

MATH 110 Problem Set 1.6 Solutions

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1. (a) The following is an example of a completely rigorous argument for evaluating limits. For an example of an argument which is not completely rigorous but would be acceptable, see the solution for part (b).

By the quotient rule,

$$\lim_{x \rightarrow 2} \frac{2x^2 + 1}{x^2 + 6x - 4} = \frac{\lim_{x \rightarrow 2} (2x^2 + 1)}{\lim_{x \rightarrow 2} (x^2 + 6x - 4)}$$

provided both limits exist *and provided* the limit in the quotient is not equal to zero. Since we don't (officially) know whether those conditions hold, the equality in the above line is provisional for the time being. We won't know whether those conditions hold until we reach the end of the calculation.

To evaluate the numerator in the above calculation, we use the sum rule

$$\lim_{x \rightarrow 2} (2x^2 + 1) = \lim_{x \rightarrow 2} 2x^2 + \lim_{x \rightarrow 2} 1$$

Again, we don't know whether the equality above holds until we finish calculating the limit. We now use the constant multiple rule:

$$\lim_{x \rightarrow 2} 2x^2 = 2 \lim_{x \rightarrow 2} x^2$$

and the power rule

$$\lim_{x \rightarrow 2} x^2 = \left(\lim_{x \rightarrow 2} x \right)^2$$

and finally the limit of x

$$\lim_{x \rightarrow 2} x = 2$$

to obtain

$$\lim_{x \rightarrow 2} (2x^2 + 1) = \lim_{x \rightarrow 2} 2x^2 + \lim_{x \rightarrow 2} 1 = 2 \left(\lim_{x \rightarrow 2} x \right)^2 + \lim_{x \rightarrow 2} 1 = 2(2)^2 + 1 = 9$$

Similarly, to evaluate the denominator, we use the sum rule, the constant multiple rule, the power rule, and the "bottom level rules" for $\lim_{x \rightarrow 2} x$ and $\lim_{x \rightarrow 2} c$ to obtain

$$\lim_{x \rightarrow 2} (x^2 + 6x - 4) = \lim_{x \rightarrow 2} x^2 + \lim_{x \rightarrow 2} 6x - \lim_{x \rightarrow 2} 4 = \left(\lim_{x \rightarrow 2} x \right)^2 + 6 \lim_{x \rightarrow 2} x - \lim_{x \rightarrow 2} 4 = 2^2 + 6(2) - 4 = 12$$

Note that the above limit is not 0, so we can apply the quotient rule

$$\lim_{x \rightarrow 2} \frac{2x^2 + 1}{x^2 + 6x - 4} = \frac{\lim_{x \rightarrow 2} (2x^2 + 1)}{\lim_{x \rightarrow 2} (x^2 + 6x - 4)} = \frac{9}{12} = \frac{3}{4}$$

- (b) We can express the same type of argument as in part (a) in a single line (provided we understand that every step is provisional until the whole argument is complete):

$$\begin{aligned}
 \lim_{t \rightarrow -1} (t^2 + 1)^3 (t + 3)^5 &= \lim_{t \rightarrow -1} (t^2 + 1)^3 \times \lim_{t \rightarrow -1} (t + 3)^5 \\
 &= \left(\lim_{t \rightarrow -1} t^2 + 1 \right)^3 \times \left(\lim_{t \rightarrow -1} (t + 3) \right)^5 \\
 &= \left(\lim_{t \rightarrow -1} t^2 + \lim_{t \rightarrow -1} 1 \right)^3 \times \left(\lim_{t \rightarrow -1} t + 3 \right)^5 \\
 &= \left(\left(\lim_{t \rightarrow -1} t \right)^2 + \lim_{t \rightarrow -1} 1 \right)^3 \times \left(\lim_{t \rightarrow -1} t + 3 \right)^5 \\
 &= ((-1)^2 + 1)^3 \times (-1 + 3)^5 = (1 + 1)^3 \times (2)^5 = 8 \times 32 = 256
 \end{aligned}$$

2. (a) First, we should substitute $x = -4$ into the denominator; if the result is non-zero, we can find the answer to the question just with a straight substitution. If the result is zero, we will probably have to factor. We have

$$x^2 + 3x - 4 \Big|_{x=-4} = (-4)^2 + 3(-4) - 4 = 16 - 12 - 4 = 0$$

so we will have to factor. Note that

$$\begin{aligned}
 x^2 + 5x + 4 &= (x + 4)(x + 1) \\
 x^2 + 3x - 4 &= (x + 4)(x - 1)
 \end{aligned}$$

so we have

$$\lim_{x \rightarrow -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} = \lim_{x \rightarrow -4} \frac{(x + 4)(x + 1)}{(x + 4)(x - 1)} = \lim_{x \rightarrow -4} \frac{x + 1}{x - 1} = \frac{-4 + 1}{-4 - 1} = \frac{-3}{-5} = \frac{3}{5}$$

- (b) As above, we check the value of the function when $x = 1$:

$$\frac{x^3 - 1}{x^2 - 1} \Big|_{x=1} = \frac{1^3 - 1}{1^2 - 1} = \frac{0}{0}$$

so we have to do more work. In this case we have to factor. The problem factor is $x - 1$, so we try to pull a factor of that type from both the numerator and the denominator. In the numerator, we have

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

which you can memorize as a common, useful factorization, or you can obtain by polynomial division. In the denominator we have

$$x^2 - 1 = (x - 1)(x + 1)$$

by the standard difference of squares factorization. Evaluating the limit we have

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)} = \lim_{x \rightarrow 1} \frac{x^2 + x + 1}{x + 1} = \frac{1^2 + 1 + 1}{1 + 1} = \frac{3}{2}$$

- (c) Again, we first check what happens when we substitute $h = 0$ into the expression. We have

$$\frac{\sqrt{1+h} - 1}{h} \Big|_{h=0} = \frac{\sqrt{1+0} - 1}{0} = \frac{0}{0}$$

so we have to do more work. To evaluate limits involving square roots, we often find it helpful to multiply by the “conjugate radical”, in this case $\sqrt{1+h} + 1$:

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} = \lim_{h \rightarrow 0} \frac{1 + h - 1}{h(\sqrt{1+h} + 1)} \\
 &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{1+h} + 1)} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2}
 \end{aligned}$$

- (d) Again, substituting $h = 0$ gives an expression of the form $0/0$, so we have to do more work. We rewrite the negative power as a fraction, and then clear fractions:

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{(3+h)^{-1} - 3^{-1}}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} = \lim_{h \rightarrow 0} \left(\frac{3}{3(3+h)} - \frac{3+h}{3(3+h)} \right) \frac{1}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{3 - (3+h)}{3(3+h)} \right) \frac{1}{h} = \lim_{h \rightarrow 0} \frac{-h}{3(3+h)} \frac{1}{h} = \lim_{h \rightarrow 0} \frac{-1}{3(3+h)} = -\frac{1}{3(3+0)} = -\frac{1}{9}\end{aligned}$$

3. This question calls for the squeeze theorem. We need to separate the function into two parts, one of which is bounded and the other of which tends to 0 (“vanishes”) as $x \rightarrow 0^+$. We have $\sin^2(2\pi/x) \geq 0$ because squares are always nonnegative, and $\sin^2(2\pi x) \leq 1$ because the sine of anything is between -1 and 1 , so we can write

$$0 \leq \sin^2(2\pi/x) \leq 1 \implies 1 \leq 1 + \sin^2(2\pi/x) \leq 2$$

which gives our bounded part. On the other hand, $\sqrt{x} \rightarrow 0$ as $x \rightarrow 0^+$ which gives our “vanishing” part, and multiplying the above inequality by \sqrt{x} we can write

$$1\sqrt{x} \leq \sqrt{x}(1 + \sin^2(2\pi/x)) \leq 2\sqrt{x}$$

Note that $\sqrt{x} \geq 0$ for $x > 0$ so the above inequality does not ever reverse. Now since $\sqrt{x} \rightarrow 0$ as $x \rightarrow 0^+$, and similarly $2\sqrt{x} \rightarrow 0$ as $x \rightarrow 0^+$, the squeeze theorem tells us that the part squeezed in the middle of the above inequality also $\rightarrow 0$ as $x \rightarrow 0^+$, so we conclude

$$\lim_{x \rightarrow 0^+} \sqrt{x}(1 + \sin^2(2\pi/x)) = 0$$

4. Just substituting $x = 2$ into the function gives the expression $0/0$ so we must do some more work before we can get an answer. In expressions involving radicals (square roots) we often find it helpful to multiply by the conjugate radical. In this case there are two radicals so we multiply by them both, one at a time. We have

$$\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} = \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} \cdot \frac{\sqrt{6-x}+2}{\sqrt{6-x}+2} = \frac{6-x-4}{(\sqrt{3-x}-1)(\sqrt{6-x}+2)} = \frac{2-x}{(\sqrt{3-x}-1)(\sqrt{6-x}+2)}$$

Repeating the procedure with the second conjugate radical,

$$\begin{aligned}\frac{2-x}{(\sqrt{3-x}-1)(\sqrt{6-x}+2)} &= \frac{2-x}{(\sqrt{3-x}-1)(\sqrt{6-x}+2)} \cdot \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1} \\ &= \frac{(2-x)(\sqrt{3-x}+1)}{(3-x-1)(\sqrt{6-x}+2)} = \frac{(2-x)(\sqrt{3-x}+1)}{(2-x)(\sqrt{6-x}+2)} = \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\end{aligned}$$

Taking the limit we have

$$\lim_{x \rightarrow 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} = \lim_{x \rightarrow 2} \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{\sqrt{3-2}+1}{\sqrt{6-2}+2} = \frac{2}{4} = \frac{1}{2}$$