MATH 110 Problem Set 2.9 Solutions

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Not part of the course

1. The best way to remember how to find the linearization is to find the equation of the tangent line and then solve for y.

   (a) First note that when $f(a) = f(0) = (2+0)^{-1/2} = 1/\sqrt{2}$, so $(a, f(a)) = (0, 1/sqrt2)$ is the point of interest on the curve. The slope of the tangent line at that point is $f'(a) = (-1/2)(2+a)^{-3/2} = (-1/2)(2+0)^{-3/2} = -1/2^{5/2} = -1/(4\sqrt{2})$. Therefore the equation of the tangent line is

   $$y - \frac{1}{\sqrt{2}} = -\frac{1}{4\sqrt{2}}(x-0)$$

   Solving for y gives

   $$y = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}}(x-0)$$

   so the linearization of $f(x)$ is

   $$L(x) = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}}x$$

   (b) We have $f(a) = 16^{3/4} = (16^{1/4})^3 = 2^3 = 8$, so $(a, f(a)) = (16, 8)$ is the point of interest on the curve. The slope of the tangent line at that point is $f'(a) = (3/4)a^{-1/4} = (3/4)(1/2) = 3/8$ so the equation of the tangent line is

   $$y - 8 = \frac{3}{8}(x-16)$$

   and the linearization of $f(x)$ at $a = 16$ is

   $$L(x) = 8 + \frac{3}{8}(x-16)$$

2. (a) By the quotient rule,

   $$\frac{dy}{ds} = \frac{(1+2s) - s(2)}{(1+2s)^2} = \frac{1}{(1+2s)^2}$$

   so

   $$dy = \frac{1}{(1+2s)^2}ds$$

   which can also be written

   $$dy = \frac{ds}{(1+2s)^2}$$

(b) Differentiating by the product rule,

$$\frac{dy}{du} = \cos u - u \sin u$$

so

$$dy = (\cos u - u \sin u)\ du$$

Don't forget the brackets!

(c) Differentiating by the chain rule,

$$\frac{dy}{dt} = 5(t + \tan t)^4 \cdot \frac{d}{dt}(t + \tan t) = 5(t + \tan t)^4(1 + \sec^2 t)$$

so

$$dy = 5(t + \tan t)^4(1 + \sec^2 t)\ dt$$

(d) Differentiating by the chain rule,

$$\frac{dy}{dz} = \frac{1}{2}(z + 1/z)^{-1/2} \cdot \frac{d}{dz}(z + 1/z) = \frac{1}{2}(z + 1/z)^{-1/2} \cdot (1 - 1/z^2)$$

so

$$dy = \frac{1}{2}(z + 1/z)^{-1/2} \cdot (1 - 1/z^2) \cdot dz$$

3. We have $\Delta y = f(x+\Delta x) - f(x) = \sqrt{1+1} - \sqrt{1} = \sqrt{2} - 1 \approx 0.4142$, and we have $dy = (1/2)x^{-1/2}\ dx = (1/2)1^{-1/2}\ 1 = 0.5$. See Figure 1.

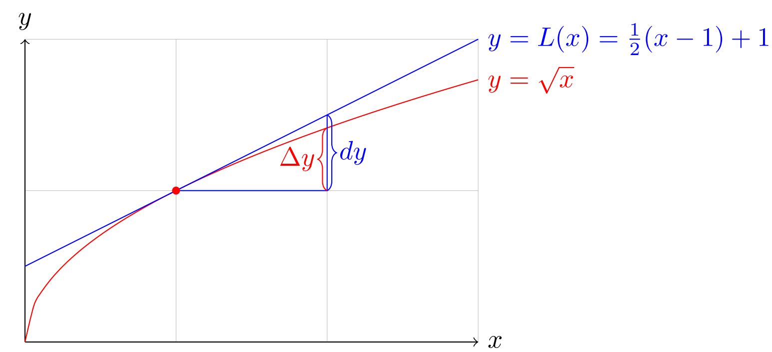

Figure 1: dy versus Δy for Problem 3

4. This type of problem is a sort of combination of related rates problems and differentials. It is like a related rates problem in that we need to find a relationship between two variables, in this case the hypoteneuse and the measured angle; it is different from related rates in that time is not a factor. So instead of differentiating with respect to time, we calculate the differentials. (If we wanted to change the problem to be about a situation in which θ were actually changing over time, we could put time back into the picture by dividing through by dt; do that, and compare with what would happen if you had applied the operator d/dt to the relationship instead of finding the differentials first and dividing by dt.)

(a) Let θ be the angle opposite the 20 cm side. Then $\sin\theta = 20/h$; if θ were exactly $30° = \pi/6$ radians, we would have $\sin\theta = 0.5$ which would mean $h = 40$ cm. We think of the potential error in measurement as a small change in the angle. We write the small change in the angle as $d\theta$, and we try to estimate the resulting change in the calculated value of the hypoteneuse $\Delta h \approx dh$.

You could figure out Δh exactly (try it), but the goal of this problem is to estimate it quickly by calculating dh instead.

Differentiating both sides of the relationship between θ and h with respect to θ gives

$$\cos\theta = -\frac{20}{h^2}\frac{dh}{d\theta}$$

by the chain rule. Multiplying by $d\theta$ gives

$$\cos\theta \, d\theta = -\frac{20}{h^2}\, dh$$

When $\theta = 30^\circ = \pi/6$ and $d\theta = \pm 1^\circ \approx \pm 0.0174$ radians we have $\cos\theta = \sqrt{3}2 \approx 0.8660$, $h = 40$ (calculated above), and so

$$0.8660 \cdot \pm 0.0174 = -\frac{20}{(40)^2}\, dh \implies dh \approx \mp 1.209$$

So an error of about 1° smaller in the measured angle means that the hypoteneuse is about 1.209 cm larger than 40 cm; an error of about 1° larger in the measured angle means that the hypoteneuse is about 1.209 cm smaller than 40 cm. See Figure 2.

Note that we had to do the calculation in radians; if you did the calculation in degrees, you would have gotten the wrong answer!

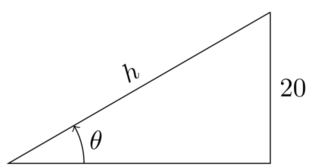

Figure 2: Triangle for problem 4

(b) The relative error is $dh/h \approx \mp 1.21/40 = 0.0302$, so the percentage error is about $\mp 3\%$.