MATH 110 Problem Set 5.1 Solutions

Edward Doolittle

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1. See Figure 1 on page 2 for the graphs.

   (a) From the graph, we see that $g(x) = x+4$ is above $f(x) = x^2 - 2x$ throughout the region we are integrating. We need to find the intersections of the two curves to determine the bounds of integration. We have

   $$x^2 - 2x = x+4 \implies x^2 - 3x - 4 = 0 \implies (x+1)(x-4) = 0$$

   so the intersections are at $x = -1$ and $x = 4$. Then the area is

   $$\int_{-1}^{4} (g(x) - f(x))\,dx = \int_{-1}^{4} (x+4-x^2+2x)\,dx = \int_{-1}^{4} (-x^2+3x+4)\,dx = -\frac{1}{3}x^3 - \frac{3}{2}x^2 - 4x\bigg|_{-1}^{4}$$
   $$= -\frac{64}{3} + \frac{48}{2} + 16 - \frac{1}{3} - \frac{3}{2} + 4 = \frac{125}{6}$$

   (b) From the graph, $f(x) = 1+\sqrt{x}$ is above $g(x) = (3+x)/3$ in the area between the curves. To find the intersection we need to solve the equation

   $$1+\sqrt{x} = \frac{3+x}{3} \implies 3+3\sqrt{x} = 3+x \implies x - 3\sqrt{x} = 0 \implies \sqrt{x}(\sqrt{x}-3) = 0$$

   with solutions $\sqrt{x} = 0$ and $\sqrt{x} = 3$, or in other words, $x = 0$ and $x = 9$. Then the area is

   $$\int_0^9 (f(x) - g(x))\,dx = \int_0^9 \left(1+\sqrt{x} - 1 - \frac{x}{3}\right)\,dx = \frac{2}{3}x^{3/2} - \frac{1}{6}x^2\bigg|_0^9 = \frac{2}{3}9^{3/2} - \frac{1}{6}9^2 = \frac{9}{2}$$

   (c) From the graph, $g(x) = 4x - x^2$ is above $f(x) = x^2$ in the region of interest to us. The curves intersect when

   $$x^2 = 4x - x^2 \implies 2x^2 - 4x = 0 \implies 2x(x-2) = 0$$

   so the curves intersect when $x = 0$ and $x = 2$. The area is

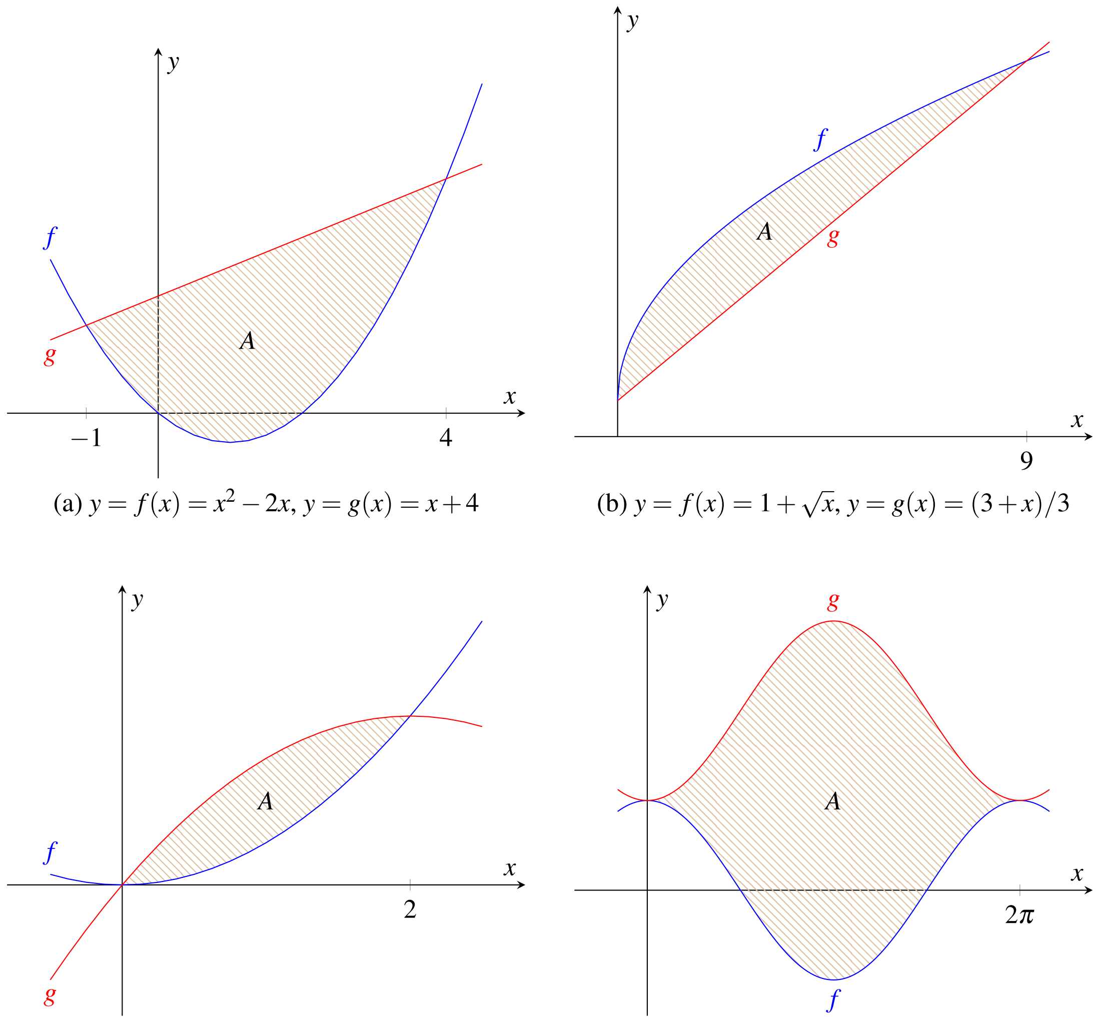
   $$\int_0^2 (4x - x^2 - x^2)\,dx = \int_0^2 (4x - 2x^2)\,dx = 2x^2 - \frac{2}{3}x^3\bigg|_0^2 = 8 - \frac{16}{3} = \frac{8}{3}$$

   (d) From the graph, $g(x) = 2 - \cos x$ is above $f(x) = \cos x$. The graphs intersect when

   $$2 - \cos x = \cos x \implies 2\cos x = 2 \implies \cos x = 1$$

   which happens when $x = 0, 2\pi, 4\pi, \ldots$. Because the domain $0 \le x \le 2\pi$ is given in the question, we ignore what happens outside of that domain, and the only intersections of interest are $x = 0$ and $x = 2\pi$. The area is

   $$\int_0^{2\pi} (2 - \cos x - \cos x)\,dx = \int_0^{2\pi} (2 - 2\cos x)\,dx = 2x - 2\sin x\bigg|_0^{2\pi} = 4\pi - 0 = 4\pi$$



(a) $y = f(x) = x^2 - 2x$, $y = g(x) = x + 4$

(b) $y = f(x) = 1 + \sqrt{x}$, $y = g(x) = (3 + x)/3$

(c) $y = f(x) = x^2$, $y = g(x) = 4x - x^2$

(d) $y = f(x) = \cos x$, $y = g(x) = 2 - \cos x$

Figure 1: Graphs for Question 1

2. These problems are harder than the previous, for various different reasons. See Figure 2 on page 5 for the graphs.

   (a) In this case, it's better to think of the curves as functions of y instead of x. For one thing, when we write $4x+y^2=12$ as a function of y, it is a polynomial, but if we write it at as a function of x, it uses a square root, which may be harder to integrate. For another, $y^2=12-4x$ has two branches, $y=\sqrt{12-4x}$ and $y=-\sqrt{12-4x}$, both of which are involved in this question, complicating the analysis. Note that $x=f(y)=3-y^2/4$ is greater than $x=g(y)=y$ in the region of interest. The intersections are when

$$3-y^2/4=y \implies y^2+4y-12=0 \implies (y+6)(y-2)=0$$

   i.e., when $y=-6$ and $y=2$, so the area is

$$\int_{-6}^{2}(f(y)-g(y))\,dy=\int_{-6}^{2}\left(3-\frac{y^2}{4}-y\right)\,dy=3y-\frac{y^3}{12}-\frac{y^2}{2}\Big|_{-6}^{2}=6-\frac{2}{3}-2+18-18+18=\frac{64}{3}$$

   (b) From the graph we see $y=f(x)=\sin(\pi x/2)$ is above $y=g(x)=x$ in the region of interest. The curves intersect when

$$\sin(\frac{\pi x}{2})=x$$

   which is an equation we don't know how to solve. Instead, we try to guess the solutions. When $x=0$, $\sin(\pi 0/2)=\sin 0=0$, so $x=0$ is a solution. When $x=1$, $\sin(\pi 1/2)=\sin\pi/2=1$, so $x=1$ another solution. There is yet another solution when $x=-1$ (check it). For now, let's just concentrate on the interval $[0,1]$. We'll deal with $[-1,0]$ later. We have

$$A_1=\int_{0}^{1}(f(x)-g(x))\,dx=\int_{0}^{1}\left(\sin(\frac{\pi x}{2})-x\right)\,dx=-\frac{2}{\pi}\cos(\frac{\pi x}{2})-\frac{x^2}{2}\Big|_{0}^{1}=0-\frac{1}{2}+\frac{2}{\pi}-0=\frac{2}{\pi}-\frac{1}{2}$$

   Now, by symmetry, we can conclude that $A_2=A_1$, so altogether the area between the two curves is

$$2A_1=2\left(\frac{2}{\pi}-\frac{1}{2}\right)=\frac{4}{\pi}-1$$

   (c) The two curves coincide when

$$\cos x=1-\cos x \implies 2\cos x=1 \implies \cos x=\frac{1}{2} \implies x=\frac{\pi}{3}$$

   There are other solutions to the equation, but they are outside of the interval $[0,\pi]$. The area is in two parts; for A_1, $y=f(x)$ is above $y=g(x)$ so the area is

$$A_1=\int_{0}^{\pi/3}(f(x)-g(x))\,dx=\int_{0}^{\pi/3}(2\cos x-1)\,dx=2\sin x-x\Big|_{0}^{\pi/3}=\sqrt{3}-\frac{\pi}{3}$$

   On the other hand, for A_2, the area is

$$A_2=\int_{\pi/3}^{\pi}(g(x)-f(x))\,dx=\int_{\pi/3}^{\pi}(1-2\cos x)\,dx=x-2\sin x\Big|_{\pi/3}^{\pi}=\pi-\frac{\pi}{3}+\sqrt{3}=\frac{2\pi}{3}+\sqrt{3}$$

   Altogether the area between the two curves in the domain $[0,\pi]$ is

$$A=A_1+A_2=\sqrt{3}-\frac{\pi}{3}+\frac{2\pi}{3}+\sqrt{3}=2\sqrt{3}+\frac{\pi}{3}$$

(d) We should write the absolute value function $|x|$ in terms of cases:

$$|x| = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Next we find the intersections between $y = f(x) = |x|$ and $y = g(x) = x^2 - 2$ by solving two equations. First, when $x \leq 0$, $f(x) = -x$ so we have

$$-x = x^2 - 2 \implies x^2 + x - 2 = 0 \implies (x+2)(x-1) = 0$$

We keep the solution $x = -2$ and drop the solution $x = 1$ because it is outside the domain $x \leq 0$. Similarly, when $x \geq 0$, $f(x) = x$ so we have

$$x = x^2 - 2 \implies x^2 - x - 2 = 0 \implies (x+1)(x-2) = 0$$

We keep the solution $x = 2$ and drop the solution $x = -1$ because it is outside the domain $x \geq 0$. Now we are ready to find the area. First we find the area

$$A_1 = \int_{-2}^{0} (f(x) - g(x))\, dx = \int_{-2}^{0} (-x - x^2 + 2)\, dx = -\frac{x^3}{3} - \frac{x^2}{2} + 2x\bigg|_{-2}^{0} = -\frac{8}{3} + \frac{4}{2} - 2(-2) = \frac{10}{3}$$

By symmetry, $A_2 = 10/3$ also. (Or you can evaluate the relevant integral.) Altogether, the area between the curves is

$$A = A_1 + A_2 = \frac{20}{3}$$

3. See Figure 3 on page 5 for the graph. The integral can be interpreted as the positive area between the two curves $y = f(x) = \sqrt{x+2}$ and $y = g(x) = x$. We find the intersection points between the curves by solving

$$\sqrt{x+2} = x \implies x + 2 = x^2 \implies x^2 - x - 2 = 0 \implies (x+1)(x-2) = 0$$

with solutions $x = 2$, which we keep, and $x = -1$, which we discard because it is outside the domain $[0,4]$. (It is also a "spurious" solution of the original equation. If you substitute it back into the original equation, you'll find that it doesn't work. Operations like squaring an equation may introduce such spurious solutions.) The area can be found as the sum of two parts:

$$A_1 = \int_{0}^{2} (f(x) - g(x))\, dx = \int_{0}^{2} (\sqrt{x+2} - x)\, dx = \frac{2}{3}(x+2)^{3/2} - \frac{x^2}{2}\bigg|_{0}^{2} = \frac{16}{3} - 2 - \frac{2}{3}2^{3/2} = \frac{10}{3} - \frac{4}{3}\sqrt{2}$$

and

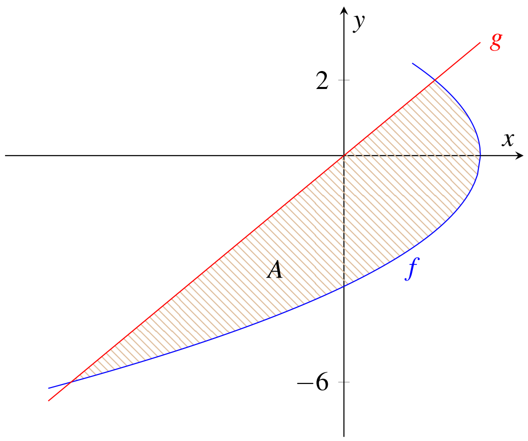
$$A_2 = \int_{2}^{4} (g(x) - f(x))\, dx = \int_{2}^{4} (x - \sqrt{x+2})\, dx = \frac{x^2}{2} - \frac{2}{3}(x+2)^{3/2}\bigg|_{2}^{4} = 8 - \frac{2}{3}6^{3/2} - 2 + \frac{16}{3} = \frac{34}{3} - 4\sqrt{6}$$

Altogether the area between the curves is

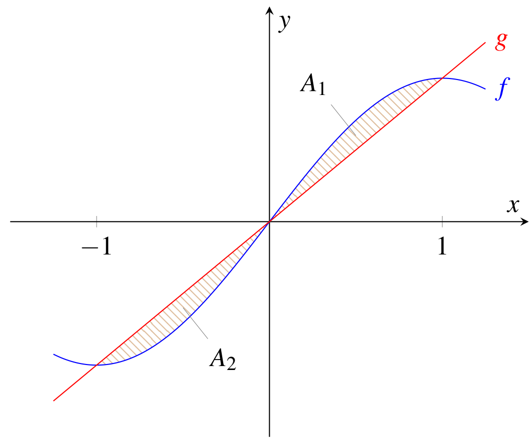
$$A = A_1 + A_2 = \frac{10}{3} - \frac{4}{3}\sqrt{2} + \frac{34}{3} - 4\sqrt{6} = \frac{44}{3} - \frac{4}{3}\sqrt{2} - 4\sqrt{6}$$

4. See Figure 4 on page 6 for the graph. The area of the region between the birth rate and death rate curves is the net change in population over the 10 year period, by the Net Change Theorem. It is

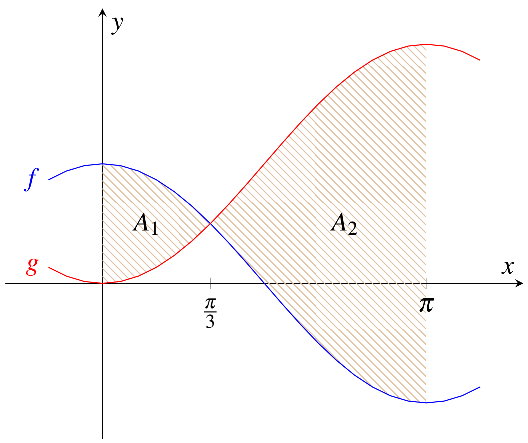
$$\int_{0}^{10} (2200 + 52.3t + 0.74t^2 - 1460 - 28.8t)\, dt = \int_{0}^{10} (0.74t^2 + 23.5t + 740)\, dt = 0.247t^3 + 11.75t^2 + 740t\bigg|_{0}^{10}$$
$$= 247 + 1175 + 7400 = 8822$$



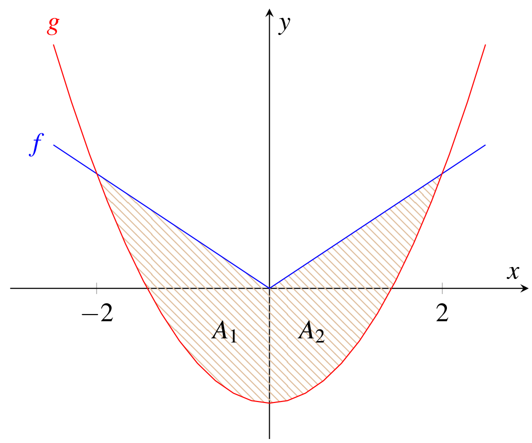
(a) $4x + y^2 = 12$, $x = y$



(b) $y = \sin(\pi x/2)$, $y = x$



(c) $y = \cos x$, $y = 1 - \cos x$



(d) $y = |x|$, $y = x^2 - 2$

Figure 2: Graphs for Question 2

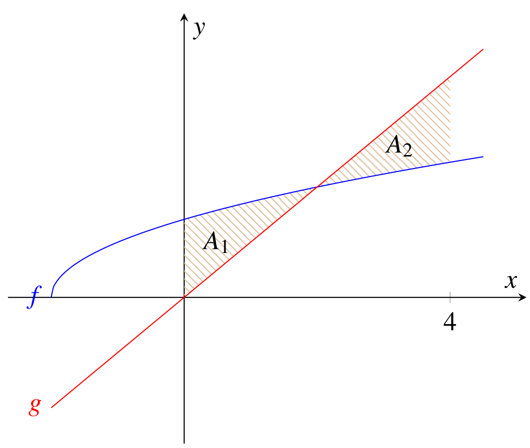


Figure 3: Graph for Question 3

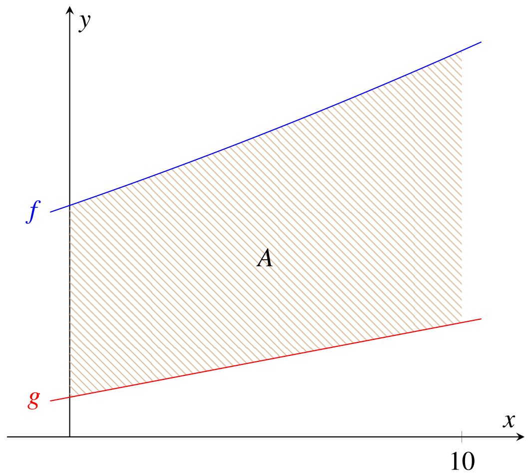


Figure 4: Graph for Question 4

5. See Figure 5 on page 7 for the graph. First, we have to find the tangent line. The slope of the tangent line is $(d/dx)\, x^2 = 2x$; the slope when $x = 1$ is $m = 2$. The point-slope equation for the tangent line is

$$y - 1 = 2(x-1) \implies y - 1 = 2x - 2 \implies y = 2x - 1$$

The tangent line only intersects the curve when $x = 1$ because

$$2x - 1 = x^2 \implies x^2 - 2x + 1 = 0 \implies (x-1)^2 = 0$$

which has only one solution, $x = 1$. Since the x-axis is also a boundary of the region in question, we need to find the intersection of the x-axis $y = 0$ with the other two curves:

$$x^2 = 0 \implies x = 0$$

is the only intersection between the parabola and the x-axis, and

$$2x - 1 = 0 \implies x = \frac{1}{2}$$

is the only intersection between the x-axis and the tangent line. From the graph we can see that the area is

$$\int_0^{1/2} (x^2 - 0)\, dx + \int_{1/2}^{1} (x^2 - (2x-1))\, dx = \left.\frac{x^3}{3}\right|_0^{1/2} + \left.\frac{x^3}{3} - x^2 + x\right|_{1/2}^{1} = \frac{1}{24} + \frac{1}{3} - 1 + 1 - \frac{1}{24} + \frac{1}{4} - \frac{1}{2} = \frac{1}{12}$$

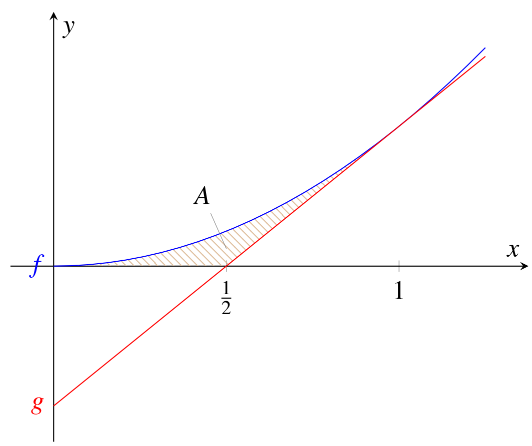


Figure 5: Graph for Question 5