MATH 110 Problem Set 1.5a Solutions

Edward Doolittle

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1. Applying the definition, the given equation means "$f(x)$ can be made arbitrarily close to 7 by making x close to (but not equal to) 3." It is possible for

$$\lim_{x\to 3} f(x) = 7$$

but for $f(3) = 4$; for example the function

$$f(x) = \begin{cases} x+4 & \text{if } x \neq 3 \\ 4 & \text{if } x = 3 \end{cases}$$

has those properties.

2. (a)

(b) We have the following table of values, where $f(x) = \dfrac{10^x - 2^x}{x}$:

x	10^x	2^x	$f(x)$
+1.00	10.0000	2.0000	8.0000
+0.50	3.1623	1.4142	3.4961
+0.10	1.2589	1.1072	1.8715
+0.05	1.1220	1.0353	1.7350
+0.01	1.0233	1.0070	1.6337
+0.005	1.0116	1.0035	1.6215
+0.001	1.0023	1.0007	1.6119
±0			
-0.001	0.9977	0.9993	1.6070
-0.005	0.9886	0.9965	1.5974
-0.01	0.9772	0.9930	1.5855
-0.05	0.8913	0.9659	1.4937

Based on the data, we could reasonably conclude that the limit is somewhere between 1.6070 and 1.6119. That may be enough accuracy, depending on the use we were going to make of the answer. If not, we could add more lines to the table. (Methods you will learn if you take MATH 111 give 1.609 as the actual answer to three decimal places.)

3. We tabulate values of $(1+x)^{1/x}$ for numbers x that are closer and closer to 0 until the values stabilize in the fourth digit. We obtain

x	$(1+x)^{1/x}$
$+1.0$	2.00000
$+0.1$	2.59374
$+0.01$	2.70481
$+0.001$	2.71692
$+0.0001$	2.71814
$+0.00001$	2.71826
$+0.000001$	2.71828

for small positive values of x and

x	$(1+x)^{1/x}$
-1.0	∞
-0.1	2.86797
-0.01	2.73200
-0.001	2.71964
-0.0001	2.71841
-0.00001	2.71829

for small negative values of x. The two tables agree to four digits, so we have (most likely) found the first four digits of the limit. It looks suspiciously like the special number $e = 2.71828\ldots$ which is the base of the natural exponential and logarithmic functions. We'll see later that the limit is exactly e.

4.

5. It's difficult to make an educated guess in this problem. If you follow the instructions in the textbook, you'll see that the limit seems to head towards 0.333, but then as x gets very small, the value of $(\tan x - x)/x^3$ suddenly seems to veer down to 0. On my calculator this happens at about $x = 1.0\times10^{-9}$.

   You should try to use a computer to graph the function at a smaller scale than the sample x values given in the textbook to see what happens when you really zoom in on the thing. Something similar should happen at a high enough zoom.

   The moral of the story is that adding more accuracy to your inputs doesn't necessarily result in more accurate outputs when using the numerical approximations provided by calculators and computers. We have to be particularly careful when we subtract two numbers which are very close to one another. You should try to arrange calculations so that you don't subtract two numbers which are close, if possible. You can learn more about the associated issues if you take a course in numerical methods.

   It is very hard for us to figure out how to evaluate this limit given what we now know. Some of the masters of the art from long ago (e.g., Madhava of Sangamagrama) were able to do it, but until the development of calculus such calculations were beyond the reach of nearly everyone. If you take MATH 111, you will learn straightforward ways to evaluate this limit; L'Hôpital's rule or Taylor series tell me that the answer really is $1/3$, so our first guess was correct.