MATH 110-003 200730 Quiz 10 Solutions

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1. **Solution 1**: We first try to find the indefinite integral $\displaystyle\int \frac{x}{\sqrt{1+4x}}\,dx$. We make the substitution $u = 1+4x$, $du = 4\,dx$, $dx = du/4$, $x = (u-1)/4$ to obtain

$$\int \frac{x}{\sqrt{1+4x}}\,dx = \int \frac{(u-1)/4}{u^{1/2}}\,\frac{du}{4} = \frac{1}{16}\int (u^{1/2} - u^{-1/2})\,du = \frac{1}{16}\left(\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} + C\right)$$
$$= \frac{1}{24}(1+4x)^{3/2} - \frac{1}{8}(1+4x)^{1/2} + C$$

You should check the integration by differentiating the final expression above to obtain the integrand. (You will need to factor the lowest power of $(1+4x)$ that occurs in the expression from each term.) Now we can evaluate the definite integral using the Fundamental Theorem of Calculus:

$$\int_2^6 \frac{x}{\sqrt{1+4x}}\,dx = \frac{1}{24}(1+4x)^{3/2} - \frac{1}{8}(1+4x)^{1/2}\bigg|_2^6 = \frac{1}{24}5^3 - \frac{1}{8}5 - \frac{1}{24}3^3 + \frac{1}{8}3$$

You can stop there for full marks, or you can continue with the arithmetic to obtain the final answer $23/6$.

Soluton 2: You can cut down somewhat on the amount of calculation required, at the expense of the ability to check your integration by differentiating, if you change the limits of integration at the same time as you make the substitution. We use the same substitution $u = 1+4x$, $du = 4\,dx$, $dx = du/4$, $x = (u-1)/4$, but now we also note that when $x = 2$, $u = 9$ and when $x = 6$, $u = 25$ to obtain

$$\int_2^6 \frac{x}{\sqrt{1+4x}}\,dx = \int_9^{25} \frac{(u-1)/4}{u^{1/2}}\,\frac{du}{4} = \frac{1}{16}\int_9^{25} (u^{1/2} - u^{-1/2})\,du = \frac{1}{16}\left(\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right)\bigg|_9^{25}$$
$$= \frac{1}{16}\frac{2}{3}5^3 - \frac{1}{16}2\cdot 5 - \frac{1}{16}\frac{2}{3}3^3 + \frac{1}{16}2\cdot 3$$

Again, you can leave the answer as-is or continue with the arithmetic to obtain $23/6$.

Note: other substitutions are possible; you may find that the substitution $u = (1+4x)^{1/2}$ works better, but it is harder to guess.

2. Differentiating both sides of the equation,

$$\frac{d}{dx}\left(6+\int_a^x \frac{f(t)}{t^2}\,dt\right)=\frac{d}{dx}2\sqrt{x}$$

or, by the Fundamental Theorem of Calculus 1,

$$\frac{f(x)}{x^2}=2\frac{1}{2}x^{-1/2}=x^{-1/2}$$

Solving for f gives $f(x)=x^{3/2}$.

Now letting $x=a$ we have

$$6+\int_a^a \frac{f(t)}{t^2}\,dt=2\sqrt{a}$$

or, using the fact that $\int_a^a$ of anything is 0,

$$6=2\sqrt{a} \implies a=9$$

In summary, $f(x)=x^{3/2}$ and $a=9$ makes the equation in the question true for all $x>0$.