Important Formulas for Midterm Test 2

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2.1

- **The definition of the derivative.** The **derivative of a function** f **at a number** a is denoted by $f'(a)$ and is given by the formula
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
if this limit exists.
- **The definition of a tangent line.** The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is defined to be the line through P with the slope $f'(a)$ provided that the derivative exists.
- **The definition of a tangent line in words.** The tangent line to $y = f(x)$ at $P(a, f(a))$ is the line through P with slope equal to the derivative of f at a.
- **The formula for the tangent line.** $y - f(a) = f'(a)(x - a)$
- **The definition of (instantaneous) velocity.** The **instantaneous velocity** of a particle with position given by $s(t)$ at time $t = a$ is denoted by $v(a)$ and is defined to be $s'(a)$.
- **The definition of instantaneous velocity in words.** The instantaneous velocity of a particle with position function $s(t)$ at time $t = a$ is defined to be the derivative of s at a.
- **Process for finding a derivative 'from first principles'.**
  1. Recall the definition of derivative:
  $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
  2. Calculate $f(a)$, reducing the answer as much as possible. (If a is an actual number, like 2 or 10 or whatever, then it should be possible to write $f(a)$ as a single number.)
  3. Calculate $f(a+h)$, reducing the answer as much as possible using the binomial theorem and/or other algebraic techniques with fractions, etc. Remember to insert the subexpression $(a+h)$ in brackets until you have processed the resulting expressions with the correct rules of algebra. **Note: you cannot simplify expressions with square roots by replacing something like $\sqrt{c+d}$ with $\sqrt{c} + \sqrt{d}$. That is 100% wrong.** (If a is an actual number, you should be able to write $f(a+h)$ as a function of h alone.)
  4. Calculate the difference $f(a+h) - f(a)$. I advise you to put brackets around $f(a)$ until you have processed the negative sign correctly using the distributive law, i.e., you should really write something like $f(a+h) - (f(a))$. Generally speaking, you should be able to reduce $f(a+h) - f(a)$ to an expression from which h can factor, although that isn't always the case, particularly with square roots.
  5. Calculate the limit $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$. Since the expression $\frac{f(a+h) - f(a)}{h}$ is undefined at $h = 0$, we can never evaluate the limit the easy way (using continuity). Instead, we will have to rearrange the numerator until a factor of h appears which will then cancel the h in the denominator.
  6. What we have at the end of this process should be the derivative $f'(a)$. If a is a number, $f'(a)$ should be a number. If a is not a definite number, but is a variable, then $f'(a)$ should be a function of a and a alone, i.e., there should be no hs left.

2.2 – **How a function can fail to be differentiable.** A function can fail to be differentiable at some number x in the following ways:

1. The function is not continuous at x. Then the limit
$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
is guaranteed not to exist because the limit
$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot h$$
doesn't exist.
2. The one-sided limits
$$\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h} \quad \text{and} \quad \lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}$$
exist but differ. We then say f has an **angle point** at x.
3. Either or both of the one-sided limits
$$\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h} \quad \text{and} \quad \lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}$$
are infinite. We then say that f has a **cusp** at x (if the one-sided limits are $+\infty$ and $-\infty$, or $-\infty$ and $+\infty$), or that f has a **vertical tangent** at x (if the one-sided limits are $-\infty$ and $-\infty$, or $+\infty$ and $+\infty$). It is also possible to have a vertical tangent on one side and a non-vertical tangent on the other, leading to a kind of angle point.
4. Either or both of the one-sided limits don't exist because the secant lines oscillate wildly in a neighborhood of x. The function $x\sin(1/x)$ near $x=0$ is an example.

– **Second derivatives.** The second derivative of a function f is defined to be the derivative of the derivative, i.e., $(f')'$ and is denoted by f''.

– **Third and higher derivatives.** The third derivative of a function f is the derivative of the second derivative, i.e., $(f'')'$, and is denoted by f'''. For higher derivatives we use notation like $f^{(4)}$ or $f^{(n)}$ (for fourth derivatives and nth derivatives, respectively).

2.3 – **The derivative of a constant.** $\frac{d}{dx}c=0$

– **The derivative of the identity function.** $\frac{d}{dx}x=1$.

– **The sum rule for derivatives.** The derivative of a sum is the sum of the derivatives: $(f+g)'=f'+g'$.

– **The difference rule for derivatives.** The derivative of a difference is a the difference of the derivatives: $(f-g)'=f'-g'$.

– **The constant multiple rule for derivatives.** The derivative of a constant multiple of a function is the constant multiple of the derivative: $(cf)'=cf'$.

– **The product rule for derivatives.** fg all prime is f prime g plus f g prime: $(fg)'=f'g+fg'$.

– **The quotient rule for derivatives.** Bottom times derivative of top minus top times derivative of bottom, over bottom squared: $\left(\frac{f}{g}\right)'=\frac{gf'-fg'}{g^2}$.

– **The power rule for derivatives.** $\frac{d}{dx}x^p=px^{p-1}$ for any real number p.

- **Slopes of perpendicular lines.** Two lines are perpendicular if the product of their slopes is -1: $m_1 m_2 = -1$ or $m_2 = -\dfrac{1}{m_1}$, i.e., m_2 is the negative reciprocal of m_1.
- **Equation of the normal line at a number a.** The normal line to a curve $y = f(x)$ at a point $x = a$ is given by $y - f(a) = -\dfrac{1}{f'(a)}(x-a)$ provided $f'(a)$ exists and is not zero. If $f'(a)$ is zero, the normal line is the vertical line $x = a$.

2.4

- **Basic trig limits: continuity.** The sin and cos functions are continuous at 0, which is the same thing as saying
$$\lim_{h\to 0} \sin h = 0 \quad \text{and} \quad \lim_{h\to 0} \cos h = 1$$
- **Basic trig limits: differentiability.** $\sin'(0) = 1$ which is the same thing as saying
$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$
Note that h must be interpreted in radians for the above result to hold.
- **Trig definitions.** The following definitions for trig functions should be reviewed.
$$\begin{aligned}
\tan x &= \frac{\sin x}{\cos x} \\
\sec x &= \frac{1}{\cos x} \\
\cot x &= \frac{\cos x}{\sin x} = \frac{1}{\tan x} \\
\csc x &= \frac{1}{\sin x} = \frac{1}{\sec x}
\end{aligned}$$
- **Trig identities.** The following formulas for trig functions should be reviewed.
$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
\sin(x+h) &= \sin x \cos h + \cos x \sin h
\end{aligned}$$
- **Derived trig limits: fractions involving sin.** We can find some limits involving sin by the following technique:
$$\lim_{h\to 0} \frac{\sin 5h}{h} = \lim_{h\to 0} \frac{\sin 5h}{5h} \cdot 5 = \lim_{h\to 0} \frac{\sin 5h}{5h} \cdot \lim_{h\to 0} 5 = 1 \cdot 5 = 5$$
- **Derived trig limits: fractions involving cos.** We find the following limit by multiplying and dividing by a conjugate expression:
$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{\cos h - 1}{h} \cdot \frac{\cos h + 1}{\cos h + 1} = \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)} = \lim_{h\to 0} \frac{\sin h}{h} \cdot \frac{-\sin h}{\cos h + 1} = \cdots$$
- **Basic trig derivatives.** These derivatives of trig functions are consequences of the definition of derivative, trig identities, and the basic trig limits:
$$\begin{aligned}
\frac{d}{dx} \sin x &= \cos x \\
\frac{d}{dx} \sin x &= -\cos x
\end{aligned}$$

- **Derived trig derivatives.** These derivatives of trig functions are consequences of the definitions of trig functions, the basic trig derivatives, and the differentiation formulas of section 3.3:

$$\frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}\csc x = -\csc x \cot x$$

- **Higher trig derivatives.** The higher derivatives of sin and cos repeat in a cyle of 4: we have

$$\begin{aligned}
\sin' x &= \cos x \\
\sin'' x &= \cos' x = -\sin x \\
\sin^{(3)} x &= -\sin' x = -\cos x \\
\sin^{(4)} x &= -\cos' x = \sin x \\
\sin^{(5)} x &= \sin' x = \cos x
\end{aligned}$$

and so on, and

$$\begin{aligned}
\cos' x &= -\sin x \\
\cos'' x &= -\sin' x = -\cos x \\
\cos^{(3)} x &= -\cos' x = \sin x \\
\cos^{(4)} x &= \sin' x = \cos x \\
\cos^{(5)} x &= \cos' x = -\sin x
\end{aligned}$$

In particular, the $4k$th derivative of sin is sin and the $4k$th derivative of cos is cos, for any positive integer k.

2.5 - **Chain rule, Newton form.** Recall $f \circ g$ is the composition of f and g, defined by $(f \circ g)(x) = f(g(x))$.

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

- **Chain rule, Leibniz form.** For the function $y(u(x))$ we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

- **Chain rule, hybrid form.**

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot \frac{d}{dx} g(x)$$

- **Chain rule for the composition of three functions.**

$$\frac{d}{dx} f(g(h(x))) = f'(g(h(x))) \cdot \frac{d}{dx} g(h(x)) = f'(g(h(x))) \cdot g'(h(x)) \cdot \frac{d}{dx} h(x)$$

2.6 Technique for finding implicit derivatives Technique for finding second derivatives

2.7 velocity acceleration increasing decreasing stationary linear density rate of a reaction marginal cost

2.8 Process for solving related rates problems