

# **MATH 110 Lecture 1.6**

## Calculating Limits Using the Limit Laws

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Edward Doolittle

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Department of Indigenous Knowledge and Science  
First Nations University of Canada

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## Limit Laws

Suppose that the limits  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  exist. Then

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If the condition in the last rule above is violated, we can't conclude anything about the limit without further work.

## The Limit Laws in Words

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If the condition in the the last rule above is violated, the rule does not apply.

## Example of the Limit Laws

Suppose that we know that  $\lim_{x \rightarrow 3} f(x) = 5$ ,  $\lim_{x \rightarrow 3} g(x) = -2$ , and  $\lim_{x \rightarrow 3} h(x) = 0$ . Then the limit laws tell us that

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## A More Complicated Example of the Limit Laws

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Then we evaluate those in order.

1.  $\lim_{x \rightarrow 3} g(x)h(x) = \lim_{x \rightarrow 3} g(x) \cdot \lim_{x \rightarrow 3} h(x) = -2 \cdot 0 = 0$
2.  $\lim_{x \rightarrow 3} [f(x) - g(x)h(x)] = \lim_{x \rightarrow 3} f(x) - \lim_{x \rightarrow 3} g(x)h(x) = 5 - 0 = 5$

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## Consequences of the Limit Laws

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- If  $f$  is a polynomial, then  $\lim_{x \rightarrow a} f(x) = f(a)$
- If  $f = g/h$  is a rational function (i.e.,  $g$  and  $h$  are polynomials),  
**and**  $h(a) \neq 0$ , then  $\lim_{x \rightarrow a} f(x) = \frac{g(a)}{h(a)}$ .

# Limit Theorems

1. **Single Discrepancy Theorem:** If  $f(x) = g(x)$  when  $x \neq a$ , then  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$ , provided the limits exist.

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4. **Squeeze Theorem:** If  $f(x) \leq g(x) \leq h(x)$  for all  $x \neq a$ , and  $\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x)$ , then  $\lim_{x \rightarrow a} g(x) = L$ .

## Evaluation of Limits of Rational Functions

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Unfortunately, that equation is not true! It fails for  $x = 1$ , in which case the left hand side is undefined but the right hand side is  $1/2$ .

## Application of the Single Discrepancy Theorem

On the other hand, the equality

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This is a common technique for evaluating limits of the form

$$\lim_{x \rightarrow a} \frac{g(x)}{h(x)}$$

where  $h(a) = 0$ .

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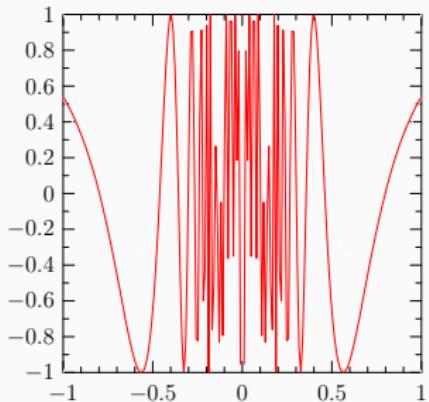
But we can't remove the question mark because

$$\lim_{x \rightarrow 0} \cos(1/x^2)$$

does not exist!

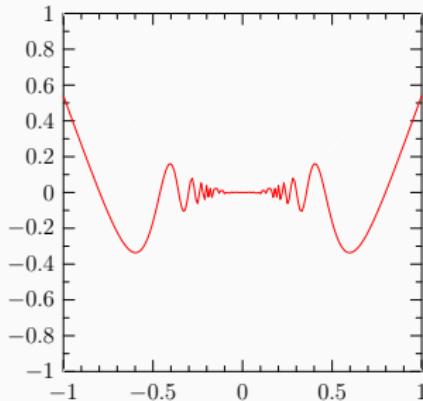
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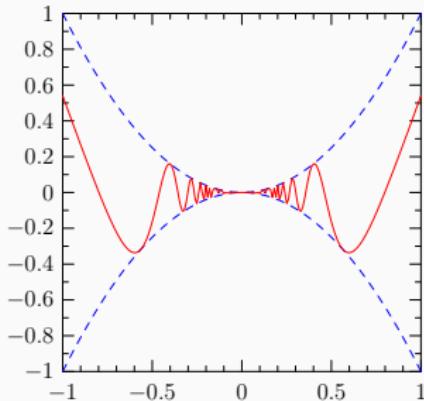
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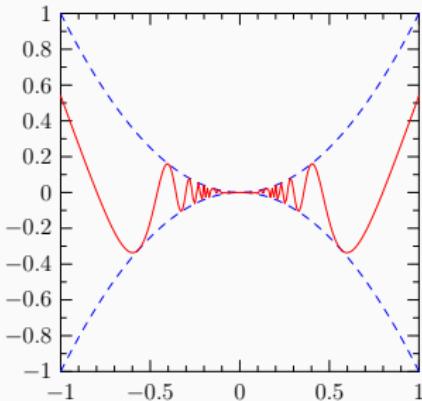
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- We have  $-x^2 \leq x^2 \cos(1/x^2) \leq x^2$  (why?) and  $\lim_{x \rightarrow 0} -x^2 = 0 = \lim_{x \rightarrow 0} x^2$ . It follows that  $\lim_{x \rightarrow 0} x^2 \cos(1/x^2) = 0$ .



## Examples

1. Evaluate

$$\lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2}$$

2. Evaluate

$$\lim_{s \rightarrow 16} \frac{4 - \sqrt{s}}{s - 16}$$

3. Find

$$\lim_{x \rightarrow 0} x \sin(1/x)$$

## Exercises

Now you should work on Problem Set 1.6. After you have finished it, you should try the following additional exercises from Section 1.6:

1.6 C-level: 1, 2, 3–9, 10, 11–26, 50, 52;

B-level: 27–30, 31–32, 33–34, 35–40, 47–48, 49, 51;

A-level: 41–46, 53–55, 56–66