MATH 110 Lecture 2.2

The Derivative as a Function

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- Doing the calculation, we find

$$\begin{aligned} s(1) &= 4.9 \\ s(1+h) &= 4.9(1+h)^2 = 4.9(1+2h+h^2) \\ &= 4.9 + 9.8h + 4.9h^2 \end{aligned}$$

A Simple Derivative Calculation

- Subtracting the previous two,

$$
\begin{aligned}
s(1+h) - s(1) &= 4.9 + 9.8h + 4.9h^2 - 4.9 \\
&= 9.8h + 4.9h^2 \\
s'(1) &= \lim_{h \to 0} \frac{9.8h + 4.9h^2}{h} \\
&= \lim_{h \to 0} 9.8 + 4.9h \\
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- So the velocity of the ball at $t=1$ seconds is 9.8 metres per second (downward).

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- I could repeat the above calculation with base time 2 instead of base time 1.
- However, I may need to repeat the calculation at $t = 3, 4$ etc. Instead of re-doing similar looking calculations over and over, I could just leave a variable a in place of the number 1 in the above calculation.

The Same Derivative Calculation with Variable Base Time

Calculating the derivative $s'(a)$, we have

$$\begin{aligned}
s(a) &= 4.9a^2 \\
s(a+h) &= 4.9(a+h)^2 = 4.9a^2 + 9.8ah + 4.9h^2 \\
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I have saved some repetitive work by using a as a variable, but now my derivative is a **function of** a.

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- That is why I prefer the $\lim\limits_{h \to 0}$ version of the definition of derivative over the $\lim\limits_{t \to a}$ version. (Try it the other way to see what happens!)

Graphing the Derivative

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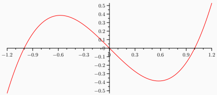
Graphing the Derivative



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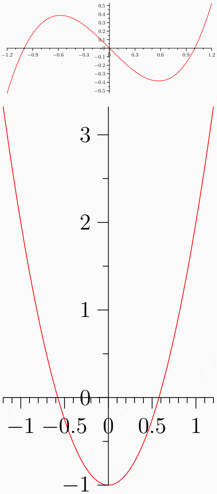
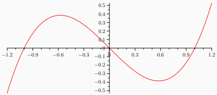
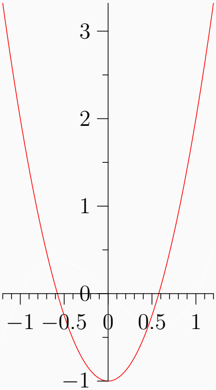
- For example, consider the graph of some function $f(x)$.
- I have graphed the derivative $f'(x)$ below that $f(x)$; the scale and position of the x-axis is the same for both graphs. I could have graphed them both on one set of axes, but for now it is less confusing to separate the graphs.



The Relationship Between Graphs of f and f'

- There is a relationship between the two graphs that we will explore in great detail later.





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  1. when f is increasing, f' is positive;
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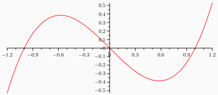

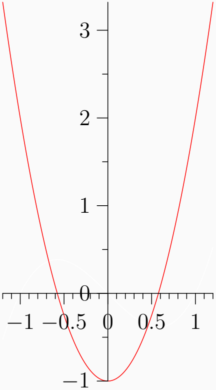

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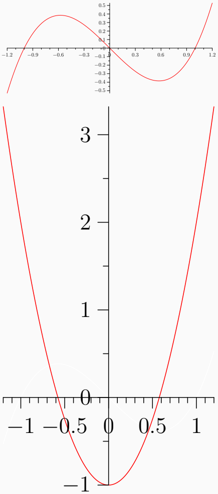
- Using that relationship, given two graphs, one of a function f and the other of its derivative f', it should be possible to determine which is which.



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- Both notation systems have their advantages and disadvantages. Newton's is clean and simple, but can be difficult to use in a 'change of variable situation' but under-emphasizes the role of x.
- Leibniz's notation behaves nicely under changes of variable and emphasizes the relationship between $\frac{df}{dx}$ and $\frac{\Delta f}{\Delta x}$, but may be misunderstood as an ordinary fraction.

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    3. A one-sided limit doesn’t exist but isn’t infinite (oscillatory behaviour).

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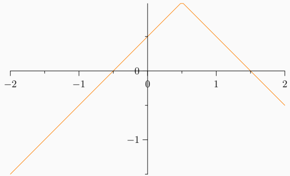

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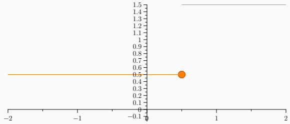

jump discontinuity

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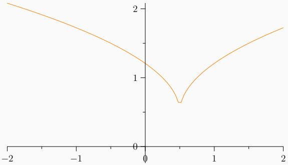

cusp

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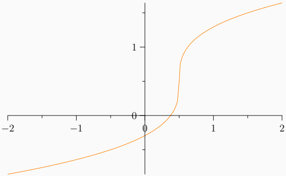

vertical tangent

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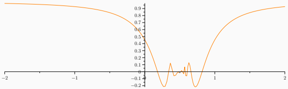

rapid oscillation

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- It follows that the domain of f' can be smaller than the domain of f.

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- In a similar way, we can find third and higher derivatives of a function.

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- Another famous use of third derivatives was by Richard Nixon, who once stated, "The rate of increase of inflation is now decreasing".

Examples

1. Find the derivative of $f(x) = |x-6|$ from first principles, and indicate all numbers a at which f is not differentiable.
2. Let $g(x) = \sqrt{3-5x}$.
    2.1 Find $g'(x)$ from first principles.
    2.2 Find the domains of g and g'.
3. Let $h(x) = \sqrt[3]{x}$.
    3.1 If $a \neq 0$, find $h'(a)$ from first principles.
    3.2 Show that $h'(0)$ does not exist.
    3.3 Show that the graph of h has a vertical tangent at 0.

Exercises

Now you should work on Problem Set 2.2. After you have finished it, you should try the following additional exercises from Section 2.2:

2.2 C-level: 1–2, 3, 4–11, 12–15, 16, 19–29, 31–32, 33–36, 37–38, 39–42, 47–50, 51–53;
B-level: 17–18, 30, 43–44, 45–46, 54, 55-56, 57–60, 63–64;
A-level: 61, 62, 65;