MATH 110 Midterm Test 2 Solutions

Edward Doolittle

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1 A Version

1. (a) We can write $y = (\tan(3x))^2$, so y is the composition of three functions. We have

$$\frac{dy}{dx} = 2(\tan(3x))^1 \cdot \frac{d}{dx}\tan(3x) = 2\tan(3x)\sec^2(3x)\frac{d}{dx}3x = 2\tan(3x)\cdot\sec^2(3x)\cdot 3$$

   (b) We can write $g(t) = (t^2+1)^{1/3}$ so we have by the chain rule

$$g'(t) = \frac{1}{3}(t^2+1)^{-2/3}\cdot\frac{d}{dt}(t^2+1) = \frac{1}{3}(t^2+1)^{-2/3}\cdot 2t = \frac{2}{3}t(t^2+1)^{-2/3}$$

   Differentiating again, by the product rule and the chain rule,

$$g''(t) = \frac{2}{3}(t^2+1)^{-2/3} + \frac{2}{3}t\frac{d}{dt}(t^2+1)^{-2/3} = \frac{2}{3}(t^2+1)^{-2/3} + \frac{2}{3}t\cdot -\frac{2}{3}(t^2+1)^{-5/3}\cdot 2t$$

2. By the quotient rule we have

$$\frac{dy}{ds} = \frac{(1+2s)-s(2)}{(1+2s)^2} = \frac{1}{(1+2s)^2}$$

   so

$$dy = \frac{1}{(1+2s)^2}\,ds$$

3. Multiplying and dividing by the conjugate radical we have

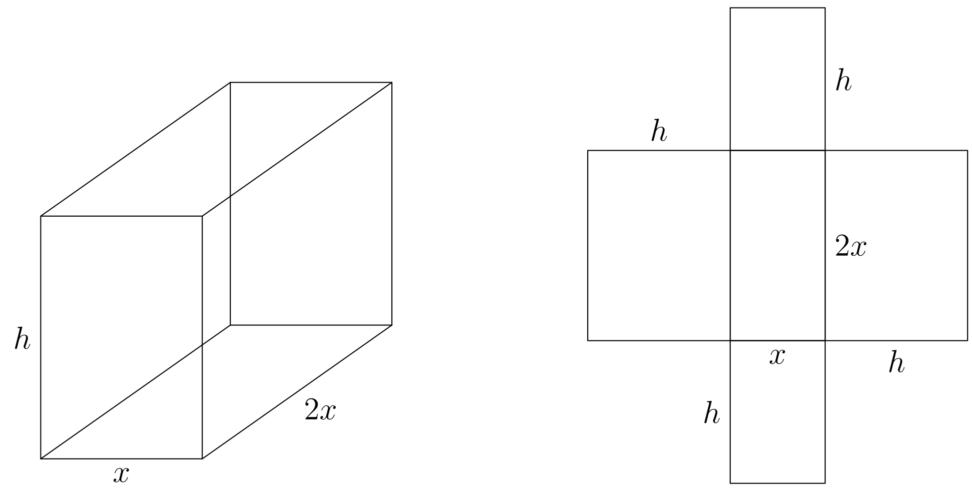
$$\lim_{x\to\infty}\left(\sqrt{4x^2+2x}-2x\right) = \lim_{x\to\infty}\left(\sqrt{4x^2+2x}-2x\right)\cdot\frac{\sqrt{4x^2+2x}+2x}{\sqrt{4x^2+2x}+2x} = \lim_{x\to\infty}\frac{2x}{\sqrt{4x^2+2x}+2x}$$

   Dividing through by the highest power of x in the denominator, namely x (because x^2 appears under the square root sign),

$$\lim_{x\to\infty}\frac{2x}{\sqrt{4x^2+2x}+2x} = \lim_{x\to\infty}\frac{2}{\sqrt{4x^2/x^2+2x/x^2}+2} = \lim_{x\to\infty}\frac{2}{\sqrt{4+2/x}+2} = \frac{2}{\sqrt{4+0}+2} = \frac{1}{2}$$

4. (a) Call the short side x and the height h; then the long side is $2x$. See Figure 1.

   (b) The amount of material on the base is $x\times 2x = 2x^2$, the amount of material on the short sides is $xh+xh = 2xh$, and the amount of material on the long sides is $2xh+2xh = 4xh$, so the objective function to be minimized is $S = 2x^2+6xh$. The constraint is that the volume must be 10 units, so $10 = x\cdot 2x\cdot h$ or $10 = 2x^2h$.



(a) oblique projection (b) unfolded

Figure 1: Two views of the box

(c) We use the constraint to eliminate one of the variables. The easiest way is to write $h = 5/x^2$, so $S(x) = 2x^2 + 30/x$ is to be minimized. Its first derivative is $S'(x) = 4x - 30/x^2$ so a local minimum will occur when $S'(x)$ is undefined (nowhere) or when $S'(x) = 0$ which is equivalent to $x^3 = 7.5$ or $x = \sqrt[3]{7.5}$.

To apply the first derivative test note $S'(x) = (4x^3 - 30)/x^2$ is negative when $4x^3 - 30 < 0$, i.e., when $x < \sqrt[3]{7.5}$, and is positive when $4x^3 - 30 > 0$, i.e., when $x > \sqrt[3]{7.5}$. Therefore $S(x)$ decreases from 0 to $\sqrt[3]{7.5}$ then increases thereafter, guaranteeing that $S(x)$ has a global minimum at $x = \sqrt[3]{7.5}$.

(d) The dimensions of the box which minimize the amount of material used are $\sqrt[3]{7.5}$ for the short side, $2\sqrt[3]{7.5}$ for the long side, and $5/(\sqrt[3]{7.5})^2$ for the height.

5. Call the point at which the cars start the origin. The distance between the cars always satisfies $z^2 = x^2 + y^2$, so the rates of change are related by $zz' = xx' + yy'$. Two hours later, we have the car moving south at position $y = -120$ and the car moving west at position $x = -50$. The distance between the cars is $z = \sqrt{(-50)^2 + (-120)^2} = 130$ and the known velocities are $y' = -60$ and $x' = -25$. Filling in all that information we have

$$z' = \frac{xx' + yy'}{z} = \frac{-50 \cdot -25 + -120 \cdot -60}{130} = 65$$

Two hours later, the distance between the cars is increasing at a rate of 65 mi/h.

6. Note that $f'(x)$ is negative when $x < 1$ and positive when $1 < x$, so $f(x)$ is decreasing when $x < 1$ and increasing when $1 < x$. Similarly, $f''(x)$ is negative when $x < 0$, positive when $0 < x < 2$, and negative when $2 < x$, so $f(x)$ is concave down when $x < 0$, concave up when $0 < x < 2$, and concave down when $2 < x$.

(a) f is increasing and concave up on the overlap of the intervals on which it is increasing ($1 < x$) and concave up ($0 < x < 2$), i.e., f is increasing and concave up on $1 < x < 2$.

(b) f is increasing and concave down on $2 < x$.

(c) f is decreasing and concave up when $0 < x < 1$.

(d) f is decreasing and concave down when $x < 0$.

(e) We have $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \dfrac{2 - 4/x + 1/x^2}{1 - 2/x + 4/x^2} = 2$, so the horizontal asymptote is $y = 2$.

See Figure 2. Although the graph isn't required to answer the question, graphing the function makes it easier to understand and visualize the situation.

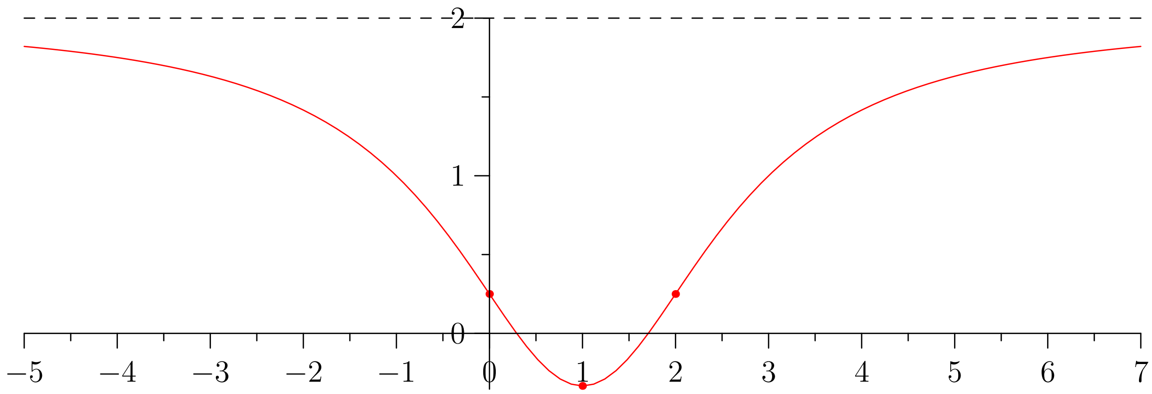


Figure 2: Graph of $f(x) = (2x^2 - 4x + 1)/(x^2 - 2x + 4)$

7. **Solution 1:** Multiplying and dividing by the "conjugate" $\cos\theta + 1$ and using $\tan\theta = \sin\theta/\cos\theta$,

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\tan\theta} = \lim_{\theta\to 0} \frac{(\cos\theta - 1)(\cos\theta + 1)\cos\theta}{\sin\theta(\cos\theta + 1)} = \lim_{\theta\to 0} \frac{(\cos^2\theta - 1)\cos\theta}{\sin\theta(\cos\theta + 1)}$$

Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$,

$$\lim_{\theta\to 0} \frac{(\cos^2\theta - 1)\cos\theta}{\sin\theta(\cos\theta + 1)} = \lim_{\theta\to 0} \frac{-\sin^2\theta\cos\theta}{\sin\theta(\cos\theta + 1)} = \lim_{\theta\to 0} \frac{-\sin\theta\cos\theta}{\cos\theta + 1} = \frac{-0\cdot 1}{1+1} = 0$$

Solution 2: Using the identity $\tan\theta = \sin\theta/\cos\theta$ and multiplying and dividing by θ gives

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\tan\theta} = \lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} \cdot \frac{\theta}{\sin\theta} \cdot \cos\theta = \lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} \cdot \lim_{\theta\to 0} \frac{\theta}{\sin\theta} \cdot \lim_{\theta\to 0} \cos\theta = 0\cdot 1\cdot 1 = 0$$

by the basic trig limits.

8. We have to show that the function $f(x) = 2\sqrt{x} - 3 + 1/x$ satisfies $f(x) > 0$ for all $x > 1$. Note $f'(x) = x^{-1/2} - 1/x^2 = (x^{3/2} - 1)/x^2 > 0$ for $x > 1$ so $f(x)$ is increasing for $x > 1$, which means $f(x) > f(1) = 0$ for $x > 1$, as required.

2 B Version

1. (a) We can write $y = (\sec(2x))^3$, so y is the composition of three functions. We have

$$\frac{dy}{dx} = 3(\sec(2x))^2 \cdot \frac{d}{dx}\sec(2x) = 3\sec^2(2x)\sec(2x)\tan(2x)\frac{d}{dx}2x = 3\sec^2(2x)\cdot\sec(2x)\tan(2x)\cdot 2$$

(b) We can write $g(t) = (t^3 + 1)^{1/2}$ so we have by the chain rule

$$g'(t) = \frac{1}{2}(t^3+1)^{-1/2} \cdot \frac{d}{dt}(t^3+1) = \frac{1}{2}(t^3+1)^{-1/2}\cdot 3t^2 = \frac{3}{2}t^2(t^3+1)^{-1/2}$$

Differentiating again, by the product rule and the chain rule,

$$g''(t) = \frac{3}{2}2t(t^3+1)^{-1/2} + \frac{3}{2}t^2\frac{d}{dt}(t^3+1)^{-1/2} = 3t(t^3+1)^{-1/2} + \frac{3}{2}t^2 \cdot -\frac{1}{2}(t^3+1)^{-3/2}\cdot 3t^2$$

2. By the quotient rule we have

$$\frac{dy}{du} = \frac{(u-1)-(u+1)}{(u-1)^2} = \frac{-2}{(u-1)^2}$$

so

$$dy = -\frac{2}{(u-1)^2}\,du$$

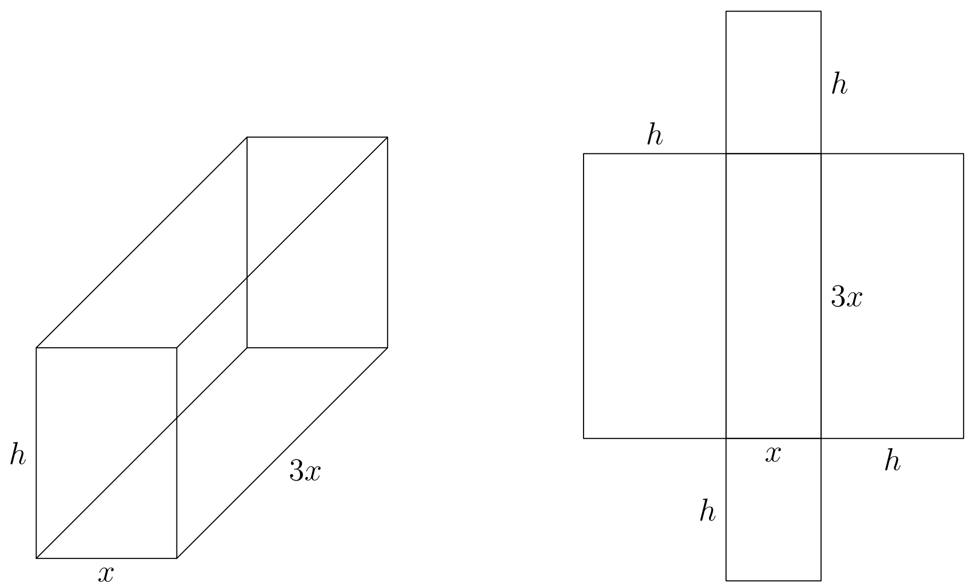
3. Multiplying and dividing by the conjugate radical we have

$$\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right) = \lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)\cdot\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x}$$

Dividing through by the highest power of x in the denominator, namely x (because x^2 appears under the square root sign),

$$\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty}\frac{1}{\sqrt{9x^2/x^2+x/x^2}+3x/x} = \lim_{x\to\infty}\frac{1}{\sqrt{9+1/x}+3} = \frac{1}{\sqrt{9+0}+3} = \frac{1}{6}$$

4. (a) Call the short side x and the height h; then the long side is $3x$. See Figure 3.



(a) oblique projection

(b) unfolded

Figure 3: Two views of the box

(b) The amount of material on the base is $x \times 3x = 3x^2$, the amount of material on the short sides is $xh + xh = 2xh$, and the amount of material on the long sides is $3xh + 3xh = 6xh$, so the objective function to be minimized is $S = 3x^2 + 8xh$. The constraint is that the volume must be 10 units, so $10 = x \cdot 3x \cdot h$ or $10 = 3x^2h$.

(c) We use the constraint to eliminate one of the variables. The easiest way is to write $h = 10/(3x^2)$, so $S(x) = 3x^2 + 80/(3x)$ is to be minimized. Its first derivative is $S'(x) = 6x - 80/(3x^2)$ so a local minimum will occur when $S'(x)$ is undefined (nowhere) or when $S'(x) = 0$ which is equivalent to $x^3 = 40/9$ or $x = \sqrt[3]{40/9}$.

To apply the first derivative test note $S'(x) = (18x^3 - 80)/x^2$ is negative when $18x^3 - 80 < 0$, i.e., when $x < \sqrt[3]{40/9}$, and is positive when $18x^3 - 80 > 0$, i.e., when $x > \sqrt[3]{40/9}$. Therefore $S(x)$ decreases from 0 to $\sqrt[3]{40/9}$ then increases thereafter, guaranteeing that $S(x)$ has a global minimum at $x = \sqrt[3]{40/9}$.

(d) The dimensions of the box which minimize the amount of material used are $\sqrt[3]{40/9}$ for the short side, $2\sqrt[3]{40/9}$ for the long side, and $10/(3(\sqrt[3]{40/9})^2)$ for the height.

5. Call the point at which the cars start the origin. The distance between the cars always satisfies $z^2 = x^2 + y^2$, so the rates of change are related by $zz' = xx' + yy'$. Two hours later, we have the car moving north at position $y = 120$ and the car moving east at position $x = 160$. The distance between the cars is $z = \sqrt{(160)^2 + (120)^2} = 200$ and the known velocities are $y' = 60$ and $x' = 80$. Filling in all that information, we have

$$z' = \frac{xx' + yy'}{z} = \frac{160 \cdot 80 + 120 \cdot 60}{200} = 100$$

Two hours later, the distance between the cars is increasing at a rate of 100 mi/h.

6. Note that $f'(x)$ is negative when $x < 0$ and positive when $0 < x$, so $f(x)$ is decreasing when $x < 0$ and increasing when $0 < x$. Similarly, $f''(x)$ is negative when $x < -1$, positive when $-1 < x < 1$, and negative when $1 < x$, so $f(x)$ is concave down when $x < -1$, concave up when $-1 < x < 1$, and concave down when $1 < x$.

(a) f is increasing and concave up on the overlap of the intervals on which it is increasing ($0 < x$) and concave up ($-1 < x < 1$), i.e., f is increasing and concave up on $0 < x < 1$.

(b) f is increasing and concave down on $1 < x$.

(c) f is decreasing and concave up when $-1 < x < 0$.

(d) f is decreasing and concave down when $x < -1$.

(e) We have $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{1 + 1/x^2}{1 + 3/x^2} = 1$, so the horizontal asymptote is $y = 1$.

See Figure 4. Although the graph isn't required to answer the question, graphing the function makes it easier to understand and visualize the situation.

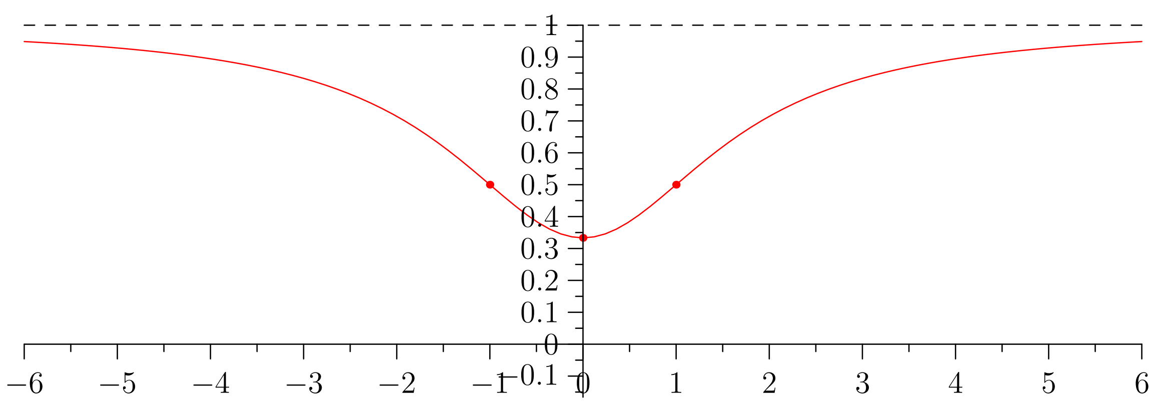


Figure 4: Graph of $f(x) = (x^2 + 1)/(x^2 + 3)$

7. **Solution 1:** Multiplying and dividing by the "conjugate" $\cos\theta + 1$ and using $\tan\theta = \sin\theta/\cos\theta$,

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\tan\theta} = \lim_{\theta\to 0} \frac{(\cos\theta - 1)(\cos\theta + 1)\cos\theta}{\sin\theta(\cos\theta + 1)} = \lim_{\theta\to 0} \frac{(\cos^2\theta - 1)\cos\theta}{\sin\theta(\cos\theta + 1)}$$

Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$,

$$\lim_{\theta\to 0} \frac{(\cos^2\theta - 1)\cos\theta}{\sin\theta(\cos\theta+1)} = \lim_{\theta\to 0} \frac{-\sin^2\theta\cos\theta}{\sin\theta(\cos\theta+1)} = \lim_{\theta\to 0} \frac{-\sin\theta\cos\theta}{\cos\theta+1} = \frac{-0\cdot 1}{1+1} = 0$$

Solution 2: Using the identity $\tan\theta = \sin\theta/\cos\theta$ and multiplying and dividing by θ gives

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\tan\theta} = \lim_{\theta\to 0} \frac{\cos\theta-1}{\theta}\cdot\frac{\theta}{\sin\theta}\cdot\cos\theta = \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin\theta}\cdot\lim_{\theta\to 0}\cos\theta = 0\cdot 1\cdot 1 = 0$$

by the basic trig limits.

8. We have to show that the function $f(x) = 2\sqrt{x} - 3 + 1/x$ satisfies $f(x) > 0$ for all $x > 1$. Note $f'(x) = x^{-1/2} - 1/x^2 = (x^{3/2}-1)/x^2 > 0$ for $x > 1$ so $f(x)$ is increasing for $x > 1$, which means $f(x) > f(1) = 0$ for $x > 1$, as required.

3 C Version

1. (a) We can write $y = (\sec(3x))^2$, so y is the composition of three functions. We have

$$\frac{dy}{dx} = 2(\sec(3x))^1\cdot\frac{d}{dx}\sec(3x) = 2\sec(3x)\sec(3x)\tan(3x)\frac{d}{dx}3x = 2\sec(3x)\cdot\sec(3x)\tan(3x)\cdot 3$$

   (b) We can write $g(t) = (t^2+1)^{1/2}$ so we have by the chain rule

$$g'(t) = \frac{1}{2}(t^2+1)^{-1/2}\cdot\frac{d}{dt}(t^2+1) = \frac{1}{2}(t^2+1)^{-1/2}\cdot 2t = t(t^2+1)^{-1/2}$$

   Differentiating again, by the product rule and the chain rule,

$$g''(t) = (t^2+1)^{-1/2} + t\frac{d}{dt}(t^2+1)^{-1/2} = (t^2+1)^{-1/2} + t\cdot -\frac{1}{2}(t^2+1)^{-3/2}\cdot 2t$$

2. By the quotient rule we have

$$\frac{dy}{dx} = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

so

$$dy = \frac{2}{(x+1)^2}\,dx$$

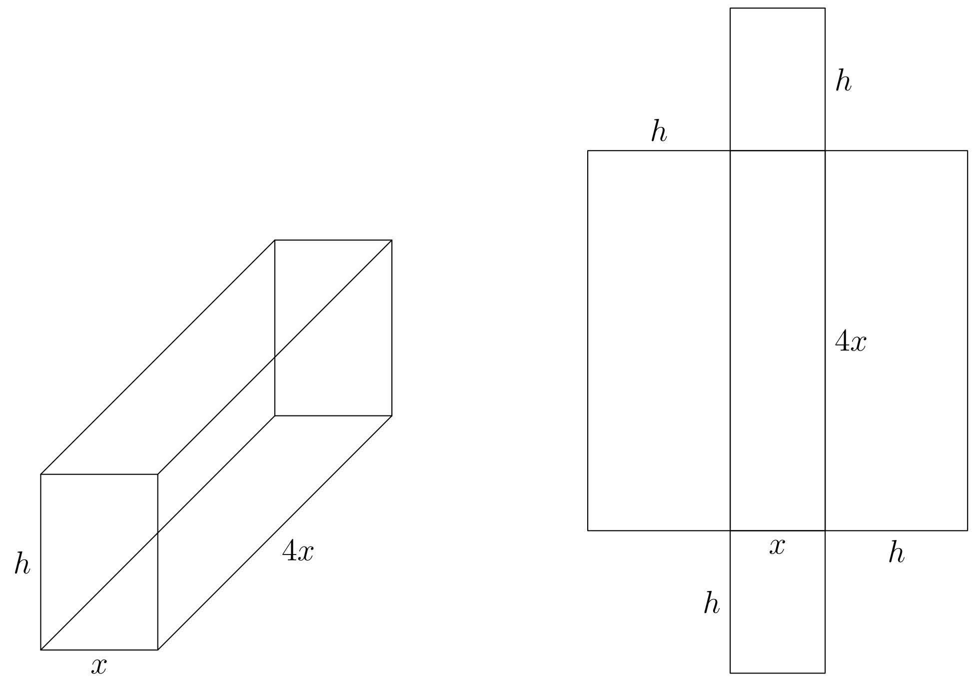
3. Multiplying and dividing by the conjugate radical we have

$$\lim_{x\to\infty}\left(\sqrt{25x^2+3x} - 5x\right) = \lim_{x\to\infty}\left(\sqrt{25x^2+3x}-5x\right)\cdot\frac{\sqrt{25x^2+3x}+5x}{\sqrt{25x^2+3x}+5x} = \lim_{x\to\infty}\frac{3x}{\sqrt{25x^2+3x}+5x}$$

Dividing through by the highest power of x in the denominator, namely x (because x^2 appears under the square root sign),

$$\lim_{x\to\infty}\frac{3x}{\sqrt{25x^2+3x}+5x} = \lim_{x\to\infty}\frac{3}{\sqrt{25x^2/x^2+3x/x^2}+5} = \lim_{x\to\infty}\frac{3}{\sqrt{25+3/x}+5} = \frac{3}{\sqrt{25+0}+5} = \frac{3}{10}$$

4. (a) Call the short side x and the height h; then the long side is $4x$. See Figure 5.



(a) oblique projection (b) unfolded

Figure 5: Two views of the box

(b) The amount of material on the base is $x \times 4x = 4x^2$, the amount of material on the short sides is $xh + xh = 2xh$, and the amount of material on the long sides is $4xh + 4xh = 8xh$, so the objective function to be minimized is $S = 4x^2 + 10xh$. The constraint is that the volume must be 10 units, so $10 = x \cdot 4x \cdot h$ or $10 = 4x^2 h$.

(c) We use the constraint to eliminate one of the variables. The easiest way is to write $h = 5/(2x^2)$, so $S(x) = 4x^2 + 25/x$ is to be minimized. Its first derivative is $S'(x) = 8x - 25/x^2$ so a local minimum will occur when $S'(x)$ is undefined (nowhere) or when $S'(x) = 0$ which is equivalent to $x^3 = 25/8$ or $x = \sqrt[3]{25/8}$. To apply the first derivative test note $S'(x) = (8x^3 - 25)/x^2$ is negative when $8x^3 - 25 < 0$, i.e., when $x < \sqrt[3]{25/8}$, and is positive when $8x^3 - 25 > 0$, i.e., when $x > \sqrt[3]{25/8}$. Therefore $S(x)$ decreases from 0 to $\sqrt[3]{25/8}$ then increases thereafter, guaranteeing that $S(x)$ has a global minimum at $x = \sqrt[3]{25/8}$.

(d) The dimensions of the box which minimize the amount of material used are $\sqrt[3]{25/8}$ for the short side, $4\sqrt[3]{25/8}$ for the long side, and $5/(2(\sqrt[3]{25/8})^2)$ for the height.

5. Call the point at which the cars start the origin. The distance between the cars always satisfies $z^2 = x^2 + y^2$, so the rates of change are related by $zz' = xx' + yy'$. Two hours later, we have the car moving north at position $y = 200$ and the car moving west at position $x = -150$. The distance between the cars is $z = \sqrt{(-150)^2 + (200)^2} = 250$ and the known velocities are $y' = 100$ and $x' = -75$. Filling in all that information we have

$$z' = \frac{xx' + yy'}{z} = \frac{-150 \cdot -75 + 200 \cdot 100}{250} = 125$$

Two hours later, the distance between the cars is increasing at a rate of 125 km/h.

6. Note that $f'(x)$ is negative when $x < -1$ and positive when $-1 < x$, so $f(x)$ is decreasing when $x < -1$ and increasing when $-1 < x$. Similarly, $f''(x)$ is negative when $x < -2$, positive when $-2 < x < 0$, and negative when $0 < x$, so $f(x)$ is concave down when $x < -2$, concave up when $-20 < x < 0$, and concave down when $0 < x$.

(a) f is increasing and concave up on the overlap of the intervals on which it is increasing ($-1 < x$) and concave up ($-2 < x < 0$), i.e., f is increasing and concave up on $-1 < x < 0$.

(b) f is increasing and concave down on $0 < x$.

(c) f is decreasing and concave up when $-2 < x < -1$.

(d) f is decreasing and concave down when $x < -2$.

(e) We have $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \dfrac{1-2/x}{1+2/x+4/x^2} = 1$, so the horizontal asymptote is $y = 1$.

See Figure 6. Although the graph isn't required to answer the question, graphing the function makes it easier to understand and visualize the situation.

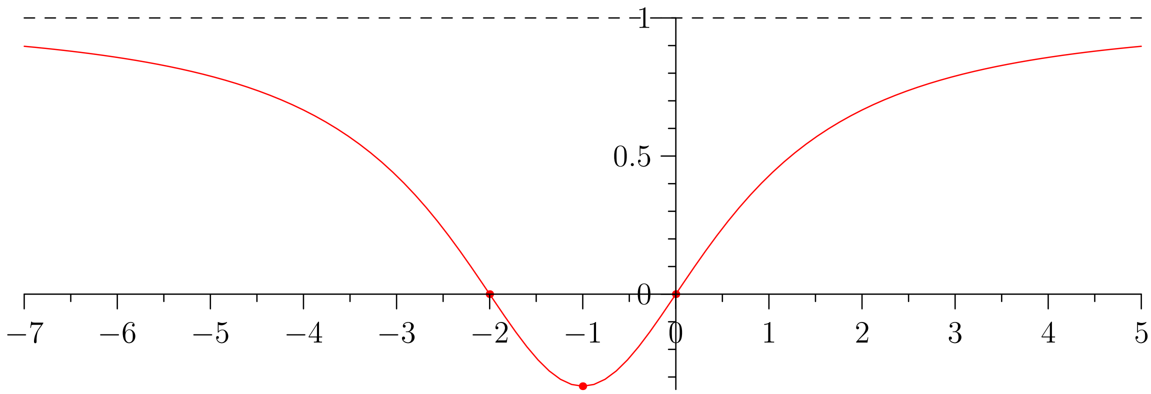


Figure 6: Graph of $f(x) = (x^2+2x)/(x^2+2x+4)$

7. **Solution 1:** Multiplying and dividing by the "conjugate" $\cos\theta + 1$ and using $\tan\theta = \sin\theta/\cos\theta$,

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\tan\theta} = \lim_{\theta\to 0} \frac{(\cos\theta-1)(\cos\theta+1)\cos\theta}{\sin\theta(\cos\theta+1)} = \lim_{\theta\to 0} \frac{(\cos^2\theta - 1)\cos\theta}{\sin\theta(\cos\theta+1)}$$

Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$,

$$\lim_{\theta\to 0} \frac{(\cos^2\theta-1)\cos\theta}{\sin\theta(\cos\theta+1)} = \lim_{\theta\to 0} \frac{-\sin^2\theta\cos\theta}{\sin\theta(\cos\theta+1)} = \lim_{\theta\to 0} \frac{-\sin\theta\cos\theta}{\cos\theta+1} = \frac{-0\cdot 1}{1+1} = 0$$

Solution 2: Using the identity $\tan\theta = \sin\theta/\cos\theta$ and multiplying and dividing by θ gives

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\tan\theta} = \lim_{\theta\to 0} \frac{\cos\theta-1}{\theta}\cdot\frac{\theta}{\sin\theta}\cdot\cos\theta = \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin\theta}\cdot\lim_{\theta\to 0}\cos\theta = 0\cdot 1\cdot 1 = 0$$

by the basic trig limits.

8. We have to show that the function $f(x) = 2\sqrt{x} - 3 + 1/x$ satisfies $f(x) > 0$ for all $x > 1$. Note $f'(x) = x^{-1/2} - 1/x^2 = (x^{3/2}-1)/x^2 > 0$ for $x > 1$ so $f(x)$ is increasing for $x > 1$, which means $f(x) > f(1) = 0$ for $x > 1$, as required.