MATH 110 Problem Set 4.5 Solutions

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Tuesday, April 7, 2026

1. (a) Let $u = x^3 + 5$. Then $du/dx = 3x^2$, $du/3 = x^2\,dx$ and
$$\int x^2(x^3+5)^9\,dx = \int u^9 \frac{du}{3} = \frac{1}{30}u^{10} + C = \frac{1}{30}(x^3+5)^{10} + C$$
Check by differentiating.

   (b) Let $u = 3t + 2$. Then $du/dt = 3$, $du/3 = dt$ and
$$\int (3t+2)^{2.4}\,dt = \int u^{2.4}\,\frac{du}{3} = \frac{1}{3}\int u^{2.4}\,du = \frac{1}{3}\frac{u^{3.4}}{3.4} + C = \frac{1}{10.2}(3t+2)^{3.4} + C$$
Check by differentiating.

   (c) Let $u = x^2 + 1$. Then $du/dx = 2x$, $du/2 = x\,dx$ and
$$\int \frac{x}{(x^2+1)^2}\,dx = \int u^{-2}\frac{du}{2} = -\frac{1}{2}u^{-1} + C = -\frac{1}{2(x^2+1)} + C$$
Check by differentiating.

   (d) Let $u = 5t + 4$, $du/dt = 5$, $du/5 = dt$. Then
$$\int \frac{1}{(5t+4)^{2.7}}\,dt = \int u^{-2.7}\frac{du}{5} = \frac{1}{5}\frac{u^{-1.7}}{-1.7} + C = -\frac{1}{8.5}(5t+4)^{-1.7} + C$$
Check by differentiating.

2. (a) Let $u = 2\theta$, $du = 2\,d\theta$, $du/2 = d\theta$:
$$\int \sec 2\theta \tan 2\theta\;d\theta = \int \sec u \tan u\,\frac{du}{2} = \frac{1}{2}\sec u + C = \frac{1}{2}\sec 2\theta + C$$
Check by differentiating.

   (b) Let $u = 1 + x^{3/2}$. Then $du/dx = 3x^{1/2}/2$, $2\,du/3 = \sqrt{x}\,dx$ and
$$\int \sqrt{x}\sin(1 + x^{3/2}\;dx = \frac{2}{3}\int \sin u\,du = -\frac{2}{3}\cos u + C = -\frac{2}{3}\cos(1+x^{3/2}) + C$$
Check by differentiating.

   (c) Let $u = \pi/x$, $du/dx = -\pi/x^2$, $-du/\pi = dx/x^2$. Then
$$\int \frac{\cos(\pi/x)}{x^2}\,dx = -\frac{1}{\pi}\int \cos u\,du = -\frac{1}{\pi}\sin u + C = -\frac{1}{\pi}\sin(\pi/x) + C$$
Check by differentiating.

(d) Let $u = 1+\tan\theta$, $du/d\theta = \sec^2\theta$, $du = \sec^2\theta d\theta$. Then

$$\int (1+\tan\theta)^5 \sec^2\theta\ d\theta = \int u^5\ du = \frac{1}{6}u^6 + C = \frac{1}{6}(1+\tan\theta)^6 + C$$

Check by differentiating.

3. These ones are a little trickier than the previous.

(a) Let $u = 1+\tan t$, $du/dt = \sec^2 t$, $du = \sec^2 t\ dt = dt/\cos^2 t$ where we have used the trig identity $\sec^2 t = 1/\cos^2 t$. Then

$$\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}} = \int u^{-1/2}\ du = 2u^{1/2} + C = 2\sqrt{1+\tan t} + C$$

Check by differentiating.

(b) Let $u = \cos t$, $du/dt = -\sin t$, $-du = \sin t\ dt$. Then

$$\int \sin t \sec^2(\cos t)\ dt = -\int \sec^2 u\ du = -\sec u + C = -\sec(\cos t) + C$$

(c) Let $u = 1-x$, $du = -dx$, $-du = dx$. Then

$$\int \frac{x^2}{\sqrt{1-x}}\ dx = -\int \frac{x^2}{\sqrt{u}}\ du$$

Unfortunately, that substitution is not complete, because not all of the xs have been eliminated from the integral. In order to complete the substitution we need to go back to $u = 1-x$ and solve for x: $x = 1-u$. So we can write

$$-\int \frac{x^2}{\sqrt{u}}\ du = -\int \frac{(1-u)^2}{u^{1/2}}\ du = -\int \frac{1-2u+u^2}{u^{1/2}}\ du = -\int (u^{-1/2} - 2u^{1/2} + u^{3/2})\ du$$

Integrating by the power rule we have

$$\int \frac{x^2}{\sqrt{1-x}}\ dx = -2u^{1/2} - \frac{4}{3}u^{3/2} + \frac{2}{5}u^{5/2} + C = -2(1-x)^{1/2} - \frac{4}{3}(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2} + C$$

You should check your answer by differentiating, then doing some algebra.

(d) Let $u = x^2+1$, $du/dx = 2x$, $du/2 = x\ dx$. Then

$$\int x^3\sqrt{x^2+1}\ dx = \frac{1}{2}\int x^2\sqrt{u}\ du$$

Again, the substitution hasn't been completed. We must eliminate x^2: $u = x^2+1$ gives $x^2 = u-1$ so

$$\frac{1}{2}\int x^2\sqrt{u}\ du = \frac{1}{2}\int (u-1)^2 u^{1/2}\ du = \frac{1}{2}\int (u^2-2u+1)u^{1/2}\ du = \frac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\ du$$

Integrating by the power rule,

$$\int x^3\sqrt{x^2+1}\ dx = \frac{1}{7}u^{7/2} - \frac{2}{5}u^{5/2} + \frac{1}{3}u^{3/2} + C = \frac{1}{7}(x^2+1)^{7/2} - \frac{2}{5}(x^2+1)^{5/2} + \frac{1}{3}(x^2+1)^{3/2} + C$$

You should check your answer by differentiating, then doing some algebra.

4. (a) To evaluate the indefinite integral, let $u = \sin x$. Then $du/dx = \cos x$, $du = \cos x\, dx$:

$$\int \cos x \, \sin(\sin x)\, dx = \int \sin u\, du = -\cos u + C = -\cos(\sin x) + C$$

You should check the indefinite integral by differentiating. Now by the Fundamental Theorem of Calculus,

$$\int_0^{\pi/2} \cos x \, \sin(\sin x)\, dx = \left. -\cos(\sin x) \right|_0^{\pi/2} = -\cos(\sin(\pi/2)) + \cos(\sin(0)) = -\cos(1) + \cos(0) = 1 - \cos(1)$$

Alternate Solution: We can combine the two steps above into one step by substituting the lower and upper bounds of integration when we do the substitution. When $x = 0$, $u = \sin x = \sin 0 = 0$, so that becomes the lower bound of integration after the substitution. Similarly when $x = \pi/2$, $u = \sin x = \sin(\pi/2) = 1$. The calculation becomes

$$\int_0^{\pi/2} \cos x \, \sin(\sin x)\, dx = \int_0^1 \sin u\, du = \left. -\cos u \right|_0^1 = -\cos 1 + \cos 0 = 1 - \cos 1$$

That is the same answer as before, but the calculation is somewhat shorter because we don't have to reverse the subsitition after the integration step. On the other hand, it gives you less opportunity to check your work by differentiating.

(b) In this problem a is a constant parameter. To evaluate the indefinite integral we make the substitution $u = a^2 - x^2$, $du/dx = -2x$, $-du/2 = x\, dx$:

$$\int x\sqrt{a^2 - x^2}\, dx = -\frac{1}{2}\int u^{1/2}\, du = -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(a^2 - x^2)^{3/2} + C$$

You can check the answer by differentiating. The definite integral is

$$\int_0^a x\,\sqrt{a^2 - x^2}\, dx = \left. -\frac{1}{3}(a^2 - x^2)^{3/2} \right|_0^a = -\frac{1}{3}(a^2 - a^2)^{3/2} + \frac{1}{3}(a^2 - 0)^{3/2} = \frac{1}{3}a^3$$

Alternate Solution: When $x = 0$, $u = a^2 - x^2 = a^2 - 0 = a^2$, so that becomes the lower bound of integration after the substitution. Similarly, when $x = a$, $u = a^2 - a^2 = 0$, so that becomes the upper bound of integration after the substition. We can write

$$\int_0^a x\,\sqrt{a^2 - x^2}\, dx = -\frac{1}{2}\int_{a^2}^0 u^{1/2}\, du = \left. -\frac{1}{2}\,\frac{2}{3}u^{3/2} \right|_{a^2}^0 = -\frac{1}{3}(0^{3/2} - (a^2)^{3/2}) = \frac{1}{3}a^3$$

Again, the alternate solution is shorter, but we can't check the indefinite integral by differentiating.

(c) Let $u = 1 + 2x$, $du/dx = 2$, $du/2 = dx$, $x = (u-1)/2$:

$$\int \frac{x}{\sqrt{1+2x}}\, dx = \int \frac{(u-1)/2}{\sqrt{u}}\,\frac{du}{2} = \frac{1}{4}\int (u^{1/2} - u^{-1/2})\, du = \frac{1}{6}u^{3/2} - \frac{1}{2}u^{1/2} + C = \frac{1}{6}(1+2x)^{3/2} - \frac{1}{2}(1+2x)^{1/2} + C$$

Check by differentiating and then doing some algebra. Now we can evaluate the definite integral:

$$\int_0^4 \frac{x}{\sqrt{1+2x}}\, dx = \left. \frac{1}{6}(1+2x)^{3/2} - \frac{1}{2}(1+2x)^{1/2} \right|_0^4 = \frac{1}{6}9^{3/2} - \frac{1}{2}9^{1/2} - \frac{1}{6}1^{3/2} + \frac{1}{2}1^{1/2} = \frac{27}{6} - \frac{3}{2} - \frac{1}{6} + \frac{1}{2} = \frac{10}{3}$$

Alternate Solution: When $x = 0$, $u = 1 + 2x = 1$, and when $x = 4$, $u = 1 + 2x = 9$:

$$\int_0^4 \frac{x}{\sqrt{1+2x}}\, dx = \frac{1}{4}\int_1^9 (u^{1/2} - u^{-1/2})\, du = \left. \frac{1}{4}\,\frac{2}{3}u^{3/2} - 2u^{1/2} \right|_1^9 = \frac{1}{6}9^{3/2} - \frac{1}{2}9^{1/2} - \frac{1}{6}1^{3/2} + \frac{1}{2}1^{1/2} = \frac{10}{3}$$

As usual, the alternate solution is shorter but has the disadvantage that you can't check the integral by differentiating.

(d) In this problem T and α are constant parameters. Let $u = 2\pi t/T - \alpha$. Then $du/dt = 2\pi/T$ and $T/(2\pi)du = dt$. The indefinite integral is

$$\int \sin(2\pi t/T - \alpha)\, dt = \frac{T}{2\pi}\int \sin u\, du = \frac{T}{2\pi}(-\cos u) + C = -\frac{T}{2\pi}\cos(2\pi t/T - \alpha) + C$$

which you can check by differentiating. Now the definite integral is

$$\begin{aligned}\int_0^{T/2} \sin(2\pi t/T - \alpha)\, dt &= -\frac{T}{2\pi}\cos(2\pi t/T - \alpha)\Big|_0^{T/2} \\ &= -\frac{T}{2\pi}\cos(2\pi(T/2)/T - \alpha) + \frac{T}{2\pi}\cos(2\pi(0)/T - \alpha) = -\frac{T}{2\pi}\left(\cos(\pi - \alpha) - \cos(-\alpha)\right)\end{aligned}$$

Alternate Solution: When $t = 0$, $u = 2\pi(0)/T - \alpha = -\alpha$, and when $t = T/2$, $u = 2\pi(T/2)/T - \alpha = \pi - \alpha$, so

$$\int_0^{T/2} \sin(2\pi t/T - \alpha)\, dt = \frac{T}{2\pi}\int_{-\alpha}^{\pi-\alpha} \sin u\, du = -\frac{T}{2\pi}\cos u\Big|_{-\alpha}^{\pi-\alpha} = -\frac{T}{2\pi}\left(\cos(\pi - \alpha) - \cos(-\alpha)\right)$$

As usual, the alternate solution is shorter, but you can't check the substitution by differentiating.

5. (a) We have

$$\int_{-2}^{2}(x+3)\sqrt{4-x^2} = \int_{-2}^{2}\left(x\sqrt{4-x^2} + 3\sqrt{4-x^2}\right)\, dx = \int_{-2}^{2} x\sqrt{4-x^2}\, dx + 3\int_{-2}^{2}\sqrt{4-x^2}\, dx$$

by the rules for definite integrals. The first integral we can do by the substitution $u = 4 - x^2$, $du/dx = -2x$, $-du/2 = x\,dx$:

$$\int x\sqrt{4-x^2}\, dx = -\frac{1}{2}\int u^{1/2}\, du = -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(4-x^2)^{3/2} + C$$

which you should check by differentiating. The definite integral is then

$$\int_{-2}^{2} x\sqrt{4-x^2}\, dx = -\frac{1}{3}(4-x^2)^{3/2}\Big|_{-2}^{2} = -\frac{1}{3}(4-(-2)^2)^{3/2} + \frac{1}{3}(4-2^2)^{3/2} = 0$$

On the other hand, the second integral

$$3\int_{-2}^{2}\sqrt{4-x^2}\, dx$$

cannot be done by a substitution (at least none that you have learned in MATH 110; if you take MATH 111, you will learn a clever substitution that will work). However, you can evaluate that definite integral geometrically; $y = \sqrt{4-x^2}$ is the upper half of a circle of radius 2 centered at the origin $(0,0)$, so the integral is the area of a semicircle of radius 2, namely 2π. Altogether,

$$\int_{-2}^{2}(x+3)\sqrt{4-x^2} = \int_{-2}^{2}\left(x\sqrt{4-x^2} + 3\sqrt{4-x^2}\right)\, dx = \int_{-2}^{2} x\sqrt{4-x^2}\, dx + 3\int_{-2}^{2}\sqrt{4-x^2}\, dx = 0 + 3(2\pi) = 6\pi$$

(b) If we try the obvious substitution $u = 1 - x^4$, we are going to have difficulty completing it. So instead, inspired by the previous problem, let's try $u = x^2$, $du/dx = 2x$, $du/2 = x\,dx$; when $x = 0$, $u = x^2 = 0$, and when $x = 1$, $u = x^2 = 1$:

$$\int_0^1 x\sqrt{1-x^4}\, dx = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\, du$$

Geometrically, the integral is the area of one quarter of a unit circle, which is $\pi/4$, so the answer is

$$\int_0^1 x\sqrt{1-x^4}\, dx = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\, du = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$$

6. (a) This one requires a trick. Recall the Pythagorean identity $\cos^2 x = 1 - \sin^2 x$. We can write

$$\int \cos^3 x \, \sin^6 x \, dx = \int \cos x \, (1 - \sin^2 x) \, \sin^6 x \, dx = \int \cos x \, \sin^6 x \, dx - \int \cos x \, \sin^8 x \, dx$$

Both of the resulting integrals can be evaluated by the substitution $u = \sin x$, $du = \cos x \, dx$:

$$\int \cos x \, \sin^6 x \, dx - \int \cos x \, \sin^8 x \, dx = \int u^6 \, du - \int u^8 \, du = \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C$$

A similar strategy works for the integral of any product of a power of cosine times a power of sine, as long as at least one of the powers is odd. If both powers are even, you need to use more advanced techniques you will learn in MATH 111.

(b) Let $u = -x$, $-du = dx$; note that when $x = -1$, $u = 1$ and when $x = 1$, $u = -1$. Then

$$\int_{-1}^{1} \frac{x^2 \, \tan x}{x^4 + 1} \, dx = \int_{1}^{-1} \frac{(-u)^2 \, \tan(-u)}{(-u)^4 + 1} (-du) = \int_{-1}^{1} \frac{u \, (-\tan u)}{u^4 + 1} \, du = -\int_{-1}^{1} \frac{x \, \tan x}{x^4 + 1} \, dx$$

where we have used the fact that $\tan x$ is odd (i.e., $\tan(-x) = -\tan x$), x^2 is even (i.e., $(-x)^2 = x^2$), x^4 is even, and we used the property

$$\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$$

of the Riemann integral. In summary,

$$\int_{-1}^{1} \frac{x^2 \, \tan x}{x^4 + 1} \, dx = -\int_{-1}^{1} \frac{x^2 \, \tan x}{x^4 + 1} \, dx$$

which implies that the value of the integral is 0.

You can also reason about this problem geometrically; the integrand is odd, and the domain of integration is balanced across 0, so the negative areas exactly balance the positive areas and the integral must be 0.

7. Let $u = x^2$, $du/dx = 2x$, $du/2 = x \, dx$; note that when $x = 0$, $u = x^2 = 0$, and when $x = 3$, $u = x^2 = 9$. So

$$\int_0^3 x f(x^2) \, dx = \frac{1}{2} \int_0^9 f(u) \, du = \frac{1}{2} \cdot 4 = 2$$

8. Let $u = \pi - x$, $du/dx = -1$, $-du = dx$, $x = \pi - u$; note that when $x = 0$, $u = \pi - x = \pi$, and when $x = \pi$, $u = \pi - \pi = 0$:

$$\int_0^\pi x f(\sin x) \, dx = -\int_\pi^0 (\pi - u) f(\sin(\pi - u)) \, du = \int_0^\pi (\pi - u) f(\sin(\pi - u)) \, du$$

By the angle subtraction identity for sin we have

$$\sin(\pi - u) = \sin \pi \cos u - \sin u \cos \pi = 0 \cos u - \sin u (-1) = \sin u$$

so

$$\int_0^\pi x f(\sin x) \, dx = \int_0^\pi (\pi - u) f(\sin(\pi - u)) \, du = \pi \int_0^\pi f(\sin u) \, du - \int_0^\pi u f(\sin u) \, du$$

Since u is a dummy variable in the integral we can replace it by x:

$$\int_0^\pi x f(\sin x) \, dx = \pi \int_0^\pi f(\sin x) \, dx - \int_0^\pi x f(\sin x) \, dx$$

We can move the last term to the left side of the equation to obtain

$$2 \int_0^\pi x f(\sin x) \, dx = \pi \int_0^\pi f(\sin x) \, dx$$

Dividing both sides of the equation above by 2 gives the result.