MATH 110 Lecture 2.3

Differentiation Formulas

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Contents

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- We would like to develop a similar strategy for evaluating derivatives.

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- The derivatives of constant functions and of the identity function will be the building blocks for most of the derivative calculations in this section.
- There are other bottom-level derivatives that we will learn later. Each time we introduce a new kind of function (e.g., sin, cos, exp, log, ...) there will be a new bottom-level derivative that will have to be calculated from first principles.

Sums, Differences, and Constant Multiples for Derivatives

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- The derivative of a constant multiple is the constant multiple of the derivative.

$$(cf)' = cf' \quad \text{or} \quad \frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

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- Now we apply the sum rule for limits to show the latter expression is the same as $f'(x)+g'(x)$.

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- What is the correct rule?

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- To understand what the derivative of a product should be, consider the meaning of the derivative. If I change x by a small amount dx, this is what happens to f:

$$f(x+dx) = f(x) + f'(x)dx + \text{terms in } dx^2, \text{ etc.}$$

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- Now consider what happens to the product $f(x)g(x)$ when we change x by a small amount dx:

$$\begin{aligned} f(x+dx)g(x+dx) &= (f(x)+f'(x)dx+\cdots)(g(x)+g'(x)dx+\cdots) \\ &= f(x)g(x) + \left(f'(x)g(x)+f(x)g'(x)\right) dx + \cdots \end{aligned}$$

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- This tells us that $(f(x)g(x))'$ should be $f'(x)g(x)+f(x)g'(x)$.

The Product Rule for Derivatives

- In summary, we have the formula in Leibniz notation

$$\frac{d}{dx}(f(x)g(x)) = \left(\frac{d}{dx}f(x)\right)g(x) + f(x)\left(\frac{d}{dx}g(x)\right)$$

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- Note that the derivative of a product **is not** the product of the derivatives!
- Derivatives differ from limits in this regard.

Discovering the Quotient Rule for Derivatives

- In a similar way, the quotient rule for derivatives is not what we might naively expect. Consider $f(x) = x$, $g(x) = x$. Then $(f/g)' = 0$ but $f'/g' = 1$.

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- The mantra "bottom times derivative of top minus top times derivative of bottom over bottom squared" may help you remember the quotient rule.

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- Continuing the pattern, we have $\frac{d}{dx}x^n = nx^{n-1}$ for $n = 1, 2, \ldots$.
- That result is found in a more complicated way in the textbook.

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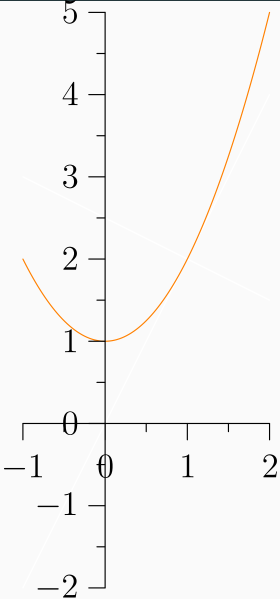
- The case $n = 0$ is special, but the result still holds.
- In general, $\dfrac{d}{dx}x^p = px^{p-1}$ for any real number p, but the proof is too complicated to give now.

Tangent Lines Revisited

- Now we can more easily solve derivative questions.



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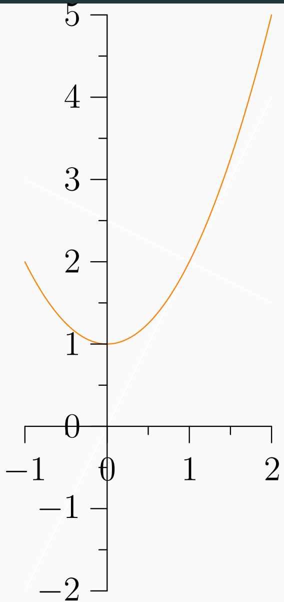

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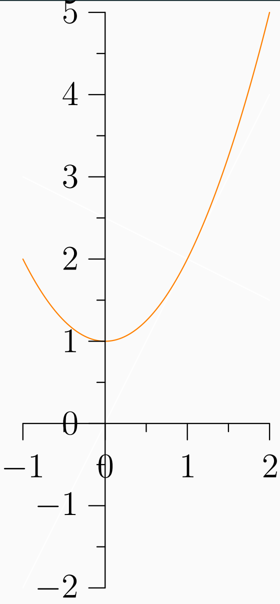

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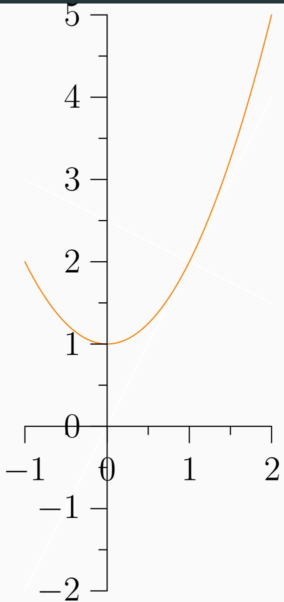

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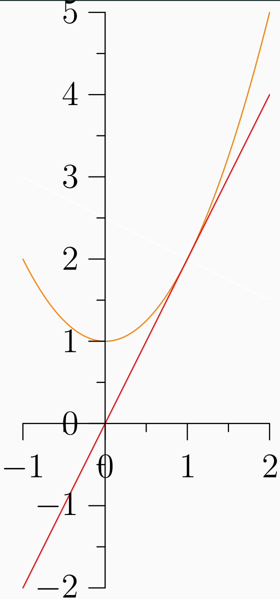
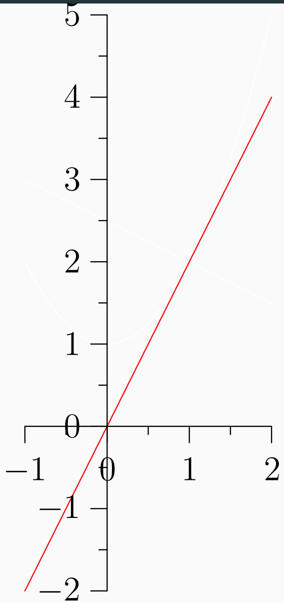
- So $f'(a) = f'(1) = 2(1) = 2$, and the tangent line is $y - 2 = 2(x - 1)$.



- We will need to find lines which are **perpendicular** to a tangent line.



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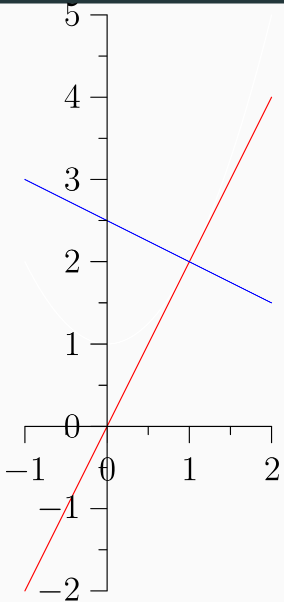

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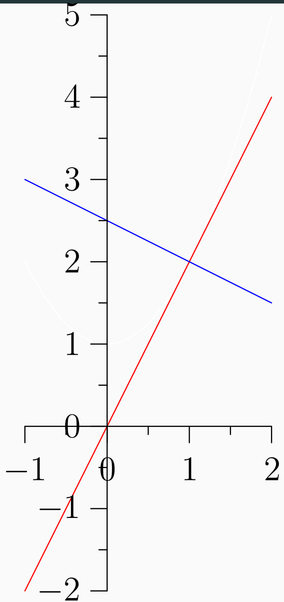

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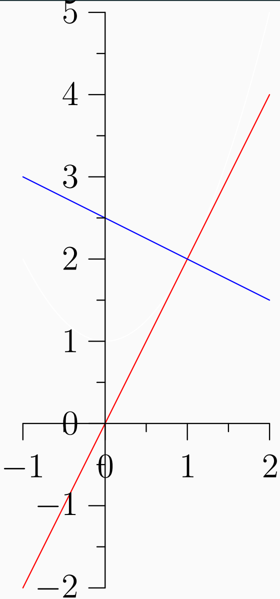
- Given the slope m_1 of a line, the slope m_2 of any perpendicular line is the **negative reciprocal** of m_1: $m_2 = -\dfrac{1}{m_1}$



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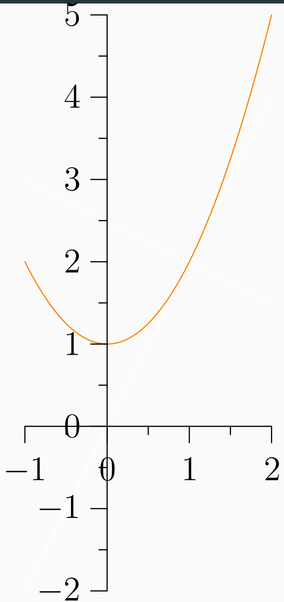

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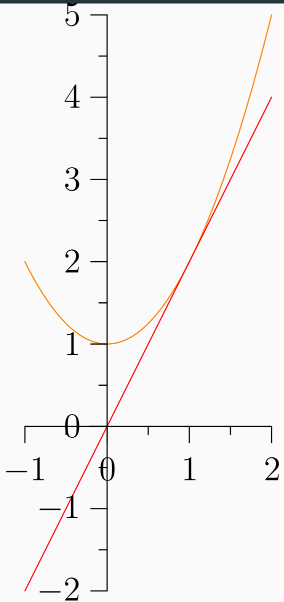

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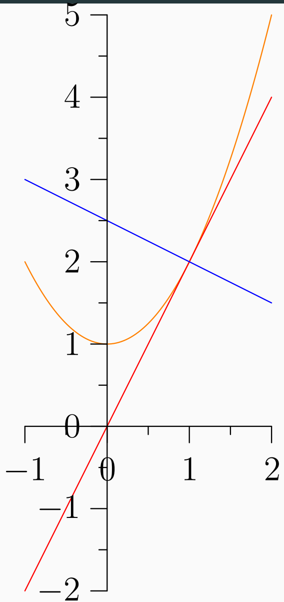
- The normal line in point-slope form is $y - f(a) = m_2(x - a)$, or filling in the blanks,



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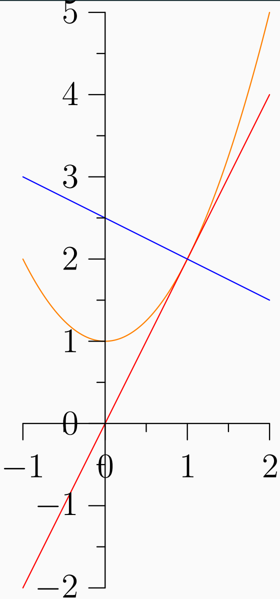

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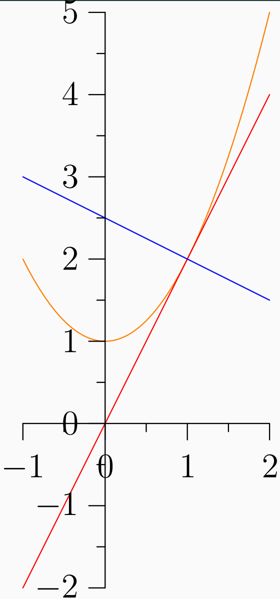
- The normal line in point-slope form is $y - f(a) = m_2(x - a)$, or filling in the blanks, $y - 2 = -\frac{1}{2}(x - 1)$.
- The normal line to $y = f(x)$ at $P(a, f(a))$ is $y - f(a) = -\dfrac{1}{f'(a)}(x - a)$.



Examples

1. Calculate y' where
    1.1 $y = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^4}}$
    1.2 $y = \dfrac{3x-2}{\sqrt{x}}$.
    1.3 $y = \dfrac{\sqrt{x}}{3x-2}$.
2. Find the equations of the tangent and normal lines to the curve $y = \dfrac{x-1}{x+1}$ at the point $(0,-1)$.
3. Find the equations of the tangent and normal lines to the curve $y = \dfrac{x^2-1}{x^2+1}$ at the point $(0,-1)$.

Solution to Example 3

- In this example we have
$y' = \dfrac{(x^2+1)2x-(x^2-1)2x}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$, so $y'(0)=0$.
- The equation of the tangent line is easy enough to find:
$y-f(a)=f'(a)(x-a)$ where $a=0, f(a)=-1$, and $f'(a)=0$
gives $y-(-1)=0(x-0)$ or in other words $y=-1$.
- The equation of the normal is not so clear in this case. Since the slope of the tangent is 0, the slope of the normal would be $-1/0$ which doesn't make any sense. However, it should seem clear that a line perpendicular to the horizontal line $y=-1$ is a vertical line. Since the vertical line must pass through $(0,1)$, the equation of the normal line is $x=0$. Draw it!

Exercises

Now you should work on Problem Set 2.3. After you have finished it, you should try the following additional exercises from Section 2.3:

2.3 C-level: 1–22, 25–44, 51–52, 53–54, 55–58, 59–62, 63–66
B-level: 23–24, 45, 46–48, 49–50, 67, 68, 69–72, 73–74, 75–76, 77–82, 83–86, 93–96,
A-level: 87, 88–92, 97–100, 101–110