MATH 110 Review 0.A

Review of Inequalities and Absolute Values

Edward Doolittle

Tuesday, January 6, 2026

Department of Indigenous Knowledge and Science
First Nations University of Canada

Contents

Numbers, Inequalities, and Absolute Values

The Natural Number System

- We begin with the natural number system, which is just the counting numbers 1, 2, 3, $\ldots$, and the associated operations of addition and multiplication.

The Natural Number System

- We begin with the natural number system, which is just the counting numbers 1, 2, 3, ..., and the associated operations of addition and multiplication.
- The study of the natural numbers is known as *arithmetic*, and *number theory*.

The Natural Number System

- We begin with the natural number system, which is just the counting numbers 1, 2, 3, $\ldots$, and the associated operations of addition and multiplication.
- The study of the natural numbers is known as *arithmetic*, and *number theory*.
- We are all familiar with arithmetic, and there's no need for us to go into number theory in this course.

The Natural Number System

- We begin with the natural number system, which is just the counting numbers 1, 2, 3, . . . , and the associated operations of addition and multiplication.
- The study of the natural numbers is known as *arithmetic*, and *number theory*.
- We are all familiar with arithmetic, and there's no need for us to go into number theory in this course.
- However, there is one subject we should consider: the opposite or inverse of addition,

The Natural Number System

- We begin with the natural number system, which is just the counting numbers 1, 2, 3, . . . , and the associated operations of addition and multiplication.
- The study of the natural numbers is known as *arithmetic*, and *number theory*.
- We are all familiar with arithmetic, and there's no need for us to go into number theory in this course.
- However, there is one subject we should consider: the opposite or inverse of addition,
- Subtraction is fine if I subtract a smaller number from a larger number: $5 - 3 = 2$.

The Natural Number System

- We begin with the natural number system, which is just the counting numbers 1, 2, 3, ..., and the associated operations of addition and multiplication.
- The study of the natural numbers is known as *arithmetic*, and *number theory*.
- We are all familiar with arithmetic, and there's no need for us to go into number theory in this course.
- However, there is one subject we should consider: the opposite or inverse of addition,
- Subtraction is fine if I subtract a smaller number from a larger number: $5 - 3 = 2$.
- But what if I subtract a number from itself: $5 - 5 = ?$

The Whole Number System

- In order to be able to do subtraction, we must adjoin additional numbers to the natural numbers. The first and most important “unnatural” addition is 0.

The Whole Number System

- In order to be able to do subtraction, we must adjoin additional numbers to the natural numbers. The first and most important "unnatural" addition is 0.
- We can now subtract any number from itself: $5 - 5 = 0$, $11 - 11 = 0, \ldots$

The Whole Number System

- In order to be able to do subtraction, we must adjoin additional numbers to the natural numbers. The first and most important “unnatural” addition is 0.
- We can now subtract any number from itself: $5 - 5 = 0$, $11 - 11 = 0, \ldots$
- We have to work out addition and multiplication rules for 0.

The Whole Number System

- In order to be able to do subtraction, we must adjoin additional numbers to the natural numbers. The first and most important "unnatural" addition is 0.
- We can now subtract any number from itself: $5-5=0$, $11-11=0, \ldots$
- We have to work out addition and multiplication rules for 0.
- After some thought, we conclude that $n+0=n$ for any number n, and $n \times 0=0$ for any number n.

The Whole Number System

- In order to be able to do subtraction, we must adjoin additional numbers to the natural numbers. The first and most important "unnatural" addition is 0.
- We can now subtract any number from itself: $5-5=0$, $11-11=0$, ...
- We have to work out addition and multiplication rules for 0.
- After some thought, we conclude that $n+0=n$ for any number n, and $n \times 0 = 0$ for any number n.
- With those rules, we have a coherent system called the *whole numbers*: 0, 1, 2, ...

The Integers

- We still cannot subtract a larger number from a smaller: $5-7=?$

The Integers

- We still cannot subtract a larger number from a smaller: $5 - 7 = ?$
- In order to be able to subtract any two numbers, we have to adjoin a whole infinite set of new numbers, the *negative numbers*: $-1, -2, -3, \ldots$

The Integers

- We still cannot subtract a larger number from a smaller: $5 - 7 = ?$
- In order to be able to subtract any two numbers, we have to adjoin a whole infinite set of new numbers, the *negative numbers*: $-1, -2, -3, \dots$
- The new number system created this way is call the *integers*: $\dots, -3, -2, -1, 0, 1, 2, \dots$

The Integers

- We still cannot subtract a larger number from a smaller: $5 - 7 = ?$
- In order to be able to subtract any two numbers, we have to adjoin a whole infinite set of new numbers, the *negative numbers*: -1, -2, -3, $\dots$
- The new number system created this way is call the *integers*: $\dots$, -3, -2, -1, 0, 1, 2, $\dots$
- Defining the result of addition or subtraction is straightforward. In particular, the sum of two negative numbers is negative.

The Integers

- We still cannot subtract a larger number from a smaller: $5 - 7 = ?$
- In order to be able to subtract any two numbers, we have to adjoin a whole infinite set of new numbers, the *negative numbers*: $-1, -2, -3, \ldots$
- The new number system created this way is call the *integers*: $\ldots$, $-3, -2, -1, 0, 1, 2, \ldots$
- Defining the result of addition or subtraction is straightforward. In particular, the sum of two negative numbers is negative.
- Defining the product of two negative numbers is more puzzling.

The Integers

- We still cannot subtract a larger number from a smaller: $5-7=?$
- In order to be able to subtract any two numbers, we have to adjoin a whole infinite set of new numbers, the *negative numbers*: -1, -2, -3, $\ldots$
- The new number system created this way is call the *integers*: $\ldots$, -3, -2, -1, 0, 1, 2, $\ldots$
- Defining the result of addition or subtraction is straightforward. In particular, the sum of two negative numbers is negative.
- Defining the product of two negative numbers is more puzzling.
- Is the product of two negative numbers positive or negative?

The Integers

- We still cannot subtract a larger number from a smaller: $5 - 7 = ?$
- In order to be able to subtract any two numbers, we have to adjoin a whole infinite set of new numbers, the *negative numbers*: $-1, -2, -3, \ldots$
- The new number system created this way is call the *integers*: $\ldots, -3, -2, -1, 0, 1, 2, \ldots$
- Defining the result of addition or subtraction is straightforward. In particular, the sum of two negative numbers is negative.
- Defining the product of two negative numbers is more puzzling.
- Is the product of two negative numbers positive or negative?
- Why?

The Rational Numbers

- Now we consider the inverse of multiplication, namely division.

The Rational Numbers

- Now we consider the inverse of multiplication, namely division.
- Sometimes we can divide two integers, e.g., $12/3 = 4$.

The Rational Numbers

- Now we consider the inverse of multiplication, namely division.
- Sometimes we can divide two integers, e.g., $12/3 = 4$.
- But most of the time we can't divide two integers.

The Rational Numbers

- Now we consider the inverse of multiplication, namely division.
- Sometimes we can divide two integers, e.g., $12/3 = 4$.
- But most of the time we can't divide two integers.
- We fix that situation by adjoining more new numbers to the integers.

The Rational Numbers

- Now we consider the inverse of multiplication, namely division.
- Sometimes we can divide two integers, e.g., $12/3 = 4$.
- But most of the time we can't divide two integers.
- We fix that situation by adjoining more new numbers to the integers.
- The new numbers are called *rational numbers* because they are ratios of the form m/n where m and n are integers.

The Rational Numbers

- Now we consider the inverse of multiplication, namely division.
- Sometimes we can divide two integers, e.g., $12/3 = 4$.
- But most of the time we can't divide two integers.
- We fix that situation by adjoining more new numbers to the integers.
- The new numbers are called *rational numbers* because they are ratios of the form m/n where m and n are integers.
- We have to tinker a bit with our idea of equality.

The Rational Numbers

- Now we consider the inverse of multiplication, namely division.
- Sometimes we can divide two integers, e.g., $12/3 = 4$.
- But most of the time we can't divide two integers.
- We fix that situation by adjoining more new numbers to the integers.
- The new numbers are called *rational numbers* because they are ratios of the form m/n where m and n are integers.
- We have to tinker a bit with our idea of equality.
- For example, $2/3 = 4/6$ even though they look different.

The Rational Numbers

- Now we consider the inverse of multiplication, namely division.
- Sometimes we can divide two integers, e.g., $12/3 = 4$.
- But most of the time we can't divide two integers.
- We fix that situation by adjoining more new numbers to the integers.
- The new numbers are called *rational numbers* because they are ratios of the form m/n where m and n are integers.
- We have to tinker a bit with our idea of equality.
- For example, $2/3 = 4/6$ even though they look different.
- We also have to define how the arithmetic operations work on rational numbers, which is something we discussed in the previous lecture.

Irrational Numbers

- We're still not done adding new numbers.

Irrational Numbers

- We're still not done adding new numbers.
- There are useful numbers which cannot be expressed as rational numbers.

Irrational Numbers

- We're still not done adding new numbers.
- There are useful numbers which cannot be expressed as rational numbers.
- For example, it is impossible to express $\sqrt{2}$ as a rational number.

Irrational Numbers

- We're still not done adding new numbers.
- There are useful numbers which cannot be expressed as rational numbers.
- For example, it is impossible to express $\sqrt{2}$ as a rational number.
- There are many other such irrational numbers, such as $\sqrt{3}$, $\sqrt[3]{7}$, $\sqrt{1+\sqrt{2}}$, and so on.

Irrational Numbers

- We're still not done adding new numbers.
- There are useful numbers which cannot be expressed as rational numbers.
- For example, it is impossible to express $\sqrt{2}$ as a rational number.
- There are many other such irrational numbers, such as $\sqrt{3}$, $\sqrt[3]{7}$, $\sqrt{1+\sqrt{2}}$, and so on.
- All of those numbers are examples of "algebraic numbers" which are roots of some polynomial.

The Real Number System

- But wait, there's more.

The Real Number System

- But wait, there’s more.
- It turns out that there are even more numbers, and that we need them for calculus.

- But wait, there's more.
- It turns out that there are even more numbers, and that we need them for calculus.
- One example is the number $\pi = 3.14159\ldots$.

The Real Number System

- But wait, there's more.
- It turns out that there are even more numbers, and that we need them for calculus.
- One example is the number $\pi = 3.14159\ldots$.
- The number π does not fit into any of the previous categories.

The Real Number System

- But wait, there's more.
- It turns out that there are even more numbers, and that we need them for calculus.
- One example is the number $\pi = 3.14159\ldots$.
- The number π does not fit into any of the previous categories.
- There are many other infinite decimal expansions that don't fit into the previous categories, for example $e = 2.718281828\ldots$.

- But wait, there's more.
- It turns out that there are even more numbers, and that we need them for calculus.
- One example is the number $\pi = 3.14159\ldots$.
- The number π does not fit into any of the previous categories.
- There are many other infinite decimal expansions that don't fit into the previous categories, for example $e = 2.718281828\ldots$.
- Once we adjoin the infinite decimal expansions, we get a "complete" system which works well without adding any other numbers.

The Number Line

- One useful tool to help us think about real numbers is the number line.

The Number Line

- One useful tool to help us think about real numbers is the number line.

- We mark off the number 0, then the number 1, then the positions of the rest of the integers follow.

The Number Line

- One useful tool to help us think about real numbers is the number line.

- We mark off the number 0, then the number 1, then the positions of the rest of the integers follow.
- Fractions are represented as numbers partway between two integers.

- One useful tool to help us think about real numbers is the number line.

- We mark off the number 0, then the number 1, then the positions of the rest of the integers follow.
- Fractions are represented as numbers partway between two integers.

- To represent infinite decimal expansions, we use reasonable approximations, e.g., 2.71 for e.

Set Notation

- Another way of representing collections of real numbers is using *set builder* notation.

Set Notation

- Another way of representing collections of real numbers is using *set builder* notation.
- The simplest way to use set notation is just to list the elements of the set in curly braces, e.g., $\{1, -2, \pi, 1.5\}$.

Set Notation

- Another way of representing collections of real numbers is using *set builder* notation.
- The simplest way to use set notation is just to list the elements of the set in curly braces, e.g., $\{1, -2, \pi, 1.5\}$.
- We can also put conditions inside the braces: $\{x | 0 < x < 5 \text{ and } x \text{ is an integer}\}$

Set Notation

- Another way of representing collections of real numbers is using *set builder* notation.
- The simplest way to use set notation is just to list the elements of the set in curly braces, e.g., $\{1, -2, \pi, 1.5\}$.
- We can also put conditions inside the braces: $\{x | 0 < x < 5 \text{ and } x \text{ is an integer}\}$
- That set can also be written $\{1, 2, 3, 4\}$

Set Notation

- Another way of representing collections of real numbers is using *set builder* notation.
- The simplest way to use set notation is just to list the elements of the set in curly braces, e.g., $\{1, -2, \pi, 1.5\}$.
- We can also put conditions inside the braces: $\{x | 0 < x < 5 \text{ and } x \text{ is an integer}\}$
- That set can also be written $\{1, 2, 3, 4\}$
- We read that notation as "the set of x *such that* 0 is less than x which is less than 5 and x is an integer".

Set Notation

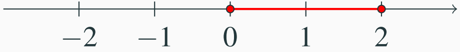
- Another way of representing collections of real numbers is using *set builder* notation.
- The simplest way to use set notation is just to list the elements of the set in curly braces, e.g., $\{1, -2, \pi, 1.5\}$.
- We can also put conditions inside the braces: $\{x | 0 < x < 5 \text{ and } x \text{ is an integer}\}$
- That set can also be written $\{1, 2, 3, 4\}$
- We read that notation as "the set of x *such that* 0 is less than x which is less than 5 and x is an integer".
- A wide variety of different conditions can go after the | "such that" bar.

Intervals

- Some sets known as *intervals* are so commonly used that we have special notation for them.

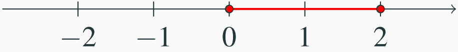
Intervals

- Some sets known as *intervals* are so commonly used that we have special notation for them.
- An example of an interval is the set $[0,2] = \{x | 0 \leq x \leq 2\}$



Intervals

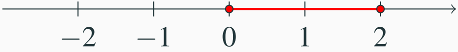
- Some sets known as *intervals* are so commonly used that we have special notation for them.
- An example of an interval is the set $[0,2] = \{x | 0 \leq x \leq 2\}$



- Another different interval is $(0,2) = \{x | 0 < x < 2\}$

Intervals

- Some sets known as *intervals* are so commonly used that we have special notation for them.
- An example of an interval is the set $[0,2] = \{x|0 \leq x \leq 2\}$

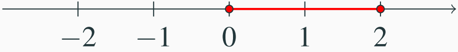


- Another different interval is $(0,2) = \{x|0 < x < 2\}$

- Note the subtle difference between the two examples.

Intervals

- Some sets known as *intervals* are so commonly used that we have special notation for them.
- An example of an interval is the set $[0,2] = \{x | 0 \leq x \leq 2\}$

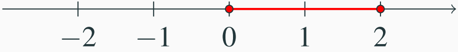


- Another different interval is $(0,2) = \{x | 0 < x < 2\}$

- Note the subtle difference between the two examples.
- The top interval is *closed*; it includes its endpoints.

Intervals

- Some sets known as *intervals* are so commonly used that we have special notation for them.
- An example of an interval is the set $[0,2] = \{x | 0 \leq x \leq 2\}$



- Another different interval is $(0,2) = \{x | 0 < x < 2\}$

- Note the subtle difference between the two examples.
- The top interval is *closed*; it includes its endpoints.
- The bottom interval is *open*; it does not include its endpoints.

Table of Intervals

- Here is a table of all possibilities:

$[a,b]$	$\{x \mid a \leq x \leq b\}$	closed
(a,b)	$\{x \mid a < x < b\}$	open
$[a,b)$	$\{x \mid a \leq x < b\}$	closed-open
$(a,b]$	$\{x \mid a < x \leq b\}$	open-closed
$[a,\infty)$	$\{x \mid a \leq x\}$	closed-infinite
(a,∞)	$\{x \mid a < x\}$	open-infinite
$(-\infty,b]$	$\{x \mid x \leq b\}$	infinite-closed
$(-\infty,b)$	$\{x \mid x < b\}$	infinite-open
$(-\infty,\infty)$	$\{x \mid x \in \mathbb{R}\}$	infinite-infinite

Table of Intervals

- Here is a table of all possibilities:

$[a,b]$	$\{x \mid a \leq x \leq b\}$	closed
(a,b)	$\{x \mid a < x < b\}$	open
$[a,b)$	$\{x \mid a \leq x < b\}$	closed-open
$(a,b]$	$\{x \mid a < x \leq b\}$	open-closed
$[a,\infty)$	$\{x \mid a \leq x\}$	closed-infinite
(a,∞)	$\{x \mid a < x\}$	open-infinite
$(-\infty,b]$	$\{x \mid x \leq b\}$	infinite-closed
$(-\infty,b)$	$\{x \mid x < b\}$	infinite-open
$(-\infty,\infty)$	$\{x \mid x \in \mathbb{R}\}$	infinite-infinite

- In the above, ∞ is *not a number* but is just a notation for writing an unbounded interval.

Rules for Inequalities

- Often we have to deal with more complicated inequalities.

Rules for Inequalities

- Often we have to deal with more complicated inequalities.
- We need rules to manipulate inequalities correctly.

Rules for Inequalities

- Often we have to deal with more complicated inequalities.
- We need rules to manipulate inequalities correctly.
- If $a < b$ then $a + c < b + c$ (add the same number to both sides)

Rules for Inequalities

- Often we have to deal with more complicated inequalities.
- We need rules to manipulate inequalities correctly.
- If $a < b$ then $a + c < b + c$ (add the same number to both sides)
- If $a < b$ and $c < d$ then $a + c < b + d$ (add two inequalities)

Rules for Inequalities

- Often we have to deal with more complicated inequalities.
- We need rules to manipulate inequalities correctly.
- If $a < b$ then $a + c < b + c$ (add the same number to both sides)
- If $a < b$ and $c < d$ then $a + c < b + d$ (add two inequalities)
- If $a < b$ and $c > 0$ then $ac < bc$ (multiply by a *positive* number)

Rules for Inequalities

- Often we have to deal with more complicated inequalities.
- We need rules to manipulate inequalities correctly.
- If $a < b$ then $a + c < b + c$ (add the same number to both sides)
- If $a < b$ and $c < d$ then $a + c < b + d$ (add two inequalities)
- If $a < b$ and $c > 0$ then $ac < bc$ (multiply by a *positive* number)
- If $a < b$ and $c < 0$ then $ac > bc$ (multiply by a *negative* number and *reverse the direction of the inequality*

Rules for Inequalities

- Often we have to deal with more complicated inequalities.
- We need rules to manipulate inequalities correctly.
- If $a < b$ then $a + c < b + c$ (add the same number to both sides)
- If $a < b$ and $c < d$ then $a + c < b + d$ (add two inequalities)
- If $a < b$ and $c > 0$ then $ac < bc$ (multiply by a *positive* number)
- If $a < b$ and $c < 0$ then $ac > bc$ (multiply by a *negative* number and *reverse the direction of the inequality*
- If $0 < a < b$ then $1/a > 1/b$ (in an inequality of positive numbers, take reciprocals and *reverse the direction of the inequality*

Multiplying Inequalities by Negatives

- When we work with equalities, we always get equalities when we perform correct algebraic manipulations.

Multiplying Inequalities by Negatives

- When we work with equalities, we always get equalities when we perform correct algebraic manipulations.
- However, when we work with inequalities, we sometimes must *reverse* the sign of the inequality.

Multiplying Inequalities by Negatives

- When we work with equalities, we always get equalities when we perform correct algebraic manipulations.
- However, when we work with inequalities, we sometimes must *reverse* the sign of the inequality.
- For example, consider the inequality $1 < 2$.

Multiplying Inequalities by Negatives

- When we work with equalities, we always get equalities when we perform correct algebraic manipulations.
- However, when we work with inequalities, we sometimes must *reverse* the sign of the inequality.
- For example, consider the inequality $1 < 2$.
- Multiplying it by -1 we get $-1 < -2$ which is incorrect!

Multiplying Inequalities by Negatives

- When we work with equalities, we always get equalities when we perform correct algebraic manipulations.
- However, when we work with inequalities, we sometimes must *reverse* the sign of the inequality.
- For example, consider the inequality $1 < 2$.
- Multiplying it by -1 we get $-1 < -2$ which is incorrect!
- When we multiply an inequality by a negative number we must reverse the inequality: $1 < 2 \implies -1 > -2$.

Multiplying Inequalities by Negatives

- When we work with equalities, we always get equalities when we perform correct algebraic manipulations.
- However, when we work with inequalities, we sometimes must *reverse* the sign of the inequality.
- For example, consider the inequality $1 < 2$.
- Multiplying it by -1 we get $-1 < -2$ which is incorrect!
- When we multiply an inequality by a negative number we must reverse the inequality: $1 < 2 \implies -1 > -2$.
- We have to be especially careful when we multiply an inequality by a variable like x: $1 < 2 \implies x < 2x$ *only if we know that x is positive*.

Multiplying Inequalities by Negatives

- When we work with equalities, we always get equalities when we perform correct algebraic manipulations.
- However, when we work with inequalities, we sometimes must *reverse* the sign of the inequality.
- For example, consider the inequality $1 < 2$.
- Multiplying it by -1 we get $-1 < -2$ which is incorrect!
- When we multiply an inequality by a negative number we must reverse the inequality: $1 < 2 \implies -1 > -2$.
- We have to be especially careful when we multiply an inequality by a variable like x: $1 < 2 \implies x < 2x$ *only if we know that x is positive*.
- If $x = 0$ then $x = 2x$.

Multiplying Inequalities by Negatives

- When we work with equalities, we always get equalities when we perform correct algebraic manipulations.
- However, when we work with inequalities, we sometimes must *reverse* the sign of the inequality.
- For example, consider the inequality $1 < 2$.
- Multiplying it by -1 we get $-1 < -2$ which is incorrect!
- When we multiply an inequality by a negative number we must reverse the inequality: $1 < 2 \implies -1 > -2$.
- We have to be especially careful when we multiply an inequality by a variable like x: $1 < 2 \implies x < 2x$ *only if we know that x is positive*.
- If $x = 0$ then $x = 2x$.
- If $x < 0$ then $x > 2x$.

Solving Linear Inequalities

- If we solve inequalities, we must be careful if we multiply or divide by a number that is negative (-3, say) or that *could be negative* (the variable a, say).

Solving Linear Inequalities

- If we solve inequalities, we must be careful if we multiply or divide by a number that is negative (-3, say) or that *could be negative* (the variable a, say).
- Keeping that in mind, we solve a linear inequality similar to how we solve a linear equality.

Solving Linear Inequalities

- If we solve inequalities, we must be careful if we multiply or divide by a number that is negative (-3, say) or that *could be negative* (the variable a, say).
- Keeping that in mind, we solve a linear inequality similar to how we solve a linear equality.
- Consider $x+1 < 3x+5$.

Solving Linear Inequalities

- If we solve inequalities, we must be careful if we multiply or divide by a number that is negative (-3, say) or that *could be negative* (the variable a, say).
- Keeping that in mind, we solve a linear inequality similar to how we solve a linear equality.
- Consider $x+1 < 3x+5$.
- Move all x's to one side, all constants to the other: $-2x < 4$

Solving Linear Inequalities

- If we solve inequalities, we must be careful if we multiply or divide by a number that is negative (-3, say) or that *could be negative* (the variable a, say).
- Keeping that in mind, we solve a linear inequality similar to how we solve a linear equality.
- Consider $x+1 < 3x+5$.
- Move all x's to one side, all constants to the other: $-2x < 4$
- Divide by the (negative!) coefficient of x: $x > -2$.

Solving Linear Inequalities

- If we solve inequalities, we must be careful if we multiply or divide by a number that is negative (-3, say) or that *could be negative* (the variable a, say).
- Keeping that in mind, we solve a linear inequality similar to how we solve a linear equality.
- Consider $x+1 < 3x+5$.
- Move all x's to one side, all constants to the other: $-2x < 4$
- Divide by the (negative!) coefficient of x: $x > -2$.
- Express the solution set as an interval if you like: $\{x| -2 < x\} = (-2, \infty)$

Solving Multiple Inequalities

- Sometimes we need to solve inequalities like $11 \geq 2 - 3x > 5$

Solving Multiple Inequalities

- Sometimes we need to solve inequalities like $11 \geq 2 - 3x > 5$
- In that case, we break it into two separate problems and recombine at the end.

Solving Multiple Inequalities

- Sometimes we need to solve inequalities like $11 \geq 2 - 3x > 5$
- In that case, we break it into two separate problems and recombine at the end.
- $11 \geq 2 - 3x \implies 9 \geq -3x \implies -3 \leq x$

Solving Multiple Inequalities

- Sometimes we need to solve inequalities like $11 \geq 2 - 3x > 5$
- In that case, we break it into two separate problems and recombine at the end.
- $11 \geq 2 - 3x \implies 9 \geq -3x \implies -3 \leq x$
- $2 - 3x > 5 \implies -3x > 3 \implies x < -1$

Solving Multiple Inequalities

- Sometimes we need to solve inequalities like $11 \geq 2 - 3x > 5$
- In that case, we break it into two separate problems and recombine at the end.
- $11 \geq 2 - 3x \implies 9 \geq -3x \implies -3 \leq x$
- $2 - 3x > 5 \implies -3x > 3 \implies x < -1$
- Altogether the solution set is $\{x | -3 \leq x < -1\} = [-3, -1)$

Solving Multiple Inequalities

- Sometimes we need to solve inequalities like $11 \geq 2 - 3x > 5$
- In that case, we break it into two separate problems and recombine at the end.
- $11 \geq 2 - 3x \implies 9 \geq -3x \implies -3 \leq x$
- $2 - 3x > 5 \implies -3x > 3 \implies x < -1$
- Altogether the solution set is $\{x | -3 \leq x < -1\} = [-3, -1)$
- Note that the "sub-goal" in each "sub-problem" was to *isolate the variable x*.

Solving Multiple Inequalities

- Sometimes we need to solve inequalities like $11 \geq 2 - 3x > 5$
- In that case, we break it into two separate problems and recombine at the end.
- $11 \geq 2 - 3x \implies 9 \geq -3x \implies -3 \leq x$
- $2 - 3x > 5 \implies -3x > 3 \implies x < -1$
- Altogether the solution set is $\{x | -3 \leq x < -1\} = [-3, -1)$
- Note that the "sub-goal" in each "sub-problem" was to *isolate the variable x*.
- That makes it possible to recombine the sub-results.

Solving Multiple Inequalities

- Sometimes we need to solve inequalities like $11 \geq 2 - 3x > 5$
- In that case, we break it into two separate problems and recombine at the end.
- $11 \geq 2 - 3x \implies 9 \geq -3x \implies -3 \leq x$
- $2 - 3x > 5 \implies -3x > 3 \implies x < -1$
- Altogether the solution set is $\{x | -3 \leq x < -1\} = [-3, -1)$
- Note that the "sub-goal" in each "sub-problem" was to *isolate the variable x*.
- That makes it possible to recombine the sub-results.
- Sometimes when we recombine the sub-results, we find that some of them are *redundant* in which case they can be dropped.

Solving Multiple Inequalities

- Sometimes we need to solve inequalities like $11 \geq 2 - 3x > 5$
- In that case, we break it into two separate problems and recombine at the end.
- $11 \geq 2 - 3x \implies 9 \geq -3x \implies -3 \leq x$
- $2 - 3x > 5 \implies -3x > 3 \implies x < -1$
- Altogether the solution set is $\{x | -3 \leq x < -1\} = [-3, -1)$
- Note that the "sub-goal" in each "sub-problem" was to *isolate the variable x*.
- That makes it possible to recombine the sub-results.
- Sometimes when we recombine the sub-results, we find that some of them are *redundant* in which case they can be dropped.
- Sometimes when we recombine the sub-results, we find that some of them are *contradictory* in which case the solution set is the *empty set* $\varnothing$.

Solving Inequalities Involving Quadratics

- Solving inequalities involving quadratics can be difficult.

Solving Inequalities Involving Quadratics

- Solving inequalities involving quadratics can be difficult.
- Consider for example $x^2 - 4x + 3 < 0$.

Solving Inequalities Involving Quadratics

- Solving inequalities involving quadratics can be difficult.
- Consider for example $x^2 - 4x + 3 < 0$.
- Factoring, we have $(x-1)(x-3) < 0$.

Solving Inequalities Involving Quadratics

- Solving inequalities involving quadratics can be difficult.
- Consider for example $x^2 - 4x + 3 < 0$.
- Factoring, we have $(x-1)(x-3) < 0$.
- If we divide by $x-1$ we have to be careful about the cases $x-1=0$ and $x-1<0$.

Solving Inequalities Involving Quadratics

- Solving inequalities involving quadratics can be difficult.
- Consider for example $x^2 - 4x + 3 < 0$.
- Factoring, we have $(x-1)(x-3) < 0$.
- If we divide by $x-1$ we have to be careful about the cases $x-1=0$ and $x-1<0$.
- Similarly, if we divide by $x-3$, we have to be careful about the cases $x-3=0$ and $x-3<0$.

Solving Inequalities Involving Quadratics

- Solving inequalities involving quadratics can be difficult.
- Consider for example $x^2 - 4x + 3 < 0$.
- Factoring, we have $(x-1)(x-3) < 0$.
- If we divide by $x-1$ we have to be careful about the cases $x-1=0$ and $x-1<0$.
- Similarly, if we divide by $x-3$, we have to be careful about the cases $x-3=0$ and $x-3<0$.
- In general, it is best to avoid dividing by an expression containing a variable.

Solving Inequalities Involving Quadratics

- Solving inequalities involving quadratics can be difficult.
- Consider for example $x^2 - 4x + 3 < 0$.
- Factoring, we have $(x-1)(x-3) < 0$.
- If we divide by $x-1$ we have to be careful about the cases $x-1 = 0$ and $x-1 < 0$.
- Similarly, if we divide by $x-3$, we have to be careful about the cases $x-3 = 0$ and $x-3 < 0$.
- In general, it is best to avoid dividing by an expression containing a variable.
- It is best to find another approach. We will talk about a graphical approach later. Now we will consider using tables.

Using Tables to Solve Inequalities

- Suppose we want to solve $(x-1)(x-3) < 0$.

Using Tables to Solve Inequalities

- Suppose we want to solve $(x-1)(x-3) < 0$.
- We look for places where the sign of the expression $(x-1)(x-3)$ changes.

Using Tables to Solve Inequalities

- Suppose we want to solve $(x-1)(x-3) < 0$.
- We look for places where the sign of the expression $(x-1)(x-3)$ changes.
- It changes at $x = 1$ and $x = 3$, so we make up the following table.

I	$(x-1)$	$(x-3)$	$(x-1)(x-3)$
$x < 1$			
$x = 1$			
$1 < x < 3$			
$x = 3$			
$3 < x$			

Using Tables to Solve Inequalities

- Suppose we want to solve $(x-1)(x-3) < 0$.
- We look for places where the sign of the expression $(x-1)(x-3)$ changes.
- It changes at $x = 1$ and $x = 3$, so we make up the following table.

I	$(x-1)$	$(x-3)$	$(x-1)(x-3)$
$x < 1$	$-$		
$x = 1$	0		
$1 < x < 3$	$+$		
$x = 3$	$+$		
$3 < x$	$+$		

Using Tables to Solve Inequalities

- Suppose we want to solve $(x-1)(x-3) < 0$.
- We look for places where the sign of the expression $(x-1)(x-3)$ changes.
- It changes at $x = 1$ and $x = 3$, so we make up the following table.

I	$(x-1)$	$(x-3)$	$(x-1)(x-3)$
$x < 1$	$-$	$-$	
$x = 1$	0	$-$	
$1 < x < 3$	$+$	$-$	
$x = 3$	$+$	0	
$3 < x$	$+$	$+$	

Using Tables to Solve Inequalities

- Suppose we want to solve $(x-1)(x-3) < 0$.
- We look for places where the sign of the expression $(x-1)(x-3)$ changes.
- It changes at $x = 1$ and $x = 3$, so we make up the following table.

I	$(x-1)$	$(x-3)$	$(x-1)(x-3)$
$x < 1$	$-$	$-$	$+$
$x = 1$	0	$-$	0
$1 < x < 3$	$+$	$-$	$-$
$x = 3$	$+$	0	0
$3 < x$	$+$	$+$	$+$

Using Tables to Solve Inequalities

- Suppose we want to solve $(x-1)(x-3) < 0$.
- We look for places where the sign of the expression $(x-1)(x-3)$ changes.
- It changes at $x = 1$ and $x = 3$, so we make up the following table.

I	$(x-1)$	$(x-3)$	$(x-1)(x-3)$
$x < 1$	$-$	$-$	$+$
$x = 1$	0	$-$	0
$1 < x < 3$	$+$	$-$	$-$
$x = 3$	$+$	0	0
$3 < x$	$+$	$+$	$+$

- Now we can easily see that the only interval on which $(x-1)(x-3)$ is negative is $1 < x < 3$.

Using Tables to Solve Inequalities

- Suppose we want to solve $(x-1)(x-3) < 0$.
- We look for places where the sign of the expression $(x-1)(x-3)$ changes.
- It changes at $x = 1$ and $x = 3$, so we make up the following table.

I	$(x-1)$	$(x-3)$	$(x-1)(x-3)$
$x < 1$	$-$	$-$	$+$
$x = 1$	0	$-$	0
$1 < x < 3$	$+$	$-$	$-$
$x = 3$	$+$	0	0
$3 < x$	$+$	$+$	$+$

- Now we can easily see that the only interval on which $(x-1)(x-3)$ is negative is $1 < x < 3$.
- Therefore the solution set to the inequality $(x-1)(x-3) < 0$ is $\{x | 1 < x < 3\} = (1,3)$.

Definition of Absolute Value

- The absolute value of a number is the magnitude of the number, ignoring its sign.

Definition of Absolute Value

- The absolute value of a number is the magnitude of the number, ignoring its sign.
- The absolute value of x is written $|x|$.

Definition of Absolute Value

- The absolute value of a number is the magnitude of the number, ignoring its sign.
- The absolute value of x is written $|x|$.
- For example, $|5| = 5$, $|-2| = 2$, $|\pi| = \pi$, etc.

Definition of Absolute Value

- The absolute value of a number is the magnitude of the number, ignoring its sign.
- The absolute value of x is written $|x|$.
- For example, $|5| = 5$, $|-2| = 2$, $|\pi| = \pi$, etc.
- Formally we define it using *cases*, as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Definition of Absolute Value

- The absolute value of a number is the magnitude of the number, ignoring its sign.
- The absolute value of x is written $|x|$.
- For example, $|5| = 5$, $|-2| = 2$, $|\pi| = \pi$, etc.
- Formally we define it using *cases*, as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- We read that cases construction from the right to the left. First we decide which condition the x we are looking at satisfies, whether $x \geq 0$ or $x < 0$. Then we look to the left of the "if" clause to decide which value we take for $|x|$.

Algebraic Properties of Absolute Value

- There is another way to define the absolute value:

$$|x| = \sqrt{x^2}$$

Algebraic Properties of Absolute Value

- There is another way to define the absolute value:

$$|x| = \sqrt{x^2}$$

- Think about what happens in that expression when $x \geq 0$ and when $x < 0$.

Algebraic Properties of Absolute Value

- There is another way to define the absolute value:

$$|x| = \sqrt{x^2}$$

- Think about what happens in that expression when $x \geq 0$ and when $x < 0$.
- It follows by the laws of exponents that the absolute value behaves “nicely” with respect to multiplication.

Algebraic Properties of Absolute Value

- There is another way to define the absolute value:

$$|x| = \sqrt{x^2}$$

- Think about what happens in that expression when $x \geq 0$ and when $x < 0$.
- It follows by the laws of exponents that the absolute value behaves “nicely” with respect to multiplication.
- We have

$$|ab| = |a|\,|b| \qquad \left|\frac{a}{b}\right| = \frac{|a|}{|b|} \qquad |a^n| = |a|^n$$

Inequalities and Absolute Value

- Often we use absolute values in conjunction with inequalities.

Inequalities and Absolute Value

- Often we use absolute values in conjunction with inequalities.
- For example, we might see an expression of the form $|x| < 2$.

Inequalities and Absolute Value

- Often we use absolute values in conjunction with inequalities.
- For example, we might see an expression of the form $|x| < 2$.
- That means x is less than 2 in magnitude.

Inequalities and Absolute Value

- Often we use absolute values in conjunction with inequalities.
- For example, we might see an expression of the form $|x| < 2$.
- That means x is less than 2 in magnitude.
- That means $-2 < x < 2$. So we have translated $|x| < 2$ into an inequality without the absolute value sign.

- Often we use absolute values in conjunction with inequalities.
- For example, we might see an expression of the form $|x| < 2$.
- That means x is less than 2 in magnitude.
- That means $-2 < x < 2$. So we have translated $|x| < 2$ into an inequality without the absolute value sign.
- In summary,

$$|x| < a \Leftrightarrow -a < x \text{ and } x < a$$

$$|x| = a \Leftrightarrow x = -a \text{ or } x = a$$

$$|x| > a \Leftrightarrow x < -a \text{ or } a < x$$

Solving Absolute Value Equations

- Often we have to solve more complicated equations involving absolute values.

Solving Absolute Value Equations

- Often we have to solve more complicated equations involving absolute values.
- For example, consider the equation $|2x+1| = 5$.

Solving Absolute Value Equations

- Often we have to solve more complicated equations involving absolute values.
- For example, consider the equation $|2x+1| = 5$.
- We use the equivalence on the previous slide to rewrite the equation as

$$2x+1 = -5 \text{ or } 2x+1 = 5$$

Solving Absolute Value Equations

- Often we have to solve more complicated equations involving absolute values.
- For example, consider the equation $|2x+1| = 5$.
- We use the equivalence on the previous slide to rewrite the equation as

$$2x+1 = -5 \text{ or } 2x+1 = 5$$

- We solve each of those equalities separately. $2x+1 = -5$ implies $2x = -6$ implies $x = -3$.

Solving Absolute Value Equations

- Often we have to solve more complicated equations involving absolute values.
- For example, consider the equation $|2x+1| = 5$.
- We use the equivalence on the previous slide to rewrite the equation as

$$2x+1 = -5 \text{ or } 2x+1 = 5$$

- We solve each of those equalities separately. $2x+1 = -5$ implies $2x = -6$ implies $x = -3$.
- Similarly, $2x+1 = 5$ implies $2x = 4$ implies $x = 2$.

Solving Absolute Value Equations

- Often we have to solve more complicated equations involving absolute values.
- For example, consider the equation $|2x+1| = 5$.
- We use the equivalence on the previous slide to rewrite the equation as
$$2x+1=-5 \text{ or } 2x+1=5$$
- We solve each of those equalities separately. $2x+1=-5$ implies $2x=-6$ implies $x=-3$.
- Similarly, $2x+1=5$ implies $2x=4$ implies $x=2$.
- Therefore the solution set is $\{-3,2\}$.

Solving Absolute Value Equations

- Often we have to solve more complicated equations involving absolute values.
- For example, consider the equation $|2x+1| = 5$.
- We use the equivalence on the previous slide to rewrite the equation as

$$2x+1 = -5 \text{ or } 2x+1 = 5$$

- We solve each of those equalities separately. $2x+1 = -5$ implies $2x = -6$ implies $x = -3$.
- Similarly, $2x+1 = 5$ implies $2x = 4$ implies $x = 2$.
- Therefore the solution set is $\{-3, 2\}$.
- Check by substituting both those numbers into the original equation $|2x+1| = 5$.

Solving Absolute Value Inequalities

- Similarly, we often have to solve more complicated inequalities involving absolute values.

Solving Absolute Value Inequalities

- Similarly, we often have to solve more complicated inequalities involving absolute values.
- For example, consider the inequality $|2x+1| < 5$.

Solving Absolute Value Inequalities

- Similarly, we often have to solve more complicated inequalities involving absolute values.
- For example, consider the inequality $|2x+1| < 5$.
- Using the equivalences on a previous slide, we can translate that into an expression that does not involve absolute values

Solving Absolute Value Inequalities

- Similarly, we often have to solve more complicated inequalities involving absolute values.
- For example, consider the inequality $|2x+1| < 5$.
- Using the equivalences on a previous slide, we can translate that into an expression that does not involve absolute values
- Namely, we have

$$-5 < 2x+1 \text{ and } 2x+1 < 5$$

Solving Absolute Value Inequalities

- Similarly, we often have to solve more complicated inequalities involving absolute values.
- For example, consider the inequality $|2x+1| < 5$.
- Using the equivalences on a previous slide, we can translate that into an expression that does not involve absolute values
- Namely, we have

$$-5 < 2x+1 \text{ and } 2x+1 < 5$$

- Solving the first inequality gives $-6 < 2x$ or $-3 < x$

Solving Absolute Value Inequalities

- Similarly, we often have to solve more complicated inequalities involving absolute values.
- For example, consider the inequality $|2x+1| < 5$.
- Using the equivalences on a previous slide, we can translate that into an expression that does not involve absolute values
- Namely, we have

$$-5 < 2x+1 \text{ and } 2x+1 < 5$$

- Solving the first inequality gives $-6 < 2x$ or $-3 < x$
- Solving the second inequality gives $2x < 4$ or $x < 2$

Solving Absolute Value Inequalities

- Similarly, we often have to solve more complicated inequalities involving absolute values.
- For example, consider the inequality $|2x+1| < 5$.
- Using the equivalences on a previous slide, we can translate that into an expression that does not involve absolute values
- Namely, we have

$$-5 < 2x+1 \text{ and } 2x+1 < 5$$

- Solving the first inequality gives $-6 < 2x$ or $-3 < x$
- Solving the second inequality gives $2x < 4$ or $x < 2$
- The condition that x must satisfy is "$-3 < x$ and $x < 2$"

Solving Absolute Value Inequalities

- Similarly, we often have to solve more complicated inequalities involving absolute values.
- For example, consider the inequality $|2x+1| < 5$.
- Using the equivalences on a previous slide, we can translate that into an expression that does not involve absolute values
- Namely, we have

$$-5 < 2x+1 \text{ and } 2x+1 < 5$$

- Solving the first inequality gives $-6 < 2x$ or $-3 < x$
- Solving the second inequality gives $2x < 4$ or $x < 2$
- The condition that x must satisfy is "$-3 < x$ and $x < 2$"
- We can write it as a single inequality: $-3 < x < 2$

The Triangle Inequality

- We will need one more general result about absolute values.

The Triangle Inequality

- We will need one more general result about absolute values.
- The *triangle inequality* says that the absolute value of a sum is less than or equal to the sum of the absolute values.

$$|x+y| \leq |x| + |y|$$

The Triangle Inequality

- We will need one more general result about absolute values.
- The *triangle inequality* says that the absolute value of a sum is less than or equal to the sum of the absolute values.

$$|x+y| \leq |x|+|y|$$

- If x and y both have the same sign, the two sides are equal. (Try some examples.)

The Triangle Inequality

- We will need one more general result about absolute values.
- The *triangle inequality* says that the absolute value of a sum is less than or equal to the sum of the absolute values.

$$|x+y| \leq |x|+|y|$$

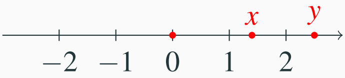
- If x and y both have the same sign, the two sides are equal. (Try some examples.)
- If x and y have opposite signs, $|x+y|$ is less than $|x|+|y|$ because x cancels off part of y in $x+y$.

The Triangle Inequality

- We will need one more general result about absolute values.
- The *triangle inequality* says that the absolute value of a sum is less than or equal to the sum of the absolute values.

$$|x+y| \leq |x|+|y|$$

- If x and y both have the same sign, the two sides are equal. (Try some examples.)
- If x and y have opposite signs, $|x+y|$ is less than $|x|+|y|$ because x cancels off part of y in $x+y$.
- It's called the *triangle* inequality because the numbers 0, x, and y make a (rather flat) triangle in the number line. If x and y have the same sign:

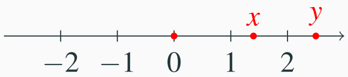


The Triangle Inequality

- We will need one more general result about absolute values.
- The *triangle inequality* says that the absolute value of a sum is less than or equal to the sum of the absolute values.

$$|x+y| \leq |x|+|y|$$

- If x and y both have the same sign, the two sides are equal. (Try some examples.)
- If x and y have opposite signs, $|x+y|$ is less than $|x|+|y|$ because x cancels off part of y in $x+y$.
- It's called the *triangle* inequality because the numbers 0, x, and y make a (rather flat) triangle in the number line. If x and y have the same sign:



- If x and y have opposite signs: