MATH 110 Lecture 3.1

Maximum and Minimum Values

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Contents

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Introduction to Applications of Differentiation

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- The major classes of applications we will study in this chapter are optimization, curve sketching, and approximation.
- In section 3.1 we will study the simplest type of optimization, finding the maximum and minumum values of a continuous function on a closed interval (an interval containing its endpoints).

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- The maximum and minimum values of f (i.e., $f(c)$ in the above, not c) are called the **extreme values** of f.

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- To be a little more precise, "x is near c" means "in some open interval containing c". An open interval is an interval which does not include its endpoints.

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- The Extreme Value Theorem is similar to the Intermediate Value Theorem in chapter 1. Both are hard to prove. The idea is to use a successive approximations process.

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- There may also be multiple numbers c at which the function attains its maximum. The Extreme Value Theorem guarantees one but there may be more.

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- **Fermat's Theorem:** If f has a local maximum or minimum at c, and $f'(c)$ exists, then $f'(c) = 0$.
- For those of you who are interested, a rigorous proof of Fermat's Theorem is in the textbook.

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    2. Find the values of f at each of those critical numbers.
    3. Find the values of f at each of the endpoints a and b.
    4. The largest of the values from the previous two steps is the absolute maximum value; the smallest of the values is the absolute minimum value.

The Closed Interval Theorem in Action

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    3. We evaluate f at the endpoints of the domain: $f(-2) = 11$, $f(3) = 66$.
    4. The largest of the above values is 66, so f has a maximum value of 66 at the endpoint $b = 3$. The smallest of the above values is 2, so f has a minimum value of 2 at both interior points $c = -1$ and $c = 1$.

Examples and Exercises

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1. Find the critical numbers of the given functions.
   1.1 $f(x) = x^3 + 3x^2 - 24x$
   1.2 $s(t) = 3t^4 + 4t^3 - 6t^2$
   1.3 $g(y) = \dfrac{y-1}{y^2 - y + 1}$
2. Find the local and absolute extreme values of the function on the given interval.
   2.1 $f(x) = x\sqrt{1-x}$, $[-1, 1]$
   2.2 $f(x) = (x^2 + 2x)^3$, $[-2, 1]$
   2.3 $f(x) = \sin x + \cos^2 x$, $[0, \pi]$
3. Between 0° C and 30° C, the volume V (in cm^3) of 1 kg of water at temperature T is given approximately by

   $$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$$

   Find the temperature at which water has its maximum density.

Exercises

Now you should work on Problem Set 3.1. After you have finished it, you should try the following additional exercises from Section 3.1:

3.1 C-level: 1–42, 45–56; B-level: 57–65, 67–68; A-level: 43–44, 69–70, 72