MATH 110 Lecture 3.4

Limits at Infinity; Horizontal Asymptotes

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Department of Indigenous Knowledge and Science
First Nations University of Canada

Contents

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- Knowledge of the long term behaviour (also called *end behaviour*) of a function completes the picture and provides a kind of a frame within which we can get a more accurate graph.

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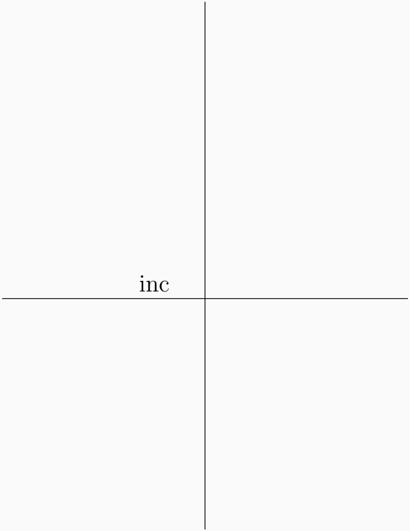

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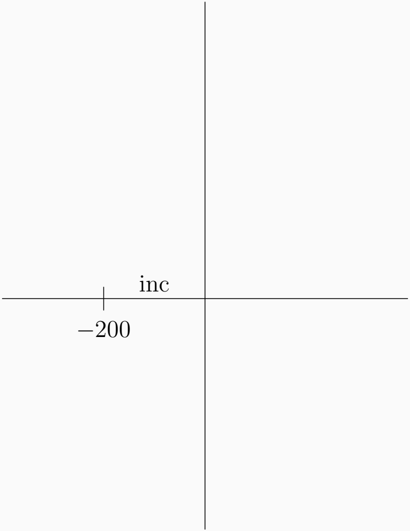

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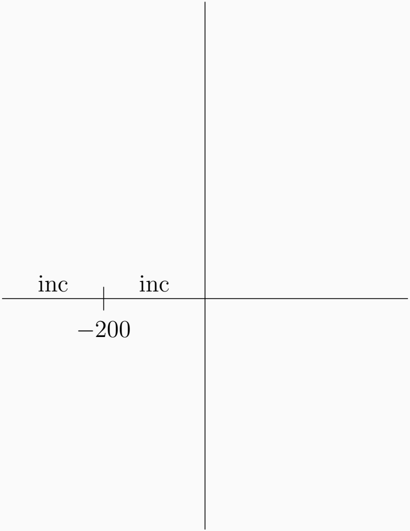

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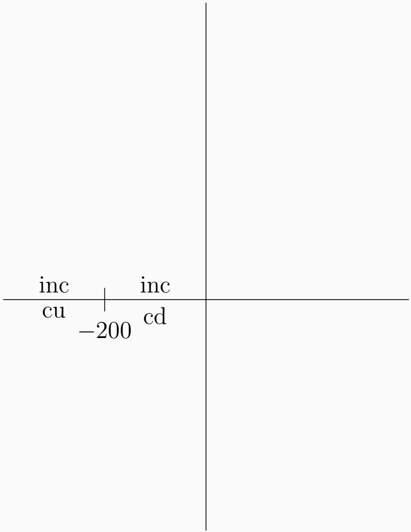

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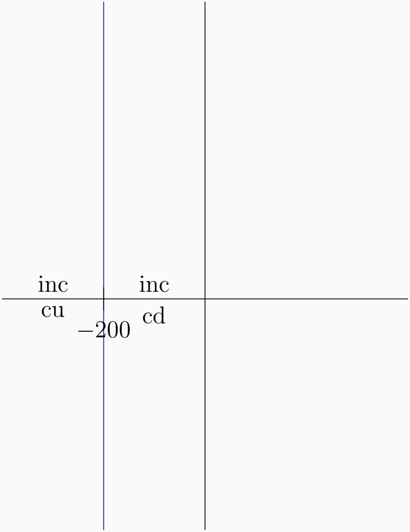

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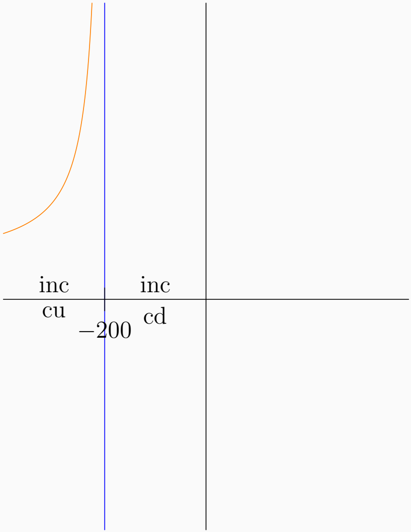

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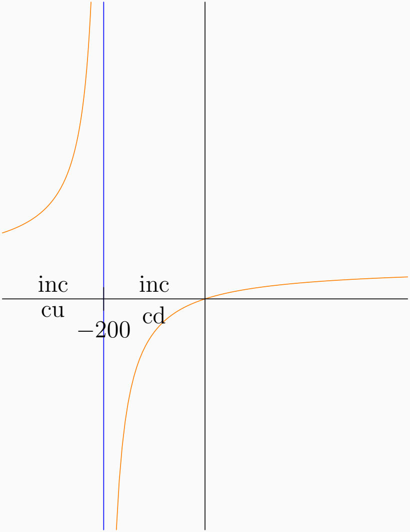

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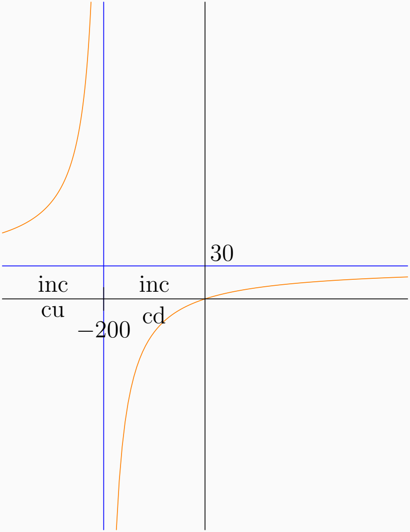
- If we could predict the horizontal asymptote $y = 30$, C would be easier to graph.



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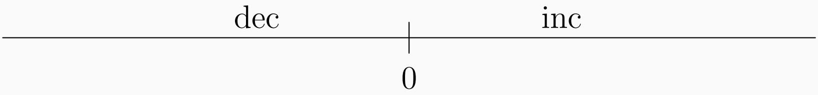

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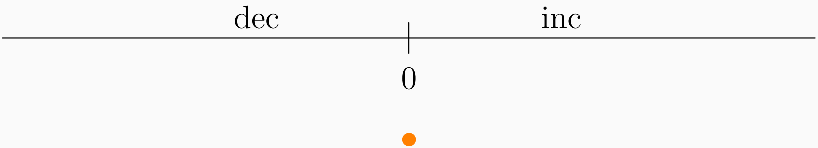

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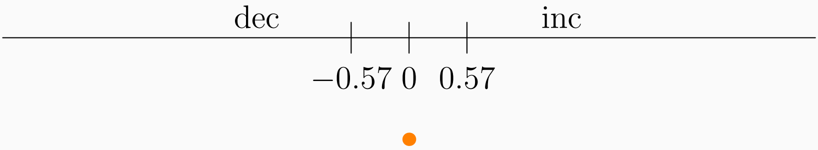

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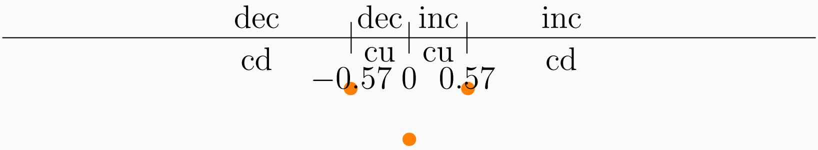

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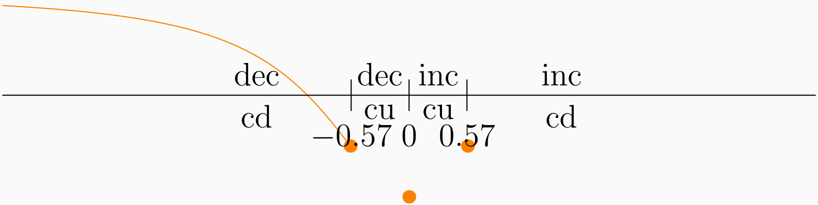

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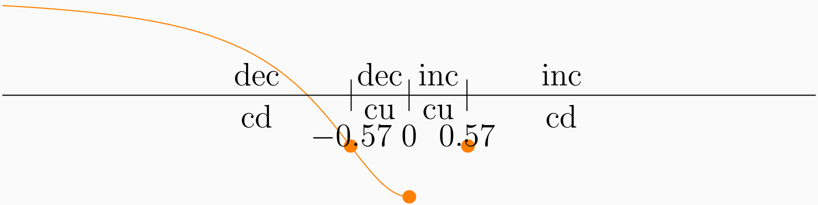

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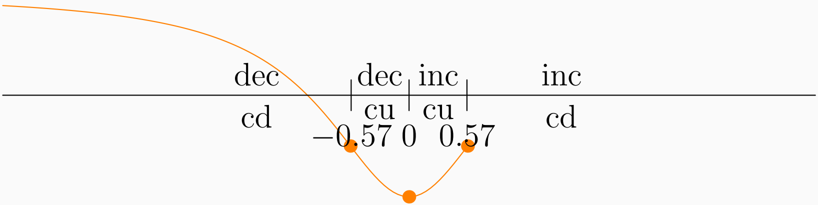

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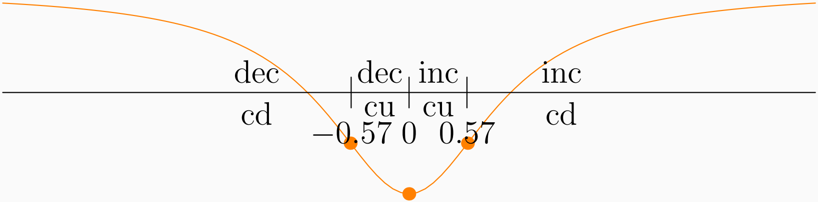

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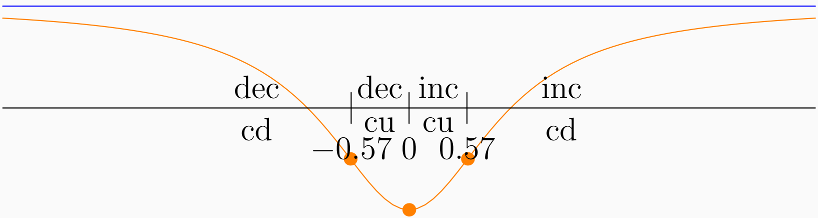
- Note the large-scale "frame" on the graph provided by $y=1$.



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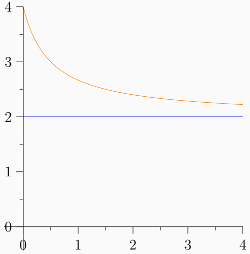

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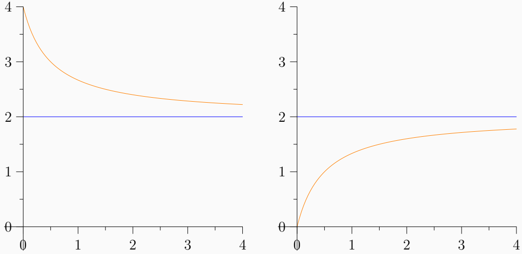

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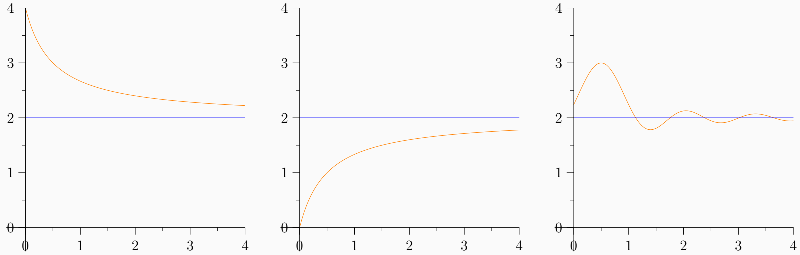

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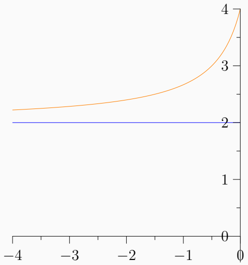

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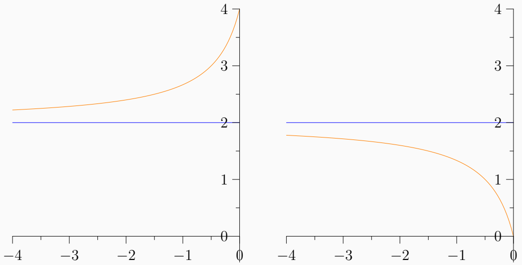

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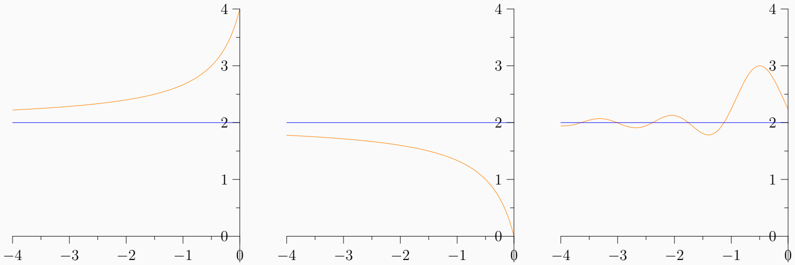

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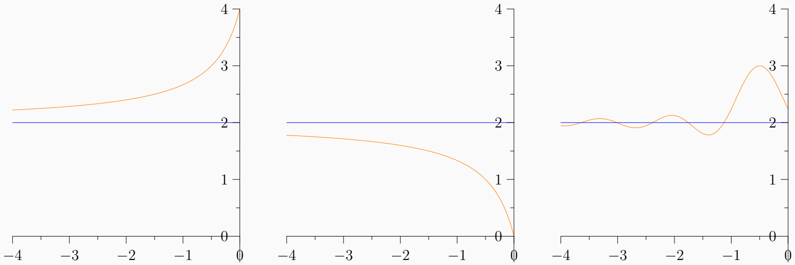

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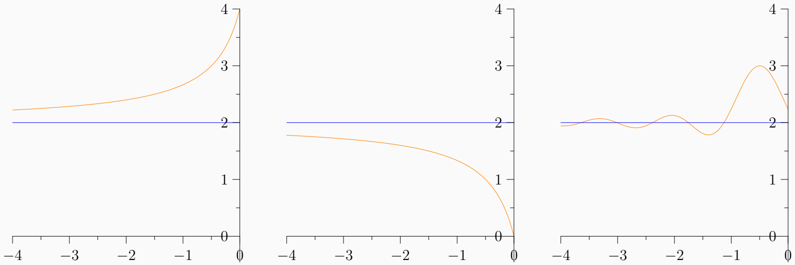

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- **Definition:** The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

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$$\lim_{x\to\infty}C(x)=\lim_{x\to\infty}\frac{30x}{200+x}=\lim_{x\to\infty}\frac{30x\cdot\frac{1}{x}}{(200+x)\cdot\frac{1}{x}}$$

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Limits at Infinity for Rational Functions

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$$= \frac{\sqrt{1+0}}{5+0+0} = \frac{1}{5}$$

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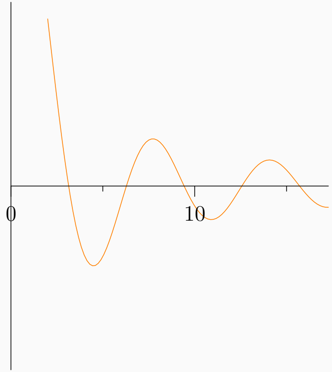

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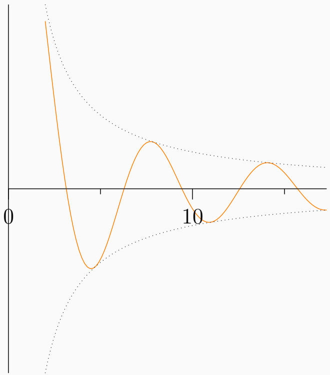

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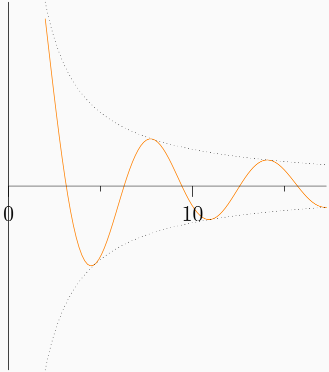

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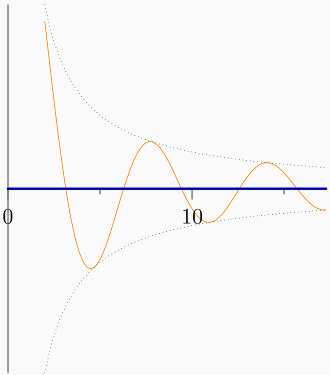
- Therefore by the Squeeze Theorem, $\lim\limits_{x \to \infty} \dfrac{\sin x}{x} = 0$.



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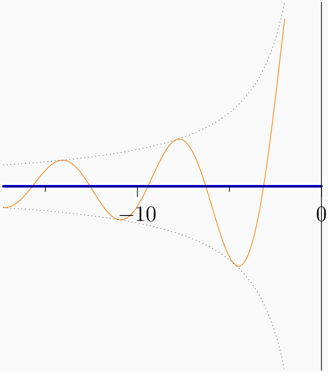
- Similarly, you can show $\lim\limits_{x\to-\infty} \dfrac{\sin x}{x} = 0$. (Try it!)



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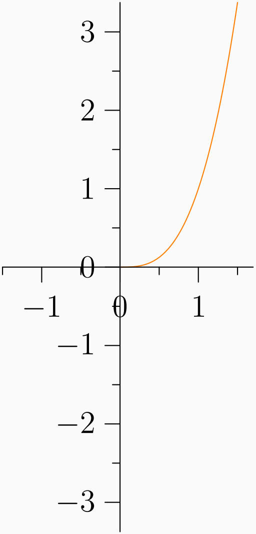

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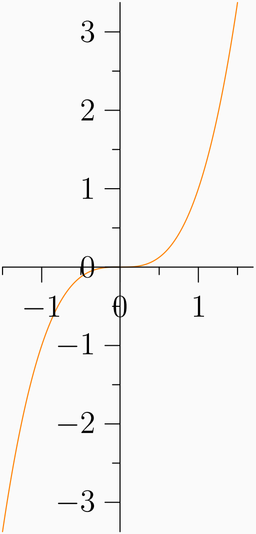

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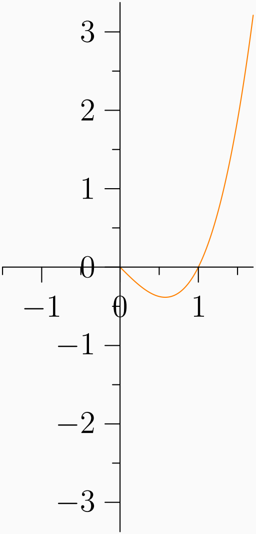
- Also $\lim_{x\to-\infty} x^3 = -\infty$.
- Also, $\lim_{x\to\infty} x^3 - x = \lim_{x\to\infty} x(x^2-1) = \infty$ because both multiplicands tend to ∞.



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- Consider $\lim_{x\to\infty} \frac{-3x^2+2}{5x-4}$.
- The highest power of x in the denominator is x^1. We have

$$\lim_{x\to\infty} \frac{-3x^2+2}{5x-4} = \lim_{x\to\infty} \frac{(-3x^2+2)\cdot\frac{1}{x}}{(5x-4)\cdot\frac{1}{x}} = \lim_{x\to\infty} \frac{-3x+\frac{2}{x}}{5-\frac{4}{x}}$$

- The numerator of the expression tends to $-\infty$, the denominator to the finite value 1, so the result is $-\infty$.

Examples and Exercises

Examples

1. Find the following limits at infinity.
   1.1 $\lim\limits_{x\to\infty} \dfrac{3x^4+x-5}{6x^4-2x^2+1}$
   1.2 $\lim\limits_{x\to-\infty} \dfrac{\sqrt{4x^2+1}}{3x-1}$
   1.3 $\lim\limits_{x\to-\infty} (x^2+x^3)$
   1.4 $\lim\limits_{x\to-\infty} \left(\sqrt{4x^2+3x}-2x\right)$
   1.5 $\lim\limits_{x\to\infty} \dfrac{\sin^4 x}{\sqrt{x}}$
2. Sketch a graph of $y = \dfrac{1-x}{1+x}$ illustrating intervals of increase/decrease, extrema, concavity, inflection points, and horizontal and vertical asymptotes.
3. The same for $y = \dfrac{\sqrt{2x^2+1}}{3x-5}$.

Now you should work on Problem Set 3.4. After you have finished it, you should try the following additional exercises from Section 3.4:

3.4 C-level: 1–40, 44–46, 48–51, 57–60;
B-level: 41–43, 47, 52–56, 61–64;
A-level: 65–74