MATH 110 Problem Set 2.1 Solutions

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1. Recall that the derivative from first principles is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

To evaluate the above expression, we first evaluate $f(x+h)$, simplifying if possible, then evaluate the limit.

(a) We have

$$\begin{aligned} f(x+h) &= (x+h)^2 - 2(x+h) - 3 = x^2 + 2xh + h^2 - 2x - 2h - 3 \\ f(x+h) - f(x) &= x^2 + 2xh + h^2 - 2x - 2h - 3 - x^2 + 2x + 3 = 2xh + h^2 - 2h \end{aligned}$$

Then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h} \\ &= \lim_{h \to 0} (2x + h - 2) \\ &= 2x - 2 \end{aligned}$$

At the point $(0, -3)$ we have $x = 0$ so $f'(0) = 2(0) - 2 = -2$, which is the slope of the tangent line at $x = 0$.

(b) We use the point-slope form for the equation of the tangent line. The point slope form is

$$y - y_0 = m(x - x_0)$$

where (x_0, y_0) is the point and m is the slope. In our case, $(x_0, y_0) = (0, -3)$ and $m = -2$ from the previous answer, so the tangent line has equation

$$y - (-3) = (-2)(x - 0) \implies y + 3 = -2x \implies y = -2x - 3$$

(c) See Figure 1.

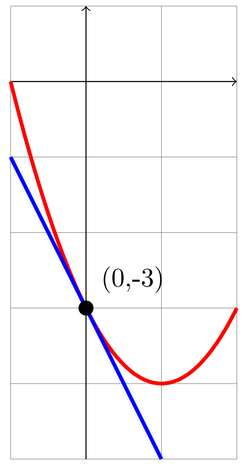


Figure 1: Graphs of $y = x^2 - 2x - 3$ and $y = -2x - 3$

2. (a) We have

$$\begin{aligned}
f(x+h) &= \frac{1}{x+h+1} \\
f(x) &= \frac{1}{x+1} \\
f(x+h) - f(x) &= \frac{1}{x+h+1} - \frac{1}{x+1} = \frac{(x+1)-(x+h+1)}{(x+h+1)(x+1)} = \frac{-h}{(x+h+1)(x+1)} \\
\frac{f(x+h)-f(x)}{h} &= \frac{-1}{(x+h+1)(x+1)} \\
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{-1}{(x+h+1)(x+1)} = \frac{-1}{(x+1)^2}
\end{aligned}$$

Substituting $x = 0$,

$$f'(0) = \frac{-1}{(0+1)^2} = -1$$

Using point-slope form, the equation of the tangent line is

$$y - y_0 = m(x - x_0) \implies y - 1 = (-1)(x-0) \implies y - 1 = -x \implies y = -x + 1$$

(b) We have

$$\begin{aligned}
f(x+h) &= \sqrt{2(x+h)-1} \\
f(x) &= \sqrt{2x-1} \\
f(x+h) - f(x) &= \sqrt{2(x+h)-1} - \sqrt{2x-1} \\
\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h} \\
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h}
\end{aligned}$$

To evaluate limits like this, multiply by the conjugate radical:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{2x+2h-1}-\sqrt{2x-1}}{h} \times \frac{\sqrt{2x+2h-1}+\sqrt{2x-1}}{\sqrt{2x+2h-1}+\sqrt{2x-1}} \\
&= \lim_{h\to 0} \frac{(2x+2h-1)-(2x-1)}{h(\sqrt{2x+2h-1}-\sqrt{2x-1})} \\
&= \lim_{h\to 0} \frac{2h}{h(\sqrt{2x+2h-1}-\sqrt{2x-1})} \\
&= \frac{2}{\sqrt{2x-1}+\sqrt{2x-1}} = \frac{2}{2\sqrt{2x-1}} \\
&= \frac{1}{\sqrt{2x-1}}
\end{aligned}$$

Substituting $x = 5$ gives $f'(5) = 1/(2(5)-1) = 1/9$. The equation of the tangent line is $y - 3 = 1/9(x-5)$, which can be rearranged to $y = (1/9)x - 22/9$.

(c) We have

$$\begin{aligned}
f(x+h) &= (x+h)^3 - 4(x+h) = x^3 + 3x^2h + 3xh^2 + h^3 - 4x - 4h \\
f(x) &= x^3 - 4x \\
f(x+h) - f(x) &= 3x^2h + 3xh^2 + h^3 - 4h \\
\frac{f(x+h)-f(x)}{h} &= \frac{3x^2h + 3xh^2 - h^3 - 4h}{h} = 3x^2 + 3xh - h^2 - 4 \\
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 3x^2 + 3xh - h^2 - 4 = 3x^2 - 4
\end{aligned}$$

At $x = -1$, $f'(-1) = 3(-1)^2 - 4 = 3 - 4 = -1$. The equation of the tangent line is $y - 3 = (-1)(x - (-1))$ or $y - 3 = -x - 1$ or $y = -x + 2$.

(d) We have

$$\begin{aligned}
f(x+h) &= \frac{(x+h)^2}{2(x+h)-1} = \frac{x^2+2xh+h^2}{2x+2h-1} \\
f(x) &= \frac{x^2}{2x-1} \\
f(x+h) - f(x) &= \frac{x^2+2xh+h^2}{2x+2h-1} - \frac{x^2}{2x-1} = \frac{(x^2+2xh+h^2)(2x-1) - (2x+2h-1)x^2}{(2x+2h-1)(2x-1)} \\
&= \frac{2x^3 - x^2 + 4x^2h - 2xh + 2xh^2 - h^2 - 2x^3 - 2x^2h + x^2}{(2x+2h-1)(2x-1)} = \frac{4x^2h - 2xh - h^2}{(2x+2h-1)(2x-1)} \\
\frac{f(x+h)-f(x)}{h} &= \frac{4x^2h - 2xh - h^2}{(2x+2h-1)(2x-1)} = \frac{4x^2 - 2x - h}{(2x+2h+1)(2x-1)} \\
f'(x) &= \lim_{h\to 0} \frac{4x^2 - 2x - h}{(2x+2h-1)(2x-1)} = \frac{4x^2 - 2x}{(2x-1)(2x-1)} = \frac{4x^2 - 2x}{(2x-1)^2}
\end{aligned}$$

3. (a) We have

$$\begin{aligned}
f(a+h) &= 10+3(a+h)^2-(a+h)^3 = 10+3a^2+6ah+3h^2-a^3-3a^2h-3ah^2-h^3\\
f(a) &= 10+3a^2-a^3\\
f(a+h)-f(a) &= 10+3a^2+6ah+3h^2-a^3-3a^2h-3ah^2-h^3-10-3a^2+a^3\\
&= 6ah+3h^2-3a^2h-3ah^2-h^3\\
\frac{f(a+h)-f(a)}{h} &= \frac{6ah+3h^2-3a^2h-3ah^2-h^3}{h}
\end{aligned}$$

Taking the limit,

$$\begin{aligned}
f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\\
&= \lim_{h\to 0}\frac{6ah+3h^2-3a^2h-3ah^2-h^3}{h}\\
&= \lim_{h\to 0}(6a+3h-3a^2-3ah-h^2)\\
&= 6a-3a^2
\end{aligned}$$

(b) Now that we have a formula for the slope of the tangent line, we can apply it to obtain the slope immediately in both cases. In the first case,

$$m = f'(1) = 6(1)-3(1)^2 = 6-3 = 3$$

so the point-slope form of the tangent line is

$$y-12 = 3(x-1)$$

You can further simplify your answer but that is not necessary. In the second case,

$$m = f'(3) = 6(3)-3(3)^2 = 18-27 = -9$$

so the point-slope form of the tangent line is

$$y-10 = -9(x-3)$$

(c) See Figure 2.

4. Instantaneous velocity is the derivative. Since we need the velocity at several moments $t=1$, $t=2$, and $t=3$, we should find a general formula for the derivative. We have

$$\begin{aligned}
f(a+h) &= \frac{1}{a+h}\\
f(a) &= \frac{1}{a}\\
f(a+h)-f(a) &= \frac{1}{a+h}-\frac{1}{a} = \frac{a-a-h}{(a+h)(a)} = \frac{-h}{(a+h)(a)}\\
\frac{f(a+h)-f(a)}{h} &= \frac{-h}{(a+h)(a)(h)}
\end{aligned}$$

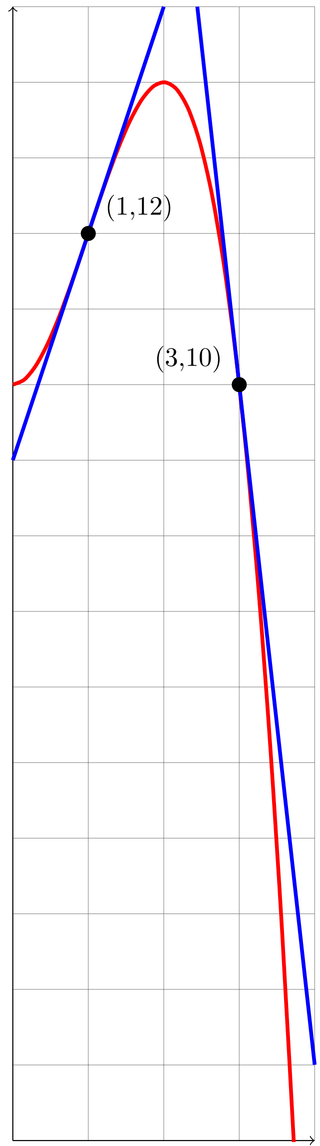


Figure 2: Graphs of $y = 10 + 3x^2 - x^3$ and $y = 3x + 9$ and $y = -9x + 37$

Taking the limit,

$$\begin{aligned}
v(a) = s'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{-h}{(a+h)(a)(h)} \\
&= \lim_{h\to 0} \frac{-1}{(a+h)(a)} \\
&= \frac{-1}{(a)(a)} \\
&= \frac{-1}{a^2}
\end{aligned}$$

Then at times $a = 1$, $a = 2$, and $a = 3$ we have

$$\begin{aligned}
v(1) &= \frac{-1}{1^2} = -1 \\
v(2) &= \frac{-1}{2^2} = -\frac{1}{4} \\
v(3) &= \frac{-1}{3^2} = -\frac{1}{9}
\end{aligned}$$

where all velocity units are m/s. Finally, the speeds are the absolute values of the velocities, so the speeds are

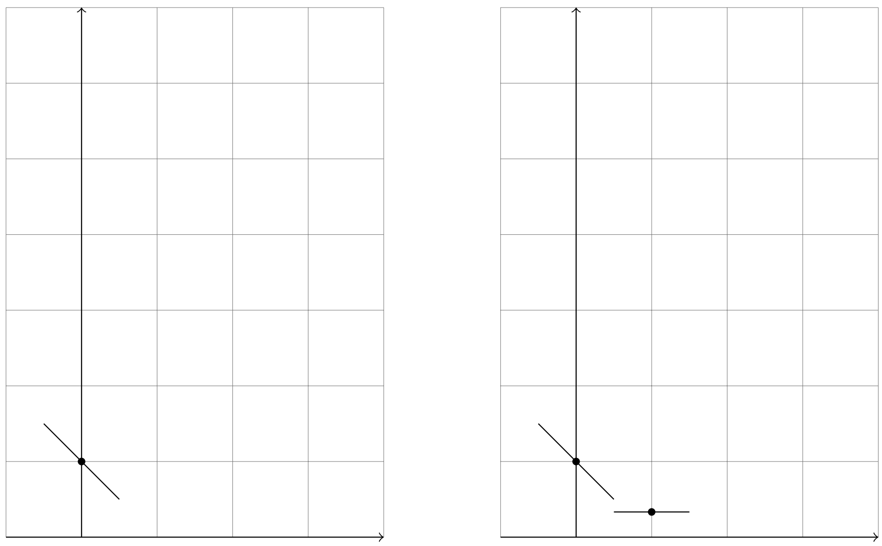
$$\begin{aligned}
|v(a)| &= \left|-\frac{1}{a^2}\right| = \frac{1}{a^2} \\
|v(1)| &= |-1| = 1 \\
|v(2)| &= \left|-\frac{1}{4}\right| = \frac{1}{4} \\
|v(3)| &= \left|-\frac{1}{9}\right| = \frac{1}{9}
\end{aligned}$$

again in m/s. Note that in $|v(a)|$ it is potentially tricky to get the sign right in general, but the fraction $1/a^2$ is always positive.

5. Note that there are many possible answers to this question. First, we place a dot on an empty grid representing the point $(0, 1)$ through which the graph passes, and we draw a small line segment with slope -1 passing through the point $(0, 1)$ representing the tangent line to the graph at the point. See Figure 3(a). Next, we draw a line segment of slope 0 at a point $(1, y_1)$. We are not given a condition on y_1 so we pick any y_1 (we should pick a value less than 1 to get a nice graph). See Figure 3(b). We do the same thing with a small line segment of slope 3 at the point $(2, y_2)$, where y_2 can be anything (but we should pick a value greater than y_1 to get a nice graph). See Figure 4(c). Finally, we sketch a nice smooth graph passing through the three points we have drawn, which is tangent to the three line segments we have drawn. See Figure 4(d).

6. (a) We have

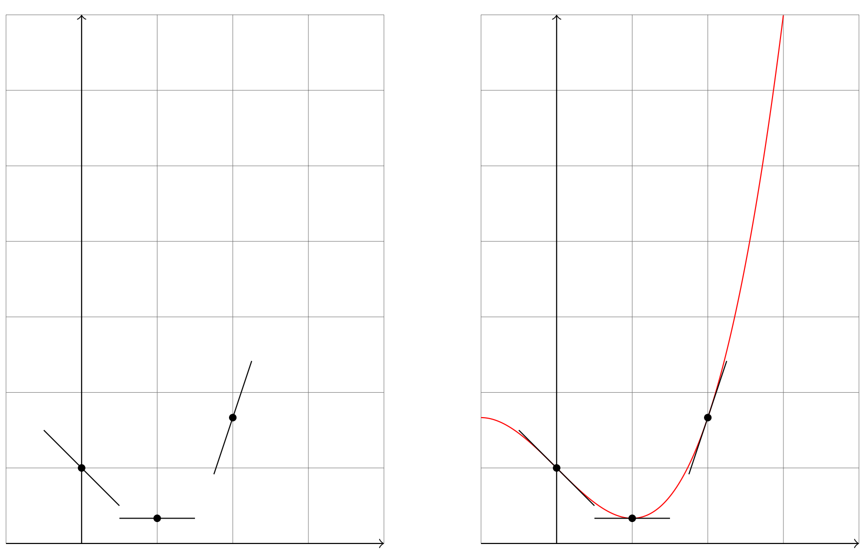
$$\begin{aligned}
f(a+h) &= 4(a+h)^2 - (a+h)^3 = 4a^2 + 8ah + 4h^2 - a^3 - 3a^2h - 3ah^2 - h^3 \\
f(a) &= 4a^2 - a^3 \\
f(a+h) - f(a) &= 4a^2 + 8ah + 4h^2 - a^3 - 3a^2h - 3ah^2 - h^3 - 4a^2 + a^3 \\
&= 8ah + 4h^2 - 3a^2h - 3ah^2 - h^3
\end{aligned}$$



(a) Conditions $f(0) = 1,\ f'(0) = -1$

(b) Adding condition $f'(1) = 0$

Figure 3: First two conditions for graph



(c) Adding Condition $f'(2) = 3$

(d) Graph Consistent with (c)

Figure 4: Third Condition, and Graph Consistent with Conditions

Taking limits of the difference quotient,

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{8ah + 4h^2 - 3a^2h - 3ah^2 - h^3}{h} \\ &= \lim_{h\to 0} (8a + 4h - 3a^2 - 3ah - h^2) \\ &= 8a - 3a^2 \end{aligned}$$

(b) By the binomial theorem we have

$$\begin{aligned} f(a+h) &= (a+h)^4 - 5(a+h) = a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + h^4 - 5a - 5h \\ f(a) &= a^4 - 5a \\ f(a+h) - f(a) &= a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + h^4 - 5a - 5h - a^4 + 5a \\ &= 4a^3h + 6a^2h^2 + 4ah^3 + h^4 - 5h \end{aligned}$$

Taking the limit of the difference quotient,

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{4a^3h + 6a^2h^2 + 4ah^3 + h^4 - 5h}{h} \\ &= \lim_{h\to 0} (4a^3 + 6a^2h + 4ah^2 + h^3 - 5) \\ &= 4a^3 - 5 \end{aligned}$$

(c) We have

$$\begin{aligned} f(a+h) &= \frac{(a+h)^2 + 1}{a+h-2} = \frac{a^2 + 2ah + h^2 + 1}{a+h-2} \\ f(a) &= \frac{a^2+1}{a-2} \\ f(a+h) - f(a) &= \frac{a^2 + 2ah + h^2 + 1}{a+h-2} - \frac{a^2+1}{a-2} \\ &= \frac{(a^2 + 2ah + h^2 + 1)(a-2) - (a^2+1)(a+h-2)}{(a+h-2)(a-2)} \\ &= \frac{a^3 + 2a^2h + ah^2 + a - 2a^2 - 4ah - 2h^2 - 2 - a^3 - a^2h + 2a^2 - h + 2}{(a+h-2)(a-2)} \\ &= \frac{a^2h + ah^2 - 4ah - 2h^2 - h}{(a+h-2)(a-2)} \end{aligned}$$

Taking the limit of the difference quotient,

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{a^2h + ah^2 - 4ah - 2h^2 - h}{h(a+h-2)(a-2)} \\ &= \lim_{h\to 0} \frac{a^2 + ah - 4a - 2h - 1}{(a+h-2)(a-2)} \\ &= \frac{a^2 - 4a - 1}{(a-2)^2} \end{aligned}$$

(d) We have

$$\begin{aligned} f(a+h) &= \sqrt{3a + 3h + 1} \\ f(a) &= \sqrt{3a+1} \\ f(a+h) - f(a) &= \sqrt{3a + 3h + 1} - \sqrt{3a+1} \end{aligned}$$

Taking the limit of the difference quotient and evaluating the limit by multiplying by the conjugate radical,

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{\sqrt{3a+3h+1}-\sqrt{3a+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3a+3h+1}-\sqrt{3a+1}}{h} \times \frac{\sqrt{3a+3h+1}+\sqrt{3a+1}}{\sqrt{3a+3h+1}+\sqrt{3a+1}} \\
&= \lim_{h\to 0} \frac{(3a+3h+1)-(3a+1)}{h(\sqrt{3a+3h+1}+\sqrt{3a+1})} \\
&= \lim_{h\to 0} \frac{3h}{h(\sqrt{3a+3h+1}+\sqrt{3a+1})} \\
&= \lim_{h\to 0} \frac{3}{\sqrt{3a+3h+1}+\sqrt{3a+1}} \\
&= \frac{3}{2\sqrt{3a+1}}
\end{aligned}$$

7. We have

$$\begin{aligned}
y(t+h) &= 5(t+h)-4.9(t+h)^2 = 5t+5h-4.9t^2-9.8th-4.9h^2 \\
y(t) &= 5t-4.9t^2 \\
y(t+h)-y(t) &= 5h-9.8th-4.9h^2 \\
\frac{y(t+y)-y(t)}{h} &= 5-9.8t-4.9h \\
y' &= \lim_{h\to 0} \frac{y(t+h)-y(t)}{h} = \lim_{h\to 0}(5-9.8t-4.9h) = 5-9.8t
\end{aligned}$$

At $t = 0.5$ seconds, $y'(0.5) = 5 - 9.8(0.5) = 5 - 4.9 = 0.1$ meters per second. Positive velocities are upward (and negative velocities would be downward) in this problem.

8. We have to figure out how to represent some of the numbers in the expressions differently.

(a) Write $32 = 2^5$. Then we have $f(x) = x^5$ and $a = 2$. Check.

(b) Write $4 = \sqrt{16}$. Then we have $f(x) = \sqrt{x}$ and $a = 16$. Check.

(c) Note that $\sin\frac{\pi}{2} = 1$ so we can write the expression as

$$\lim_{t\to\pi/2} \frac{\sin t - \sin(\pi/2)}{t-\pi/2}$$

which is an alternate form for the derivative of $f(t) = \sin t$ at $a = \pi/2$.

Alternatively, you can convert the limit so it is a limit as $h \to 0$. Let $h = t - \pi/2$, in which case $t = h + \pi/2$. We have

$$\lim_{t\to\pi/2} \frac{\sin t - 1}{t-\pi/2} = \lim_{h\to 0} \frac{\sin(h+\pi/2)-\sin(\pi/2)}{h}$$

so we have $f(t) = \sin t$ and $a = \pi/2$.

9. In business applications, the derivative is usually interpreted as the “marginal”.

(a) The marginal cost is the cost of making one more widget after you have made x widgets. It can be written as

$$f(x+1) - f(x) = \frac{f(x+1) - f(x)}{1} = \frac{f(x+h) - f(x)}{h}$$

where $h = 1$. If you are making large numbers of widgets, 1 is considered close to 0, so we have the approximation

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \approx \frac{f(x+1) - f(x)}{1}$$

so the derivative is approximately equal to the marginal. The units are dollars per widget.

(b) $f'(100) = 12$ means the cost of making one more widget, after you have already made 100 widgets, is 12 dollars. (Usually the first few widgets are very expensive because they carry the fixed costs like renting factory space, etc. After you get rolling, the cost of one more widget just includes the materials and labour, not the fixed costs.)

(c) For small values of x, the marginal cost will decrease as you make more widgets, because more of the fixed costs will be absorbed into the previously made widgets. For large values of x we may see an increase in the marginal cost because extra manufacturing capacity will need to be added beyond a certain level, and materials costs will increase because of the increasing demand caused by your manufacturing.

10. This question is hard. When $x \neq 0$,

$$\begin{aligned} f(0+h) &= (0+h)^2 \cos \frac{1}{0+h} \\ f(0) &= 0 \end{aligned}$$

where we have used the first part of the definition of f for values $x = 0 + h$ away from 0, and the second part of the definition of f for the value $x = 0$.

$$\begin{aligned} f(0+h) - f(0) &= h^2 \cos \frac{1}{h} \\ \frac{f(0+h) - f(0)}{h} &= h \cos \frac{1}{h} \\ f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} h \cos \frac{1}{h} \end{aligned}$$

We have to be careful evaluating this limit. We can't use the product rule for limits because

$$\lim_{h \to 0} \cos \frac{1}{h}$$

does not exist. Instead we have to use the squeeze theorem. Note that cos is bounded between -1 and 1:

$$-1 \leq \cos \frac{1}{h} \leq 1$$

for any $h \neq 0$. Assuming $h > 0$, multiply by h to obtain

$$-h \leq h \cos \frac{1}{h} \leq h$$

Both $-h \to 0$ and $h \to 0$, squeezing the value in the middle, so we can conclude that the one sided limit

$$\lim_{h \to 0^+} h \cos \frac{1}{h} = 0$$

You should make a similar (but slightly different) argument to show that $\lim_{h \to 0^-} h \cos(1/h) = 0$ also. Since the one-sided limits exist and are equal, we conclude that

$$f'(0) = \lim_{h \to 0} h \cos \frac{1}{h} = 0$$