MATH 110 Problem Set 2.2 Solutions

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1. (a) The function is a polynomial so its domain is the entire real line. We have

$$f(x+h) = \frac{3}{2}(x+h)^2 - (x+h) + \frac{37}{10} = \frac{3}{2}x^2 + 3xh + \frac{3}{2}h^2 - x - h + \frac{37}{10}$$

so

$$\begin{aligned} f(x+h) - f(x) &= \frac{3}{2}x^2 + 3xh + \frac{3}{2}h^2 - x - h + \frac{37}{10} - \left(\frac{3}{2}x^2 - x + \frac{37}{10}\right) \\ &= \frac{3}{2}x^2 + 3xh + \frac{3}{2}h^2 - x - h + \frac{37}{10} - \frac{3}{2}x^2 + x - \frac{37}{10} \\ &= 3xh + \frac{3}{2}h^2 - h \end{aligned}$$

and therefore

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{3xh + (3/2)h^2 - h}{h} \\ &= \lim_{h\to 0} \left(3x + \frac{3}{2}h - 1\right) \\ &= 3x - 1 \end{aligned}$$

which you should check by applying the short cut rules we have learned. Note that the above limit exists and equals $3x - 1$ for any value of x, so the domain of the derivative is also the entire real line.

(b) The domain of $f(x)$ is $x \geq 0$, because those are the x-values for which the square root is defined. Note that

$$f(x+h) = x + h + \sqrt{x+h}$$

so

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{x+h+\sqrt{x+h}-(x+\sqrt{x})}{h} \\
&= \lim_{h\to 0} \frac{h+\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0} 1+\frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= 1+\lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} \\
&= 1+\lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= 1+\lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\
&= 1+\lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= 1+\frac{1}{2\sqrt{x}}
\end{aligned}$$

You should check the derivative using the short-cut rules that we have learned. The above limit is defined when $x > 0$, which is the domain of $f'(x)$. Note that in this case, the domain of $f'(x)$ is slightly smaller than the domain of $f(x)$: the former does not include the number $x = 0$.

(c) The domain of f is every number except where the denominator is 0, i.e., every number except 1/3. Differentiating,

$$\begin{aligned}
f(x+h)-f(x) &= \frac{3+x+h}{1-3x-3h} - \frac{3+x}{1-3x} \\
&= \frac{(3+x+h)(1-3x)-(3+x)(1-3x-3h)}{(1-3x-3h)(1-3x)} \\
&= \frac{3+x+h-9x-3x^2-3xh-(3-9x-9h+x-3x^2-3xh}{(1-3x-3h)(1-3x)} \\
&= \frac{3+x+h-9x-3x^2-3xh-3+9x+9h-x+3x^2+3xh}{(1-3x-3h)(1-3x)} \\
&= \frac{10h}{(1-3x-3h)(1-3x)}
\end{aligned}$$

so

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{10h}{h(1-3x-3h)(1-3x)} \\
&= \lim_{h\to 0} \frac{10}{(1-3x-3h)(1-3x)} \\
&= \frac{10}{(1-3x)^2}
\end{aligned}$$

which you can check by using the quotient rule. The above limit for the derivative exists at every value of x where the denominator exists, i.e., at every value of x except $x = 1/3$, so the domain of f' is the same as the domain of f.

(d) The domain of g is the set of all t where (1) $\sqrt{t}$ exists and (2) $\sqrt{t} \neq 0$, in other words, at all values of $t > 0$. Differentiating,

$$\begin{aligned} g(t+h) - g(t) &= \frac{1}{\sqrt{t+h}} - \frac{1}{\sqrt{t}} \\ &= \frac{\sqrt{t} - \sqrt{t+h}}{\sqrt{t+h}\sqrt{t}} \end{aligned}$$

so

$$\begin{aligned} g'(t) &= \lim_{h \to 0} \frac{\sqrt{t} - \sqrt{t+h}}{h\sqrt{t+h}\sqrt{t}} \\ &= \lim_{h \to 0} \frac{(\sqrt{t} - \sqrt{t+h})(\sqrt{t} + \sqrt{t+h})}{h\sqrt{t+h}\sqrt{t}(\sqrt{t} + \sqrt{t+h})} \\ &= \lim_{h \to 0} \frac{t - (t+h)}{h\sqrt{t+h}\sqrt{t}(\sqrt{t} + \sqrt{t+h})} \\ &= \lim_{h \to 0} \frac{-h}{h\sqrt{t+h}\sqrt{t}(\sqrt{t} + \sqrt{t+h})} \\ &= \lim_{h \to 0} \frac{-1}{\sqrt{t+h}\sqrt{t}(\sqrt{t} + \sqrt{t+h})} \\ &= \frac{-1}{(\sqrt{t})^2(2\sqrt{t})} \\ &= \frac{-1}{2t^{3/2}} \end{aligned}$$

which you can check by using the power rule for derivatives. Note that the above limit exists for all $t > 0$ so the domain of g' is the same as the domain of g.

2. By the definition of derivative,

$$\begin{aligned} f(x) &= x^3 - 2x^2 \\ f(x+h) &= (x+h)^3 - 2(x+h)^2 = x^3 + 3x^2h + 3xh^2 + h^3 - 2x^2 - 4xh - 2h^2 \\ f(x+h) - f(x) &= 3x^2h + 3xh^2 + h^3 - 4xh - 2h^2 \\ f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(3x^2 + 3xh + h^2 - 4x - 2h)h}{h} = 3x^2 - 4x \end{aligned}$$

See Figure 1. The function f is graphed in red, and its derivative f' is graphed in orange. Note that where f is increasing, f' is positive, and where f is decreasing, f' is negative.

3. By the definition of derivative we have

$$\begin{aligned} f(x) &= x^2 - 6x + 5 \\ f(x+h) &= (x+h)^2 - 6(x+h) + 5 = x^2 + 2xh + h^2 - 6x - 6h + 5 \\ f(x+h) - f(x) &= 2xh + h^2 - 6h \\ f'(x) &= \lim_{h \to 0} \frac{(2x + h - 6)h}{h} = 2x - 6 \end{aligned}$$

Differentiating the derivative,

$$\begin{aligned} f'(x+h) &= 2x + 2h - 6 \\ f'(x+h) - f'(x) &= 2h \\ f''(x) &= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{2h}{h} = 2 \end{aligned}$$

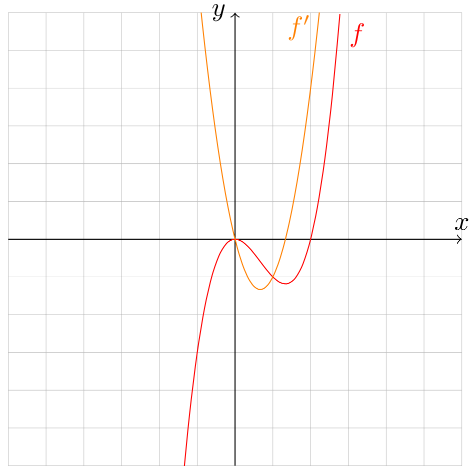


Figure 1: Graphs of $f(x) = x^3 - 2x^2$ and $f'(x) = 3x^2 - 4x$

Differentiating the second derivative,

$$f'''(x) = \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h \to 0} \frac{2-2}{h} = 0$$

Differentiating the third derivative,

$$f^{(4)}(x) = \lim_{h \to 0} \frac{f'''(x+h) - f'''(x)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

See Figure 2. The function is in red, the first derivative in orange, the second derivative in yellow, and the third and higher derivatives in green. Note that where f is decreasing, f' is negative, and where f is increasing, f' is positive. Similar relationships hold between f' and f'', etc.

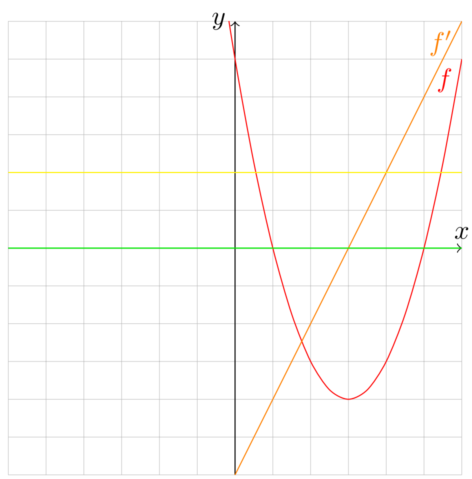


Figure 2: Graphs of $f(x) = x^2 - 6x + 5$, $f'(x) = 2x - 6$, $f''(x) = 2$, $f'''(x) = 0$, $f^{(4)}(x) = 0$

4. See Figure 3 for the graph of $\cos x$ in red and its derivative in orange. The derivative appears to be another trigonometric function, a shifted version of the cosine function. We will learn more about it in Section 2.4.

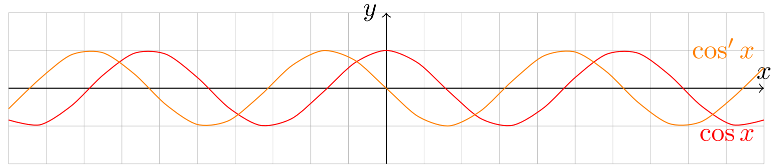


Figure 3: Graphs of $\cos x$ and its derivative

5. The function $y = x^{2/3}$ has a cusp at $(0, 0)$, but is differentiable at $(8, 4)$. Zooming in on the cusp with an online calculator like Desmos, the graph will eventually look like a vertical half-line; since the zoomed-in graph does not look like a whole line (and since the zoomed-in graph is vertical) we know that the function is not differentiable at $(0, 0)$. Zooming in on the point $(8, 4)$, the function will eventually look like a whole non-vertical line, which means that the function is differentiable at $(8, 4)$.

6. Note that if $x < 0$, $|x| = -x$, so $f(x) = |x| - x = -x - x = -2x$; if $x = 0$ then $f(0) = |0| - 0 = 0 - 0 = 0$; and if $x > 0$ then $|x| = x$ so $f(x) = |x| - x = x - x = 0$. In summary, we can write

$$f(x) = \begin{cases} -2x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

According to the definition of derivative,

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$

It is difficult to work with $f(h) = |h| - h$ as it was given in the problem statement, but we note by the above discussion that $f(h)$ has simpler expressions when $h < 0$ and when $h > 0$, so we analyze the above limit using one-sided limits.

$$\lim_{h \to 0^-} \frac{f(h)}{h} = \lim_{h \to 0^-} \frac{-2h}{h} = -2$$

while on the other hand

$$\lim_{h \to 0^+} \frac{f(h)}{h} = \lim_{h \to 0^+} \frac{0}{h} = 0$$

Since the left and right-sided limits do not agree, the two sided limit does not exist, so $f(x)$ is not differentiable at $x = 0$.

If $x < 0$ then

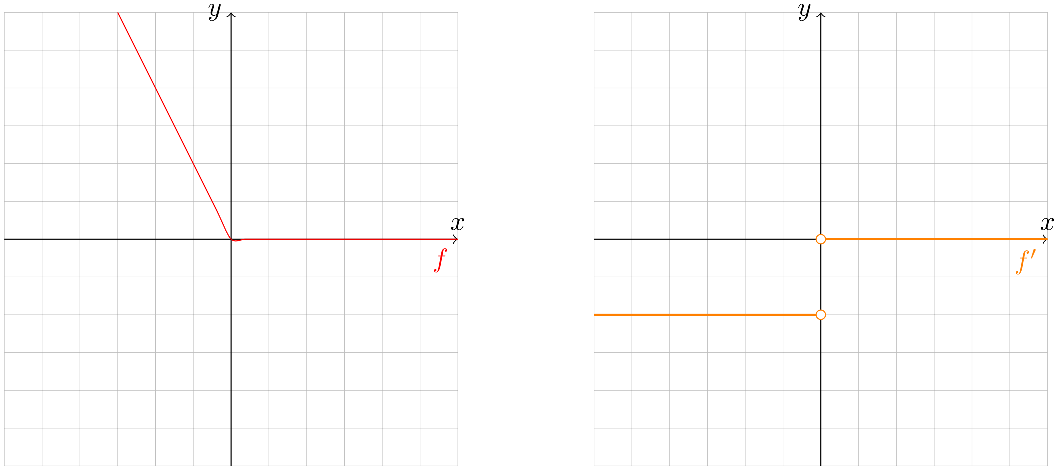
$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{-2(x + h) - (-2x)}{h} = \lim_{h \to 0} \frac{-2h}{h} = -2$$

because we can assume that $x < 0$ implies $x + h < 0$ for small values of h in the limit $h \to 0$, and so $f(x + h) = -2(x + h)$. If $x > 0$ then similar reasoning shows $f'(x) = 0$. In summary, we have

$$f'(x) = \begin{cases} -2 & \text{if } x < 0 \\ \text{undef} & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

See Figure 4 for the graph of $f'(x)$.

7. (a) Write $f(x) = x^{1/3}$. Then $f'(x) = (1/3)x^{-2/3}$.



(a) Graph of $y = f(x) = |x| - x$ (b) Graph of $y = f'(x)$ where $f(x) = |x| - x$

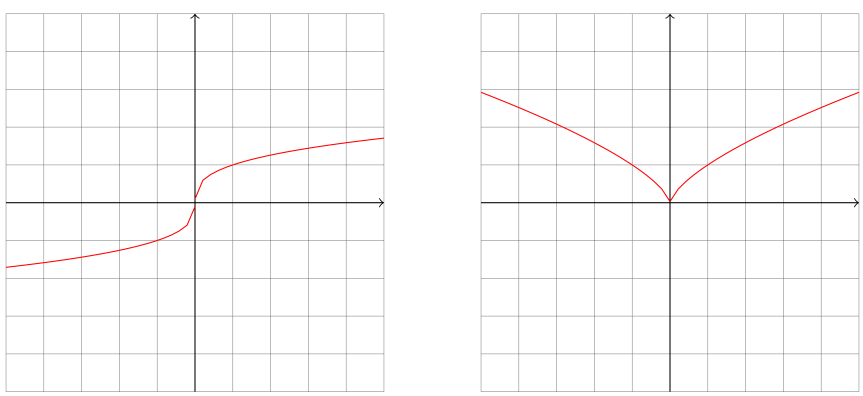
Figure 4: Graphs for Problem 6

(b) By the result above, $f'(x)$ is undefined for $x = 0$ because of the negative power of x.

(c) To show that $f(x)$ has a vertical tangent at $x = 0$, we need to perform a more careful analysis of the derivative of $f(x)$ at $x = 0$ using the definition of the derivative. We have

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{1/3} - 0}{h} = \lim_{h \to 0} h^{-2/3} = +\infty$$

because $h^{-2/3} = 1/(h^{1/3})^2$ is a large positive number when h is a small positive or negative number. Since $f'(0) = +\infty$ we have a vertical tangent, a tangent with slope $+\infty$.
Compare the above result to the result for $g(x) = x^{2/3}$, which does not have a vertical tangent at $x = 0$ but instead has a cusp. See Figure 5.



(a) Graph of $y = x^{1/3}$ (b) Graph of $y = x^{2/3}$

Figure 5: Graphs for Problem 7

8. First we find out all we can about the tangent line. We know a point on the line, namely $(1, 1)$, so all we need is the slope of the line to determine the line. The slope is the derivative of the function evaluated at $x = 1$, i.e.,

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 3(1+h) + 3 - 1}{h} = \lim_{h \to 0} \frac{(2+h-3)h}{h} = -1$$

So the equation of the tangent line is

$$y - y_0 = m(x - x_0) \implies y - 1 = (-1)(x - 1) \implies y = -x + 2$$

See Figure 6. Forming a right triangle with the tangent line as hypoteneuse and the angle of inclination as the angle, we see that the tangent of the angle of inclination is the opposite over the adjacent, or the rise over the run, in other words the slope. In general we have $\tan\phi = m$. In this case, we have $\tan\phi = -1$ so $\phi = \tan^{-1}(-1)$. My calculator says $\phi \approx -0.79$ radians, or $\phi = -45$ degrees.

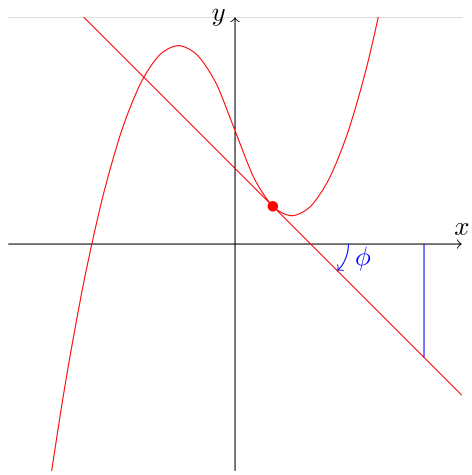


Figure 6: Graphs of $f(x) = x^2 - 3x + 3$ and Tangent Line