MATH 110 Lecture 2.6

Implicit Differentiation

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Department of Indigenous Knowledge and Science
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Contents

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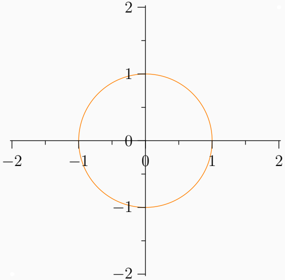
Implicit and Explicit Functions

- Consider the curve defined by $x^2 + y^2 = 1$.



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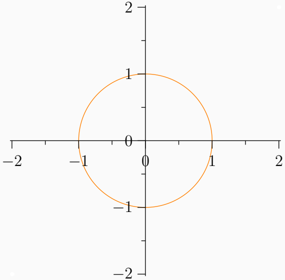

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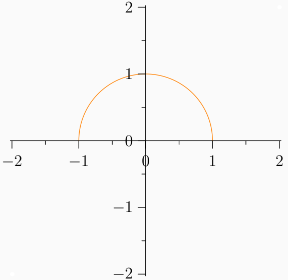

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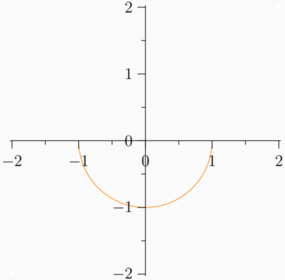

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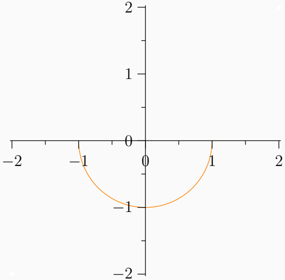

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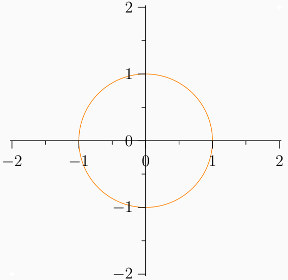
- The formula $x^2 + y^2 = 1$ is an **implicit** definition of those functions.



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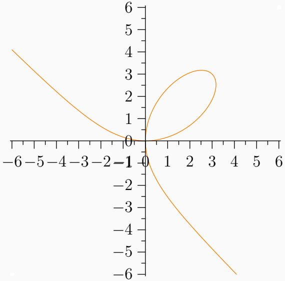

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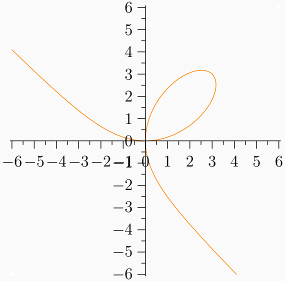

Explicit Functions are Hard to Find

- Consider the curve defined by $x^3 + y^3 = 6xy$.
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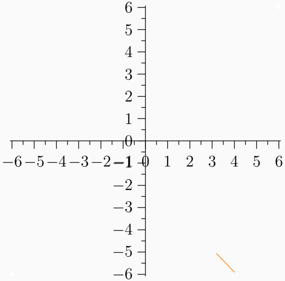
- For example, the simplest explicit function determine by $x^3 + y^3 = 6xy$ is

$$y = \sqrt[3]{-\frac{1}{2}x^3 + \sqrt{\frac{1}{4}x^6 - 8x^3}} + \sqrt[3]{-\frac{1}{2}x^3 - \sqrt{\frac{1}{4}x^6 - 8x^3}}$$



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- The two methods agree. (What happens if $y = -\sqrt{1-x^2}$?)

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- Note that to complete the calculation I had to solve for dy/dx.

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\end{aligned}$$

- We can now construct tangents on the curve, *if we know a point on the curve*. Points on the curve will be given, or special x values where y can be solved explicitly will be chosen, or points on the curve can be approximated by numerical methods.

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- Compare that with finding the second derivative of the explicit function.

Examples

1. Find y' where $x \tan y = y - 1$.
2. Find y'' if $x^6 + y^6 = 1$.
3. Find the equations of the tangent line and normal line to the curve $x^2 + 4xy + y^2 = 13$ at the point $(2,1)$.
4. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.
5. Show that the length of the portion of any tangent line to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ cut off by the coordinate axes is constant.

Exercises

Now you should work on Problem Set 2.6. After you have finished it, you should try the following additional exercises from Section 2.6:

2.6 C-level: 1–4, 5–20, 25–30, 31–32, 33–34, 35–38, 39–40;
B-level: 43, 44–45, 46–47;
A-level: 48, 49–52, 53–54, 55, 57–62