# MATH 110 Lecture 3.2

## The Mean Value Theorem

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Department of Indigenous Knowledge and Science
First Nations University of Canada

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# The Intermediate and Extreme Value Theorems

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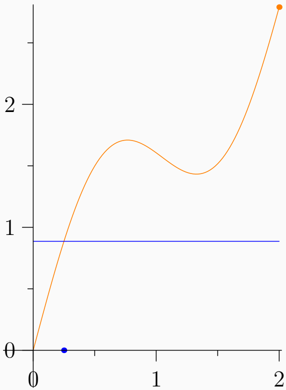

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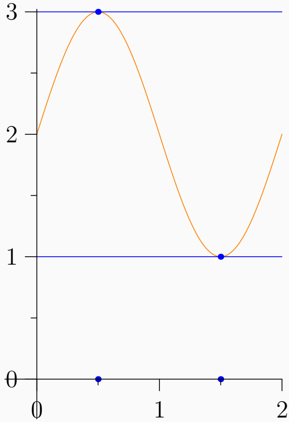

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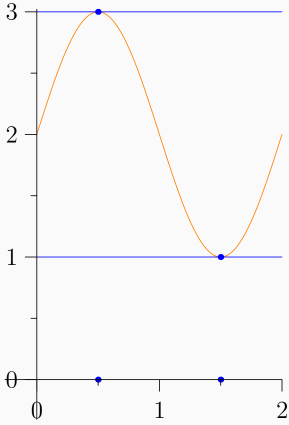
- The *values* in those theorems are values taken on by the function $f(x)$.



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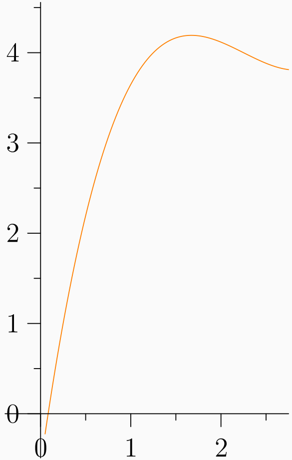

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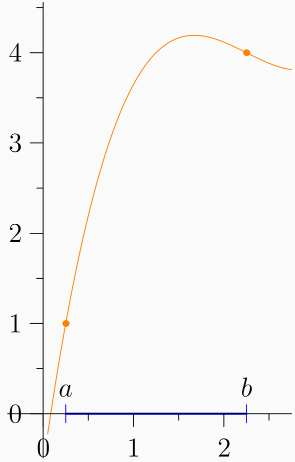

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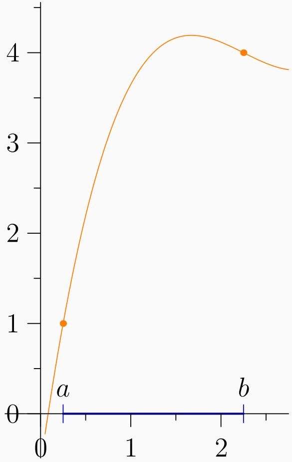

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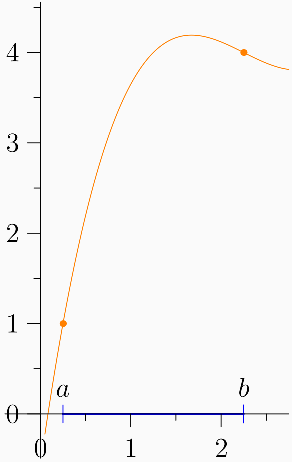

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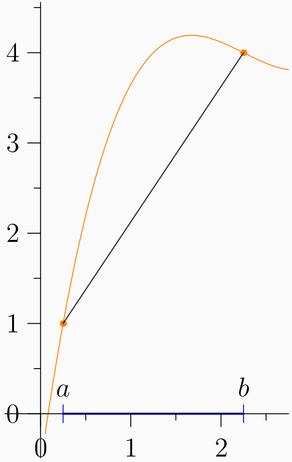

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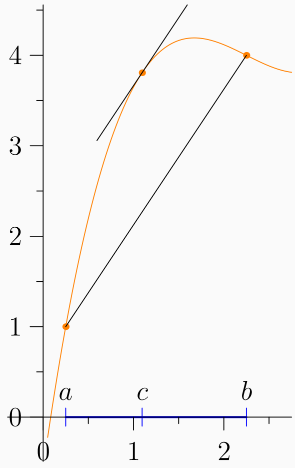
- The mean value is the slope of the secant line connecting $(a, f(a))$ and $(b, f(b))$ on the graph.
- The Mean Value Theorem says that there is a tangent line through $(c, f(c))$ parallel to the secant line where $c$ is between $a$ and $b$.



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- The Mean Value Theorem then says that the average velocity over an interval $[t_0, t_1]$ must be equal to the instantaneous velocity at some moment $c$ between $t_0$ and $t_1$.
- For example, if you drive 130 km in 1 h, your mean velocity is 130 km/h. The MVT says that at some moment during your trip your instantaneous velocity must have been 130 km/h.

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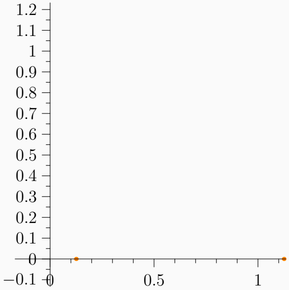
- We will only consider the concept, outline of the proofs, and simple examples in this lecture.

# Rolle's Theorem

- We begin by looking at the special case of $f(a) = f(b)$.



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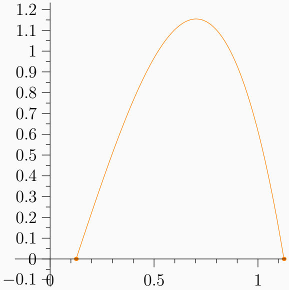

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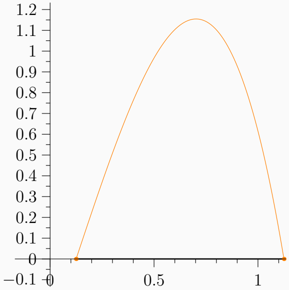

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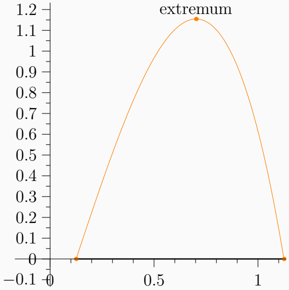

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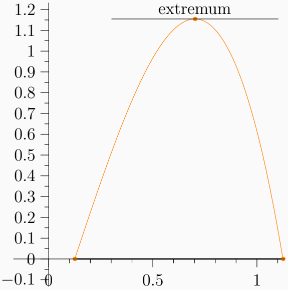

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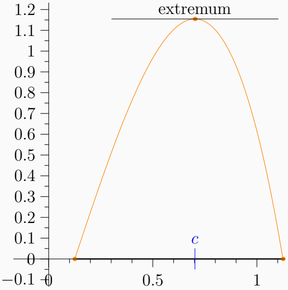
- Fermat's theorem says $f'(c) = 0$ or $f'(c)$ does not exist.
- Since $f'$ exists everywhere in $(a,b)$ we must have $f'(c) = 0$, and we're done.



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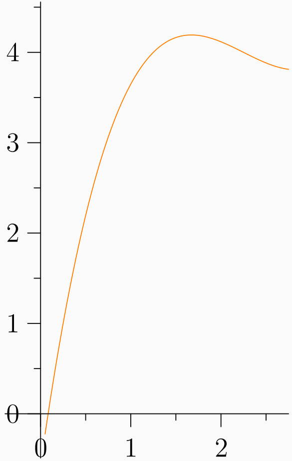

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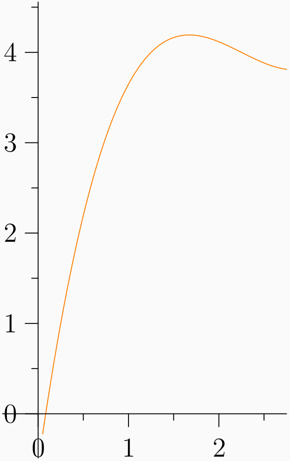

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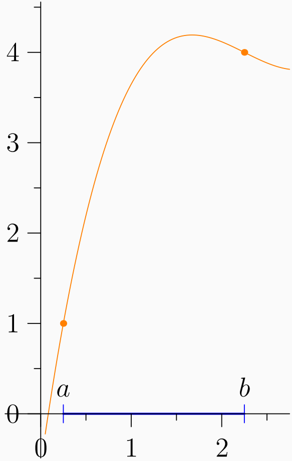

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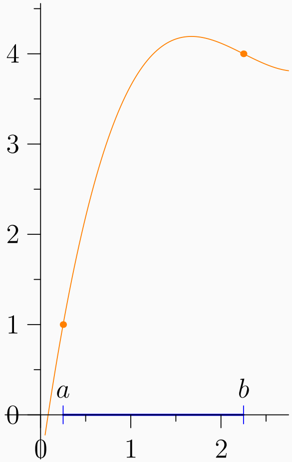

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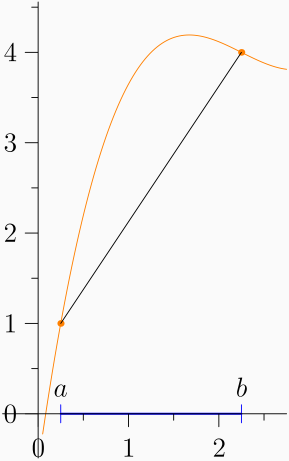

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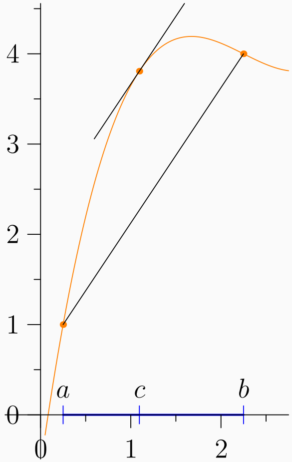

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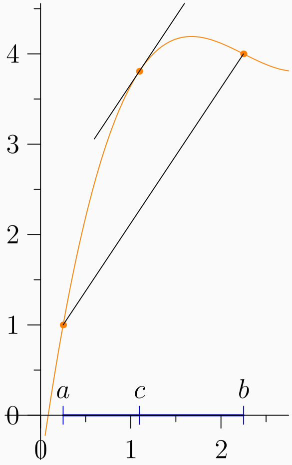
  Alternatively, we can write

  $$f(b)-f(a) = f'(c)(b-a)$$



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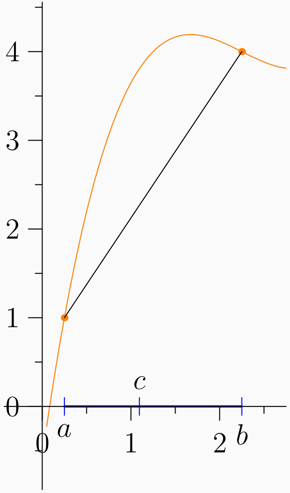

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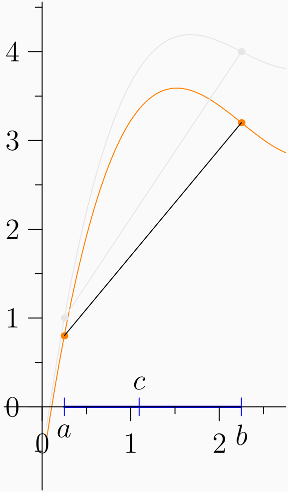

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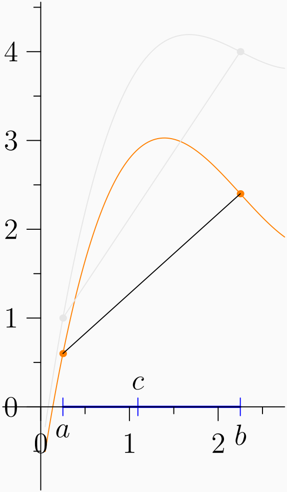

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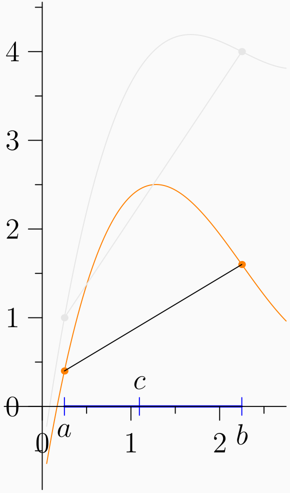

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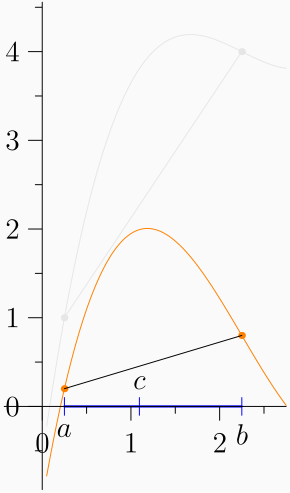

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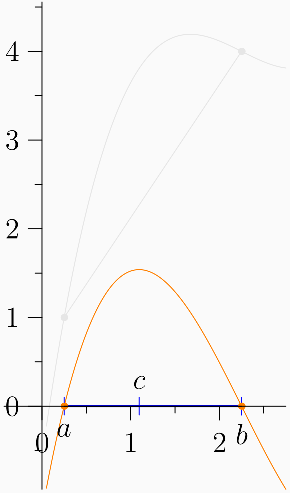

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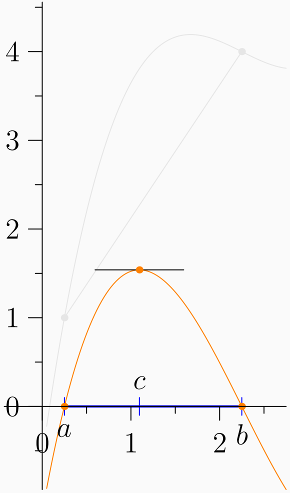

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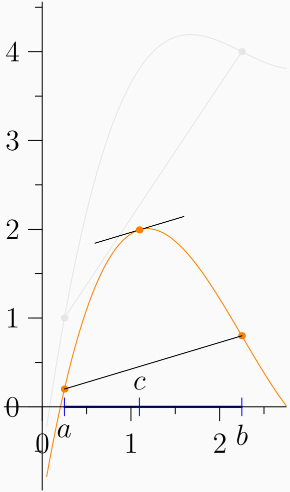

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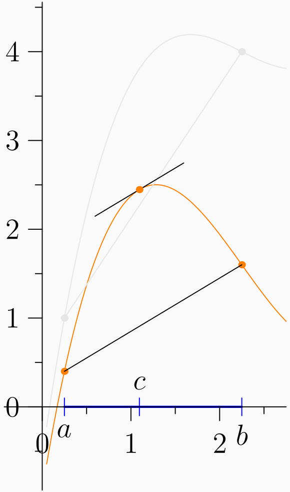

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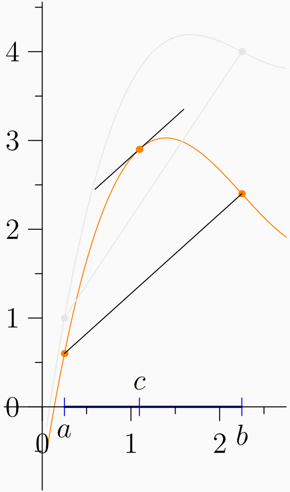

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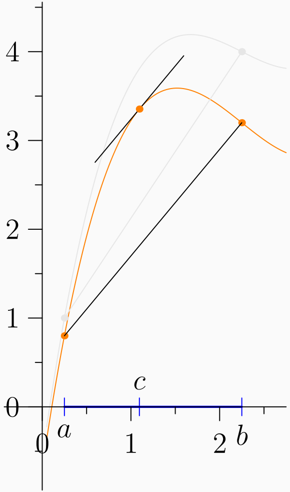

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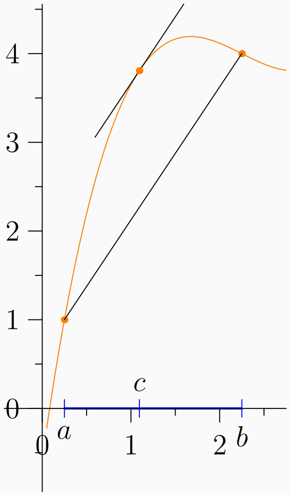

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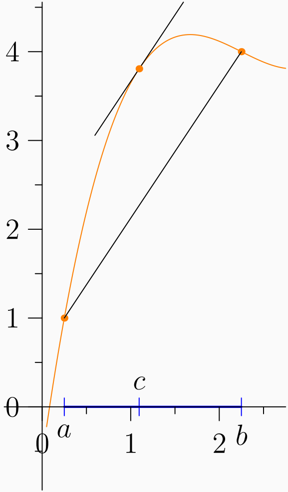
- To reduce the MVT to Rolle's theorem, we just apply a *shear transformation* to the graph of $f$.
- In symbols, we define a new function

$$h(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right)$$



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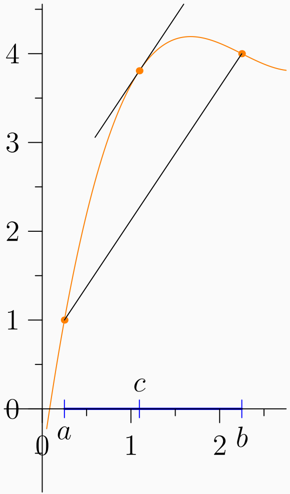
- The MVT can be reduced to Rolle's Theorem.
- Since we know that Rolle's Theorem is true, it follows that the MVT is true.
- To reduce the MVT to Rolle's theorem, we just apply a *shear transformation* to the graph of $f$.
- In symbols, we define a new function

$$h(x) = f(x) - \left( f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right)$$

- The details are in the textbook if you are interested.



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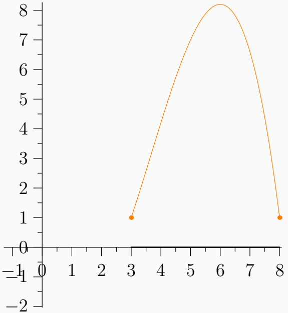

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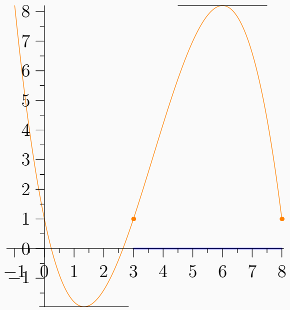

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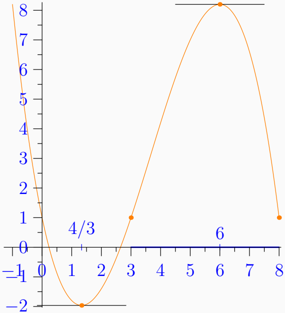

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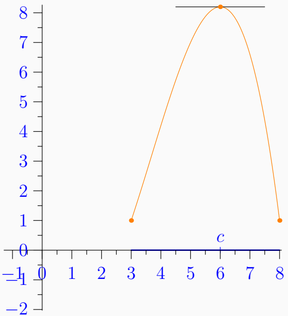

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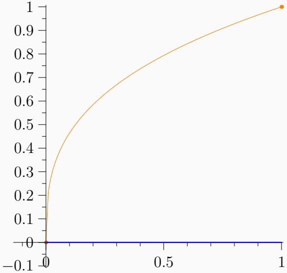

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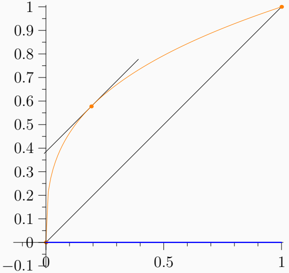

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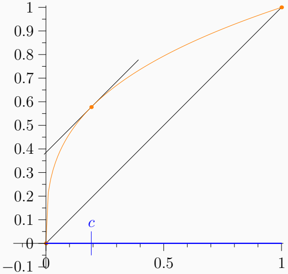

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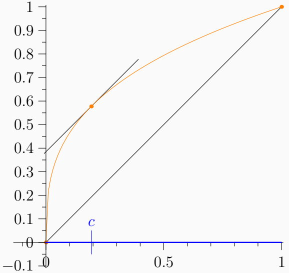
- **Note:** Finding $c$ explicitly is often not possible! We use Rolle's Theorem and the MVT *to avoid* finding $c$.



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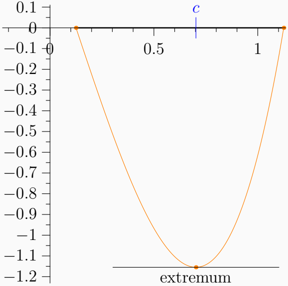

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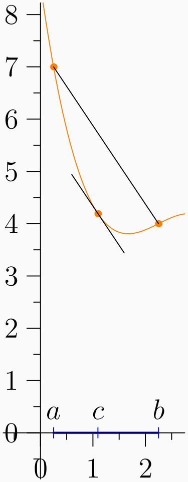

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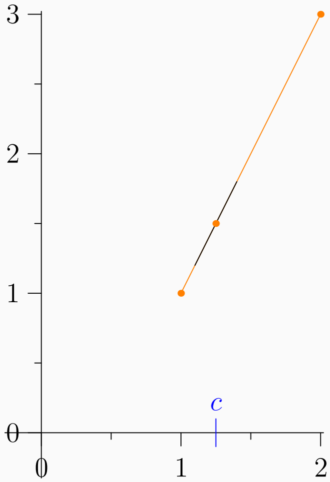

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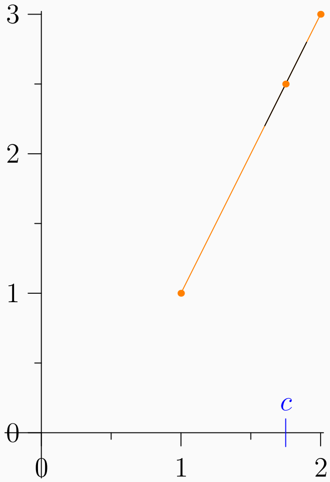

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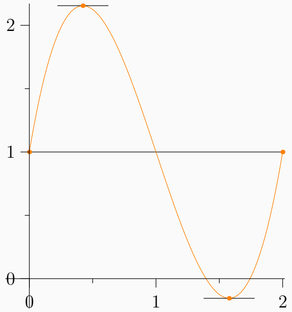

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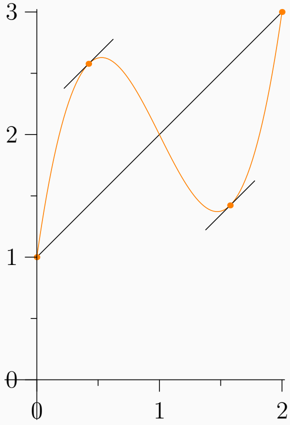

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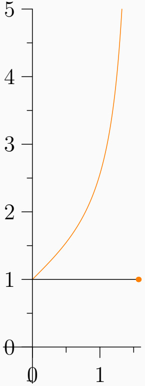

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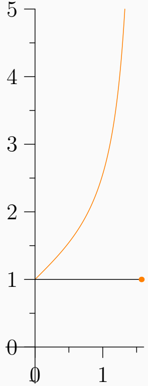

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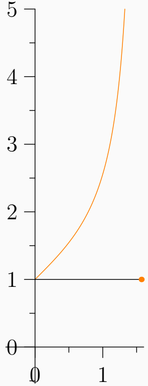

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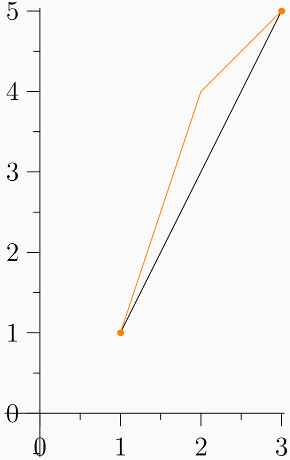

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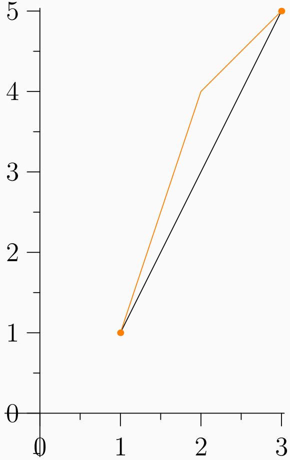

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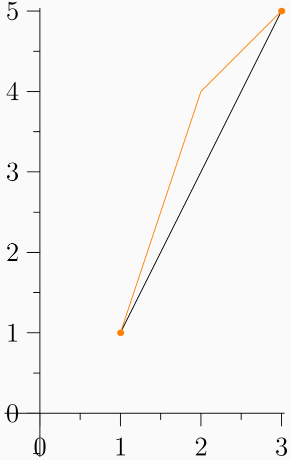

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1. Verify that $f(x) = x^3 - x^2 - 6x + 2$ satisfies the hypotheses of Rolle's theorem on the interval $[0,3]$. Then find all numbers $c$ that satisfy the conclusion of Rolle's theorem.
2. Let $f(x) = \tan x$. Show that $f(0) = f(\pi)$ but there is no number $c$ in $(-1,1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem?
3. Let $f(x) = \dfrac{x}{x+2}$. Show that $f$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1,4]$. Then find all numbers $c$ that satisfy the conclusion of the MVT.
4. Let $f(x) = 2 - |2x-1|$. Show that there is no value of $c$ such that $f(3) - f(0) = f'(c)(3-0)$. Why does this not contradict the MVT?

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- The MVT is also used to show that if $f'$ is constant, then $f$ cannot be anything other than linear.
- It is best to think of those applications as saying what $f$ cannot be; that is what I mean when I say the MVT is applied in a negative way.

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- Now we have both the smallest positive number $r$ and a smaller, different positive number $r/2$, which is absurd.
- Therefore our assumption that there is a smallest positive number must be false.

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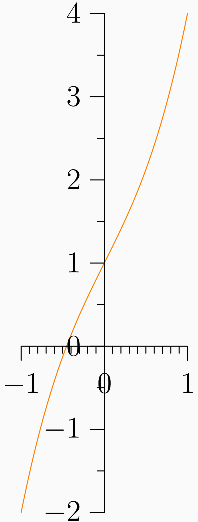

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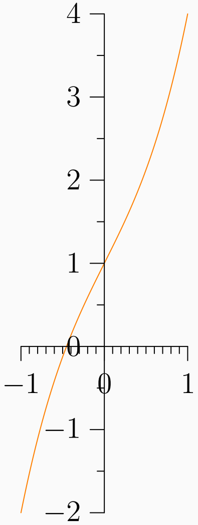

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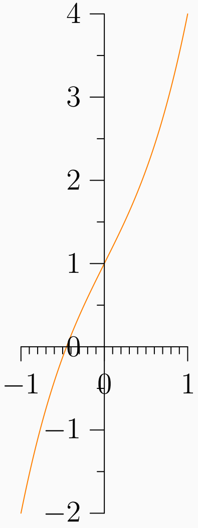

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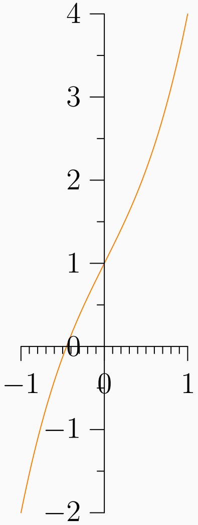
- From the graph, we suspect there is just one root but graphs can be misleading.



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- Then Rolle's Theorem now tells us that $f'(c) = 0$ for some $c$ between $a$ and $b$. Since $a$ and $b$ are in $(-1, 1)$, so must be $c$.

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- Furthermore, since $f$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$, Rolle's Theorem applies.
- Then Rolle's Theorem now tells us that $f'(c) = 0$ for some $c$ between $a$ and $b$. Since $a$ and $b$ are in $(-1, 1)$, so must be $c$.
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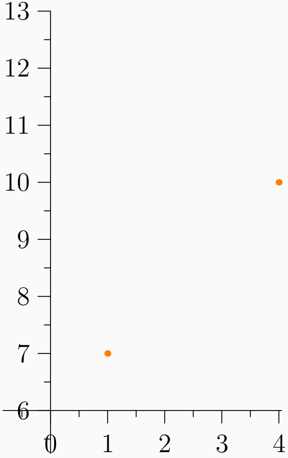

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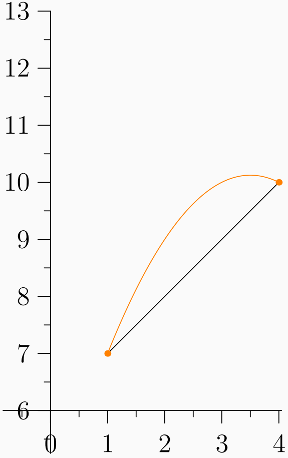

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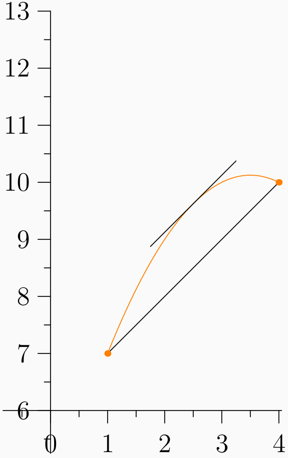

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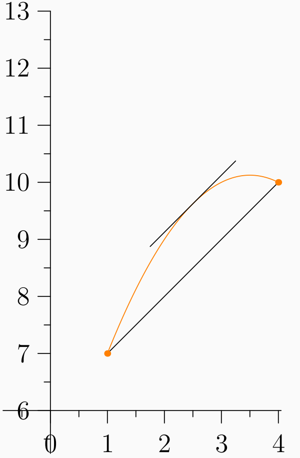

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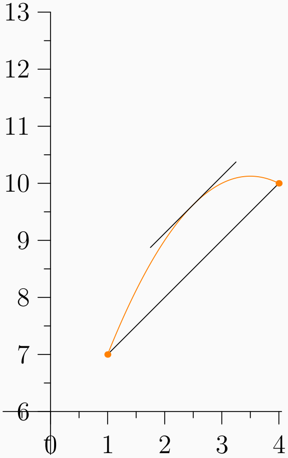

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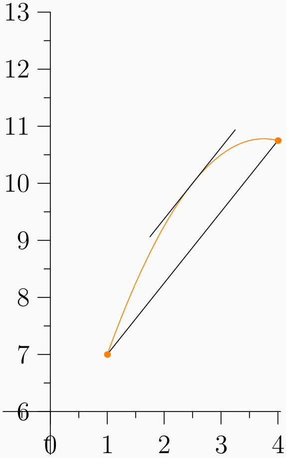

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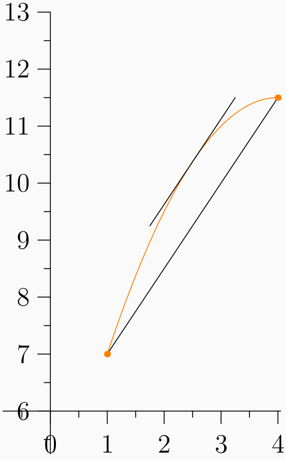

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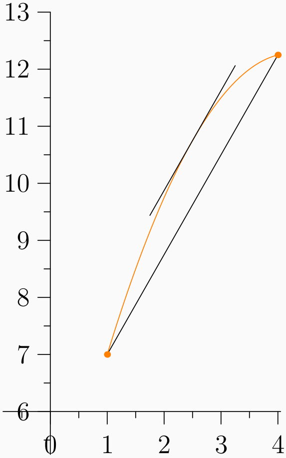

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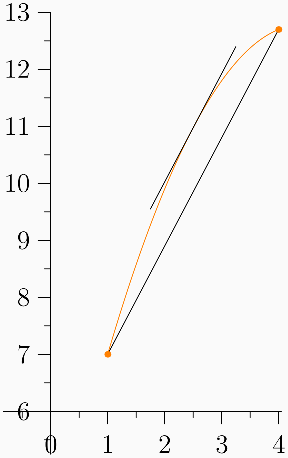

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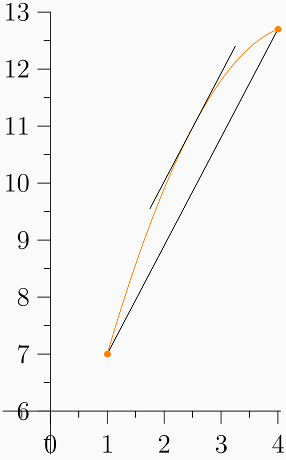

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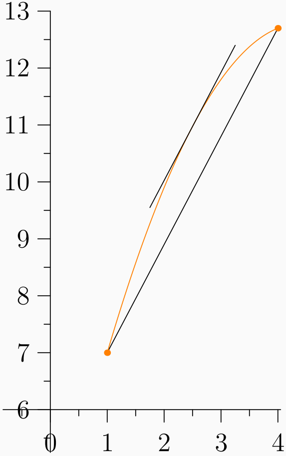

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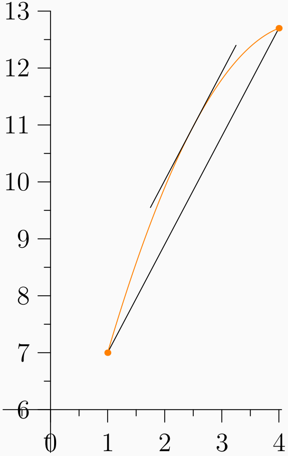

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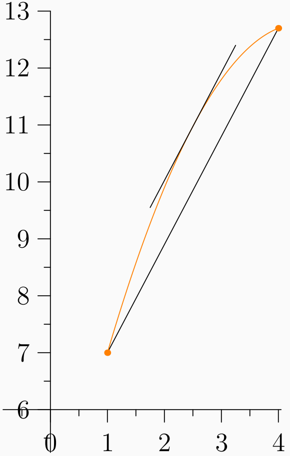

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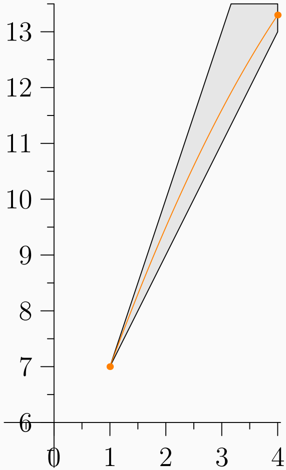

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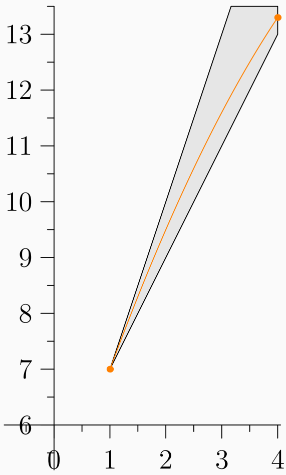

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- It follows that there is no non-constant solution to $f'(x) = 0$.

# Examples and Exercises

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1. Show that the function $3x+2\cos x+5=0$ has exactly one real root.
2. Suppose that $f$ is continuous on $[0,4]$, $f(0)=1$, and $2 \le f'(x) \le 5$ for all $x$ in the interval $(0,4)$. Show that $9 \le f(4) \le 21$.
3. By applying the Mean Value Theorem to the function $f(x)=x^{1/5}$ on the interval $[32,33]$, show that $2 < \sqrt[5]{33} < 2.0125$.
4. Show that $\sqrt{1+x} < 1+\frac{1}{2}x$ for $x > 0$.
5. Show that $|\sin b - \sin a| \le |b-a|$ for all $a$ and $b$.

# Exercises

Now you should work on Problem Set 3.2. After you have finished it, you should try the following additional exercises from Section 3.2:

3.2 C-level: 1–20, 34;
B-level: 25–27, 29, 33, 35;
A-level: 21–22, 23–24, 28, 30–32, 36