MATH 110 Lecture 1.5

The Limit of a Function

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Department of Indigenous Knowledge and Science
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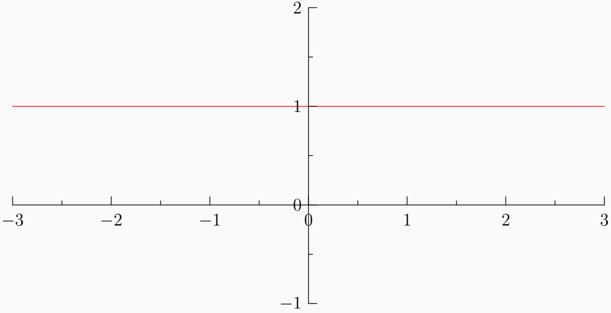
Contents

The Limit of a Function

Functions with Holes

Consider the graph of the function

$$f(x) = 1$$



$f(x) = 1$

Every time I give an input to f, the function answers back with the number 1. So

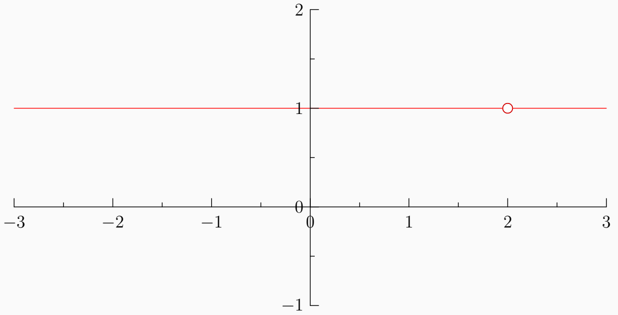
- $f(7) = 1$
- $f(0.5) = 1$
- $f(1) = 1$
- $f(-\sqrt{2}) = 1$

and so on. Simple.

Undefined values

Now consider the similar graph of the function

$$g(x) = \begin{cases} 1, & x \neq 2 \\ \text{undefined} & x = 2 \end{cases}$$



$g(x)$

Most of the time, when I give an input to g, the function replies with the output 1. So,

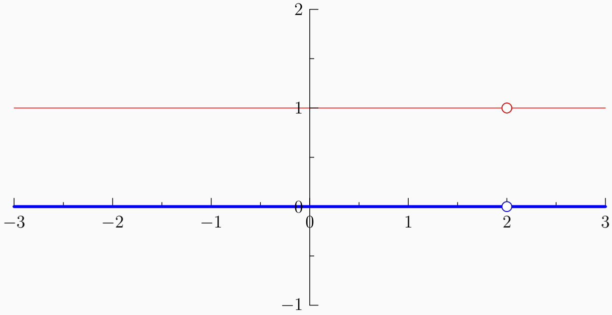
- $g(7) = 1$
- $g(0.5) = 1$
- $g(1) = 1$
- $g(-\sqrt{2}) = 1$

as with f. However, if I give the input 2 to g, it doesn't respond at all.

Domain of a Function

We say that the domain of g doesn't include 2; more precisely, the domain of g is the set

$$\{x \in \mathbb{R} : x \neq 2\}$$



Defining an Undefined Value

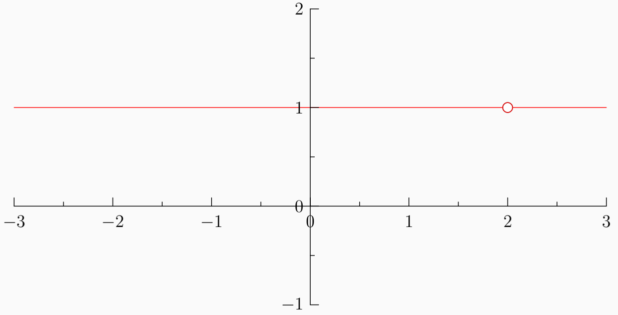
If someone were to insist on us assigning a value to $g(2)$, we can say two things:

- Strictly speaking, $g(2)$ is undefined!
- However, if you **insist**, the only value that makes sense for $g(2)$ is $g(2) = 1$.

The Concept of a Limit

The two statements on the previous slide are inconsistent. We can't have that in mathematics, so we work around it by saying

- $g(2)$ is undefined, but
- $\lim_{x\to 2} g(x) = 1$



Another Example of Limits

Consider the function

$$h(x) = \frac{x(x-3)}{x-3}$$

The domain of the function is $\{x \in \mathbb{R} : x \neq 3\}$. So the answer to the question "What is $h(3)$?" is, "It's undefined". But now, we can say more.

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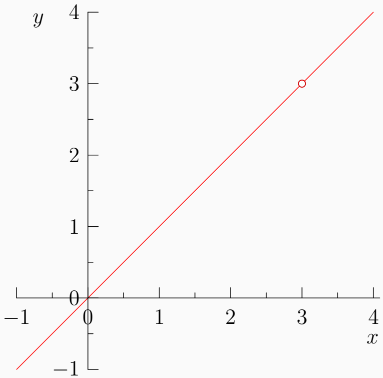

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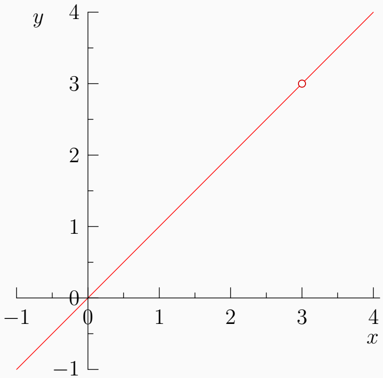
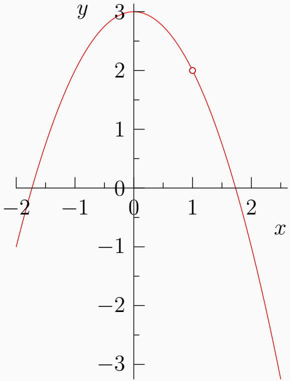
The domain of the function is $\{x \in \mathbb{R} : x \neq 3\}$. So the answer to the question "What is $h(3)$?" is, "It's undefined". But now, we can say more. From the graph,

$$\lim_{x \to 3} h(x) = 3$$



Three More Examples of Limits

Consider the function $f(x)$ represented by the graph on the right.
For this function, we say that

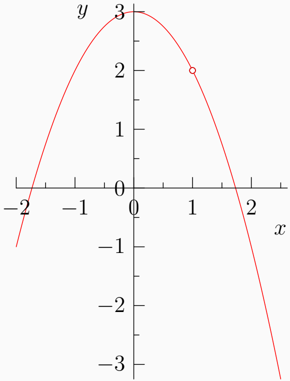


Three More Examples of Limits

Consider the function $f(x)$ represented by the graph on the right.

For this function, we say that

$$\lim_{x\to 1} f(x) = 2$$



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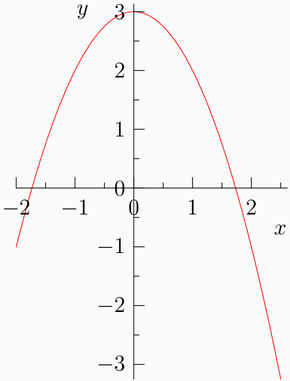

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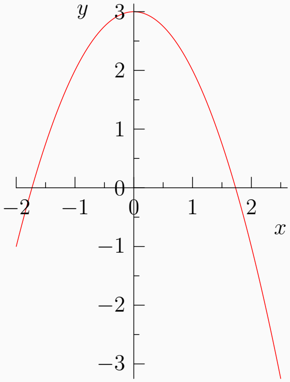
We can say that $f(1) = 2$, but can we say anything about limits?



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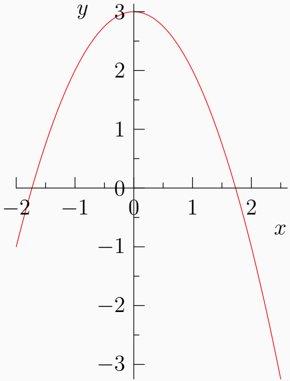
Yes, in this case, we can still say

$$\lim_{x \to 1} f(x) = 2$$



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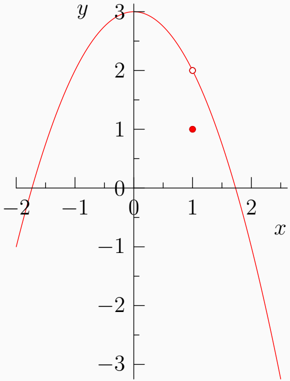

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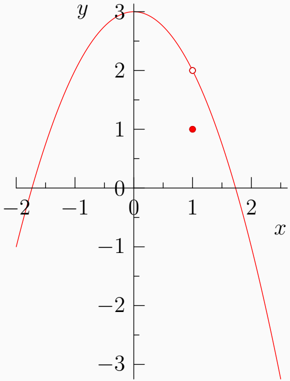
And what about this function?

In this case, we say that $f(1) = 1$, but can we say anything about limits?



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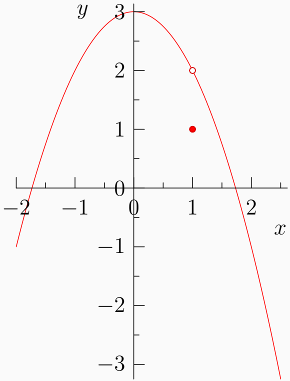
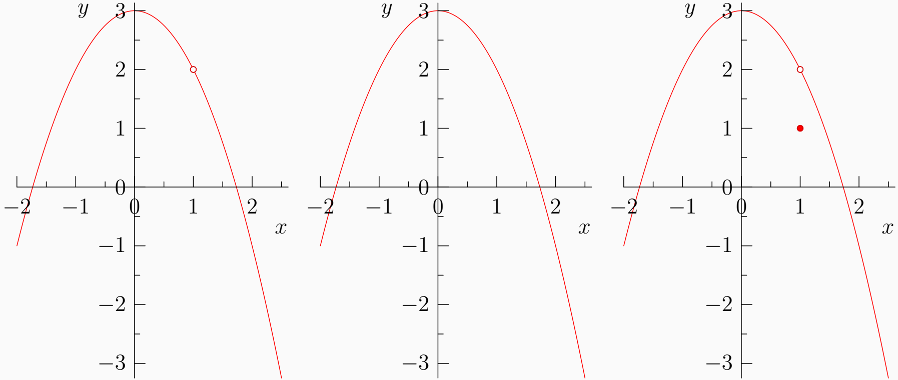


The Three Situations in which Limits Exist

The following three graphs illustrate the three situations in which we can talk about limits.



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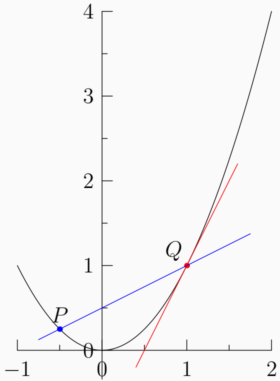
- You may be wondering whether this idea of limits is really very useful.
- But we have already seen an example where limits can help us:
- the tangent problem.
- Limits can also help us solve the area problem, as we will see in chapter 5.

Limits and the Tangent Problem

- To see how limits can help us solve the tangent problem, consider the graph of $f(x) = x^2$ at the right.



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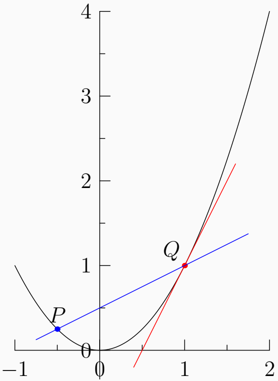

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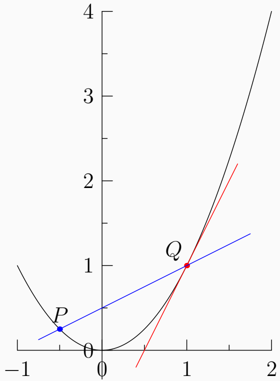

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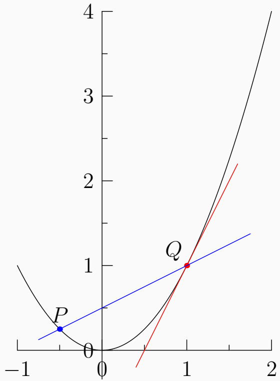
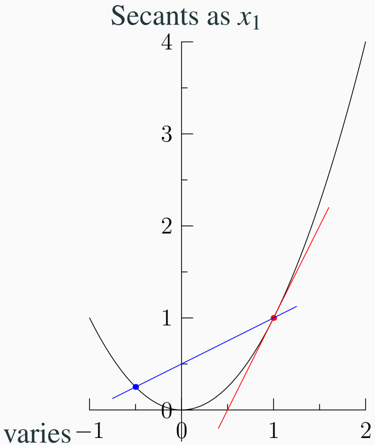
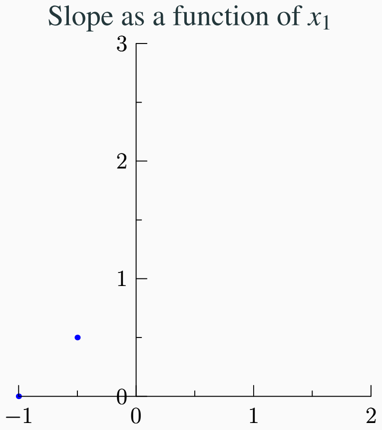
- We want to find the slope of the tangent line at the base point $Q = (1, 1)$.
- To find it, we find the slopes of the secant lines PQ, where $P \neq Q$.
- Let's graph the slope of the secant line as a function of the x-value of the second point P.



The Tangent Problem for $f(x) = x^2$





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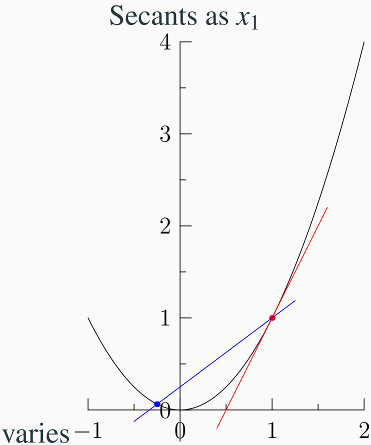

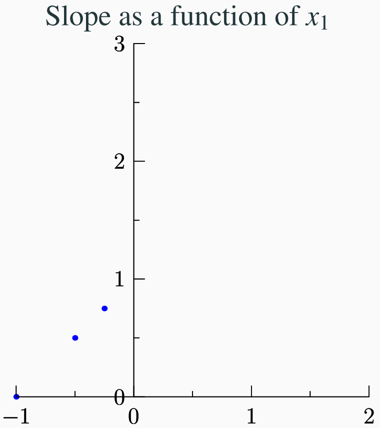

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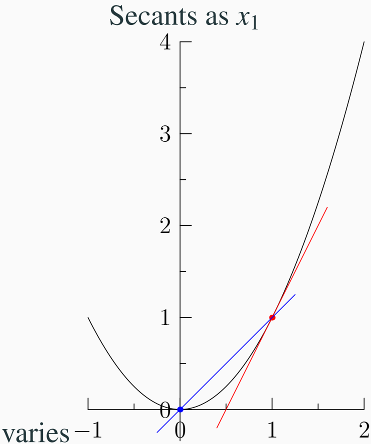
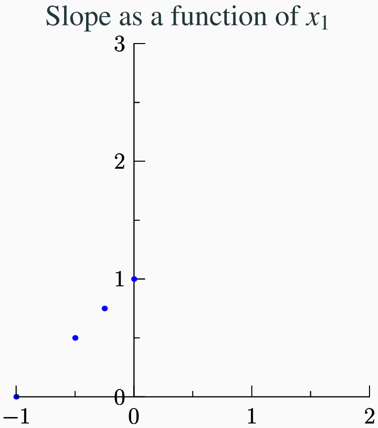
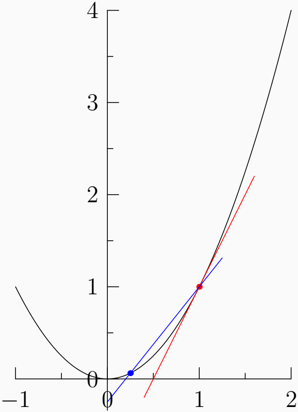


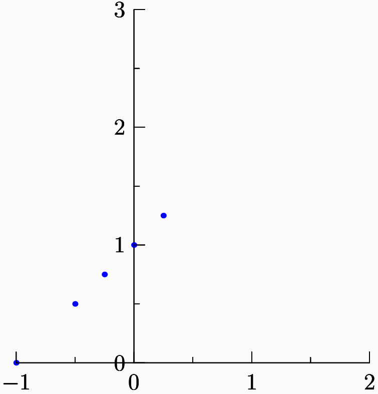


The Tangent Problem for $f(x) = x^2$

Secants as x_1 varies



Slope as a function of x_1



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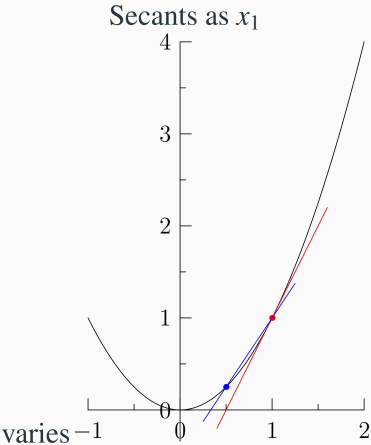

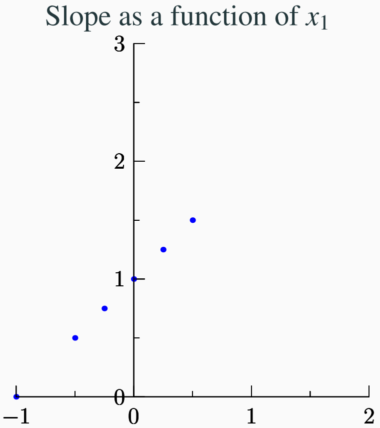

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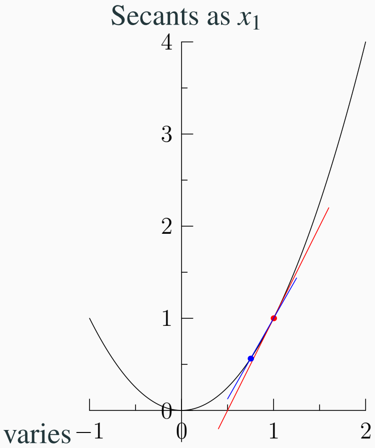

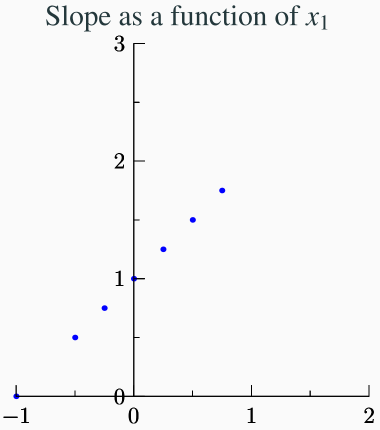

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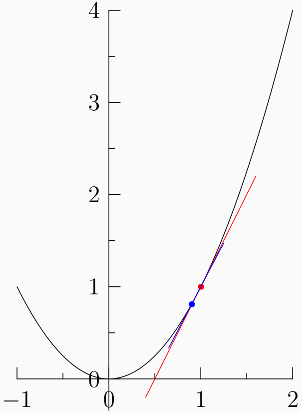
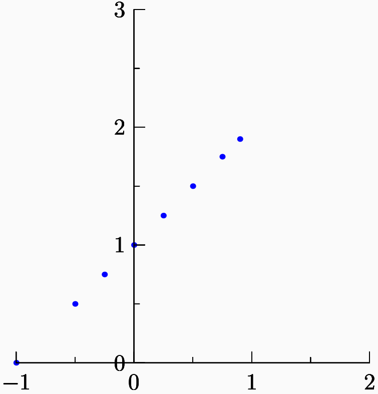
Secants as x_1



varies

Slope as a function of x_1



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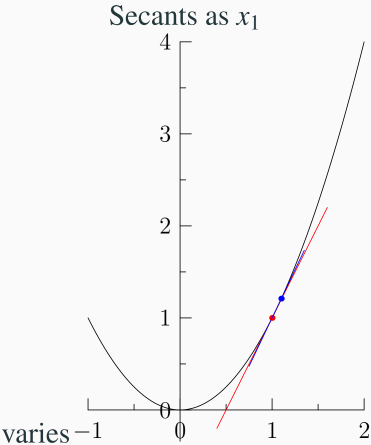

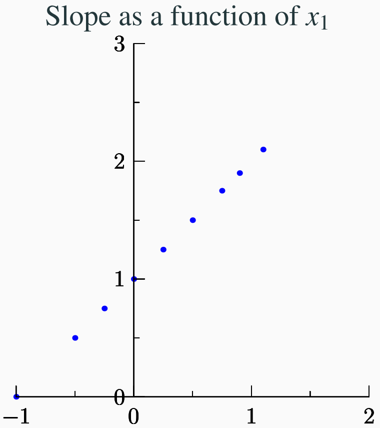

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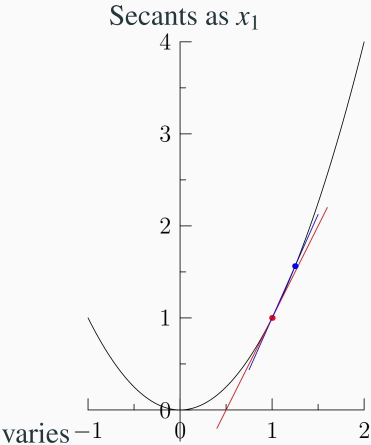

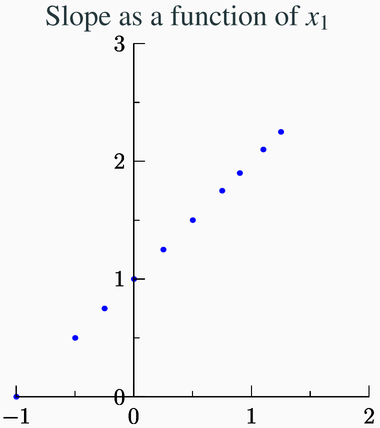

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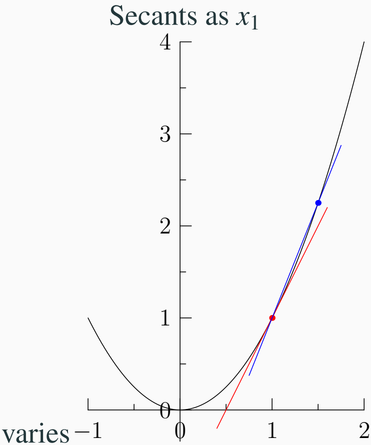

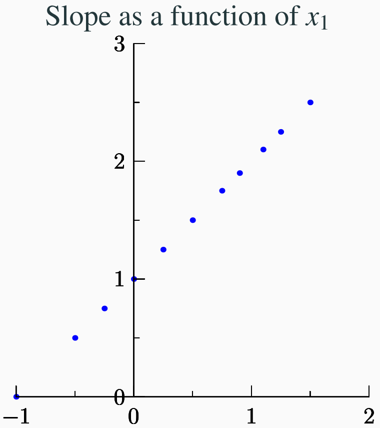

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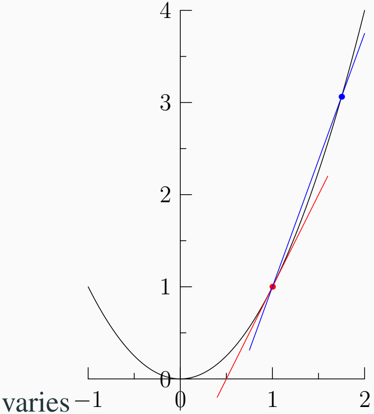

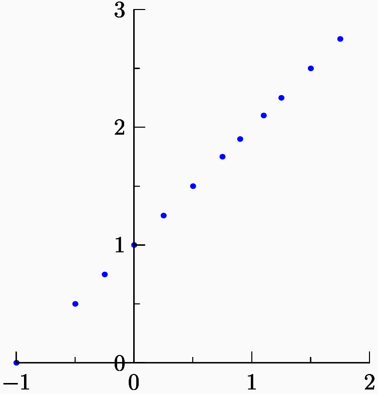

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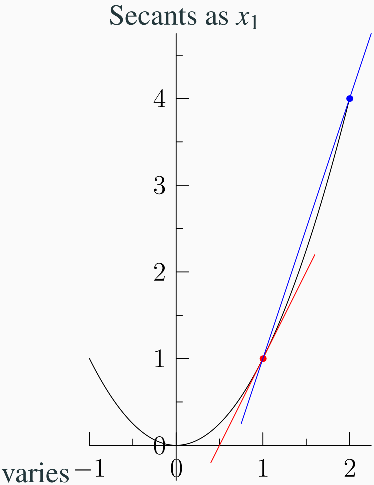

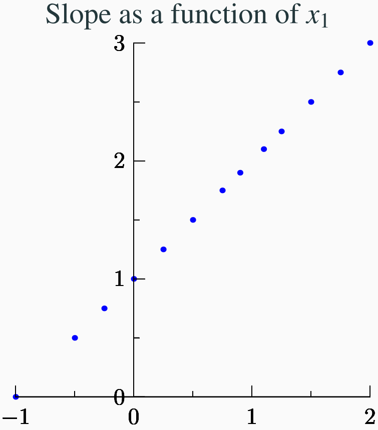

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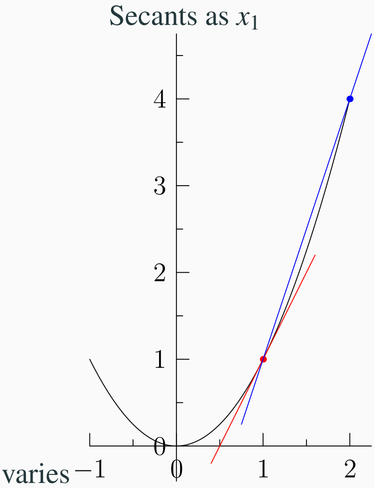
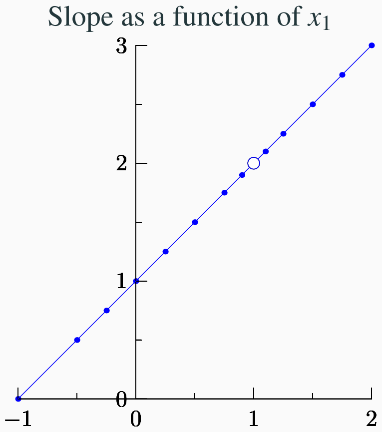
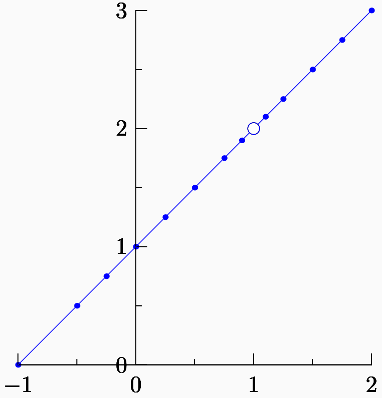




The Tangent Problem for $f(x) = x^2$

- The slope of the secant line appears to be a nice linear function of the x-value of P.
- However, there is one x-value of P which is not allowed: $x = 1$, where $P = Q$.
- But that is the most interesting x-value!
- We need to use limits to find slopes of tangents.

Slope as a function of x_1



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- See examples 1–5 in the textbook.
- Note, however, that those methods are not 100% reliable. We need reliable ways of calculating limits.
- We will learn more reliable methods for calculating limits in section 1.6.

The Heaviside Step Function

Consider the function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

This function was studied by electrical engineer Oliver Heaviside as a model for throwing a switch in an electric circuit.

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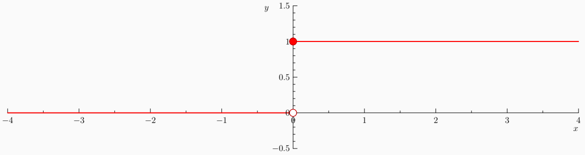

Limits for the Heaviside Step Function

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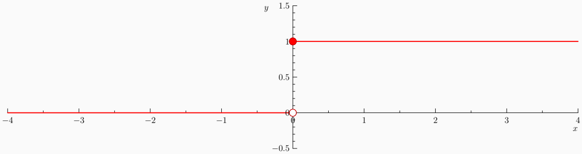

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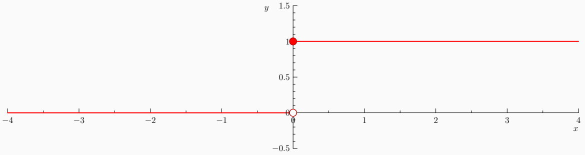

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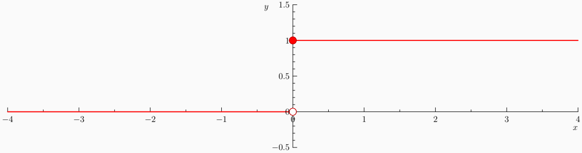

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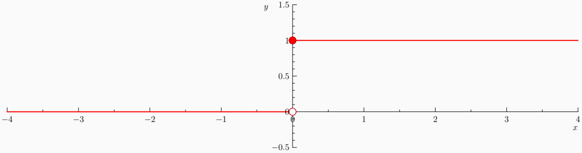
- But what about $\lim_{x \to 0} H(x)$?
- Since there is no sensible value we could give $H(0)$, we must say $\lim_{x \to 0} H(x)$ is **undefined**.



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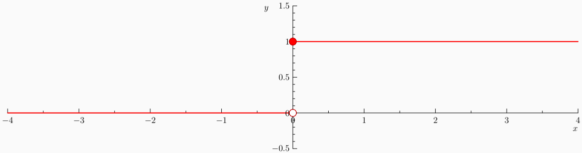

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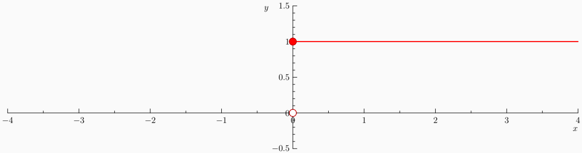

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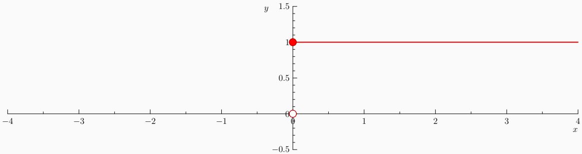

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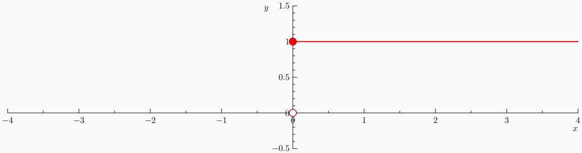

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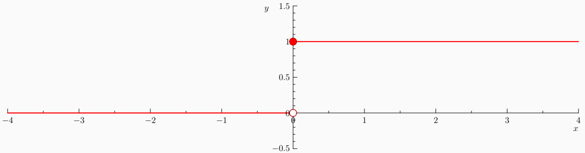

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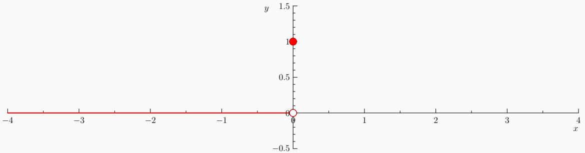

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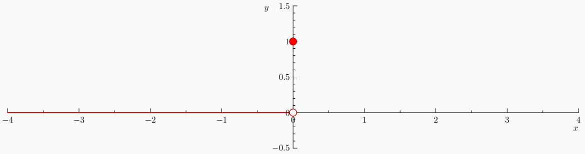

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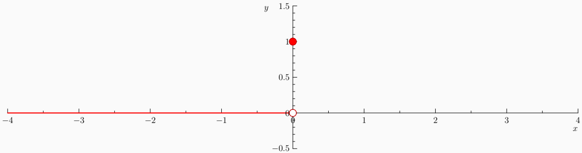

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- So one way to show that a (two-sided) limit does not exist is to show that the corresponding one-sided limits are not equal.

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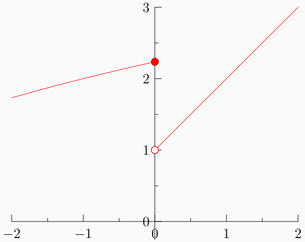

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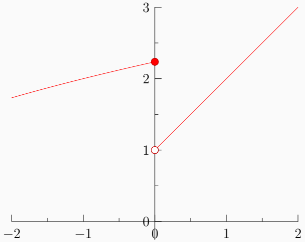

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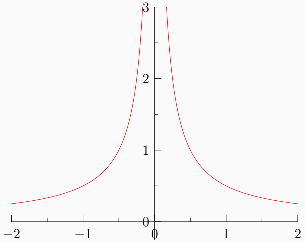
- Another way for a limit to fail to exist is where f gets really large as x approaches some value.



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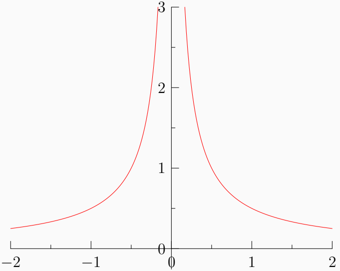

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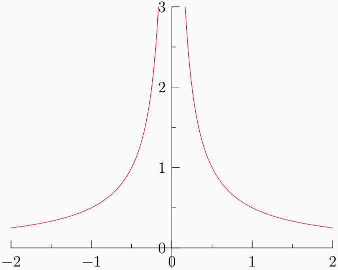
- Writing this as a limit expression just means that the limit fails to exist in a particular way.

Negative and One-sided Infinite Limits

There are six variations on infinite limits.



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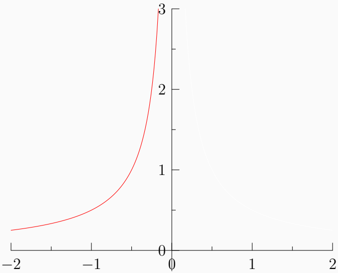

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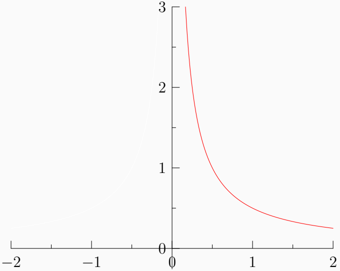

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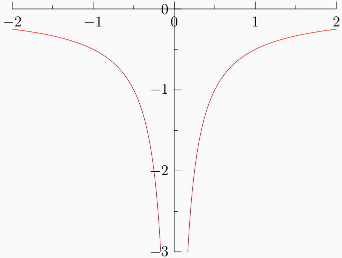

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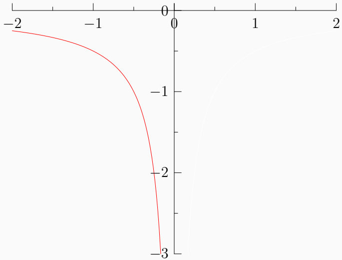

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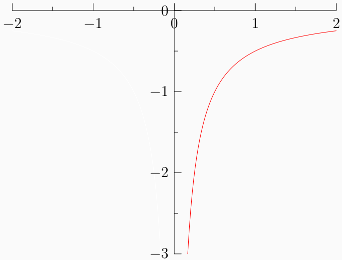

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- Infinite limits may occur at the values a where $h(a) = 0$. Notice that these values a are not in the domain of f.
- We determine which type of limit occurs at a by investigating the size and sign of f on each side of a.

Vertical Asymptotes

The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \quad \lim_{x\to a^-} f(x) = \infty \quad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \quad \lim_{x\to a^-} f(x) = -\infty \quad \lim_{x\to a^+} f(x) = -\infty$$

In other words, if any of the limits as x approaches a from the left or the right is $\pm\infty$, we say that $y = f(x)$ has a vertical asymptote at a.

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- A more rigourous approach to limits was formulated by Cauchy.
- Cauchy's definition is based on the idea of controlling the range of values a function can take by limiting the domain.
- Cauchy's definition is still somewhat vague. A completely satisfactory definition is Weirstrass's epsilon-delta definition, which can be found in section 1.7. We will not be studying the epsilon-delta definition in this course.

The Definition of an Ordinary Limit

- We write

  $$\lim_{x \to a} f(x) = L$$

  and say

  the limit as x approaches a of $f(x)$ is L

  if we can make the values of $f(x)$ as close to L as we like by taking x to be sufficiently close to **(but not equal to)** a.

The Definition of a One-sided Limit

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- There is a similar definition for the right-hand limit $\lim_{x \to a^+} f(x) = L$. (Try writing out the definition yourself.)

The Definition of an Infinite Limit

- Let f be a function defined on both sides of a, except possibly at a itself. Then we can write

$$\lim_{x \to a} f(x) = \infty$$

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$f(x)$ increases without bound as x approaches a

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- There are similar definitions for $\lim_{x \to a} f(x) = -\infty$ and for the corresponding one-sided limits. (Try writing out the definitions yourself.)

Examples

1. Sketch the graph of the following function and use it to determine the values of a for which $\lim_{x\to a} f(x)$ exists:

$$f(x) = \begin{cases} 2 - x & \text{if } x < -1 \\ x & \text{if } -1 \le x < 1 \\ (x-1)^2 & \text{if } 1 \le x \end{cases}$$

2. Use a table of values to estimate the value of the following limits.
   2.1 $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$
   2.2 $\displaystyle\lim_{x\to 0} \frac{\tan 3x}{\tan 5x}$

3. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leq x < 3 \\ (x-3)^2 & \text{if } 3 \leq x \end{cases}$$

   Determine whether the limits $\lim_{x \to 0} f(x)$ and $\lim_{x \to 3} f(x)$ exist. If so, find the values of those limits.

4. Find the vertical asymptotes of the function

$$y = \frac{x}{x^2 + x - 2}$$

Exercises

Now you should work on Problem Set 1.5a and Problem Set 1.15b. After you have finished them, you should try the following additional exercises from Section 1.5 for two-sided limits:

1.5 C-level: 1, 19–22, 23, 26, 35;
B-level: 32–44;
A-level: 49

and the following additional exercises from Section 1.5 for one-sided and infinite limits:

1.5 C-level: 2–3, 4–5, 7, 8–9, 10, 11–12, 15–18, 24–25, 29–34, 38–39, 40;
B-level: 6, 11, 13–14, 35–37, 41, 46;
A-level: 47–48