MATH 110 Problem Set 2.3 Solutions

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1. We use the quotient rule first in all of the parts of this question

   (a) We have

   $$f'(t) = \frac{d}{dt}\frac{2t}{4+t^2} = \frac{(4+t^2)\frac{d}{dt}2t - 2t\frac{d}{dt}(4+t^2)}{(4+t^2)^2} = \frac{(4+t^2)(2) - 2t(2t)}{(4+t^2)^2}$$

   where we have used the sum, constant multiple, and power rules in the last step above. Further simplification is possible, but is not necessary in this question:

   $$f'(t) = \frac{8+2t^2-4t^2}{(4+t^2)^2} = \frac{8-2t^2}{(4+t^2)^2}$$

   (b) By the quotient rule,

   $$\frac{dy}{dx} = \frac{(x^3+x-3)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x^3+x-3)}{(x^3+x-3)^2} = \frac{(x^3-x-3)(1) - (x+1)(3x^2+1)}{(x^3+x-3)^2}$$

   Further simplification is optional:

   $$\frac{dy}{dx} = \frac{x^3-x-3-3x^3-x-3x^2-1}{(x^3+x-3)^2} = \frac{-2x^3-3x^2-2x-4}{(x^3+x-3)^2}$$

   Note that I have not expanded the denominator, which is seldom a useful simplification.

   (c) By the quotient rule,

   $$\frac{dy}{dt} = \frac{(t-1)^2\frac{d}{dt}t - t\frac{d}{dt}(t-1)^2}{((t-1)^2)^2}$$

   In order to take the derivative of $(t-1)^2$ in the above expression we must first expand it (in the numerator, not the denominator) to obtain

   $$\frac{dy}{dt} = \frac{(t-1)^2\frac{d}{dt}t - t\frac{d}{dt}(t^2-2t+1)}{((t-1)^2)^2} = \frac{(t-1)^2(1) - t(2t-2)}{(t-1)^4}$$

   where we have used the rules of exponents in the denominator. Further simplification is optional:

   $$\frac{dy}{dt} = \frac{t^2-2t+1-2t^2+2t}{(t-1)^4} = \frac{1-t^2}{(t-1)^4}$$

   (d) We write all the powers in exponential notation and then apply the quotient rule:

   $$\begin{aligned} g'(t) = \frac{d}{dt}\frac{t-t^{1/2}}{t^{1/3}} &= \frac{t^{1/3}\frac{d}{dt}(t-t^{1/2} - (t-t^{1/2})\frac{d}{dt}t^{1/3}}{(t^{1/3})^2} \\ &= \frac{t^{1/3}(1-(1/2)t^{-1/2}) - (t-t^{1/2})(1/3)t^{-2/3}}{t^{2/3}} \end{aligned}$$

Further simplification is optional but is not necessary. First we clear fractions by multiplying the numerator and denominator through by 6:

$$g'(t) = \frac{t^{1/3}(6 - 3t^{-1/2}) - (t - t^{1/2})2t^{-2/3}}{t^{2/3}} = \frac{6t^{1/3} - 3t^{-1/6} - 2t^{1/3} + 2t^{-1/6}}{t^{2/3}}$$

2. We are given a point on the tangent line, so we must find its slope in order to write the equation in point-slope form.

   (a) By habit, I first do the optional check that the given point is actually on the line: when $x = 1$, $y = (1 + 2(1))^2 = 3^2 = 9$, so the point $(1, 9)$ actually is on the line. To find the slope, we can use the product rule:

   $$y = (1 + 2x)^2 = (1 + 2x)(1 + 2x) \implies y' = 2(1 + 2x) + (1 + 2x)(2) = 2 + 4x + 2 + 4x = 4 + 8x$$

   so $y'(1) = 4 + 8(1) = 12$, i.e., the slope of the tangent line at the point $(x, y) = (1, 9)$ is $y'(1) = 12$. Then the equation of the tangent line in point-slope form is

   $$y - 9 = 12(x - 1)$$

   Further simplification is possible but is not necessary. Finally, if the slope of a line is 12, the slope of the perpendicular line is the negative reciprocal of 12, i.e., $-1/12$, so the equation of the normal line is

   $$y - 9 = -\frac{1}{12}(x - 1)$$

   (b) We have

   $$y(4) = \frac{\sqrt{4}}{4 + 1} = \frac{2}{5} = 0.4$$

   so the point $(4, 0.4)$ actually is a point on the curve. The slope of the tangent line at that point is found by taking the derivative:

   $$y'(x) = \frac{(x + 1)\frac{d}{dx}x^{1/2} - x^{1/2}\frac{d}{dx}(x + 1)}{(x + 1)^2} = \frac{(x + 1)(1/2)x^{-1/2} - x^{1/2}(1)}{(x + 1)^2}$$

   so

   $$y'(4) = \frac{5(1/2)(1/2) - 2}{25} = \frac{5 - 8}{100} = -\frac{3}{100}$$

   Therefore the equation of the tangent line is

   $$y - \frac{2}{5} = -\frac{3}{100}(x - 4)$$

   and the equation of the normal line, which has the same point but negative reciprocal slope, is

   $$y - \frac{2}{5} = \frac{100}{3}(x - 4)$$

3. (a) By the difference and constant multiple rules, $h'(x) = 3f'(x) - 5g'(x)$ so $h'(1) = 3f'(1) - 5g'(1) = 3(-2) - 5(2) = -16$

   (b) By the product rule, $h'(x) = f'(x)g(x) + f(x)g'(x)$. Evaluating at $x = 1$ we have $h'(1) = f'(1)g(1) + f(1)g'(1) = (-2)(-1) + (3)(2) = 2 + 6 = 8$

   (c) By the quotient rule,

   $$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

   Evaluating at $x = 1$,

   $$h'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} = \frac{(-2)(-1) - (3)(2)}{(-1)^2} = \frac{2 - 6}{1} = -4$$

(d) By the quotient rule,

$$h'(x) = \frac{D(2f(x)-1)(g(x)+3)-(2f(x)-1)D(g(x)+3)}{(g(x)+3)^2} = \frac{f'(x)(g(x)+3)-(2f(x)-1)g'(x)}{(g(x)+3)^2}$$

using the sum, difference, and constant multiple rules as well. Evaluating at $x=1$,

$$h'(1) = \frac{f'(1)(g(1)+3)-(2f(1)-1)g'(1)}{(g(1)+3)^2} = \frac{-2(-1+3)-(2(3)-1)(2)}{(-1+3)^2} = \frac{-4-10}{2^2} = \frac{7}{2}$$

4. Two lines are parallel if and only if they have the same slope. The slope of the line $x-2y=2$ can be obtained by solving for y and examining the coefficient of x: $2y=x-2$, $y=(1/2)x-1$, so the slope of the given line is $1/2$, hence the slope of all parallel lines is $1/2$, and we are looking for tangent lines with slope $1/2$.

   The slope of a tangent line to the given curve can be obtained by taking the derivative:

$$y' = \frac{(x+1)\frac{d}{dx}(x-1)-(x-1)\frac{d}{dx}(x+1)}{(x+1)^2} = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

   We are looking for x values at which $y'=1/2$; in other words, we want to solve the equation

$$y'=1/2 \implies \frac{2}{(x+1)^2}=1/2 \implies (x+1)^2=4 \implies x+1=\pm 2 \implies x=1,-3$$

   The number $x=1$ gives the point on the curve $(1,y(1))=(1,0)$ and of course slope $y'(1)=1/2$, so the equation of the tangent line is $y-0=(1/2)(x-1)$. The other number $x=-3$ gives the point on the curve $(-3,2)$ and tangent line $y-2=(1/2)(x+3)$. See Figure 1.

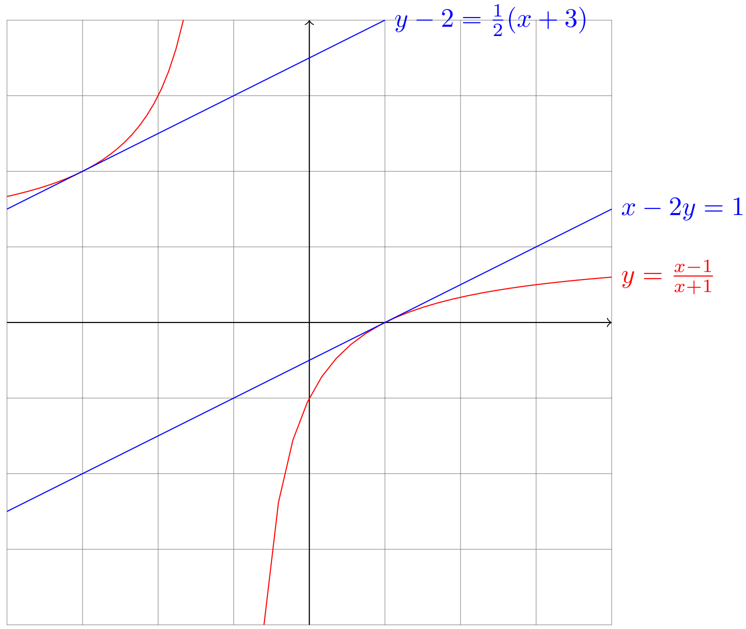

Figure 1: Graph for Problem 4

5. By the product rule applied twice we have

$$(fgh)' = ((fg)h)' = (fg)'h+(fg)h' = (f'g+fg')h+fgh' = f'gh+fg'h+fgh'$$

Applying that result to the expression $(f'gh)'$ we have

$$(f'gh)' = f''gh + f'g'h + f'gh'$$

Similarly,

$$(fg'h)' = f'g'h + fg''h + fg'h'$$
$$(fgh')' = f'gh' + fg'h' + fgh''$$

Assembling everything we know,

$$(fgh)'' = (f'gh + fg'h + fgh')' = f''gh + f'g'h + f'gh' + f'g'h + fg''h + fg'h' + f'gh' + fg'h' + fgh''$$

Some simplification may be helpful:

$$(fgh)'' = f''gh + fg''h + fgh'' + 2f'g'h + 2fg'h' + 2f'gh'$$