

MATH 110 Lecture 1.6

Calculating Limits Using the Limit Laws

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Limit Laws

Consequences of the Limit Laws

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Examples and Exercises

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Limit Laws

Suppose that the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then

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5. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ **under the condition that** $\lim_{x \rightarrow a} g(x) \neq 0$.

If the condition in the last rule above is violated, we can't conclude anything about the limit without further work.

The Limit Laws in Words

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If the condition in the the last rule above is violated, the rule does not apply.

Example of the Limit Laws

Suppose that we know that $\lim_{x \rightarrow 3} f(x) = 5$, $\lim_{x \rightarrow 3} g(x) = -2$, and $\lim_{x \rightarrow 3} h(x) = 0$. Then the limit laws tell us that

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because $\lim_{x \rightarrow 3} h(x) = 0$. **You can't divide by zero!**

A More Complicated Example of the Limit Laws

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Then we evaluate those in order.

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2. $\lim_{x \rightarrow 3} [f(x) - g(x)h(x)] = \lim_{x \rightarrow 3} f(x) - \lim_{x \rightarrow 3} g(x)h(x) = 5 - 0 = 5$

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Consequences of the Limit Laws

The following results are immediate consequences of the limit laws.

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- If $f = g/h$ is a rational function (i.e., g and h are polynomials),
and $h(a) \neq 0$, then $\lim_{x \rightarrow a} f(x) = \frac{g(a)}{h(a)}$.

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 provided the limits exist.
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3. **Limits preserve inequalities:** If $f(x) \leq g(x)$ for all $x \neq a$, $\lim_{x \rightarrow a} f(x) = L$, and $\lim_{x \rightarrow a} g(x) = M$, then $L \leq M$.

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4. **Squeeze Theorem:** If $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$, and $\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x)$, then $\lim_{x \rightarrow a} g(x) = L$.

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Unfortunately, that equation is not true! It fails for $x = 1$, in which case the left hand side is undefined but the right hand side is $1/2$.

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and now we can apply the quotient rule for limits (or the theorem on the evaluation of limits of rational functions) to conclude that

$$\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = \lim_{x \rightarrow 1} \frac{1}{x+1} = \frac{1}{2}$$

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This is a common technique for evaluating limits of the form

$$\lim_{x \rightarrow a} \frac{g(x)}{h(x)}$$

where $h(a) = 0$.

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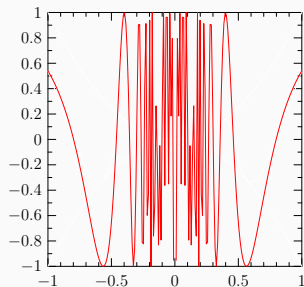
But we can't remove the question mark because

$$\lim_{x \rightarrow 0} \cos(1/x^2)$$

does not exist!

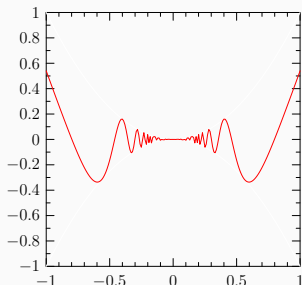
Application of the Squeeze Theorem

- The function $\cos(1/x^2)$ oscillates wildly as x approaches 0.



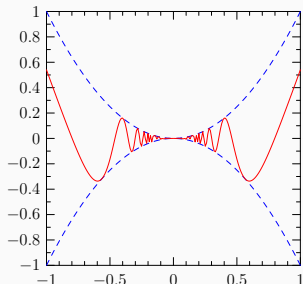
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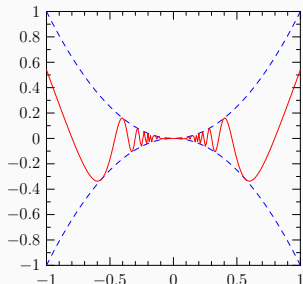
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- We can find the upper and lower **envelopes** x^2 and $-x^2$ for the function and use the Squeeze Theorem.
- We have $-x^2 \leq x^2 \cos(1/x^2) \leq x^2$ (why?) and $\lim_{x \rightarrow 0} -x^2 = 0 = \lim_{x \rightarrow 0} x^2$. It follows that $\lim_{x \rightarrow 0} x^2 \cos(1/x^2) = 0$.



Examples

1. Evaluate

$$\lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2}$$

2. Evaluate

$$\lim_{s \rightarrow 16} \frac{4 - \sqrt{s}}{s - 16}$$

3. Find

$$\lim_{x \rightarrow 0} x \sin(1/x)$$

Now you should work on Problem Set 1.6. After you have finished it, you should try the following additional exercises from Section 1.6:

1.6 C-level: 1, 2, 3–9, 10, 11–26, 50, 52;

B-level: 27–30, 31–32, 33–34, 35–40, 47–48, 49, 51;

A-level: 41–46, 53–55, 56–66