MATH 110 Review 0.D

Review of Trigonometry

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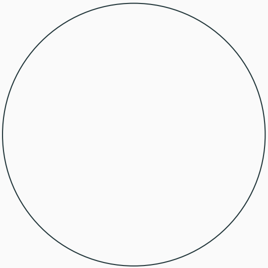
Department of Indigenous Knowledge and Science
First Nations University of Canada

Contents

Trigonometry

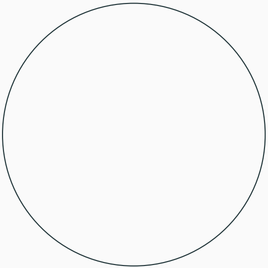
Radian Measure

- The system of degrees we often use to measure angles is based on an arbitrary choice of 360° for a whole circle.



Radian Measure

- The system of degrees we often use to measure angles is based on an arbitrary choice of 360° for a whole circle.
- It is better to use a more natural measure for angles, *radians*.



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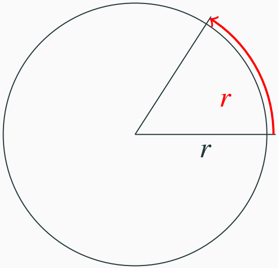
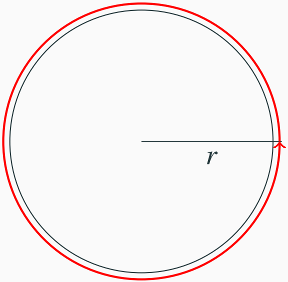
- The system of degrees we often use to measure angles is based on an arbitrary choice of 360° for a whole circle.
- It is better to use a more natural measure for angles, *radians*.
- One radian is the angle subtended by a length of one radius marked out around the circle.



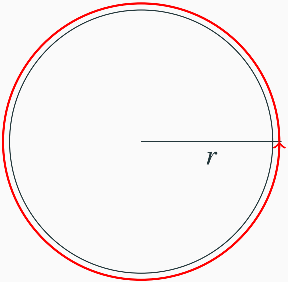
Radian Measure

- The system of degrees we often use to measure angles is based on an arbitrary choice of 360° for a whole circle.
- It is better to use a more natural measure for angles, *radians*.
- One radian is the angle subtended by a length of one radius marked out around the circle.
- The circumference of a circle is $2\pi r$, so a full circle is 2π radians.



Common Conversions

- A full circle is 360° which is 2π radians.



Common Conversions

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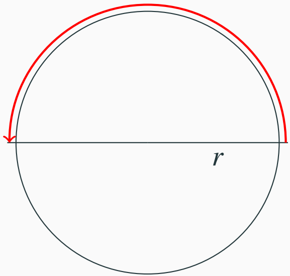

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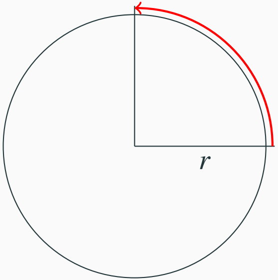

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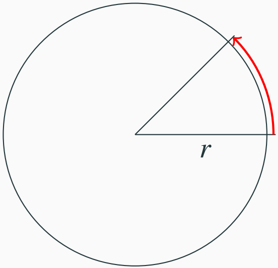

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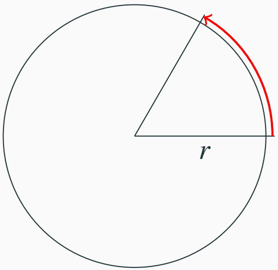
- A sixth of a circle is 60° which is $2\pi/6 = \pi/3$ radians.



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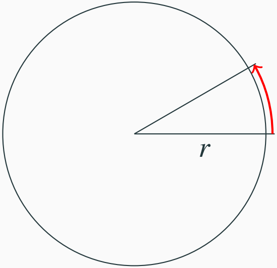
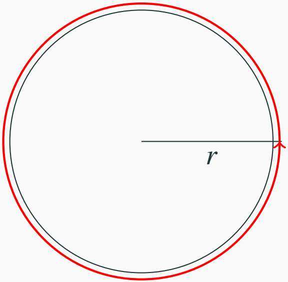
- A sixth of a circle is 60° which is $2\pi/6 = \pi/3$ radians.
- A twelfth of a circle is 30° which is $2\pi/12 = \pi/6$ radians.



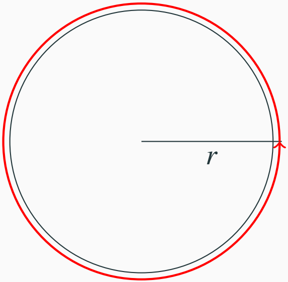
Length of a Circular Arc

- Similar proportional reasoning applies to the length of a circular arc.



Length of a Circular Arc

- Similar proportional reasoning applies to the length of a circular arc.
- The arc length of a whole circle is $2\pi r$, the angle in radians times the radius.



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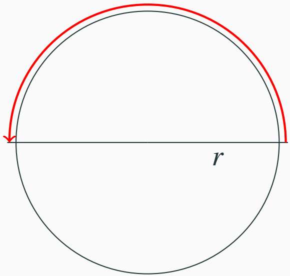

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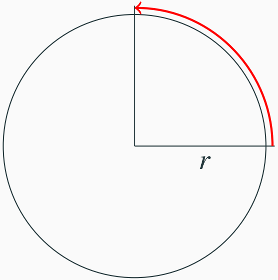

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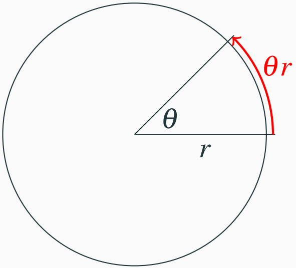
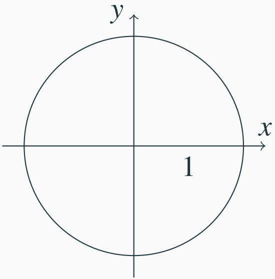
- The arc length of half a circle is πr, the angle in radians times the radius.
- The arc length of a quarter circle is $(\pi/2)r$, the angle in radians times the radius.
- In general, the arc length of any arc is the angle subtended by the arc in radians times the radius.



Angles in Standard Position

- We often work with a particular circle, the unit circle on the Cartesian plane.



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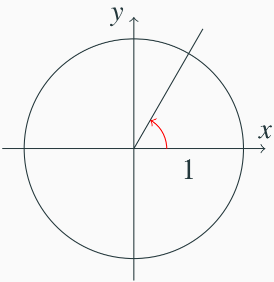
- We often work with a particular circle, the unit circle on the Cartesian plane.
- In that context, we measure angles from the x-axis counter-clockwise.



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- We often work with a particular circle, the unit circle on the Cartesian plane.
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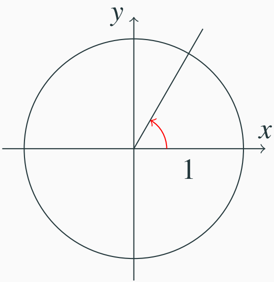
- Angles from 0 to $\pi/2$ radians are said to be in the first quadrant.



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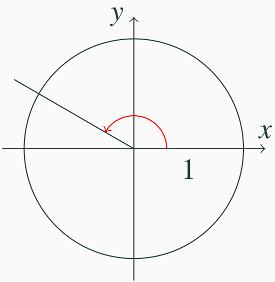
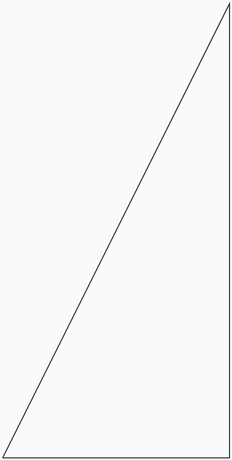
- Angles from $\pi/2$ to π radians are said to be in the second quadrant.



Right Triangles

- The easiest way to explore angles is to use right triangles.



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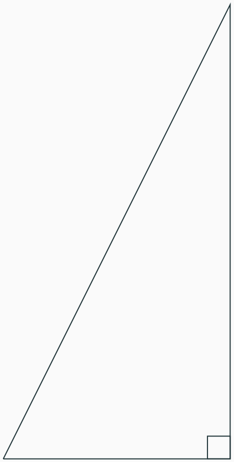
- We orient the base of the triangle horizontally, with the right angle (usually) on the right.



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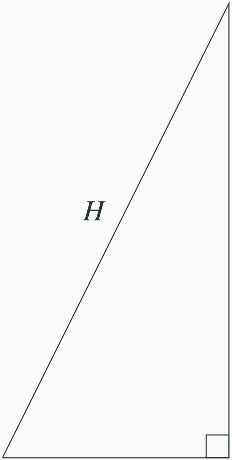
- The easiest way to explore angles is to use right triangles.
- We orient the base of the triangle horizontally, with the right angle (usually) on the right.
- The longest side is called the *hypoteneuse*. The other sides are called *legs*.



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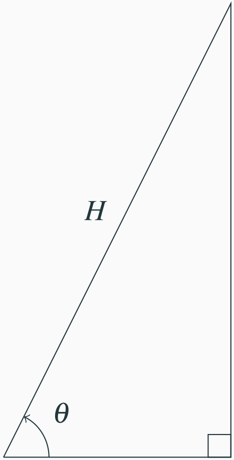

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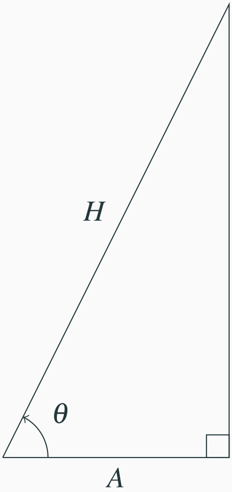
- The leg between the angle and the right angle is called *adjacent*.



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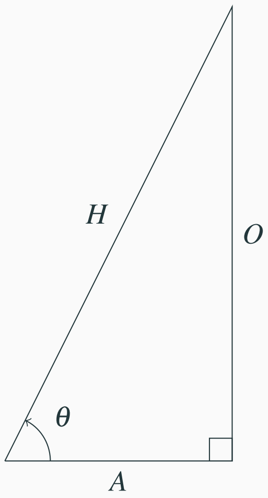
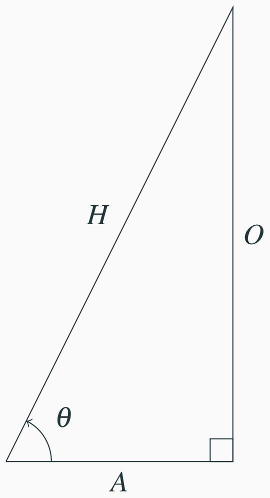
- We focus on the non-right angle on the base, we call that *the angle*.
- The leg between the angle and the right angle is called *adjacent*.
- The other leg is called *opposite*.



The Six Trigonometric Ratios

- By similar triangles, the ratio between any two sides of a right triangle depends only on the angle, not the size of the triangle.



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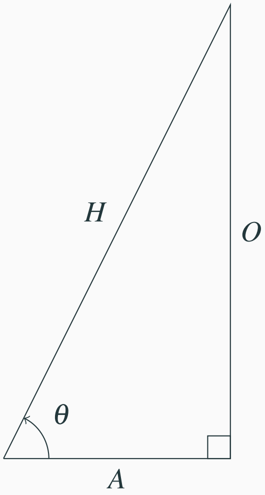

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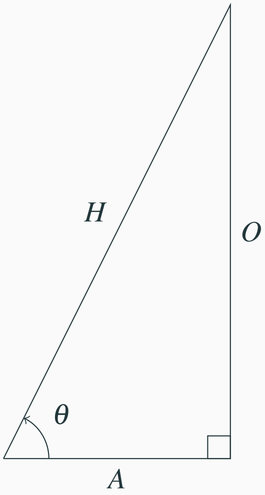

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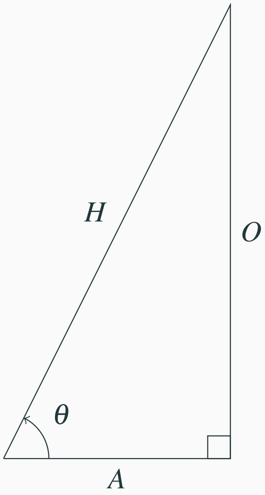

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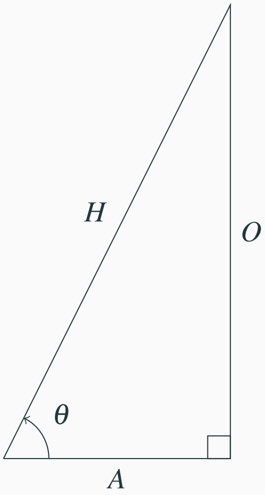
- Tangent (tan) is O/A



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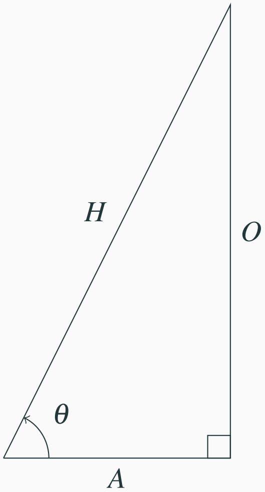

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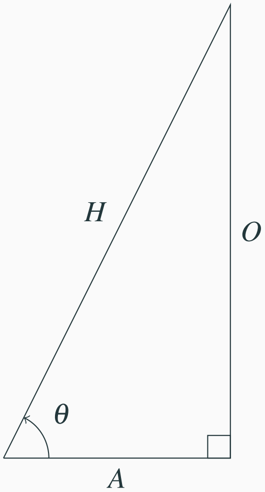
- secant (sec) is H/A



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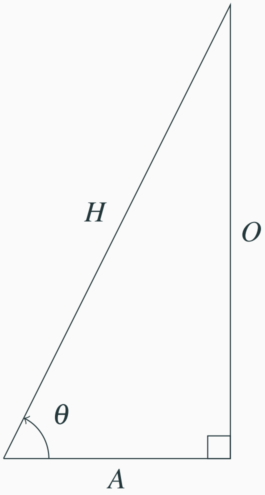
- cosecant (csc) is H/O



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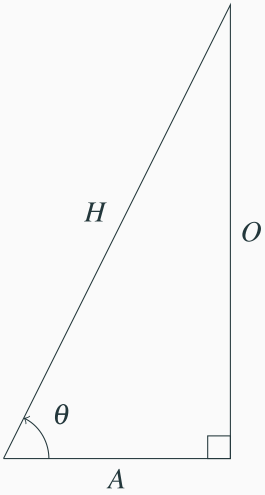
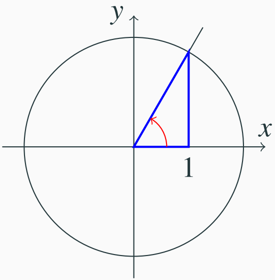
- cotangent (cot) is A/O



Trigonometry in Cartesian Coordinates

- To deal with angles outside of the range $[0, \pi/2]$ we will find it convenient to fit our right triangle inside the unit circle.



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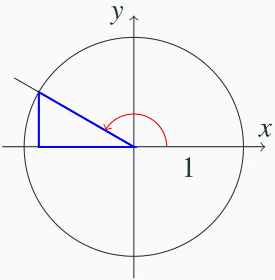
- To deal with angles outside of the range $[0, \pi/2]$ we will find it convenient to fit our right triangle inside the unit circle.
- If an angle is greater than $\pi/2$ but less than π, the adjacent side is negative but the opposite side is still positive.



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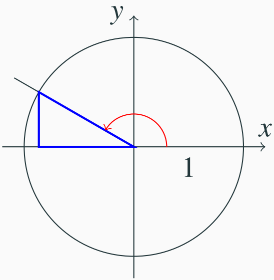
- In general, we have the translation adjacent $\to x$, opposite $\to y$, hypoteneuse $\to 1$.



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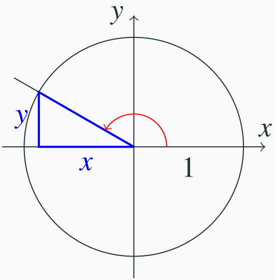
- It follows that in quadrant I, all ratios are positive; in quadrant II, sine is positive but cosine and tangent are negative, etc.



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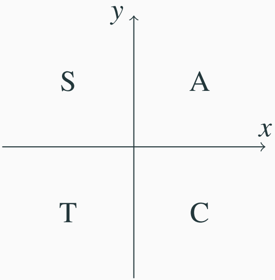
- All Students Take Calculus



Exact Values for $\theta = \pi/4$

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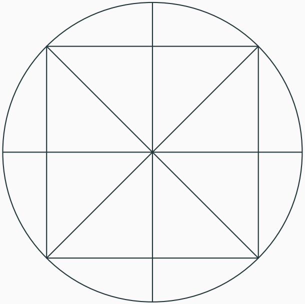

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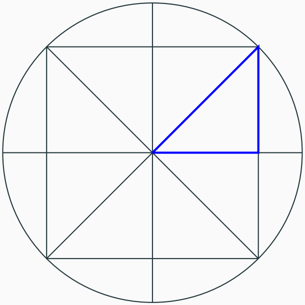
- There are 8 identical right triangles, so the angle is $2\pi/8 = \pi/4$ radians.



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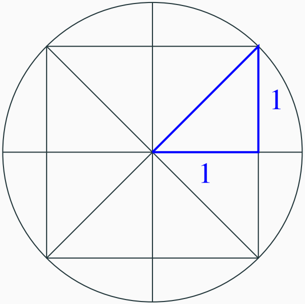
- There are 8 identical right triangles, so the angle is $2\pi/8 = \pi/4$ radians.
- By the little squares, the legs are equal. Assume they have side length 1 unit.



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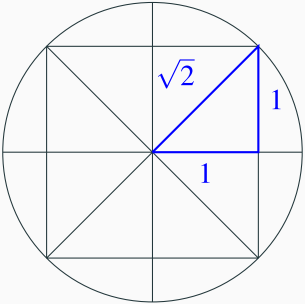
- By the Pythagorean Theorem, the hypoteneuse is $\sqrt{1^2 + 1^2} = \sqrt{2}$ units.



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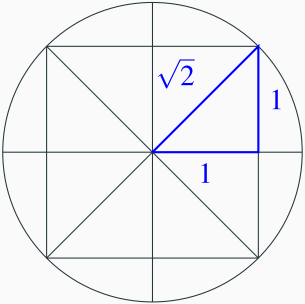
- By the little squares, the legs are equal. Assume they have side length 1 unit.
- By the Pythagorean Theorem, the hypoteneuse is $\sqrt{1^2+1^2} = \sqrt{2}$ units.
- Therefore $\sin\pi/4 = 1/\sqrt{2}$, $\cos\pi/4 = 1/\sqrt{2}$, $\tan\pi/4 = 1/1 = 1$.



Exact Values for $\theta = \pi/3$

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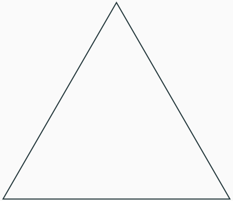

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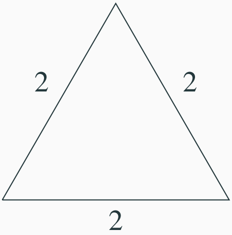
- Take an equilateral triangle of side 2.



Exact Values for $\theta = \pi/3$

- Using an equilateral triangle, we can get the ratios for $\pi/3$.
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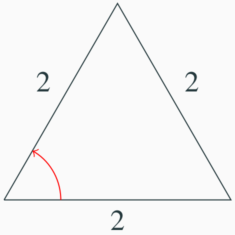

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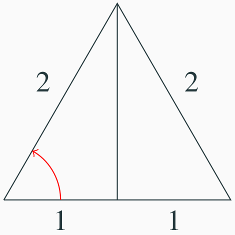

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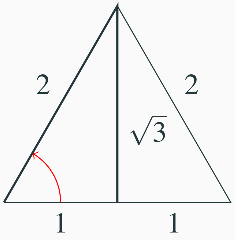

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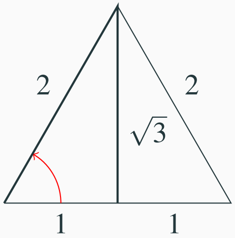

Exact Values for $\theta = \pi/6$

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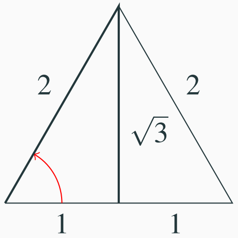

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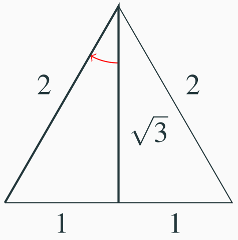

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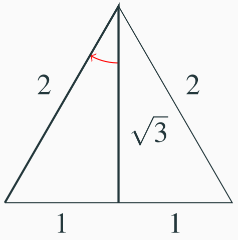

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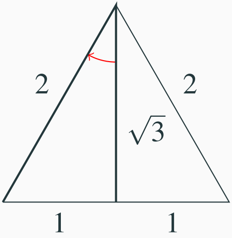

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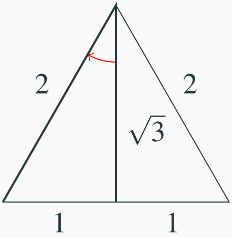

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- The ratios are $\sin \pi/6 = 1/2$, $\cos \pi/6 = \sqrt{3}/2$, $\tan \pi/6 = (1/2)/(\sqrt{3}2) = 1/\sqrt{3}$.
- For all three "special triangles" we can also find the other three ratios.
- For example, $\sec \pi/6 = H/A = 2/\sqrt{3}$.

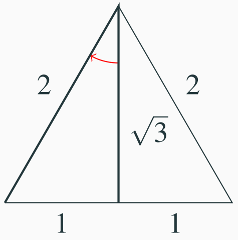


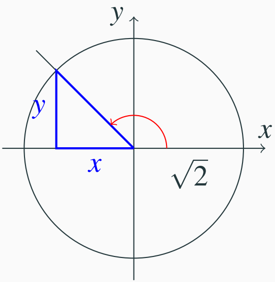
Table of Exact Values

angle	$\frac{0\pi}{12}$	$\frac{1\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\frac{6\pi}{12}$
aka	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{6}$	$\frac{\pi}{2}$
deg	0°	15°	30°	45°	60°	75°	90°
sin	0		$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$		1
cos	1		$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$		0
tan	0		$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$		∞
csc	∞		2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$		1
sec	1		$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2		∞
cot	∞		$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$		0

- Note the entries for csc, sec, and cot are reciprocals of the corresponding entries for sin, cos, tan, respectively.
- We will fill in the missing entries when we learn about trig identities.

Extending Exact Values to Other Quadrants

- Consider the angle $3\pi/4$ radians. It lies in quadrant II.



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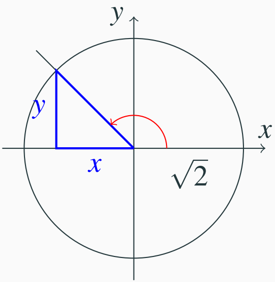

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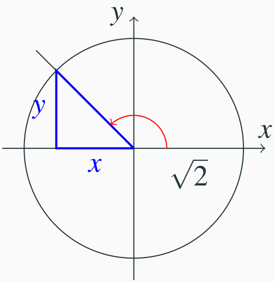

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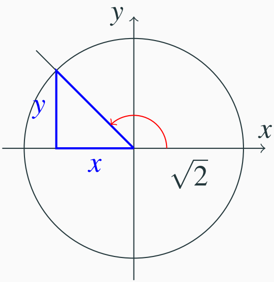

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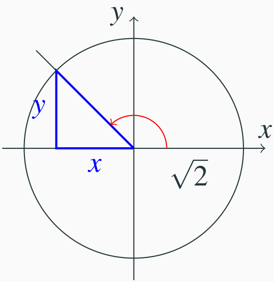
- It follows that $\sin 3\pi/4 = 1/\sqrt{2}$, $\cos 3\pi/4 = -1/\sqrt{2}$, $\tan 3\pi/4 = 1/-1 = -1$.
- We can obtain similar results for all the other special triangles in the other quadrants.



Graph of Sine

Graph of Cosine

Graph of Tangent

Graph of Secant

Graph of Cosecant

Graph of Cotangent

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- Identities give us a way to do algebra with the trig functions.

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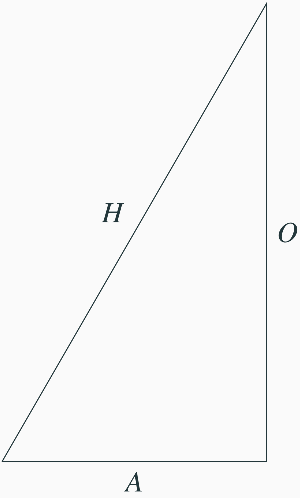
$$\frac{\sin\theta}{\cos\theta} = \frac{O/H}{A/H} = \frac{O}{A} = \tan\theta$$
- Since they hold for any angle θ, they are identities.

- Consider our typical right triangle.



The Pythagorean Identity

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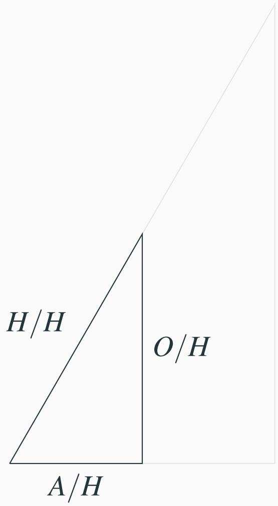
- Rescale by dividing through by H. That does not change the values of the ratios.



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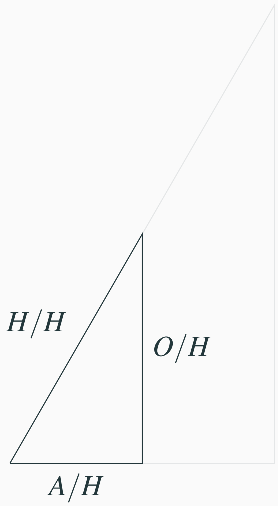

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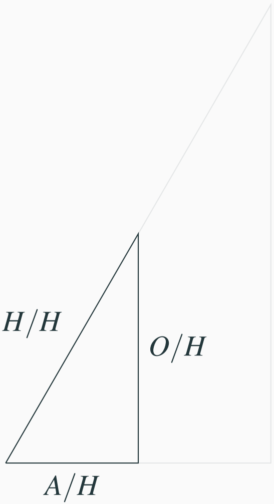

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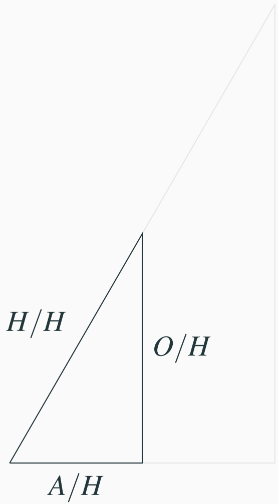
- A common shorthand is to write $\sin^2\theta + \cos^2\theta = 1$



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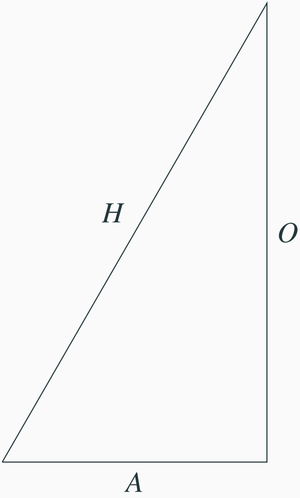

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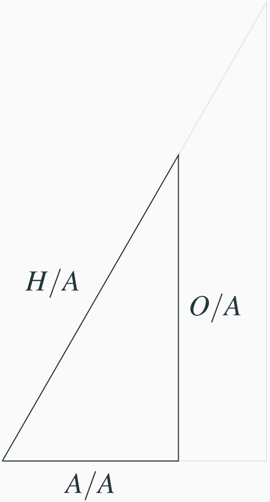

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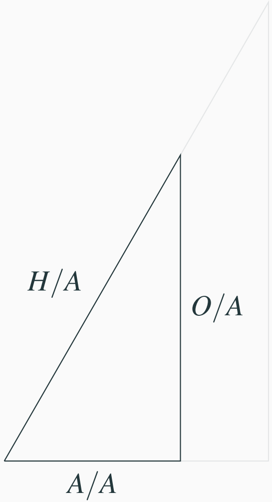

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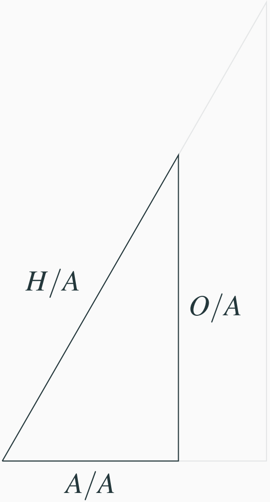

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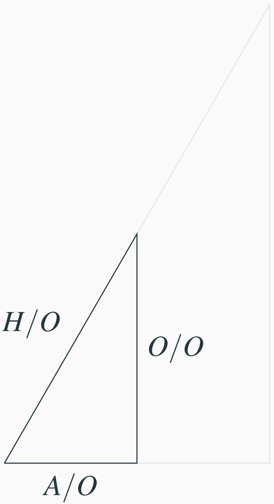

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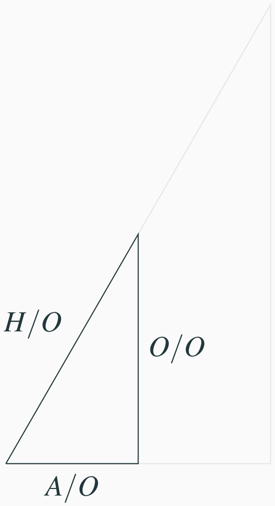
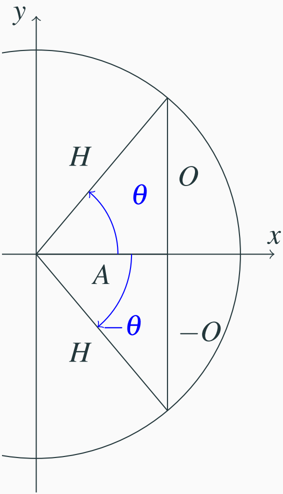
- In terms of trig functions, $1 + \cot^2\theta = \csc^2\theta$



Even and Odd Identities

- Note that for a negative angle $-\theta$, $O \to -O$, $A \to A$, and $H \to H$.



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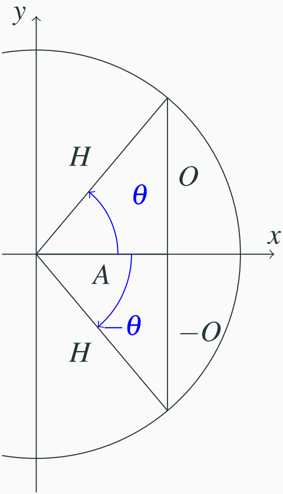

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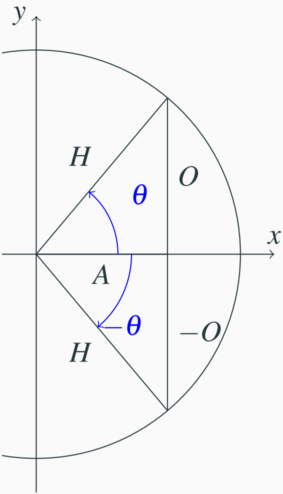

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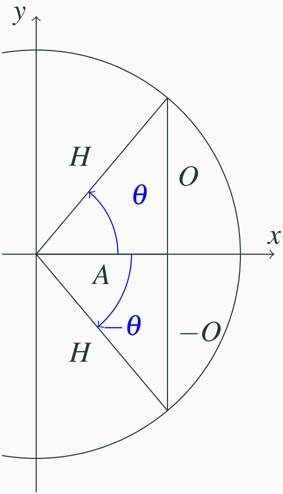

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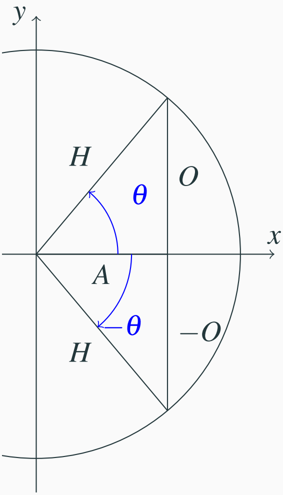
- We call cos an even function because of the analogy with even powers: $(-x)^2 = x^2$, $(-x)^4 = x^4$, ...



Periodic Identities

- If we go to an angle θ, then go further all the way around the circle, we get the same final result.

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- So we get the identities

$$\sin(\theta+2\pi)=\sin\theta \quad \cos(\theta+2\pi)=\cos\theta$$

$$\tan(\theta+2\pi)=\tan\theta \quad \cot(\theta+2\pi)=\cot\theta$$

$$\sec(\theta+2\pi)=\sec\theta \quad \csc(\theta+2\pi)=\csc\theta$$

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- For the sum of angles in the cosine function, we have

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- Similarly,

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Double Angle Formulas

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- Using $\sin^2 x + \cos^2 x = 1$ we also have

$$\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

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$$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2} \qquad \cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$$

- Taking square roots we have

$$\left|\sin \frac{x}{2}\right| = \sqrt{\frac{1 - \cos x}{2}} \qquad \left|\cos \frac{x}{2}\right| = \sqrt{\frac{1 + \cos x}{2}}$$

Product Formulas

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- By similar means

$$\cos x \cos y = \tfrac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$$
$$\sin x \sin y = \tfrac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$