MATH 110 Problem Set 4.2 Solutions

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1. (a) Note that $a = 1$, $b = 4$, and $f(x) = x^2 + 2x - 5$. First, we calculate Δx:

$$\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$$

Now we find x_i, the boundaries of the sub-intervals:

$$x_i = a + i\Delta x = 1 + i\frac{3}{n} = 1 + \frac{3i}{n}$$

Now we are ready to write the definite integral as a limit of Riemann sums:

$$\begin{aligned}
\int_1^4 (x^2 + 2x - 5)\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} (x_i^2 + 2x_i - 5)\, \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(1 + \frac{3i}{n}\right)^2 + 2\left(1 + \frac{3i}{n}\right) - 5 \right) \frac{3}{n} \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(1 + \frac{6i}{n} + \frac{9i^2}{n^2} + 2 + \frac{6i}{n} - 5 \right) \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(-2 + \frac{12}{n}i + \frac{9}{n^2}i^2 \right) \\
&= \lim_{n\to\infty} \frac{3}{n} \left(\sum_{i=1}^{n} -2 + \sum_{i=1}^{n} \frac{12}{n}i + \sum_{i=1}^{n} \frac{9}{n^2}i^2 \right) \\
&= \lim_{n\to\infty} \frac{3}{n} \left(-2\sum_{i=1}^{n} 1 + \frac{12}{n}\sum_{i=1}^{n} i + \frac{9}{n^2}\sum_{i=1}^{n} i^2 \right)
\end{aligned}$$

Now we use the formulas

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

to obtain

$$\begin{aligned}
\int_1^4 (x^2 + 2x - 5)\, dx &= \lim_{n\to\infty} \frac{3}{n} \left(-2n + \frac{12}{n} \cdot \frac{n(n+1)}{2} + \frac{9}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} \right) \\
&= \lim_{n\to\infty} \left(-6 + \frac{36}{2} \cdot \frac{n^2+n}{n^2} + \frac{27}{6} \cdot \frac{2n^3 + 3n^2 + n}{n^3} \right) \\
&= \lim_{n\to\infty} -6 + \lim_{n\to\infty} 18 \cdot \frac{1 + 1/n}{1} + \lim_{n\to\infty} \frac{9}{2} \cdot \frac{2 + 3/n + 1/n^2}{1} \\
&= -6 + 18 \cdot \frac{1+0}{1} + \frac{9}{2} \cdot \frac{2+0+0}{1} \\
&= -6 + 18 + 9 = 21
\end{aligned}$$

(b) Here we have $a = 0$, $b = 5$, and $f(x) = 1 + x^3$. Then

$$\Delta x = \frac{b-a}{n} = \frac{5-0}{n} = \frac{5}{n}$$

and

$$x_i = a + i\Delta x = 0 + i\frac{5}{n} = \frac{5i}{n}$$

The integral as a limit of a Riemann sum is

$$\begin{aligned}
\int_0^5 (1+2x^3)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}(1+2x_i^3)\,\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+2\left(\frac{5i}{n}\right)^3\right)\frac{5}{n} \\
&= \lim_{n\to\infty}\frac{5}{n}\left(\sum_{i=1}^{n}1 + 2\cdot\frac{5^3}{n^3}\sum_{i=1}^{n}i^3\right)
\end{aligned}$$

Now using the formulas

$$\sum_{i=1}^{n}1 = n \qquad \sum_{i=1}^{n}i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^4+2n^3+n^2}{4}$$

we have

$$\begin{aligned}
\int_0^5 (1+2x^3)\,dx &= \lim_{n\to\infty}\frac{5}{n}\left(n + \frac{250}{n^3}\cdot\frac{n^4+2n^3+n^2}{4}\right) \\
&= \lim_{n\to\infty}\left(5+\frac{625}{2}\cdot\frac{n^4+2n^3+n^2}{n^4}\right) \\
&= \lim_{n\to\infty}\left(5+\frac{625}{2}\cdot\frac{1+2/n+1/n^2}{1}\right) \\
&= 5+\frac{625}{2} = \frac{635}{2}
\end{aligned}$$

2. We have

$$\int_2^5 (1+3x^4)\,dx = \int_2^5 1\,dx + \int_2^5 3x^4\,dx = \int_2^5 1\,dx + 3\int_2^5 x^4\,dx$$

by the rules for the integral of a sum and the integral of a constant multiple. Now using

$$\int_a^b c\,dx = c(b-a)$$

and the given information we have

$$\int_2^5 (1+3x^4)\,dx = 1(5-2)+3(618.6) = 3+1855.8 = 1858.8$$

3. We have $a=0$, $b=2\pi$, so

$$\Delta x = \frac{b-a}{n} = \frac{2\pi}{n} \qquad x_i = a+i\Delta x = \frac{2\pi i}{n}$$

Then the integral is

$$\int_0^{2\pi} x^2\sin x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}x_i^2\sin(x_i)\frac{2\pi}{n}$$

You can substitute $x_i = 2\pi i/n$ in the above expression if you like. The limit would be very difficult to evaluate given our current knowledge, but is not impossible.

4. (a) We have $a=\pi$, $b=2\pi$, and $f(x)=\dfrac{\cos x}{x}$ so the integral is

$$\int_\pi^{2\pi}\frac{\cos x}{x}\,dx$$

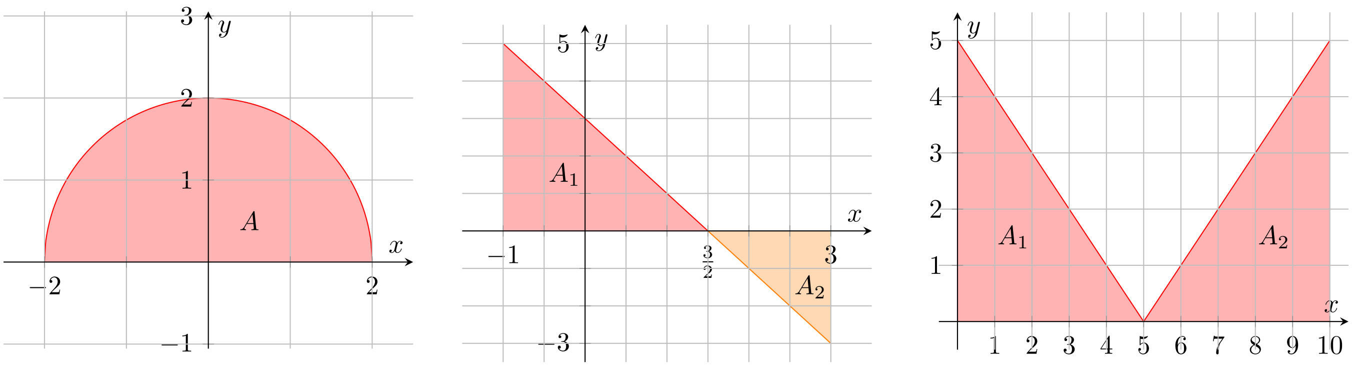
(b) We have $a=0$, $b=2$, and $f(x)=4-3x^2+6x^5$ so the integral is

$$\int_0^2 (4-3x^2+6x^5)\,dx$$

5. (a) We have $y = \sqrt{4 - x^2}$ which implies $y^2 = 4 - x^2$ or $x^2 + y^2 = 4$, which is the equation of a circle with center $(0, 0)$ and radius $\sqrt{4} = 2$. Since $y = \sqrt{4 - x^2}$ is the positive square root we take the upper half of the circle for the function we are integrating (see Figure 1(a)). Using the formula for the area of a circle, which is πr^2, the area A is the area of a semi-circle of radius 2, which is $\pi 2^2/2 = 2\pi$.

   (b) We graph the line $y = 3 - 2x$ from $x = -1$ to $x = 3$. Note that the line crosses the x-axis, so the area A_1 is positive while the area A_2 is negative (see Figure 1(b)). A_1 is the area of a triangle with base $3/2 - (-1) = 5/2$ and height 5, so is 1/2 base $\times$ height $= 1/2 \times 5/2 \times 5 = 25/4$. A_2 is the area of a triangle with base $3 - 3/2 = 3/2$ and height 3, so $A_2 = 1/2 \times 3/2 \times 3 = 9/4$. Altogether the integral is the net area $A_1 - A_2 = 25/4 - 9/4 = 16/4 = 4$ units.

   (c) We graph the function $y = |x - 5|$ from $x = 0$ to $x = 10$ by breaking it into two cases, from $x = 0$ to $x = 5$ and then from $x = 5$ to $x = 10$ (see Figure 1(c)). The integral is the sum of two positive areas, both of which are triangles. $A_1 = 1/2$ base $\times$ height $= 1/2 \times 5 \times 5 = 25/2$. Similarly $A_2 = 25/2$. Altogether the integral is $A_1 + A_2 = 25/2 + 25/2 = 25$.



(a) $y = \sqrt{4 - x^2}$ (b) $y = 3 - 2x$ (c) $y = |x - 5|$

Figure 1: Graphs for Solution 5

6. In this case a and b are unspecified so we can't evaluate Δx explicitly. We must write

$$\Delta x = \frac{b - a}{n}$$

and

$$x_i = a + i\frac{b - a}{n}$$

Evaluating the function x^2 at x_i,

$$x_i^2 = \left(a + i\frac{b - a}{n}\right)^2 = a^2 + 2ai\frac{b - a}{n} + i^2\frac{(b - a)^2}{n^2}$$

Substituting into the formula for definite integral,

$$\begin{aligned}
\int_a^b x^2\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} x_i^2 \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a^2 + 2ai\frac{b - a}{n} + i^2\frac{(b - a)^2}{n^2}\right)\frac{b - a}{n} \\
&= \lim_{n\to\infty} \left(a^2\frac{b - a}{n}\sum_{i=1}^{n} 1 + 2a\left(\frac{b - a}{n}\right)^2 \sum_{i=1}^{n} i + \left(\frac{b - a}{n}\right)^3 \sum_{i=1}^{n} i^2\right)
\end{aligned}$$

Now using the formulas

$$\sum_{i=1}^{n} 1 = n, \qquad \sum_{i=1}^{n} i = \frac{n^2 + n}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6},$$

we have

$$\begin{aligned}
\int_a^b x^2\,dx &= \lim_{n\to\infty}\left(a^2\frac{b-a}{n}n + 2a\left(\frac{b-a}{n}\right)^2\frac{n^2+n}{2} + \left(\frac{b-a}{n}\right)^3\frac{2n^3+3n^2+n}{6}\right)\\
&= a^2(b-a) + a(b-a)^2\lim_{n\to\infty}\frac{n^2+n}{n^2} + (b-a)^3\lim_{n\to\infty}\frac{2n^3+2n^2+n}{6n^3}\\
&= a^2(b-a) + a(b-a)^2 + \frac{1}{3}(b-a)^3\\
&= a^2b - a^3 + ab^2 - 2a^2b + a^3 + \frac{1}{3}b^3 - ab^2 + a^2b - \frac{1}{3}a^3\\
&= \frac{b^3-a^3}{3}
\end{aligned}$$

Alternate Solution: You can simplify the above calculation by noting that

$$\int_0^b x^2\,dx = \int_0^a x^2\,dx + \int_a^b x^2\,dx \implies \int_a^b x^2\,dx = \int_0^b x^2\,dx - \int_0^a x^2\,dx$$

so we only have to evaluate the definite integral when 0 is the lower limit of integration. In that case,

$$\Delta x = \frac{b-0}{n} = \frac{b}{n} \qquad x_i = a + i\Delta x = 0 + i\frac{b}{n} = \frac{bi}{n}$$

which are simpler than in the previous solution. Then

$$\begin{aligned}
\int_0^b x^2\,dx &= \lim_{n\to\infty}\sum_{i=1}^n x_i^2\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^n\left(\frac{bi}{n}\right)^2\frac{b}{n} = \lim_{n\to\infty}\frac{b^3}{n^3}\sum_{i=1}^n i^2\\
&= \lim_{n\to\infty}\frac{b^3}{n^3}\frac{2n^3+3n^2+n}{6} = \lim_{n\to\infty} b^3\frac{2n^3+3n^2+n}{6n^3}\\
&= b^3\frac{2}{6} = \frac{b^3}{3}
\end{aligned}$$

Substituting b with a in the above calculation gives

$$\int_0^a x^2\,dx = \frac{a^3}{3}$$

so finally

$$\int_a^b x^2\,dx = \int_0^b x^2\,dx - \int_0^a x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$$