MATH 110 Lecture 2.7

Rates of Change in the Natural and Social Sciences

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Contents

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- We will explore applications of these ideas in physics, chemistry, biology, and (micro-)economics.

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- The derivative of the amount of radioactive substance is called item the **rate of decay**.

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- Similarly, we can figure out when the particle is speeding up by solving the inequality $s''(t) > 0$.

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- In the above example, the linear density is $d\sqrt{x}/dx = 1/(2\sqrt{x})$.

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- In a reaction like $2H_2$ + O_2 $\rightarrow$ $2H_2O$, we write

$$\frac{1}{2}\frac{d[\mathrm{H_2O}]}{dt} = -\frac{d[\mathrm{O_2}]}{dt} = -\frac{1}{2}\frac{d[\mathrm{H_2}]}{dt}$$

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- The simplest model of population growth is exponential, $P(t) = P_0 2^t$. You will learn about exponential functions in MATH 111.
- Typically, we have some relationship between dP/dt and P, e.g.,

$$\frac{dP}{dt} = r_0 \left(1 - \frac{P(t)}{P_c}\right) P(t) - \beta P(t)$$

  and we have to determine something about $P(t)$, e.g., when is $P(t)$ constant.

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- The formula for marginal cost is $C(x+1) - C(x)$, where x is the production level and $C(x)$ is the cost for an entire production run of x units. For example, the marginal cost of producing one more unit after you have produced 100 units is $C(101) - C(100)$.

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- The word *marginal* comes from the practice of keeping track of the differences $C(x+1)-C(x)$ in the margin of an account.

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- The marginal cost at production level 50 is approximately $C'(50) = 30 - 0.2(50) = 20$.

Examples and Exercises

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1. A particle moves along a horizontal line so that its coordinate at time t is $x = \sqrt{b^2 + c^2 t^2}$, $t \geq 0$, where b and c are positive constants.
   1.1 Find the velocity and acceleration functions.
   1.2 Show that the particle always moves in a positive direction.
2. The mass of part of a wire is $x(1+\sqrt{x})$ kilograms, where x is measured in meters from one end of the wire. Find the linear density of the wire when $x = 4$ m.
3. The cost, in dollars, of producing x units of a certain commodity is $C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3$.
   3.1 Find the marginal cost function.
   3.2 Find $C'(100)$ and explain its meaning.
   3.3 Compare $C'(100)$ with the cost of producing the 101st item.

Exercises

Now you should work on Problem Set 2.7. After you have finished it, you should try the following additional exercises from Section 2.7:

2.7 C-level: 1–4, 6–10, 17–18, 22, 27–30;
B-level: 11–16, 19–21, 23–24, 25–26;
A-level: 31–36