MATH 110 Problem Set 3.5 Solutions

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1. (a) A Domain: a polynomial has domain all of $\mathbb{R}$.

B Intercepts: $x = 0$ implies $y = 0^4 - 4(0)^2 + 4 = 4$, so $(0, 4)$ is the y-intercept. $y = 0$ implies $x^4 - 4x^2 + 4 = 0$. Factoring, $(x^2 - 2)^2 = 0$ which implies $x^2 - 2 = 0$ so $x = \pm\sqrt{2}$. The x-intercepts are $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$.

C Symmetry: Since the function only contains even powers of x, it follows that $y(x) = y(-x)$, in other words the function is even, with mirror symmetry in the y-axis.

D Asymptotes: non-constant polynomials do not have asymptotes.

E Intervals of increase/decrease: the derivative is $y' = 4x^3 - 8x = 4x(x^2 - 2) = 4(x + \sqrt{2})x(x - \sqrt{2})$. The derivative is negative for $x < -\sqrt{2}$, positive for $-\sqrt{2} < x < 0$, negative for $0 < x < \sqrt{2}$, and positive for $\sqrt{2} < x$, which gives the intervals of increase/decrease.

F Local max/min: the derivative is 0 at $x = -\sqrt{2}$, $x = 0$, and $x = \sqrt{2}$. From E above we see that the derivative changes sign at each of those points, and we conclude that there is a local min at $x = -\sqrt{2}$, a local max at $x = 0$, and a local min at $x = \sqrt{2}$.

G Concavity and inflection points: $y'' = 12x^2 - 8 = 12(x^2 - 2/3) = 12(x + \sqrt{2/3})(x - \sqrt{2/3})$. The second derivative is positive on $x < -\sqrt{2/3}$ (concave up), negative on $-\sqrt{2/3} < x < \sqrt{2/3}$ (concave down), and positive on $\sqrt{2/3} < x$. Since the second derivative changes sign at $x = \pm\sqrt{2/3}$, those are both inflection points.

H Graph: see Figure 1.

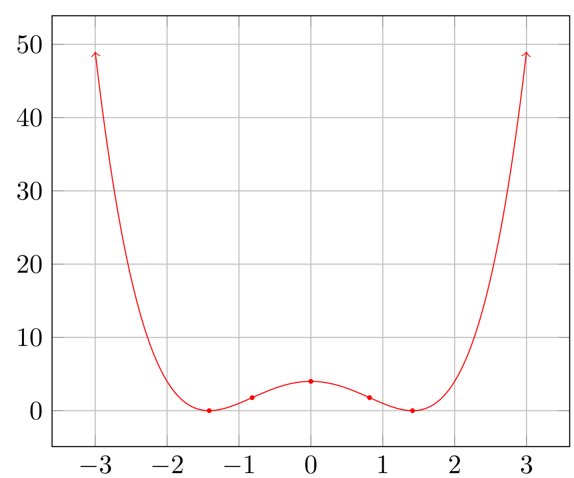

Figure 1: Graph of $y = x^4 - 4x^2 + 4$

(b) A Domain: a polynomial has domain all of $\mathbb{R}$.

B Intercepts: $x = 0$ implies $y = (9 - 0^2)^3 = 729$. $y = 0$ implies $(9 - x^2)^3 = 0$ which implies $(9 - x^2) = 0$. Factoring, $(3 + x)(3 - x) = 0$ giving the x-intercepts $(-3, 0)$ and $(3, 0)$.

C Symmetry: the function has only even powers of x, so it is an even function, with mirror symmetry in the y-axis; in other words, $y(-x) = y(x)$.

D Asymptotes: non-constant polynomials do not have asymptotes.

E Intervals of increase/decrease: $y' = 3(9-x^2)^2(-2x) = -6(3+x)^2 x(3-x)^2$. The derivative is positive on $x < -3$, remains positive on $-3 < x < 0$, becomes negative on $0 < x < 3$, and remains negative on $3 < x$.

F Local max/min: the derivative only changes sign at $x = 0$, which gives a local max.

G Concavity and inflection points: the second derivative is

$$\begin{aligned} y'' &= 24(9-x^2)x^2 - 6(9-x^2)^2 = 6(9-x^2)(4x^2 - 9 + x^2) \\ &= 30(3+x)(x + 3\sqrt{1/5})(x - 3\sqrt{1/5})(3-x) \end{aligned}$$

which is negative on $x < -3$, positive on $-3 < x < -3\sqrt{1/5}$, negative on $-3\sqrt{1/5} < x < 3\sqrt{1/5}$, positive on $3\sqrt{1/5} < x < 3$, and negative on $3 < x$. Since the concavity changes at each of the roots of y'', they are all inflection points.

H Graph: see Figure 2.

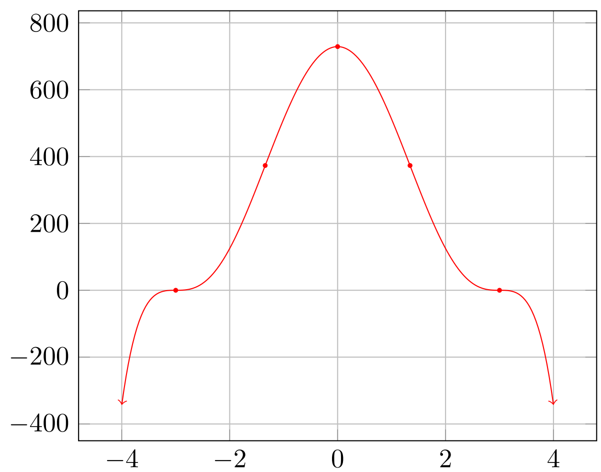

Figure 2: Graph of $y = (9 - x^2)^3$

2. (a) A Domain: rational functions are defined wherever the denominator is non-zero, so the domain is all of $\mathbb{R}$ except for solutions to $9 - x^2 = 0$ which are $x = -3$ and $x = 3$.

B Intercepts: when $x = 0$, $y = 0/9 = 0$. When $y = 0$, $x^2 - 3x = 0$. Factoring, $x(x-3) = 0$ with roots $x = 0$ and $x = 3$. However, $x = 3$ is not in the domain of the function, so the only x-intercept is at $x = 0$.

C Symmetry: there is no obvious relationship between $y(x)$ and $y(-x)$ so there is no obvious symmetry.

D Asymptotes: $\lim_{x\to\pm\infty}(x^2+3x)/(9-x^2) = \lim_{x\to\pm\infty}(1+3/x)/(9/x^2-1) = -1$, so the only horizontal asymptote is at $y = 1$. For vertical asymptotes, we take one-sided limits as x approaches the locations at which the denominator is zero:

$$\lim_{x\to-3^-} \frac{x^2+3x}{9-x^2} = \lim_{x\to-3^-} \frac{x(x+3)}{(3+x)(3-x)} = \lim_{x\to-3^-} \frac{x}{3-x} = \frac{-3}{6} = -\frac{1}{2}$$

$$\lim_{x\to-3^+} \frac{x^2+3x}{9-x^2} = \lim_{x\to-3^+} \frac{x(x+3)}{(3+x)(3-x)} = \lim_{x\to-3^+} \frac{x}{3-x} = \frac{-3}{6} = -\frac{1}{2}$$

$$\lim_{x\to 3^-} \frac{x^2+3x}{9-x^2} = \lim_{x\to 3^-} \frac{x(x+3)}{(3+x)(3-x)} = \lim_{x\to 3^-} \frac{x}{3-x} = \frac{3}{+0} = +\infty$$
$$\lim_{x\to 3^+} \frac{x^2+3x}{9-x^2} = \lim_{x\to 3^+} \frac{x(x+3)}{(3+x)(3-x)} = \lim_{x\to 3^+} \frac{x}{3-x} = \frac{3}{-0} = -\infty$$

The only vertical asymptote is at $x = 3$. The function has a removable discontinuity at $x = -3$.

E Intervals of increase/decrease: to simplify the calculation, remove the common factor of $x+3$ from the numerator and denominator and then apply the quotient rule:

$$y' = \frac{(3-x) - x(-1)}{(3-x)^2} = \frac{3}{(3-x)^2}$$

The first derivative y' is always positive on the domain of the function, so the function is increasing on its domain.

F Local max/min: since y' never changes sign, there are no local min/max.

G Concavity and inflection points: by the chain rule the second derivative is

$$y'' = -6(3-x)^{-3}(-1) = 6(3-x)^{-3}$$

The second derivative is positive for $x < 3$, so concave up; the second derivative is negative for $x > 3$, so concave down. The only candidate for an inflection point would be at $x = 3$ where the concavity changes, but $x = 3$ is not in the domain of the function, so no inflection points.

H Graph: see Figure 3.

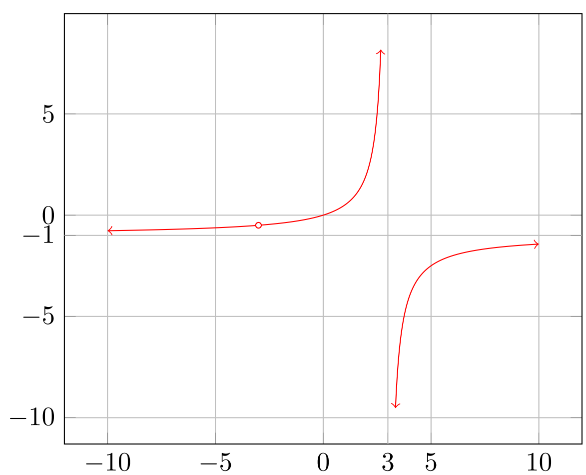

Figure 3: Graph of $y = \dfrac{x^2+3x}{9-x^2}$

(b) A Domain: the denominator is always positive, so the domain is all of $\mathbb{R}$.

B Intercepts: when $x = 0$, $y = (0-2)^2/(0^2+4) = 4/4 = 1$, so $(0, 1)$ is the y-intercept. When $y = 0$, $(x-2)^2/(x^+4) = 0$ which implies $(x-2)^2 = 0$ so $x = 2$. The point $(2, 0)$ is the only x-intercept.

C Symmetry: there is no obvious relationship between $y(x)$ and $y(-x)$, so there is no obvious symmetry.

D Asymptotes:

$$\lim_{x\to\pm\infty} \frac{(x-2)^2}{x^2+4} = \lim_{x\to\pm\infty} \frac{(x-2)^2/x^2}{(x^2+4)/x^2} = \lim_{x\to\pm\infty} \frac{(1-2/x)^2}{1+4/x^2} = \frac{(1-0)^2}{1+0} = 1$$

so $y = 1$ is the only horizontal asymptote. Since the domain of this rational function is $\mathbb{R}$, there are no vertical asymptotes.

E Intervals of increase/decrease: the derivative is

$$\begin{aligned} y' &= \frac{(x^2+4)\cdot 2(x-2) - (x-2)^2 \cdot 2x}{(x^2+4)^2} = \frac{2x^3 - 4x^2 + 8x - 16 - 2x^3 + 8x^2 - 8x}{(x^2+4)^2} \\ &= \frac{4x^2 - 16}{(x^2+4)^2} = \frac{4(x+2)(x-2)}{(x^2+4)^2} \end{aligned}$$

The first derivative is positive for $x < -2$, negative for $-2 < x < 2$, and positive for $2 < x$.

F Local max/min: the derivative changes sign at $x = -2$ (local max) and at $x = 2$ (local min).

G Concavity and inflection points: the second derivative is

$$\begin{aligned} y'' &= \frac{(x^2+4)^2 \cdot 8x - 4(x^2-4)\cdot 2(x^2+4)\cdot 2x}{(x^2+4)^4} \\ &= \frac{(x^2+4)\cdot 8x - 4(x^2-4)\cdot 4x}{(x^2+4)^3} \\ &= \frac{8x^3 + 32x - 16x^3 + 64x}{(x^2+4)^3} \\ &= \frac{-8x^3 + 96x}{(x^2+4)^3} = -\frac{8x(x+2\sqrt{3})(x-2\sqrt{3})}{(x^2+4)^3} \end{aligned}$$

The second derivative is positive on $x < -2\sqrt{3}$, negative on $-2\sqrt{3} < x < 0$, positive on $0 < x < 2\sqrt{3}$, and negative on $2\sqrt{3} < x$. Because the concavity changes at $x = -2\sqrt{3}, 0, 2\sqrt{3}$, they are all inflection points.

H Graph: see Figure 4.

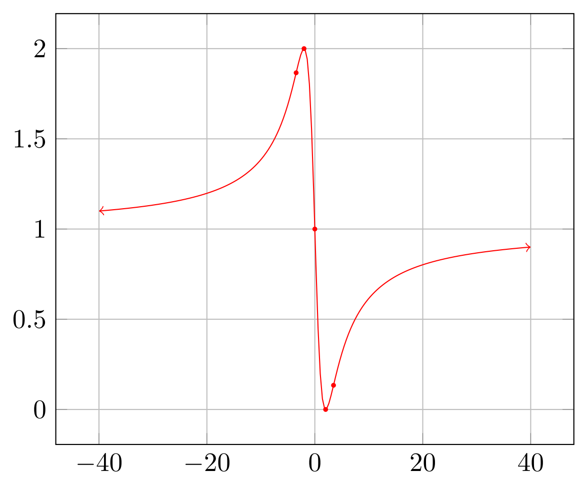

Figure 4: Graph of $y = \dfrac{(x-2)^2}{x^2+4}$

3. (a) A Domain: the function is defined where the radicand $x^2+2x \geq 0$. Factoring, we need $(x+2)x \geq 0$, which is true when both factors are negative or when both factors are positive, in other words, on $x \leq -2$ and $0 \leq x$.

B Intercepts: $x = 0$ implies $y = \sqrt{0^2 + 2(0)} - 0 = 0$, giving the y-intercept at $(0, 0)$. $y = 0$ implies $\sqrt{x^2 - 2x} - x = 0$ which implies $\sqrt{x^2 - 2x} = x$ and $x^2 - 2x = x^2$ and finally $x = 0$, so the only x-intercept is $(0, 0)$.

C Symmetry: there is no obvious relationship between $y(x)$ and $y(-x)$ so there are no obvious symmetries.

D Asymptotes: we evaluate the limit at infinity:

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+2x} - x &= \lim_{x\to\infty} (\sqrt{x^2+2x} - x) \cdot \frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x} \\
&= \lim_{x\to\infty} \frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{x^2+2x}/\sqrt{x^2}+1} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{1+2/x}+1} \\
&= \frac{2}{\sqrt{1+0}+1} = \frac{2}{2} = 1
\end{aligned}$$

On the other hand, the limit at $-\infty$ is

$$\begin{aligned}
\lim_{x\to-\infty} \sqrt{x^2+2x} - x &= \lim_{x\to-\infty} (\sqrt{x^2+2x} - x) \cdot \frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x} \\
&= \lim_{x\to-\infty} \frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x} \\
&= \lim_{x\to-\infty} \frac{2}{-\sqrt{x^2+2x}/\sqrt{x^2}+1} \\
&= \lim_{x\to-\infty} \frac{2}{-\sqrt{1+2/x}+1} \\
&= \frac{2}{-\sqrt{1+0}+1} = \frac{2}{+0} = +\infty
\end{aligned}$$

so there is no horizontal asymptote as $x \to -\infty$. Note that a negative sign appears in the calculation because $x = -\sqrt{x^2}$ for negative x, and that the value of the denominator is positive $+0$ for large negative x because $1 + 2/x < 1$ for such x. There are no vertical asymptotes because there is no denominator to go to 0 in the function.

E Intervals of increase/decrease: the derivative is

$$y' = \frac{1}{2}(x^2+2x)^{-1/2} \cdot (2x+2) - 1 = \frac{x+1}{\sqrt{x^2+2x}} - 1 = \frac{\pm\sqrt{x^2+2x+1}}{\sqrt{x^2+2x}} - 1$$

For $0 < x$, we use the $+$ sign in front of the square root, and then the fraction is always slightly larger than 0 so $y' > 0$, so y is increasing. For $x < -2$ we use the $-$ sign in front of the square root which makes $y' < 0$, so y is decreasing.

F Local max/min: the first derivative does not change signs in the domain of the function so there are no local max/min in the interior of the domain. There is a local min at both endpoints because in the neighborhood of $x = -2$ the function is decreasing and in the neighborhood of $x = 0$ the function is increasing.

G Concavity and inflection points: the second derivative is

$$y'' = \frac{(x^2+2x)^{1/2}(1) - (x+1)\frac{1}{2}(x^2+2x)^{-1/2}(2x+2)}{x^2+2x} = \frac{x^2+2x-(x+1)^2}{(x^2+2x)^{3/2}} = -\frac{1}{(x^2+2x)^{3/2}}$$

where we have simplified by multiplying the numerator and denominator by $(x^2+2x)^{1/2}$. The second derivative is always negative so the graph is always concave down.

H Graph: see Figure 5.

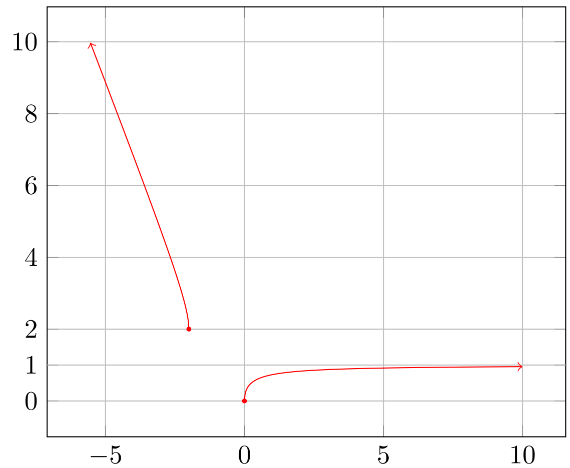

Figure 5: Graph of $y = \sqrt{x^2+2x} - x$

(b) A Domain: the radicand x^2-4 must be greater than 0 so that the square root can be taken and so that the denominator is not 0. $x^2-4>0$ implies $(x+2)(x-2)>0$ which is true for $x<-2$ and $2<x$.

B Intercepts: $x=0$ is not in the domain of the function so there is no y-intercept. For x-intercepts, $x/\sqrt{x^2-4}=0$ implies $x=0$ which is again outside of the domain, so there are no x-intercepts.

C Symmetry: $y(-x) = -x/\sqrt{(-x)^2-4} = -(x/\sqrt{x^2-4}) = -y(x)$ so the function is odd, with rotational symmetry about the origin $(0,0)$.

D Asymptotes: this function has interesting asymptotes. Taking the limit as $x\to\infty$ and dividing through by $x=\sqrt{x^2}$,

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2-4}} = \lim_{x\to\infty}\frac{x/x}{\sqrt{x^2-4}/\sqrt{x^2}} = \lim_{x\to\infty}\frac{1}{\sqrt{1-4/x^2}} = \frac{1}{\sqrt{1-0}} = 1$$

Taking the limit as $x\to-\infty$ and dividing through by $x=-\sqrt{x^2}$,

$$\lim_{x\to-\infty}\frac{x}{\sqrt{x^2-4}} = \lim_{x\to-\infty}\frac{x/x}{-\sqrt{x^2-4}/\sqrt{x^2}} = \lim_{x\to-\infty}-\frac{1}{\sqrt{1-4/x^2}} = -\frac{1}{\sqrt{1-0}} = -1$$

So this function has two different horizontal asymptotes, $y=1$ and $y=-1$, which is a phenomenon that does not occur with rational functions.

Candidates for vertical asymptotes are x-values at which the denominator of the function goes to 0, namely $x=-2$ and $x=2$. Those values are not in the domain of the function, but we can still take one-sided limits as x approaches those values:

$$\lim_{x\to-2^-}\frac{x}{\sqrt{x^2-4}} = \frac{-2}{+0} = -\infty$$

$$\lim_{x\to 2^+} \frac{x}{\sqrt{x^2-4}} = \frac{2}{+0} = +\infty$$

which show that both $x = -2$ and $x = 2$ are vertical asymptotes.

E Intervals of increase/decrease: the derivative is

$$y' = \frac{(x^2-4)^{1/2}(1) - x \cdot \frac{1}{2}(x^2-4)^{-1/2}(2x)}{x^2-4} = \frac{x^2-4-x^2}{(x^2-4)^{3/2}} = -\frac{4}{(x^2-4)^{3/2}}$$

which is always negative so the function is always decreasing.

F Local max/min: since the function is always decreasing, there can be no local max/min.

G Concavity and inflection points: the second derivative is

$$y'' = 6(x^2-4)^{-5/2} \cdot 2x$$

which has the same sign as x, so the function is concave down for $x < -2$ and concave up for $2 < x$.

H Graph: see Figure 6.

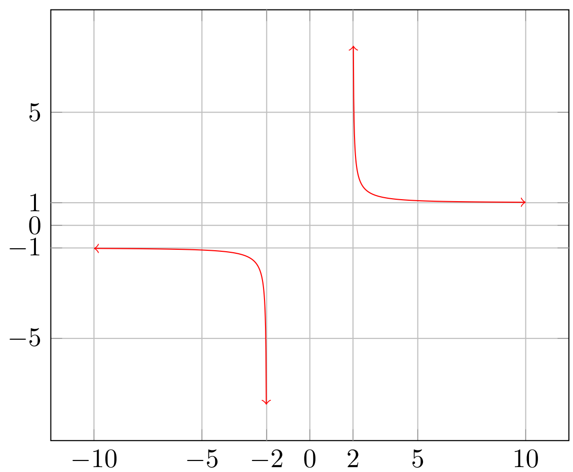

Figure 6: Graph of $y = \dfrac{x}{\sqrt{x^2-4}}$

4. (a) A Domain: the domain is given in the problem statement: $[-2\pi, 2\pi]$.

B Intercepts: when $x = 0$, $y = 0 + \sin 0 = 0 + 0 = 0$, so $(0,0)$ is the y-intercept. For the x-intercept we need to solve $0 = x + \sin x$. Solving such trigonometric equations can be difficult, but we have done most of the work already in Problem Set 3.2 where we used the mean value theorem to show that $|\sin x| \le |x|$ for all x. A slight refinement of the solutions to Problem Set 3.2 shows that $|\sin x| < |x|$ for $-2\pi \le x < 0$ and for $0 < x \le 2\pi$. So the only possible solution in our domain is $x = 0$, and a quick check shows that $x = 0$ is a solution.

C Symmetry: we have $y(-x) = -x + \sin(-x) = -x - \sin x = -y(x)$ so the function is odd, with rotational symmetry about the origin $(0,0)$.

D Asymptotes: the domain is finite so there are no horizontal asymptotes. The values of the function are bounded on its domain (because $-2\pi - 1 \le x + \sin x \le 2\pi + 1$ so there are no vertical asymptotes.

E Intervals of increase/decrease: the derivative is

$$y' = 1 + \cos x$$

Since $-1 \le \cos x \le 1$, it follows that $y' \ge 0$, and the function is non-decreasing.

F Local max/min: the points where $y' = 0$ are solutions to $\cos x = -1$, which we know from the graph of cosine are $x = -\pi$ and $x = \pi$. (There are more solutions: $x = k\pi$, where k is an odd integer, but $\pm\pi$ are the only solutions in the domain $[-2\pi, 2\pi]$.) However, the derivative does not change sign at $\pm\pi$, so they are not locations of local max/min. Since the function is defined on a finite interval, the endpoints are candidates for local max/min: $x = -2\pi$ is a local min because the function is increasing in its neighborhood, and similarly $x = 2\pi$ is a local max because the function is increasing in its neighborhood.

G Concavity and inflection points: the second derivative is $y'' = -\sin x$, which one can easily graph to obtain $y'' < 0$ on the interval $-2\pi < x < -\pi$, $y'' > 0$ on the interval $-\pi < x < 0$, $y'' < 0$ on the interval $0 < x < \pi$, and $y'' < 0$ on the interval $\pi < x < 2\pi$. Since the concavity changes at each of $x = -\pi, 0, \pi$, those are all inflection points.

H Graph:

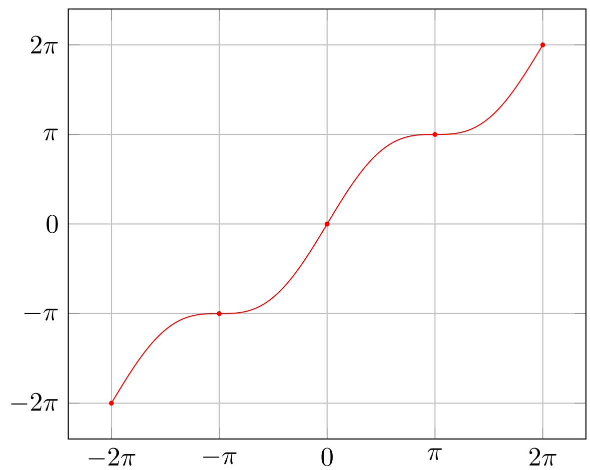

Figure 7: Graph of $y = x + \sin x$

(b) A Domain: the function is defined on $\mathbb{R}$ except where the denominator is 0. However, $-1 \leq \cos x \leq 1$ for all x, so $2 + \cos x$ is never 0.

B Intercepts: when $x = 0$, $y = \sin(0)/(2 + \cos(0)) = 0/3 = 0$ so the y-intercept is $(0, 0)$. When $y = 0$, $\sin x/(2 + \cos x) = 0$ which implies $\sin x = 0$. From a graph of $\sin x$ we see that the solutions are at $x = k\pi$ where k is any integer. So there are x-intercepts at the points $(k\pi, 0)$ where k is any integer.

C Symmetry: there is no obvious relationship between $y(x)$ and $y(-x)$ so there is no obvious mirror or rotational symmetry. However, with trigonometric functions we often see another kind of symmetry, "translational" symmetry, also called "periodicity". In this case

$$y(x + 2\pi) = \frac{\sin(x + 2\pi)}{2 + \cos(x + 2\pi)} = \frac{\sin x}{2 + \cos x} = y(x)$$

so the function is periodic with period 2π. We only have to graph a single period of the function to get a complete picture of its behaviour (but we will graph several periods below so that we can see what periodicity means geometrically).

D Asymptotes: because of periodicity, there are no horizontal asymptotes. Because the denominator never goes to 0 there are no vertical asymptotes.

E Intervals of increase/decrease: the first derivative is

$$y' = \frac{(2 + \cos x) \cdot \cos x - \sin x(-\sin x)}{(2 + \cos x)^2} = \frac{2\cos x + \cos^2 x + \sin^2 x}{(2 + \cos x)^2} = \frac{2\cos x + 1}{(2 + \cos x)^2}$$

where we have used the Pythagorean identity to simplify in the last step. The sign of the derivative is the same as the sign of $2\cos x + 1$. To find where it changes sign we solve $2\cos x + 1 = 0$ which is equivalent to $\cos x = -1/2$. From our knowledge of trigonometric functions we have solutions $x = \pm 2\pi/3 + 2k\pi$; in other words, $2\cos x + 1$ is positive for $0 \le x < 2\pi/3$, negative on $2\pi/3 < x < 4\pi/3$, positive on $4\pi/3 < x \le 2\pi$, and the pattern recurs with a period of 2π.

F Local max/min: the first derivative goes from positive to negative at $x = 2\pi/3$ so there is a local max there (and consequently at every x value of the form $2\pi/3 + 2k\pi$ where k is an integer). The first derivative goes from negative to positive at $x = 4\pi/3$ so there is a local min there (and consequently at ever x value of the form $4\pi/3 + 2k\pi$ where k is an integer).

G Concavity and inflection points: the second derivative is

$$\begin{aligned}
y'' &= \frac{(2+\cos x)^2 \cdot 2(-\sin x) - (2\cos x + 1)\cdot 2(2+\cos x)(-\sin x)}{(2+\cos x)^4} \\
&= \frac{-8\sin x - 8\sin x\cos x - 2\sin x\cos^2 x + 8\sin x\cos x + 4\sin x\cos^2 x + 4\sin x + 2\sin x\cos x}{(2+\cos x)^4} \\
&= \frac{-4\sin x + 2\sin x\cos^2 x + 2\sin x\cos x}{(2+\cos x)^4} \\
&= \frac{2\sin x(\cos^2 x + \cos x - 2)}{(2+\cos x)^4} \\
&= \frac{2\sin x(\cos x + 2)(\cos x - 1)}{(2+\cos x)^4} \\
&= \frac{2\sin x(\cos x - 1)}{(2+\cos x)^3}
\end{aligned}$$

The second derivative is 0 whenever $\sin x = 0$ or $\cos x = 1$. The solutions to those equations are actually the same, namely $x = k\pi$ where k is an integer, so those are our potential points of inflection. Note that $\cos x - 1 \le 0$ and $2 + \cos x > 0$ so the sign of the second derivative depends on $\sin x$ only; the latter goes from negative to positive at $x = 0$, from positive to negative at $x = \pi$, and so on, so all the potential inflection points are actual inflection points.

H Graph: see Figure 8.

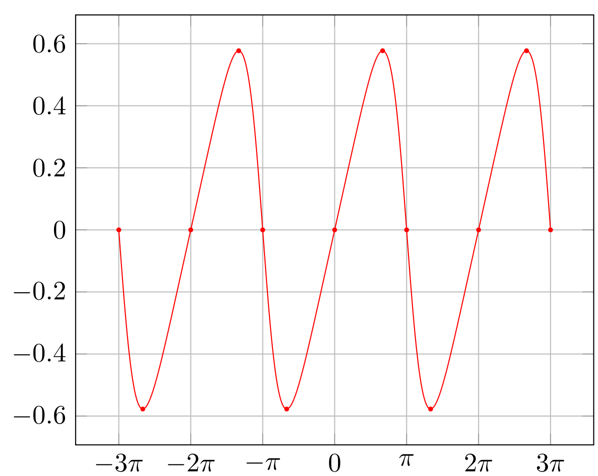

Figure 8: Graph of $y = \dfrac{\sin x}{2+\cos x}$

5. The functions both are rational with numerator of degree one higher than denominator, so they should both have slant asymptotes. We do polynomial division to find those slant asymptotes.

   (a) By polynomial division,
   $$5x^2 - 7x + 8 = (x-3)(5x+8) + 32$$
   so
   $$\frac{5x^2 - 7x + 8}{x-3} = 5x + 8 + \frac{32}{x-3} \implies \lim_{x\to\pm\infty} \frac{5x^2-7x+8}{x-3} - (5x+8) = \lim_{x\to\pm\infty} \frac{32}{x-3} = 0$$
   It follows that the slant asymptote is $y = 5x + 8$.

   (b) By polynomial division,
   $$2x^3 - 4x^2 + 5 = (x^2 - 3x - 1)(2x+2) + 8x + 7$$
   so
   $$\lim_{x\to\pm\infty} \frac{2x^3 - 4x^2 + 5}{x^2 - 3x - 1} - (2x+2) = \lim_{x\to\pm\infty} \frac{8x+7}{x^2-3x-1} = 0$$
   and $y = 2x + 2$ is the slant asymptote.

6. We follow the same set of steps as always, except we look for slant asymptotes instead of horizontal asymptotes in step D.

   (a) A Domain: the domain of the rational function is all real numbers except those where the denominator is 0, namely $x = -2$.

   B Intercepts: when $x = 0$, $y = 1/2$. When $y = 0$, $1 - 2x - 3x^2 = 0$. Factoring, $(1+x)(1-3x) = 0$ which gives x-intercepts at $(-1, 0)$ and $(1/3, 0)$.

   C Symmetry: there is no obvious relationship between $y(x)$ and $y(-x)$ so no obvious symmetry.

   D Asymptotes: since the degree of the numerator is one more than the degree of the denominator, there is no horizontal asymptote, but there may be a slant asymptote. By polynomial division,
   $$1 - 2x - 3x^2 = (x+2)(-3x+4) - 7$$
   so
   $$\frac{1-2x-3x^2}{x+2} = -3x + 4 - \frac{7}{x+2} \implies \lim_{x\to\pm\infty} \frac{1-2x-3x^2}{x+2} - (-3x+4) = \lim_{x\to\pm\infty} -\frac{7}{x+2} = 0$$
   so the slant asymptote is $y = -3x + 4$.
   There is also a vertical asymptote at $x = -2$ because
   $$\lim_{x\to-2^-} \frac{1-2x-3x^2}{x+2} = \lim_{x\to-2^-} \frac{(1+x)(1-3x)}{x+2} = \frac{(-1)(7)}{-0} = +\infty$$
   and
   $$\lim_{x\to-2^+} \frac{1-2x-3x^2}{x+2} = \lim_{x\to-2^+} \frac{(1+x)(1-3x)}{x+2} = \frac{(-1)(7)}{+0} = -\infty$$

   E Intervals of increase/decrease: the derivative is
   $$y' = \frac{(x+2)(-2-6x) - (1-2x-3x^2)(1)}{(x+2)^2} = \frac{-2x - 6x^2 - 4 - 12x - 1 + 2x + 3x^2}{(x+2)^2}$$
   $$= \frac{-3x^2 - 12x - 5}{(x+2)^2}$$

The roots of the quadratic in the numerator can be found using the quadratic formula:

$$x = -2 \pm \sqrt{\frac{7}{3}}$$

The numerator is negative to the left of the first root (y is decreasing), is positive between the roots (y is increasing), and is negative to the right of the second root (y is decreasing).

F Local max/min: since the derivative changes sign at its two roots, they are a local min and local max respectively.

G Concavity and inflection points: after some calculation, the second derivative is

$$y'' = -\frac{14}{(x+2)^3}$$

which has the opposite sign of $x+2$, so the function is concave up on $x < -2$ and concave down on $-2 < x$. The point $x = -2$ is not an inflection point because the function is not defined there.

H Graph: see Figure 9.

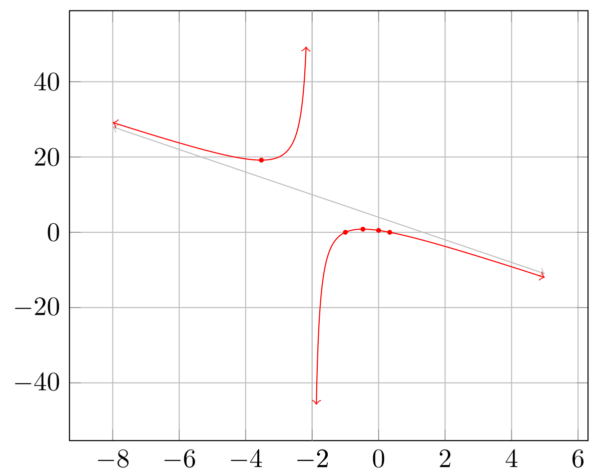

Figure 9: Graph of $y = \dfrac{1 - 2x - 3x^2}{x+2}$

(b) A Domain: the domain of this rational function is all of $\mathbb{R}$ except for x values where the denominator is 0, namely $x = -2$.

B Intercepts: $x = 0$ implies $y = -8/4 = -2$. $y = 0$ implies $(x-2)^3 = 0$ with solution $x = 2$.

C Symmetry: there is no obvious relationship between $y(x)$ and $y(-x)$, so no obvious symmetry.

D Asymptotes: since the degree of the numerator is one more than the degree of the denominator, there is no horizontal asymptote, but there may be a slant asymptote. By polynomial division,

$$x^3 - 2x^2 + 4x - 8 = (x^2 + 4x + 4)(x - 6) + 24x + 16$$

By a calculation similar to that of the previous problem, the slant asymptote is $y = x - 6$. There is a potential vertical asymptote at $x = -2$ which we can easily verify by taking limits:

$$\lim_{x \to -2^-} \frac{(x-2)^3}{(x+2)^2} = \frac{-64}{+0} = -\infty$$
$$\lim_{x \to -2^+} \frac{(x-2)^3}{(x+2)^2} = \frac{-64}{+0} = -\infty$$

E Intervals of increase/decrease: the derivative is

$$y' = \frac{(x+2)^2 \cdot 3(x-2)^2 - (x-2)^3 \cdot 2(x+2)}{(x+2)^4} = \frac{(x-2)^2(3(x+2)-2(x-2))}{(x+2)^3} = \frac{(x-2)^2(x+10)}{(x+2)^3}$$

The sign of y' changes from positive to negative at $x = -10$, and then changes from negative to positive at $x = -2$. It then goes to 0 at $x = 2$ but stays positive for $2 < x$. The function increases on $x < -10$, decreases on $-10 < x < -2$, increases on $-2 < x < 2$, and increases on $2 < x$.

F Local max/min: since the derivative changes sign from positive to negative at $x = -10$ there is a local max there. There is no local min at $x = -2$ because the function is not defined there. There is no local max/min at $x = 2$ because the derivative does not change sign there.

G Concavity and inflection points: after some calculating,

$$y'' = \frac{96(x-2)}{(x+2)^4}$$

which is negative (concave down) for $x < 2$ and positive (concave up) for $2 < x$.

H Graph: see Figure 10.

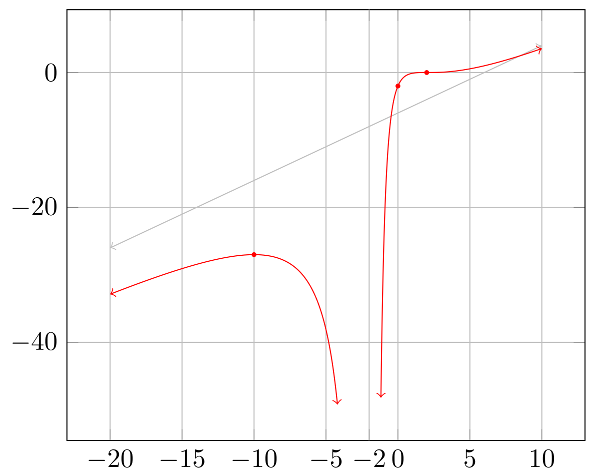

Figure 10: Graph of $y = \dfrac{(x-2)^3}{(x+2)^2}$

7. This problem is similar to the other graphing problems we have done, except that it involves the parameters m_0 and c instead of just numbers.

A Domain: the radicand $1-v^2/c^2$ must be greater than 0 for the function to be defined; $1-v^2/c^2 > 0$ implies $v^2 < c^2$, or in other words, $-c < v < c$.

B Intercepts: when $v = 0$, $m = m_0$. It is not possible to have $m = 0$ (unless $m_0 = 0$, which is an uninteresting special case).

C Symmetry: $m(-v) = m_0/\sqrt{1-(-v)^2/c^2} = m_0/\sqrt{1-v^2/c^2} = m(v)$ so the function has mirror symmetry in the y-axis.

D Asymptotes: the domain is bounded, so there can be no horizontal or slant asymptotes. The denominator goes to 0 when $v \to \pm c$ so there may be vertical asymptotes at $v = \pm c$. Checking the relevant limits,

$$\lim_{v \to -c^+} \frac{m_0}{\sqrt{1-v^2/c^2}} = \frac{m_0}{+0} = +\infty$$

$$\lim_{v \to c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}} = \frac{m_0}{+0} = +\infty$$

E Intervals of increase/decrease: the derivative is

$$y' = m_0 \cdot -\frac{1}{2}(1 - v^2/c^2)^{-3/2} \cdot -2v/c^2 = \frac{m_0 v}{c^2(1 - v^2/c^2)^{3/2}}$$

which has the same sign as v, $m(v)$ is decreasing on $-c < v < 0$ and increasing on $0 < v < c$.

F Local max/min: the derivative changes sign from negative to positive at $v = 0$, so that is a local min.

G Concavity and inflection points: the second derivative is

$$\begin{aligned} y'' &= \frac{m_0}{c^2}\left((1 - v^2/c^2)^{-3/2} + v \cdot -\frac{3}{2}(1 - v^2/c^2)^{-5/2} \cdot -2v/c^2\right) \\ &= \frac{m_0}{c^2}\left((1 - v^2/c^2) + 3v^2/c^2\right)(1 - v^2/c^2)^{-5/2} \\ &= \frac{m_0}{c^2}\frac{1 + 2v^2/c^2}{(1 - v^2/c^2)^{5/2}} \end{aligned}$$

The second derivative is always positive so the function is concave up everywhere in its domain.

H Graph: see Figure 11.

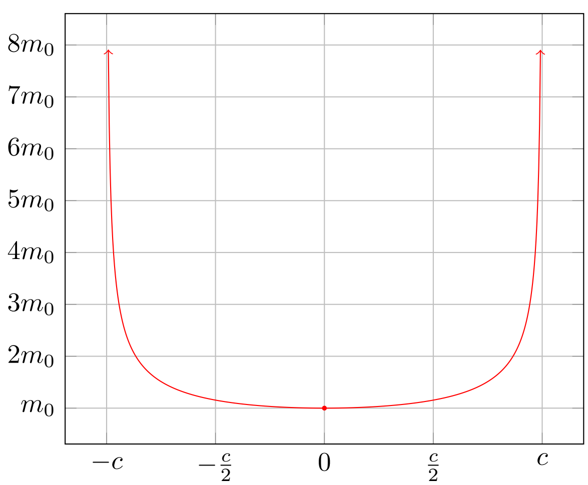

Figure 11: Graph of $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$

8. The slant asymptotes are checked in step D below.

A Domain: the radicand must be greater than or equal to 0, so $x^2 - 4x \geq 0$. Factoring, $x(x - 4) \geq 0$ which is true when $x \leq 0$ or when $4 \geq x$.

B Intercepts: $x = 0$ implies $y = 0$. $y = 0$ implies $x^2 - 4x = 0$ with solutions $x = 0$, $x = 4$.

C Symmetry: there is no obvious relationship between $y(x)$ and $y(-x)$ so there is no obvious symmetry.

D Asymptotes: as suggested, we will investigate two potential slant asymptotes by taking the limit as $x \to \pm\infty$ of the difference between the function and the proposed asymptote. In the limit calculation, we will multiply and divide by conjugate radicals.

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2-4x}-(x-2) &= \lim_{x\to\infty} \left(\sqrt{x^2-4x}-(x-2)\right)\cdot\frac{\sqrt{x^2-4x}+(x-2)}{\sqrt{x^2-4x}+(x-2)} \\
&= \lim_{x\to\infty} \frac{x^2-4x-x^2+4x-4}{\sqrt{x^2-4x}+(x-2)} \\
&= \lim_{x\to\infty} \frac{-4}{\sqrt{x^2-4x}+(x-2)} = \frac{4}{\infty} = 0
\end{aligned}$$

which shows that $y = x-2$ is a slant asymptote as $x \to \infty$. The limit as $x \to -\infty$ is similar:

$$\begin{aligned}
\lim_{x\to-\infty} \sqrt{x^2-4x}-(-x+2) &= \lim_{x\to-\infty} \left(\sqrt{x^2-4x}-(-x+2)\right)\cdot\frac{\sqrt{x^2-4x}+(-x+2)}{\sqrt{x^2-4x}+(-x+2)} \\
&= \lim_{x\to-\infty} \frac{x^2-4x-x^2+4x-4}{\sqrt{x^2-4x}+(-x+2)} \\
&= \lim_{x\to-\infty} \frac{-4}{\sqrt{x^2-4x}+(-x+2)} = \frac{4}{\infty} = 0
\end{aligned}$$

where in the last line we have used $-x$ is positive. (If we had used $x-2$ instead of $-x+2$ in this calculation, the $\sqrt{x^2-4x}$ and $x-2$ in the denominator would cancel and we would not get the denominator tending to ∞.) That shows that $y = -x+2$ is a slant asymptote as $x \to -\infty$.

E Intervals of increase/decrease: the derivative is

$$y' = \frac{1}{2}(x^2-4x)^{-1/2}\cdot(2x-4) = \frac{x-2}{\sqrt{x^2-4x}}$$

which has the same sign as $x-2$ so the function is decreasing on $x < 0$ and increasing on $4 < x$.

F Local max/min: the derivative does not change sign in the domain so there are no local max/min except for the endpoints of the domain, $x = 0$ and $x = 4$, which are both local minimums.

G Concavity and inflection points: the second derivative of the function is

$$y'' = (x^2-4x)^{-1/2} + (x-2)\cdot -\frac{1}{2}(x^2-4x)^{-3/2}\cdot(2x-4) = \frac{x^2-4x-(x-2)^2}{(x^2-4x)^{3/2}} = -\frac{4}{(x^2-4x)^{3/2}}$$

which is always negative so the function is always concave down.

H Graph: see Figure 12.

Now that we have a new technique for checking slant asymptotes, you should go back to Problem 3a to try to figure out what the slant asymptote is, and then to check it by taking the limit $x \to -\infty$.

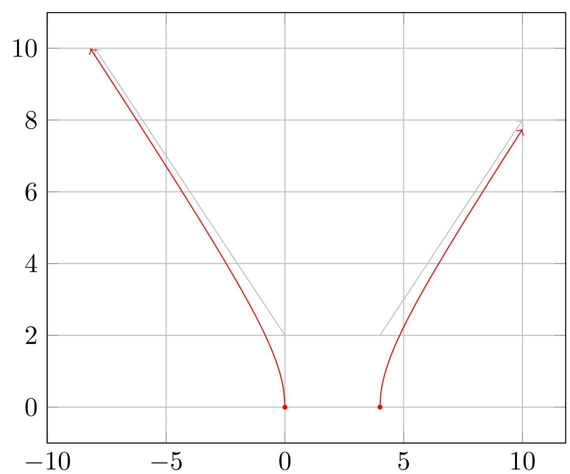

Figure 12: Graph of $y = \sqrt{x^2 - 4x}$