MATH 110 Midterm 1B Solutions

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1. (a) Write $u(t) = t^{1/2} - t^{-1/2}$. Then $u'(t) = \frac{1}{2}t^{-1/2} - -\frac{1}{2}t^{-3/2} = \frac{1}{2}t^{-1/2} + \frac{1}{2}t^{-3/2}$ and $u'(1) = \frac{1}{2}1^{-1/2} + \frac{1}{2}1^{-3/2} = \frac{1}{2} + \frac{1}{2} = 1$.

   (b) Easiest is to write $g(t) = \frac{t}{t^{1/3}} - \frac{t^{1/2}}{t^{1/3}} = t^{1-1/3} - t^{1/2-1/3} = t^{2/3} - t^{1/6}$. Then $g'(t) = \frac{2}{3}t^{-1/3} - \frac{1}{6}t^{-5/6}$. Alternatively, you can use the quotient rule:

   $$g'(t) = \frac{t^{1/3}(t - t^{1/2})' - (t - t^{1/2})(t^{1/3})'}{(t^{1/3})^2} = \frac{t^{1/3}(1 - (1/2)t^{-1/2}) - (t - t^{1/2})((1/3)t^{-2/3})}{t^{2/3}}$$

   It is not necessary to simplify. However, if you want to check by comparing the latter method with the former, we can do a little algebra:

   $$g'(t) = \frac{t^{1/3} - (1/2)t^{-1/6} - (1/3)t^{1/3} + (1/3)t^{-1/6}}{t^{2/3}} = \frac{2}{3}t^{1/3}t^{-2/3} - \frac{1}{6}t^{-1/6}t^{-2/3} = \frac{2}{3}t^{-1/3} - \frac{1}{6}t^{-5/6}$$

   That is how you can use extra time on a test: checking by solving a problem in a different way and comparing.

   (c) By the quotient rule,

   $$\frac{d}{dx}\frac{x^2}{1+2x} = \frac{(1+2x)2x - x^2(2)}{(1+2x)^2}$$

   Normally I recommend doing as little algebra as possible, but since we have to differentiate a second time, we should simplify somewhat. Furthermore, normally one wants to keep the denominator in factored form, but since the chain rule was not on the material covered by the test, it is convenient to expand the denominator. So we have

   $$\frac{d}{dx}\frac{x^2}{1+2x} = \frac{2x + 2x^2}{1 + 4x + 4x^2}$$

   Differentiating again

   $$\frac{d^2}{dx^2}\frac{x^2}{1+2x} = \frac{d}{dx}\frac{2x+2x^2}{1+4x+4x^2} = \frac{(1+4x+4x^2)(2+4x) - (2x+2x^2)(4+8x)}{(1+4x+4x^2)^2}$$

   I recommend you leave it alone now. However, if you want to check, you might want to simplify and compare with the second derivative of $x^2(1+2x)^{-1}$ calculated by the product rule and the chain rule.

2. Checking $x = 7$ gives a result of $\frac{\sqrt{7^2+9} - 5}{7+4}$ which makes sense, so 7 is in the domain of the function. Since the function can be formed from simple continuous functions using compositions, products,

quotients, sums, and differences, it is continuous on its domain, therefore

$$\lim_{x\to 7}\frac{\sqrt{x^2+9}-5}{7+4}=\frac{\sqrt{7^2+9}-5}{7+4}$$

You can evaluate the function numerically, but that is not necessary.

3. We can break this function into steps as follows:

$$\begin{aligned}
f(a) &= \frac{2a+1}{a+3}\\
f(a+h) &= \frac{2(a+h)+1}{(a+h)+3}\\
f(a+h)-f(a) &= \frac{2a+2h+1}{a+h+3}-\frac{2a+1}{a+3}\\
&= \frac{(2a+2h+1)(a+3)-(a+h+3)(2a+1)}{(a+h+3)(a+3)}\\
&= \frac{2a^2+6a+2ah+6h+a+3-2a^2-a-2ah-h-6a-3}{(a+h+3)(a+3)}\\
&= \frac{5h}{(a+h+3)(a+3)}\\
f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\\
&= \lim_{h\to 0}\frac{5h}{(a+h+3)(a+3)}\cdot\frac{1}{h}\\
&= \lim_{h\to 0}\frac{5}{(a+h+3)(a+3)}\\
&= \frac{10}{(a+3)^2}
\end{aligned}$$

Avoid the urge to take bad shortcuts by cancelling incorrectly in the fourth line of the calculation.

You can check by differentiating using the quotient rule.

4. It is a good idea, but not necessary, to check that $y(4)=0.4$. Now to get an equation of the tangent line we need a point on the line, which we have (namely $(4, 0.4)$) and the slope of the line, which comes from the derivative. By the quotient rule,

$$y'=\frac{(x+1)(1/2)x^{-1/2}-x^{1/2}(1)}{(x+1)^2}$$

Algebraic simplifications are possible at this point but are counter-productive. Substituting $x=4$ gives

$$y'(4)=\frac{(4+1)(1/2)(1/2)-2(1)}{(4+1)^2}=\frac{5/4-8/4}{5^2}=-\frac{3}{100}$$

An equation of the tangent line is $y-0.4=-\dfrac{3}{100}(x-4)$. The normal line passes through the same point $(1,2)$ but has negative reciprocal slope $m=100/3$, so the equation of the normal line is $y-0.4=\frac{100}{3}(x-4)$.

Note that if you use the point-slope form of the equations of the lines instead of the slope-intercept form $y=mx+b$, you don't have to solve for b.

5. (a) The function is continuous on $(-\infty, 2)$ because it is given by the formula cx^2+2x on that domain; similarly it is continuous on $(2, \infty)$, so the only point in question is $x = 2$. The condition for continuity at that point is $\lim\limits_{x\to 2} f(x) = f(2)$. We have

$$\begin{aligned}
\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} cx^2 + 2x = 4c + 4 \\
\lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} x^3 - cx = 8 - 2c \\
f(2) &= 2^3 - c(2) = 8 - 2c
\end{aligned}$$

Note that we use the $x^3 - cx$ definition when we evaluate $f(2)$ because that definition applies on $2 \le x < \infty$, an interval which includes 2. For $\lim_{x\to 2} f(x)$ to exist, the above two limits must be equal, so $4c+4 = 8-2c$ which implies $6c = 4$ or $c = 2/3$, in which case the limit is $20/3$. Since we also have $f(2) = 20/3$ for that value of c, it follows that $f(x)$ is continuous if and only if $c = 2/3$.

(b) A function can be differentiable only if it is continuous, so the only possible value of c for $f(x)$ to be differentiable is $c = 2/3$. When $c = 2/3$ we have the left and right sided derivatives

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^-} \frac{(8/3)+(8/3)h+(2/3)h^2+4+2h-20/3}{h} \\
&= \lim_{h\to 0^-} \frac{(14/3)h+(2/3)h^2}{h} = \frac{14}{3} \\
\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^+} \frac{8+12h+6h^2+h^3-(4/3)-(2/3)h-20/3}{h} \\
&= \lim_{h\to 0^+} \frac{(32/3)h+6h^2+h^3}{h} = \frac{32}{3}
\end{aligned}$$

Since the left and right derivatives disagree at $x = 2$ we have a corner point there and the function is not differentiable for $c = 2/3$, so it is never differentiable.

6. Substituting $x = -2$ into the expression gives $\dfrac{15-a}{0}$. That gives an infinite limit (which doesn't exist, of course) for all values of a except possibly when $15 - a = 0$, so the only possible value of a which can give us a limit which exists is $a = 15$. Checking,

$$\lim_{x\to -2} \frac{3x^2+15x+18}{x^2+x-2} = \lim_{x\to -2} \frac{(3x+9)(x+2)}{(x+2)(x-1)} = \lim_{x\to -2} \frac{3x+9}{x-1} = \frac{3}{-3} = -1$$

In conclusion, the limit exists if and only if $a = 15$, in which case the limit is -1.