MATH 110 Review 1.1

Review of Functions

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Department of Indigenous Knowledge and Science
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Four Ways to Represent a Function

Functions: Examples

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- The marginal income tax rate depends on the income level. Although there is no (simple) formula for determining the marginal tax, there is a rule.
- An EKG gives in graphical form some idea of the electrical activity of the heart as a function of time in seconds.

Functions: Formal Definition

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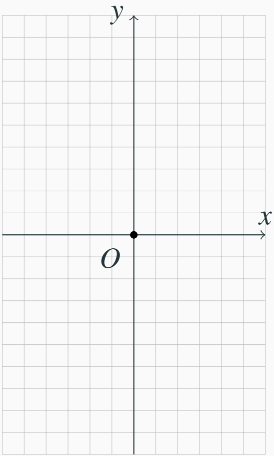
- $y = f(x)$ is called the *dependent variable*.

Functions: Range

- The *range* of a function is the set of all possible values of $y = f(x)$ as x varies over the domain.



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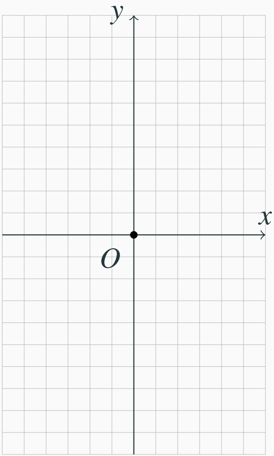

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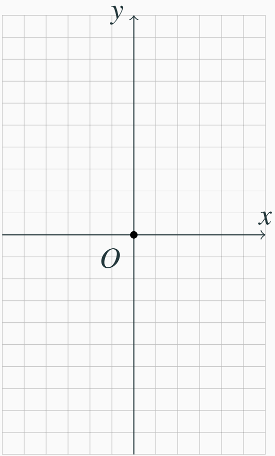

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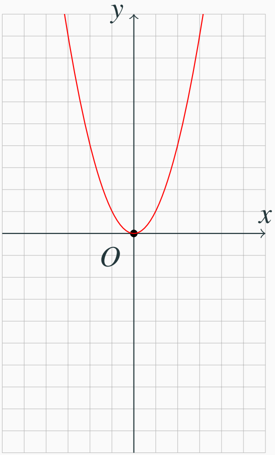

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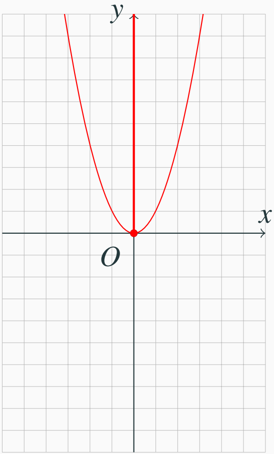
- The graph of the function $y = f(x)$ is the parabola $y = x^2$.
- The range of the function is $y \geq 0$.



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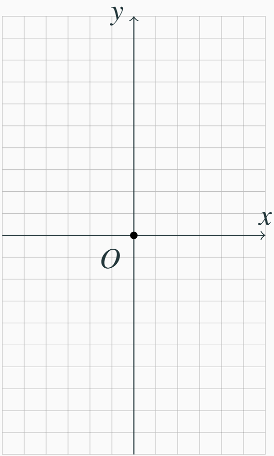

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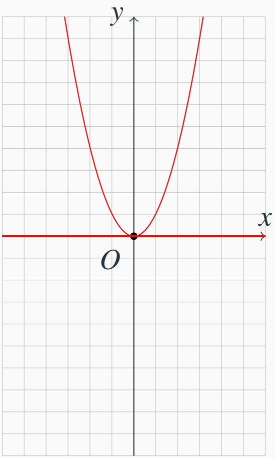

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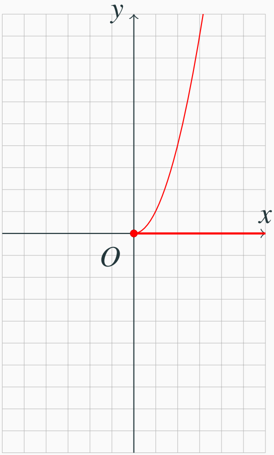

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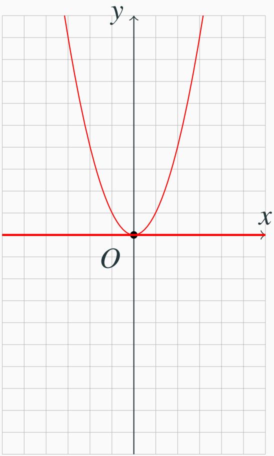
- If the domain isn't given, use the *natural domain*, the largest set D for which the formula is defined.



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- Some functions can be expressed in several or all of those formats.
- For other functions we may have difficulty translating.

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- A disadvantage is that language is sometimes imprecise.
- Often when we apply calculus to real-world problems, we start with functions represented verbally, then translate them into another form.

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Source: Statistics Canada, CANSIM, table 051-0001

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- On the other hand, tables are very concrete: the values of the function are clear.

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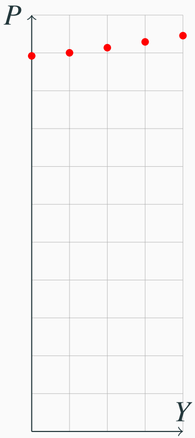

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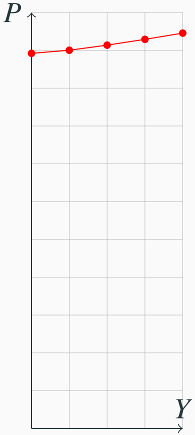
- We can also "interpolate" to make a reasonable guess at other values.



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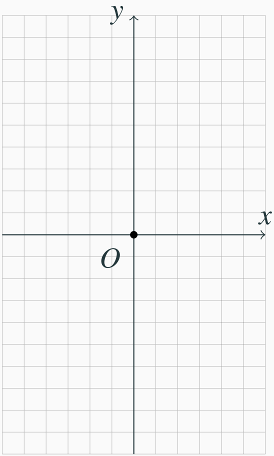
- Formulas also give us an infinite level of precision, as opposed to tables (limited by the amount of data that went into the table) or graphs (limited by the thickness of the lines)
- However, formulas can be hard to find if they are not given.

The Vertical Line Test

- For a given input to a function, you should only ever get one output.



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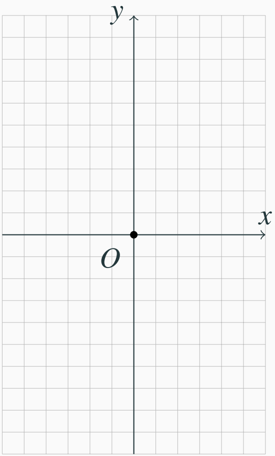

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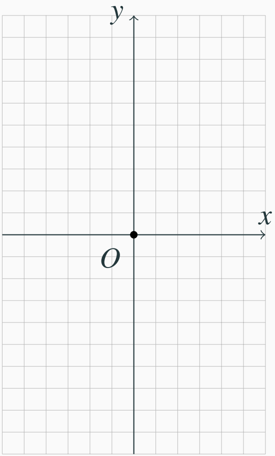

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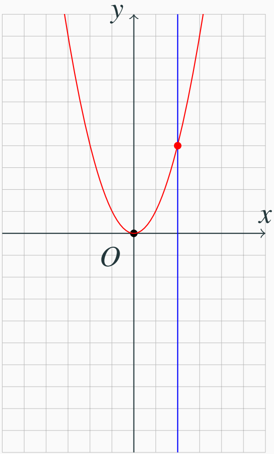
- If a vertical line intersects a curve in two or more point, that curve cannot be the graph of a function.



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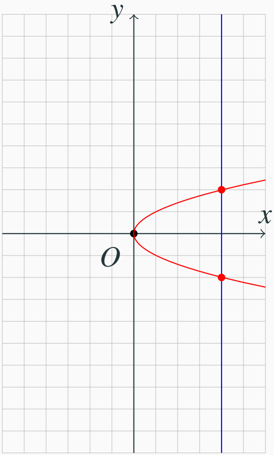
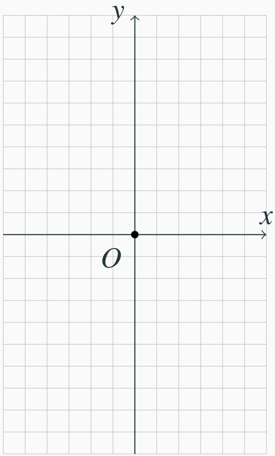


Example of a Piecewise Defined Function

- We have used cases in formulas.



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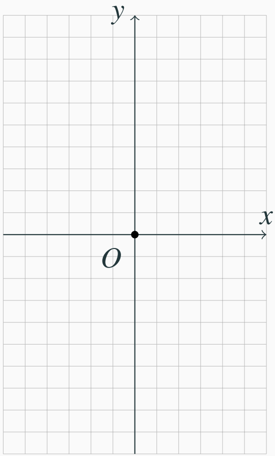

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- We have used cases in formulas.
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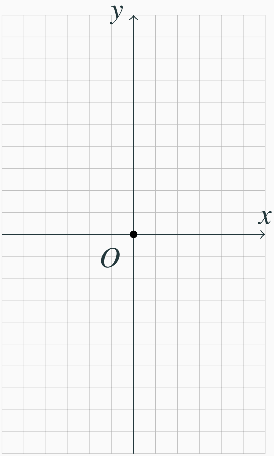

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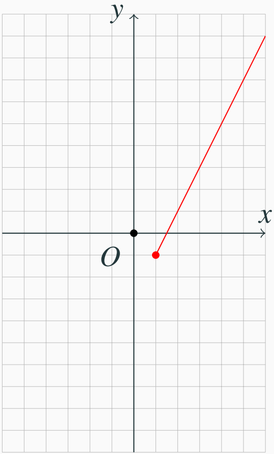

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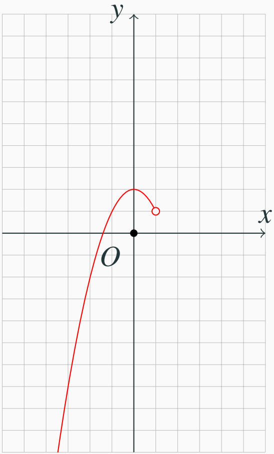

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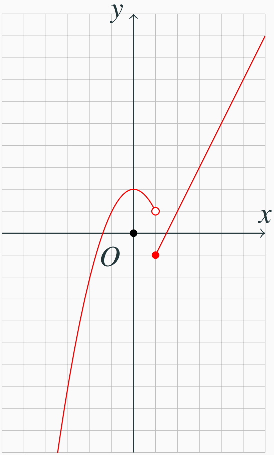
- Combine the graphs.



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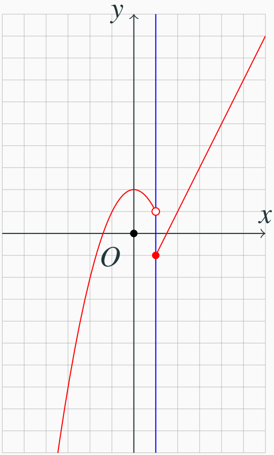
- Combine the graphs.
- Note the resulting graph passes the vertical line test.



The Absolute Value Function

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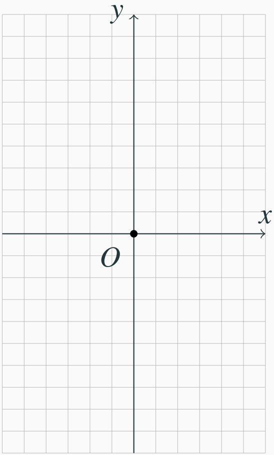

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- Recall

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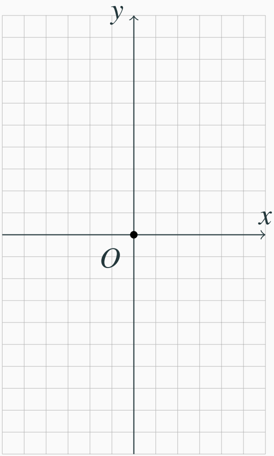

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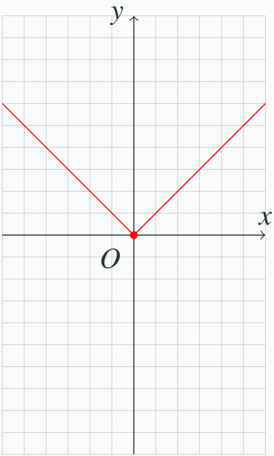
- The final graph is shown on the right.

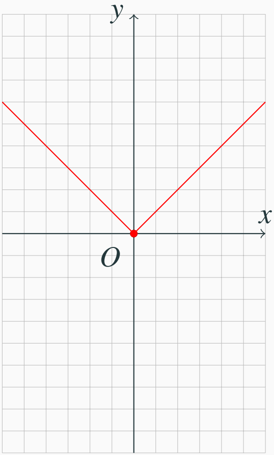


The Absolute Value Function

- One important function defined by cases is the *absolute value* function.
- Recall

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

- The final graph is shown on the right.
- Note the point $(0,0)$ is filled in because it is filled in on one of the branches.

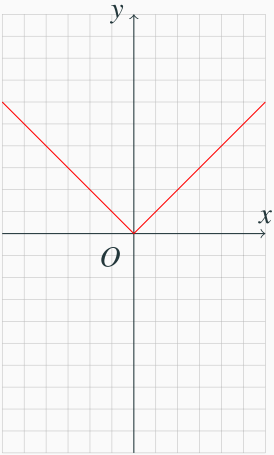


The Absolute Value Function

- One important function defined by cases is the *absolute value* function.
- Recall

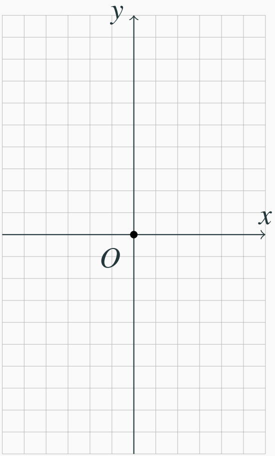
$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

- The final graph is shown on the right.
- Note the point $(0,0)$ is filled in because it is filled in on one of the branches.
- Usually the graph is drawn more simply.



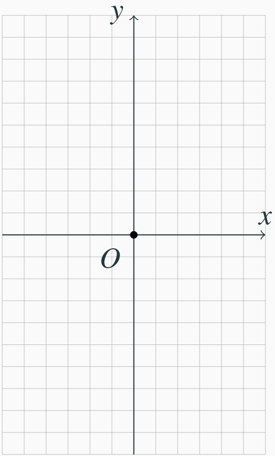
Step Functions

- X



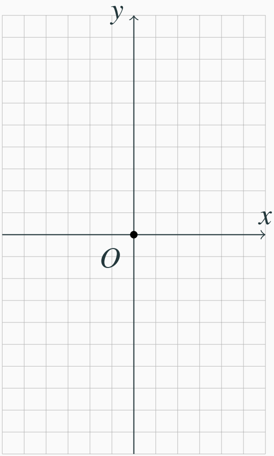
Even Functions

- X



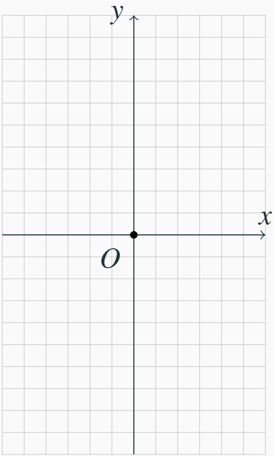
Odd Functions

- X



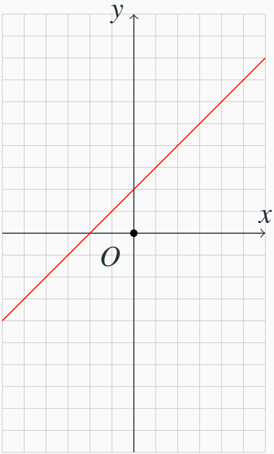
A Function Increasing on an Interval

- Graphically, we know what it means for a function to increase.



A Function Increasing on an Interval

- Graphically, we know what it means for a function to increase.
- The function "goes up" as we move from left to right along its graph.



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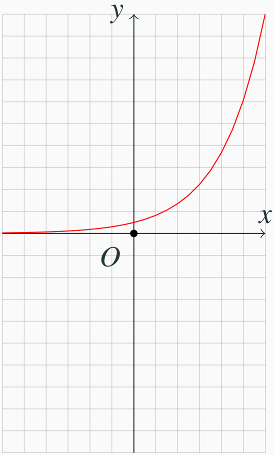

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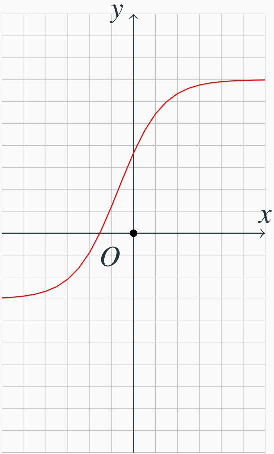
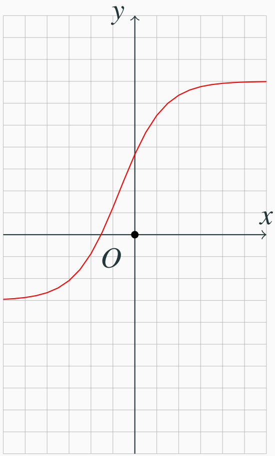


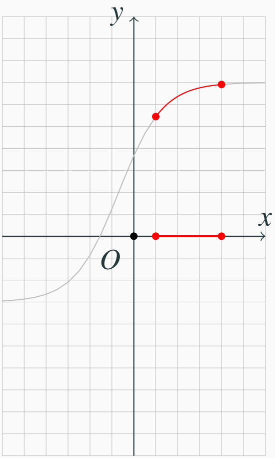
A Function Increasing on an Interval

- Graphically, we know what it means for a function to increase.
- The function "goes up" as we move from left to right along its graph.
- We don't say anything about the rate at which it increases, which may vary.



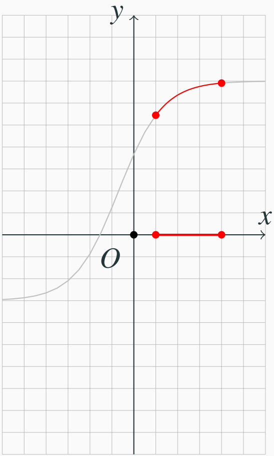
A Function Increasing on an Interval

- Graphically, we know what it means for a function to increase.
- The function "goes up" as we move from left to right along its graph.
- We don't say anything about the rate at which it increases, which may vary.
- Sometimes, we may focus our attention just on a particular interval on the x-axis and say a function is increasing there.



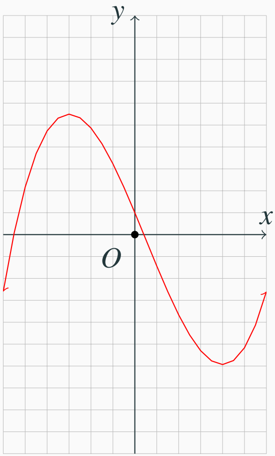
A Function Increasing and Decreasing

- We focus on intervals so we can discuss more complicated behaviour.



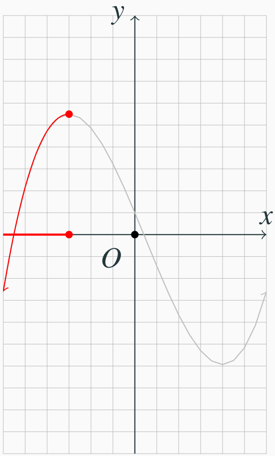
A Function Increasing and Decreasing

- We focus on intervals so we can discuss more complicated behaviour.
- Consider the function shown, which is neither consistently increasing nor decreasing.



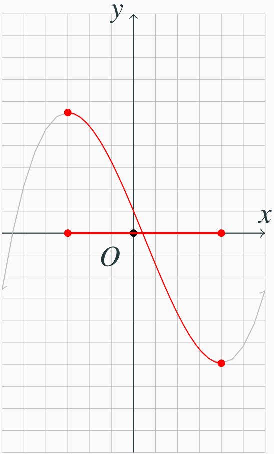
A Function Increasing and Decreasing

- We focus on intervals so we can discuss more complicated behaviour.
- Consider the function shown, which is neither consistently increasing nor decreasing.
- We say it is increasing on $(-\infty, -3] \dots$



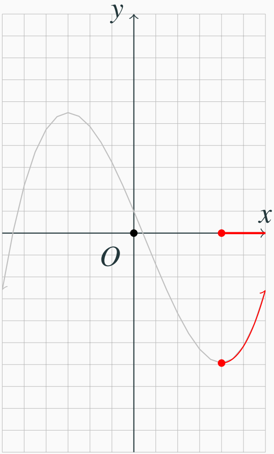
A Function Increasing and Decreasing

- We focus on intervals so we can discuss more complicated behaviour.
- Consider the function shown, which is neither consistently increasing nor decreasing.
- We say it is increasing on $(-\infty, -3]$ …
- decreasing on $[-3, 4]$ …



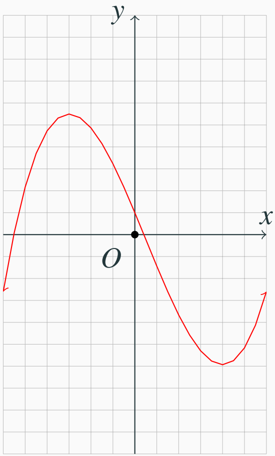
A Function Increasing and Decreasing

- We focus on intervals so we can discuss more complicated behaviour.
- Consider the function shown, which is neither consistently increasing nor decreasing.
- We say it is increasing on $(-\infty, -3]$ ...
- decreasing on $[-3, 4]$ ...
- and increasing on $[4, \infty)$



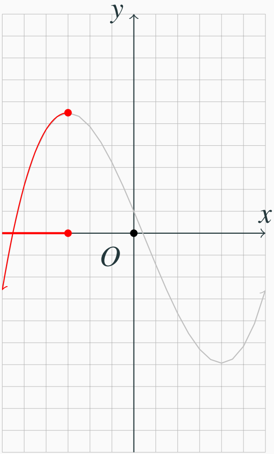
Definition of Increasing and Decreasing

- We will find it beneficial to move from our graphical understanding of increasing (or decreasing) on an interval to a more precise algebraic definition.



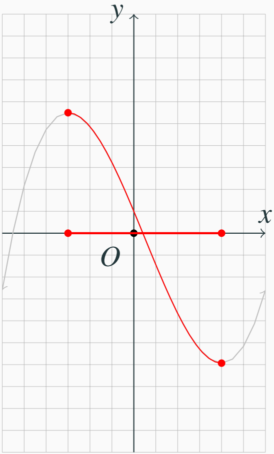
Definition of Increasing and Decreasing

- We will find it beneficial to move from our graphical understanding of increasing (or decreasing) on an interval to a more precise algebraic definition.
- We say that a function f is increasing on an interval I if, given any two numbers $a, b \in I$ with $a < b$, we have $f(a) < f(b)$.



Definition of Increasing and Decreasing

- We will find it beneficial to move from our graphical understanding of increasing (or decreasing) on an interval to a more precise algebraic definition.
- We say that a function f is increasing on an interval I if, given any two numbers $a, b \in I$ with $a < b$, we have $f(a) < f(b)$.
- Similarly, we say that a function f is decreasing on an interval I if, given any two numbers $a, b \in I$ with $a < b$, we have $f(a) > f(b)$.



Exercises

Now you should work on Problem Set 1.1. After you have finished it, you should try the following additional exercises from Section 1.1:

1.1 C-level: 1–4, 7–10, 25–30, 31–37, 38–40;
B-level: 5–6, 11–24, 41–44, 45–50, 51–56, 57–61, 62–66;
A-level: 67–80