MATH 110 Problem Set 3.4 Solutions

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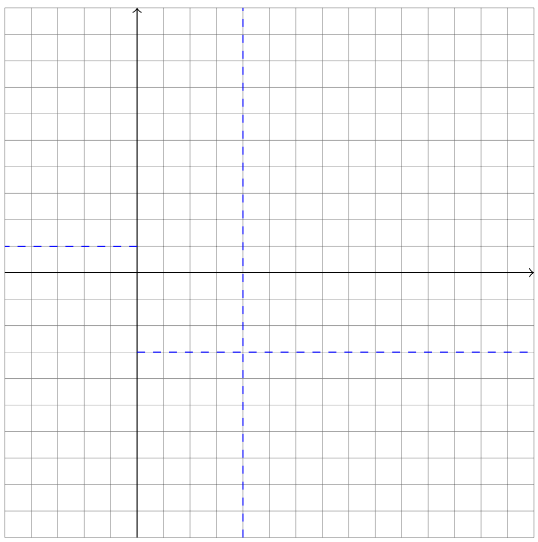
1. We draw the asymptote lines as a frame (Figure 1(a)), and then sketch any function that is consistent with the asymptotes (Figure 1(b)). There are many, many possible correct answers.


(a) Asymptotes as Frame

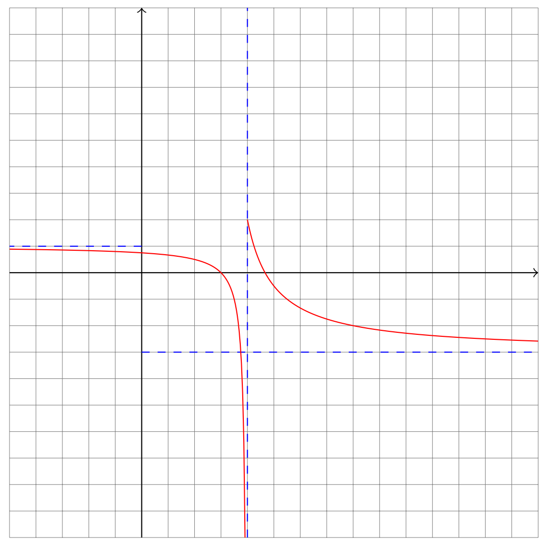

(b) Graph Consistent with (a)

Figure 1: Asymptotes and Example of Graph Consistent with Asymptotes

2. We first divide through by the highest power of x in the denominator, namely x^2, to obtain

$$\lim_{x\to\infty} \frac{3x^2 - 8x + 2}{x^2 + 6x - 5} = \lim_{x\to\infty} \frac{3 - \frac{8}{x} + \frac{2}{x^2}}{1 + \frac{6}{x} - \frac{5}{x^2}}$$

As $x \to \infty$, terms $8/x$, $2/x^2$, etc., tend to 0 so we have

$$\lim_{x\to\infty} \frac{3x^2 - 8x + 2}{x^2 + 6x - 5} = \lim_{x\to\infty} \frac{3 - \frac{8}{x} + \frac{2}{x^2}}{1 + \frac{6}{x} - \frac{5}{x^2}} = \frac{3 - 0 + 0}{1 + 0 - 0} = 3$$

3. (a) The highest power of t in the denominator is t^3. Dividing through,

$$\lim_{t\to-\infty} \frac{t^2 + 2}{t^3 + t^2 - 1} = \lim_{t\to-\infty} \frac{(1/t) + (2/t^3)}{1 + (1/t) - (1/t^3)} = \frac{0 + 0}{1 + 0 - 0} = 0$$

   (b) As usual, we divide through by the highest power of x in the denominator. The highest power appears to be x^2, but since the x^2 is under a square root sign we use $\sqrt{x^2}$ instead:

$$\lim_{x\to\infty} \frac{x + 2}{\sqrt{9x^2 + 1}} = \lim_{x\to\infty} \frac{1 + 2/x}{\sqrt{9x^2 + 1}/x} = \lim_{x\to\infty} \frac{1 + 2/x}{\sqrt{(9x^2 + 1)/x^2}} = \lim_{x\to\infty} \frac{1 + 2/x}{\sqrt{9 + 1/x^2}} = \frac{1 + 0}{\sqrt{9 + 0}} = \frac{1}{3}$$

(c) The highest power of x in the denominator is x^3. We divide through by x^3:

$$\lim_{x\to-\infty}\frac{\sqrt{9x^6-x}}{x^3+1}=\lim_{x\to-\infty}\frac{(9x^6-x)^{1/2}\cdot(1/x^3)}{1+(1/x^3)}$$

We need to bring $1/x^3$ into the square root. We would like to write

$$\frac{1}{x^3}=\left(\frac{1}{x^6}\right)^{1/2}$$

but in this case that is not quite correct. Since $x\to-\infty$, we can assume that x^3 is negative, so $1/x^3$ is negative, but $(1/x^6)^{1/2}$ is positive, so its sign is wrong. Instead we must write

$$\frac{1}{x^3}=-\left(\frac{1}{x^6}\right)^{1/2}$$

so our limit becomes

$$\begin{aligned}\lim_{x\to-\infty}\frac{(9x^6-x)^{1/2}\cdot(1/x^3)}{1+(1/x^3)}&=\lim_{x\to-\infty}\frac{(9x^6-x)^{1/2}\cdot-(1/x^6)^{1/2}}{1+(1/x^3)}\\&=\lim_{x\to-\infty}\frac{-(9-1/x^5)^{1/2}}{1+(1/x^3)}=\frac{-(9-0)^{1/2}}{1+0}=-3\end{aligned}$$

The correct answer is negative, not positive.

(d) We know how to take infinite limits of fractions, so we try to convert the expression into a fraction. Multiply and divide by the conjugate radical:

$$\begin{aligned}\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)&=\lim_{x\to-\infty}\frac{\left(x+\sqrt{x^2+2x}\right)\left(x-\sqrt{x^2+2x}\right)}{x-\sqrt{x^2+2x}}\\&=\lim_{x\to-\infty}\frac{x^2-(x^2+2x)}{x-\sqrt{x^2+2x}}=\lim_{x\to-\infty}\frac{-2x}{x-\sqrt{x^2+2x}}\end{aligned}$$

Now divide through by the highest power of x in the denominator. The x^2 in the denominator is under a square root sign so we think of it as $(x^2)^{1/2}\approx x$ so we divide through by x:

$$\lim_{x\to-\infty}\frac{-2x}{x-\sqrt{x^2+2x}}=\lim_{x\to-\infty}\frac{-2}{1-\sqrt{x^2+2x}/x}$$

It is tempting to just write $x=\sqrt{x^2}$ to move it under the square root sign, but as with the previous problem, x is negative while $\sqrt{x^2}$ is positive so we really have $x=-\sqrt{x^2}$ and we write

$$\begin{aligned}\lim_{x\to-\infty}\frac{-2}{1-\sqrt{x^2+2x}/x}&=\lim_{x\to-\infty}\frac{-2}{1+\sqrt{x^2+2x}/\sqrt{x^2}}=\lim_{x\to-\infty}\frac{-2}{1+\sqrt{(x^2+2x)/x^2}}\\&=\lim_{x\to-\infty}\frac{-2}{1+\sqrt{1+2/x}}=\frac{-2}{1+\sqrt{1+0}}=-1\end{aligned}$$

As with the previous problem, if you are not careful with this one you will get it wrong, in this case with a negative sign in front of the square root in the denominator which will lead to an infinite answer which is completely incorrect.

4. (a) The horizontal asymptotes are found by evaluating

$$\lim_{x\to-\infty}\frac{3x-1}{x+2}=\lim_{x\to-\infty}\frac{3-\frac{1}{x}}{1+\frac{2}{x}}=3,\qquad\lim_{x\to\infty}\frac{3x-1}{x+2}=\lim_{x\to\infty}\frac{3-\frac{1}{x}}{1+\frac{2}{x}}=3$$

giving $y = 3$ as the only horizontal asymptote. Candidates for the vertical asymptotes of a rational function are the roots of the denominator: $x + 2 = 0 \implies x = -2$. We check by evaluating the limits

$$\lim_{x\to-2^-} \frac{3x+1}{x+2} = \frac{7}{-0} = -\infty, \qquad \lim_{x\to-2^+} \frac{3x+1}{x+2} = \frac{7}{+0} = +\infty$$

which shows that $x = -2$ is a vertical asymptote. See Figure 2(a).

(b) The horizontal asymptotes are found by evaluating

$$\lim_{x\to-\infty} \frac{x^2+x-6}{x^2-3x+2} = \lim_{x\to-\infty} \frac{1+\frac{1}{x}-\frac{6}{x^2}}{1-\frac{3}{x}+\frac{2}{x^2}} = \frac{1+0-0}{1-0+0} = 1$$

and

$$\lim_{x\to\infty} \frac{x^2+x-6}{x^2-3x+2} = \lim_{x\to\infty} \frac{1+\frac{1}{x}-\frac{6}{x^2}}{1-\frac{3}{x}+\frac{2}{x^2}} = \frac{1+0-0}{1-0+0} = 1$$

which give the only horizontal asymptote $y = 1$. The candidates for vertical asymptotes are x values where the denominator of the rational function is zero: $x^2-3x+2 = 0 \implies (x-1)(x-2) = 0$ which gives candidates $x = 1$, $x = 2$. To check whether they are actual vertical asymptotes we must evaluate the following limits:

$$\begin{aligned}
\lim_{x\to1^-} \frac{x^2+x-6}{x^2-3x+2} &= \lim_{x\to1^-} \frac{(x+3)(x-2)}{(x-1)(x-2)} = \lim_{x\to1^-} \frac{x+3}{x-1} = \frac{4}{-0} = -\infty \\
\lim_{x\to1^+} \frac{x^2+x-6}{x^2-3x+2} &= \lim_{x\to1^+} \frac{(x+3)(x-2)}{(x-1)(x-2)} = \lim_{x\to1^+} \frac{x+3}{x-1} = \frac{4}{+0} = +\infty \\
\lim_{x\to2^-} \frac{x^2+x-6}{x^2-3x+2} &= \lim_{x\to2^-} \frac{(x+3)(x-2)}{(x-1)(x-2)} = \lim_{x\to2^-} \frac{x+3}{x-1} = \frac{5}{1} = 5 \\
\lim_{x\to2^+} \frac{x^2+x-6}{x^2-3x+2} &= \lim_{x\to2^+} \frac{(x+3)(x-2)}{(x-1)(x-2)} = \lim_{x\to2^+} \frac{x+3}{x-1} = \frac{5}{1} = 5
\end{aligned}$$

which show that $x = 1$ is a vertical asymptote but $x = 2$ is not. See Figure 2(b).

(c) The horizontal asymptotes are found by evaluating

$$\lim_{x\to-\infty} \frac{1+2x^3}{x^3-x} = \lim_{x\to-\infty} \frac{\frac{1}{x^3}+2}{1-\frac{1}{x^2}} = \frac{0+2}{1-0} = 2$$

and

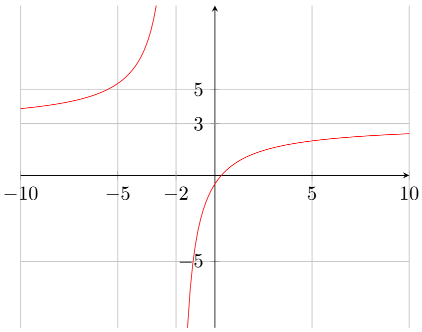
$$\lim_{x\to\infty} \frac{1+2x^3}{x^3-x} = \lim_{x\to\infty} \frac{\frac{1}{x^3}+2}{1-\frac{1}{x^2}} = \frac{0+2}{1-0} = 2$$

giving $y = 2$ as the only horizontal asymptote. The candidates for vertical asymptotes are x values at which the denominator is 0, in other words solutions to the equation $x^3 - x = 0$. Generally it is difficult to solve cubic equations, but this one can be solved by factoring: $x^3 - x = x(x^2-1) = x(x-1)(x+1) = 0 \implies x = -1, 0, 1$. Evaluating left and right limits at each of those x values gives

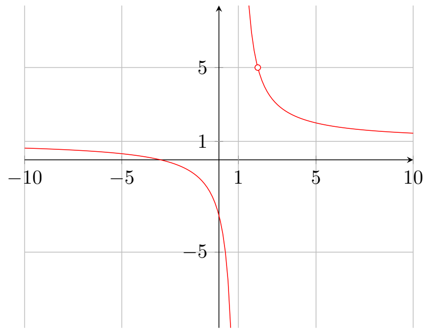
$$\begin{aligned}
\lim_{x\to-1^-} \frac{1+2x^3}{x^3-x} &= \lim_{x\to-1^-} \frac{1+2x^3}{(x+1)x(x-1)} = \frac{1+2(-1)^3}{-0(-1)(-2)} = \frac{-1}{-0} = +\infty \\
\lim_{x\to-1^+} \frac{1+2x^3}{x^3-x} &= \lim_{x\to-1^+} \frac{1+2x^3}{(x+1)x(x-1)} = \frac{1+2(-1)^3}{+0(-1)(-2)} = \frac{-1}{+0} = -\infty \\
\lim_{x\to0^-} \frac{1+2x^3}{x^3-x} &= \lim_{x\to0^-} \frac{1+2x^3}{(x+1)x(x-1)} = \frac{1+2(0)^3}{(1)(-0)(-1)} = \frac{1}{+0} = +\infty
\end{aligned}$$

$$\lim_{x\to 0^+} \frac{1+2x^3}{x^3-x} = \lim_{x\to 0^+} \frac{1+2x^3}{(x+1)x(x-1)} = \frac{1+2(0)^3}{(1)(+0)(-1)} = \frac{1}{-0} = -\infty$$
$$\lim_{x\to 1^-} \frac{1+2x^3}{x^3-x} = \lim_{x\to 1^-} \frac{1+2x^3}{(x+1)x(x-1)} = \frac{1+2(1)^3}{(2)(1)(-0)} = \frac{3}{-0} = -\infty$$
$$\lim_{x\to 1^+} \frac{1+2x^3}{x^3-x} = \lim_{x\to 1^+} \frac{1+2x^3}{(x+1)x(x-1)} = \frac{1+2(1)^3}{(2)(1)(+0)} = \frac{3}{+0} = +\infty$$

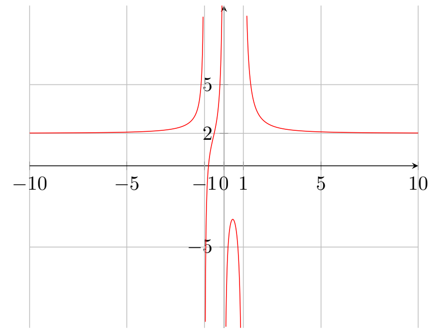
showing that there is a vertical asymptote at each of $x = -1, 0, 1$. See Figure 2(c).



(a) $y = \frac{3x+1}{x+2}$



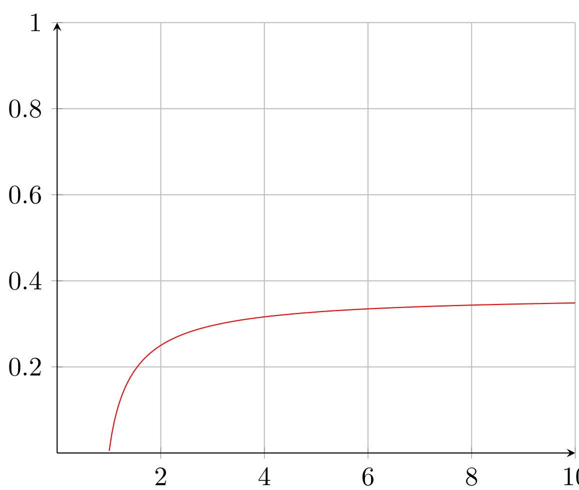
(b) $y = \frac{x^2+x-6}{x^2-3x+2}$



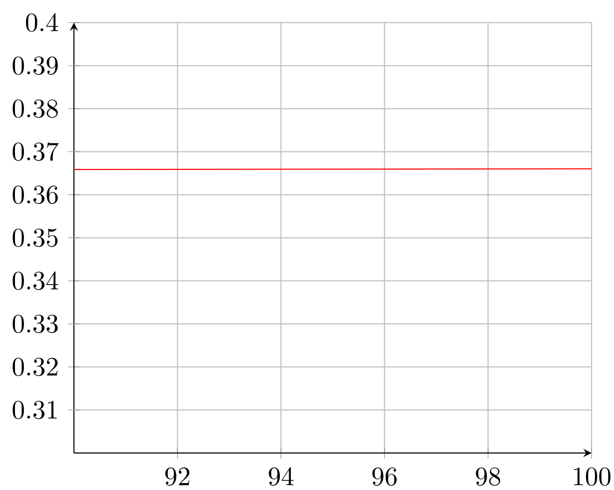
(c) $y = \frac{1+2x^3}{x^3-x}$

Figure 2: Asymptotes for Three Functions

5. In Figure 3 you can see the graph of the function plotted in two windows; in Figure 3(a) the x-axis goes from 0 to 10 and the y-axis goes from 0 to 1. From that graph it appears that the function has a horizontal asymptote somewhere between $y = 0.3$ and $y = 0.4$. In Figure 3(b) the x-axis is shifted to go from 90 to 100 and the y-axis is magnified to go from 0.3 to 0.4. From that graph it seems likely that the horizontal asymptote rounded to two decimal points is $y = 0.37$, so we conclude that the limit rounded to two decimal points is 0.37.



(a) Domain $0 < x \leq 10$



(b) Domain $90 \leq x \leq 100$

Figure 3: Two Graphs of $f(x) = \left(1 - \frac{1}{x}\right)^x$

The conclusion is supported by Table 1 where the function is calculated at successive powers of 10. In

x	$1/x$	$1-1/x$	$(1-1/x)^x$
1	1.0	0	0
10	0.1	0.9	0.3486
100	0.01	0.99	0.3660
1000	0.001	0.999	0.3677
10000	0.0001	0.9999	0.3679
100000	0.00001	0.99999	0.3679

Table 1: Table of Values for $\left(1-\frac{1}{x}\right)^x$ to Four Decimal Points

MATH 111 you will learn that the exact value of the limit is $1/e = 0.367879\ldots$.

6. In order to calculate the limits, we divide through by the highest power of x in the denominators. For the limit as $x \to \infty$,

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x-3} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}/x}{2-3/x} = \lim_{x\to\infty} \frac{\sqrt{(x^2+1)/x^2}}{2-3/x} = \lim_{x\to\infty} \frac{\sqrt{1+1/x^2}}{2-3/x} = \frac{\sqrt{1+0}}{2-0} = \frac{1}{2}$$

where we square x when we bring it under the square root. On the other hand, for the limit as $x \to -\infty$,

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x-3} = \lim_{x\to-\infty} \frac{\sqrt{x^2+1}/x}{2-3/x} = \lim_{x\to-\infty} \frac{-\sqrt{(x^2+1)/x^2}}{2-3/x} = \lim_{x\to-\infty} \frac{-\sqrt{1+1/x^2}}{2-3/x} = -\frac{\sqrt{1+0}}{2-0} = -\frac{1}{2}$$

Note the appearance of the negative sign in the second step which may at first be puzzling. This phenomenon, which may occur when taking limits at infinity of functions involving roots, is unlike what happens with ordinary rational functions which always have equal limits as $x \to \pm\infty$. The negative sign appears because x is negative in the limit $x \to -\infty$, but $\sqrt{x^2}$ is positive. For equality in this case we must write $x = -\sqrt{x^2}$. See Figure 4.

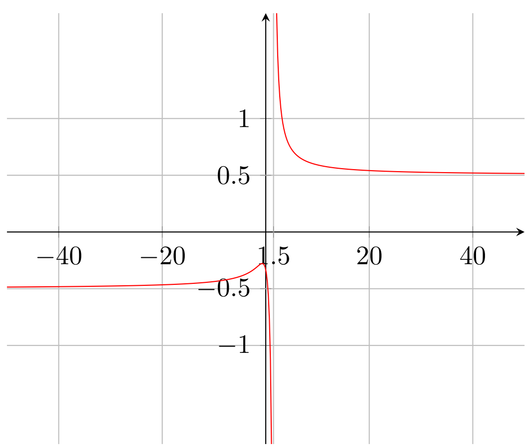


Figure 4: Graph of $y = \dfrac{\sqrt{x^2+1}}{2x-3}$

7. We will find a rational function satisfying the given conditions. The denominator of the function should contain factors $x+2$ and $x-3$; $(x+2)(x-3) = x^2 - x - 6$ is the simplest choice. In order for the function to have a horizontal asymptote the highest power of x in the numerator must match

the highest power of x in the denominator; x^2 is the simplest choice for the numerator. (We must also ensure that neither $x+2$ nor $x-3$ is a factor of the numerator.) Finally, to scale the horizontal asymptote to $5/2$ from its current value of 1, we just multiply the function by $5/2$ to obtain

$$f(x) = \frac{5x^2}{2x^2 - 2 - 12}$$

You should check that the resulting function satisfies the conditions of the problem. You should also try to find other answers to the problem.

8. This question is like Example 11 in Section 3.4 of the textbook. The y-intercept is

$$y = f(0) = 0^2(0+2)^3(1-0) = 0$$

so $(0,0)$ is a point on the graph. The x-intercepts are roots of the equation

$$f(x) = 0 \implies x^2(x+2)^3(1-x) = 0 \implies x = -2, 0, 1$$

so $(-2,0)$, $(0,0)$, and $(1,0)$ are points on the graph. In addition, a careful analysis shows that the function does not change sign at $x=0$ because the x^2 factor has an even exponent, but the function does change sign at $x=-2$ and $x=1$ because of the odd exponents in the factors $(x+2)^3$ and $(1-x)^1$. As $x \to \infty$,

$$\lim_{x\to\infty} x^2(x+2)^3(1-x) = -\infty$$

because for large positive x the x^2 and $(x+2)^3$ are positive, while $(1-x)$ is negative. Similarly, as $x \to -\infty$,

$$\lim_{x\to-\infty} x^2(x+2)^3(1-x) = -\infty$$

because for large negative x the x^2 term is positive, the $(x+2)^3$ term is negative, and the $(1-x)$ term is positive. Putting all the information together, your graph should resemble Figure 5.

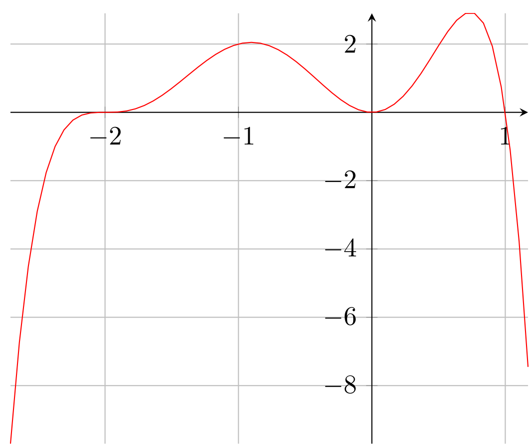


Figure 5: Graph of $y = x^2(x+2)^3(1-x)$