MATH 110 Problem Set 1.8 Solutions

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Thursday, January 22, 2026

1. (a) Each of the multiplicands $\sqrt[3]{x}$ and $1+x^2$ is defined and continuous for all real numbers x, the former because it is a root function with an odd index 3, the latter because it is a polynomial, so the product function G is also defined and continuous for all real numbers.

   (b) The sine function is defined and continuous for all real numbers (that is just a fact you should memorize), and the function $x+1$ is defined for all real numbers, but we have to be careful about when the denominator $x+1$ is 0. Solving the equation $x+1=0$ we get $x=-1$ as a value for which the quotient $\sin x/(x+1)$ is undefined, and hence not continuous. However, $x=-1$ is outside of the domain of the function, so we can still say that the function is continuous for all values of x on the domain.

   (c) The sine and cosine functions are defined and continuous everywhere, so the composition

   $$\sin \circ \cos \circ \sin$$

   is also defined and continuous everywhere.

2. (a) When $x \neq 3$, the function $f(x)$ is the same as

   $$\frac{2x^2-5x-3}{x-3}$$

   which is continuous for $x \neq 3$, so it follows that $f(x)$ is continuous for $x \neq 3$. So the only number in question is $x=3$. Calculating the limit by factoring,

   $$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{2x^2-5x-3}{x-3} = \lim_{x\to 3} \frac{(x-3)(2x+1)}{x-3} = \lim_{x\to 3}(2x+1) = 7$$

   while on the other hand $f(3)=6$. Since $\lim\limits_{x\to 3} f(x) \neq f(3)$, f is not continuous at $x=3$. (You should be able to graph f easily. How could you change the definition of f to make it continuous?)

   (b) The function $x+1$ is continuous everywhere, so it is continuous on the interval $-\infty < x < 1$. Similarly, $1/x$ is continuous everywhere except $x=0$, so it is continuous on the interval $1<x<3$. Finally, $\sqrt{x-3}$ is continuous wherever it is defined, so it is continuous on the interval $3<x<\infty$. So the only places where g might not be continuous are $x=1$ and $x=3$. At $x=1$ we have

   $$\begin{aligned} \lim_{x\to 1^-} g(x) &= \lim_{x\to 1^-}(x+1) = 2 \\ g(1) &= x+1\big|_{x=1} = 1+1 = 2 \\ \lim_{x\to 1^+} g(x) &= \lim_{x\to 1^+} \frac{1}{x} = \frac{1}{1} = 1 \end{aligned}$$

Since not all of the above results agree, the function $g(x)$ is not continuous at $x = 1$. At $x = 3$ we have

$$\begin{aligned}
\lim_{x\to 3^-} g(x) &= \lim_{x\to 3^-} \frac{1}{x} = \frac{1}{3} \\
g(3) &= \sqrt{3-3} = \sqrt{0} = 0 \\
\lim_{x\to 3^+} g(x) &= \lim_{x\to 3^+} \sqrt{x-3} = \sqrt{3-3} = 0
\end{aligned}$$

Again, not all of the above results agree, so the function $g(x)$ is not continuous at $x = 3$.

3. (a) We convert the problem to one of showing the existence of a zero of the function by rewriting the equation as $\sqrt[3]{x} + x - 1 = 0$. Then we have to show that the function $f(x) = \sqrt[3]{x} + x - 1$ has a zero on the interval $(0, 1)$.

   The Intermediate Value Theorem does not apply directly to the above situation because the interval $(0, 1)$ is not a closed interval, but an open interval. However, in this case we can widen the interval slightly to $[0, 1]$ because we can't add any roots of the function because

   $$\begin{aligned}
   f(0) &= \sqrt[3]{0} + 0 - 1 = 0 + 0 - 1 = -1 \neq 0 \\
   f(1) &= \sqrt[3]{1} + 1 - 1 = 1 + 1 - 1 = 1 \neq 0
   \end{aligned}$$

   So if f has a root on $[0, 1]$ it must have a root on $(0, 1)$.

   Note that f is continuous (why?) and $f(0) < 0$, $f(1) > 0$, so by the IVT f has a root in the interval $(0, 1)$.

   (b) As above, we want to be able to show the existence of a zero of a function by rewriting the equation as $\tan x - 2x = 0$. So we have to show that the function $f(x) = \tan x - 2x$ has a zero on the interval $(0, 1.4)$. As above, the interval is not a closed interval, but in this case it does matter if we widen it slightly because including $x = 0$ in the interval would introduce a new root. So instead, we shrink the interval a little and consider instead $[0.1, 1.4]$. (If that interval doesn't work, we could consider $[0.01, 1.4]$, etc.) We have

   $$\begin{aligned}
   f(0.1) &= \tan 0.1 - 2(0.1) \approx -0.0997 < 0 \\
   f(1.4) &= \tan 1.4 - 2(1.4) \approx 2.9979 > 0
   \end{aligned}$$

   So the function changes sign on the interval $[0.1, 1.4]$. It is continuous in that interval because $2x$ is continuous everywhere, and tan is continuous where it is defined, such as on the interval $(-\pi/2, \pi/2) \approx (-1.5708, 1.5708)$ which contains the interval $[0.1, 1.4]$, so f is continuous on the latter interval. The IVT applies, and we conclude that f has a root in that interval.

   (c) We must show that the function $f(x) = x^5 - x^2 + 2x + 3$ has a root on the interval $(-\infty, \infty)$. In this case we can't widen the interval by including $-\infty$ and ∞ because those are not numbers. So we must consider a smaller closed interval inside $(-\infty, \infty)$. We just try to guess appropriate bounds for the interval, and if they don't work, we guess again. Note that $f(0) = 0^5 - 0^2 + 2(0) + 3 > 0$ so if we can find a value a at which $f(a) < 0$ then we can apply the IVT. Try $f(-1) = (-1)^5 - (-1)^2 + 2(-1) + 3 = -1 - 1 - 2 + 3 = -1 < 0$ so we're in luck. The function $f(x)$ changes sign on the closed interval $[-1, 0]$, and it is continuous because it's a polynomial, so the IVT applies and $f(x)$ has a root in the interval $(-1, 0)$ so it has a root in the bigger interval $(-\infty, \infty)$.

4. As in a previous question, $f(x)$ is continuous in the open intervals $(-\infty, 2)$, $(2, 3)$, and $(3, \infty)$ because the functions $(x^2 - 4)/(x - 2)$, $ax^2 - bx + 3$, and $2x - a + b$ are continuous on those intervals, respectively.

So we just have to pay attention to what happens near $x = 2$ and $x = 3$. Near $x = 2$ we have

$$\begin{aligned}
\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^-} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2^-} (x+2) = 4 \\
f(2) &= a(2)^2 - b(2) + 3 = 4a - 2b + 3 \\
\lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} (ax^2 - bx + 3) = a(2)^2 - b(2) + 3 = 4a - 2b + 3
\end{aligned}$$

so the condition for f to be continuous at $x = 2$ is $4a - 2b + 3 = 4$ or $4a - 2b = 1$. Similarly, near $x = 3$ we have

$$\begin{aligned}
\lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-} (ax^2 - bx + 3) = a(3)^2 - b(3) + 3 = 9a - 3b + 3 \\
f(3) &= 2(3) - a + b = 6 - a + b \\
\lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} (2x - a + b) = 2(3) - a + b = 6 - a + b
\end{aligned}$$

so the condition for f to be continuous at $x = 3$ is $9a - 3b + 3 = 6 - a + b$ or $10a - 4b = 3$. In summary our conditions are

$$\begin{aligned}
4a - 2b &= 1 \\
10a - 4b &= 3
\end{aligned}$$

which is a linear system in the two variables a and b. Multiplying the first equation by -2 and adding, $2a = 1$ so $a = 1/2$, which implies $b = 1/2$.