MATH 110 Lecture 2.1

Derivatives and Rates of Change

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Department of Indigenous Knowledge and Science
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Contents

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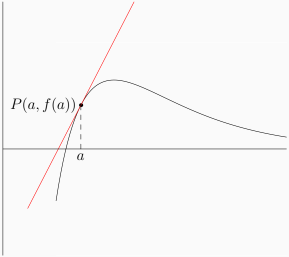

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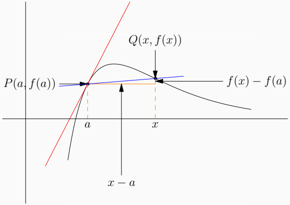

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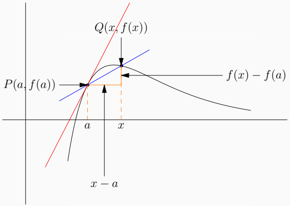

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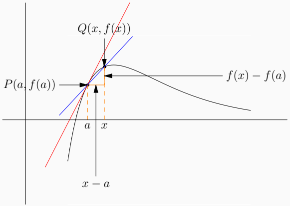

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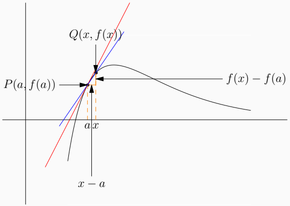

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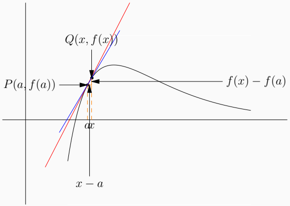

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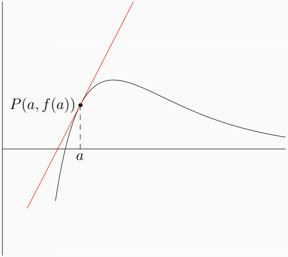
- We did that by finding secant lines through P and $Q(x, f(x))$ and letting x approach a.
- We found the slope of the tangent line as a limit. The tangent line could then be described in point-slope form $y - y_0 = m(x - x_0)$.



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If the above limit does not exist, the tangent line doesn't exist (the case where the limit is infinite is an exception that we'll study later).

Alternative Expressions for the Tangent

- In words, we can say
  The tangent line at $P(a, f(a))$ is the line through P with slope equal to the limit of the slopes of the secant lines through $P(a, f(a))$ and $Q(x, f(x))$ as x approaches a.

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- In summary, we have built up a system for completely solving the tangent problem!

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limit as $h \to 0$	$\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ slope of tangent	$\lim\limits_{h \to 0} \dfrac{s(a+h) - s(a)}{h}$ instantaneous velocity

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The instantaneous velocity is the limit of the average velocities as the length of the time interval $\Delta t = h = t - a$ on which the average velocities are taken approaches 0.

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Definition: The **derivative of a function** f **at a number** a is denoted by $f'(a)$ and given by the formula

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

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The tangent line to $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope $f'(a)$.

- We can write the equation of the tangent line in a single expression:

$$y - f(a) = f'(a)(x - a)$$

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- We also have the concept of average **speed** and **instantaneous speed** which are given by the absolute values of the velocities: $|\Delta s/\Delta t|$ and $|s'(a)|$ respectively.

Rates of Change

- In general, for any process represented by a function $y = f(x)$, we can talk about the average rate of change and the instantaneous rate of change,

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- Remember that both those concepts depend on our base number a, although a doesn't appear in the above formulas.
- The concept of instantaneous rate of change is more useful than average rate of change; the former just depends on a while the latter depends on a and Δx.

Examples

1. Let $f(x) = x^3 + 5x + 4$. Find $f'(2)$ from first principles (that is, from the definition of the derivative).
2. Let $g(x) = \dfrac{4-x}{3+x}$. Find the equation of the tangent line to the curve $y = g(x)$ at the point $P(1, g(1))$ on the curve.
3. If a ball is thrown upward starting at the ground with a velocity of 40 ft/s, its height (in feet) after t seconds is (approximately) given by $s(t) = 40t - 16t^2$.
   3.1 Find its (instantaneous) velocity after 1 second.
   3.2 Find its velocity at a seconds.
   3.3 When will the ball hit the ground?
   3.4 What will its velocity be at that time?

Solution to Example 3

1 Try this on your own; the answer is similar to that of the previous two questions. The answer will follow from part 2 with $a = 1$, so this part is actually redundant.

2 This is an example of calculating a derivative from first principles. We need to calculate the limit

$$v'(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}.$$

It is best to split the calculation into parts.

$$\begin{aligned} s(a) &= 40a - 16a^2 \\ s(a+h) &= 40(a+h) - 16(a+h)^2 = 40a + 40h - 16(a^2 + 2ah + h^2) \\ &= 40a + 40h - 16a^2 - 32ah - 16h^2 \end{aligned}$$

Solution to Example 3, continued

2 Subtracting the previous two gives

$$\begin{aligned} s(a+h)-s(a) &= 40a+40h-16a^2-32ah-16h^2-(40a-16a^2) \\ &= 40a+40h-16a^2-32ah-16h^2-40a+16a^2 \\ &= 40h-32ah-16h^2 \end{aligned}$$

Note that all the terms without an h have cancelled from the above expression, which is a useful check. Now we can take the limit:

$$\begin{aligned} s'(a) &= \lim_{h\to 0}\frac{s(a+h)-s(a)}{h} = \lim_{h\to 0}\frac{40h-32ah-16h^2}{h} \\ &= \lim_{h\to 0}(40-32a-16h) = 40-32a \end{aligned}$$

Solution to Example 3, continued

2 In summary, the instantaneous velocity at a seconds is $40 - 32a$ ft/s. The answer to question 1 is $s'(1) = 40 - 32(1) = 8$ ft/s.

3 The ball is at ground level when $s = 0$, so we must have $0 = s(t) = 40t - 16t^2$. Solving for t we get $t = 0$ (rejected in this case; why?) and $40 - 16t = 0$ which implies $t = 2.5$.

4 The instantaneous velocity at 2.5 seconds is $s'(2.5) = 40 - 32(2.5) = -40$ ft/s, which agrees with what we would expect from conservation of energy or some other physical reasoning: it hits the ground with the same speed with which it left the ground, but in the opposite direction.

Exercises

Now you should work on Problem Set 2.1. After you have finished it, you should try the following additional exercises from Section 2.1:

2.1 C-level: 1–4, 5–8, 9–10, 15–16, 17–18, 20–23, 27–30, 31–36, 37–42, 43–44, 47–50;
B-level: 11–12, 13–14, 19, 24–26, 45–46, 51–52, 53–56, 57–58, 61;
A-level: 59–60

You should skip example 7 of this section, and don't do example 6 until you understand everything else.