MATH 110 Problem Set 3.3 Solutions

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1. (a) The derivatives are

$$f'(x) = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(x+1)(2x-1) = 12(x+1)\left(x - \frac{1}{2}\right)$$
$$f''(x) = 24x + 6 = 24\left(x + \frac{1}{4}\right)$$

The critical numbers are where $f'(x)$ does not exist (nowhere) and where $f'(x) = 0$, namely $x = -1$ and $x = 1/2$. With those critical numbers we create Table 1, from which we can see that f is increasing on $(-\infty, -1)$ and $(1/2, \infty)$ and decreasing on $(-1, 1/2)$, and furthermore by the First Derivative Test that the critical number $x = -1$ is a local maximum and the critical number $x = 1/2$ is a local minimum. Note that the question asks for the *values* at the local maxima and

Interval	$x+1$	$x-1/2$	$f'(x)$	f
$-\infty < x < -1$	$-$	$-$	$+$	increasing
$x = -1$	0	$-$	0	stationary
$-1 < x < 1/2$	$+$	$-$	$-$	decreasing
$x = 1/2$	$+$	0	0	stationary
$1/2 < x < \infty$	$+$	$+$	$+$	increasing

Table 1: Intervals of Increase and Decrease for problem 1a

minima, which we calculate using $f(x) = 4x^3 + 3x^2 - 6x + 1$:

$$f(-1) = 4(-1)^3 + 3(-1)^2 - 6(-1) + 1 = -4 + 3 + 6 + 1 = 6$$
$$f(1/2) = 4(1/2)^3 + 3(1/2)^2 - 6(1/2) + 1 = 1/2 + 3/4 - 3 + 1 = -3/4$$

The potential inflection numbers are where f'' does not exist (nowhere) and where $f''(x) = 0$, i.e., at $x = -1/4$. The table for the second derivative, Table 2, shows that f is concave down on $(-\infty, -1/4)$, concave up on $(-1/4, \infty)$, and has a point of inflection at $x = -1/4$ (because the concavity changes across that point). Note that the question asks for the *inflection points* of f,

Interval	$x+1/4$	$f''(x)$	f
$-\infty < x < -1/4$	$-$	$-$	concave down
$x = -1/4$	0	0	inflection
$-1/4 < x < \infty$	$+$	$+$	concave up

Table 2: Intervals of Concavity for problem 1a

which requires evaluating f at the inflection number $x = -1/4$:

$$f(-1/4) = 4(-1/4)^3 + 3(-1/4)^2 - 6(-1/4) + 1 = -1/16 + 3/16 + 3/2 + 1 = 1/8 + 12/8 + 8/8 = 21/8$$

so the inflection point of f is $(-1/4, 21/8)$.

(b) The derivatives are

$$f'(x) = \frac{2x(x^2+3) - x^2(2x)}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}$$
$$f''(x) = \frac{6(x^2+3)^2 - 6x(2)(x^2+3)(2x)}{(x^2+3)^4} = \frac{6(x^2+3) - 24x^2}{(x^2+3)^3} = \frac{18(1-x^2)}{(x^2+3)^3} = -18\frac{(x+1)(x-1)}{(x^2+3)^3}$$

The critical numbers for f are where $f'(x)$ does not exist (i.e., where the denominator of f' is 0, i.e., nowhere because $x^2+3 \geq 3 > 0$) and where $f'(x) = 0$ (i.e., where the numerator of f' is 0, i.e., at $x = 0$). We make a table as usual, except that we don't need to include a column for the factor $x^2 + 3$ because that factor and all its powers is always positive so does not influence the sign of f'. See Table 3. From the table we can see that f is decreasing on $(-\infty, 0)$ and increasing

Interval	x	$f'(x)$	f
$-\infty < x < 0$	$-$	$-$	decreasing
$x = 0$	0	0	stationary
$0 < x < \infty$	$+$	$+$	increasing

Table 3: Intervals of Increase/Decrease for problem 1b

on $(0, \infty)$, and by the First Derivative Test the critical number at $x = 0$ is a local minimum. The value of f at the local min is $f(0) = 0^2/(0^2 + 3) = 0$.

The potential inflection numbers are where f' is undefined (nowhere, for the same reasons as above) and where $f'(x) = 0$, i.e., at $x = -1$ and $x = 1$. Using those numbers we prepare Table 4. Note that in addition to the factors $(x+1)$ and $(x-1)$, $f''(x)$ has a negative coefficient which affects its sign and the positive factor $(x^2+3)^{-3}$ which doesn't affect its sign. From the

Interval	-18	$x+1$	$x-1$	$f''(x)$	f
$-\infty < x < -1$	$-$	$-$	$-$	$-$	concave down
$x = -1$	$-$	0	$-$	0	inflection
$-1 < x < 1$	$-$	$+$	$-$	$+$	concave up
$x = 1$	$-$	$+$	0	0	inflection
$1 < x < \infty$	$-$	$+$	$+$	$-$	concave down

Table 4: Intervals of Concavity for problem 1b

table we see that f is concave down on $(-\infty, -1)$ and $(1, \infty)$, and f is concave up on $(-1, 1)$. Since the concavity changes across the potential inflection numbers $x = -1$ and $x = 1$, those are actual inflection numbers. The corresponding points of inflection are $(-1, f(-1)) = (-1, 1/4)$ and $(1, f(1)) = (1, 1/4)$.

(c) The derivatives of f are

$$f'(x) = 2\cos x(-\sin x) - 2\cos x = -2\cos x(\sin x + 1)$$
$$f''(x) = 2\sin x(\sin x + 1) - 2\cos x(\cos x) = 2\sin^2 x + 2\sin x - 2\cos^2 x = 2\sin^2 x + 2\sin x - 2(1 - \sin^2 x)$$
$$= 4\sin^2 x + 2\sin x - 2 = 4(\sin x - 1/2)(\sin x + 1)$$

The critical numbers of $f(x)$ are where $f'(x)$ is undefined (nowhere) and where $f'(x) = 0$, i.e., where $\cos x = 0$ or where $\sin x + 1 = 0$. The former is at $x = \pi/2$ and $x = 3\pi/2$ and the latter at $x = 3\pi/2$. (Note that we only have to solve those equations on the interval $[0, 2\pi]$ because that restriction is given in the problem.) We use those numbers to produce Table 5. In this case it's not so easy to determine the signs of the factors; it helps to draw quick graphs of the sin and cos

Interval	-2	$\cos x$	$\sin x + 1$	$f'(x)$	f
$0 < x < \pi/2$	$-$	$+$	$+$	$-$	decreasing
$x = \pi/2$	$-$	0	$+$	0	stationary
$\pi/2 < x < 3\pi/2$	$-$	$-$	$+$	$+$	increasing
$x = 3\pi/2$	$-$	0	0	0	stationary
$3\pi/2 < x < 2\pi$	$-$	$+$	$+$	$-$	decreasing

Table 5: Intervals of Increasing/Decreasing for problem 1c

functions. In summary, f is decreasing on the intervals $(0, \pi/2)$ and $(3\pi/2, 2\pi)$ and increasing on the interval $(\pi/2, 3\pi/2)$. By the first derivative test the critical number $x = \pi/2$ is a local minimum and the critical number $x = 3\pi/2$ is a local maximum. The value of f at the local min is $f(\pi/2) = \cos^2(\pi/2) - 2\sin(\pi/2) = (0)^2 - 2(1) = -2$. The value of f at the local max is $f(3\pi/2) = \cos^2(3\pi/2) - 2\sin(3\pi/2) = (0)^2 - 2(-1) = 2$.

The potential inflection numbers are where f'' does not exist (nowhere) and where $f''(x) = 0$, i.e., where $\sin x = 1/2$ or $\sin x = -1$. From a graph of sin for $0 \le x \le 2\pi$, we have $x = \pi/6$ or $x = 5\pi/6$ for the first case and $x = 3\pi/2$ for the second. See Table 6. In summary, f is concave up

Interval	$\sin x - 1/2$	$\sin x + 1$	$f''(x)$	f
$0 < x < \pi/6$	$-$	$+$	$-$	conc. down
$x = \pi/6$	0	$+$	0	inflection
$\pi/6 < x < 5\pi/6$	$+$	$+$	$+$	conc. up
$x = 5\pi/6$	0	$+$	0	inflection
$5\pi/6 < x < 3\pi/2$	$-$	$+$	$-$	conc. down
$x = 3\pi/2$	$-$	0	0	non-inflection
$3\pi/2 < x < 2\pi$	$-$	$+$	$-$	conc. down

Table 6: Intervals of Concavity for problem 1c

on $(\pi/6, 5\pi/6)$ and concave down on the intervals $(0, \pi/6)$, $(5\pi/6, 3\pi/2)$, and $(3\pi/2, 2\pi)$. There are inflection numbers at $x = \pi/6$ and $x = 5\pi/6$, but the potential inflection number at $x = 3\pi/2$ is not an inflection number because the concavity does not change as we cross $x = 3\pi/2$. (In fact, the function is considered to be concave down on the entire interval $(5\pi/6, 2\pi)$ including the point $x = 3\pi/2$, but we can't justify that statement given what we now know.) The inflection points are

$$(\pi/6, f(\pi/6)) = (\pi/6, \cos^2(\pi/6) - 2\sin(\pi/6)) = (\pi/6, 3/4 - 2(1/2)) = (\pi/6, -1/4)$$
$$(5\pi/6, f(5\pi/6)) = (5\pi/6, \cos^2(5\pi/6) - 2\sin(5\pi/6)) = (5\pi/6, 3/4 - 2(1/2)) = (5\pi/6, -1/4)$$

2. (a) The derivatives are

$$h'(x) = 5x^4 - 6x^2 + 1$$
$$h''(x) = 20x^3 - 12x$$

We don't know a general method to factor a fourth degree polynomial (or a third degree polynomial) so we have to rely on tricks to factor h' and h''. For h', note that we have no x or x^3 term, just terms with even powers of x. A trick we can use in this situation is to make a change of variable $u = x^2$ and then factor the resulting quadratic:

$$\begin{aligned} h'(x) &= 5x^4 - 6x^2 + 1 = 5u^2 - 6u + 1 = (5u - 1)(u - 1) = (5x^2 - 1)(x^2 - 1) \\ &= 5(x + 1)\left(x + \frac{1}{\sqrt{5}}\right)\left(x - \frac{1}{\sqrt{5}}\right)(x - 1) \end{aligned}$$

We factor h'' by taking out the common factor x first:

$$h''(x) = 20x\left(x^2 - \frac{3}{5}\right) = 20\left(x + \frac{\sqrt{3}}{\sqrt{5}}\right)x\left(x - \frac{\sqrt{3}}{\sqrt{5}}\right)$$

The critical numbers are the numbers x where $h'(x)$ does not exist (nowhere) and where $h'(x) = 0$ ($x = -1$, $x = -1/\sqrt{5}$, $x = 1/\sqrt{5}$, and $x = 1$). We obtain Table 7. We see that f is decreasing on

Interval	$x+1$	$x+1/\sqrt{5}$	$x-1/\sqrt{5}$	$x-1$	$h'(x)$	h
$-\infty < x < -1$	$-$	$-$	$-$	$-$	$+$	increasing
$x = -1$	0	$-$	$-$	$-$	0	stationary
$-1 < x < -1/\sqrt{5}$	$+$	$-$	$-$	$-$	$-$	decreasing
$x = -1/\sqrt{5}$	$+$	0	$-$	$-$	0	stationary
$-1/\sqrt{5} < x < 1/\sqrt{5}$	$+$	$+$	$-$	$-$	$+$	increasing
$x = 1/\sqrt{5}$	$+$	$+$	0	$-$	0	stationary
$1/\sqrt{5} < x < 1$	$+$	$+$	$+$	$-$	$-$	decreasing
$x = 1$	$+$	$+$	$+$	0	0	stationary
$1 < x < \infty$	$+$	$+$	$+$	$+$	$+$	increasing

Table 7: Intervals of Increase/Decrease for problem 2a

the intervals $(-1, -1/\sqrt{5})$ and and $(1/\sqrt{5}, 1)$ and increasing on $(-\infty, -1)$, $(-1/\sqrt{5}, 1/\sqrt{5})$, and $(1, \infty)$. By the First Derivative Test there are local maxima at $x = -1$ and $x = 1/\sqrt{5}$, and local minima at $x = -1/\sqrt{5}$ and $x = 1$.

The potential inflection points are where h'' is undefined (nowhere) and where $h''(x) = 0$ ($x = -\sqrt{3}/\sqrt{5}$, $x = 0$, and $x = \sqrt{3}/\sqrt{5}$). Using those numbers we develop Table 8. In summary,

Interval	$x+\sqrt{3}/\sqrt{5}$	x	$x-\sqrt{3}/\sqrt{5}$	$h'(x)$	h
$-\infty < x < -\sqrt{3}/\sqrt{5}$	$-$	$-$	$-$	$-$	concave down
$x = -\sqrt{3}/\sqrt{5}$	0	$-$	$-$	0	inflection
$-\sqrt{3}/\sqrt{5} < x < 0$	$+$	$-$	$-$	$+$	concave up
$x = 0$	$+$	0	$-$	0	inflection
$0 < x < \sqrt{3}/\sqrt{5}$	$+$	$+$	$-$	$-$	concave down
$x = \sqrt{3}/\sqrt{5}$	$+$	$+$	0	0	inflection
$\sqrt{3}/\sqrt{5} < x < \infty$	$+$	$+$	$+$	$+$	concave up

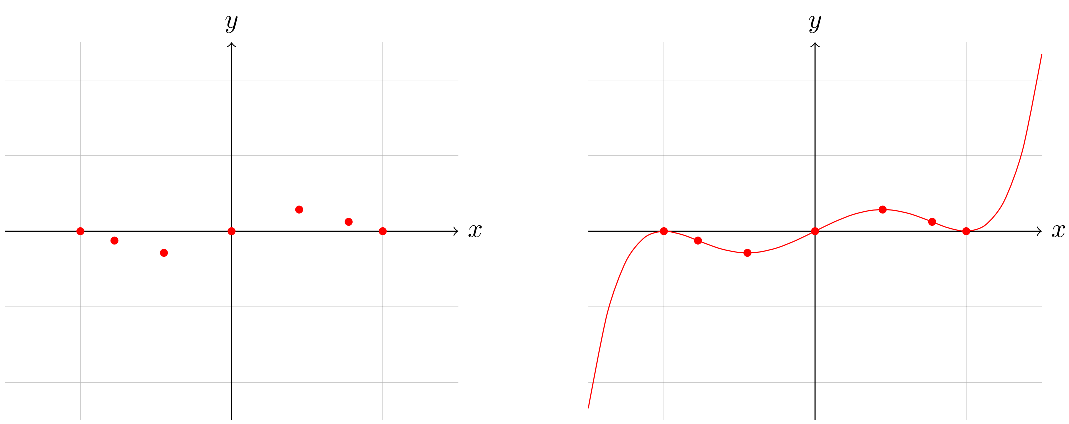
Table 8: Intervals of Concavity for problem 2a

we have h concave down on $(-\infty, -\sqrt{3}/\sqrt{5})$ and $(0, \sqrt{3}/\sqrt{5})$, concave up on $(-\sqrt{3}/\sqrt{5}, 0)$ and $(\sqrt{3}/\sqrt{5}, \infty)$, and that each of the potential inflection numbers is an actual inflection number.

Now we graph the function. We first plot each of the interesting points we found above, by calculating the corresponding y value for the function:

$$
\begin{aligned}
h(-1) &= (-1)^5 - 2(-1)^3 + (-1) = 0 \\
h(-\sqrt{3}/\sqrt{5}) &= -\frac{\sqrt{3}}{\sqrt{5}}\left(\left(-\frac{\sqrt{3}}{\sqrt{5}}\right)^4 - 2\left(-\frac{\sqrt{3}}{\sqrt{5}}\right)^2 + 1\right) = -\frac{\sqrt{3}}{\sqrt{5}}\left(\frac{9}{25} - \frac{6}{5} + 1\right) = -\frac{4\sqrt{3}}{25\sqrt{5}} \\
h(-1/\sqrt{5}) &= -\frac{1}{\sqrt{5}}\left(\left(-\frac{1}{\sqrt{5}}\right)^4 - 2\left(-\frac{1}{\sqrt{5}}\right)^2 + 1\right) = -\frac{1}{\sqrt{5}}\left(\frac{1}{25} - \frac{2}{5} + 1\right) = -\frac{16}{25\sqrt{5}} \\
h(0) &= 0 \\
h(1/\sqrt{5}) &= \frac{1}{\sqrt{5}}\left(\left(\frac{1}{\sqrt{5}}\right)^4 - 2\left(\frac{1}{\sqrt{5}}\right)^2 + 1\right) = \frac{1}{\sqrt{5}}\left(\frac{1}{25} - \frac{2}{5} + 1\right) = \frac{16}{25\sqrt{5}} \\
h(\sqrt{3}/\sqrt{5}) &= \frac{\sqrt{3}}{\sqrt{5}}\left(\left(\frac{\sqrt{3}}{\sqrt{5}}\right)^4 - 2\left(\frac{\sqrt{3}}{\sqrt{5}}\right)^2 + 1\right) = \frac{\sqrt{3}}{\sqrt{5}}\left(\frac{9}{25} - \frac{6}{5} + 1\right) = \frac{4\sqrt{3}}{25\sqrt{5}} \\
h(1) &= (1)^5 - 2(1)^3 + 1 = 0
\end{aligned}
$$

We plot the critical points and inflection points on graph paper as in Figure 1(a) and then join those points with appropriate arcs after consulting the tables of derivatives, as in Figure 1(b). Note that the information in the table of first derivatives is redundant: if you are connecting a point to another point higher and to the right of the first point, the arc must be increasing; the table of first derivatives only needs to be checked to see whether it is consistent with that information. The table of second derivatives, on the other hand, gives new information about whether each arc should be concave up or down.



(a) Critical and Inflection Points (b) Points Joined by Appropriate Arcs

Figure 1: Graphs for problem 2a

(b) The derivatives are

$$
\begin{aligned}
B'(x) &= 3 \cdot \frac{2}{3}x^{-1/3} - 1 = 2x^{-1/3} - 1 \\
B''(x) &= -\frac{2}{3}x^{-4/3}
\end{aligned}
$$

The critical numbers are where $B'(x)$ doesn't exist (at $x = 0$ because we can't take a negative power of 0) and where $B'(x) = 0$:

$$B'(x) = 0 \implies 2x^{-1/3} - 1 = 0 \implies x^{-1/3} = \frac{1}{2} \implies x^{1/3} = 2 \implies x = 2^3 = 8$$

So the list of critical numbers is $c = 0$, $c = 8$. With those critical numbers as boundaries, we obtain the following table:

Interval	$h'(x)$	h
$-\infty < x < 0$	$-$	decreasing
$x = 0$	∞	stationary
$0 < x < 8$	$+$	decreasing
$x = 8$	0	stationary
$8 < x < \infty$	$+$	increasing

Table 9: Intervals of Increase/Decrease for problem 2b

(c) The derivatives are

$$\begin{aligned} G'(x) &= 1 - 4 \cdot \frac{1}{2}x^{-1/2} = 1 - 2x^{-1/2} \\ G''(x) &= -2 \cdot -\frac{1}{2}x^{-3/2} = x^{-3/2} \end{aligned}$$

(d) The derivatives are

$$\begin{aligned} f'(t) &= 1 - \sin t \\ f''(t) &= -\cos t \end{aligned}$$

3. By the product and chain rules, the first derivative of f is

$$f'(x) = 4x^3(x-1)^3 + x^4 \cdot 3(x-1)^2 \frac{d}{dx}(x-1) = 4x^3(x-1)^3 + 3x^4(x-1)^2$$

We will find it helpful (to answer this question and for taking the second derivative) to factor f' by taking out the lowest power of each factor before we go any further:

$$f'(x) = x^3(x-1)^2 [4(x-1) + 3x] = x^3(x-1)^2(7x-4) = 7x^3(x - 4/7)(x-1)^2$$

The critical numbers are the zeros of $f'(x)$, namely $x = 0$, $x = 4/7$, and $x = 1$. Taking the second derivative of f by applying the product rule twice, we have

$$\begin{aligned} f''(x) &= \left(\frac{d}{dx}x^3\right)\left[(x-1)^2(7x-4)\right] + x^3 \frac{d}{dx}\left[(x-1)^2(7x-4)\right] \\ &= \left(\frac{d}{dx}\right)(x-1)^2(7x-4) + x^3\left(\frac{d}{dx}(x-1)^2\right)(7x-4) + x^3(x-1)^2\left(\frac{d}{dx}(7x-4)\right) \\ &= 3x^2(x-1)^2(7x-4) + x^3 \cdot 2(x-1)(7x-4) + x^3(x-1)^2 \cdot 7 \end{aligned}$$

There is no need to simplify f'', but you can if you want by taking out the lowest power of each factor that appears the terms of the above expression:

$$\begin{aligned} f''(x) &= x^2(x-1)\left[3(x-1)(7x-4) + 2x(7x-4) + 7x(x-1)\right] \\ &= x^2(x-1)\left[21x^2 - 33x + 4 + 14x^2 - 8x + 7x^2 - 7x\right] = x^2(x-1)(42x^2 - 48x + 4) \end{aligned}$$

You might be able to factor further, but there is no need.

Applying the second derivative test at the critical number $x = 0$ we have $f''(0) = 0$ so the test is inconclusive there. Similarly, the second derivative test is inconclusive at $x = 1$ because $f''(1) = 0$. However, the second derivative test is conclusive at $x = 4/7$ because

$$f''(4/7) = (4/7)^2(4/7-1)(42(4/7)^2-48(4/7)+4) = (+)(-)(+4)(42/49-48/7+1) = (+)(-)(+)(-) \geq 0$$

so by the second derivative test f has a local minimum at $x = 4/7$.

The first derivative test is better in this case. Using the factors of the first derivative

$$f'(x) = 7x^3(x - 4/7)(x - 1)^2$$

and the critical numbers we obtain Table 10. According to the table, $f(x)$ has a local maximum at

Interval	x^3	$x - 4/7$	$(x - 1)^2$	$f'(x)$	f
$-\infty < x < 0$	$-$	$-$	$+$	$+$	increasing
$x = 0$	0	$-$	$+$	0	stationary
$0 < x < 4/7$	$+$	$-$	$+$	$-$	decreasing
$x = 4/7$	$+$	0	$+$	0	stationary
$4/7 < x < 1$	$+$	$+$	$+$	$+$	increasing
$x = 1$	$+$	$+$	0	0	stationary
$1 < x < \infty$	$+$	$+$	$+$	$+$	increasing

Table 10: Intervals of Increase/Decrease for problem 3

the critical number $x = 0$ (which the second derivative test didn't detect), a local minimum at the critical number $x = 4/7$ (which agrees with the second derivative test) and neither a minimum nor a maximum at the critical number $x = 1$ (where again the second derivative test was inconclusive).

4. We don't know what f is, but we only need f' to construct our usual table for intervals of increase/decrease. The critical numbers are at the zeros of f', namely $x = -1$, $x = 3$, and $x = 6$, which gives us Table 11. Note that the minus signs in the $(x + 1)^2$ and $(x - 6)^4$ columns have flipped

Interval	$(x + 1)^2$	$(x - 3)^5$	$(x - 6)^4$	$f'(x)$	f
$-\infty < x < -1$	$+$	$-$	$+$	$-$	decreasing
$x = -1$	0	$-$	$+$	0	stationary
$-1 < x < 3$	$+$	$-$	$+$	$-$	decreasing
$x = 3$	$+$	0	$+$	0	stationary
$3 < x < 6$	$+$	$+$	$+$	$+$	increasing
$x = 6$	$+$	$+$	0	0	stationary
$6 < x < \infty$	$+$	$+$	$+$	$+$	increasing

Table 11: Intervals of Increase/Decrease for problem 4

to plus signs because the power is an even number in both cases. We conclude that f is increasing on just the intervals $(3, 6)$ and $(6, \infty)$.

5. The derivatives are

$$y' = \frac{(1 + x^2) - (1 + x)(2x)}{(1 + x^2)^2} = (1 - 2x - x^2)(1 + x^2)^{-2}$$
$$y'' = (-2 - 2x)(1 + x^2)^{-2} + (1 - 2x - x^2)(-2)(1 + x^2)^{-3}(2x)$$

Simplifying the second derivative,

$$\begin{aligned} y'' &= -2(1+x^2)^{-3}\left[(x+1)(1+x^2)+(2x)(1-2x-x^2)\right] \\ &= -2(1+x^2)^{-3}\left[x^3+x^2+x+1-2x^3-4x^2+2x\right] \\ &= 2(1+x^2)^{-3}(x^3+3x^2-3x-1) \end{aligned}$$

We need to further factor y''. We guess roots which divide the constant term in the cubic $-x^3-3x^2+3x+1$, namely 1, so our guesses for roots are ± 1. The root $x=1$ works so we can pull out a factor $(x-1)$ by polynomial division or otherwise:

$$y'' = 2(1+x^2)^{-3}(x-1)(x^2+4x+1)$$

We complete factoring by using the quadratic formula to get the roots $x=-2-\sqrt{3}$ and $x=-2+\sqrt{3}$, so we can write

$$y'' = 2(1+x^2)^{-3}(x-(-2-\sqrt{3}))(x-(-2+\sqrt{3}))(x-1)$$

The potential inflection numbers are the numbers x where y'' is undefined (nowhere, since the denominator $(1+x^2)^3$ is never 0) and where $y''=0$, namely $x=-2-\sqrt{3}$, $x=-2+\sqrt{3}$ and $x=1$. You can make a table for the sign of y'' if you wish, but since each of the factors in the numerator is not to an even power, the sign of y'' will change across each of those potential inflection numbers, so the concavity will change at each of those potential inflection numbers, so each of those potential inflection numbers is an actual inflection number. (Make a table if you are confused by the above discussion.)

To find the actual inflection points, we evaluate y at each of the inflection numbers:

$$y(-2-\sqrt{3}) = \frac{1+-2-\sqrt{3}}{1+(-2-\sqrt{3})^2} = \frac{-1-\sqrt{3}}{1+4+4\sqrt{3}+3} = \frac{-1-\sqrt{3}}{8+4\sqrt{3}} = -\frac{1}{4}\frac{1+\sqrt{3}}{2+\sqrt{3}}$$

It will be convenient to rationalize the denominator by multiplying through by the conjugate of the denominator:

$$y(-2-\sqrt{3}) = -\frac{1}{4}\frac{(1+\sqrt{3})(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} = -\frac{1}{4}\frac{-1+\sqrt{3}}{1} = \frac{1}{4}(1-\sqrt{3})$$

Similarly (just changing some of the plus signs to minus signs in the above two calculations) we obtain

$$y(-2+\sqrt{3}) = \frac{1}{4}(1+\sqrt{3})$$

and obviously $y(1)=(1+1)/(1+1)=1$. So the three points of inflection are

$$(-2-\sqrt{3}, 1/4-(1/4)\sqrt{3}),\ (-2+\sqrt{3}, 1/4+(1/4)\sqrt{3}),\text{ and }(1,1)$$

There are various ways to show that three points lie on a straight line, the most straightforward of which is to find the line through two of the points and show that the third point lies on that line. We find the line through the first two inflection points. The slope of the line is

$$m = \frac{y_2-y_1}{x_2-x_1} = \frac{(1/4)(1+\sqrt{3})-(1/4)(1-\sqrt{3})}{(-2+\sqrt{3})-(-2-\sqrt{3})} = \frac{1}{4}\frac{2\sqrt{3}}{2\sqrt{3}} = \frac{1}{4}$$

So one way to write the equation of the line through the first two inflection points is

$$y-y_1 = m(x-x_1) \implies y-\frac{1}{4}(1-\sqrt{3}) = \frac{1}{4}(x-(-2-\sqrt{3}))$$

Obviously the second inflection point lies on that line, so we only need to check whether the third inflection point $(1, 1)$ also lies on it. The left hand side of the equation for the line is

$$1 - \frac{1}{4}(1 - \sqrt{3}) = \frac{3}{4} + \frac{1}{4}\sqrt{3}$$

while the right hand side is

$$\frac{1}{4}(1 - (-2 - \sqrt{3}) = \frac{1}{4}(3 + \sqrt{3}) = \frac{3}{4} + \frac{1}{4}\sqrt{3}$$

Since the LHS and the RHS agree, $(1, 1)$ also lies on the line, so all three points lie on the same straight line.