MATH 110 Lecture 1.8

Continuity

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Thursday, January 22, 2026

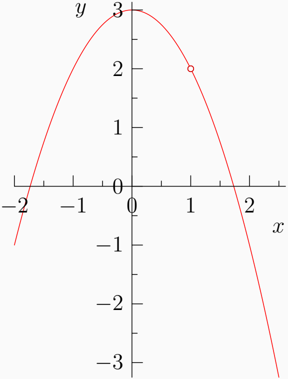
Department of Indigenous Knowledge and Science
First Nations University of Canada

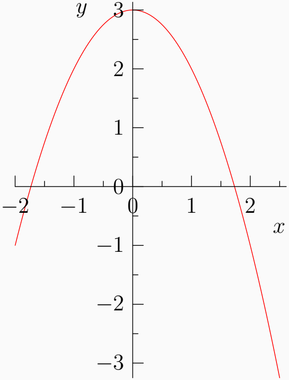
Contents

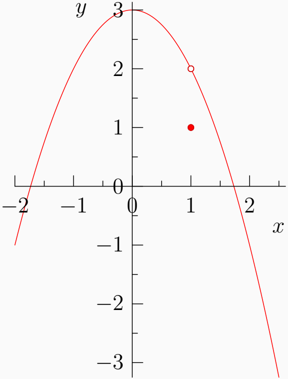
Continuity (I)

Examples of Continuous and Discontinuous Functions

Recall the following three graphs from Lecture 2.







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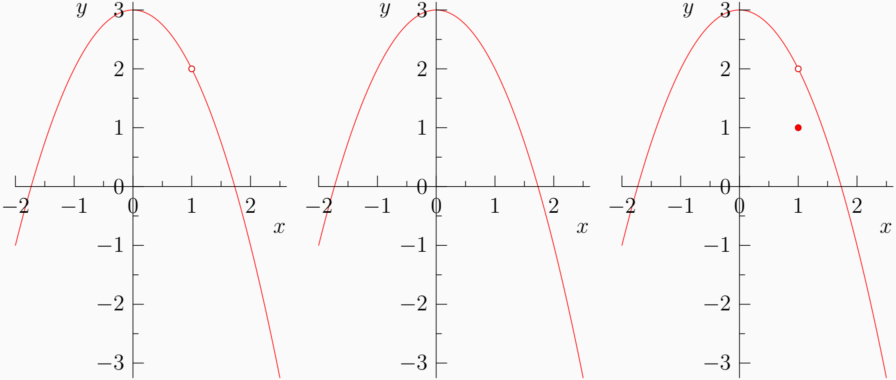

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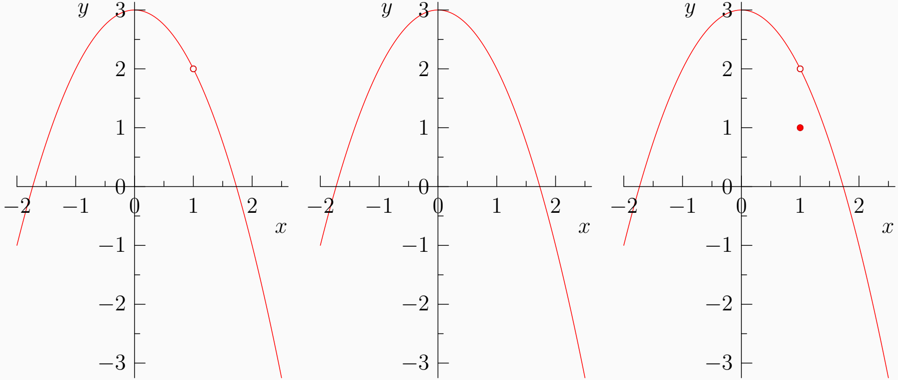
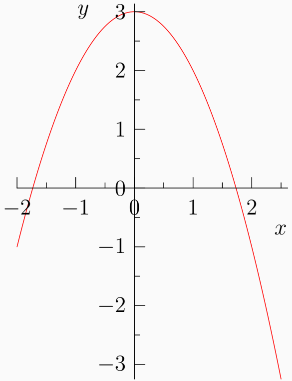


The middle graph is special.

The Idea of Continuity

- In the middle graph on the previous slide, there were ‘no surprises.’



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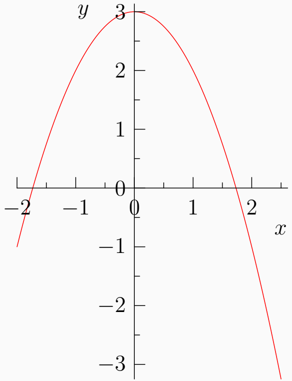

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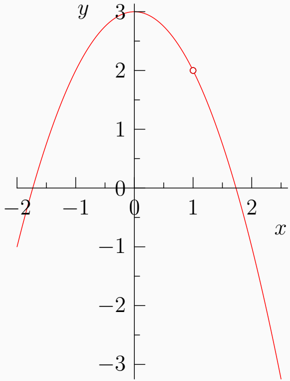

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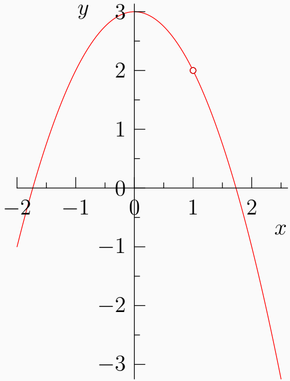

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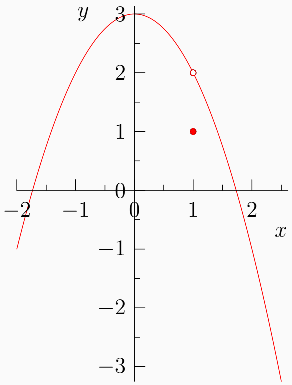
- In the other two graphs, the limits of the function are inconsistent with its values.
- In the first case, $\lim_{x \to 1} f(x) = 2$ but $f(1)$ is undefined.
- In the second case, $\lim_{x \to 1} f(x) = 2$ and $f(1)$ is defined, but $f(1) \neq 2$.



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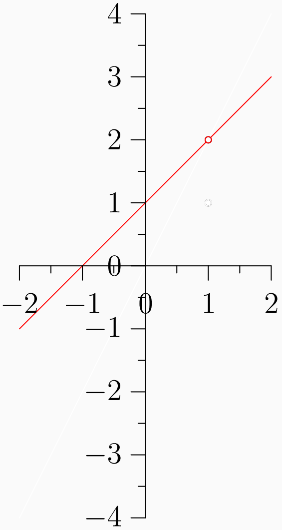
If any of those conditions is violated, we say f is **discontinuous at a**.

The discontinuities of a function can be classified into four categories.

1. **Removable discontinuities**: where the limit $\lim_{x \to a}$ exists but is not equal to $f(a)$.



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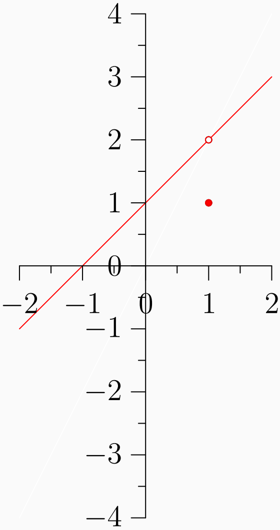

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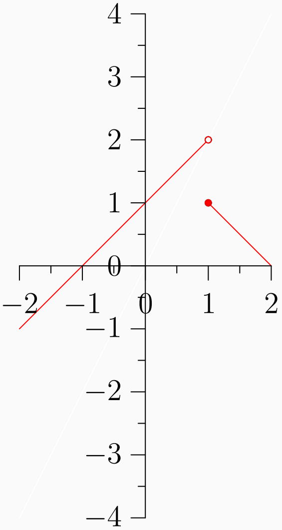

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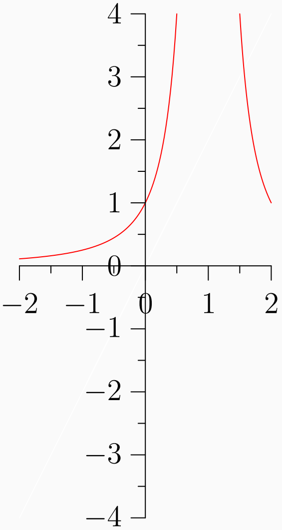

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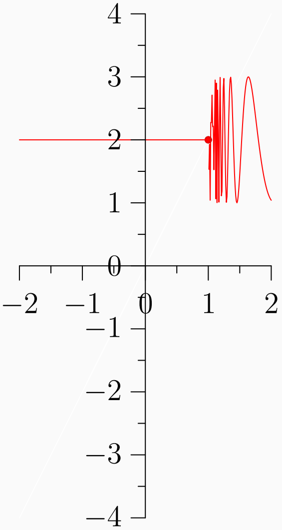
4. All other cases: where either of the one-sided limits doesn't exist (i.e., the function oscillates rapidly).



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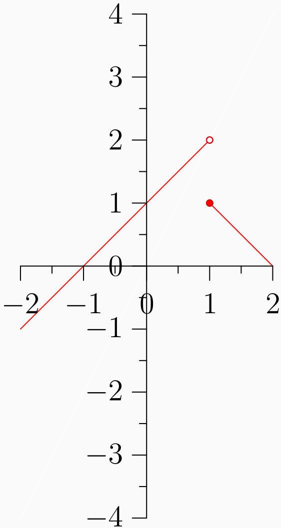

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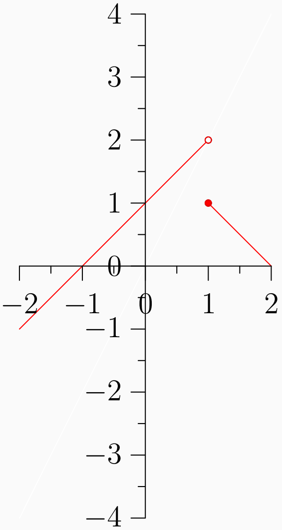

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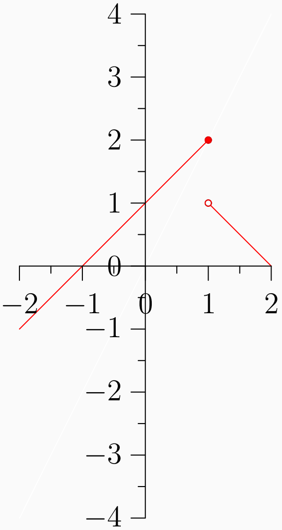

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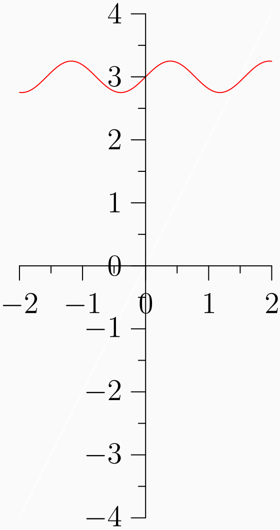

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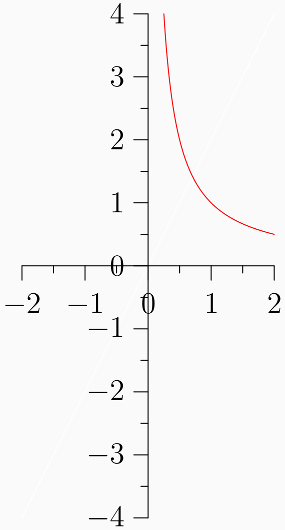

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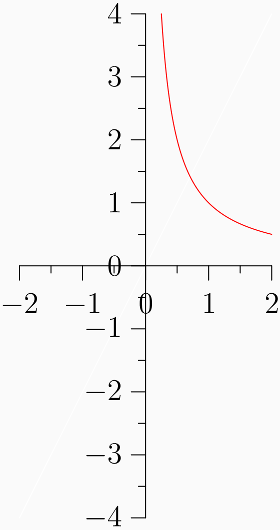

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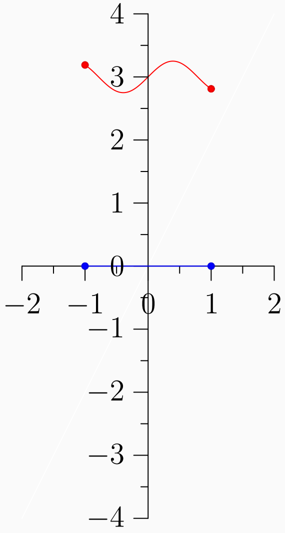

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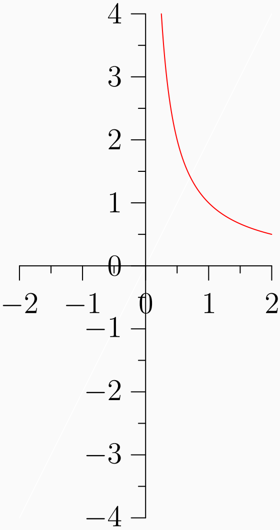

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3. A function which is continuous on an interval can be drawn without lifting the pencil from the paper.

Combining Continuous Functions

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In summary, any algebraic combination of continuous functions is continuous, provided we don't divide by zero.

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- It also follows that rational functions like $q(x) = \dfrac{x^4 - 3x^2 + 5}{x^2 - 1}$ are continuous everywhere except where their denominators are 0.
- We can summarize those statements by saying that polynomials and rational functions are continuous on their domains.

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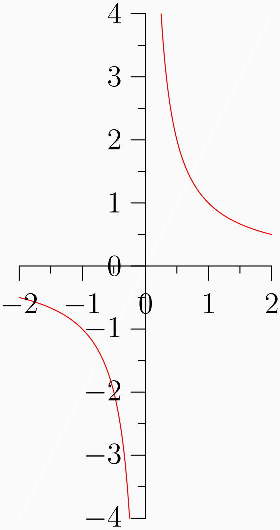

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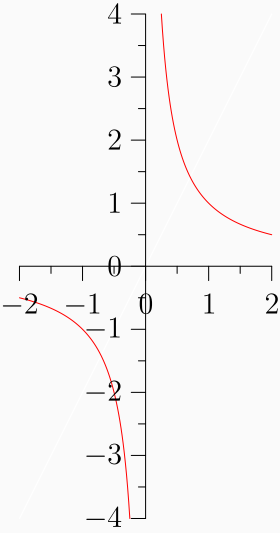

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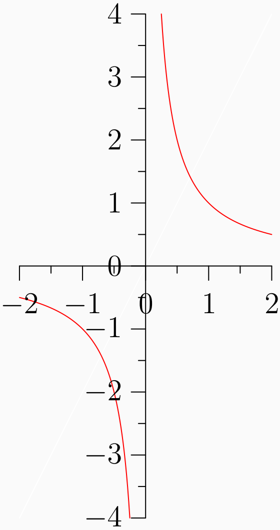

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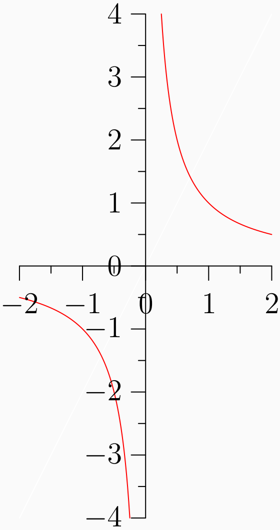

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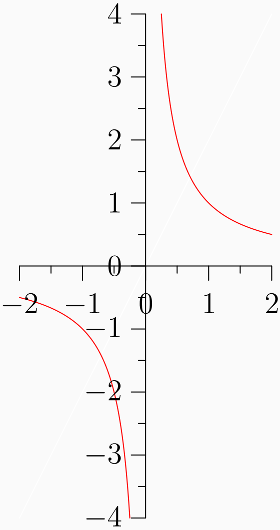
- But the function is not continuous on $(-\infty, \infty)$.



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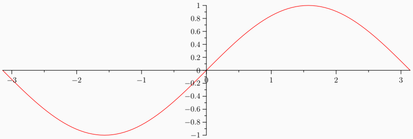

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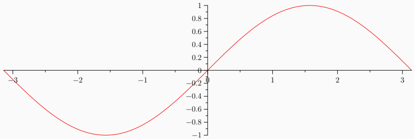

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which are obvious from the graphs of sin and cos or from basic trigonometry. From those limits and the identities

$$\sin(x+h) = \sin x \cos h + \cos x \sin h$$
$$\cos(x+h) = \cos x \cos h - \sin x \sin h$$

it follows that sin and cos are continuous on the whole real line.

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$$\lim_{x \to 3} \frac{\sin x + 1}{x^2 - 4} = \frac{\sin 3 + 1}{3^2 - 4} \approx 0.2282$$

Continuity (II)

Composition of Functions

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The resulting function $h(x)$ is called the **composition of f and g** and is sometimes written as $h = f \circ g$ when we don't want to put the xs in.

Composition of Functions: Order Matters

Note that $f \circ g \neq g \circ f$ in general. In the above case we have

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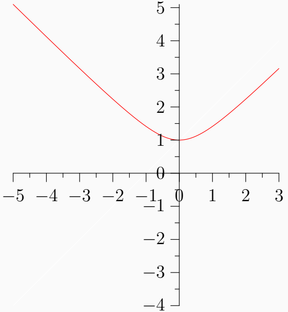

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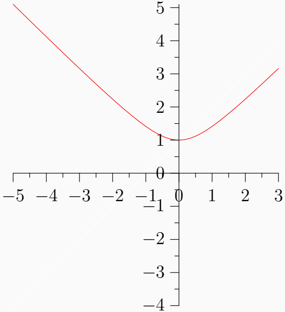

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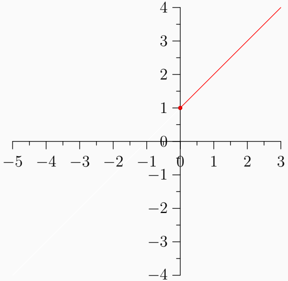

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Aside: A Rant on Nonlinearity

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Functions for which $f(a+b)=f(a)+f(b)$ are not typical. You study them in linear algebra, but calculus was invented because most useful functions are not like that.

Another Aside: $x+1$ vs. $(\sqrt{x})^2+1$ vs. $\sqrt{x^2}+1$

We can't even say $x+1$ is the same thing as $(\sqrt{x})^2+1$ if we are being completely correct; the domains are different. Also, $\sqrt{x^2}+1$ is different yet again:

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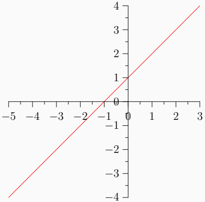

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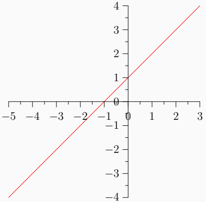

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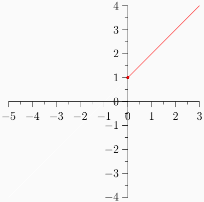

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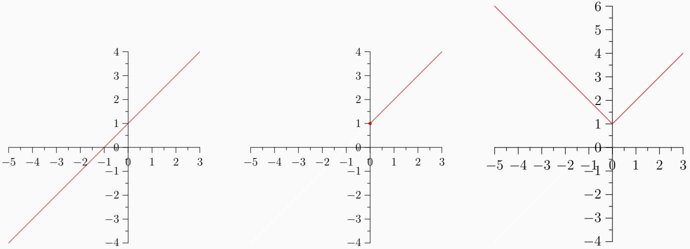

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Moral of this story: keep track of not on the formulas describing functions, but also their domains.

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The idea is that we can exchange the lim symbol with any continuous function f.

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We say that 'the composition of continuous functions is continuous'.

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$$\lim_{x\to 4} \sqrt{\frac{x^2-1}{x+3}} = \sqrt{\frac{4^2-1}{4+3}}$$

provided the latter expression makes sense.

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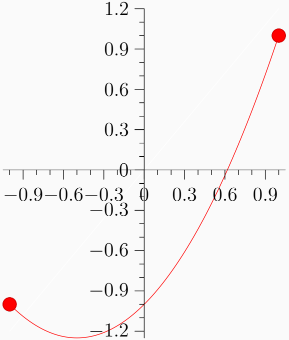

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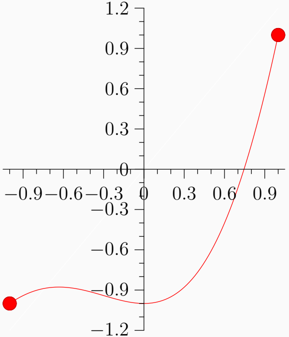

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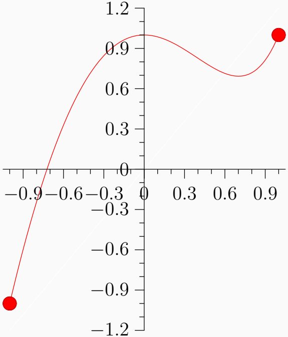

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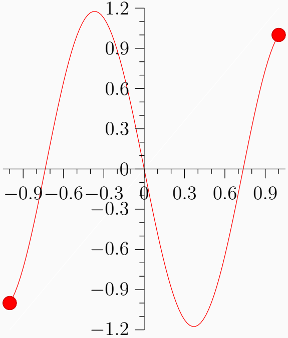

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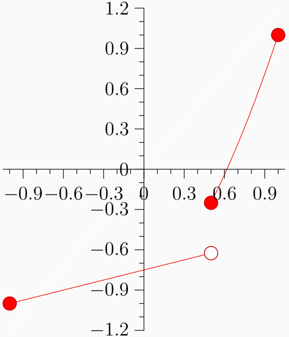

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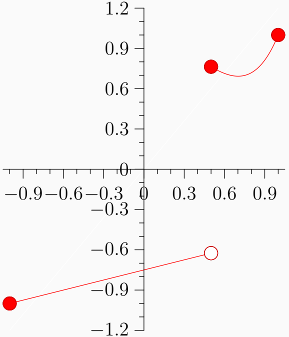

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  Theorem: Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number satisfying $f(a) \leq N \leq f(b)$. Then there exists a number c in the interval (a,b) such that $f(c) = N$.

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- or $f(a)$ any postive number, $f(b)$ any negative number, and $N = 0$.
- In simpler words, we can say:

  If a continuous function changes sign from $x = a$ to $x = b$, it must have a root between a and b.

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You probably don't know a method for solving this equation exactly. However, we can use the IVT to establish the existence of a solution. Note that $f(1) = (1)^3 - 3(1) + 1 = -1$ and $f(2) = 2^3 - 3(2) + 1 = 1$. Since f is continuous and we have

$$f(1) < 0 < f(2)$$

it follows from the IVT that there is some number c in the interval $(1,2)$ such that $f(c) = 0$.

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We can then continue to subdivide the interval to get better and better approximations.

Examples

1. Consider the function

$$f(x) = \frac{\sin x}{x+1}$$

   1.1 Find the domain of f.
   1.2 Show that f is continuous in its domain.
   1.3 Evaluate $\lim_{x \to \pi} f(x)$.
   1.4 Evaluate $\lim_{x \to -1^+} f(x)$.

2. Show that the function

$$g(x) = \begin{cases} \sin x & \text{if } -\pi \leq x \leq \pi/4 \\ \cos x & \text{if } \pi/4 < x \leq \pi \end{cases}$$

   is continuous on the interval $[-\pi, \pi]$.

More Examples

1. Let

$$f(x) = \begin{cases} 2x - x^2 & \text{if } 0 \leq x \leq 2 \\ 2 - x & \text{if } 2 < x \leq 3 \\ x - 4 & \text{if } 3 < x < 4 \\ \pi & \text{if } 4 \leq x \end{cases}$$

   Determine whether f is continuous, continuous from the left, and/or continuous from the right at each point in its domain, and sketch the graph of f.

2. Let $g(x) = \lfloor x \rfloor$, the greatest integer less than or equal to x, and let $h(x) = \lfloor x \rfloor + \lfloor -x \rfloor$.
   - 2.1 For what values of a does $\lim_{x \to a} g(x)$ exist?
   - 2.2 For what values of a is $g(x)$ continuous at a?
   - 2.3 For what values of a does $\lim_{x \to a} h(x)$ exist?
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Yet More Examples

1.
   1.1 Find the domain of $f(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 2}$.

   1.2 Show that f is continuous on its domain.

   1.3 Find $\lim_{x \to 5} f(x)$.

2. Use the Intermediate Value Theorem to show that the equation $2\sin x = 3 - 2x$ has a solution on the interval $(0, 1)$.

3.
   3.1 Show that there is a root of the equation $2x^3 + x^2 + 2 = 0$ in the interval $(-2, -1)$.

   3.2 Find the root to an accuracy of two decimal points.

Exercises

Now you should work on Problem Set 1.8. After you have finished it, you should try the following additional exercises from Section 1.8:

1.8 C-level: 1–2, 3–4, 5–6, 7–8, 9–10, 11–14, 15–16, 17–20, 22, 23–24, 35–38, 39–40, 41–43, 49, 53–56;
B-level: 21, 19, 25–32, 33–34, 44–45, 47, 48, 50–52, 57–58, 59–60, 61–62;
A-level: 46, 63–73