MATH 110 Lecture 1.4

The Tangent and Velocity Problems

Edward Doolittle

Department of Indigenous Knowledge and Science
First Nations University of Canada

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Contents

The Big Idea of Differential Calculus

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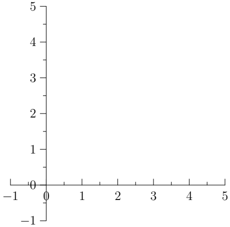
- We will try to study curves and other complicated, varying phenomena by approximating them by the simplest shapes we know:
- Lines.

Describing Lines

- We use a Cartesian coordinate system for everything we do in calculus.



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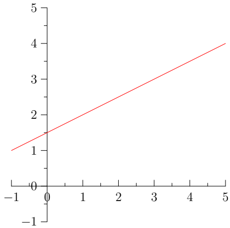

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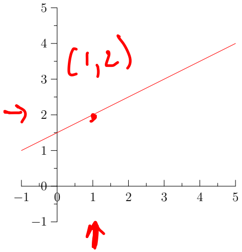

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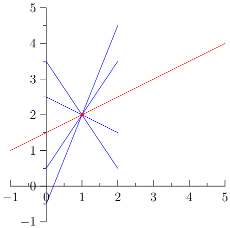

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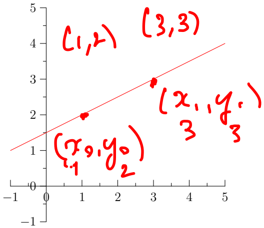
- We could describe a line completely by giving two points on the line:

$$\boxed{\frac{y-y_0}{x-x_0}=\frac{y_1-y_0}{x_1-x_0}}$$

Two-point form of a line



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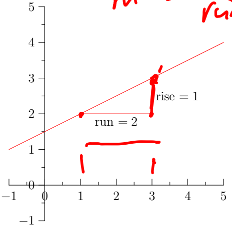
slope

$m = \frac{\text{rise}}{\text{run}} = \frac{1}{2}$



The Tangent Problem

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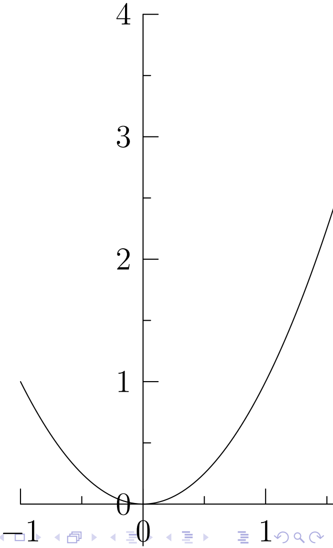

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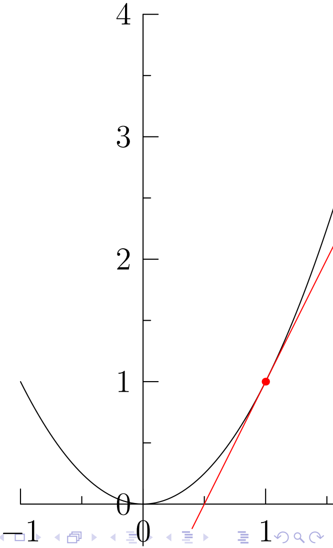

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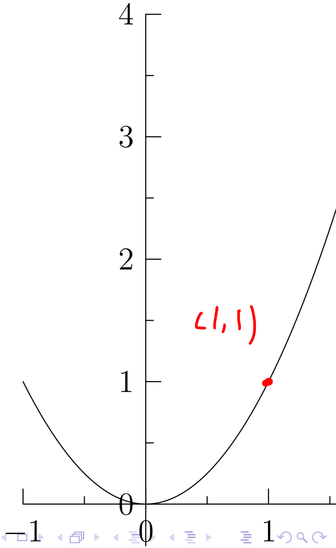

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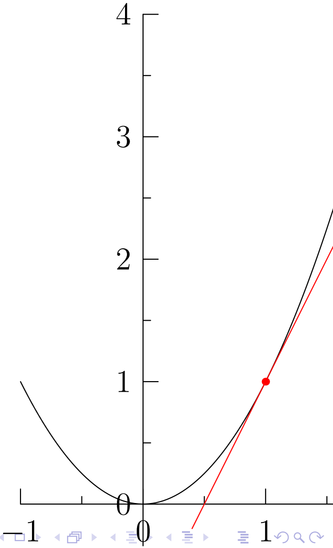
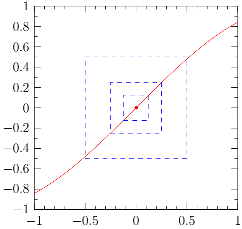
- Many phenomena in nature are actually best represented by curves.
- The basic idea of differential calculus is to use a line to approximate a given curve in the “neighborhood” of a given point.
- That point we will call the “base point” for the time being.
- The best linear approximation is called the tangent to the curve at the base point.



Is Linear Approximation Possible?

You may wonder whether linear approximation is always possible. The answer is 'sometimes': only when zooming in on the base point makes the curve look more and more straight.

A familiar function, base point $(0, 0)$



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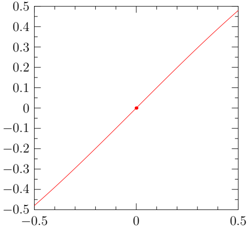

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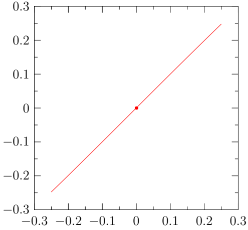

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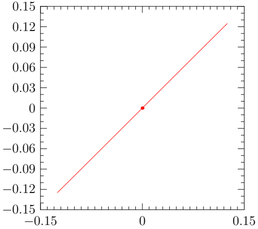
Zooming in by another factor of 2. The curve is almost indistinguishable from a straight line at this scale. We can say that $\sin(\theta) \approx \theta$ for small θ (when the angle θ is measured in radians).

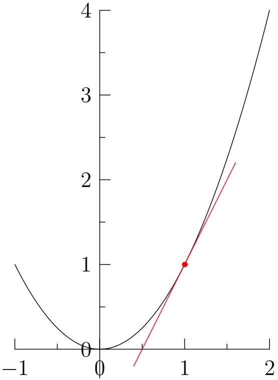


Is Linear Approximation Possible?

- Linear approximation is possible for a wide range of curves.
- Curves which can be linearly approximated are called **differentiable** for a reason we will explore shortly.
- Most of the curves we encounter in modern science are differentiable.
- There are many interesting functions which are not differentiable, however, such as fractals.
- Calculus doesn't apply to non-differentiable functions. You likely won't study non-differentiable functions in detail unless you go to graduate school.

Finding the Tangent Line

- Over the centuries, many techniques have been tried to find the tangent line to a curve at a point.



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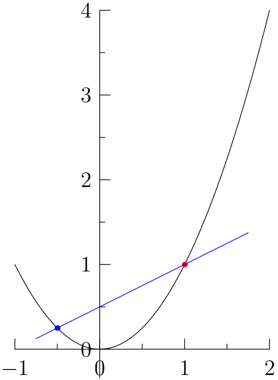
- The best general purpose solution to the problem is, paradoxically, to look at curves which are not tangents: secants.



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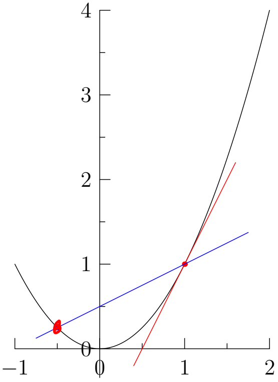

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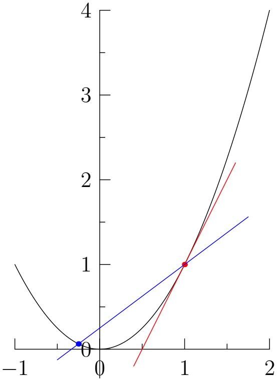

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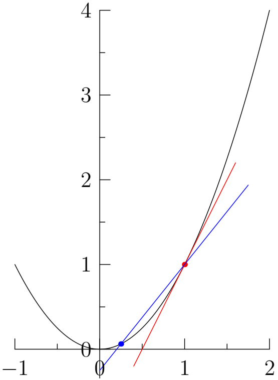

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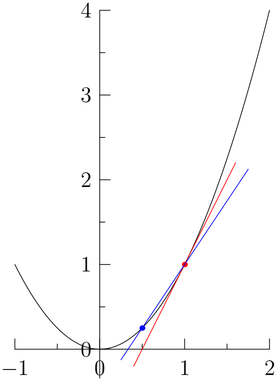

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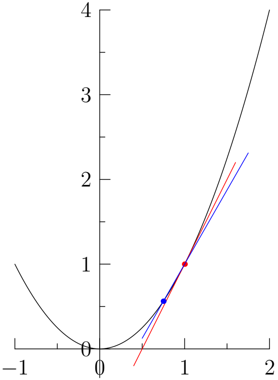
- As the second point on the secant approaches the base point ...
- the secant approaches the tangent.



Finding the Slope of the Tangent Line

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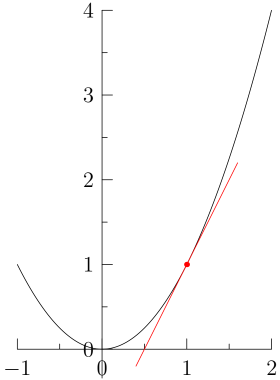

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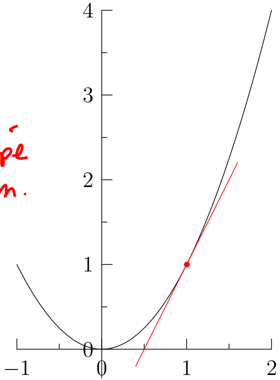
point-slope form.



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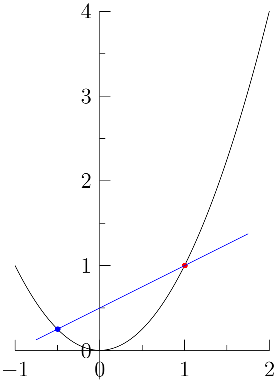

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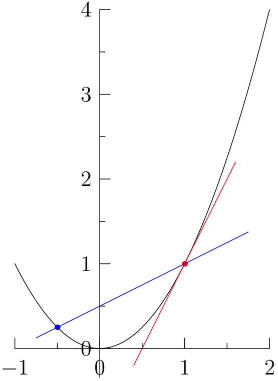
- Slope $m = 0.5$



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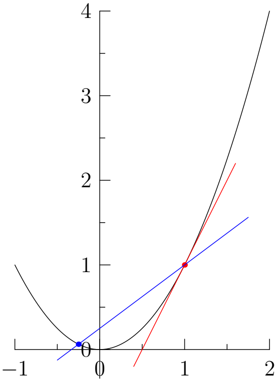
- Slope $m = 0.75$



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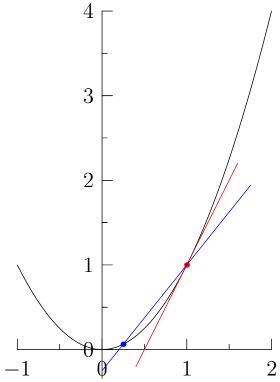

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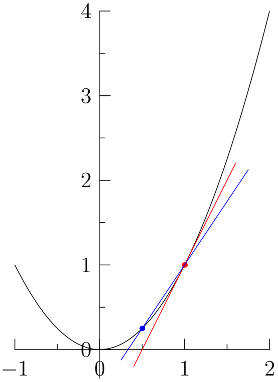

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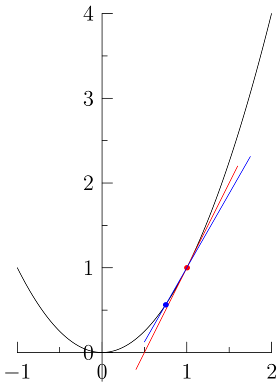

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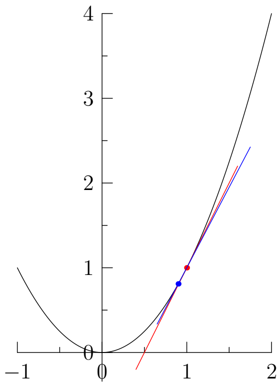
- Slope $m = 1.9$, and so on.



The Slope of the Tangent as a Limit

- We can make a table giving the slope of the secant line as a function of the x-value of the second point on the secant:

x	m	x	m
0	1	2	3
0.5	1.5	1.5	2.5
0.9	1.9	1.1	2.1
0.99	1.99	1.01	2.01
0.999	1.999	1.001	2.001

Note: I have considered secants both above and below the tangent in the table.

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  Note: I have considered secants both above and below the tangent in the table.
- The above gives us strong reason to suspect that the slope of the tangent line is 2.
- How can we be sure? We must learn more about limits. That will be in section 2.2.

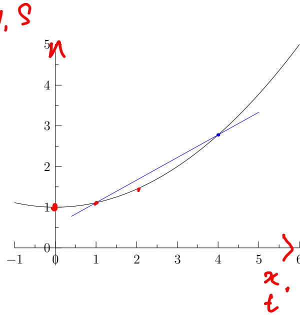
Estimating Tangents from Tabular Data

- Sometimes functions are not given to us by formulas, but by a series of data points.
- See Example 2 of section 2.1 in the textbook.
- We will not be covering Example 2 in detail in this course. If you're interested, take a course in numerical analysis.

The Velocity Problem

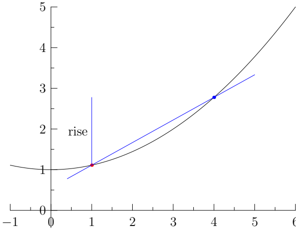
Rectilinear Motion.

- When the x axis represents time, and the y axis represents position on a line, there is a simple interpretation of the geometric analysis of the previous section.



The Velocity Problem

- When the x axis represents time, and the y axis represents position on a line, there is a simple interpretation of the geometric analysis of the previous section.
- In that case, the rise of a secant segment corresponds to the distance travelled.



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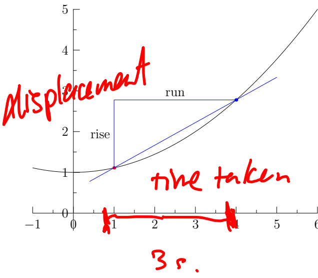
- The run of a secant segment corresponds to the time taken.



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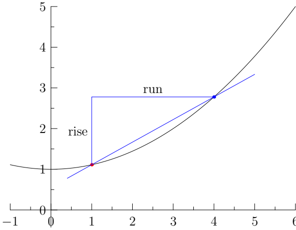

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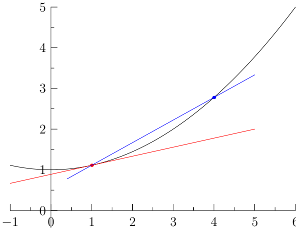
- Therefore the slope of a secant segment corresponds to the distance divided by the time, or the **average velocity**.
- The slope of the tangent line is the **instantaneous velocity**.



Examples

$t=0$

positive displacement is down.

$t=3$. $t=3.1$. $t=3.5$. 3.05. 3.01.

1. Suppose a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.
    1. Find the average velocity of the ball for an interval beginning at $t = 3$ seconds and lasting (i) 0.5 seconds, (ii) 0.1 second, (iii) 0.05 seconds, (iv) 0.01 second. (Use Galileo's law: the distance travelled by a falling ball in t seconds is $\boxed{s(t) = 4.9t^2}$ metres, neglecting air resistance.)
    2. Estimate the instantaneous velocity when $t = 3$.
2. Estimate $\sqrt{99}$.

Solution to Example 1(a)(i)

$s(t) = 4.9t^2$

(a)(i) The interval begins at $t = 3$ and ends 0.5 seconds later at $t = 3.5$. The position of the ball at $t = 3$ is $s(3) = 4.9(3)^2 = 44.1$ metres below the observation deck. The position of the ball at $t = 3.5$ is $s(3.5) = 4.9(3.5)^2 = 60.0$ metres below the observation deck, approximately.
The distance travelled (the rise on the position vs. time graph) is $s(3.5) - s(3) = 60.0 - 44.1 = 15.9$ metres. The time taken (the run on the position vs. time graph) is $3.5 - 3 = 0.5$. The average velocity (the slope on the position vs. time graph) is rise/run $= 15.9/0.5 = 31.8$ meters per second.

Answers to Examples 1(a)(ii,iii,iv)

Similar calculations give us the following:

(a)(ii) The average velocity on the interval $t \in (3, 3.1)$ is

$$\frac{s(3.1) - s(3)}{3.1 - 3} = \frac{47.1 - 44.1}{0.1} = 30\text{m/s}$$

(a)(iii) The average velocity on the interval $t \in (3, 3.05)$ is

$$\frac{s(3.05) - s(3)}{3.05 - 3} = \frac{45.58 - 44.1}{0.05} = 29.64\text{m/s}$$

(a)(iv) The average velocity on the interval $t \in (3, 3.01)$ is

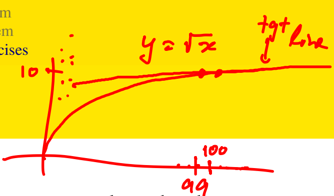
$$\frac{s(3.01) - s(3)}{3.01 - 3} = \frac{44.39 - 44.1}{0.01} = 29.45\text{m/s}$$

(Note that I had to use more digits of accuracy as the time measurement became smaller and more accurate.)

Answers to Example 1(b)

(b) From the above data, it seems that the slopes of the secant lines (i.e., the average velocities) are declining below 29.45. A reasonable guess would be 29.4. However we don't have enough data to be really sure. We really need to look at time intervals before $t = 3$, such as the interval $t \in (2.9, 3)$, to get a good bound on the instantaneous velocity.
However, we will soon learn an easier way of calculating the instantaneous velocity, from which it will follow that the instantaneous velocity is exactly 29.4 m/s.

Solution to Example 2



- This question is difficult, but shows us that we can already do something interesting with what we've learned.
- The big idea is that we are going to try to approximate the function $y = f(x) = \sqrt{x}$ by a tangent line $y = t(x) = m(x - x_0) + y_0$. We can then easily approximate $\sqrt{99} = f(99)$ by $t(99)$.

(100, 10) base point.

- To describe the tangent line, we need to find a base point and a slope. The base point is easy: we choose a nearby x value for which we know the square root. Let's choose $x_0 = 100$, $y_0 = 10$.
- For greater accuracy, we could choose a base point closer to $x = 99$ if we wanted. Think about how you would pick a base point closer to $x = 99$.

Solution to Example 2: points on curve

- All that remains is to find the slope of the tangent line.
- The only way we know how to do that now is by finding a set of points on the curve sliding in to the base point, finding the slopes of the secants, and guessing the slope of the tangent.
- It isn't immediately clear how to find a point on the curve. However, if we're being clever, we might come up with the idea of picking y values instead of picking x values. For example, if we pick $y = 10.1$, we get $x = y^2 = 10.1^2 = 102$. That gives us the point $(102.01, 10.1)$ on the curve.
- Picking $y = 10.05$, 10.01, and 10.005 gives us the points $(101.0025, 10.05)$ $(100.2001, 10.01)$, and (100.100025) respectively.

Solution to Example 2: slopes of secants

We can now find the slopes of the secant lines. Recall that the base point is $(100, 10)$.

x_1	y_1	$y_1 - y_0$	$x_1 - x_0$	m
102.01	10.1	0.1	2.01	0.04975
101.0025	10.05	0.05	1.0025	0.04988
100.2001	10.01	0.01	0.2001	0.04998
100.100025	10.005	0.005	0.100025	0.04999

tends to ~~0.5~~ 0.05

From the table, we see that the slopes of the secant lines seem to be converging to 0.05; that would be a reasonable guess for the slope of the tangent line.

Solution to Example 2: tangent line and answer

We can now fill in the blanks and write down the equation of the tangent line. We have $x_0 = 100$, $y_0 = 10$, and $m = 0.05$ giving

$$y = t(x) = m(x - x_0) + y_0 = 0.05(x - 100) + 10.$$

$\boxed{y = 0.05(x-100) + 10}$

Using the tangent line to approximate $\sqrt{99}$, we have

$$\sqrt{99} \approx t(99) = 0.05(99 - 100) + 10 = \boxed{9.95.}$$

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My calculator says that $\sqrt{99} = \boxed{9.9499}$ to four digits of accuracy, so we have an excellent approximation.

Note: there is an easier way to do this. Also, we don't yet have error bounds on our approximation. We'll learn more about those later.

Exercises

Now you should work on Problem Set 1.4. After you have finished it, you should try the following additional exercises from Section 1.4:

1.4 C-level: 1–2, 3–4, 5–6, 7–8;
B-level: 9

You should skip Example 2 of Section 1.4.