MATH 110 Problem Set 2.6 Solutions

Edward Doolittle

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1. (a) Differentiating both sides of the equation,

$$\begin{aligned}\frac{d}{dx}(x^2+xy-y^2) &= \frac{d}{dx}4\\ \frac{d}{dx}x^2+\frac{d}{dx}xy-\frac{d}{dx}y^2 &= 0\end{aligned}$$

We apply the power rule to the first term, the product rule to the second, and the power-chain rule to the third:

$$\begin{aligned}2x+\left(\frac{d}{dx}x\right)y+x\left(\frac{d}{dx}y\right)-2y\frac{d}{dx}y &= 0\\ 2x+y+x\frac{dy}{dx}-2y\frac{dy}{dx} &= 0\end{aligned}$$

Solving for dy/dx,

$$\begin{aligned}x\frac{dy}{dx}-2y\frac{dy}{dx} &= -2x-y\\ (x-2y)\frac{dy}{dx} &= -2x-y\\ \frac{dy}{dx} &= \frac{-2x-y}{x-2y}\end{aligned}$$

(b) Write the roots in terms of powers and differentiate both sides of the equation:

$$\begin{aligned}x^{1/2}+2y^{1/2} &= 3\\ \frac{d}{dx}(x^{1/2}+2y^{1/2}) &= \frac{d}{dx}3\\ \frac{d}{dx}x^{1/2}+2\frac{d}{dx}y^{1/2} &= 0\end{aligned}$$

By the power rule and the power-chain rule,

$$\frac{1}{2}x^{-1/2}+2\frac{1}{2}y^{-1/2}\frac{dy}{dx}=0$$

Solving for dy/dx,

$$\begin{aligned}y^{-1/2}\frac{dy}{dx} &= -\frac{x^{-1/2}}{2}\\ \frac{dy}{dx} &= -\frac{x^{-1/2}}{2y^{-1/2}}\end{aligned}$$

The following simplification is nice but not necessary:

$$\frac{dy}{dx} = -\frac{y^{1/2}}{2x^{1/2}}$$
$$\frac{dy}{dx} = -\frac{\sqrt{y}}{2\sqrt{x}}$$

(c) Differentiating both sides,

$$\frac{d}{dx}\left(x^3(x-y)\right) = \frac{d}{dx}\left(y^2(x+3y)\right)$$
$$\left(\frac{d}{dx}x^3\right)(x-y) + x^3\left(\frac{d}{dx}(x-y)\right) = \left(\frac{d}{dx}y^2\right)(x+3y) + y^2\left(\frac{d}{dx}(x+3y)\right)$$

where we have used the product rule. By the power rule, difference rule, power-chain rule, and sum rule,

$$3x^2(x-y) + x^3\left(1 - \frac{dy}{dx}\right) = 2y\frac{dy}{dx}(x+3y) + y^2\left(1 + 3\frac{dy}{dx}\right)$$

Solving for dy/dx,

$$3x^2(x-y) + x^3 - x^3\frac{dy}{dx} = 2y(x+3y)\frac{dy}{dx} + y^2 + 3y^2\frac{dy}{dx}$$
$$(-x^3 - 2xy - 6y^2 - 3y^2)\frac{dy}{dx} = y^2 - 3x^3 + 3x^2y - x^3$$
$$\frac{dy}{dx} = \frac{-4x^3 + y^2 + 3x^2y}{-x^3 - 2xy - 9y^2}$$

2. (a) Differentiating both sides,

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

Solving for dy/dx,

$$(x+2y)\frac{dy}{dx} = -2x - y \implies \frac{dy}{dx} = \frac{-2x-y}{x+2y}$$

At the point $(x, y) = (1, 2)$ we have

$$\frac{dy}{dx} = \frac{-2x-y}{x+2y} = \frac{-2(1)-(2)}{(1)+2(2)} = \frac{-4}{5}$$

The slope of the tangent line is $-4/5$, so the equation of the tangent line is

$$y - y_0 = m(x - x_0) \implies y - 2 = -\frac{4}{5}(x-1)$$

(b) Differentiating both sides,

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \implies x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

Solving for dy/dx,

$$y^{-1/3}\frac{dy}{dx} = -x^{-1/3} \implies \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$$

At the point $\left(-\frac{64}{27}, -1\right)$ we have

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3} = -\left(\frac{-64/27}{-1}\right)^{1/3} = -\left(\frac{64}{27}\right)^{1/3} = -\frac{4}{3}$$

So an equation for the tangent line is

$$y + 1 = -\frac{4}{3}\left(x + \frac{64}{27}\right)$$

3. Differentiating both sides,

$$\begin{aligned} \frac{d}{dx}2(x^2+y^2)^2 &= \frac{d}{dx}25(x^2-y^2) \\ 2\frac{d}{dx}(x^2+y^2)^2 &= 25\frac{d}{dx}(x^2-y^2) \end{aligned}$$

By the chain rule we have

$$2 \cdot 2(x^2+y^2) \cdot \frac{d}{dx}(x^2+y^2) = 25\frac{d}{dx}(x^2-y^2)$$

By the sum and difference rules, the power rule, and the power-chain rule,

$$4(x^2+y^2)\left(2x + 2y\frac{dy}{dx}\right) = 25\left(2x - 2y\frac{dy}{dx}\right)$$

Expanding and gathering,

$$\begin{aligned} 8x(x^2+y^2) + 8y(x^2+y^2)\frac{dy}{dx} &= 50x - 50y\frac{dy}{dx} \\ 8y(x^2+y^2)\frac{dy}{dx} + 50y\frac{dy}{dx} &= 50x - 8x(x^2+y^2) \\ \frac{dy}{dx} &= \frac{50x - 8x(x^2+y^2)}{8y(x^2+y^2)+50y} \end{aligned}$$

At the point $(x, y) = (3, -1)$ we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{50(3) - 8(3)(3^2+(-1)^2)}{8(-1)(3^2+(-1)^2)+50(-1)} \\ &= \frac{150 - 24(9+1)}{-8(9+1)-50} = \frac{150-240}{-80-50} = \frac{-90}{-130} \\ &= \frac{9}{13} \end{aligned}$$

It follows that the equation of the tangent line is

$$y - y_0 = m(x - x_0) \implies y + 1 = \frac{9}{13}(x-3)$$

and the equation of the normal line is

$$y - y_0 = -\frac{1}{m}(x - x_0) \implies y + 1 = -\frac{13}{9}(x-3)$$

4. (a) Differentiating both sides,

$$8x + 18yy' = 0 \implies 9yy' = -4x \implies y' = -\frac{4x}{9y}$$

Differentiating again,

$$y'' = -\frac{4(9y) - 4x(9y')}{y^2} = -\frac{36y - 36xy'}{y^2}$$

Substituting y' from above,

$$y'' = -\frac{36y - 36x\left(-\frac{4x}{9y}\right)}{y^2}$$

Further simplification is nice, but is not necessary:

$$y'' = -\frac{36y + 16\frac{x}{y}}{y^2} = -\frac{36y^2 + 16x^2}{y^3} = -\frac{4(9y^2 + 4x^2)}{y^3} = -\frac{4(36)}{y^3}$$

where we have used the equation $4x^2 + 9y^2 = 36$ to simplify the above expression.

(b) (Note that a is a constant.) Differentiating both sides,

$$3x^2 + 3y^2y' = 0 \implies y' = -\frac{x^2}{y^2}$$

Differentiating again by the quotient rule,

$$y'' = -\frac{2xy^2 - x^2 \cdot 2yy'}{y^4}$$

Substituting y' from above,

$$y'' = -\frac{2xy^2 - 2x^2yy'}{y^4} = -\frac{2xy^2 - 2x^2y\left(-\frac{x^2}{y^2}\right)}{y^4}$$

Further simplification is possible (but not necessary). Multiplying through by y^2 to clear fractions,

$$y'' = -\frac{2xy^4 - 2x^2y(-x^2)}{y^6} = -\frac{2xy(y^3 + x^3)}{y^6} = -\frac{2xa^3}{y^5}$$

where we have used $x^3 + y^3 = a^3$ to simplify.

5. For this problem, we need the results

$$(\sin^{-1})'(u) = \frac{1}{\sqrt{1-u^2}} \text{ and } (\tan^{-1})'(u) = \frac{1}{1+u^2}$$

(a) By the chain rule,

$$\frac{d}{dx}\sin^{-1}\sqrt{x} = (\sin^{-1})'(\sqrt{x}) \cdot \frac{d}{dx}\sqrt{x} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

(b) By the chain rule

$$\frac{d}{dx}\sqrt{\sin^{-1}(x)} = \frac{1}{2}(\sin^{-1}(x))^{-1/2} \cdot \frac{1}{\sqrt{1-x^2}}$$

(c) By the chain rule applied to the term $\tan^{-1}(1/t) = \tan^{-1}(t^{-1})$,

$$y' = \frac{1}{1+t^2} + \frac{1}{1+(1/t)^2} \cdot (-1)t^{-2} = \frac{1}{1+t^2} - \frac{1}{t^2+1} = 0$$

(d) By the chain rule,

$$f'(\theta) = \frac{1}{\sqrt{1-(2\sin\theta)^2}} \cdot 2\cos\theta$$

6. Differentiating by the chain and product rules,

$$2f(x)f'(x) + f(x) + xf'(x) = 0$$

Solving for $f'(x)$,

$$(2f(x)+x)f'(x) = -f(x) \implies f'(x) = -\frac{f(x)}{2f(x)+x}$$

Substituting $x = 2$,

$$f'(2) = -\frac{f(2)}{2f(2)+2}$$

Substituting $f(2) = 1$,

$$f'(2) = -\frac{1}{2(1)+2} = -\frac{1}{4}$$

7. Differentiating by d/dy,

$$\frac{d}{dy}(x^3y^2) - \frac{d}{dy}(xy^3) + 3\frac{d}{dy}(xy) = 0$$

By the product rule,

$$\frac{d}{dy}x^3 \cdot y^2 + x^3\frac{d}{dy}y^2 - \frac{d}{dy}x \cdot y^3 - x\frac{d}{dy}y^3 + 3\frac{d}{dy}x \cdot y + 3x\frac{d}{dy}y = 0$$

Since x is dependent on y, the chain rule says

$$\frac{d}{dy}x^3 = 3x^2\frac{dx}{dy}$$

etc., so we have

$$3x^2y^2\frac{dx}{dy} + 2x^3y - y^3\frac{dx}{dy} - 3xy^2 + 3y\frac{dx}{dy} + 3x = 0$$

Solving for dx/dy,

$$(3x^2y^2 - y^3 + 3y)\frac{dx}{dy} = -3x^3y + 3xy^2 - 3x \implies \frac{dx}{dy} = \frac{-3x^3y + 3xy^2 - 3x}{3x^2y^2 - y^3 + 3y}$$

You might compare with the result of finding dy/dx. How are the two results related?

8. Differentiating with respect to x,

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

Solving for y',

$$\frac{yy'}{b^2} = -\frac{x}{a^2} \implies y' = -\frac{b^2x}{a^2y}$$

At a point (x_0, y_0) on the ellipse, the slope of the tangent line is

$$y' = -\frac{b^2 x_0}{a^2 y_0}$$

and the slope of the normal line is therefore the negative reciprocal

$$-\frac{1}{y'} = \frac{a^2 y_0}{b^2 x_0}$$

An equation of the normal line is therefore

$$y - y_0 = \frac{a^2 y_0}{b^2 x_0}(x - x_0)$$

9. The slope of the tangent at (x, y) to one of the parabolas is found by

$$1 = 2cyy' \implies y_p' = \frac{1}{2cy}$$

The slope of the tangent at (x, y) to one of the ellipses is found by

$$4x + 2yy' = 0 \implies y_e' = -\frac{2x}{y}$$

The product of the two slopes is

$$y_p' y_e' = \frac{1}{2cy} \cdot -\frac{2x}{y} = -\frac{x}{cy^2}$$

Since the point (x, y) lies on the parabola $x = cy^2$ we have $cy^2 = x$ so

$$y_p' y_e' = -\frac{x}{cy^2} = -\frac{x}{x} = -1$$

Therefore y_e' is the negative reciprocal of y_p', i.e., the tangents to the two curves are perpendicular.

10. First we find an equation of a tangent line through an arbitrary point (x_0, y_0) on the ellipse. Differentiating,

$$\frac{2x}{3^2} + \frac{2yy'}{6^2} = 0 \implies \frac{yy'}{6^2} = -\frac{x}{3^2} \implies y' = -\frac{6^2 x}{3^2 y} = -\frac{4x}{y}$$

So an equation of the tangent line at (x_0, y_0) is

$$y - y_0 = -\frac{4x_0}{y_0}(x - x_0) \implies y_0 y - y_0^2 + 4x_0 x - 4x_0^2 = 0 \implies 4x_0 x + y_0 y = 4x_0^2 + y_0^2 = 36$$

(the latter equality is a consequence of (x_0, y_0) being on the ellipse). Since the tangent line also passes through $(3, 12)$, the point $(x, y) = (3, 12)$ satisfies the equation of the tangent line, i.e.,

$$12x_0 + 12y_0 = 36$$

We therefore have a system of equations for (x_0, y_0):

$$\begin{aligned} 4x_0^2 + y_0^2 &= 36 \\ 12x_0 + 12y_0 &= 36 \end{aligned}$$

Solving the second equation for y_0,

$$y_0 = 3 - x_0$$

Substituting that into the first equation,

$$4x_0^2 + y_0^2 = 36 \implies 4x_0^2 + (3 - x_0)^2 = 36$$

Expanding and gathering,

$$4x_0^2 + 9 - 6x_0 + x_0^2 = 36 \implies 5x_0^2 - 6x_0 - 27 = 0$$

The equation factors as

$$(5x_0 + 9)(x_0 - 3) = 0 \implies x = -\frac{9}{5} \quad \text{or} \quad x = 3$$

The corresponding y_0 values are respectively $y_0 = 3 + (9/5) = 24/5$ and $y_0 = 3 - 3 = 0$. So the two lines are

$$\begin{aligned} \frac{24}{5}y - \frac{24^2}{5^2} - 4\frac{9}{5}x - 4\frac{9^2}{5^2} &= 0 \\ 0y - 0 - 4(3)x - 4(3^2) &= 0 \end{aligned}$$

You can simplify the equations of the lines.