MATH 110 Problem Set 3.7 Solutions

Edward Doolittle

Thursday, March 19, 2026

1. See Figure 1. The constraint is the volume of the box, which is $x^2h = 32000$. The objective is the amount of material in the box, which is four sides of size $x \times h$ and one base of size $x \times x$. So we must minimize $f(x, h) = 4xh + x^2$, subject to $0 < x, h$. It is easier to solve the constraint for h: $h = 32000/x^2$. Substituting the constraint into the objective, we must minimize

$$f(x) = 4x \cdot \frac{32000}{x^2} + x^2 = \frac{128000}{x} + x^2$$

on the domain $0 < x$. Taking the first derivative,

$$f'(x) = -\frac{128000}{x^2} + 2x = 2\frac{x^3 - 64000}{x^2} = 2\frac{(x-40)(x^2+40x+1600)}{x^2}$$

On the interval $0 < x < 40$, we have $f'(x) < 0$ so $f(x)$ is decreasing. On the interval $40 < x$, we have $f'(x) > 0$ so $f(x)$ is increasing. It follows that $x = 40$ is the global minimum of the function. When $x = 40$, $h = 32000/1600 = 20$, so the dimensions of the box which minimize the amount of material used are $x = 40$ cm for the square base and 20 cm for the height.

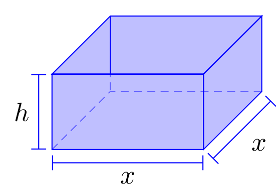

Figure 1: Box with Square Base and Open Top

2. See Figure 2. (It is convenient to make the width of the window $2x$ instead of x, so that you have to deal with fewer fractions.) The constraint on the perimeter of the window is $2h + 2x + \pi x = 30$. The objective function is $f(x, h) = 2xh + \pi x^2/2$ where $0 < x, h$. Solving the constraint for h, $h = 15 - x - \pi x/2$. Substituting into the objective function we have

$$f(x) = 2x(15 - x - \pi x/2) + \pi x^2/2 = 30x - 2x^2 - \pi x^2 + \pi x^2/2 = 30x - 2x^2 - \pi x^2/2$$

on the domain $0 \leq x$. The derivative is $f'(x) = 30 - 4x - \pi x$, so $f(x)$ is increasing on $0 \leq x < 30/(\pi+4)$ and decreasing on $30/(\pi+4) < x$. The function has an absolute maximum at $x = 30/(\pi+4)$. The dimensions of the window admitting the greatest possible amount of light are width of $2x = 60/(\pi+4)$ ft and height of

$$h = 15 - \frac{30}{\pi+4} - \frac{15\pi}{\pi+4} = \frac{15\pi + 60 - 30 - 15\pi}{\pi+4} = \frac{30}{\pi+4}$$

feet. For the optimal window, the width of the rectangular portion is twice its height.

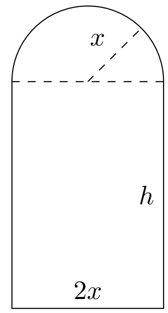

Figure 2: A Norman Window

3. See Figure 3. The length of the circular arc from A to B is $r\theta$ where r is the radius of the lake and θ is the angle in radians. The time spent on the circular arc is distance divided by speed or $r\theta/4$.

   The length of the chord from B to C is $2r\sin((\pi-\theta)/2) = 2r\cos(\theta/2)$. The time spent on the chord is distance divided by speed or $2r\cos(\theta/2)/2 = r\cos(\theta/2)$.

   Using $r = 2$, the total time of the path is

   $$f(\theta) = \theta/2 + 2\cos(\theta/2)$$

   on the domain $0 \le \theta \le \pi$. The derivative is

   $$f'(\theta) = \frac{1}{2} - 2\sin(\theta/2)\cdot\frac{1}{2} = \frac{1}{2} - \sin(\theta/2)$$

   The derivative is 0 when $\sin(\theta/2) = 1/2$ which occurs when $\theta/2 = \pi/3$, $\theta = 2\pi/3$. The derivative is positive on $0 < \theta < 2\pi/3$ and negative on $2\pi/3 < x < \pi$, so $\theta = 2\pi/3$ is a local maximum. The local minimum occurs at one or the other of the endpoints. $f(0) = 0/2 + 2\cos(0/2) = 2$ hours, and $f(\pi) = \pi/2 + 2\cos(\pi/2) = \pi/2 \approx 1.5708$ hours, so the absolute minimum is $\pi/2$ hours which occurs when the woman walks all the way around the lake and doesn't use the boat at all.

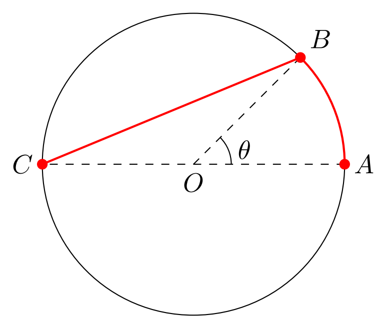

Figure 3: Path on a Circular Lake

4. Let x be the rent charged. We need to write a formula for the demand function, the number of units rented at price x. We are told that for every 10 increase in price, one fewer unit is rented, so the demand function is linear with slope $-1/10$. We also know that $(800, 100)$ is a point on the line (when rent is \$800, all 100 units can be rented). So the demand line is

   $$y - 100 = -\frac{1}{10}(x - 800) \implies y = p(x) = -\frac{1}{10}x + 180$$

The revenue is the rent charged times the number of units rented, or

$$R(x) = x \cdot p(x) = -\frac{1}{10}x^2 + 180x$$

The derivative of the revenue is

$$R'(x) = -\frac{1}{5}x + 180$$

which is positive for $0 \le x < 900$ and negative for $900 < x$, so $R(x)$ is maximized when $x = 900$.

5. See Figure 4. The height of the dashed triangle is $10 \sin \theta$, and the width of the dashed triangle is $10 \cos \theta$. The area of the two dashed triangles together is $100 \sin \theta \cos \theta$, and the area of the dashed rectangle is $100 \sin \theta$. The area of the rain gutter cross section is

$$A(\theta) = 100 \sin \theta (1 + \cos \theta)$$

on the domain $0 \le \theta \le \pi/2$. Its derivative is

$$A'(\theta) = 100 \cos \theta (1 + \cos \theta) + 100 \sin \theta (-\sin \theta) = 100(\cos \theta + \cos^2 \theta - \sin^2 \theta)$$

We want to solve the equation $A'(\theta) = 0$, but that is difficult in its current form. It will be easier if we eliminate $\sin^2 \theta$ using the Pythagorean identity $-\sin^2 \theta = \cos^2 \theta - 1$:

$$A'(\theta) = 100(2 \cos^2 \theta + \cos \theta - 1) = 100(2 \cos \theta - 1)(\cos \theta + 1)$$

where we have factored the trigonometric polynomial. The factor $\cos \theta + 1$ is always positive on the domain, so the sign of $A'(\theta)$ is the same as the sign of $2 \cos \theta - 1$. The latter is 0 when $\cos \theta = 1/2$ at $\theta = \pi/3$, is positive for $0 \le \theta < \pi/3$, and is negative for $\pi/3 < \theta \le \pi/2$. The function has a global maximum at $\theta = \pi/3$ radians (60 degrees).

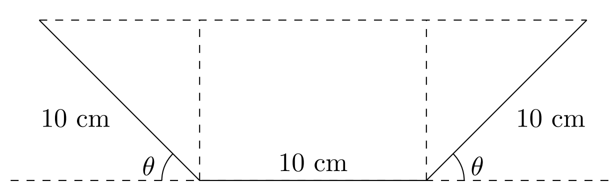

Figure 4: Cross Section of a Rain Gutter

6. See Figure 5. Let the coordinates of A be (x_1, y_1), the coordinates of B be (x_2, y_2), and the coordinates of C be $(x, 0)$. The distance from A to C is $\sqrt{(x - x_1)^2 + y_1^2}$, and the time for the ray of light to travel that segment is $\sqrt{(x - x_1)^2 + y_1^2}/v_1$. Similarly, the time from C to B is $\sqrt{(x - x_2)^2 + y_2^2}/v_2$. The total time of the path is

$$T(x) = \frac{1}{v_1}((x - x_1)^2 + y_1^2)^{1/2} + \frac{1}{v_2}((x - x_2)^2 + y_2^2)^{1/2}$$

Note that the function is differentiable for all real x, and tends to $+\infty$ as x tends to $\pm\infty$. That means that $T(x)$ must have a global minimum at which its derivative is 0. The derivative is

$$T'(x) = \frac{1}{2v_1}((x - x_1)^2 + y_1^2)^{-1/2} \cdot 2(x - x_1) + \frac{1}{2v_2}((x - x_2)^2 + y_2^2)^{-1/2} \cdot 2(x - x_2)$$

which is defined for all real x. To search for critical numbers we set $T'(x) = 0$ to obtain

$$0 = \frac{1}{v_1} \cdot \frac{x - x_1}{\sqrt{(x - x_1)^2 + y_1^2}} - \frac{1}{v_2} \cdot \frac{x_2 - x}{\sqrt{(x - x_2)^2 + y_2^2}}$$

By basic trigonometry (opposite over hypoteneuse),

$$\frac{x - x_1}{\sqrt{(x - x_1)^2 + y_1^2}} = \sin\theta_1 \quad \text{and} \quad \frac{x_2 - x}{\sqrt{(x - x_2)^2 + y_2^2}} = \sin\theta_2$$

so at any critical point (in particular, at the global minimum) we have

$$0 = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

which can be rearranged to give Snell's law.

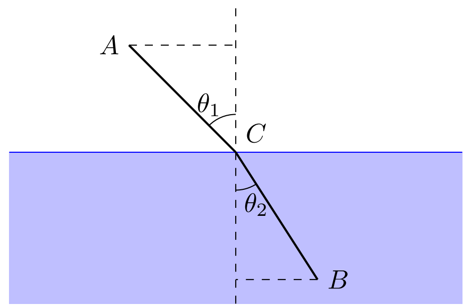

Figure 5: Ray of Light Passing from Air into Water