MATH 110 Midterm 1A Solutions

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1. (a) We write $u(t) = t^{1/5} + 4(t^5)^{1/2} = t^{1/5} + 4t^{5/2}$ so by the sum, constant multiple, and power rules we have $u'(t) = \dfrac{1}{5}t^{1/5-1} + 4\dfrac{5}{2}t^{5/2-1} = \dfrac{1}{5}t^{-4/5} + 10t^{3/2}$. Then $u'(1) = \frac{1}{5}t^{-4/5} + 101^{3/2} = 10.2$.

   (b) By the quotient rule

$$y' = \frac{(1+r^{1/2})(r^2)' - r^2(1+r^{1/2})'}{(1+r^{1/2})^2} = \frac{(1+r^{1/2})2r - r^2((1/2)r^{-1/2})}{(1+r^{1/2})^2}$$

   (c) By the quotient rule,

$$\frac{df}{dx} = \frac{(3-x)1' - 1(3-x)'}{(3-x)^2} = \frac{(3-x)0 - 1(-1)}{(3-x)^2} = \frac{1}{(3-x)^2} = \frac{1}{9-6x+x^2}$$

   Differentiating again,

$$\frac{d^2f}{dx^2} = \frac{(9-6x+x^2)1' - 1(9-6x+x^2)'}{(9-6x+x^2)^2} = \frac{6-2x}{(9-6x+x^2)^2}$$

   Various simplifications are possible but are not necessary. However, if you want to check, you might want to simplify and compare with the second derivative of $(3-x)^{-1}$ calculated by the chain rule and the product rule.

2. Checking $x = 7$ gives a result of $0/0$ so further work is necessary to evaluate the limit. The usual strategy is to multiply top and bottom by the conjugate radical, i.e.,

$$\begin{aligned}
\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} &= \lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} \cdot \frac{\sqrt{x+2}+3}{\sqrt{x+2}+3} \\
&= \lim_{x\to 7} \frac{(x+2)-9}{(x-7)(\sqrt{x+2}+3)} \\
&= \lim_{x\to 7} \frac{1}{\sqrt{x+2}+3} = \frac{1}{\sqrt{7+2}+3} = \frac{1}{6}
\end{aligned}$$

because the function under the final limit above is continuous at $x = 7$.

3. We can break this function into steps as follows:

$$\begin{aligned}
f(a) &= \frac{3+a}{1-3a} \\
f(a+h) &= \frac{3+(a+h)}{1-3(a+h)} \\
f(a+h)-f(a) &= \frac{3+a+h}{1-3a-3h} - \frac{3+a}{1-3a} \\
&= \frac{(3+a+h)(1-3a)-(1-3a-3h)(3+a)}{(1-3a-3h)(1-3a)} \\
&= \frac{3+a+h-9a-3a^2-3ha-3+9a+9h-a+3a^2+3ah}{(1-3a-3h)(1-3a)} \\
&= \frac{10h}{(1-3a-3h)(1-3a)} \\
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{10h}{(1-3a-3h)(1-3a)} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{10}{(1-3a-3h)(1-3a)} \\
&= \frac{10}{(1-3a)^2}
\end{aligned}$$

Avoid the urge to take bad shortcuts by cancelling incorrectly in the fourth line of the calculation.

You can check by differentiating using the quotient rule.

4. It is a good idea, but not necessary, to check that $y(1) = 2$. Now to get an equation of the tangent line we need a point on the line, which we have (namely $(1, 2)$) and the slope of the line, which comes from the derivative. By the quotient rule,

$$y' = \frac{(x^2+1)(3)-(3x+1)(2x)}{(x^2+1)^2}$$

Algebraic simplifications are possible at this point but are counter-productive. Substituting $x = 1$ gives

$$y'(1) = \frac{(1^2+1)(3)-(3(1)+1)(2(1))}{(1^2+1)^2} = \frac{6-8}{2^2} = -\frac{1}{2}$$

An equation of the tangent line is $y - 2 = -\frac{1}{2}(x-1)$. The normal line passes through the same point $(1, 2)$ but has negative reciprocal slope $m = 2$, so the equation of the normal line is $y - 2 = 2(x-1)$.

Note that if you use the point-slope form of the equations of the lines instead of the slope-intercept form $y = mx + b$, you don't have to solve for b.

5. (a) The function is continuous on $(-\infty, 2)$ because it is given by the formula cx^2+2x on that domain; similarly it is continuous on $(2, \infty)$, so the only point in question is $x = 2$. The condition for continuity at that point is $\lim_{x\to 2} f(x) = f(2)$. We have

$$\begin{aligned}
\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} cx^2+2x = 4c+4 \\
\lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} x^3-cx = 8-2c \\
f(2) &= 2^3-c(2) = 8-2c
\end{aligned}$$

Note that we use the $x^3 - cx$ definition when we evaluate $f(2)$ because that definition applies on $2 \le x < \infty$, an interval which includes 2. For $\lim_{x\to 2} f(x)$ to exist, the above two limits must be equal, so $4c+4 = 8-2c$ which implies $6c = 4$ or $c = 2/3$, in which case the limit is $20/3$. Since we also have $f(2) = 20/3$ for that value of c, it follows that $f(x)$ is continuous if and only if $c = 2/3$.

(b) A function can be differentiable only if it is continuous, so the only possible value of c for $f(x)$ to be differentiable is $c = 2/3$. When $c = 2/3$ we have the left and right sided derivatives

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^-} \frac{(8/3)+(8/3)h+(2/3)h^2+4+2h-20/3}{h} \\
&= \lim_{h\to 0^-} \frac{(14/3)h+(2/3)h^2}{h} = \frac{14}{3} \\
\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^+} \frac{8+12h+6h^2+h^3-(4/3)-(2/3)h-20/3}{h} \\
&= \lim_{h\to 0^+} \frac{(32/3)h+6h^2+h^3}{h} = \frac{32}{3}
\end{aligned}$$

Since the left and right derivatives disagree at $x = 2$ we have a corner point there and the function is not differentiable for $c = 2/3$, so it is never differentiable.

6. Substituting $x = -2$ into the expression gives $\dfrac{15-a}{0}$. That gives an infinite limit (which doesn't exist, of course) for all values of a except possibly when $15 - a = 0$, so the only possible value of a which can give us a limit which exists is $a = 15$. Checking,

$$\lim_{x\to -2} \frac{3x^2+15x+18}{x^2+x-2} = \lim_{x\to -2} \frac{(3x+9)(x+2)}{(x+2)(x-1)} = \lim_{x\to -2} \frac{3x+9}{x-1} = \frac{3}{-3} = -1$$

In conclusion, the limit exists if and only if $a = 15$, in which case the limit is -1.