MATH 110 Lecture 2.9

Linear Approximations and Differentials

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Not part of the course

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Contents

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- The function on the right side of the above expressions is called the **linearization** of f at a and is sometimes denoted $L(x)$.

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- So we can say $\sqrt{25 - x^2} \approx 4 - \frac{3}{4}(x - 3)$.

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- For now, we will just make some educated guesses.

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- For example, gravitational force on a pendulum is $-mg\sin x$, but for small x the approximation $-mgx$ is much easier to work with.
- In optics, $\sin x \approx x$ and $\cos x \approx 1$ are used to make lens grinding much easier. Those approximations lead to lenses with spherical surfaces, which are very easy to manufacture and work reasonably well near the center of the field of vision.

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- Instead of "differences" Δx and Δy we talk of "differentials" dx and dy.

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- Then we have $A = \pi r^2 = 9\pi$ with an error of $dA = 2\pi r\,dr = 0.6\pi$ or so.

Examples and Exercises

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1. Find the linear approximation of $f(x) = \sqrt{1-x}$ near $a = 0$ and use it to approximate $\sqrt{0.9}$ and $\sqrt{1.01}$.
2. For the function $y = 2x - x^2$, compute Δy and dy for $x = 2$ and $dx = \Delta x = -0.4$. Sketch a diagram illustrating dx, dy, and Δy.
3. The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm
   3.1 Use differentials to estimate the maximum error in the calculated area of the disk.
   3.2 What is the relative error? What is the percentage error?
4. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

Exercises

Now you should work on Problem Set 2.9. After you have finished it, you should try the following additional exercises from Section 2.9:

2.9 C-level: 1–30;
B-level: 31–34;
A-level: 35–42