t) = 0$ when $t = 0$ is noon. Then at 4 pm, ship A is $150 - 4(35) = 10$ km west of O, i.e., $x(4) = -10$, and ship B is $4(25) = 100$ km north of O, i.e., $y(4) = 100$. Let $z(t)$ be the distance between the two ships at time t. By the Pythagorean theorem,

$$z^2 = x^2 + y^2$$
$$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

At 4 pm we have $x = -10$, $y = 100$, $dx/dt = 35$, $dy/dt = 25$ and we can calculate $z = \sqrt{10^2 + 100^2} = \sqrt{10100} \approx 100.5$ (we take the positive square root because distances are always positive). Filling in all that information,

$$\frac{dz}{dt} \approx \frac{1}{100.5}(-10 \cdot 35 + 100 \cdot 25) = 21.4$$

So at 4 pm the distance between the two ships is increasing at a rate of 21.4 km/h. See Figure 1.



(a) Situation at noon (b) Situation at 4pm

Figure 1: Diagrams for Problem 5

6. Note: in the problem statement there is a mix of different units (cm and m); the easiest way to manage the different units is to convert every meaurement into the same system of units. In this case, we'll convert everything to meters.

The volume of a cone is $V = (1/3)\pi r^2 h$. The radius and height at the top of the cone are 2 m and 6 m respectively, so by similar triangles, we have $r/h = 2/6$ or $r = h/3$. Substituting into the volume formula, $V = (1/3)\pi(h/3)^2 h$ or $V = (\pi/27)h^3$. Differentiating with respect to time,

$$\frac{dV}{dt} = \frac{\pi}{27} \cdot 3h^2 \cdot \frac{dh}{dt}$$
$$\frac{dV}{dt} = \frac{\pi}{9} \cdot h^2 \cdot \frac{dh}{dt}$$

We are given $h = 2$ and $dh/dt = 0.2$ (in meters per minute), so $dV/dt = (\pi/9)2^2 \cdot 0.2 \approx 0.2793$ (in cubic meters per minute). Since water is running out of the tank at 0.01 cubic meters per minute, the rate at which water is entering the tank must be $0.01 + 0.2793 = 0.2893$ cubic meters per minute.

7. The formula connecting the variables is $R^{-1} = R_1^{-1} + R_2^{-1}$. When $R_1 = 80$ and $R_2 = 100$ we have $1/R = 1/80 + 1/100 = 5/400 + 4/400 = 9/400$ so $R = 400/9$. Differentiating the formula with respect to time gives

$$R^{-2}\frac{dR}{dt} = R_1^{-2}\frac{dR_1}{dt} + R_2^{-2}\frac{dR_2}{dt}$$

Filling in all the information we know,

$$\frac{9^2}{400^2}\frac{dR}{dt} = \frac{1}{80}^{2} 0.3 + \frac{1}{100^2}0.2 \implies 81\frac{dR}{dt} = 25 \cdot 0.3 + 16 \cdot 0.2 \implies \frac{dR}{dt} = 0.132$$

Under the given conditions R is increasing at a rate of 0.132 ohms per second.

8. Say the lighthouse is at point L. Then QPL is a right triangle with right angle at P. Let $QP = x$ and let the angle QLP be θ. Since $PL = 3$ we have $\tan\theta = x/3$ or $x = 3\tan\theta$. Differentiating with respect to time, $dx/dt = 3\sec^2\theta \cdot d\theta/dt$. Since the light makes four revolutions per minute, and once complete circle is 2π radians, the light circles through an angle of 8π radians in one minute, so $d\theta/dt = 8\pi$. We also have $x = 1$ so $\tan\theta = 1/3$. We could figure out θ from that equation (using the $\tan^{-1}$ button on our calculator), but it is easier to note that we really want $\sec^2\theta$. From our triangle, the hypoteneuse QL is $\sqrt{1^2 + 3^2} = \sqrt{10}$ so $\cos\theta = 3/\sqrt{10}$ so $\sec\theta = \sqrt{10}/3$ so $\sec^2\theta = 10/9$. Filling that information in to our differential relation,

$$\frac{dx}{dt} = 3\sec^2\theta \cdot \frac{d\theta}{dt} = 3 \cdot \frac{10}{9} \cdot 8\pi \approx 83.8$$

So the light is moving at about 83.8 km per minute at point Q on the shore.