MATH 110 Lecture 4.1

Areas and Distances

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Tuesday, March 24, 2026

Department of Indigenous Knowledge and Science
First Nations University of Canada

Contents

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Integral Calculus

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- Now we are going to study two other problems, the *area* and *distance* problems.
- The area problem is a problem in geometry. The distance problem is a problem in physics.
- We are going to solve them now, which will lead to a useful new limiting process called *integration*.

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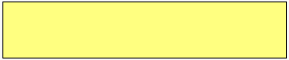

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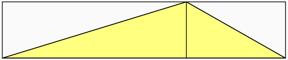

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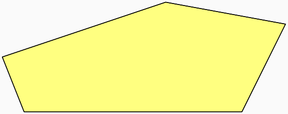

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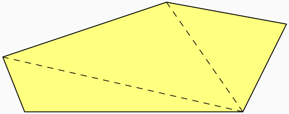
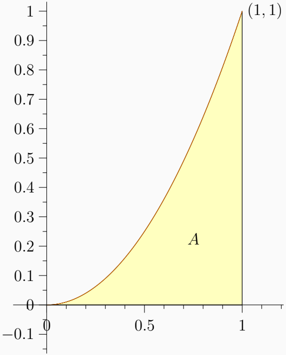
- We just subdivide a polygonal region into triangles.



Estimating Areas of Curved Regions

- However, we need new ideas to find the area under a parabola.



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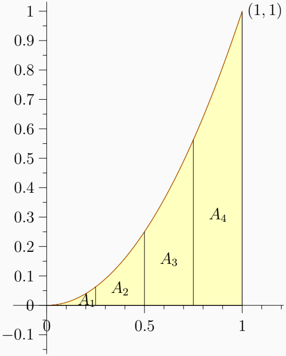

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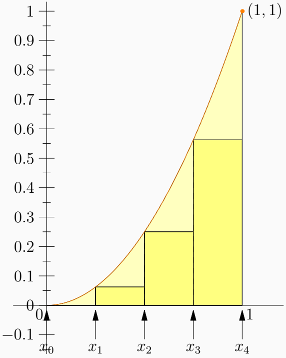

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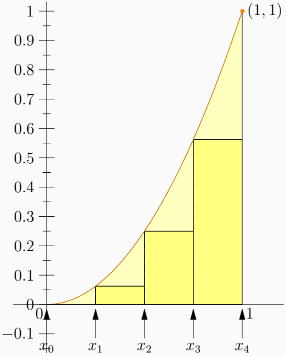
- We obtain an under-estimate of the area.



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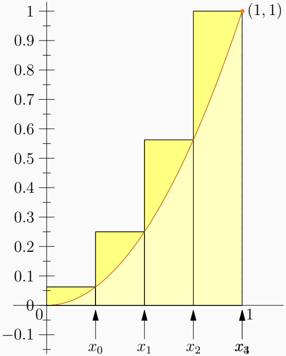

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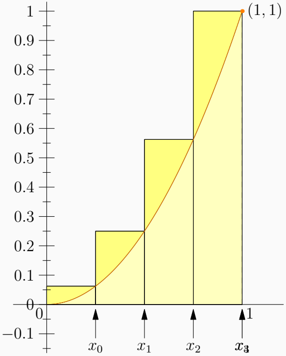

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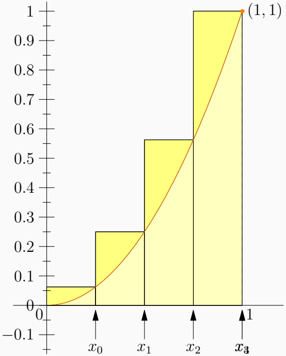
- The under- and over-estimates give us an estimate.



Under-estimate for the Area Under a Parabola

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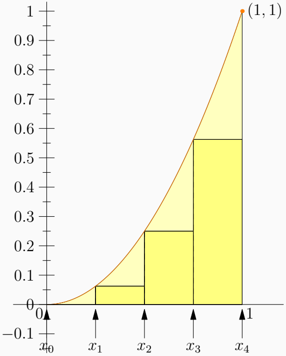
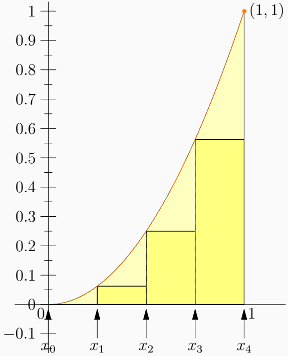


Under-estimate for the Area Under a Parabola

- We can work out a formula for the under-estimate.
- Label the points at which we subdivide the interval $x_0, \ldots, x_4$.



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- Label the points at which we subdivide the interval $x_0, \ldots, x_4$.
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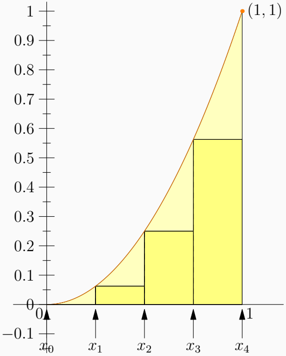

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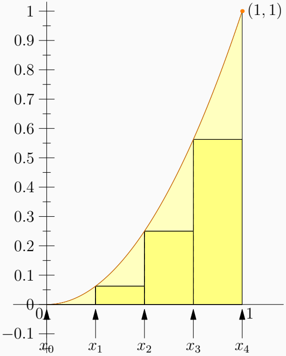
- The width of rectangle i is the width of the region divided by the number of rectangles, in this case $1/4$.



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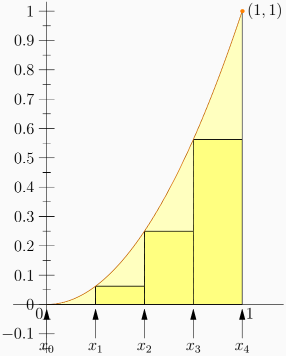
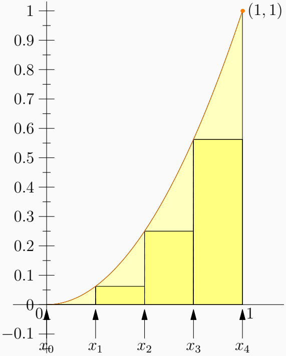
- The width of rectangle i is the width of the region divided by the number of rectangles, in this case $1/4$.
- The area of rectangle i is $f(x_{i-1}) \cdot 1/4$.



Under-estimate for the Area Under a Parabola, ctd.

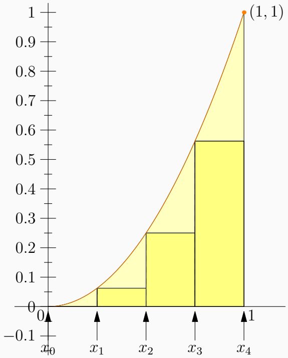
- Rectangle i has area $f(x_{i-1}) \cdot 1/4$.



Under-estimate for the Area Under a Parabola, ctd.

- Rectangle i has area $f(x_{i-1}) \cdot 1/4$.
- Adding the areas of the four rectangles, the underestimate for the total area is

$$f(x_0)\cdot\frac{1}{4}+f(x_1)\cdot\frac{1}{4}+f(x_2)\cdot\frac{1}{4}+f(x_3)\cdot\frac{1}{4}$$



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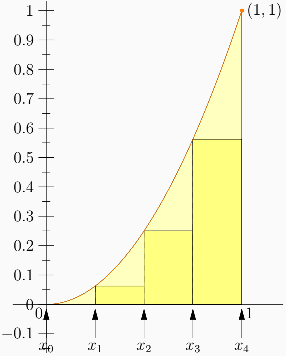
- Since we know $f(x) = x^2$, and $x_0 = 0/4$, $x_1 = 1/4$, $x_2 = 2/4$, $x_3 = 3/4$ we have $A >$

$$\left(\frac{0}{4}\right)^2 \frac{1}{4} + \left(\frac{1}{4}\right)^2 \frac{1}{4} + \left(\frac{2}{4}\right)^2 \frac{1}{4} + \left(\frac{3}{4}\right)^2 \frac{1}{4}$$



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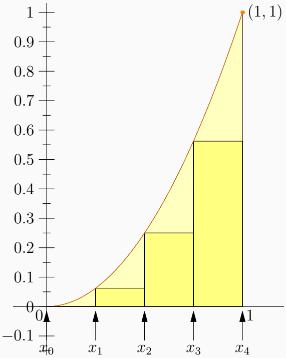
- Numerically, that gives $0.21875 < A$.



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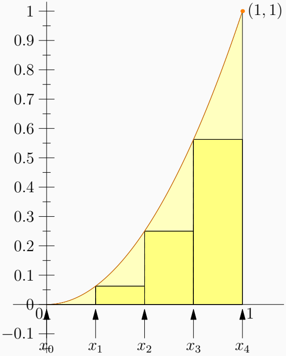

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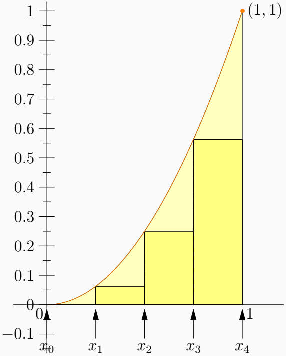

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- Since we know $f(x) = x^2$, and $x_1 = 1/4$, $x_2 = 2/4$, $x_3 = 3/4$, $x_4 = 4/4$ we have $A <$

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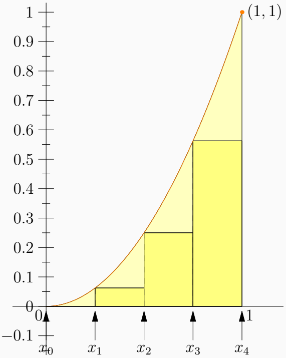

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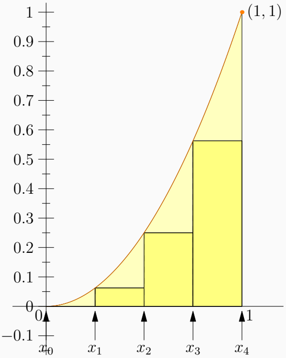
- Numerically, that gives $A < 0.46875$.



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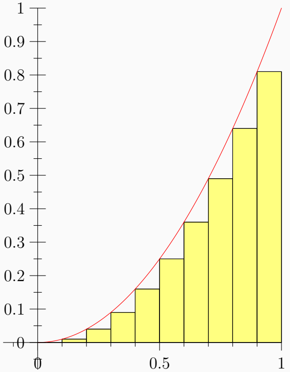

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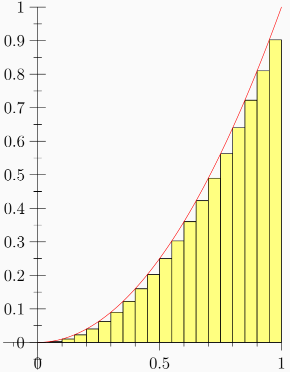

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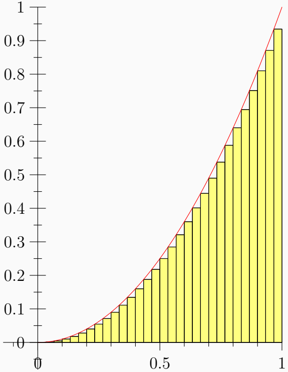

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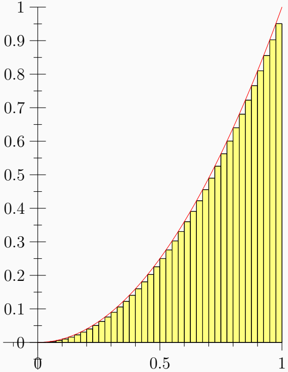

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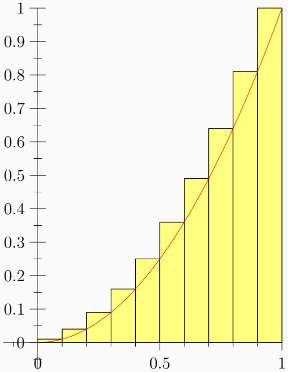

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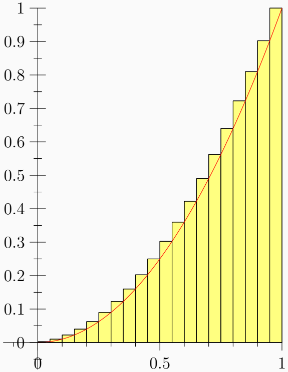

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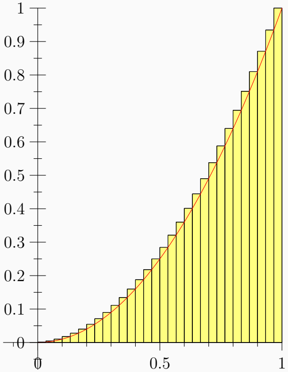

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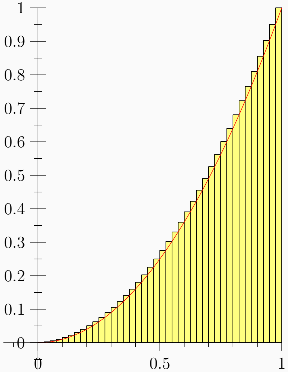

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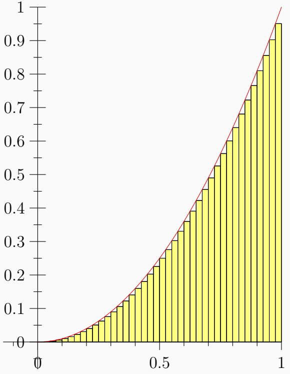
$$= \left(\frac{0}{n}\right)^2 \frac{1}{n} + \left(\frac{1}{n}\right)^2 \frac{1}{n} + \cdots + \left(\frac{n-1}{n}\right)^2 \frac{1}{n}$$



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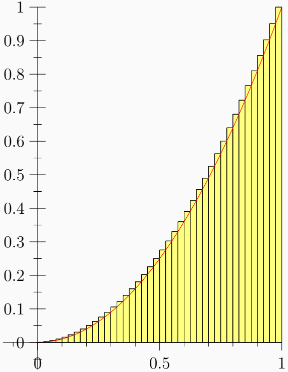

Estimating Areas for non-Increasing Functions

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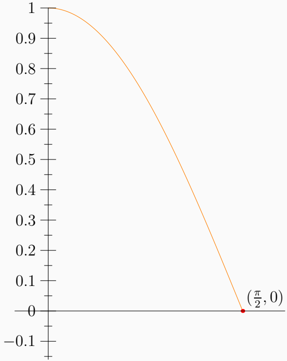

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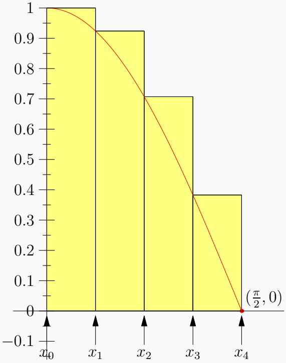

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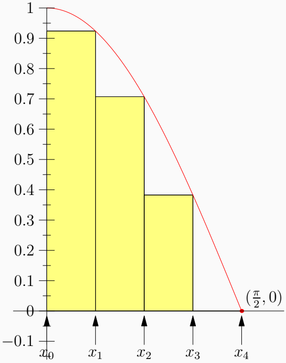

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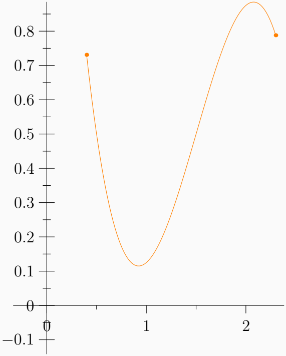

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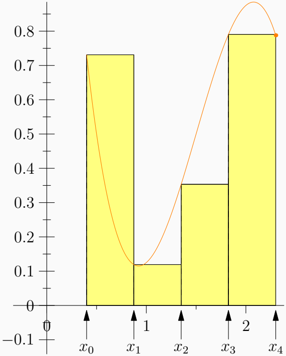

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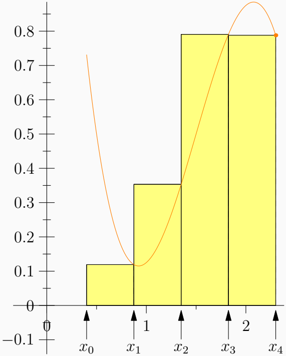

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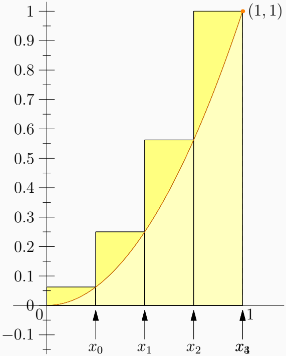

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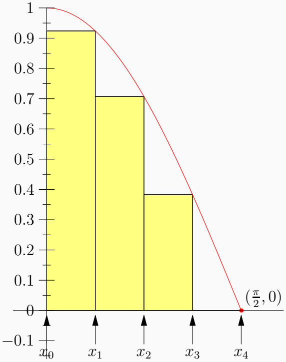

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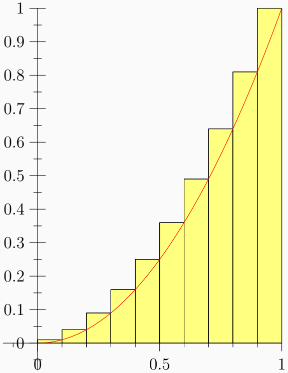

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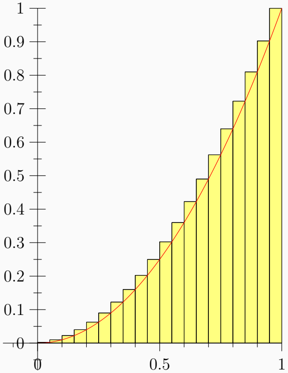

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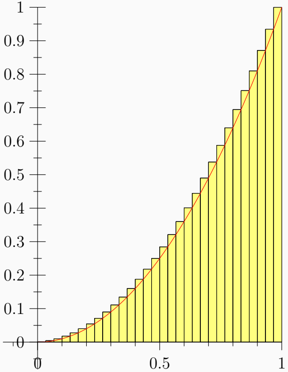

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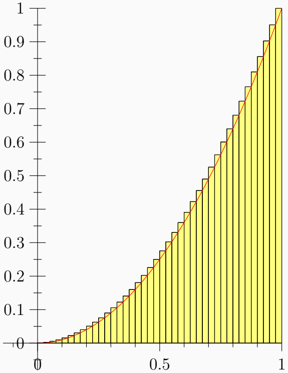

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- We *define* the area under a curve as

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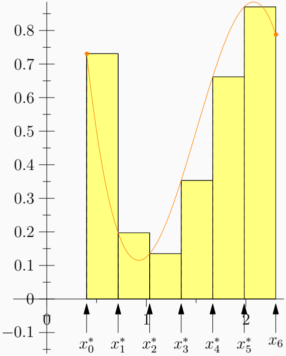

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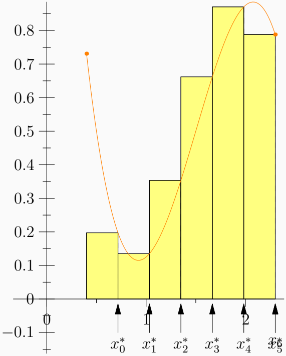

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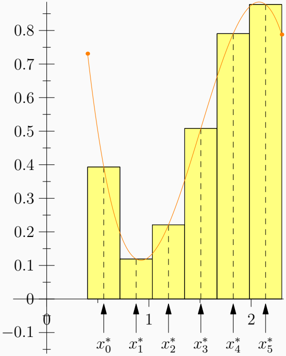

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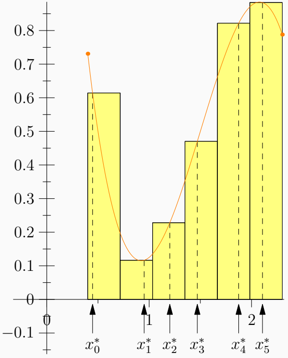

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- It is possible to prove that the above limit always exists for continuous functions.
- It is also possible to prove that

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  using any points $x_i^* \in [x_{i-1}, x_i]$ as sample points.

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- Next, $(b-a)/n = (1-0)/n = 1/n$. (We usually call that Δx.)

Example: Area from First Principles, continued

- Substituting everything we know into the above expression for A we have

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

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- Given the instantaneous velocity of a particle at all times, we would like to calculate the distance it has travelled (i.e., the displacement).
- Just as the velocity problem is directly analogous to the tangent problem in geometry, the distance problem is directly analogous to the area problem in geometry.

The Distance Problem Example

- Suppose the odometer on our car is broken, but we want to determine the distance travelled over a 30 second interval. We take speedometer readings every 5 seconds and convert them to ft/s as in the following table:

Time (s)	0	5	10	15	20	25	30
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- Altogether the distance over 30 seconds is approximately $d \approx 25 \times 5 + 31 \times 5 + 35 \times 5 + 43 \times 5 + 47 \times 5 + 45 \times 5 = 1130$ ft.

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- Alternatively, we could use the velocity at the end of every 5 second subinterval to approximate the distance travelled

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- However the two approximations are close so the answer is near the two approximations.
- If we want to improve the approximations, we could take velocity readings every 2 seconds, every 1 second, every 0.5 seconds, and so on.

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$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

- Those expressions appear so often (in area and distance calculations, but also in work calculations in physics, cardiac output in physiology, and so on) that they have a special name: Riemann sums.

Examples and Exercises

Examples

1. Find an expression for the area under $f(x) = \sqrt[4]{x}$, $1 \leq x \leq 16$.
2. 2.1 Evaluate the Riemann sum for $x^2 + x$, $0 \leq x \leq 2$ with $n = 4$, taking the sample points to be right endpoints.
   2.2 Use the definition of area to calculate the area under the curve $f(x) = x^2 + x$, $0 \leq x \leq 2$.
3. 3.1 Find an expression for the area under the curve $y = x^3$ from 0 to 1 as a limit.
   3.2 Use the formula

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

   to evaluate the limit.

Exercises

Now you should work on Problem Set 4.1. After you have finished it, you should try the following additional exercises from Section 4.1:

4.1 C-level: 1–8, 13–20, 21–23, 24–25;
B-level: 7–8, 20–21;
A-level: 27–28, 32