MATH 110 Lecture 2.5

The Chain Rule

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Review of Composition of Functions

- Recall the concept of chaining or composing functions:

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- Consider $h(x) = \sin(2x + \pi)$. We read the expression on the right as

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  The separator between f and g is usually the word 'of'. In this case $f(u) = \sin u$ and $g(x) = 2x + \pi$.
- Another way of thinking of it is that $g(x) = 2x + \pi$ is the 'inner function', i.e., the function inside brackets. So $f(u) = \sin u$ is the outer function. The u is just a placeholder.

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- This suggests that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$.
- The result is proven in the textbook, but you should skip the proof unless you are majoring in math.

The Chain Rule in Newton and Leibniz Notation

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- In the Leibniz rule, y and u have changing meanings.
- The way I personally think of the chain rule is

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot \frac{d}{dx} g(x)$$

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- The chain rule tells us the derivative of $h(x) = f(g(x))$ is $h'(x) = f'(g(x)) \cdot g'(x) = \cos(5x) \cdot 5$.
- Conventionally, we rearrange expressions like that to the form $h'(x) = 5\cos 5x$.

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- We can rearrange if we like:

$$y' = -3\cos^2 x \sin x$$

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- Another example is $h(x) = (3x+2)^{2.5}$, with derivative

$$\begin{aligned} h'(x) &= 2.5(3x+2)^{1.5} \cdot \frac{d}{dx}(3x+2) = 2.5(3x+2)^{1.5} \cdot 3 \\ &= 7.5(3x+2)^{1.5} \end{aligned}$$

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- The special case power-chain rule can be written

$$\frac{d}{dx}[g(x)]^p = p[g(x)]^{p-1} \cdot \frac{d}{dx} g(x)$$

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- If you have a function of the form $y = \cos^3 x^2$, you might try re-writing it before applying the chain rule.
- $y = f(\cos(h(x)))$ where $f(u) = u^3$ and $h(x) = x^2$.

Examples

1. Calculate y' where $y = \cos(\tan x)$.
2. If $f(t) = \sqrt{4t+1}$, find $f''(2)$.
3. Find an equation for the tangent line and normal line to $y = 4\sin^2 x$ at the point $(\pi/6, 1)$ on the curve.
4. Differentiate the trig identities
    4.1 $\cos 2x = \cos^2 x - \sin^2 x$
    4.2 $\sin(x+a) = \sin x \cos a + \cos x \sin a$

    to obtain new trig identities.

Exercises

Now you should work on Problem Set 2.5. After you have finished it, you should try the following additional exercises from Section 2.5:

2.5 C-level: 1–6, 7–46, 47–50, 51–54, 55–57, 59–60, 61–64, 67–68, 69–72
B-level: 58, 65–66, 73–74, 75–76, 77–80
A-level: 83–90