MATH 110 Lecture 2.4

Derivatives of Trigonometric Functions

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Department of Indigenous Knowledge and Science
First Nations University of Canada

Contents

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Strategy for Finding Derivatives of Trig Functions

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- We still need to differentiate those functions. In order to do so, we will have to use the definition of the derivative in terms of limits.
- That means we will need to be able to calculate limits of sin and cos, which is the problem to which we now turn.

The Basic Trig Limits

- We have already seen two basic trig limits in connection with our study of the continuity of sin and cos:

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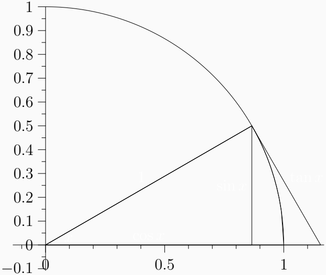
- All of our results for trig functions will follow from those limits.

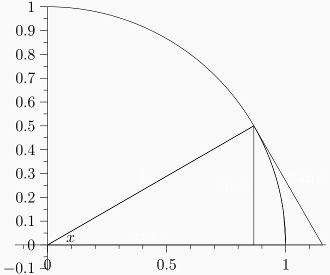
The Big Idea Behind the Proof of a Basic Trig Limit

Consider the diagram on the right.



The Big Idea Behind the Proof of a Basic Trig Limit

We calculate the lengths of various lines in the diagram in terms of the angle x.



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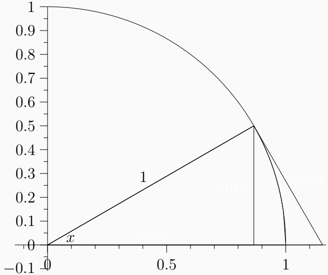

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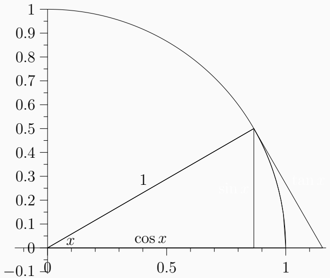

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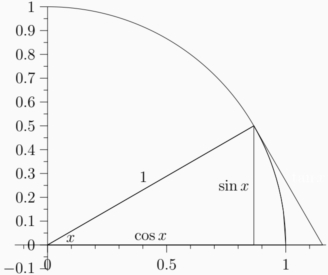

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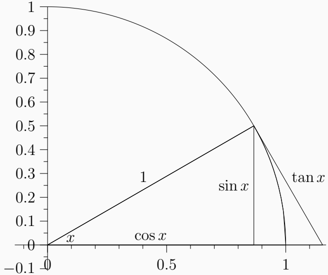

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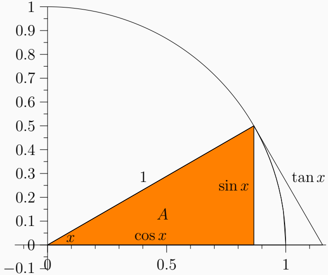
Now consider the areas of various regions in the diagram.



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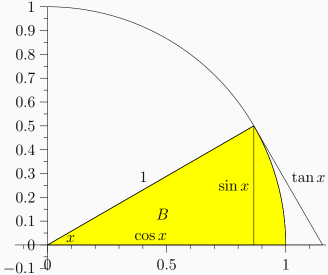

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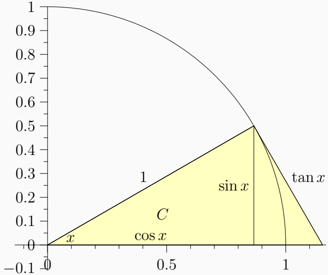

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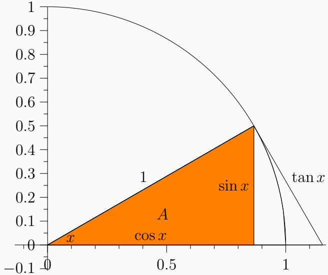
It should be clear from the diagram that

$$\text{area}(A) \leq \text{area}(B) \leq \text{area}(C)$$



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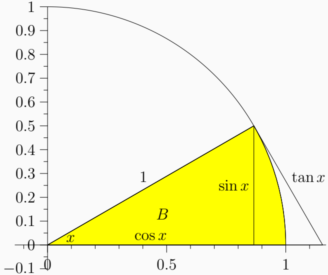

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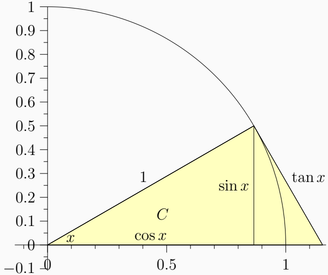

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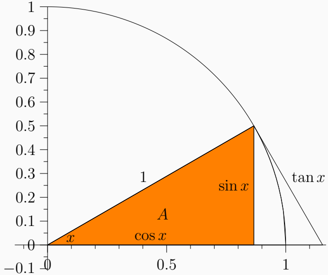

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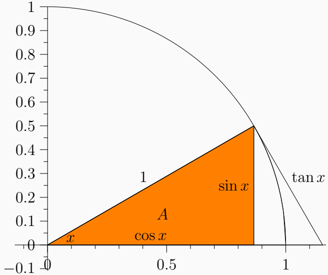
$$\frac{1}{2}\sin x \cos x \leq \text{area}(B) \leq \text{area}(C)$$



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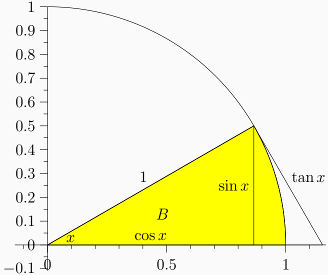

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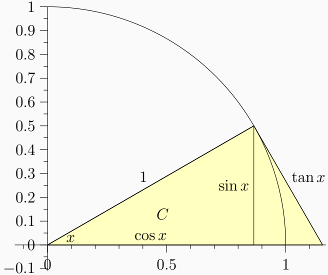
$$\frac{1}{2}\sin x\cos x \leq \frac{1}{2}x \leq \frac{1}{2}\tan x$$



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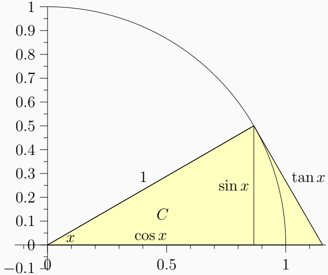

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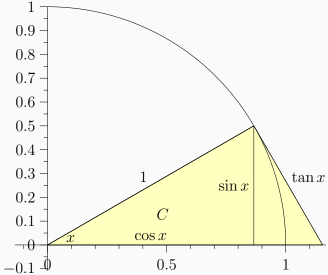

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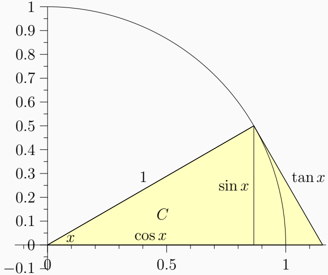
$$\sin x \cos x \leq x \leq \frac{\sin x}{\cos x}$$



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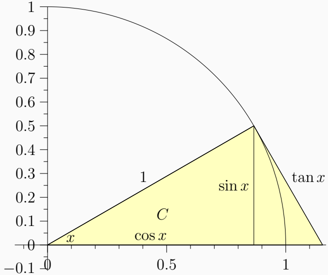
Simplifying,

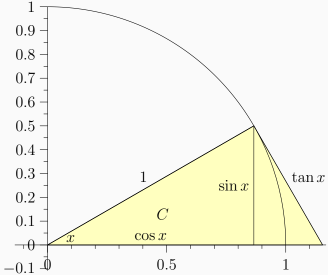
$$\cos x \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$



The Big Idea Behind the Proof of a Basic Trig Limit

Taking limits,

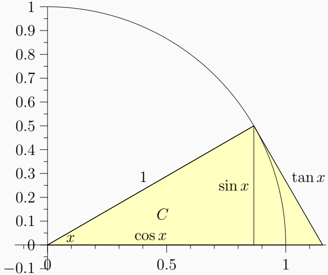
$$\lim_{x\to 0} \cos x \leq \lim_{x\to 0} \frac{x}{\sin x} \leq \lim_{x\to 0} \frac{1}{\cos x}$$



The Big Idea Behind the Proof of a Basic Trig Limit

Applying the squeeze theorem,

$$1 \leq \lim_{x \to 0} \frac{x}{\sin x} \leq 1$$



The Limit $\lim_{h \to 0} \frac{\tan h}{h}$

- The basic trig limits

$$\lim_{h \to 0} \sin h = 0 \qquad \lim_{h \to 0} \cos h = 1 \qquad \lim_{h \to 0} \frac{\sin h}{h} = 1$$

allow us to compute more complicated limits of trig functions.

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- In general, we try to use algebra to rearrange expressions so that the basic trig limits appear.

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- In general, we want to divide and multiply by the arguments of any sin functions that appear.

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&= -\lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{\sin h}{\cos h + 1} = -1 \cdot \frac{0}{1+1} = 0
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- An analogous argument gives $\cos' x = -\sin x$. Try it!

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- For another example, try differentiating sindeg. We'll take another look at that in section 3.5.

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The traditional formulas for the derivatives of the basic trig functions are as follows:

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You should memorize that table as well as you can.

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$\tan' x = \sec^2 x$	$\cot' x = -\csc^2 x$

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Examples

1. Find the limit $\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\sin\theta}$.
2. Find the derivative of $y = \dfrac{\sec x}{1 + \tan x}$.
3. Find equations of the tangent and normal lines to $y = \sec x$ at the point $(\pi/3, 2)$.
4. Find the numbers x at which the tangent to the curve $y = \sin x + \cos x$ is horizontal.

Exercises

Now you should work on Problem Set 2.4. After you have finished it, you should try the following additional exercises from Section 2.4:

2.4 C-level: 1–16, 21–24, 25–26, 27–30, 33–34, 39–50;
B-level: 17–20, 31, 32, 35–38;
A-level: 51–52, 53–58