

# MATH 110 Problem Set 3.2 Solutions

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1. (a) The function is a polynomial so it is continuous and differentiable everywhere, in particular on the given interval; furthermore we have

$$\begin{aligned}f(0) &= (0)^3 - (0)^2 - 6(0) + 2 = 2 \\f(3) &= 3^3 - 3^2 - 6(3) + 2 = 27 - 9 - 18 + 2 = 2\end{aligned}$$

so  $f(0) = f(3)$ , and  $f$  satisfies all the hypotheses of Rolle's Theorem. So we know there is a number  $c$  in the interval  $(0, 3)$  at which  $f'(c) = 0$ . We can find  $c$  explicitly in this case by solving  $f'(x) = 3x^2 - 2x - 6 = 0$  or

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)} = \frac{2 \pm \sqrt{76}}{6} \approx 1.7863, -1.1196$$

Only the positive root above is in the interval  $[0, 3]$ , so  $c \approx 1.7863$  is the only number satisfying the conclusion of Rolle's Theorem.

- (b) By our theorems on the continuity and differentiability of trig functions,  $f(x)$  is continuous and differentiable everywhere, so it certainly is in the given closed interval. Furthermore,

$$\begin{aligned}\cos(2(\pi/8)) &= \cos(\pi/4) = \frac{\sqrt{2}}{2} \\ \cos(2(7\pi/8)) &= \cos(7\pi/4) = \cos(2\pi - (\pi/4)) = \cos(\pi/4) = \frac{\sqrt{2}}{2}\end{aligned}$$

so  $f(\pi/8) = f(7\pi/8)$ . Therefore  $f$  satisfies the hypotheses of Rolle's Theorem.

Rolle's Theorem guarantees the existence of a number  $c \in [\pi/8, 7\pi/8]$  such that  $f'(c) = 0$ . In this special case we can find  $c$  exactly. Taking the derivative we have

$$f'(x) = -2 \sin 2x \implies -2 \sin 2c = 0 \implies \sin 2c = 0$$

The roots of the sine function are at  $k\pi$ ,  $k = 0, \pm 1, \pm 2, \dots$ . That means that  $c$  is among the set of numbers  $k\pi/2$ ,  $k = 0, \pm 1, \pm 2, \dots$ . We need to figure out which of those values of  $c$  is in the interval  $[\pi/8, 7\pi/8]$ . Note that if  $k \leq 0$ , then  $k\pi/2 \leq 0 < \pi/8$ , i.e.  $k\pi/2$  is outside (to the left of) the given interval. Furthermore if  $k > 2$  then  $k\pi/2 > \pi > 7\pi/8$  so  $k\pi/2$  is again outside (to the right of) the given interval. Therefore  $k$  must be 1, and in fact  $c = \pi/2$  is inside the interval and satisfies  $f'(c) = 0$ .

2. (a) The function  $f(x) = x^3 - x^2 - 6x + 2$  is a polynomial so is continuous and differentiable everywhere, and the interval  $[0, 3]$  is a closed interval, so the situation satisfies the hypotheses of the MVT. The slope of the secant line connecting the points  $(0, f(0))$  and  $(3, f(3))$  is

$$m = \frac{f(3) - f(0)}{3 - 0} = \frac{27 - 9 - 18 + 2 - 2}{3} = 0$$

So to satisfy the conclusion of the MVT, we must find a number  $c$  such that  $f'(c) = 0$ . The derivative is

$$f'(x) = 3x^2 - 2x - 6$$

so  $f'(c) = 0$  means

$$3c^2 - 2c - 6 = 0 \implies c = \frac{2 \pm \sqrt{4 - 4(3)(-6)}}{2(3)} = \frac{1 \pm \sqrt{19}}{3} \approx 1.7863 \text{ or } -1.1196$$

The value  $c = -1.1196$  lies outside the interval  $[0, 3]$  so we throw it away. Therefore there is only one number  $c$  satisfying the conclusion of the MVT, namely  $c = (1 + \sqrt{19})/3 \approx 1.7863$ . See Figure 1. Note that in Figure 1(a), I have graphed the tangent line at  $(c, f(c))$  so you can see that it is parallel to the secant line. I have graphed the tangent line to show

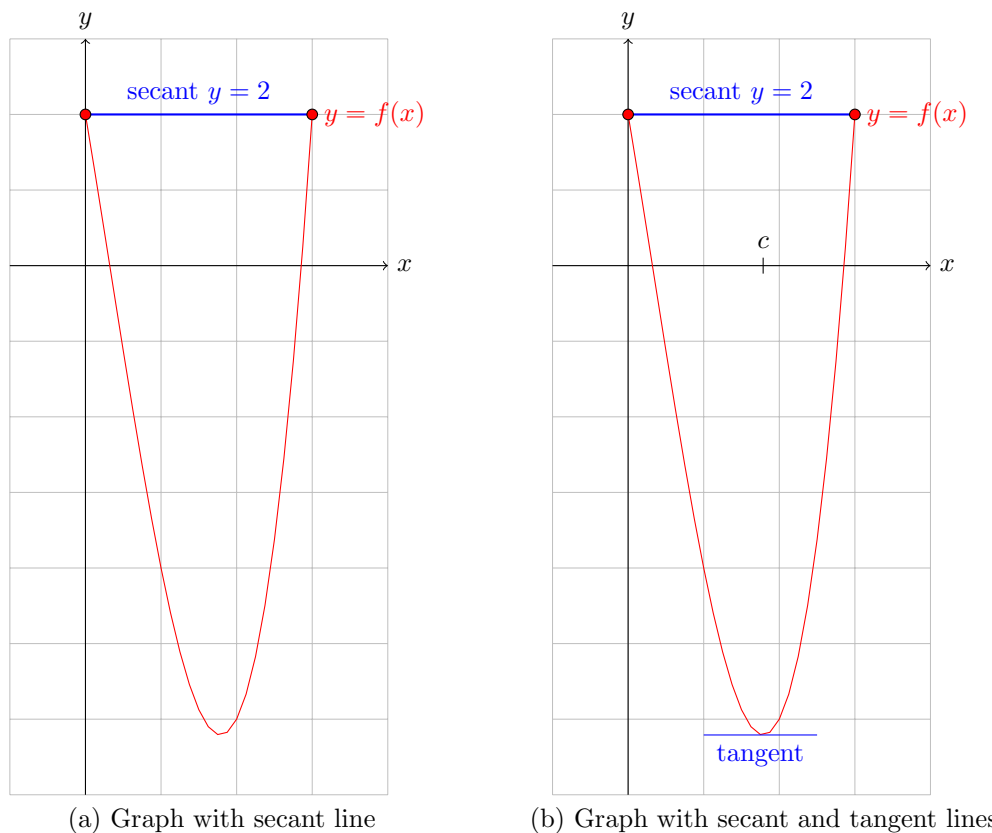


Figure 1: Graph of  $f(x) = x^3 - x^2 - 6x + 2$  for problem 2a

- (b) The function  $f(x) = x/(x+2)$  is continuous and differentiable everywhere except at  $x = -2$  which is outside the given interval  $[1, 4]$ , so  $f$  is continuous and differentiable on that interval, so satisfies the conditions of the MVT. The slope of the secant line connecting the endpoints  $(1, f(1))$  and  $(4, f(4))$  of the graph is

$$m = \frac{f(4) - f(1)}{4 - 1} = \frac{\frac{4}{4+2} - \frac{1}{1+2}}{3} = \frac{\frac{2}{3} - \frac{1}{3}}{3} = \frac{1}{9}$$

To satisfy the conclusion of Rolle's Theorem we need to find a number  $c \in [1, 4]$  such that

$f'(c) = 1/9$ . Calculating the derivative,

$$f'(x) = \frac{(x+2) - x}{(x+2)^2} = \frac{2}{(x+2)^2}$$

so we have to find  $c$  satisfying

$$f'(c) = \frac{1}{9} \implies \frac{2}{(c+2)^2} = \frac{1}{9} \implies (c+2)^2 = 18 \implies c = -2 \pm 3\sqrt{2} \approx 2.2426 \text{ or } -6.2426$$

Since the negative solution is outside the interval  $[1, 4]$ , we throw it away, and obtain just one value  $c \approx 2.2426$  which satisfies the conclusion of the mean value theorem. See Figure 2.

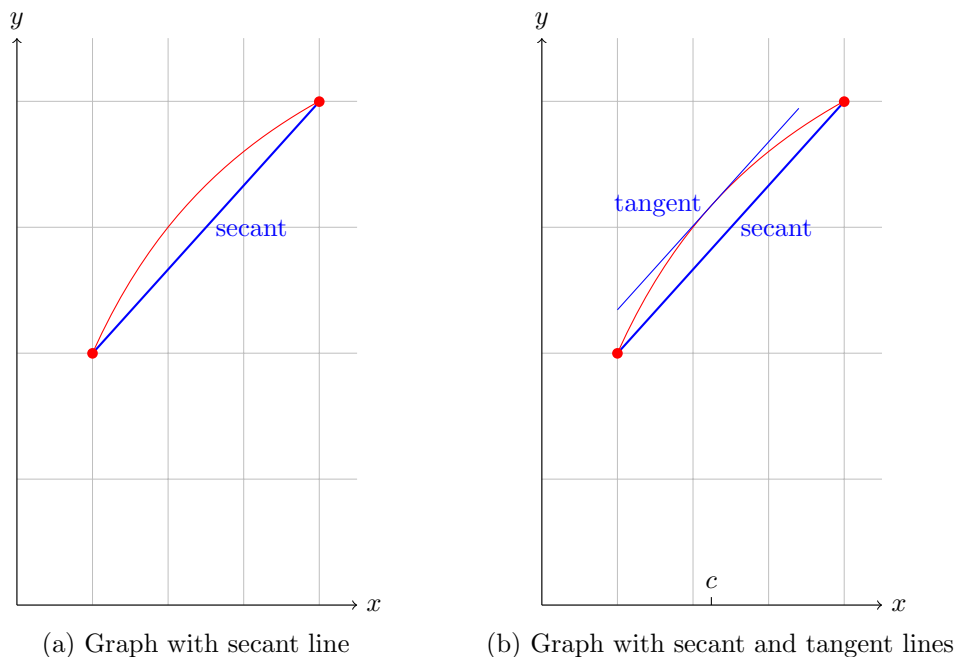


Figure 2: Graph of  $f(x) = x/(x+2)$  for problem 2b

3. Note that your calculator may have trouble with  $(-1)^{2/3}$ . We have

$$f(-1) = 1 - (-1)^{2/3} = 1 - ((-1)^2)^{1/3} = 1 - (1)^{1/3} = 1 - 1 = 0$$

$$f(1) = 1 - (1)^{2/3} = 1 - ((1)^2)^{1/3} = 1 - (1)^{1/3} = 1 - 1 = 0$$

so  $f(-1) = f(0)$ . Taking the derivative, we have

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

Note that

$$f'(c) = 0 \implies \frac{2}{3\sqrt[3]{c}} = 0$$

Multiplying through by  $3\sqrt[3]{c}$ ,

$$\frac{2}{3\sqrt[3]{c}} = 0 \implies 2 = 0$$

which is impossible, leading to the conclusion that the equation  $f'(c) = 0$  has no solution. On the other hand, Rolle's Theorem would suggest that  $f(-1) = f(1)$  implies  $f'(c) = 0$  for some  $c$  in  $[-1, 1]$ , so there is an apparent paradox here.

The solution is that  $f(x)$  does not satisfy all the hypotheses of Rolle's theorem. The function is continuous everywhere, but its derivative  $f'(x) = 2/(3\sqrt[3]{x})$  does not exist at  $x = 0$  (because we would end up dividing by 0). That single point in  $[-1, 1]$  where  $f'(x)$  does not exist is enough to void the application of Rolle's Theorem. This *counterexample* shows that the hypothesis that  $f'(x)$  is differentiable at every point in the interval  $(a, b)$  is required for Rolle's Theorem. See Figure 3. Note

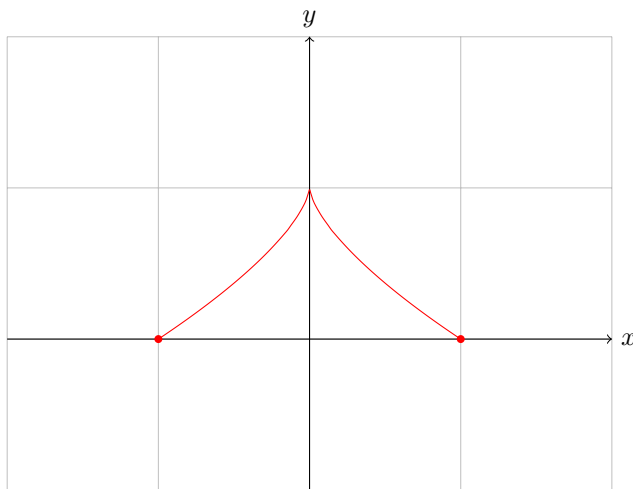


Figure 3: Graph of  $f(x) = 1 - x^{2/3}$  for problem 3

that  $f(x)$  has a cusp at  $x = 0$  because the one-sided limits of  $f'(x)$  are  $+\infty$  on the left and  $-\infty$  on the right.

4. We have

$$\begin{aligned} f(4) &= (4 - 3)^{-2} = 1^{-2} = 1 \\ f(1) &= (1 - 3)^{-2} = (-2)^{-2} = \frac{1}{(-2)^2} = \frac{1}{4} \\ f(4) - f(1) &= \frac{3}{4} \end{aligned}$$

Calculating the derivative,

$$f'(x) = -2(x - 3)^{-3} \frac{d}{dx}(x - 3) = -2(x - 3)^{-3}$$

So the MVT would seem to predict that there is a number  $c \in (1, 4)$  such that

$$f(4) - f(1) = f'(c)(4 - 1) \implies \frac{3}{4} = -2(c - 3)^{-3}(3) \implies (c - 3)^{-3} = -\frac{1}{8}$$

Taking both sides to the power  $-1/3$ ,

$$\implies c - 3 = \left(-\frac{1}{8}\right)^{-1/3} = -2 \implies c = 3 - 2 = 1$$

However, that number  $c = 1$  is *not* in the open interval  $(1, 4)$  because an open interval excludes its end points. This would seem to violate the MVT, which predicts a number  $c$  in the *open* interval  $(1, 4)$  satisfying  $f(4) - f(1) = f'(c)(4 - 1)$ .

Again, the solution is that the hypotheses of the MVT are not satisfied. In this case, two hypotheses are not satisfied:  $f(x)$  is not continuous on the interval  $[1, 4]$  and  $f(x)$  is not differentiable on the interval  $(1, 4)$ . Let's see why. We rewrite

$$f(x) = (x - 3)^{-2} = \frac{1}{(x - 3)^2}$$

and in that fraction form it is clear that  $f(3)$  is undefined at  $3 \in [1, 4]$  because there is a division by 0. Similarly,

$$f'(x) = -2(x - 3)^{-3} = -\frac{2}{(x - 3)^3}$$

is again undefined at  $3 \in (1, 4)$  because there is division by 0. The moral of this story is that it may be better to rewrite expressions involving negative exponents in terms of positive exponents and fractions if we are checking for continuity or differentiability. See Figure

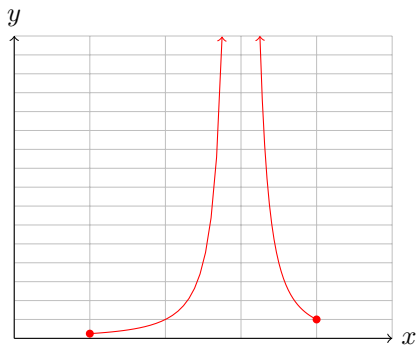


Figure 4: Graph of  $f(x) = (x - 3)^{-2}$  for problem 4

5. Let  $f(x) = 2x - 1 - \sin x$ . Note that  $f(x)$  is continuous everywhere. We first use the Intermediate Value Theorem to show that  $f(x)$  has at least one root. We need to find numbers  $a$  and  $b$  such that  $f(a) < 0$  and  $f(b) > 0$ . There are probably a lot of numbers in each category so we can try to pick ones that are convenient; picking numbers of the form  $k\pi$ ,  $k = 0, \pm 1, \pm 2, \dots$  will simplify our calculations because  $\sin(k\pi) = 0$ . We have

$$f(-\pi) = 2(-\pi) - 1 - \sin(-\pi) = -2\pi - 1 \approx -7.2832 < 0$$

$$f(\pi) = 2\pi - 1 - \sin(\pi) = 2\pi - 1 \approx 5.2832 > 0$$

so by the IVT,  $f(x)$  has at least one root (on the interval  $(-\pi, \pi)$ , in fact).

To show that  $f(x)$  has at most one root, we use the Mean Value Theorem (or Rolle's Theorem) and argue by contradiction. Note that  $f(x)$  is not only continuous, it is differentiable everywhere as well. Suppose that there are two distinct numbers  $a$  and  $b$  where  $f(a) = f(b)$ . Then by the MVT, there is some number  $c$  in the interval  $(a, b)$  such that  $f'(c) = 0$ . On the other hand,

$$f'(c) = 2 + \cos c \geq 2 - 1 > 0$$

because  $\cos c > -1$  for all  $c$ . We have both  $f'(c) = 0$  and  $f'(c) > 0$ , which is a contradiction, so our supposition that  $f$  has two distinct roots  $a$  and  $b$  must be incorrect.

We will learn a more straightforward way of solving problems like this when we learn about increasing and decreasing functions.

6. This problem is actually surprisingly straightforward. Suppose  $b < a$ ; a simple modification to the argument below can handle the case  $b > a$ . (What if  $b = a$ ?) Note that  $\sin x$  is continuous and differentiable everywhere, the Mean Value Theorem applies on every interval. Applying it on the interval  $[b, a]$ , the MVT says that there is a number  $c \in (b, a)$  such that

$$f'(c) = \frac{\sin a - \sin b}{a - b}$$

However,  $f'(c) = \cos c$ . Since we know that  $-1 \leq \cos x \leq 1$  for any  $x$ , we know that

$$-1 \leq f'(c) \leq 1 \implies |f'(c)| \leq 1 \implies \left| \frac{\sin a - \sin b}{a - b} \right| \leq 1 \implies \frac{|\sin a - \sin b|}{|a - b|} \leq 1$$

by the rules for working with the absolute value function. Multiplying the inequality through by the positive number  $|a - b|$  gives the required result.